DENSITY OF ACCESSIBILITY FOR PARTIALLY HYPERBOLIC DIFFEOMORPHISMS WITH ONE-DIMENSIONAL CENTER

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Abstract. It is shown that stable accessibility property is C^r-dense among partially hyperbolic diffeomorphisms with one-dimensional center bundle, for $r \geq 2$, volume preserving or not. This answers a conjecture by Pugh and Shub for these systems.

1. Introduction

Partially hyperbolic systems are diffeomorphisms $f\colon M \to M$ with a Tf-invariant splitting $TM = E^s \oplus E^c \oplus E^u$ such that Tf is contracting on E^s, expanding on E^u, and has an intermediate behavior on E^c. For more details, see §2. The "hyperbolic part" of the system, and hence the most relevant dynamical information, is given by the plane fields E^s and E^u. It is natural, then, to study (E^s, E^u) accessibility.

Accessibility is a concept arising from control theory (see for instance [13] and [22]). In this setting, one has two plane fields X, Y and states that the system (X, Y) is *accessible* if one can join any two points in the manifold by a path which is piecewise tangent to either X or Y. See also [17] for an account of this. Essential accessibility is the weaker property that if A and B are measurable sets with positive measure, then some point of A must be joined to some point of B by such a path.

It was Brin and Pesin [1] (see also Sacksteder [21]) who first suggested that accessibility (i.e. (E^s, E^u) accessibility) should be relevant in the context of ergodic theory, more precisely, to study ergodic properties of partially hyperbolic systems.

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Around 1995, Pugh and Shub developed a program to obtain ergodicity for (at least) a C^1 open and C^r dense set of partially hyperbolic systems [16, 17]. More precisely they formulated the following:

CONJECTURE 1. *Stable ergodicity is C^r dense among partially hyperbolic diffeomorphisms for $r \geq 2$*

A stably ergodic diffeomorphism is a C^2 diffeomorphism such that all C^1-perturbations among C^2 volume preserving diffeomorphisms are ergodic.

Pugh and Shub divided this conjecture into two:

CONJECTURE 2. *Essential accessibility implies ergodicity for a C^2 volume preserving partially hyperbolic diffeomorphism.*

CONJECTURE 3. *Stable accessibility is C^r-dense among partially hyperbolic diffeomorphisms, volume preserving or not.*

Conjectures 2 and 3 have been attacked by many authors, and there are now many partial results about them, an account of which may be found, for instance, in [10]. Let us recall some of these advances:

Conjecture 2 was proved by Brin and Pesin in [1] under the additional hypotheses of *dynamical coherence* (that is, unique integrability of the center bundle), a technical condition on the rates of contraction/expansion of the invariant bundles, called *center bunching*, which requires Tf to behave close to conformally on E^c, and Lipschitzness of E^c.

It took another 20 years until, Grayson, Pugh and Shub [9] obtained the first result without using Lipschitzness of E^c by proving Conjecture 2 for perturbations of the time-one map of the geodesic flow of a surface of constant negative curvature. This provided the first non-hyperbolic stably ergodic example. Their result was extended by A. Wilkinson in her PhD thesis to non-constant curvature [23].

In [17] and [18], Pugh and Shub proved Conjecture 2 under the additional hypotheses of dynamical coherence, and a rather strong center bunching condition requiring that the action of Tf on E^c be close to isometric. This result was used, for instance, by Burns, Pugh, Wilkinson in [3] to establish stable ergodicity of time-one maps of Anosov flows that are not suspensions of Anosov diffeomorphisms, and by Burns and Wilkinson in [4] to establish denseness of stable ergodicity among compact Lie group extensions of Anosov diffeomorphisms.

Finally, Burns and Wilkinson in [5] removed the dynamical coherence hypothesis and proved Conjecture 2 with a weaker center bunching hypothesis, proving the strongest version of the conjecture known so far. This center bunching hypothesis requires that the action of Tf on E^c be close enough to conformal, and is always satisfied if $\dim E^c = 1$. Another proof of the conjecture in the case $\dim E^c = 1$ was obtained by Rodriguez Hertz, Rodriguez Hertz and Ures in [19].

With respect to Conjecture 3, Brin and Pesin showed in [1] that the time-one map of a k-frame flow on a manifold of constant negative curvature has the accessibility property. Grayson, Pugh and Shub [9] and Wilkinson [23] proved the accessibility property for perturbations of the time one map of the geodesic flow of negatively curved surfaces. Katok and Kononenko [12] proved accessibility for all diffeomorphisms such that $E^s \oplus E^u$ is a contact bundle, and for certain perturbations, in particular, for C^2 perturbations of time-one maps of contact Anosov flows. This was used by Pugh and Shub in [17] to show that the time-one maps of the geodesic flows of manifolds of negative sectional curvature are stably accessible. In [3], Burns, Pugh and Wilkinson proved that the time-one map of any 3-dimensional Anosov flow that is not a suspension is stably accessible.

In [15], Niţică and Török found a C^r-dense set of stably accessible diffeomorphisms among the following ones: r-normally hyperbolic diffeomorphisms with one-dimensional center distribution, having two close compact periodic leaves, volume preserving or not. Dolgopyat and Wilkinson proved Conjecture 3 with C^r density weakened to C^1 density [7]. When the center distribution is one-dimensional, Didier showed that accessibility is C^1-open [6].

Rodriguez Hertz, Rodriguez Hertz and Ures found a C^∞-dense set of stably accessible diffeomorphisms among the C^r volume preserving partially hyperbolic diffeomorphisms with one-dimensional center distribution, proving the volume preserving part of Conjecture 3 for this case [19].

In this paper we extend the arguments in [19] to show that accessible diffeomorphisms are C^r dense in the space of all C^r partially hyperbolic diffeomorphisms with one-dimensional center, thereby completing the proof of Conjecture 3 for the case of one-dimensional center.

2. Preliminaries

Let $f\colon M \to M$ be a diffeomorphism of a compact manifold M. We say that f is *partially hyperbolic* if the following holds. First, there is a nontrivial splitting of the tangent bundle, $TM = E^s \oplus E^c \oplus E^u$ that is invariant under the derivative map Tf. Further, there is a Riemannian metric for which we can choose continuous positive functions ν, $\hat\nu$, γ and $\hat\gamma$ with

$$\nu, \hat\nu < 1 \quad \text{and} \quad \nu < \gamma < \hat\gamma^{-1} < \hat\nu^{-1} \tag{2.1}$$

such that, for any unit vector $v \in T_pM$,

$$\|Tfv\| < \nu(p), \qquad \text{if } v \in E^s(p), \tag{2.2}$$

$$\gamma(p) < \|Tfv\| < \hat{\gamma}(p)^{-1}, \qquad \text{if } v \in E^c(p), \tag{2.3}$$

$$\hat{\nu}(p)^{-1} < \|Tfv\|, \qquad \text{if } v \in E^u(p). \tag{2.4}$$

Denote by $PHD_1^r(M)$ the set of (not necessarily volume-preserving) C^r partially hyperbolic diffeomorphisms of M with 1-dimensional center distribution. Unless otherwise specified we give $PHD_1^r(M)$ the C^r topology. It is convenient to let s, c and u denote the dimensions of E^s, E^c, and E^u, respectively. When necessary we use a subscript to indicate the dependence of the bundles on the diffeomorphism.

We say that f is *center bunched* if the functions $\nu, \hat{\nu}, \gamma$, and $\hat{\gamma}$ can be chosen so that:

$$\max\{\nu, \hat{\nu}\} < \gamma\hat{\gamma}. \tag{2.5}$$

Center bunching means that the hyperbolicity of f dominates the nonconformality of Tf on the center. Inequality (2.5) always holds when $Tf|_{E^c}$ is conformal. For then we have $\|T_pfv\| = \|T_pf|_{E^c(p)}\|$ for any unit vector $v \in E^c(p)$, and hence we can choose $\gamma(p)$ slightly smaller and $\hat{\gamma}(p)^{-1}$ slightly bigger than

$$\|T_pf|_{E^c(p)}\|.$$

By doing this we may make the ratio $\gamma(p)/\hat{\gamma}(p)^{-1} = \gamma(p)\hat{\gamma}(p)$ arbitrarily close to 1, and hence larger than both $\nu(p)$ and $\hat{\nu}(p)$. In particular, center bunching holds whenever E^c is one-dimensional.

The bundles E^u and E^s are uniquely integrable. As usual $\mathcal{W}^u$ and $\mathcal{W}^s$ will denote the foliations to which they are tangent. There are partially hyperbolic diffeomorphisms for which E^c is not integrable, but none of the known examples has one dimensional center. The question of whether the center distribution must be uniquely integrable if it is one dimensional is still open, even for partially hyperbolic diffeomorphisms of three dimensional manifolds.

We assume that we have a Riemannian metric on M adapted to f so that the inequalities at the beginning of this section hold. Distance with respect to this metric will be denoted by $d(\cdot, \cdot)$.

If $\mathcal{W}$ is a foliation of M, $\mathcal{W}_\rho(x)$ will denote the set of points that can be reached from x by a C^1 path of length less than ρ tangent to the foliation; this set is a disc for small enough ρ. We define $\mathcal{W}_{loc}(x)$ to be $\mathcal{W}_R(x)$ for a suitably small R. The radii such as ϵ and δ considered in the paper are, of course, much smaller than R.

3. Accessibility

A partially hyperbolic diffeomorphism has the accessibility property if any two points are joined by a *us-path.* A *us*-path from x to y is a finite sequence of points $z_0, \dots, z_m$ such that $z_0 = x$, $z_m = y$ and $z_i \in \mathcal{W}^u(z_{i-1}) \cup \mathcal{W}^s(z_{i-1})$ for $1 \leq i \leq m$.

Denote by $\mathcal{A}$ the set of all diffeomorphisms in $PHD_1^r(M)$ with the accessibility property. Didier [6] showed that $\mathcal{A}$ is a C^1 open subset of $PHD_1^r(M)$. Note that the assumption of one dimensional center is crucial in Didier's work. It is not known whether accessibility is an open property when the center is higher dimensional.

In this paper we prove the following result.

Theorem 1. *$\mathcal{A}$ is C^r dense in $PHD_1^r(M)$.*

We extend the arguments in [19] where the analogous result is proved for the subspace of volume preserving diffeomorphisms in $PHD_1^r(M)$. The proof in this paper can also be adapted to the volume preserving case; all of the perturbations that we need can be made in a volume preserving way. Together with Didier's result, Theorem 1 and its analogue in [19] establish the conjecture of Pugh and Shub about the density of accessibility (Conjecture 3) in the case when the center bundle E^c is one dimensional.

Given a diffeomorphism $f \in PHD_1^r(M)$, the accessibility class $AC(x, f)$ of a point x is the set of all points that can be joined to x by us-paths for f.

We denote by $\Gamma(f)$ the set of points $x \in M$ for which the accessibility class is not open. The map f has the accessibility property if and only if $\Gamma(f) = \emptyset$. The set $\Gamma(f)$ is a compact invariant subset of M.

Denote by $\mathcal{K}(M)$ the set of compact subsets of M with Hausdorff distance. In Section 4 we prove the following:

Theorem 2. *The map $\Gamma\colon PHD_1^r(M) \to \mathcal{K}(M)$ is upper-semicontinuous with respect to the C^r topology on $PHD_1^r(M)$.*

Upper semi-continuity means that if $f_n \to f$, $x_n \to x$ in M and $x_n \in \Gamma(f_n)$ for each n, then $x \in \Gamma(f)$.

Remark 3. Recall that $\Gamma(f) = \emptyset$ is equivalent to accessibility of f. Theorem 2 above implies Didier's result in [6] that the set of diffeomorphisms in $PHD_1^r(M)$ with the accessibility property is open. Indeed, if this were not the case, we could find $f \in PHD_1^r(M)$ with $\Gamma(f) = \emptyset$ and a sequence $f_n \to f$ in $PHD_1^r(M)$ such that $\Gamma(f_n) \neq \emptyset$ for each n. But then Theorem 2 and compactness of M give $\Gamma(f) \neq \emptyset$, which is a contradiction.

It is a classical result that the set of continuity points of an upper-semicontinous function such as Γ is dense; see e.g. §39.IV.2 in [11]. Theorem 1 now follows immediately from the next result.

Theorem 4. *If f is a continuity point of Γ, then $\Gamma(f) = \emptyset$.*

From Section 5 on, this paper is dedicated to proving Theorem 4. Here is an outline of its proof:

In the first place, we show that there is a C^r dense set of diffeomorphisms of $PHD^r_1(M)$ for which the accessibility class of every periodic point is open, this is, $\Gamma(g) \cap \mathrm{Per}(g) = \emptyset$ for a C^r dense set of $g \in PHD^r_1(M)$ (Proposition 12). In order to get this dense set we use an unweaving method (see Lemma 11), which allows us to break up the joint integrability of E^s and E^u on periodic orbits. In this way, we "open" the accessibility class of a periodic point by means of a C^r small perturbation. The unweaving method, in turn, is based on the Keepaway Lemma (Lemma 9) which may be found in Section 5.

On the other hand, in Section 8, we assume there exists a continuity point f of Γ with $\Gamma(f) \neq \emptyset$. Under this hypothesis, we find an open set $\mathcal{N}$ in $PHD^r_1(M)$ such that every $h \in \mathcal{N}$ has a periodic point with nonopen accessibility class, that is, $\Gamma(h) \cap \mathrm{Per}(h) \neq \emptyset$ for every $h \in \mathcal{N}$ (Lemma 16). Therefore, we obtain a contradiction.

4. Theorem 2

Let us say that a diffeomorphism f in $PHD^r_1(M)$ is *jointly integrable* at a point $x \in M$ if there exists $\epsilon > 0$ such that for all $y \in \mathcal{W}^u_\epsilon(x)$ and $z \in \mathcal{W}^s_\epsilon(x)$ we have

$$\mathcal{W}^s_{loc}(y) \cap \mathcal{W}^u_{loc}(z) \neq \emptyset.$$

See Figure 1.

Proposition 5. *If $x \in \Gamma(f)$, then f is jointly integrable at x.*

Proof. Lemma 2 of [6]; see also Remark 3.1 of [19]. □

Conversely, if $x \notin \Gamma(f)$, we have a point of non-joint integrability in $\Gamma(f)$.

Proposition 6. *If $x \notin \Gamma(f)$, there is a point in $AC(x, f)$ at which f is not jointly integrable.*

Proof. See Proposition 3 of [6] in the case where $\Gamma(f)$ is empty and Section 3.1 of [19] for the general case. □

Proposition 7. *For $x \notin \Gamma(f)$ there exist a C^1-neighborhood $\mathcal{U}$ of f, and $\epsilon > 0$ such that*

$$B(x, \epsilon) \subset AC(x, g) \qquad \text{for all} \quad g \in \mathcal{U}.$$

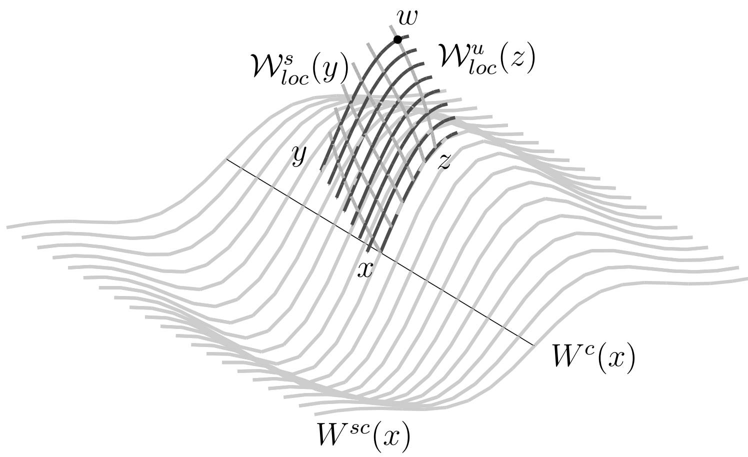

FIGURE 1. f is jointly integrable at x

Proof. Combining Proposition 6 above with Proposition 5 of [6], we obtain that for $x \notin \Gamma(f)$ there exist a C^1-neighborhood $\mathcal{V}$ of f, $\alpha > 0$ and $z \in M$ such that $B(z, \alpha) \subset AC(x, g)$ for all $g \in \mathcal{V}$.

Strong foliations vary continuously, in the following sense: if $x_n \to x$ in M, and $f_n \to f$ in the C^1-topology, then for each $\rho > 0$

$$\mathcal{W}^s_{f_n,\rho}(x_n) \to \mathcal{W}^s_{f,\rho}(x) \quad \text{and} \quad \mathcal{W}^u_{f_n,\rho}(x_n) \to \mathcal{W}^u_{f,\rho}(x) \quad \text{in} \quad \mathcal{K}(M). \tag{4.6}$$

Let $z = y_1, y_2, \ldots, y_m = x$ be a *us*-path for f joining z to x such that $y_i \in \mathcal{W}^s_{f,\rho}(y_{i-1}) \cup \mathcal{W}^s_{f,\rho}(y_{i-1})$ for all i. An inductive argument using (4.6) shows that if we start with any small enough $\epsilon_1 > 0$ we can find C^1-neighborhoods $\mathcal{V}_i$ of f and $\epsilon_i > 0$ such that for $2 \le i \le m$ we have

$$B(y_i, \epsilon_i) \subset \mathcal{W}^s_{g,\rho}(B(y_{i-1}, \epsilon_{i-1})) \cup \mathcal{W}^u_{g,\rho}(B(y_{i-1}, \epsilon_{i-1})) \quad \text{for all} \quad g \in \mathcal{V}_i.$$

We obtain the proposition by considering $\epsilon_1 < \alpha$ and taking $\epsilon = \epsilon_m$ and $\mathcal{U} = \bigcap_{i=1}^m \mathcal{V}_i$. □

Proof of Theorem 2. Let us note that the accessibility class of a point x is open if it has non-empty interior (see for instance [19, Proposition A.4.]). If $x \notin \Gamma(f)$, then for all $y \in B(x, \epsilon)$, and all $g \in \mathcal{U}$ as above, we have $B(x, \epsilon) \subset AC(y, g)$, so $y \notin \Gamma(g)$. This proves Theorem 2. □

Finally, let us emphasize the fact that the compact invariant set $\Gamma(f)$ is a lamination [19, Proposition A.3], whose laminae are codimension one immersed submanifolds, everywhere tangent to $E^s \oplus E^u$ [6, Proposition 3]. Each lamina is an accessibility class. We will denote by $\Gamma(x,f)$ the lamina of $\Gamma(f)$ that contains x. Note that $\Gamma(x,f) = AC(x,f)$ for $x \in \Gamma(f)$.

Now, compactness of $\Gamma(f)$ and semicontinuity of the operator Γ guarantee the uniformity of jointly integrability in a C^r-neighborhood of f:

Proposition 8. *For $f \in PHD_1^r(M)$, there are a C^r neighborhood $\mathcal{U}_0$ and an $\epsilon > 0$ such that if $x \in \Gamma(g)$, $y \in \mathcal{W}^u_{g,\epsilon}(x)$ and $z \in \mathcal{W}^s_{g,\epsilon}(x)$ then*

$$\mathcal{W}^s_{g,loc}(y) \cap \mathcal{W}^u_{g,loc}(z) \neq \emptyset$$

for all $g \in \mathcal{U}_0$. See Figure 1.

5. The Keepaway Lemma

Let f be a diffeomorphism preserving a foliation $\mathcal{W}$ tangent to a continuous sub-bundle E of TM. Call $\mathcal{W}(x)$ the leaf of $\mathcal{W}$ through x and $\mathcal{W}_\epsilon(x)$ the set of points that are reached from x by a curve contained in $\mathcal{W}(x)$ of length less than ϵ. We are interested in the case where the bundle E is uniformly expanded by Tf. This means that there is a constant $\mu < 1$ such that $\|Tf^{-1}|_E\| < \mu < 1$.

The following lemma was already proved by R. Mañé [14, Lemma 5.2.] when the dimension of E is 1. The general case is presented in [19]. We reproduce the proof since it is quite short and the lemma is fundamental to this paper.

Given a (small) embedded manifold V transverse to $\mathcal{W}$ whose dimension equals the codimension of E and $\delta > 0$, define

$$B_\delta(V) = \bigcup_{y \in V} \mathcal{W}_\delta(y).$$

We will always assume that V and δ are chosen so that the discs $\mathcal{W}_{5\delta}(y)$ for $y \in V$ are pairwise disjoint. There is no need for V to be connected.

Lemma 9 (Keepaway Lemma)**.** *Assume that the bundle E is uniformly expanded by Tf, i.e. there is a constant $\mu < 1$ such that $\|Tf^{-1}|_E\| < \mu < 1$. Let $N > 0$ be such that $\mu^{-N} > 5$ and let V be a small manifold transverse to $\mathcal{W}$ whose dimension is complementary to that of the leaves of $\mathcal{W}$. Suppose that for some $\epsilon > 0$ we have*

$$f^n(B_{5\epsilon}(V)) \cap B_\varepsilon(V) = \emptyset \qquad \text{for } n = 1, \dots, N.$$

Then for each $x \in M$ there is a point $z \in \mathcal{W}_\varepsilon(x)$ such that $f^n(z) \notin B_\varepsilon(V)$ for all $n \geq 1$.

Proof. We shall construct a sequence of closed discs $D_0, D_1, D_2, \dots$ starting from $D_0 = \overline{\mathcal{W}_\epsilon(x)}$ such that $f^{-1}(D_n) \subset D_{n-1}$ for all $n > 0$ and $D_n \cap B_\epsilon(V) = \emptyset$.

Then z can be chosen to be any point in

$$\bigcap_{n=0}^{\infty} f^{-n}(D_n).$$

In fact this intersection will consist of a unique point in our construction.

Observe that for any $\delta > 0$ and any point $p \in M$ we have

$$\overline{\mathcal{W}_\delta(f(p))} \subset \mathcal{W}_{\delta/\mu}(f(p)) \subset f(\mathcal{W}_\delta(p)). \tag{5.7}$$

The construction is as follows:

(0) Set $D_0 = \overline{\mathcal{W}_\epsilon(w_0)}$, where $w_0 = x$.
(1) If $n < N$, put $D_n = f^n(D_0)$.
(2) For the N^{th} iterate, observe that we still have $f^N(D_0) \cap B_\epsilon(V) = \emptyset$ and now we also know that $f^N(D_0)$ contains the round ball of radius 5ϵ centered at $f^N(w)$, that is,

$$\overline{\mathcal{W}_{5\epsilon}(f^N(w))} \subset f^N(D_0).$$

We set $D_N = \overline{\mathcal{W}_{5\epsilon}(f^N(w_0))}$.
(3) For $n > N$, we continue to set $D_n = \overline{\mathcal{W}_{5\epsilon}(f^n(w))}$ until we reach $n = n_1$, where n_1 is the first n such that

$$\overline{\mathcal{W}_{5\epsilon}(f^{n_1}(w_0))} \cap B_\epsilon(V) \neq \emptyset.$$

We get $D_n \subset f(D_{n-1})$ for $N < n < n_1$ from (5.7) with $\delta = 5\epsilon$ and $p = f^{n-1}(w_0)$.
(4) For the n_1^{th} iterate, we do not take $\overline{\mathcal{W}_{5\epsilon}(f^{n_1}(w_0))}$, since this disc intersects $B_\epsilon(V)$. But there is a point $w_{n_1} \in \overline{\mathcal{W}_{4\epsilon}(f^{n_1}(w_0))}$ such that $\overline{\mathcal{W}_\epsilon(w_{n_1})} \subset B_{5\epsilon}(V) \setminus B_\epsilon(V)$. Indeed, let $y_{n_1} \in V$ be the center of the leaf in $B_\epsilon(V)$ that intersects $\overline{\mathcal{W}_{5\epsilon}(f^{n_1}(w_0))}$. If y_{n_1} lies outside $\overline{\mathcal{W}_{2\epsilon}(f^{n_1}(w))}$, then we can take $w_{n_1} = f^{n_1}(w_0)$. If y_{n_1} lies outside $\overline{\mathcal{W}_{2\epsilon}(f^{n_1}(w))}$, then w_{n_1} can be any point in $\overline{\mathcal{W}_{5\epsilon}(f^{n_1}(w_0))}$ whose distance from $f^{n_1}(w_0))$ is 4ϵ.

Choose $D_{n_1} = \overline{\mathcal{W}_\epsilon(w_{n_1})}$. We get $D_{n_1} \subset f(D_{n_1-1})$ using (5.7) with $\delta = 5\epsilon$ and $p = f^{n_1-1}(w_0)$ and the fact that $D_{n_1} \subset \overline{\mathcal{W}_{5\epsilon}(f^{n_1}(w_0))}$.
(5) Now, go to Step 1, replace D_0 by D_{n_1}, and continue the construction with the obvious modifications.

This algorithm gives the desired sequence of discs, and then the point z, proving the lemma. □

The Keepaway Lemma gives us an abundance of nonrecurrent points.

Corollary 10. *Suppose $f\colon M \to M$ has an invariant foliation $\mathcal{W}$ tangent to a bundle that is uniformly expanded by Tf. Then the set $\{z : z \notin \omega(z)\}$ of points that are nonrecurrent in the future is dense in every leaf of $\mathcal{W}$.*

Proof. Let y be a point in M. If y is not periodic, we can choose a transversal V to $\mathcal{W}$ that passes through y such that the hypothesis of the previous lemma is satisfied for any small enough $\epsilon > 0$; the lemma then gives us a point $z \in \mathcal{W}_\epsilon(y)$ that is not forward recurrent. If y is periodic, no other point of $\mathcal{W}(y)$ is recurrent, so for any small $\epsilon > 0$ we can choose a nonrecurrent point $y' \in \mathcal{W}_{\epsilon/2}(y)$ and then find a point $z \in \mathcal{W}_{\epsilon/2}(y')$ that is not forward recurrent. □

6. Unweaving

The results of the previous section allow us to break up integrability of E^u and E^s.

Lemma 11. *Let $K \subset \Gamma(f)$ be a minimal set for a diffeomorphism $f \in PHD_1^r(M)$. Then we can find $g \in PHD_1^r(M)$ as close to f in the C^r topology as we wish such that $f|_K = g|_K$ and $AC(x, g)$ is open for some point $x \in K$.*

Note that this lemma includes the case when K is a periodic orbit, which was treated in [19].

Proof. We construct g by perturbing f in the complement of the closed f-invariant set K. This ensures that K remains invariant under g.

The construction is an application of the Brin quadrilateral argument. We choose a closed us-quadrilateral with corners x, y, z, w such that $x \in K$ and $y \notin K$, as in Figure 1. The quadrilateral is constructed so that there are radii $\rho, \rho_1, \rho_2, \rho_3, \rho_4 > 0$ such that $B(y, \rho) \cap K = \emptyset$ and:

(1) $w \in \overline{\mathcal{W}^s_{\rho_1}(y)}$ and $f^n(\overline{\mathcal{W}^s_{\rho_1}(y)}) \cap B(y, \rho) = \emptyset$ for any $n \geq 1$;
(2) $y \in \overline{\mathcal{W}^u_{\rho_2}(x)}$ and $f^{-n}(\overline{\mathcal{W}^u_{\rho_2}(x)}) \cap B(y, \rho) = \emptyset$ for any $n \geq 1$;
(3) $z \in \overline{\mathcal{W}^s_{\rho_3}(x)}$ and $f^n(\overline{\mathcal{W}^s_{\rho_3}(x)}) \cap B(y, \rho) = \emptyset$ for any $n \geq 0$;
(4) $w \in \overline{\mathcal{W}^u_{\rho_4}(z)}$ and $f^{-n}(\overline{\mathcal{W}^u_{\rho_4}(z)}) \cap B(y, \rho) = \emptyset$ for any $n \geq 0$.

A perturbation that changes f only inside $B(y, \rho)$ leaves x, z and w joined by a us-path. It is easy to break the us-connection from x to w through y by composing f with a "push" in the central direction that is restricted to $B(y, \rho)$ (see Figure 2).

In order to create the desired quadrilateral, we first choose $x_0 \in K$. We can then apply Corollary 10 with $\mathcal{W} = \mathcal{W}^u$ to find a point $y \in \mathcal{W}^u_{loc}(x_0)$ that is not forward recurrent and is as close to x as we wish. We make sure that $y \in \mathcal{W}^u_\beta(x_0)$, where β is very small compared to the radius R of the local stable and unstable manifolds. Since y is not forward recurrent, it does not belong to the minimal set K. Choose $\delta > 0$ small enough so that $K \cap B(y, \delta) = \emptyset$ and $f^n(y) \notin B(y, \delta)$ for $n \geq 1$.

We now choose the point x. It must belong to $\mathcal{W}^u_{loc}(y) \cap K$ and have the property that no other point of $\mathcal{W}^u_{loc}(y) \cap K$ is closer to y. Note that $f^n(x) \notin B(y, \delta)$ for all n.

Next we apply Lemma 9 with f replaced by f^{-1} to choose a point $z \in \mathcal{W}^s_{loc}(x)$ very close to x. We choose V to be a disc tranverse to $\mathcal{W}^s$ that contains $\mathcal{W}^u_{loc}(x)$ and $\epsilon \ll \delta$ small enough so that the stable discs of radius 5ϵ centered at points of V are pairwise disjoint. We may assume that V was chosen so that $B_\epsilon(V)$ contains $\mathcal{W}^u_{2\beta}(y')$ for all points y' close enough to y. Lemma 9 gives us a point $z \in \mathcal{W}^s_\epsilon(x)$ whose backward orbit under f avoids $B_\epsilon(V)$.

Since $x \in K \subset \Gamma(f)$, the points y and z are in the immersed codimension one submanifold $\Gamma(x, f)$ and $\mathcal{W}^s_{loc}(y)$ and $\mathcal{W}^u_{loc}(z)$ intersect in a unique point w. We may assume that ϵ was chosen small enough so that $w \in \mathcal{W}^u_{2\beta}(z)$. We now verify properties (1)–(4) above.

Let ρ_3 be the distance in $\mathcal{W}^s_{loc}(x)$ from x to z. Then $\rho_3 \leq \epsilon \ll \delta$. The stable manifold $\mathcal{W}^s_{loc}(x)$ contracts under forward iteration of f, so we have $d(f^n(x), f^n(z)) \ll \delta$ for all $n \geq 0$. Since $f^n(x) \notin B(y, \delta)$ for $n \geq 0$, we see that (3) holds as long as $\rho < \delta/2$.

The proof of (1) is similar. Let ρ_1 be the distance in $\mathcal{W}^s_{loc}(y)$ from y to w. We may assume that ϵ was chosen small enough so that $\rho_1 \ll \delta$. Since iteration of f contracts $\mathcal{W}^s_{loc}(y)$ and $f^n(y) \notin B(y, \delta)$ for $n \geq 1$, we see as before that (1) will hold if $\rho < \delta/2$.

To prove (4), let ρ_4 be the distance in $\mathcal{W}^u_{loc}(z)$ from z to w. Then $\rho_4 < 2\beta$. Since $\mathcal{W}^u_{loc}(z)$ contracts under iteration of f^{-1}, we see from the choices made above that (4) will hold if ρ is small enough.

Finally let ρ_2 be the distance in $\mathcal{W}^u_{loc}(x)$ from x to y. Then $\beta \geq \rho_2 > \delta$. Iteration of f^{-1} contracts $\mathcal{W}^u_{\rho_2}(x)$. Choose n_0 so that the diameter of $f^{-n}(\mathcal{W}^u_{\rho_2}(x))$ is less than $\delta/2$ for $n \geq n_0$. Then $f^{-n}((\overline{\mathcal{W}^u_{\rho_2}(x)}) \cap B_{\delta/2}(y) = \emptyset$ for $n \geq n_0$, since otherwise $f^{-n}(x)$ would be a point of K in $B_\delta(y)$.

On the other hand, there is $\rho_2' < \rho$ such that such that $f^{-n}((\overline{\mathcal{W}^u_{\rho_2}(x)}) \subset \mathcal{W}^u_{\rho_2'}(y)$ for all $n \geq 1$. This means that $f^{-n}((\overline{\mathcal{W}^u_{\rho_2}(x)}) \cap \overline{\mathcal{W}^u_{\rho_2 - \rho_2'}(y)} = \emptyset$, for otherwise $f^{-n}(x)$ would be a point of $K \cap \mathcal{W}^u_{loc}(y)$ closer to y than x. It now follows from a compactness argument that we can choose a positive $\rho < \delta/2$ such that $f^{-n}((\overline{\mathcal{W}^u_{\rho_2}(x)}) \cap B_\rho(y) = \emptyset$ for $1 \leq n \leq n_0$. Property (2) holds for any such $\rho > 0$.

Let us consider a C^r perturbation of f of the form $g = f \circ h$, where $\text{supp}(h) \subset B(y, \rho)$ see Figure 2. We can choose h so that g be in the C^r-neighborhood $\mathcal{U}_0$

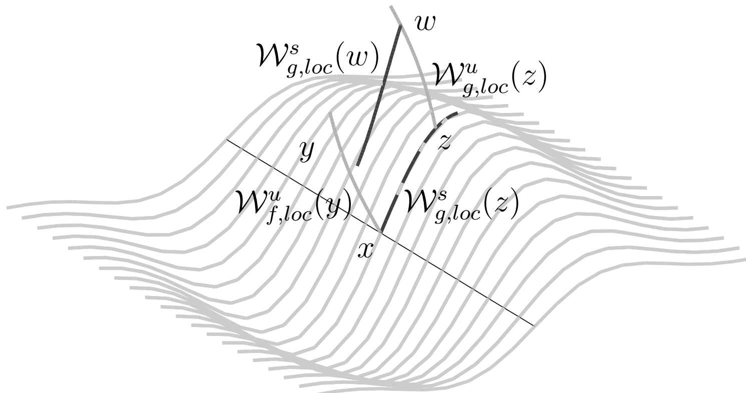

FIGURE 2. Lemma 11: Opening the accessibility class of x

found in Proposition 8. We produce a push so that $W^u_{f,loc}(y) \cap W^s_{g,loc}(w) = \emptyset$. See Figure 2. Now, Properties (1)–(4) above imply that $\mathcal{W}^u_{g,loc}(y) = \mathcal{W}^u_{f,loc}(y)$, $\mathcal{W}^s_{g,loc}(z) = \mathcal{W}^s_{f,loc}(z)$ and $\mathcal{W}^u_{g,loc}(z) = \mathcal{W}^s_{g,loc}(z)$, so we do not change the fact that there is a us-path from x to z to w, but we do change the local stable disc of w. Now, we have $x \in \mathcal{W}^s_{g,\epsilon}(z)$ and $w \in \mathcal{W}^u_{g,\epsilon}(z)$. If z belonged to $\Gamma(g)$, we would have $\mathcal{W}^s_{g,loc}(w) \cap \mathcal{W}^u_{g,loc}(x) \neq \emptyset$. But this does not happen. Hence $z \notin \Gamma(g)$, and thus $AC(z,g) = AC(x,g)$ is open. □

Proposition 12. *$PHD^r_1(M)$ contains a C^r dense set of diffeomorphisms with the property that the accessibility class of every periodic point is open.*

Proof. For $k \geq 1$ let $\mathcal{U}_k$ denote the set of all diffeomorphisms in $PHD^r_1(M)$ with the property that the periodic points of period k are finite in number and are all hyperbolic. Each $\mathcal{U}_k$ is open and C^r dense by the Kupka-Smale theorem. The number of points of period k is constant on each component of $\mathcal{U}_k$. It is immediate from the previous lemma that $\mathcal{U}_k$ has a C^r dense subset $\mathcal{U}'_k$ such that the accessibility class of every periodic point with period k for every diffeomorphism in $\mathcal{U}'_k$ is open. The set $\mathcal{U}'_k$ is C^1 open, by Proposition 7. The diffeomorphisms in $\bigcap_{k\geq 1} \mathcal{U}'_k$ have the property that the accessibility class of every period point is open. This set is dense by Baire's theorem. □

7. Preliminary lemmas

Henceforth we consider a fixed diffeomorphism $f \in PHD^r_1(M)$. Here we present two lemmas which will be used in the next section to show that if f is a continuity point of Γ, then $\Gamma(f) = \emptyset$. The lemmas apply to all diffeommorphisms close enough to f in the C^r topology.

The first lemma is an application of the Anosov Closing Lemma. Let us denote by $\Gamma_\rho(x, h)$ the set of points in the lamina $\Gamma(x, h)$ that can be reached from x by a C^1 path of length less than ρ.

Lemma 13. *There are a neighborhood $\mathcal{N}_1$ of f in $PHD_1^r(M)$, an integer $n_0 > 0$, and a radius $\rho > 0$ such that the following property holds for any $h \in \mathcal{N}_1$: If there is a point $y \in \Gamma(h)$ with $h^n(y) \in \Gamma_\rho(y, h)$ for some $n \geq n_0$, then there is a periodic point of h in $\Gamma(y, h)$ with period n.*

Proof. This follows from Proposition 8 and the Anosov Closing Lemma. □

We define a *central curve* to be a C^1 curve with unit speed that is tangent to E^c at all times. The following lemma states that if a (short) central curve hits the disc $\Gamma_\rho(y, h)$, then all sufficiently near central curves also hit $\Gamma_\rho(y, h)$. The length of the central curves and the proximity of their origins are uniform over a neighborhood of f in $PHD_1^r(M)$. This lemma involves an orientation for the one dimensional bundle E^c. This bundle may not be globally orientable, but all that is needed in the lemma is a local orientation in the neighborhood of a point.

Lemma 14. *For each $\rho > 0$, there are a neighborhood $\mathcal{N}_2$ of f in $PHD_1^r(M)$, and $\Delta > 0$ such that the following holds for any h in $\mathcal{N}_2$: Suppose that $x_0 \notin \Gamma(h)$ and $\sigma_0 : [0, \Delta] \to M$ is a central curve with $\sigma_0(0) = x_0$ and $y_0 = \sigma_0(t_0) \in \Gamma(h)$ for some $t_0 \in (0, \Delta]$. Suppose $\sigma : [0, 2\Delta] \to M$ is a central curve such that $d(\sigma(0), x_0) < d(x, \Gamma(h))$ and $\dot{\sigma}(0)$ is oriented in the same direction as $\dot{\sigma}_0(0)$. Then σ intersects $\Gamma_\rho(y_0, h)$ in a unique point y. Moreover if y_0 is the first point of σ_0 that is in $\Gamma(h)$, then y is the first point of σ that is in $\Gamma(h)$. See Figure 3.*

Proof. This is a consequence of the continuity of $h \mapsto E_h^c, E_h^s, E_h^u$ and of the transversality of central curves and the laminae of the set $\Gamma(h)$. □

8. Creating a periodic point with non open accessibility class

We now wish to show that for the continuity point $f \in PHD_1^r(M)$ of the function $\Gamma : PHD_1^r(M) \to \mathcal{K}(M)$, we have $\Gamma(f) = \emptyset$. In order to do this, we assume that $\Gamma(f) \neq \emptyset$ and obtain a contradiction to Proposition 12, by eventually showing that there is an open set of diffeomorphisms with a periodic point whose accessibility class is not open (Lemma 16).

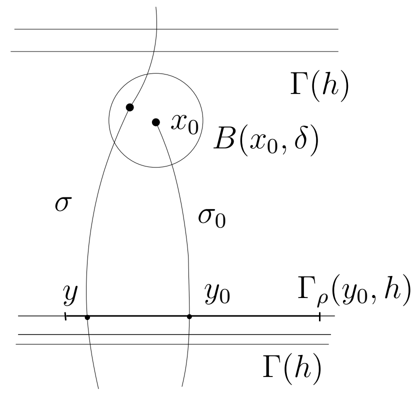

FIGURE 3. Lemma 14: Central curves hitting $\Gamma(y, h)$

By continuity of Γ at f there is a neighborhood $\mathcal{N}_3$ of f in the C^r topology on $PHD_1^r(M)$ such that for any $g \in \mathcal{N}_3$, we have $dist(x, \Gamma(g)) < \Delta$, for all $x \in \Gamma(f)$, where Δ is the constant from Lemma 14.

Now, the set $\Gamma(f)$ is closed and invariant, hence it contains a minimal set K. Applying Lemma 11, we can choose g in $\mathcal{N}_1 \cap \mathcal{N}_2 \cap \mathcal{N}_3$, where $\mathcal{N}_1$ and $\mathcal{N}_2$ are the neighborhoods of f defined in the previous section, such that $AC(x, g)$ is open for some $x \in K$. Recall that $dist(x, \Gamma(g)) < \Delta$.

Let n_0, ρ and Δ be the numbers defined in Lemma 14 and Lemma 13, and choose an orientation for the one dimensional bundle E^c on the ball $B(x, \Delta)$.

Lemma 15. *There is $n > n_0$ such that $g^n(x) \in B(x, \delta/2)$ and $Tg^n(x)$ preserves orientation of E^c.*

Proof. The point x is recurrent because $x \in K$ and $f|_K = g|_K$. Hence there is an integer $n_1 > n_0$ such that $g^{n_1}(x) \in B(x, \delta/2)$. If $Tg^{n_1}(x)$ preserves orientation, we can take $n = n_1$.

If $Tg^{n_1}(x)$ reverses orientation, we then pick $n_2 > n_0$ such that $g^{n_2}(x)$ is in $B(x, \delta/2)$ and is close enough to x so that $g^{n_1}(g^{n_2}(x)) \in B(x, \delta/2)$. If $Tg^{n_2}(x)$ preserves orientation, we can take $n = n_2$; if not we can take $n = n_1 + n_2$. Figure 4 illustrates the case in which $n = n_1 + n_2$. □

Lemma 16. *There is a C^r open neighborhood $\mathcal{N}$ of g, such any $h \in \mathcal{N}$ has a periodic point in $\Gamma(h)$.*

Proof. Any h close enough to g in the C^r topology satisfies the following properties:

(1) $B(x, \delta) \subset AC(x, h)$ for some $\delta < \Delta$.
(2) $h \in \mathcal{N}_1 \cap \mathcal{N}_2 \cap \mathcal{N}_3$.

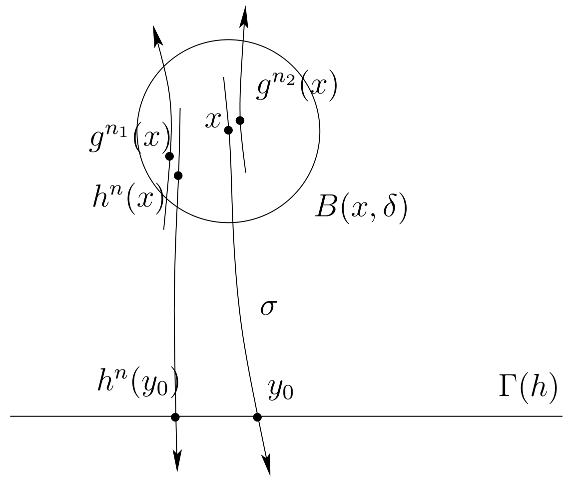

FIGURE 4. Lemmas 15 and 16

(3) There is $n > n_0$ such that $h^n(x) \in B(x,\delta)$ and Th^n preserves the orientation of E^c near x.

The first property follows from Proposition 7. The second holds because $g \in \mathcal{N}_1 \cap \mathcal{N}_2 \cap \mathcal{N}_3$. The third follows from Lemma 15.

We now show that any diffeomorphism h satisfying these three properties has a periodic point in $\Gamma(h)$. Let σ be a central arc connecting x and $\Gamma(h)$. Choose a point $y_0 \in \sigma \cap \Gamma(h)$ such that σ contains no point of $\Gamma(h)$ except y_0. Choose $n > n_0$ such that $h^n(x) \in B(x,\delta)$ and $Th^n(x)$ preserves orientation of E^c.

Then the central arc $h^n(\sigma)$ connects $h^n(x)$ to a point $y \in \Gamma_\rho(y_0,h)$. Observe that there are no points of $\Gamma(h)$ on $h^n(\sigma)$ between $h^n(x)$ and y. Since the set $\Gamma(h)$ is invariant, the image under h^n of the point y_0 where the curve σ first hits $\Gamma(h)$ must be the point where $h^n(\sigma)$ first hits $\Gamma(h)$. Hence $h^n(y_0) = y$. We can now apply Lemma 13 to obtain a periodic point of h in $\Gamma(y_0,h)$. □

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