A quadratic programming model for topology optimization in electromagnetic casting

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Partial differential equations, optimal design and numerics
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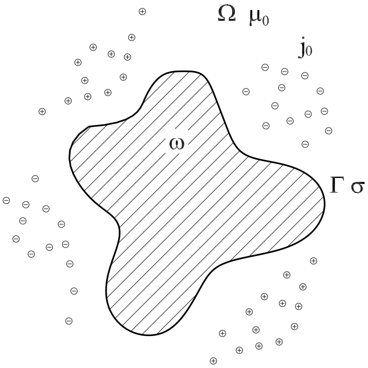
Electromagnetic Casting Problem



We assume that the electric current frequency is so high that the magnetic field penetrates a negligible distance into the liquid metal (skin effect).

Electromagnetic Casting Problem

Magnetic Field Equations

Michel Pierre, Jean R. Roche (1991)

$$\begin{cases} \nabla \times \mathbf{B} &= \mu_0 \mathbf{j}_0 & \text{in } \Omega \\ \nabla \cdot \mathbf{B} &= 0 & \text{in } \Omega \\ \mathbf{B} \cdot n &= 0 & \text{on } \Gamma \\ \|\mathbf{B}\| &= O(\|x\|^{-1}) & \text{as } \|x\| \to \infty \text{ in } \Omega \end{cases} \quad (1)$$

ω: domain occupied by the liquid metal.
Γ: boundary of ω.
$\Omega = \mathbb{R} \setminus \omega$ is the exterior of the liquid metal.
$\mathbf{j}_0 = (0, 0, j_0)$ is the electric current density.
$\mathbf{B} = (B_1, B_2, 0)$ is the magnetic field vector.
μ_0: magnetic permeability of the vacuum.
n: outward-pointing unit normal vector of Γ.

Equilibrium and constraints

In addition to the field equations we have the equilibrium equation:

$$\frac{1}{2\mu_0}\|\mathbf{B}\|^2 + \sigma\,\mathcal{C} = p_0 \quad \text{constant on } \Gamma \tag{1}$$

And the volume constraint:

$$\int_\omega \mathrm{d}\Omega = S_0 \tag{2}$$

We also assume that j_0 has a compact support in Ω and satisfies:

$$\int_\Omega j_0 \,\mathrm{d}\Omega = 0 \tag{3}$$

Magnetic flux function

There exists the magnetic flux funtion $\varphi : \Omega \to \mathbb{R}$ such that $\mathbf{B} = (\frac{\partial \varphi}{\partial x_2}, -\frac{\partial \varphi}{\partial x_1}, 0)$ where φ is solution to the state equation:

$$\begin{cases} -\Delta\varphi & = \mu_0 j_0 & \text{in } \Omega \\ \varphi & = 0 & \text{on } \Gamma \\ \varphi(x) & = c + o(1) & \text{as } \|x\| \to \infty \end{cases} \tag{1}$$

φ has a unique solution in $W_0^1(\Omega) = \{u : \ \rho u \in L^2(\Omega) \text{ and } \nabla u \in L^2(\Omega)\}$ with $\rho(x) = [\sqrt{1 + \|x\|^2} \log(2 + \|x\|^2)]^{-1}$. c is unique in $\mathbb{R}$.

The equilibrium in terms of the flux function φ becomes:

$$\frac{1}{2\mu_0} \left| \frac{\partial \varphi}{\partial n} \right|^2 + \sigma \mathcal{C} = p_0 \quad \text{constant on } \Gamma \tag{2}$$

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Design of inductors

- Find the configuration of inductors to have the liquid metal in equilibrium occupying certain known target domain ω.
- Find j_0 and c such that $\int_\Omega j_0 \, dx = 0$ and that for the target domain ω the solution φ of the state equation

$$\begin{cases} -\Delta\varphi = \mu_0 j_0 & \text{in } \Omega \\ \varphi = 0 & \text{on } \Gamma \\ \varphi(x) = c + o(1) & \text{as } \|x\| \to \infty \end{cases} \tag{1}$$

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$$\frac{1}{2\mu_0}\left|\frac{\partial\varphi}{\partial n}\right|^2 + \sigma\mathcal{C} = p_0 \quad \text{constant on } \Gamma \tag{1}$$

Then:

$$\frac{\partial\varphi}{\partial n} = \varkappa\sqrt{2\mu_0(p_0 - \sigma\mathcal{C})} \quad \text{with} \quad \varkappa = \pm 1\,. \tag{2}$$

Therefore $p_0 \geq \max_\Gamma \sigma\mathcal{C}$. However, $\frac{\partial\varphi}{\partial n}$ is zero at some points, therefore:

$$p_0 = \max_\Gamma \sigma\mathcal{C}\,. \tag{3}$$

Calling $\bar{p} = \sqrt{2\mu_0(p_0 - \sigma\mathcal{C})}$ we have:

$$\frac{\partial\varphi}{\partial n} = \varkappa\,\bar{p} \quad \text{on } \Gamma\,, \tag{4}$$

with the sign changes of $\varkappa$ at the zeros of $(p_0 - \sigma\mathcal{C})$, that is at the points of maximum curvature.

Design of inductors - formulation

We can formulate the problem as: find j_0 and c such that the overdetermined problem

$$\begin{cases} -\Delta\varphi = \mu_0 j_0 & \text{in } \Omega\,, \\ \varphi = 0 & \text{on } \Gamma\,, \\ \dfrac{\partial\varphi}{\partial n} = \varkappa\,\bar{p} & \text{on } \Gamma\,, \\ \varphi(x) = c + o(1) & \text{as } \|x\| \to \infty\,, \end{cases} \tag{1}$$

has a solution $\varphi \in W_0^1(\Omega)$.

We known that for a simply connected ω, with Γ an analitic Jordan curve satisfying a compatibility condition, and $p_0 = \max_\Gamma \sigma\mathcal{C}$ (the maximum must be attained at an even number of points) then (Henrot and Pierre 1989):

(i) there exists a solution for the design problem,
(ii) the solution j_0 is not unique.

Design Problem - formulation

We propose to minimize the Kohn–Vogelius functional:

$$J(\phi) = \frac{1}{2}\|\phi\|^2_{L^2(\Gamma)} = \frac{1}{2}\int_\Gamma |\phi|^2 \, \mathrm{d}s \,, \tag{1}$$

where the auxiliary function ϕ depends implicitly on j_0 and c through the solution of the problem:

$$\begin{cases} -\Delta\phi = \mu_0 j_0 & \text{in } \Omega \,, \\ \dfrac{\partial\phi}{\partial n} = \varkappa \, \bar{p} & \text{on } \Gamma \,, \\ \phi(x) = c + o(1) & \text{as } \|x\| \to \infty \,. \end{cases} \tag{2}$$

There is a compatibility condition:

$$\int_\Gamma \varkappa \, \bar{p} \, \mathrm{d}s = 0 \,, \tag{3}$$

Design Problem - possible formulation

SAND formulation of the design problem:

$$\min_{j_0,\phi,c} \quad J(\phi)\,,$$

$$\text{s.t.} \quad \begin{cases} -\Delta\phi &= \mu_0 j_0 & \text{in } \Omega\,, \\ \dfrac{\partial\phi}{\partial n} &= \varkappa\,\bar{p} & \text{on } \Gamma\,, \\ \phi(x) &= c + o(1) & \text{as } \|x\| \to \infty\,, \end{cases} \tag{1}$$

$$\int_\Omega j_0 \,\mathrm{d}x = 0\,,$$

$$j_0(x) \in \{-I, 0, +I\} \quad \forall x \in \Omega\,,$$

where I is a given constant value for the electric current density.

Design Problem - proposed formulation

We propose the following penalized SAND formulation of the design problem:

$$\min_{j_0,\phi,c} \quad J(\phi) + \rho\psi(j_0)\,,$$

$$\text{s.t.} \quad \begin{cases} -\Delta\phi & = \mu_0 j_0 & \text{in } \Omega\,, \\ \dfrac{\partial\phi}{\partial n} & = \varkappa\,\bar{p} & \text{on } \Gamma\,, \\ \phi(x) & = c + o(1) & \text{as } \|x\| \to \infty\,, \end{cases} \tag{1}$$

$$\int_\Omega j_0 \, \mathrm{d}x = 0\,,$$

$$j_0(x) \in [-I, +I]$$

We relax the last constraint, now I is a given bound for the electric current density, and $\rho\psi(j_0)$ acts as a penalty term $\psi(j_0) = \int_\Omega |j_0| \, \mathrm{d}x$.

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Discretization

$$j_0 = I \sum_{p=1}^{m} \alpha_p \chi_{\Theta_p} , \quad \Theta = \cup_{p=1}^{m} \Theta_p \subset \Omega . \tag{2}$$

$\alpha_p \in [-1, 1]$: dimensionless coefficients (continuous project variables).

$$c(\xi)\phi(\xi) + \int_\Gamma q^* \phi \, \mathrm{d}s - \int_\Gamma u^* \varkappa \bar{p} \, \mathrm{d}s = c + \int_\Omega u^* \mu_0 j_0(x) \, \mathrm{d}x , \tag{3}$$

where u^* is the fundamental solution of the problem,
$u^*(\xi, x) = -\log \|\xi - x\|/(2\pi)$, q^* is the normal derivative of u^*.

After application of the BEM:

$$\mathbf{H}\phi - \mathbf{G}\bar{\mathbf{p}} = c\mathbf{d} + \mathbf{A}\boldsymbol{\alpha} , \tag{4}$$

Discretization

$$J(\phi) = \frac{1}{2}\int_\Gamma \phi^2 \, \mathrm{d}s = \frac{1}{2}\boldsymbol{\phi}^T \mathbf{M}\boldsymbol{\phi}\,, \tag{1}$$

$$\int_\Omega j_0 \, \mathrm{d}s = \mathbf{e}^T \boldsymbol{\alpha}\,, \tag{2}$$

$$\psi(j_0) = \int_\Omega |j_0| \, \mathrm{d}s = \mathbf{e}^T |\boldsymbol{\alpha}|\,, \tag{3}$$

Where **M** is obtained by integrating the interpolation functions and $\mathbf{e}_p = I|\Theta_p|$.

To address the absolute value we use the positive and negative parts: $\boldsymbol{\alpha}_p^+ = \max\{0, \boldsymbol{\alpha}_p\}$ and $\boldsymbol{\alpha}_p^- = \max\{0, -\boldsymbol{\alpha}_p\}$ so that

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}^+ - \boldsymbol{\alpha}^-, \quad \text{and} \quad |\boldsymbol{\alpha}| = \boldsymbol{\alpha}^+ + \boldsymbol{\alpha}^-. \tag{4}$$

Contents

Quadratic programming formulation

$$
\begin{aligned}
\min_{\boldsymbol{\alpha}^+,\boldsymbol{\alpha}^-,\boldsymbol{\phi},c} \quad & \tfrac{1}{2}\boldsymbol{\phi}^T\mathbf{M}\boldsymbol{\phi} + \rho\mathbf{e}^T\left(\boldsymbol{\alpha}^+ + \boldsymbol{\alpha}^-\right)\,, \\
\text{s.t.} \quad & \mathbf{H}\boldsymbol{\phi} - \mathbf{G}\bar{\mathbf{p}} = c\mathbf{d} + \mathbf{A}\left(\boldsymbol{\alpha}^+ - \boldsymbol{\alpha}^-\right)\,, \\
& \mathbf{e}^T\left(\boldsymbol{\alpha}^+ - \boldsymbol{\alpha}^-\right) = 0\,, \\
& 0 \leq \boldsymbol{\alpha}^+ \leq 1\,, \\
& 0 \leq \boldsymbol{\alpha}^- \leq 1\,.
\end{aligned}
\tag{1}
$$

The boundary element matrices **H**, **G** and **A** are full. However if the number of cells is much larger than the number of boundary elements, then Problem (1) is sparse.

We have implemented a simple variable mesh approach: After solving Problem (1), we subdivide the cells whose dimensionless electric current density α_p differs more than a specified tolerance of the corresponding value of any of the adjacent cells.

Quadratic programming formulation

We have formulated the Design of inductors problem in Electromagnetic Casting as a Convex quadratic programming problem:

- There are efficient interior-point techniques of solution.
- The problem is sparse.
- A variable mesh approach was developed.

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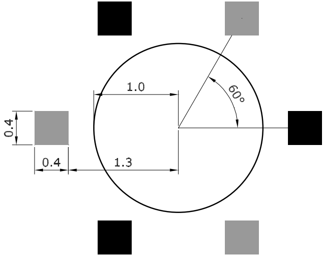
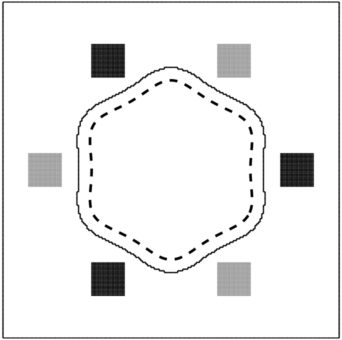
Example 1



(a)



(b)

Figure : Example 1 – (a) initial configuration of the direct free-surface problem, (b) target shape of area $S_0 = \pi$.

Example 1

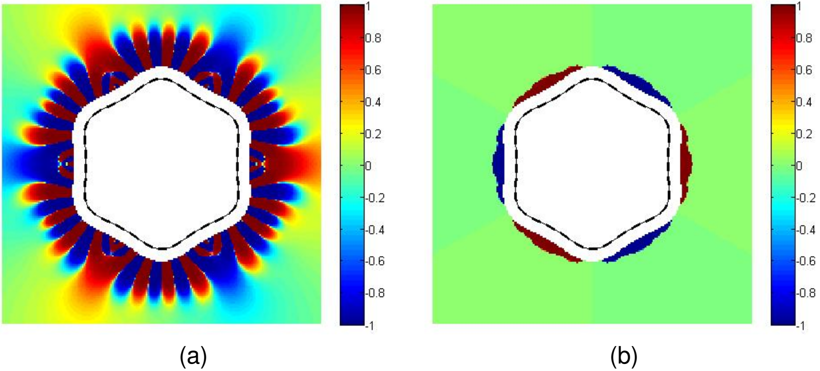
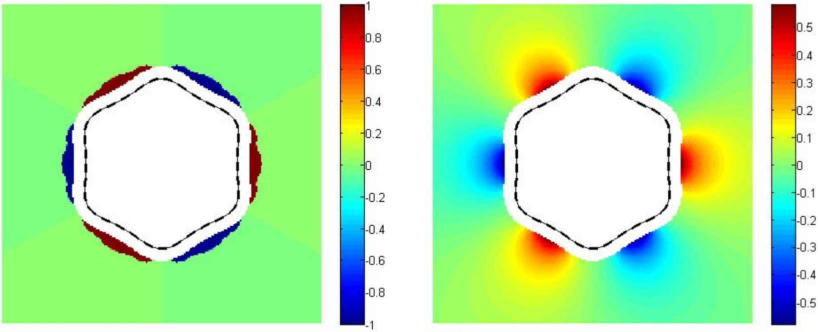


Figure : Example 1 – contour plot of $I^{-1}j_0$, (a) $\rho = 0$, (b) $\rho = 1 \times 10^{-7}$.

Example 1



(a) (b)

Figure : Example 1 – contour plot of $I^{-1} j_0$, (a) penalizing $\|j_0\|_{L_1(\Omega)}$, (b) penalizing $\|j_0\|^2_{L_2(\Omega)}$.

Example 2

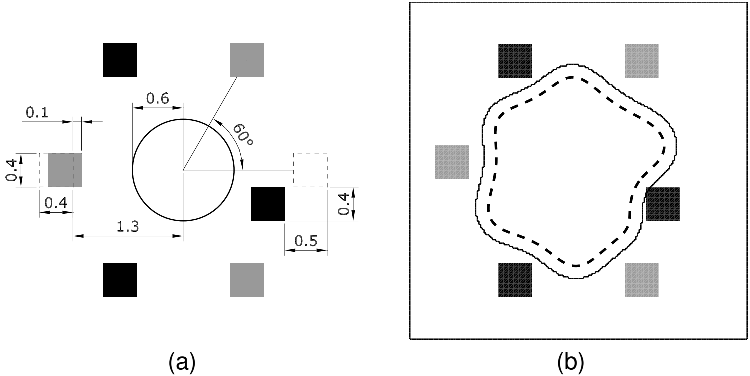
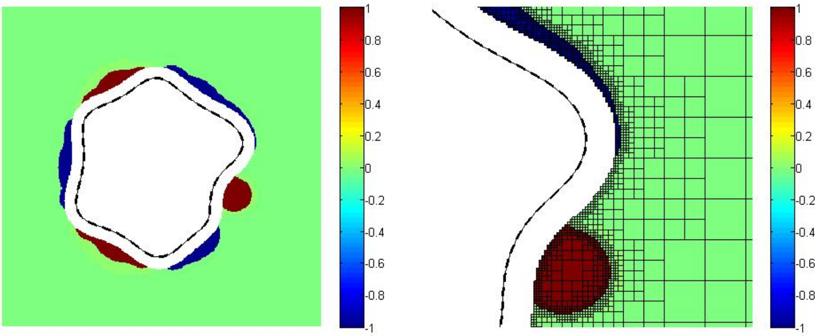


Figure : Example 2 – (a) initial configuration of the direct free-surface problem, (b) target shape of area $S_0 = \pi$.

Example 2



(a) (b)

Figure : Example 2 – contour plot of $I^{-1} j_0$, (a) solution obtained using a fixed mesh of 75433 cells, 15.5 minutes, (b) detail of the solution obtained using a variable mesh of 5728 cells (the finest), 1.5 minutes.

Example 3 – Interior problem

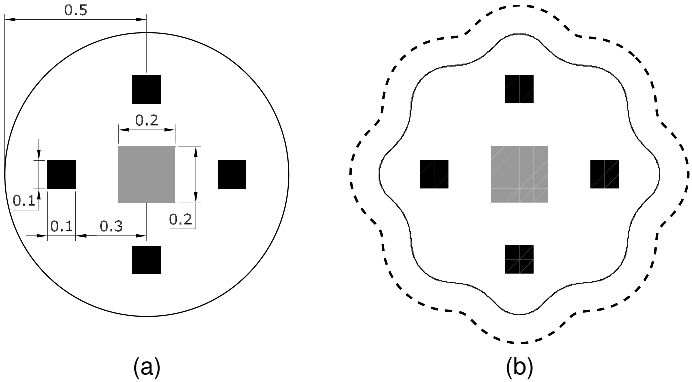


Figure : Example 3 – (a) initial configuration of the direct free-surface problem, (b) target shape of area $S_0 = 1$.

Example 3 – Interior problem

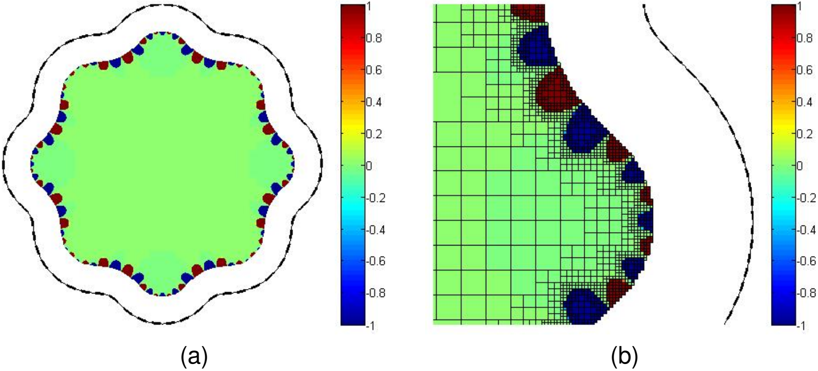
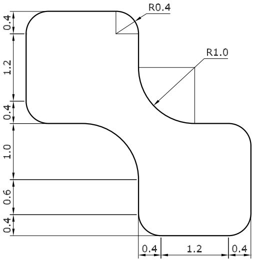
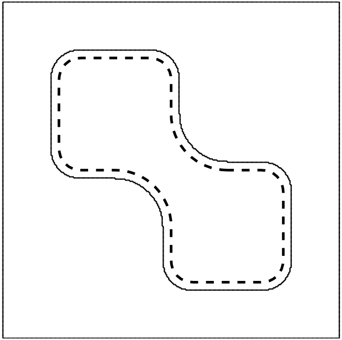


Figure : Example 3 – contour plot of $I^{-1}j_0$.

Example 4



(a)



(b)

Figure : Example 4 – (a) description of the problem geometry, (b) target shape.

Example 4

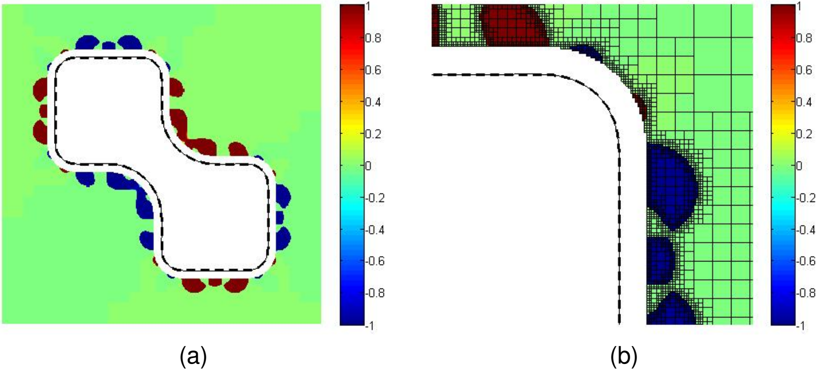
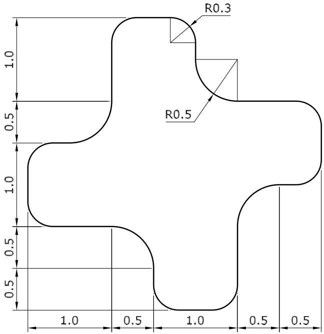
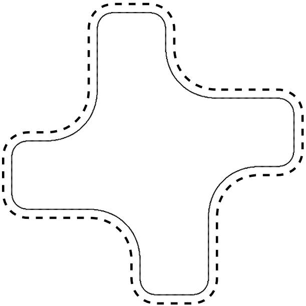


Figure : Example 4 – contour plot of $I^{-1} j_0$.

Example 5



(a)



(b)

Figure : Example 5 – (a) description of the problem geometry, (b) target shape.

Example 5

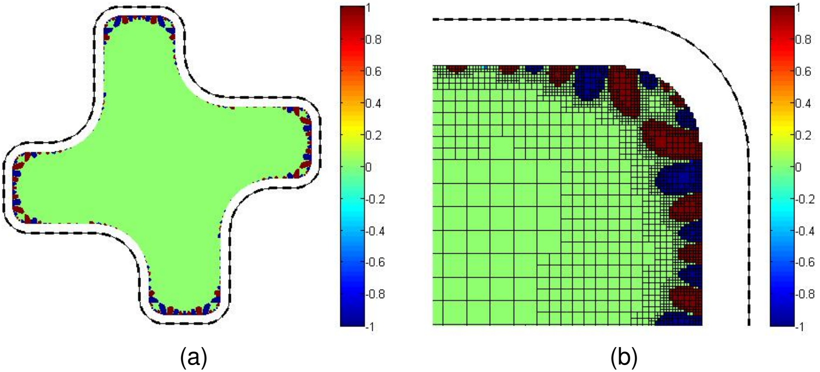


Figure : Example 5 – contour plot of $I^{-1}j_0$.

Contents

Conclusions

- A convex quadratic programming formulation was stated for solving the design of inductors problem in Electromagnetic Casting.
- The problem can be solved efficiently using interior-point optimization algorithms (we have used *quadprog* of MATLAB). In addition, the problem is sparse and a variable mesh approach was developed.
- Some examples were solved successfully.

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