Partially Hyperbolic Dynamics

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Preface

In this book we present some aspects of the theory of partially hyperbolic diffeomorphisms. A diffeomorphism on a compact manifold is partially hyperbolic if it preserves a splitting of the tangent bundle into three sub-bundles in such a way that one of them is uniformly contracted, other one is uniformly expanded and the last one, called the center bundle, has an intermediate behavior, that is, it is neither as contracting as the first one nor as expanding as the second one. This concept is a natural generalization of the notion of uniformly hyperbolicity and its study goes back to the early seventies (see for instance [**73, 26**]) but surely these issues were under discussion since before. Hyperbolic behavior has proved to be a powerful tool to get different types of chaotic properties from the ergodic and topological viewpoints but at that time the need of relaxing the full hyperbolicity hypothesis appeared (see for instance Shub's examples of non-hyperbolic robustly transitive diffeomorphisms [**114**])

Some works that appeared in the nineties opened the way for making partial hyperbolicity one of the most active topics in dynamics over the last decade. These works relate partial hyperbolicity with two robust fundamental properties: stable ergodicity (see [**53, 103**]) and robust transitivity (see [**13, 43**]) This book will mainly be concentrated in the ergodic properties of these systems. We will maintain the exposition of the topics in the simplest possible cases. Most of the time in the simplest cases appears already the main insight of the theory.

The book is divided in five chapters. The general intention is that each chapter be self-contained. In the first chapter we give the very basic definitions and we present the main examples of partially hyperbolic diffeomorphisms (we follow the presentation of the examples in [**66**]) Second chapter is devoted to the Pugh-Shub conjecture about the abundance of ergodicity among the conservative partially hyperbolic diffeomorphism. In particular, we explain the proof of the conjecture for one dimensional center. In the third chapter we study the relationship between partial hyperbolicity and entropy, entropy maximizing measures, etc. The research in this area is growing recently and there are many interesting open problems. Fourth chapter is about co-cycles with partially hyperbolic (or even hyperbolic) base dynamics. In this area there are new interesting results that relate the regularity of the center "foliation" wit rigidity phenomena. Finally, the last chapter is dedicated to explain the advances on our conjecture about the ergodicity of conservative partially hyperbolic diffeomorphisms in dimension 3. Roughly speaking, this conjecture asserts that non-ergodic partially hyperbolic diffeomorphisms can exist only on a few 3-dimensional manifolds (essentially on those manifolds that are torus bundles that fiber over the circle)

Significant advances in many aspects were recently obtained in the theory of partial hyperbolicity through the work of many authors. These advances deserve a systematic presentation. These notes are an attempt to do so with part of this material.

CHAPTER 1

Introduction

1.1. First definitions

Throughout this book we shall work with a *partially hyperbolic diffeomorphism* $f : M \to M$ where M is a compact riemannian manifold.

DEFINITION 1.1.1. *A diffeomorphism f is **partially hyperbolic** if it admits a nontrivial Tf-invariant splitting of the tangent bundle $TM = E^s \oplus E^c \oplus E^u$, such that all unit vectors $v^\sigma \in E^\sigma_x$ $(\sigma = s, c, u)$ with $x \in M$ satisfy:*

$$\|T_x f v^s\| < \|T_x f v^c\| < \|T_x f v^u\|$$

*for some suitable Riemannian metric. f also must satisfy that $\|Tf|_{E^s}\| < 1$ and $\|Tf^{-1}|_{E^u}\| < 1$. There is also a stronger type of partial hyperbolicity. We will say that f is **absolutely partially hyperbolic** if it is partially hyperbolic and*

$$\|T_x f v^s\| < \|T_y f v^c\| < \|T_z f v^u\|$$

for all $x, y, z \in M$ and $v^\sigma \in E^\sigma_w$ unit vectors, $\sigma = s, c, u$ and $w = x, y, z$ respectively.

There are two invariant foliations $\mathcal{F}^s$ and $\mathcal{F}^u$, the *strong stable* and the *strong unstable foliations*, that are tangent, respectively, to E^s and E^u. These are the only foliations with these features. But, in general, there is no invariant foliation tangent to E^c; and, in case there were, in general, it is not unique. We will discuss properties of these foliations many times trough these notes.

Other important fact is that partial hyperbolicity is an open property in the C^1 topology. That is, if f is partially hyperbolic there exists a neighborhood $\mathcal{U} \subset \mathrm{Diff}^1(M)$ of f such that $\forall g \in \mathcal{U}$, g

is partially hyperbolic. A proof of this fact can be obtained by using an argument with cones like in the hyperbolic case. Observe that the invariance of suitable defined cones depends only in the relation between the derivatives restricted to each invariant bundle.

1.2. Examples

In studying partially hyperbolic systems, one of the problems is that it is not clear if the amount of existing examples is small, or if it essentially includes all the examples. Thus we get two parallel problems: the search of examples and the classification problem. We would like to split the examples into two categories in nature, a grosser or topological one and another finer or geometric one; or even a measure theoretic one.

For the topological type we would be interested in knowing in which manifolds and in which homotopy classes the partially hyperbolic dynamics can occur. For example, we say that two partially hyperbolic systems $f : M \to M$ and $g : N \to N$, both having a central foliation $\mathcal{F}$ are *centrally conjugated* or conjugated modulo the central direction [**74**] if there is a homeomorphism $h : M \to N$ such that

i) $h\left(\mathcal{F}_f(x)\right) = \mathcal{F}_g\left(h(x)\right)$
ii) $h\left(f\left(\mathcal{F}_f(x)\right)\right) = g\left(h\left(\mathcal{F}_f(x)\right)\right)$ or, which is equivalent, $\mathcal{F}_g\left(h(f(x))\right) = \mathcal{F}_g\left(g(h(x))\right)$

It would be useful to classify partially hyperbolic systems modulo central conjugacy. Some cases were indeed studied and will appear in future chapters. It would be also interesting to have an analogous concept when the central distribution is not integrable.

Bellow we give a list of some of the existing examples. We hope that we had put there most of them.

The second type of examples typically live within the first type and will be appearing along this notes.

1.2.1. Anosov Diffeomorphisms.

A diffeomorphism $f : M \to M$ is an *Anosov diffeomorphism* if its derivative Df leaves the splitting $TM = E^s \oplus E^u$ invariant, where Df contracts vectors in E^s exponentially fast, and Df expands vectors in E^u exponentially fast.

Anosov systems are the hallmark of hyperbolic and chaotic behaviors.

From the ergodic point of view, Anosov diffeomorphisms are very much better understood.

THEOREM 1.2.1. [**3**] *Volume preserving Anosov diffeomorphisms are ergodic.*

The partially hyperbolic systems share lots of their properties with the Anosov systems. Let us describe some of those properties. There are two invariant foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ tangent to E^s and E^u. Both foliations have smooth leaves (as smooth as the diffeomorphism), but the foliations themselves are not smooth a priori. In fact, although there are some interesting cases where the invariant foliations are smooth, the general case is that they are rarely smooth [**5**], [**60**]. Thus, it became an interesting problem to study the transversal regularity of these foliations. For example, it turned out that the holonomies of these foliations are *absolutely continuous* [**3**], [**4**], [**6**], [**117**] i.e. we say that a map $h : \Sigma_1 \to \Sigma_2$ is absolutely continuous if it sends zero measure sets into zero measure sets. The importance of absolute continuity of the holonomies is that it implies that Fubini's theorem is true for these foliations, that is, a measurable set A has zero measure if and only if for a.e. point x in M the intersection of A with the leaf through x has zero leaf-wise measure. It is worth mentioning that in smooth ergodic theory, when dealing with any kind of hyperbolicity, the smooth regularity of the system is typically required to be at least $C^{1+\text{Hölder}}$. In fact, in the C^1 category the following is still unknown:

PROBLEM 1.2.2. *Are there examples of non ergodic volume preserving Anosov diffeomorphisms?*

Despite these results, Anosov diffeomorphisms are far from being completely understood. For example, the following problem is still open.

PROBLEM 1.2.3. [**118**] *Is every Anosov diffeomorphism conjugated to an infra-nil-manifold automorphism?*

When the manifold underlying the dynamics is a nil-manifold or if the unstable foliation has codimension one the answer is yes, [**48**], [**88**], [**93**]. For expanding maps (when every vector is expanded by the derivative) the answer also is yes, they are always conjugated to infra-nil-manifold endomorphisms, [**113**], [**54**]. It would be interesting to get analogous results for partially hyperbolic diffeomorphisms, or at least to have an answer to the following:

PROBLEM 1.2.4. *Let f be a partially hyperbolic diffeomorphism on a nilmanifold. Is it true that its action in homology is partially hyperbolic?*

In dimension three the answer is yes, see [**21**],[**27**], [**96**].

1.2.2. Anosov Flows. We say that a flow ϕ_t on a manifold M is an *Anosov flow* if admits an invariant splitting $TM = E^s \oplus E^0 \oplus E^u$, where, as usual, vectors in E^s and E^u are respectively exponentially contracted and exponentially expanded, and E^0 is the space spanned by the vector-field. One of the main difference between Anosov flows and Anosov diffeomorphisms is that there are known examples of Anosov flows where the non-wandering set is not the whole manifold, [**49**]. In fact, this changes completely the hope of finding a complete classification of Anosov flows like that stated in Problem 1.2.3. On the other hand, when dealing with transitive Anosov flows, there is a dichotomy, either they are mixing or else the bundle $E^s \oplus E^u$ is jointly integrable. In fact, in [**98**], it is proven that either the strong unstable manifold is minimal or $E^s \oplus E^u$ is integrable. In this second case J. Plante also proved that the flow is conjugated to a suspension but possibly changing the time. In fact it is still an open problem to know if the $su-$foliation is by compact leafs, and this is closely related to the following long-standing problem

PROBLEM 1.2.5. *Is the action in homology of an Anosov diffeomorphism hyperbolic?*

As already mentioned, volume preserving Anosov flows are ergodic. Moreover, the following is proven in [**31**]:

THEOREM 1.2.6. *Let ϕ_1 be the time-one map of a volume preserving Anosov flow ϕ. If ϕ is mixing then it is stably ergodic.*

Of course, the time-one map of the suspension of an Anosov diffeomorphism by a constant roof function is not stably ergodic.

1.2.3. Geodesic Flows. Let V be an n-dimensional manifold and let g be a metric on V. On TV it is defined the *geodesic flow* as follows. Given a point $x \in V$ and a vector $v \in T_xV$ there is a unique geodesic γ with $\gamma(0) = x$ and $\dot{\gamma}(0) = v$. For $t \in \mathbb{R}$ we define $\phi_t(x, v) = (\gamma(t), \dot{\gamma}(t))$. It follows that $|\dot{\gamma}(t)| = |v|$ for every $t \in \mathbb{R}$ or more precisely, $g_{\gamma(t)}(\dot{\gamma}(t)) = g_x(v)$ for every $t \in \mathbb{R}$. Thus the geodesic flow preserves the vectors of a given magnitude. Let $M = T_1V$ be the bundle of unit vectors tangent to V and let us restrict the geodesic flow to M. It turns out that if the sectional curvature is negative then the geodesic flow is in fact an Anosov flow [**3**]. Indeed, for every unit vector v in M, T_vM may be identified with the orthogonal Jacobi fields. Thus, if we call E^s the set of orthogonal Jacobi fields that are bounded for the future and E^u the set of orthogonal Jacobi fields that are bounded for the past, then negative sectional curvature implies that $TM = E^s \oplus E^0 \oplus E^u$ and the vectors in E^s are exponentially contracted in the future, E^0 is the one dimensional space spanned by the vector-field defining the geodesic flow and the vectors in E^u are exponentially contracted in the past.

The geodesic flow preserves a natural measure defined on M, the Liouville measure *Liou*. Let us first define a one-form η over TV as follows: if $\omega \in TV$ and $\chi \in T_\omega TV$ then we define $\eta_\omega(\chi)$ as being $\omega \cdot d_\omega p(\chi)$ where $x = p(\omega)$, $p : TV \to V$ is the canonical projection. It turns out that $d\eta$ is a symplectic 2-form on TV and that the geodesic flow preserves this symplectic form. Thus, $L = d\eta \wedge \cdots \wedge d\eta$ (n-times) is a $2n$-form. The restriction of L to M is the $(2n-1)$-form defining *Liou*.

It was for the geodesic flows on surfaces of negative curvature that E. Hopf [**75**] developed the machinery now called the Hopf argument to prove ergodicity w.r.t. *Liou* and the antecedent for D. Anosov work. For general manifolds of negative sectional curvature D. Anosov proved that the geodesic flow is ergodic w.r.t. *Liou*. In fact, he proved more generally that C^2 volume preserving Anosov systems are ergodic, thus, since being an Anosov flow is an open condition, they form an open set of ergodic flows [**3**], [**4**], [**6**].

The time-one map of the geodesic flow on negative curvature, i.e. ϕ_1, is naturally a partially hyperbolic diffeomorphism. It was not until 1992 that M. Grayson, C. Pugh and M. Shub, [**53**] proved that the time-one map of the geodesic flow on a surface of constant negative curvature is a *stably ergodic* diffeomorphism, that is, as in the Anosov case, their perturbations remain ergodic, see Chapter 2.

Later, A. Wilkinson proved the same result but for variable curvature [**125**].

There is also a related topological question about robust transitivity for partially hyperbolic systems that remains widely open. In [**13**] it is proven that close to the time-one map of the geodesic flow on a negatively curved surface there are whole open sets of transitive diffeomorphisms. But the following is still open:

PROBLEM 1.2.7. *Is the time-one map of the geodesic flow on a negatively curved surface robustly transitive?*

1.2.4. Frame Flows. [**18**], [**23**], [**24**], [**25**], [**30**]. The *frame flow* on a Riemannian manifold (V, g) fibers over its geodesic flow. Let $\hat{M}$ be the space of positively oriented orthonormal n-frames in TV. Thus $\hat{M}$ naturally fibers over $M = T_1V$, where the projection takes a frame to its first vector. The associated structure group $SO(n-1)$ acts on fibres by rotating the frames keeping the first vector fixed. In particular, we can identify each fiber with $SO(n-1)$. Let $\hat{\phi}_t : \hat{M} \to \hat{M}$ denote the frame flow, which acts on frames by moving their first vectors according to the geodesic flow and moving the other vectors by parallel transport along the geodesic defined by the first vector. The projection is a semi-conjugacy from $\hat{\phi}_t$ to ϕ_t. In particular, $\hat{\phi}_t$ is an $SO(n-1)$-group extension of ϕ_t. The frame flow preserves the measure $\mu = Liou \times \nu_{SO(n-1)}$, where $\nu_{SO(n-1)}$ is the (normalized) Haar measure on $SO(n-1)$. It turns out that the time-t map of the frame flow is a partially hyperbolic diffeomorphism [**26**]. The neutral direction has dimension $1 + dimSO(n-1)$ and is spanned by the flow direction and the fibre direction.

The frame flow on manifolds of negative sectional curvature is known to be ergodic in lots of cases. The study of the ergodicity of the frame flow restricts to the study of its accessibility classes (see Chapter 2 for the notion of accessibility) and is a very interesting example to begin with, in order to learn how to manage them. Finally the frame flow is stably ergodic in the cases it is known to be ergodic. But it is not always ergodic, Kähler manifolds with negative curvature and real dimension at least 4 have non-ergodic frame flows because the complex structure is invariant under parallel translation. We suggest the reader to see [**30**] for a good account of the existing results, problems and conjectures.

1.2.5. Affine diffeomorphisms. Let G be a Lie group and $B \subset G$ a subgroup. Given a one parameter subgroup of G it defines an *homogeneous flow* on G/B. Examples of homogeneous flows are geodesic flows of hyperbolic surfaces. There are lots of interplays between the dynamics of homogeneous flows and the algebraic properties of the groups involving it, see for example [**120**] for an account.

The time-t map of an homogeneous flow is a particular case of an *affine diffeomorphism.* In fact affine diffeomorphisms and homogeneous flows are typically treated in a similar way. Let G be a connected Lie group, $A : G \to G$ an automorphism, B a closed subgroup of G with $A(B) = B$, and $g \in G$. Then we define the affine diffeomorphism $f : G/B \to G/B$ as $f(xB) = gA(x)B$. We shall assume that G/B supports a finite left $G-$invariant measure and call, in this case, G/B a finite volume homogeneous space. If G/B is compact and B is discrete the existence of such a measure is immediate, but if B is not discrete the assumption is nontrivial.

The affine diffeomorphism f is covered by the diffeomorphism $\bar{f} = L_g \circ A : G \to G$; where $L_g : G \to G$ the left multiplication by g. If $\mathfrak{g}$ is the Lie algebra of G, we may identify $T_eG = \mathfrak{g}$ where e is the identity map. Let us fix a right invariant metric on G, i.e. R_g is an isometry for every g where R_g is right multiplication by g. Let us define the naturally associated automorphism $\mathfrak{a}(f) : \mathfrak{g} \to \mathfrak{g}$ by $\mathfrak{a}(f) = Ad(g) \circ D_eA$ where $Ad(g)$ is the adjoint automorphism of g, that is the derivative at e of $x \to gxg^{-1}$. In other words, $\mathfrak{a}(f)$ is essentially the derivative of $\bar{f}$, but after right multiplication by g^{-1} (which is an isometry) in order to send T_gG to T_eG. So we have the splitting $\mathfrak{g} = \mathfrak{g}^s \oplus \mathfrak{g}^c \oplus \mathfrak{g}^u$ w.r.t the eigenvalues of $\mathfrak{a}(f)$ being of modulus less than one, one, or bigger than one respectively and similarly, $\mathfrak{g}^s$ is formed by the vectors going exponentially to 0 in the future, $\mathfrak{g}^u$ is formed by the vectors going exponentially to 0 in the past and $\mathfrak{g}^c$ is formed by the vectors that grow at most polynomially for the future and the past. Observe that if v_λ and v_σ are eigenvectors for $\mathfrak{a}(f)$ w.r.t. λ and σ respectively then we have that

$$\mathfrak{a}(f)\left([v_\lambda, v_\sigma]\right) = [\mathfrak{a}(f)(v_\lambda), \mathfrak{a}(f)(v_\sigma)] = \lambda\sigma[v_\lambda, v_\sigma]$$

and hence if $[v_\lambda, v_\sigma] \neq 0$ then it is an eigenvector for $\lambda\sigma$. As a consequence we get that $\mathfrak{g}^s$, $\mathfrak{g}^u$, $\mathfrak{g}^c$, $\mathfrak{g}^{cs} = \mathfrak{g}^c \oplus \mathfrak{g}^s$ and $\mathfrak{g}^{cu} = \mathfrak{g}^c \oplus \mathfrak{g}^u$ are subalgebras tangent to connected subgroups G^s, G^u, G^c, G^{cs} and

G^{cu} of G and their translates will define the stable, unstable, center, center-stable and center-unstable foliations respectively.

Let $\mathfrak{h}$ denote the smallest Lie subalgebra of $\mathfrak{g}$ containing $\mathfrak{g}^s$ and $\mathfrak{g}^u$. Using Jacobi identity it is not hard to see that it is an ideal, $\mathfrak{h}$, called the *hyperbolic subalgebra* of $\bar{f}$. Moreover, let us denote $H \subset G$ the connected subgroup tangent to $\mathfrak{h}$ and call it the hyperbolic subgroup of $\bar{f}$. As $\mathfrak{h}$ is an ideal in $\mathfrak{g}$, H is a normal subgroup of G. Finally let us denote with $\mathfrak{b} \subset \mathfrak{g}$ the Lie algebra of $B \subset G$. Then we have the following:

THEOREM 1.2.8. [**105**] *Let $f : G/B \to G/B$ be an affine diffeomorphism as above, then f is partially hyperbolic if and only if $\mathfrak{h} \not\subset \mathfrak{b}$. Moreover, if f is partially hyperbolic then the left action of G^σ, $\sigma = s, u, c, cs, cu$ on G/B foliates G/B into the stable, unstable, center, center-stable and center-unstable foliations respectively.*

PROBLEM 1.2.9. *Is there an example of a non-Anosov affine diffeomorphism that is robustly transitive? Are they exactly the same as the stably ergodic ones?.*

1.2.6. Linear Automorphisms on Tori. A special case of affine diffeomorphisms are the affine automorphisms on tori. In fact, the torus $\mathbb{T}^N$ may be seen as the quotient $\mathbb{R}^N/\mathbb{Z}^N$. Integer entry $N \times N$ matrices with determinant ± 1 define what we shall call linear automorphisms of tori simply via matrix multiplication. Thus, given such a matrix A and a vector $v \in \mathbb{R}^N$, it is defined an affine diffeomorphism of the torus f by $f(x) = Ax + v$. It is quite easy to see that, conjugating by a translation, it is enough to study the case where v belongs to the eigenspace corresponding to the eigenvalue 1, E_1. Observe also that E_1 is a rational space, that is, it has a basis formed by vectors of rational coordinates.

The corresponding splitting of the tangent bundle here, is the splitting given by the eigenspaces of A. Thus, a not quite involved argument proves that f is partially hyperbolic unless all the eigenvalues of A are roots of unity. Moreover, using a little bit of harmonic analysis (Fourier series) it is seen [**56**] that f is ergodic if and only if A has no eigenvalues that are roots of the identity other than one itself and v has irrational slope inside E_1. Finally, notice that if E_1 is not trivial, we may always perturb in order to make v of rational slope, thus in order to get that perturbations remain ergodic it is necessary that also 1 be not in the spectrum of A. Thus we reach to the following problem:

PROBLEM 1.2.10. [**74**], [**64**], *Are the ergodic linear automorphisms stably ergodic?*

Of course, an analogous problem may be posed in the topological category, that is, are their perturbations also transitive? [**74**].

1.2.7. Direct Products. Given a partially hyperbolic diffeomorphism $f : M \to M$ and $g : N \to N$ a diffeomorphism, the product $f \times g : M \times N \to M \times N$ is partially hyperbolic if the dynamics of g is less expanding and contracting, respectively, than the expansions and contractions of f. This is essentially the most trivial way a partially hyperbolic dynamics appears, Anosov$\times$identity. Besides, we can also make the product of two partially hyperbolic diffeomorphisms.

It is quite interesting that by making perturbations of this product dynamics, lots of nontrivial examples arises. For instance, the first example of a robustly transitive non-Anosov diffeomorphism constructed by M. Shub [**114**], although not a product, is a large perturbation of a product. In fact direct products as well as the construction of M. Shub are part of a more general type of construction, the partially hyperbolic systems that fiber over other partially hyperbolic systems.

1.2.8. Fiberings over partially hyperbolic diffeomorphisms. Let $f : B \to B$ be a partially hyperbolic diffeomorphism with splitting $TM = E^s_f \oplus E^c_f \oplus E^u_f$. Let $p : N \to B$ be a fibration with fiber F, let us call $F(x)$ the fiber through x. Then any lift $g : N \to N$ of f is a partially hyperbolic diffeomorphism if

$$\left|D_{p(x)}f|E^s_f\right| < m\left(D_x g|T_xF(x)\right) \leq |D_x g|T_xF(x)| < m\left(D_{p(x)}f|E^s_f\right).$$

where $m(A) = |A^{-1}|^{-1}$. As we said, M. Shub's example of a robustly transitive diffeomorphism is of this kind, and, in fact, many of the existing examples are of this kind. It would be interesting to find the minimal pieces over which partially hyperbolic systems are built. For example:

PROBLEM 1.2.11. *Find all the partially hyperbolic diffeomorphisms f such that no partially hyperbolic diffeomorphism g homotopic to f^n,*

$n > 0$, fibers over a lower dimensional partially hyperbolic diffeomorphism. The geodesic flow on negative curvature as well as the ergodic automorphisms of tori defined in [**64**] *are examples of that building blocks. Find other types of gluing technics to generate new partially hyperbolic systems.*

1.2.9. Skew products. Another type of systems that fiber over lower dimensional partially hyperbolic diffeomorphisms are the *skew products*: Let $f : M \to M$ be a partially hyperbolic diffeomorphism, G a Lie group and $\theta : M \to G$ a function. Define the skew product $f_\theta : M \times G \to M \times G$ by $f_\theta(x, g) = (f(x), \theta(x)g)$. Skew products where extensively studied in the context of partially hyperbolic diffeomorphisms, see for example [**1**], [**18**], [**19**], [**26**], [**32**], [**47**].

CHAPTER 2

Stable ergodicity of partially hyperbolic diffeomorphisms

2.1. Introduction

One particularly relevant topic in partially hyperbolic dynamics concerns the frequency of ergodicity among conservative diffeomorphisms. Let $\mathrm{Diff}^1_m(M)$ denote the set of C^1-diffeomorphisms preserving a smooth volume. It is not known yet wether there exist open sets in $\mathrm{Diff}^1_m(M)$ of ergodic or of non-ergodic diffeomorphisms if $\dim M \geq 2$. However, there are examples of stably ergodic diffeomorphisms in $\mathrm{Diff}^1_m(M)$: A diffeomorphism $f \in \mathrm{Diff}^{1+\alpha}_m(M)$ is called *stably ergodic* in $\mathrm{Diff}^1_m(M)$ if there exists a C^1-neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1_m(M)$ such that all $C^{1+\alpha}$-diffeomorphisms in $\mathcal{U}$ are ergodic.

Examples of stably ergodic diffeomorphisms are $C^{1+\alpha}$ Anosov diffeomorphisms. Indeed, any $C^{1+\alpha}$ Anosov diffeomorphism f is ergodic [**4**], [**6**]. But the set of Anosov diffeomorphisms is C^1-open, so all nearby $C^{1+\alpha}$-diffeomorphisms are Anosov and hence ergodic, too.

Until 1993, Anosov diffeomorphisms were the only known examples of stably ergodic diffeomorphisms, but Grayson, Pugh and Shub showed that the time-one map of the geodesic flow of a surface of negative curvature is stably ergodic [**53**]. This example is a particular case of a partially hyperbolic diffeomorphism, and inspired the following conjecture, which we shall develop in Section 2.2:

Conjecture 2.1.1. [**106**] [**104**] *Stable ergodicity is open and dense among conservative partially hyperbolic diffeomorphisms.*

Now it is known that there are also examples of stably ergodic diffeomorphisms that are *not* partially hyperbolic [**121**], though all stably ergodic diffeomorphism have a global dominated splitting [**7**], that is, there exists an invariant decomposition of the tangent bundle $TM = E \oplus F$, and a Riemannian metric for which all unit vectors $v_E \in E_x$ and $v_F \in F_x$ satisfy

$$\|Df(x)v_E\| \leq \frac{1}{2}\|Df(x)v_F\|.$$

2.2. Pugh-Shub Conjecture

The first place where Charles Pugh and Mike Shub stated their Conjecture 2.1.1 about the frequency of stable ergodicity among conservative partially hyperbolic diffeomorphisms was in the International Congress on Dynamical Systems, held in Montevideo in 1995, in the memory of Ricardo Mañé [**106**]. Actually, we had completely forgotten this fact, but it was reminded to us by Keith Burns in one of his visits to Uruguay, while we were eating some chivitos in a small bar by the sea.

The Pugh-Shub Conjecture states that stable ergodicity is C^1-open and C^r-dense among conservative partially hyperbolic diffeomorphisms. The C^1-openness condition is trivial by definition.

In [**103**], Pugh and Shub propose a program to prove their conjecture. They claim that there is a property called accessibility that implies ergodicity, and this property is essentially open and dense. A diffeomorphism has the *accessibility property* if any two points of the manifold can be joined by a path that is the concatenation of segments that are contained in either in stable or unstable manifolds. As long as we know, Sacksteder was the first to use accessibility to establish ergodicity [**111**]. It was also used by Brin and Pesin in [**25**]. We describe better this phenomenon in Section 2.6, see also at the end of this section.

As we have said, the plan of Pugh and Shub to establish Conjecture 2.1.1 is to split it into Conjecture 2.2.1 and Conjecture 2.2.2 below:

Conjecture 2.2.1 (Pugh-Shub A). *Accessibility implies ergodicity.*

After [**63**], however, it became plausible that a weaker property, named essential accessibility would be enough to establish ergodicity. A diffeomorphism is *essentially accessible* if the set of points x, y that can be joined by paths that are piecewise tangent to either the stable or the unstable bundle have either full or zero measure. The other conjecture of the program is:

CONJECTURE 2.2.2 (Pugh-Shub B). *Stable accessibility is C^r-dense.*

A diffeomorphism is *stably accessible* if it belongs to a C^1-neighborhood of accessible diffeomorphisms.

In general, it is extremely difficult to work with C^r-perturbations for $r \geq 2$. We believe that the following certainly helps to establish Conjecture 2.2.2:

CONJECTURE 2.2.3. *Accessibility is C^1-open.*

The idea behind the Pugh-Shub program to establish stable ergodicity is to extend in an audacious way the *Hopf argument*, originally used to prove ergodicity of the geodesic flow of a compact negatively curved surface [**75**], and which we succinctly describe below, see also Section 2.7.

2.2.1. The Hopf argument. It is not hard to see that the diffeomorphism f is ergodic if and only for every *continuous* observable $\varphi : M \to \mathbb{R}$, its Birkhoff average

$$\tilde{\varphi}(x) = \lim_{|n| \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ f^k(x) \tag{2.1}$$

is almost everywhere constant. But $\tilde{\varphi}(x)$ coincides almost everywhere with

$$\varphi^+(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ f^k(x) \tag{2.2}$$

which is constant on stable manifolds, see Section 2.7 for details. Analogously, $\tilde{\varphi}(x)$ coincides almost everywhere with $\varphi^-(x)$ (defined likewise), which is constant on unstable manifolds.

To simplify ideas, assume f is a conservative Anosov C^2 diffeomorphism, and suppose f is not ergodic. Then there would be a continuous observable φ for which $\tilde{\varphi}$, and hence φ^+ and φ^- are not almost everywhere constant. That is, there would be two invariant sets A and B of positive measure such that $\varphi^+(x) \geq \alpha$ for all $x \in A$, and $\varphi^-(x) < \alpha$ for all $x \in B$.

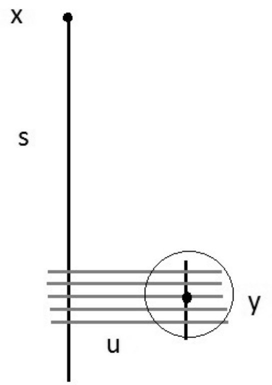

FIGURE 1. The Hopf argument

Let x be a point of A, such that almost all points w in its stable manifold satisfy $\varphi^-(w) = \varphi^+(w) = \varphi^+(x)$. Such an x exists since the stable foliation is *absolutely continuous* [**6**] and $\varphi^+(w) = \varphi^-(w)$ almost everywhere (more details in Section 2.7). And let y be a Lebesgue density point of B. Since the stable foliation is *minimal*, that is, every stable leaf is dense, the stable leaf of x gets very close to y, and so the local unstable manifold of y intersects the stable leaf of x. Since y is a density point of B, the 99% of points in a small ball around y also belong to B, that is there is a set of measure $0.99m(B_\delta(y))$ in $B_\delta(y)$ of points belonging to B. The local unstable manifold of all these points intersect the stable manifold of x. See Figure 1.

As we said before, the local stable foliation is an absolutely continuous partition of the ball $B_\delta(y)$, this means that the measure of a set A in $B_\delta(y)$ is the sum of the conditional measures $m^s_x(A)$ over all leaves of the partition. This can be also written as $m(A) = \int_{W^u_\delta(x)} m^s_x(A)dm(x)$. In our particular case, this means that there will be at least a point z in $B_\delta(y)$ such that the 99% of the points in its local stable manifold belong to B. Call T_1 the local stable manifold of z, the local unstable manifold of z intersects the stable manifold of x at a point z'. Hence there is a local stable manifold T_2 of z' (contained in the stable manifold of x), such that the unstable holonomy between T_1 and T_2 is well defined (see Section 2.6 for the definition of holonomy). But the unstable foliation is *transversely absolutely continuous*. This means that it takes positive measure sets in T_1 into positive measure sets in T_2.

We have that $m^s_z(T_1 \cap B) > 0$ and, since the unstable holonomy h^u is absolutely continuous, $m^s_{z'}(h^u(T_1 \cap B)) > 0$. But B was the set of points z such that $\varphi^-(z) < \alpha$. Since φ is continuous, φ^- is constant over unstable manifolds. Hence, if $z \in B$, then $W^u(z) \subset B$. In particular, $h^u(T_1 \cap B) = T_2 \cap B$. So, we have $m^s_{z'}(T_2 \cap B) > 0$. This means, there is a positive measure set of points in the stable manifold of x which belong to B. But this is absurd, since x was chosen so that almost all points in its stable manifold belong to A!

In brief, there are three fundamental steps in the Hopf argument:

(1) every pair of points can be joined by a a concatenation of stable and unstable leaves
(2) the stable and unstable foliations are *absolutely continuous*, in the sense that the measure of a set is the sum of conditional measures in the, respectively, stable or unstable leaves.
(3) the stable and unstable foliations are *transversely absolutely continuous*, in the sense that the stable and unstable leaves take positive measure sets in a transversal, into a positive measure set in another (close) transversal. That is, the stable and unstable holonomy maps are absolutely continuous.

We give more details of the Hopf argument in Section 2.7.

2.2.2. The Hopf argument for partially hyperbolic diffeomorphisms. If we liked to mimic the Hopf argument for the case of partially hyperbolic diffeomorphisms, we would have to pay attention to Steps (1), (2) and (3) listed above. If the partially hyperbolic system has the *accessibility property*, then item (1) is satisfied, that is, every pair of points can be joined by an *su*-path.

The other two steps are more delicate. In fact, items (2) and (3) are satisfied, in the sense that, indeed, stable and unstable foliations are absolutely continuous and transversely absolutely continuous for partially hyperbolic systems [**26**], see also [**102**]. But these absolutely continuous foliations are not transverse, due to the existence of a center bundle, obstructing the direct application of the Hopf argument in this case.

This problem can be overcome though, if the holonomies are *rigid* enough. For instance, Sacksteder uses accessibility and Lipschitzness of the stable and unstable holonomies to prove ergodicity of linear partially hyperbolic automorphisms of nil-manifolds [**111**]. More generally, Brin and Pesin proved that accessibility and Lipschitzness of the stable and unstable foliations imply ergodicity (in fact, Kolmogorov), in the following way [**26**, Theorem 5.2,p.204], see also [**53**]: if A and B are defined as in the previous subsection, consider a density point x in A, and a density point y in B. Take an *su*-path joining x and y. Call h a global holonomy map from x to y, that is, h is a local homeomorphism that takes points in a neighborhood U of x, slides them first along a stable segment, then along an unstable, then along a stable again, etc. until reaching a neighborhood V of y, all the *su*-paths are near the original *su*-path joining x and y. Since A is essentially *su*-saturated, we have that $h(A \cap U) = A \cap V$ modulo a zero set. Since h can be chosen to be Lipschitz, there exists a constant $C > 1$ such that, for each measurable set $E \subset U$, and for each sufficiently small $r > 0$, we have

$$\begin{aligned} \frac{1}{C} m(E) &< m(h(E)) < & Cm(E) \qquad (2.3)\\ B_{\frac{r}{C}}(y) &\subset h(B_r(x)) \subset & B_{Cr}(y). \qquad (2.4) \end{aligned}$$

This implies that

$$\frac{m(B_{Cr}(y) \cap A)}{m(B_{Cr}(y) \setminus A)} \geq \frac{m(h(B_r(x) \cap A))}{m(h(B_{C^2 r}(x) \setminus A))} \geq \frac{1}{C^2.C'} \frac{m(B_r(x) \cap A)}{m(B_r(x) \setminus A)} \to \infty$$

since $m(B_{C^2 r}(x) \setminus A) \leq C' m(B_r(x) \setminus A)$ for some positive constant C'. From this we get that y is also a density point of A. This is absurd, since y was a density point of B, complementary to A modulo a zero set.

This is essentially how the Hopf argument would work in the partially hyperbolic setting. However, Lipschitzness of the holonomy maps is a very strong hypothesis, not satisfied for most of the partially hyperbolic diffeomorphisms.

The idea of Grayson, Pugh and Shub [**53**], later improved by [**125**], [**67**], [**33**] is to show that the stable and unstable holonomies do, in fact, preserve density points, but they preserve density points according to another base, different from round balls $B_r(x)$. Assume M is 3-dimensional for simplicity, and for a point x consider a small center segment, locally saturate it first in a dynamic way by unstable leaves (better explained in Section 2.6), then by stable leaves. This small prism is called *s-julienne*, and denoted by $J_n^{suc}(x)$. An s-julienne density point of a set E is a point x such that:

$$\lim_{n\to\infty}\frac{m(J_n^{suc}(x)\cap E)}{m(J_n^{suc}(x))}=1 \tag{2.5}$$

The scheme is to consider the sets A and B we considered above, and prove:

(1) the s-julienne density points of A (and of any essentially u-saturated set) coincide with the Lebesgue density points of A (Theorem 2.6.2)
(2) the s-julienne density points of A (and of any essentially s-saturated set) are preserved by stable holonomies (Theorem 2.6.3)

Analogous statement is proved for A with respect to u-julienne density points, which are defined with respect to the local basis obtained by locally saturating a small center segment first in a dynamic way by stable leaves, and then by unstable leaves. Now, we have that the stable and unstable holonomies preserve the Lebesgue density points of A, hence, if the diffeomorphism has the accessibility property A is all M modulo a zero set. This proves the system is ergodic. See more details in Section 2.6.

The result above holds under an extra hypothesis on the center bundle called *center bunching* [**33**], which essentially states that the non-conformality of $Df|_{E^c}$ can be bounded by the hyperbolicity of the strong bundles. This condition is always satisfied when the center dimension is one. It is not known yet if accessibility implies ergodicity for non-center bunched partially hyperbolic diffeomorphisms.

2.3. Accessibility and accessibility classes

To simplify all the arguments, from now on, M will be a closed Riemannian 3-dimensional manifold. For any point x in M, $AC(x)$, the *accessibility class* of x consists of all the points y such that x and y can be joined by a concatenation of arcs tangent to either the stable bundle E^s or the unstable bundle E^u. This path is called an *su*-path from x to y. Note that the accessibility classes form a partition of M: if $AC(x) \cap AC(y) \neq \emptyset$, then $AC(x) = AC(y)$. A diffeomorphism $f \in \mathrm{Diff}^1(M)$ has the *accessibility property* if $AC(x) = M$, for some x. A diffeomorphism $f \in \mathrm{Diff}^1_m(M)$ has the *essential accessibility property* if any set $E \subset M$ consisting of accessibility classes, satisfies either $m(E) = 0$ or $m(E) = 1$.

For any set A, let us denote by $W^s(A)$ the set of all stable leaves $W^s(x)$, with $x \in A$, we call this set the *s-saturation* of A. Define analogously $W^u(A)$. A set A is *s-saturated* if $W^s(A) = A$, and *u-saturated* if $W^u(A) = A$. For instance, $AC(x)$ is the minimal set containing x which is both s- and u-saturated.

2.3.1. Properties of the accessibility classes.

In this subsection we shall basically show the following property of accessibility classes:

THEOREM 2.3.1. *Given $f \in \mathrm{Diff}^1(M)$, for each x in M, the accessibility class $AC(x)$ of x is either an open set or an immersed manifold. Moreover, $\Gamma(f)$, the set of non-open accessibility classes of f is a compact laminated set.*

This theorem depends strongly on the hypothesis we made on the dimension of M. It would be interesting to solve the following question:

QUESTION 2.3.2. *Theorem 2.3.1 holds for partially hyperbolic diffeomorphisms whose center bundle is one-dimensional* [**67**]. *Does it apply for diffeomorphisms with higher dimensional center bundle?*

Let us begin by a local description of open accessibility classes, which is valid for partially hyperbolic diffeomorphisms with center bundle of any dimension:

PROPOSITION 2.3.3. *For any point x in M, the following statements are equivalent:*

(1) *$AC(x)$ is open*
(2) *$AC(x)$ has non-empty interior*
(3) *$AC(x) \cap W^c_{loc}(x)$ has non-empty interior for any choice of $W^c_{loc}(x)$*

When the center bundle has higher dimension, $W^c_{loc}(x)$ does not necessarily exist; however, statement (3) can be replaced by (4) $AC(x) \cap D$ has non-empty interior, for any disc $D \ni x$ transverse to $E^s_x \oplus E^u_x$.

Proof. (2) $\Rightarrow$ (1) Let y be in the interior of $AC(x)$, and consider any point z in $AC(x)$. Then there is an su-path from y to z of the form $y = x_0, x_1, \dots, x_N = z$ such that x_n and x_{n+1} are either in the same s-leaf or in the same u-leaf. Let U be a neighborhood of y contained in $AC(x)$, and suppose that, for instance $y = x_0$ and x_1 belong to the same s-leaf. Then $U_1 = W^s(U)$ is an open set contained in $AC(x)$, that contains x_1, so x_1 is in the interior of $AC(x)$. Indeed, W^s is a C^0-foliation, so the s-saturation of an open set is open.

Now, x_1 and x_2 belong to the same u-leaf. If we consider $U_2 = W^u(U_1)$, then U_2 is an open set contained in $AC(x)$ and containing x_2 in its interior. Defining inductively U_n as $W^s(U_{n-1})$ or $W^u(U_{n-1})$ according to whether x_n belongs to the s- or the u-leaf of x_{n-1}, we obtain that all x_n belong to the interior of $AC(x)$. In particular, z. This proves that $AC(x)$ is open.

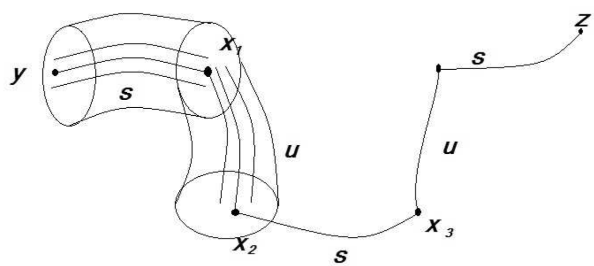

Figure 2. An su-path from y to z

(1) $\Rightarrow$ (3) is obvious, follows from the definition of relative topology.

(3) $\Rightarrow$ (2) Let V be an open set in $AC(x) \cap W^c_{loc}(x)$, relative to the topology of $W^c_{loc}(x)$. Then $W^s(V)$ is contained in $AC(x)$, and contains a disc D^{sc} of dimension $s+c$ transverse to E^u. This implies that $W^u(D^{sc})$ is contained in $AC(x)$ and contains an open set. Therefore, $AC(x)$ has non-empty interior. □

Let $O(f)$ be the set of open accessibility classes, which is, obviously, an open set. Then its complement, $\Gamma(f)$ is a compact set. Let us see that is laminated by the accessibility classes of its points.

For any point $x \in M$, consider a local center leaf $W^c_{loc}(x)$. Locally saturate it by stable leaves, that is, take the local stable manifolds of all points $y \in W^c_{loc}(x)$, to obtain a small $(s+c)$-disc $W^{sc}_{loc}(x)$. Now, locally saturate $W^{sc}_{loc}(x)$ by unstable leaves to obtain a small neighborhood $W^{usc}_{loc}(x)$. See, for instance, Figure 3. On $W^{usc}_{loc}(x)$,

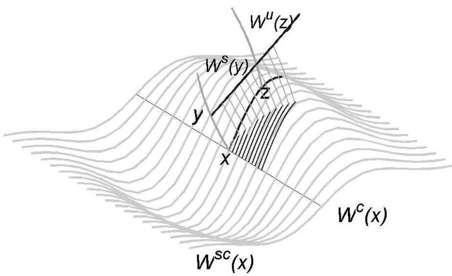

FIGURE 3. An open accessibility class

consider the map

$$p_{us} : W^{usc}_{loc}(x) \to W^c_{loc}(x) \tag{2.6}$$

defined in the following way: given $y \in W^{usc}_{loc}(x)$, there exists a unique point $p_u(y)$ in the disc $W^{sc}(x)$ that belongs to the local unstable manifold of y. Since $W^{sc}_{loc}(x)$ is the local stable saturation of $W^c_{loc}(x)$, then $p_u(y) \in W^{sc}_{loc}(x)$ is in the local stable manifold of a unique point $p^{us}(y)$ in $W^c_{loc}(x)$. That is, $p_{us}(y)$ is the point obtained by first projecting along unstable manifolds onto $W^{sc}_{loc}(x)$, and then projecting along stable manifolds onto $W^c_{loc}(x)$. Since the local stable and unstable foliations are continuous, p_{su} is obviously continuous.

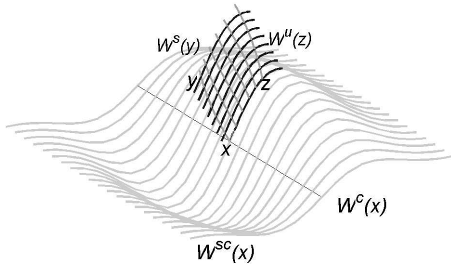

FIGURE 4. An accessibility class in $\Gamma(f)$

Let $AC_x(y)$ be the connected component of $AC(y) \cap W^{usc}_{loc}(x)$ that contains y. The points of $AC_x(y)$ are the points that can be accessed by su-paths from y without getting out from $W^{usc}_{loc}(x)$, see Figures 3 and 4. Then we have the following local description of accessibility classes of points in $\Gamma(f)$:

LEMMA 2.3.4. *For any $y \in W^c_{loc}(x)$ such that $y \in \Gamma(f)$, we have* $AC_x(y) = p^{-1}_{su}(y)$

PROOF. Let y be a point in $W^c_{loc}(x)$. Then $p^{-1}_{su}(y) = W^u_{loc}(W^s_{loc}(y))$, which is clearly contained in $AC_x(y)$. But also, we have $p_{su}(AC_x(y)) = y$. Indeed, if $p_{su}(z)$ were different from y, for some $z \in AC_x(y)$, we would have a situation as described in Figure 3. For, since p_{su} is continuous, and $AC_x(y)$ is connected, $p_{su}(AC_x(y))$ is connected. If $p_{su}(AC_x(y))$ contained another point, then it would contain a segment, which has non-empty interior in $W^c_{loc}(x)$. Proposition 2.3.3 then would imply that $AC(y)$ is open, which is absurd, since $y \in \Gamma(f)$. This proves that also $AC_x(y)$ is contained in $p^{-1}_{su}(y)$. $\square$

Hence, due to Lemma 2.3.4 above, we have that, for each $x \in \Gamma(f)$:

$$AC_x(x) = p^{-1}_{su}(x) = W^u_{loc}(W^s_{loc}(x)) \approx W^u_{loc}(x) \times W^s_{loc}(x)$$

$W^s_{loc}(x)$ and $W^u_{loc}(x)$ are (evenly sized) embedded segments that vary continuously with respect to $x \in M$ (see Hirsch, Pugh, Shub [**74**]). this implies that $\Gamma(f) \ni x \mapsto AC_x(x)$ is a continuous map that assigns to each x an evenly sized 2-disc. More precisely:

To simplify ideas, let us see the local stable and unstable manifolds as orbits of flows ψ^s and ψ^u, in such a way that $\psi^s_{(-\varepsilon,\varepsilon)}(x) = W^s_\varepsilon(x)$ and $\psi^u_{(-\varepsilon,\varepsilon)}(x) = W^u_\varepsilon(x)$. [**74**] implies that $\psi^s_t(x)$ and $\psi^u_t(x)$ are continuous with respect to (t,x). $\varepsilon > 0$ does not depend on x.

Now, for each $x \in \Gamma(f)$, take the neighborhood $W^{usc}_{loc}(x) \approx W^c_{loc}(x) \times W^s_{loc}(x) \times W^u_{loc}(x)$, that is $W^{usc}_{loc}(x) \approx W^c_{loc}(x) \times \psi^s_I(x) \times \psi^u_I(x)$, where $I = (-\varepsilon,\varepsilon)$. And define $\phi_x : W^c_{loc}(x) \times I \times I \to W^{usc}_{loc}(x)$ such that

$$\phi_x(z,t,r) = \psi^u_r(\psi^s_t(z))$$

that is, $\phi_x(z,t,r)$ consists in taking $z \in W^c_{loc}(x)$, sliding time t in the direction of the flow ψ^s and then sliding time r in the direction of the flow ψ^u. It is easy to see that for each x, ϕ_x is continuous and injective. It is also clear that $p_{su}(\phi_x(z,t,r)) = z$ for all $r,t \in I$. See Figures 3 and 4. Moreover, for all $x,z \in \Gamma(f)$ $\phi_x(\{z\} \times I^2) = p^{-1}_{su}(z) = AC_x(z)$. Hence

$$\phi_x : [\Gamma(f) \cap W^c_{loc}] \times I^2 \to W^{usc}_{loc}(x)$$

is a chart of the lamination of $\Gamma(f)$. To finish the description of accessibility classes, let us introduce the following definition:

Definition 2.3.5. *The foliations W^s and W^u are* jointly integrable *at a point $x \in M$ if there exists $\delta > 0$ such that for each $z \in W^s_\delta(x)$ and $y \in W^u_\delta(x)$, we have*

$$W^u_{loc}(z) \cap W^s_{loc}(y) \neq \emptyset$$

See Figure 4 for an illustration of a point of joint integrability of W^s and W^u.

Then Lemma 2.3.4 and discussion above imply the following:

Proposition 2.3.6. *A point x belongs to $\Gamma(f)$ if and only if W^s and W^u are jointly integrable at all points of $AC(x)$.*

Indeed, if x belongs to $\Gamma(f)$, then for all $y \in AC(x) \subset \Gamma(f)$, we have $p_{su}(AC_y(x)) = \{y\}$. In particular, if $z \in W^u_\delta(y)$ and $w \in W^s_\delta(y)$, then $W^s_{loc}(z) \cap W^u_{loc}(w) \neq \emptyset$. On the other hand, if W^s and W^u are jointly integrable at all points of $AC(x)$, then $AC(x)$ is a lamina, due to the explanation above (the coherence of the charts ϕ_x defined above depend only on the joint integrability of W^s and W^u). Moreover, this 2-dimensional lamina is transverse to $W^c_{loc}(x)$, $AC(x) \cap W^c_{loc}(x)$

cannot be open. Proposition 2.3.3 implies $AC(x)$ is not open, so $x \in \Gamma(f)$.

The following lemma shows that, in fact, the laminae of $\Gamma(f)$, that is, the accessibility classes of points in $\Gamma(f)$ are C^1.

LEMMA 2.3.7. [**44**, Lemma 5] *If W^s and W^u are jointly integrable at x, then the set*

$$W^{su}_{loc}(x) = \{W^u(z) \cap W^s(y) : \textit{with } z \in W^s_\delta(x) \textit{and} y \in W^u_\delta(x)\}$$

where $\delta > 0$ is as in the definition of joint integrability (Definition 2.3.5), is a 2-dimensional C^1-disc that is everywhere tangent to $E^s \oplus E^u$.

In order to prove Lemma 2.3.7 we shall use the following result by Journé:

THEOREM 2.3.8. [**79**] *Let F^h and F^v be two transverse foliations with uniform smooth leaves on an open set U. If $\eta : U \to M$ is uniformly C^1 along F^h and F^v, then η is C^1 on U.*

PROOF OF LEMMA 2.3.7. Let D be a small smooth 2-dimensional disc containing x and transverse to E^c_x. Consider a one-dimensional smooth foliation of a small neighborhood N of x, transverse to D. If D is sufficiently small, there is a smooth map $\pi : N \to D$, which consists in projecting along this smooth one-dimensional foliation. Note that $W^{su}_{loc}(x)$ can be seen as the graph of a continuous function $\eta : D \to N$.

We produce a grid on D in the following way: the horizontal lines are the projections of the stable manifolds $W^s(y)$, with $y \in W^u_\delta(x)$, that is, the horizontal lines are of the form $\pi(W^s_{loc}(\eta(v)))$, with $v \in D$. Analogously, the vertical lines are the projections of the unstable manifolds $W^u_{loc}(z)$, with $z \in W^s_\delta(x)$, that is, the vertical lines are of the form $\pi(W^u_{loc}(\eta(w)))$, with $w \in D$.

Now, $v \mapsto W^s_{loc}(\eta(v))$ and $w \mapsto W^u_{loc}(\eta(w))$ are continuous in the C^1-topology, that is, for close v we obtain close $W^s_{loc}(\eta(v))$ in the C^1-tolopology (E^s is a continuous bundle). Since π is smooth, we also obtain that $F^h = \{\pi(W^s_{loc}(\eta(v)))\}_{v\in D}$, the horizontal partition of D, and $F^v = \{W^u_{loc}(\eta(w))\}_{w\in D}$, the vertical partition of D, are transverse foliations continuous in the C^1-topology.

But η is uniformly C^1 along F^h, since η along a leaf $F^h(v_0) = \pi(W^s_{loc}(\eta(v_0)))$ is exactly $W^s_{loc}(\eta(v_0))$. Indeed, $\eta\circ\pi : W^{su}_{loc}(x) \to W^{su}_{loc}(x)$

is the identity map, and $W^s_{loc}(\eta(v_0))$ is a smooth manifold. Analogously, we obtain that η is uniformly C^1 along F^v. Hence, by Theorem 2.3.8 η is C^1. $\square$

2.4. Accessibility is C^1-open

Let us assume, as in the previous section, that M is a closed Riemannian 3-manifold. Call $\mathrm{PH}^r(M)$ and $\mathrm{PH}^r_m(M)$ the set of partially hyperbolic diffeomorphisms in $\mathrm{Diff}^r(M)$ and $\mathrm{Diff}^r_m(M)$ respectively. $\mathrm{PH}^r(M)$ and PH^r_m are open subsets of $\mathrm{Diff}^r(M)$ and $\mathrm{Diff}^r_m(M)$ respectively. This section is devoted to the following theorem:

THEOREM 2.4.1 (Didier, [**44**]). *The set of $f \in \mathrm{PH}^1(M)$ satisfying the accessibility property is open.*

As we mentioned in Section 2.2, this result is not known for partially hyperbolic diffeomorphisms with arbitrary center bundle dimension. It only known to be true for center dimension equal one. We shall not follow the proof of Didier [**44**], but the scheme in Burns, Rodriguez Hertz, Rodriguez Hertz, Talitskaya and Ures [**29**].

Recall that $\Gamma(f)$, the set of non-open accessibility classes is a compact laminated set (Section 2.6, Theorem 2.3.1). The strategy of our proof is to show that the set $\Gamma(f)$ varies semi-continuously:

PROPOSITION 2.4.2. *Let $\mathcal{K}(M)$ be the set of compact sets endowed with the Hausdorff metric. For each $r \in [1, \infty]$ the assignment*

$$\Gamma : \mathrm{PH}^r(M) \to \mathcal{K}(M)$$

such that $f \mapsto \Gamma(f)$ is upper semi-continuous.

Proposition 2.4.2 implies Theorem 2.4.1, due to the following: if f has the accessibility property, then there is only one accessibility class, which is open $AC(x) = M$; hence $\Gamma(f) = \emptyset$. Now assume there exists $f_n \to f$ in $\mathrm{PH}^r(M)$ such that $\Gamma(f_n) \neq \emptyset$. Since $\mathcal{K}(M)$ is a compact space when endowed with the Hausdorff topology, there exists a subsequence n_k such that $\Gamma(f_{n_k})$ converges to a compact set $K \neq \emptyset$. Since $f \mapsto \Gamma(f)$ is upper semi-continuous, we have $\emptyset \neq K \subset \Gamma(f)$, which is absurd. So Theorem 2.4.1 is reduced to proving Proposition 2.4.2.

To prove Proposition 2.4.2, let $f_n \to f$ in $\mathrm{PH}^r(M)$, and consider $x_n \in \Gamma(f_n)$ such that $x_n \to x_0$. If $x_0 \notin \Gamma(f)$, then by Proposition

2.3.6 there exists $x \in AC(x_0)$ such that W^s_f and W^u_f are not jointly integrable at x.

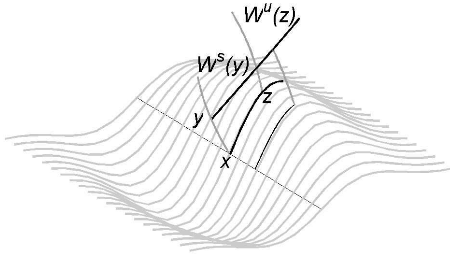

FIGURE 5. W^s and W^u are non-jointly integrable at x

By definition of joint integrability, if W^s and W^u are non-integrable at x, there are $y \in W^u_\delta(x)$ and $z \in W^s_{loc}(x)$ such that $W^s_{loc}(y) \cap W^u_{loc}(z) = \emptyset$. See Figure 5. Consider a fixed local center manifold $W^c_{loc}(x)$. Then, as it can be clearly seen in Figure 5, if we take the local unstable manifold of any point in $W^s_{loc}(y) \setminus \{y\}$, it will not meet $W^s_{loc}(x)$. Moreover this local unstable manifold of a point in $W^s_{loc}(y)$ will meet the disc $W^{sc}_{loc}(x)$ at a point w, and then the local stable manifold of w, $W^s_{loc}(w) \subset W^{sc}_{loc}(x)$ will meet $W^c_{loc}(x)$ at a point $x_1 \neq x$, see Figure 5. As we saw in last section, continuity of p_{su} and connectedness of $AC_x(x)$ implies that the whole segment $[x, x_1] \subset W^c_{loc}(x)$ is contained in $AC_x(x)$, and then in the accessibility class of x.

On the other hand, if we call $W^s_n(y) = W^s_{f_n,loc}(y)$ and $W^u_n(y) = W^u_{f_n,loc}(y)$, we have that if $\varepsilon > 0$ is small, then for all sufficiently large n, if $B_\varepsilon(x) = U$, then $V_n = W^s_n(W^u_n(W^s_n(W^u_n(U))))$ satisfies $V_n \cap U \neq \emptyset$. In particular, if $\xi \in U = B_\varepsilon(x)$, then $W^s_{f_n}$ and $W^u_{f_n}$ are not jointly integrable at ξ.

Now, since $x \in AC(x_0)$, there is an su-path joining x_0 with x. This implies that if n is large enough, there is an su-path joining x_n with a point ξ_n belonging to $B_\varepsilon(x)$. But then we would have a point of non-joint integrability in $AC_n(x_n)$, which would imply that $x_n \notin \Gamma(f_n)$, absurd. This ends the proof of Proposition 2.4.2, and then of Theorem 2.4.1, which was the goal of this section.

For further use, we state the following corollaries:

COROLLARY 2.4.3. *If $AC_f(x)$ is open, there exists an open neighborhood $\mathcal{U} \subset \mathrm{PH}^1(M)$, and $\varepsilon > 0$ such that all $AC_g(\xi)$ is open for all $g \in \mathcal{U}$ and $\xi \in B_\varepsilon(x)$.*

Since $\mathrm{PH}^r(M)$ and $\mathrm{PH}^r_m(M)$ are Baire spaces for all $r \in [1, \infty]$, and Γ is an upper semi-continuous function when restricted to each of these spaces, we also obtain:

COROLLARY 2.4.4. *The continuity points of $f \mapsto \Gamma(f)$ are residual in $\mathrm{PH}^r(M)$ and $\mathrm{PH}^r_m(M)$ for each $r \in [1, \infty]$.*

2.5. Accessibility is C^∞-dense

The result in this section, Theorem 2.5.1, holds for partially hyperbolic diffeomorphisms with one-dimensional center bundle. For simplicity, we shall consider a 3-dimensional ambient manifold M, so that the three invariant bundles E^s, E^c and E^u are one-dimensional. In [**67**], Theorem 2.5.1 was proved for volume preserving partially hyperbolic diffeomorphisms, and in [**29**], the result was extended for the non-conservative case. We follow a combination of both papers in this section.

A partially hyperbolic diffeomorphism f satisfies the *stable accessibility property* if there exists a neighborhood $\mathcal{U} \subset \mathrm{PH}^1(M)$ of f such that all diffeomorphisms g in $\mathcal{U}$ satisfy the accessibility property.

THEOREM 2.5.1. [**67**], [**29**] *Stable accessibility is C^∞-dense both in $\mathrm{PH}^1(M)$ and in $\mathrm{PH}^1_m(M)$.*

For higher dimensional center bundle, this result is only known in the C^1-topology [**46**]. See Theorem 2.8.5 in Section 2.8.

The strategy of the proof is to show the following theorem:

THEOREM 2.5.2. *If f is a continuity point of $f \mapsto \Gamma(f)$, then $\Gamma(f) = \emptyset$.*

Since, by Corollary 2.4.4, the set of continuity points of Γ is residual in $\mathrm{PH}^\infty(M)$ and $\mathrm{PH}^\infty_m(M)$; we have a residual, and then dense, set of smooth diffeomorphisms for which $\Gamma(f)$ is empty, or, equivalently, f has the accessibility property. In this section, let us denote $\mathrm{PH}^r_*(M)$ to mean indistinctly $\mathrm{PH}^r(M)$ or $\mathrm{PH}^r_m(M)$.

In the first place, we show that there is a C^r-dense set of diffeomorphisms of $\mathrm{PH}^r_*(M)$ for which the accessibility class of every periodic point is open, that is, $\Gamma(g) \cap \mathcal{P}er(g) = \emptyset$ for a C^r-dense set of $g \in \mathrm{PH}^r_*(M)$ (Subsections 2.5.1 and 2.5.2). On the other hand, in Subsections 2.5.3 and 2.5.4, we prove that if f is a continuity point of Γ with $\Gamma(f) \neq \emptyset$, then there is an open set $\mathcal{U} \subset \mathrm{PH}^r_*(M)$ such that every $h \in \mathcal{U}$ has a periodic point with non-open accessibility class, that is, $\Gamma(h) \cap \mathcal{P}er(h) \neq \emptyset$ for every $h \in \mathcal{U}$. We therefore obtain a contradiction.

In order to prove that there is a C^r-dense set of g in $\mathrm{PH}^r_*(M)$ such that $\Gamma(g) \cap \mathcal{P}er(g) = \emptyset$, recall that that if a point x is in $\Gamma(g)$, then W^s_g and W^u_g are jointly integrable at x. So, in order to get this dense set we use an unweaving method (Subsection 2.5.2), which allows us to break up the joint integrability of W^s and W^u on periodic orbits. In this way, we "open" the accessibility class of a periodic point by means of a C^r-small perturbation. The unweaving method, in turn, is based on the Keepaway Lemma (Lemma 2.5.3) which may be found in Subsection 2.5.1.

2.5.1. The Keepaway Lemma. The keepaway lemma is essential in unweaving the accessibility class of periodic points and, in fact, of an arbitrary point $x \in M$. It essentially states that if a certain ball B does not return to itself too soon in the future, then all local unstable manifolds contain a point that never enters B in the future.

We say that W^u is *uniformly expanded* by f if $\|Df^{-1}(x)|_{E^u}\| < \mu^{-1} < 1$. Observe that if W^u is μ-uniformly expanded by f, then for each $x \in M$, $k \in \mathbb{N}$ and $\delta > 0$, we have

$$W^u_\delta(f^k(x)) \subset W^u_{\mu^k\delta}(f^k(x)) \subset f^k(W^u_\delta(x)) \tag{2.7}$$

Given a point x_0, and given local center-stable manifold of x_0, $W^{sc}_{loc}(x_0)$, we denote by $V_\varepsilon(x_0)$ the set $W^u_\varepsilon(W^{sc}_{loc}(x_0))$.

LEMMA 2.5.3 (Keepaway Lemma). *Let $\mu > 1$ be such that W^u is μ-uniformly expanded by f. And let $x_0 \in M$, $\varepsilon > 0$ and $N > 0$ be such that*

(1) $\mu^N > 5$
(2) $f^n(V_{5\varepsilon}(x_0)) \cap V_\varepsilon(x_0) = \emptyset$ *for all* $n = 1, \dots, N$

then there exists $z \in W^u_\varepsilon(x_0)$ such that $o^+(z) \cap V_\varepsilon(x_0) = \emptyset$, that is, $f^n(z) \notin V_\varepsilon(x_0)$ for all $n \geq 1$.

PROOF. We construct inductively a family of discs $D_n \subset W^u(f^n(x_0))$ such that

- $D_0 = W^u_\varepsilon(x_0)$
- $D_n \subset f(D_{n-1})$ for all $n \in \mathbb{N}$, and
- $\overline{D_n} \cap V_\varepsilon(x_0) = \emptyset$.

Then, any point

$$z \in \bigcap_{n=0}^{\infty} f^{-n}(\overline{D_n})$$

will satisfy the claim.

We shall proceed inductively, in the following way:

(1) Let $D_0 = W^u_\varepsilon(x_0)$.
(2) For all $n < N$ take $D_n = f(D_{n-1})$. By hypothesis we have $\overline{D_n} \cap V_\varepsilon(x_0) = \emptyset$.
(3) For $n = N$, we have, due to (2.7) and the fact that $\mu^N > 5$, that

$$\overline{W^u_{5\varepsilon}(f^N(x_0))} \subset W^u_{\mu^N\varepsilon}(f^N(x_0)) \subset f^N(D_0)$$

and by hypothesis $f^N(D_0) \cap V_\varepsilon(x_0) = \emptyset$. Set $D_N = W^u_{5\varepsilon}(f^N(x_0))$. Then:
- $D_N \subset f(D_{N-1}) = f^N(D_0)$
- $\overline{D_N} \cap V_\varepsilon(x_0) = \emptyset$.

(4) Let $n_1 > N$ be the first integer such that $\overline{W^u_{5\varepsilon}(f^{n_1}(x_0))} \cap V_\varepsilon(x_0) \neq \emptyset$. For all $N \leq n < n_1$, set

$$D_n = W^u_{5\varepsilon}(f^n(x_0))$$

Then, for all $N \leq n \leq n_1$
- by (2.7), we have $D_n \subset f(D_{n-1})$.
- by the choice of n_1, we have $\overline{D_n} \cap V_\varepsilon(x_0) = \emptyset$.

(5) For $n = n_1$, there exists $x_{n_1} \in W^u_{4\varepsilon}(f^{n_1}(x_0))$ such that $\overline{W^u_\varepsilon(x_{n_1})} \cap V_\varepsilon(x_0) = \emptyset$. This is because of the choice of n_1, see Figure 6. Set $D_{n_1} = W^u_\varepsilon(x_{n_1})$. Then:
- by the choice of x_{x_1}, we have $D_{n_1} = W^u_\varepsilon(x_{n_1}) \subset W^u_{5\varepsilon}(f^{n_1}(x_0))$. Then,

$$f(D_{n_1-1}) = f(W^u_{5\varepsilon}(f^{n_1-1}(x_0))) \supset W^u_{5\varepsilon}(f^{n_1}(x_0)) \supset D_{n_1}$$

- Also, by the choice of x_{n_1}, we have $\overline{D_{n_1}} \cap V_\varepsilon(x_0)$.

(6) Now, go to Step (1), replace D_0 by D_{n_1}, and continue the construction with the obvious modifications.

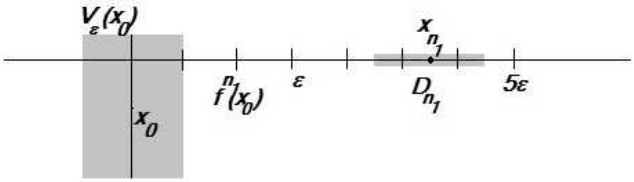

FIGURE 6. Keepaway lemma

This algorithm gives the sequence of discs D_n, and a point z (which in fact is unique), proving the lemma. □

REMARK 2.5.4. *With the same hypothesis, the proof of the Keepaway Lemma gives that, in fact, all local unstable manifolds contain a point that never enters $V_\varepsilon(x_0)$ in the future.*

We have the following corollary:

COROLLARY 2.5.5. *For all $f \in \mathrm{PH}^1_*(M)$,*

(1) *the set of non-recurrent points in the future, $\{z : z \in \omega(z)\}$ are dense in the unstable leaf $W^u(x)$ of each $x \in M$, and*
(2) *the set of non-recurrent points in the past, $\{y : y \in \alpha(y)\}$ are dense in the stable leaf $W^s(x)$ of each $x \in M$.*

PROOF. If x_0 is not periodic, then there are $\varepsilon > 0$ and $N > 0$ such that we are in the hypothesis of Lemma 2.5.3. So x_0 can be approximated by points z_n in its unstable local leaf that never enter $V_\varepsilon(x_0)$ in the future, hence z_n cannot be recurrent in the future.

If x_0 is periodic, then it is approximated by non-periodic points in its local unstable leaf, so it is also approximated by points which are non-recurrent in the future. □

2.5.2. The Unweaving Lemma. The proof of the Unweaving Lemma uses the existence of a certain quadrilateral satisfying some technical conditions (see Lemma 2.5.6), which, in turn, are based on the Keepaway lemma. We defer the proof of Lemma 2.5.6 until the end of this subsection, and show how to use it to prove the Unweaving Lemma.

LEMMA 2.5.6 (Existence of a quadrilateral). *Let K be a minimal set contained in $\Gamma(f)$. Then there exist $x \in K$, $y \notin K$, $z, w \in M$, and $\varepsilon > 0$ such that*

(1) $B_\varepsilon(y) \cap K = \emptyset$
(2) $y \in W^u_{loc}(x)$,
(3) $z \in W^s_{loc}(x)$,
(4) $w \in W^s_{loc}(y) \cap W^u_{loc}(z)$. *See Figure 7*
(5) $f^n(W^s_{loc}(y)) \cap B_\varepsilon(y) = \emptyset$ *for all* $n \geq 1$
(6) $f^{-n}(W^u_{loc}(z)) \cap B_\varepsilon(y) = \emptyset$ *for all* $n \geq 0$
(7) $f^{-n}(W^u_{loc}(x)) \cap B_\varepsilon(y) = \emptyset$ *for all* $n \geq 1$
(8) $f^n(W^s_{loc}(x)) \cap B_\varepsilon(y) = \emptyset$ *for all* $n \geq 0$

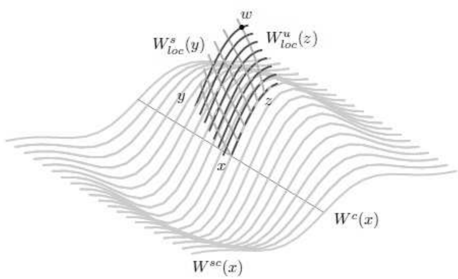

FIGURE 7. Unweaving Lemma: before perturbing (Lemma 2.5.6)

Using the points obtained in the lemma above, we can establish the following lemma:

LEMMA 2.5.7 (Unweaving Lemma). *Let K be a minimal set contained in $\Gamma(f)$, then f can be C^r-approximated in $\mathrm{PH}^r_*(M)$ by diffeomorphisms g such that*

- *$f|_K = g|_K$, and*
- *$AC_g(x)$ is open for each $x \in K$.*

Proof of the Unweaving Lemma. Let $x \in K$, $y, z, w \in M$ and $\varepsilon > 0$ be chosen as in Lemma 2.5.6. Then, there is a C^r-perturbation in $\mathrm{PH}^r_*(M)$ of the form $g = f \circ h$, with $\mathrm{supp}(h) \subset B_\varepsilon(y)$, such that

$$W^s_{g,loc}(w) \cap W^u_{f,loc}(y) = \emptyset \tag{2.8}$$

(see Figure 8). The fact that $\mathrm{supp}(h) \subset B_\varepsilon(y)$ implies that $f|_K = g|_K$.

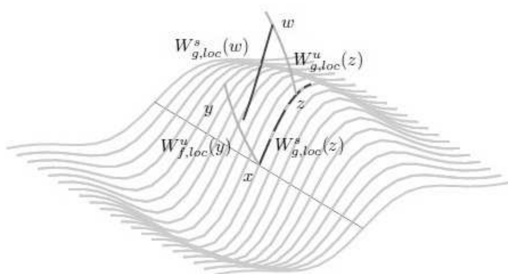

Figure 8. Unweaving Lemma: after perturbing

Now Properties (5)-(8) in Lemma 2.5.6 imply:

- $W^u_{f,loc}(x) = W^u_{g,loc}(x)$
- $W^s_{f,loc}(x) = W^s_{g,loc}(x)$, and
- $W^u_{f,loc}(z) = W^u_{g,loc}(z)$.

Hence $y \in W^u_{f,loc}(x) = W^u_{g,loc}(x)$ and $w \in W^u_{f,loc}(z) = W^u_{g,loc}(z)$ are close points that are in the same accessibility class $AC_g(x)$. But, due to (2.8), and first item above, we have $W^s_{g,loc}(w) \cap W^u_{g,loc}(y) = \emptyset$. Hence x is a point of non-joint integrability of W^s_g and W^u_g. Proposition 2.3.6 implies that x is not in $\Gamma(g)$, so $AC_g(x)$ is open.

Now, since K is minimal for f, and g coincides with f on K, then K is minimal for g and all points ξ in K have an iterate $g^n(\xi)$ in $AC_g(x)$. Then $g^n(\xi) \notin \Gamma(g)$ for some $n \in \mathbb{Z}$. This implies $K \subset M \setminus \Gamma(g)$, due to the fact that $\Gamma(g)$ is g-invariant. $\square$

Let us finish this subsection with the proof of Lemma 2.5.6. This is a technical proof, and is independent of the rest of the topics, so it can be skipped.

Proof of Lemma 2.5.6. Let x be any point of K. By Corollary 2.5.5, there exists $y \in W^u_{loc}(x)$ such that y is not forward recurrent (in particular, $y \notin K$). Hence, there exists $\varepsilon > 0$ such that $f^n(y) \notin B_\varepsilon(y)$ for all $n \geq 1$ and $B_\varepsilon(y) \cap K = \emptyset$. Moreover, since y is not periodic, we can consider $\varepsilon > 0$ such that $f^n(W^s_{loc}(y)) \cap B_\varepsilon(y) = \emptyset$ for all $n \geq 1$. This proves items (1) and (5).

Now, $f^n(x) \notin B_\varepsilon(y)$ for all $n \in \mathbb{Z}$, since $f^n(B_\varepsilon(y)) \cap K = \emptyset$. We can reduce $\varepsilon > 0$ so that $f^n(W^s_{loc}(x)) \cap B_\varepsilon(y) = \emptyset$ for all $n \geq 0$ and $f^{-n}(W^u_{loc}(x)) \cap B_\varepsilon(y) = \emptyset$ for all $n \geq 1$. This proves items (2), (7) and (8).

Since y satisfies item (1), y is not periodic, then we can reduce $\varepsilon > 0$ and $W^{uc}_{loc}(y)$ so that $V_\varepsilon(y)$ (as constructed after Equation (2.7)) is in the hypothesis of Lemma 2.5.3 applied to f^{-1}. Remark 2.5.4 implies that there is $z \in W^s_{loc}(x)$ such that $f^{-n}(z) \notin V_\varepsilon(y)$ for all $n \geq 0$, and so $f^{-n}(W^u_{loc}(z)) \cap B_\varepsilon(y) = \emptyset$ for all $n \geq 0$. This proves items (3) and (6).

We have that x belongs to $\Gamma(f)$, and hence it is a point of joint integrability (Proposition 2.3.6). On the other hand, $y \in W^u_{loc}(x)$ and $z \in W^s_{loc}(x)$ are arbitrarily close to x, hence there exists $w = W^s_{loc}(y) \cap W^u_{loc}(z)$. This proves item (4), and the lemma. □

2.5.3. The conservative setting. In Subsection 2.5.4, we give the general proof of Theorem 2.5.1. However, since it is illustrative and there are some interesting particularities that are used in other settings (see, for instance, Chapter 5). Let us begin by proving the following theorem:

Theorem 2.5.8. *The set of diffeomorphisms f such that $\Gamma(f) = M$ is a closed set with empty interior in $\mathrm{PH}^r_*(M)$ for all $r \in [1, \infty]$*

Proof. Since $f \mapsto \Gamma(f)$ is upper semi-continuous in $\mathrm{PH}^r_*(M)$, for all $r \in [1, \infty]$, due to Proposition 2.4.2, the set of diffeomorphisms satisfying $\Gamma(f) = M$ is clearly closed.

Now let f be a diffeomorphism with $\Gamma(f)$, and let K be a minimal set in $\Gamma(f)$. Then, by the Unweaving Lemma 2.5.7, f is approximated in $\mathrm{PH}^r_*(M)$ by diffeomorphisms g for which $AC_g(x)$ is open if $x \in K$. In particular, $\Gamma(g) \neq M$. $\square$

DEFINITION 2.5.9. *If $\Gamma(f)$ is a strict non-empty subset of M, the* accessible boundary *of $\Gamma(f)$ is the set of points $x \in \Gamma(f)$ for which there exists an arc α with $\alpha(0) = x$ and $\alpha \cap \Gamma(f) = \{x\}$. The accessible boundary of $\Gamma(f)$ is denoted by $\partial_a \Gamma(f)$.*

Observe that the name accessible boundary does not have to do with the accessibility property, but with the fact of being accessed by an arc from the exterior of the set. Also, note that the accessible boundary is an invariant set, but it is not closed. In particular, it does not coincide with the boundary of $\Gamma(f)$. Finally, note that α in Definition 2.5.9 can always be chosen to be contained in $W^c_{loc}(x)$.

THEOREM 2.5.10. *If $f \in \mathrm{PH}^r_m(M)$, and $\emptyset \neq \Gamma(f) \neq M$, then $\mathcal{P}er(f) \cap \Gamma(f) \neq \emptyset$. Moreover, $\mathcal{P}er(f)$ is dense in each local leaf $W^{su}_{loc}(x)$ of the accessible boundary $\partial_a \Gamma(f)$.*

We shall assume that f preserves the orientation of a local foliation transverse to $\Gamma(f)$ at $x \in \partial_a \Gamma(f)$.

PROOF. Let $x \in \partial_a \Gamma(f)$ and consider a small arc $\alpha_x \subset W^c_{loc}(x)$ such that $\alpha_x(0) = x$, and $\alpha_x \cap \Gamma(f) = \{x\}$, as in Definition 2.5.9. We can assume α_x is so small that if $U = W^s_{loc}(W^u_{loc}(\alpha)(x))$, then

$$\overline{U} \cap \Gamma(f) = W^{su}_{loc}(x).$$

We can also consider U so small that the local center manifolds of all $y \in U$ meet $\Gamma(f)$.

Since f is conservative, the non-wandering set of f is M. So, there exists $y \in U$ and $n > 0$ arbitrarily large such that $f^n(y) \in U$. Let $a_y \in W^{su}_{loc}(x)$ be such that $(a_y, y)^c \subset U \cap W^c_{loc}(y)$, then $f^n(a_y, y)^c = (f^n(a_y), f^n(y))^c \subset U \cap W^c_{loc}(f^n(y))$. But $a_y \in \partial_a \Gamma(f)$, and the accessible boundary of $\Gamma(f)$ is invariant, so $f^n(a_y) \in \partial_a \Gamma(f)$, hence $f^n(a_y) \in W^{su}_{loc}(x) = \overline{U} \cap \Gamma(f)$.

This implies that there are arbitrarily large iterates of a_y in $W^{su}_{loc}(x)$. The proof of Theorem 2.5.10 finishes now with the following:

LEMMA 2.5.11 (Anosov Closing Lemma). *There exists $n_0 > 0$ and $\varepsilon > 0$ such that if $z \in W^{su}_\varepsilon(x)$ is such that $f^n(z) \in W^{su}_\varepsilon(x)$ with $n \geq n_0$, then there is a periodic point in $W^{su}_\varepsilon(x)$ of period n.*

□

The following proposition holds both in the conservative and the non-conservative setting:

PROPOSITION 2.5.12. *For each $r \in [1, \infty]$ there exists a C^r-dense set of diffeomorphisms in $\mathrm{PH}^r_*(M)$ with the property that the accessibility class of every periodic point is open.*

PROOF. For $k \geq 1$, let $\mathcal{U}_k$ denote the set of all diffeomorphisms in $\mathrm{PH}^r_*(M)$ with the property that the periodic points of period k are all periodic. Each $\mathcal{U}_k$ is an open and dense set in $\mathrm{PH}^r_*(M)$ by the Kupka-Smale theorem. The number of periodic points of period k is finite and constant on each component of $\mathcal{U}_k$. From the Unweaving Lemma 2.5.7 it follows that $\mathcal{U}_k$ has a C^r-dense subset $\mathcal{V}_k$ such that the accessibility class of every periodic point with period k is open if the diffeomorphism is in $\mathcal{V}_k$. The set $\mathcal{V}_k$ is open, by Corollary 2.4.3. Then $\mathcal{V}_k$ is open and dense in $\mathrm{PH}^r_*(M)$ for each $k \geq 1$. Then the set $\mathcal{R} = \bigcup_{k \geq 1} \mathcal{V}_k$ is a residual set since $\mathrm{PH}^r_*(M)$ is a Baire space, in particular it is C^r-dense. All $f \in \mathcal{R}$ have the property that the accessibility class of every periodic point is open. □

Let us show the conservative version of Theorem 2.5.2:

THEOREM 2.5.13 (Conservative setting). *If f is a continuity point of $f \mapsto \Gamma(f)$, then $\Gamma(f) = \emptyset$.*

PROOF. Let us first see that $\Gamma(f)$ can not be a non-empty strict subset of M. If f is a continuity point of $f \mapsto \Gamma(f)$ such that $\emptyset \neq \Gamma(f) \neq M$, then there is a neighborhood $\mathcal{U}$ in $\mathrm{PH}^r_*(M)$ such that all $g \in \mathcal{U}$ satisfy $\emptyset \neq \Gamma(g) \neq M$. Now, Theorem 2.5.10 implies that $\mathcal{P}er(g) \cap \Gamma(g) \neq \emptyset$ for all $g \in \mathcal{U}$, but, on the other hand, by Proposition 2.5.12 there is a dense set in $\mathrm{PH}^r_*(M)$ for which the accessibility class of every periodic orbit is open. This is a contradiction.

Then, if f is a continuity point, we have that either $\Gamma(f) = M$ or $\Gamma(f) = \emptyset$, but in Theorem 2.5.8, we have shown that the set of f for which $\Gamma(f) = M$ has empty interior. Since, by Corollary 2.4.4, the continuity points of Γ are a residual, and in particular, dense, f is approximated by f_n for which $\Gamma(f_n) = \emptyset$ (otherwise, semi-continuity would imply there is an open set of g such that $\Gamma(g) = M$). But this implies that if $\Gamma(f) = M$ then f is not a continuity point. This proves the theorem. $\square$

2.5.4. The non-conservative setting. Let us state the following stronger version of the Anosov Closing Lemma. It holds since the global stable and unstable bundle vary continuously in a neighborhood of any $f \in \mathrm{PH}^r(M)$, and they form uniform angles.

LEMMA 2.5.14 (Anosov Closing Lemma). *For each $f \in \mathrm{PH}^r(M)$ there exists a neighborhood $\mathcal{U} \subset \mathrm{PH}^r(M)$ of f, an integer $N > 0$ and a small number $\varepsilon > 0$, such that if $x \in \Gamma(g)$ is such that $g^n(x) \in W^{su}_{g,\varepsilon}(x)$, with $n \geq N$, then there exists a point of period n in $W^{su}_{g,\varepsilon}(x)$.*

PROPOSITION 2.5.15. *If $f \in \mathrm{PH}^r(M)$ is a continuity point of Γ, for which $\Gamma(f) \neq \emptyset$, there exists an open set $\mathcal{V} \subset \mathrm{PH}^r(M)$ arbitrarily close to f such that for all $h \in \mathcal{V}$:*

$$\mathcal{P}er(f) \cap \Gamma(h) \neq \emptyset$$

After proving this proposition, the proof of Theorem 2.5.1 follows exactly as the proof of Theorem 2.5.13, using Proposition 2.5.15 instead of Theorem 2.5.10.

PROOF. Let f be a continuity point of Γ. And consider $\mathcal{U} \subset \mathrm{PH}^f(M)$, $N > 0$ and $\varepsilon > 0$ as in the Anosov Closing Lemma 2.5.14. We can reduce $\mathcal{U}$, so that for some $\delta > 0$ we have the following properties:

(1) $d_H(\Gamma(f), \Gamma(g)) < \delta/2$ for all $g \in \mathcal{U}$, where d_H is the Hausdorff distance.
(2) $W^c_{g,loc}(x) \cap \Gamma(g) \neq \emptyset$ if $d(x, \Gamma(g)) < \delta$. Call $b_x \in \Gamma(g)$ the first point in $W^c_{g,loc}(x) \cap \Gamma(g)$, once we have chosen a local center leaf and an orientation (coherent in a whole neighborhood of x)
(3) if $x, y \in B_\delta(\Gamma(g))$, $d(x,y) < \delta$, then $b_y \in W^{su}_{g,\varepsilon}(b_x)$.

$\Gamma(f)$ contains a minimal set K. By the Unweaving Lemma 2.5.7, there exists $g \in \mathcal{U}$ coinciding with f over K, for which $AC_g(x)$ is open for all $x \in K$.

There exists $x_0 \in K$ and $n > N$ such that $g^n(x_0) \in B_{\delta/2}(x_0)$, and g^n preserves the orientation of the local center leaves near x_0. Now, there is $\mathcal{V}(g) \subset \mathcal{U}$ so that all $h \in \mathcal{V}(g)$ satisfy that $h^n(x_0) \in B_\delta(x_0)$, and $AC_h(x_0)$ is open (Corollary 2.4.3).

Choose an orientation of the center leaves in $B_\delta(x_0)$. Then, by (3), $b_{h^n(x_0)} \in W^{su}_{h,\varepsilon}(b_{x_0})$. Now, the center arc $[x_0, b_{x_0}]^c$ meets $\Gamma(h)$ only at b_{x_0}. Hence $h^n(x_0, b_{x_0})^c = \emptyset$, and $h^n(b_{x_0}) \in \Gamma(h)$, due to the invariance of $\Gamma(h)$. This implies that $b_{h^n(x_0)} = h^n(b_{x_0})$. So, by the Anosov Closing Lemma 2.5.14, there exists an n-periodic point in $W^{su}_{h,\varepsilon}(b_{x_0}) \subset \Gamma(h)$. This proves the claim. $\square$

2.6. Accessibility implies ergodicity

In this Section we shall consider $f \in \mathrm{Diff}^{1+\alpha}_m(M)$, and M of dimension 3. The hypothesis on the differentiability can not be removed so far, since it is essential to the absolute continuity of the stable and unstable foliations. Theorem 2.6.1 as it is stated here is valid for any $C^{1+\alpha}$ volume preserving diffeomorphism with one-dimensional center bundle [**67**]. The strongest version so far, is Theorem 2.8.3, by Burns and Wilkinson [**33**].

THEOREM 2.6.1. *If $f \in \mathrm{Diff}^{1+\alpha}_m(M^3)$ satisfies the accessibility property, then it is ergodic.*

In fact, the weaker property of essential accessibility property (see its definition at the beginning of Section) is already enough to establish Theorem 2.6.1. See below.

As we said in the introduction, in order to prove Theorem 2.6.1, we shall introduce two new Vitali bases, called s-juliennes and u-juliennes. This is done in Subsection 2.6.1. We shall prove that the density points according to these bases are the Lebesgue density points if we consider essentially s- and essentially u-saturated sets. A is an *essentially s-saturated set* if it coincides modulo a zero set with an s-saturated set A_s, that is $m(A \triangle A_s) = 0$. Essentially u-saturated sets are defined analogously. An *essentially bi-saturated set* is a set that is both essentially s- and essentially u-saturated. The density points according to these new bases bases satisfy the following theorems:

THEOREM 2.6.2. *Let $f \in \mathrm{PH}^{1+\alpha}_m(M^3)$. Then*

(1) *The s-julienne density points of an essentially u-saturated set coincide with the Lebesgue density points of this set.*
(2) *The u-julienne density points of an essentially s-saturated set coincide with the Lebesgue density points of this set.*

which will be proved in Subsection 2.6.3

THEOREM 2.6.3. *Let* $f \in \mathrm{Diff}^{1+\alpha}_m(M^3)$. *Then:*

(1) *The s-julienne density points of an essentially s-saturated set are preserved by stable holonomies; hence, they form an s-saturated set.*
(2) *The u-julienne density points of an essentially u-saturated set are preserved by stable holonomies; hence, they form an u-saturated set.*

which will be proved in Subsection 2.6.2.

As a conclusion, for each continuous observable $\varphi : M \to \mathbb{R}$, since the set

$$A = \{x : \hat{\varphi}(x) \geq \int \varphi dm\}$$

is essentially s- and essentially u-saturated, its Lebesgue density points are s- and u-saturated. Then essential accessibility implies that the Lebesgue density points of A have measure 1, thus $\hat{\varphi}(x) \geq \int \varphi dm$ for m-almost every x. But $\int \hat{\varphi} dm = \int \varphi dm$, so $\hat{\varphi}(x) = \int \varphi dm$ for m-almost every x. This implies that f is ergodic, and proves Theorem 2.6.1.

2.6.1. Juliennes. In this section we shall construct a Vitali basis $\{J^{suc}_n(x)\}_{x\in M}$ which is dynamically defined. The density points of a set A according to this basis, will be called *s-julienne density points*; namely, the points x satisfying:

$$\lim_{n\to\infty} \frac{m(J^{suc}_n(x) \cap A)}{m(J^{suc}_n(x))} = 1 \tag{2.9}$$

As we said above, in Subsection 2.6.2 we shall prove that the s-julienne density points of an essentially u-saturated set form an s-saturated set, that is, are preserved under stable holonomies (Theorem 2.6.3); and in Subsection 2.6.3 we shall see that the s-julienne density points are the Lebesgue density points of any s-saturated set.

For the sake of simplicity, we shall assume that f is *absolutely partially hyperbolic*, that is, there exist constants $\lambda < 1$, $\mu > 1$, and γ, ν such that for all unit vectors $v^s \in E^s$, $v^c \in E^c$ and $v^u \in E^u$, we have:

$$\|Df(x)v^s\| < \lambda < \gamma < \|Df(x)v^c\| < \nu < \mu < \|Df(x)v^u\| \tag{2.10}$$

The difference with a (non-absolutely) partially hyperbolic diffeomorphism is that in the latter λ, μ, γ and ν are not necessarily fixed. The proof of the general case is not much more complicated, in fact, there are almost no differences; but the notation becomes substantially lighter, helping to see the essential ideas.

Since E^c is one-dimensional, there is essentially one unit vector in E^c, $\pm v^c$, so γ and ν in (2.10) can be taken so that $\frac{\gamma}{\nu} \sim 1$, in particular, we may require:

$$\lambda < \frac{\gamma}{\nu} < 1 < \frac{\nu}{\gamma} < \mu \tag{2.11}$$

This is called the *center bunching condition.* Now, let $\sigma > 0$ be chosen so that:

$$\frac{\lambda}{\gamma} < \sigma < \min\left(1, \frac{1}{\nu}\right) \tag{2.12}$$

Let us denote by $W^c_n(x)$ the set $W^c_{\sigma^n}(x)$, for any local center manifold. Also define:

$$J^u_n(x) = f^{-n}(W^u_{\lambda^n}(f^n(x))) \tag{2.13}$$

Then we introduce the following dynamically defined local u-saturation of $W^c_n(x)$:

$$J^{uc}_n(x) = J^u_n(W^c_n(x)) = \bigcup_{y \in W^c_n(x)} J^u_n(y) \tag{2.14}$$

that is, over each point $y \in W^c_n(x)$, we consider a local unstable manifold of a variable length, which dynamically depends on y. The Vitali basis that gives the s-julienne density points defined in Equation (2.9) is given by:

$$J^{suc}_n(x) = W^s_{\sigma^n}(J^{cu}_n(x)) \tag{2.15}$$

That is, the s-juliennes are the local stable saturation of the dynamically defined cu-disc $J^{uc}_n(x)$, given in (2.14).

2.6.2. Proof of Theorem 2.6.3. First of all let us recall the notion of *stable holonomy.* Given $y \in W^s_{loc}(x)$, there exists a homeomorphism $h^s : W^{uc}_{loc}(x) \to W^{uc}_{loc}(y)$ such that, for each $\xi \in W^{uc}_{loc}(x)$

$$h^s(\xi) = W^s_{loc}(\xi) \cap W^{uc}_{loc}(y) \tag{2.16}$$

For each fixed $W^{uc}_{loc}(x)$, let us denote $m_{uc}(A)$ the induced Riemannian volume of $A \cap W^{uc}_{loc}(x)$ in the local manifold $W^{uc}_{loc}(x)$. As we have mentioned in the introduction, stable foliation is absolutely continuous, as proven by Brin and Pesin [**26**]. Concretely, we have the following:

PROPOSITION 2.6.4 (Absolute continuity of the stable foliation [**26**]). *There exists $K > 1$ such that for each $x \in M$ and for all measurable sets A,*

$$\frac{1}{K} \leq \frac{m_{uc}(h^s(A \cap W^{uc}_{loc}(x)))}{m_{uc}(A \cap W^{uc}_{loc}(x))} \leq K \tag{2.17}$$

In fact, the stable holonomy can be defined between any two small discs transverse to $W^s_{loc}(x)$, and it continues to be absolutely continuous; this means, (2.17) continues to hold if we replace m_{uc} by m_D and $m_{D'}$, where D and D' are two small discs transverse to $W^s_{loc}(x)$. As a corollary, we have the following lemma:

LEMMA 2.6.5. *For each $x \in M$, and $K > 1$ obtained in Proposition 2.6.4, we have, for sufficiently large n, and any s-saturated set A_s:*

$$\frac{1}{K} \leq \frac{m(A_s \cap J^{suc}_n(x))}{2\sigma^n m_{uc}(A_s \cap J^{uc}_n(x))} \leq K \tag{2.18}$$

To see this, take a foliation of a neighborhood of x, that is transverse to $W^s_{loc}(x)$, and contains $W^{uc}_{loc}(x)$. Abusing notation, call m_{uc} the induced Riemannian volume on each leaf of this local foliation. Call m_s the induced Riemannian volume on $W^s_{loc}(x)$. We have:

$$m(A_s \cap J^{suc}_n(x)) = \int_{W^s_{\sigma^n}(x)} m_{cu}(h^s(A_s \cap J^{uc}_n(x)))dm_s(y)$$

Then, by Proposition 2.6.4 above, we get (2.18).

Let us note that if A is an essentially s-saturated set, and A_s is an s-saturated set such that $m(A \triangle A_s) = 0$, then the s-julienne density points of A coincide with the set of s-julienne density points of A_s. Indeed, for any $x \in M$, $m(A \cap J^{suc}_n(x)) = m(A_s \cap J^{suc}_n)$, so the quotients (2.9) are the same for A and for A_s. So, Theorem 2.6.3 is reduced to proving that the s-julienne density points of an s-saturated set form an s-saturated set, that is, they are invariant under stable holonomies h^s.

Lemma 2.6.5 and the previous discussion tell us that in order to prove Theorem 2.6.3 it suffices to control the distortion of $A_s \cap J_n^{uc}(x)$ under the action of stable holonomies. This is essentially the content of Proposition 2.6.6 below, whose proof we defer for a while. This proposition is the most substantial step in the proof of Theorem 2.6.3.

PROPOSITION 2.6.6. *There exists $k \geq 1$ such that, for every $x \in M$, the stable holonomy map between $W_{loc}^{uc}(x)$ and $W_{loc}^{uc}(h^s(x))$ satisfies:*

$$J_{n+k}^{uc}(h^s(x)) \subset h^s(J_n^{uc}(x)) \subset J_{n-k}^{uc}(h^s(x)) \qquad \forall n \geq k \tag{2.19}$$

Assume the validity of Proposition 2.6.6 for now, and let us finish the proof of Theorem 2.6.3. As we stated above, it suffices to see that the s-julienne density points of an s-saturated set A_s are invariant under stable holonomies. We shall instead consider an s-julienne density point x of the complement of A_s and see that $h^s(x)$ is also an s-julienne density point of $M \setminus A_s$. By our assumption we have:

$$\lim_{n \to \infty} \frac{m(A_s) \cap J_n^{suc}(x)}{J_n^{suc}(x)} = 0$$

so, by our bounds (2.18), we obtain:

$$\lim_{n \to \infty} \frac{m_{uc}(A_s \cap J_n^{uc}(x))}{m_{uc}(J_n^{uc}(x))} = 0$$

Now, Proposition 2.6.6 implies that, for some fixed $k \geq 1$, and all $n \geq k$

$$m_{uc}(h^s(A_s \cap J_{n+k}^{uc}(x))) \leq m_{uc}(A_s \cap J_n^{uc}(h^s(x))) \leq m_{uc}(h^s(A_s \cap J_{n-k}^{uc}(x)))$$

Then, by the absolute continuity of the stable holonomy, Lemma 2.6.4, we obtain that

$$\frac{1}{K} m_{uc}(A_s \cap J_{n+k}^{uc}(x)) \leq m_{uc}(A_s \cap J_n^{uc}(h^s(x))) \leq K m_{uc}(A_s \cap J_{n-k}^{uc}(x))$$

These inequalities hold for the particular case $A_s = M$, hence we obtain that:

$$\frac{m_{uc}(A_s \cap J_{n+k}^{uc}(x))}{K m_{uc}(J_{n-k}^{uc}(x))} \leq \frac{m_{uc}(A_s \cap J_n^{uc}(h^s(x)))}{m_{uc}(J_n^{uc}(h^s(x)))} \leq \frac{K m_{uc}(A_s \cap J_{n-k}^{uc}(x))}{m_{uc}(J_{n+k}^{uc}(x))}$$

Since $k \geq 1$ is fixed, by definition of $J_n^{cu}(x)$ there exists $N > 0$ such that for all $n \geq N$, we have

$$\frac{m_{uc}(J_{n-k}^{uc}(x))}{m_{uc}(J_{n+k}^{uc}(x))} \leq K$$

Hence we obtain that

$$\lim_{n \to \infty} \frac{m_{uc}(A_s \cap J_n^{uc}(h^s(x)))}{m_{uc}(J_n^{uc}(h^s(x)))} = 0$$

Lemma 2.6.5 again, implies that

$$\lim_{n \to \infty} \frac{m(A_s \cap J_n^{suc}(x))}{m(J_n^{suc}(x))} = 0$$

what ends the proof of Theorem 2.6.3.

To finish this section let us proceed to prove Proposition 2.6.6.

PROOF OF PROPOSITION 2.6.6. The proof consists in two main steps:

(1) showing that h^s does not distort much the central base $W_n^c(x)$ of $J_n^{uc}(x)$
(2) showing that h^s does not distort much each fiber $J_n^u(y)$ with $y \in W_n^c(x)$

Obviously these two claims together imply $h^s(J_n^{uc}(x)) \subset J_{n-k}^{uc}(h^s(x))$ for some fixed k independent of x, and for every stable holonomy map. Since the inverse of the holonomy map is another holonomy map, we get the claim.

The proof of Step (1) involves the following crucial lemma, which we shall not prove, see for instance [**33**].

LEMMA 2.6.7 (C^1-stable holonomies within local center stable leaves). *The stable holonomy is C^1 with uniform bounds (not depending on x), when restricted to each local center stable leaf $W_{loc}^{sc}(x)$. In particular, there exists $L > 1$ such that for all $x \in M$*

$$d(h^s(y), h^s(z)) \leq Ld(x, y) \qquad \forall y, z \in W_{loc}^c(x) \tag{2.20}$$

As a consequence of this we get:

PROPOSITION 2.6.8 (Step 1). *There exists $k \geq 1$ such that for all $x \in M$ and $n \geq k$, we have:*

$$h^s(W_n^s(x)) \subset W_{n-k}^c(h^s(x))$$

PROOF. Let $y \in W_n^c(x)$, then by definition $y \in W_{loc}^c(x)$ with $d(x,y) \leq \sigma^n$. Lemma 2.6.7 above implies:

$$d(h^s(x), h^s(y)) \leq Ld(x,y) \leq L\sigma^n \leq \sigma^{n-k}$$

for some fixed $k \geq 1$, which can be obviously chosen independent of x. $\square$

$\square$

PROPOSITION 2.6.9 (Step 2). *There exists $k \geq 1$ such that for all $x \in M$ and $n \geq k$, if $z \in W_n^c(x)$, then*

$$h^s(J_n^u(z)) \subset J_{n-k}^{uc}(h^s(x)) \tag{2.21}$$

As a consequence, $h^s(J_n^{uc}(x)) \subset J_{n-k}^{uc}(h^s(x))$.

PROOF. Let $W_{loc}^c(x)$ be given, and choose a local center leaf $W_{loc}^c(h^s(x))$ contained in $W_{loc}^{sc}(x)$. This always exists, it suffices to consider any local center unstable leaf $W_{loc}^{uc}(h^s(x))$, then its intersection with $W_{loc}^{sc}(x)$ gives the desired local center leaf.

Let $z \in W_n^c(x)$ and $y \in J_n^u(z)$, then by definition we have:

- $d(f^n(y), f^n(z)) \leq \lambda^n$ with $f^n(y) \in W_{loc}^u(f^n(z))$, and
- $d(z,x) \leq \sigma^n$ with $W_{loc}^c(x)$

Recall that $h^s(y) \in W_{loc}^s(y)$ and $W_{loc}^s(z)$, so

$$\begin{aligned} d(f^n(h^s(x)), f^n(h^s(z))) \quad \leq \quad & d(f^n(h^s(y)), f^n(y)) + d(f^n(y), f^n(z)) \\ & + d(f^n(z), f^n(h^s(z))) \leq 3\lambda^n \end{aligned}$$

Let $h^s(w) \in W_{loc}^u(h^s(y)) \cap W_{loc}^c(h^s(x))$, with $w \in W_{loc}^c(x)$. Since the angle between E^c and E^u is uniformly bounded from below, and since the unstable holonomy is uniformly C^1 in the local center unstable leaves, in particular in $W_{loc}^{uc}(f^n(h^s(x)))$, by Lemma 2.6.7, there exists $C > 0$ such that:

- $d(f^n(h^s(y)), f^n(h^s(w))) \leq C\lambda^n$
- $d(f^n(h^s(w)), f^n(h^s(z))) \leq C\lambda^n$

Note that this happens for this fixed n, both y and w depend on n, also that $C > 0$ does not depend on n or x. Let $k > 0$ be greater than the one found in Proposition 2.6.8, and such that:

$$\max(C+1, 3) \leq \lambda^{-k} \tag{2.22}$$

Then, since $f^n(h^s(w)) \in W^u_{C\lambda^n}(f^n(h^s(y)))$,

$$d(f^{n-k}(h^s(w)), f^{n-k}(h^s(y))) \leq d(f^n(h^s(w)), f^n(h^s(y))) \leq \lambda^{n-k}$$

which implies that $h^s(y) \in J^u_{n-k}(h^s(w))$. Also, by the second item above and Equations (2.10) and (2.12), we have:

$$d(h^s(w), h^s(z)) \leq C\lambda^n \nu^{-n} \leq C\sigma^n$$

Now, since $z \in W^c_n(x)$, Proposition 2.6.8 implies $d(h^s(z), h^s(x)) \leq \sigma^{n-k}$, from our choice of k it follows:

$$d(h^s(w), h^s(x)) \leq (C+1)\sigma^n \leq \sigma^{n-k}$$

Hence, we got that $h^s(y) \in J^u_{n-k}(h^s(w))$, with $h^s(w) \in W^c_{n-k}(h^s(x))$, which means that $h^s(y) \in J^{uc}_{n-k}(h^s(x))$. This finishes the proof of Step 2, and of Theorem 2.6.3. $\square$

2.6.3. Proof of Theorem 2.6.2. For each $n \geq 1$, and each $x \in M$, define

$$W^{sc}_n(x) = W^s_\sigma(W^c_n(x)) \sim W^{sc}_{\sigma^n}(x) \tag{2.23}$$

Use definition above to introduce the following Vitali basis:

$$JW^{usc}_n(x) = J^u_n(W^{sc}_n(x)) = \bigcup_{y \in W^{sc}_n(x)} J^u_n(y) \tag{2.24}$$

Let us first prove the following proposition

PROPOSITION 2.6.10. *The s-julienne density points of any set coincide with the JW^{usc}_n-density points*

PROOF. We shall find $k \geq 1$ and $K > 11$ such that for each $x \in M$ and $n \geq k$

(1) $J^{suc}_{n-k}(x) \subset JW^{usc}_n(x) \subset J^{suc}_{n-k}(x)$, and
(2) $\frac{m(JW^{usc}_{n+k}(x))}{JW^{usc}_n(x)} \geq \frac{1}{K}$

Let $z \in JW^{usc}_n(x)$, then $z \in J^u_n(y)$, with $y \in W^{sc}_n(x)$. If we choose k greater than the one obtained in Propositions 2.6.8 and 2.6.9, then we have that $y \in W^c_{n-k}(h^s(x))$, where h^s is the holonomy map from $W^c_{loc}(x)$ to $W^c_{loc}(y)$. In particular, $z \in J^{cu}_{n-k}(h^s(x))$. If we consider now the stable holonomy map going from $W^{cu}_{loc}(h^s(x))$ to $W^{cu}_{loc}(x)$, and call it h^s abusing notation, we have, by our choice of k, that $h^s(J^{uc}_{n-k}(h^s(x))) \subset J^{cu}_{n-2k}(x)$. Since the angle between E^s, E^c and E^u is bounded from below, there exists $j \geq 2k$ such that

$z \in J^{uc}_{n-k}(h^s(x)) \subset J^{suc}_{n-j}(x)$. Rename j and call it our new k. This proves one of the inclusions in (1).

To prove the other one, take $z \in J^{suc}_n(x)$, then $z \in W^s_{\sigma^n}(y)$, with $y \in J^{uc}_n(x)$. If we call $h^s(x) = W^s_{loc}(x) \cap W^{uc}_{loc}(z)$, then by Proposition 2.6.9 we have that $z \in h^s(J^{uc}_n(x)) \subset J^{uc}_{n-k}(h^s(x))$. Then $z \in J^u_{n-k}(\xi)$, with $\xi \in W^c_{n-k}(h^s(x))$. Since the angles between E^s, E^c and E^u are uniformly bounded from below, there exists $l \geq k$ such that $W^c_{n-k}(h^s(x)) \subset W^{sc}_{n-l}(x)$. Rename l and call it our new k. This finishes the proof of (1).

To prove (2), let us use that, by the absolute continuity of the stable and unstable holonomies (Proposition 2.6.4), there is $K > 1$ such that:

$$\frac{1}{K} < \frac{m(JW^{usc}_n(x))}{m_u(J^u_n(x))m_s(W^s_{\sigma(x)})m(W^c_n(x))} < K \tag{2.25}$$

we shall modify K, but will keep its name. Now, due to (2.25), it suffices to find a bound independent of x and n for

$$\frac{m_u(J^u_{n+k}(x))m_s(W^s_{\sigma^{n+k}}(x))m_c(W^c_{n+k}(x))}{m_u(J^u_n(x))m_s(W^s_{\sigma^n}(x))m_c(W^c_n(x))}$$

But

$$\frac{m_s(W^s_{n+k})(x)}{m_s(W^s_{\sigma^n}(x))} = \frac{2\sigma^{n+k}}{2\sigma^n} = \sigma^k$$

analogously, $m_c(W^c_{n+k}(x))/m_c(W^c_n(x)) = \sigma^k$, and to bound the last quotient observe that, by definition of $J^u_n(x)$ there exists $K > 1$ such that:

$$\frac{1}{K}\frac{2\lambda^n}{Jac(f^n)'(x)|_{E^u}} \leq m_u(J^u_n(x)) \leq K\frac{2\lambda^n}{Jac(f^n)'(x)|_{E^u}} \tag{2.26}$$

So, taking $K > 1$ such that all the bounds work, we get (2). It is left as an exercise to the reader to show that this implies that the density points of both bases are equal. (Hint: show that points of density zero coincide) $\square$

To finish the proof of Theorem 2.6.2, let us introduce the following Vitali basis, for each $x \in M$ and $n \geq 1$:

$$W^{usc}_n(x) = W^u_{\sigma^n}(W^{sc}_n(x)) \tag{2.27}$$

Since the angles between E^s, E^c and E^u are uniformly bounded from below, it is easy to see that density points according to the Vitali basis defined in (2.27) are the same as the Lebesgue density points. The proof is then finished with the following proposition:

PROPOSITION 2.6.11. *The W_n^{usc}-density points of an essentially u-saturated set coincide with the JW_n^{usc} density points of this set.*

PROOF. Let us assume that X_u is an u-saturated set (see Exercise 2.6.12). Call $m_{sc}(A)$ the induced Riemannian volume of A on W_{loc}^{sc}. Then, due to the absolute continuity of the local unstable foliation (Proposition 2.6.4) we have:

(1) $m(X_u \cap W_n^{usc}(x)) = \int_{X_u \cap W_n^{sc}(x)} m_u(W_{\sigma^n}^u(y))dm_{sc}(y)$
(2) $m(X_u \cap JW_n^{usc}(x)) = \int_{X_u \cap W_n^{sc}(x)} m_u(J_n^u(y))dm_{sc}(y)$

Now, by Equation (2.26), there exists $K > 1$ such that for all $y \in W_{loc}^{sc}(x)$:

$$\frac{1}{K} < \frac{m_u(J_n^u(x))}{m_u(J_n^u(y))} < K$$

This implies that:

$$\cdot\frac{1}{K^2}\frac{m_{sc}(X_u \cap W_n^{sc}(x))}{m_{sc}(W_n^{sc}(x))} \leq \frac{m(X_u \cap JW_n^{usc}(x))}{m(JW_n^{usc}(x))} \leq K^2\frac{m_{sc}(X_u \cap W_n^{sc}(x))}{m_{sc}(W_n^{sc}(x))} \tag{2.28}$$

On the other hand, from Item (1), it is easy to see that, by possibly modifying $K > 1$, we get:

$$\frac{1}{K^2}\frac{m_{sc}(X_u \cap W_n^{sc}(x))}{m_{sc}(W_n^{sc}(x))} \leq \frac{m(X_u \cap W_n^{usc}(x))}{m(W_n^{usc}(x))} \leq K^2\frac{m_{sc}(X_u \cap W_n^{sc}(x))}{m_{sc}(W_n^{sc}(x))} \tag{2.29}$$

Then, joining Equations (2.28) and (2.29), we obtain

$$\frac{1}{K^4}\frac{m(X_u \cap W_n^{usc}(x))}{m(W_n^{usc}(x))} \leq \frac{m(X_u \cap JW_n^{usc}(x))}{m(JW_n^{usc}(x))} \leq K^2\frac{m(X_u \cap W_n^{usc}(x))}{m(W_n^{usc}(x))}$$

This proves that their density points are the same (see Exercise 2.6.13) $\square$

EXERCISE 2.6.12. *Prove that the density points of X according to any Vitali basis V_n coincide with the density points of any set Y such that $m(X \triangle Y) = 0$.*

In particular, to calculate the density points of an essentially u-saturated set according to any Vitali basis, we may always assume it is u-saturated.

EXERCISE 2.6.13. *Finish the details of Proposition 2.6.11. To prove that the density points coincide, show that the zero density points according to both bases coincide.*

2.7. A Criterion to establish ergodicity

Here we present a refinement of the Hopf argument (Subsection 2.2.1) which provides a more accurate description of the hyperbolic ergodic components of a smooth invariant measure.

An invariant set E is an *ergodic component* of f if $m(E) > 0$ and $f|E$ is ergodic, that is, if $F \subset E$ is a positive measure invariant set, then $m(E \triangle F) = 0$. Equivalently, all f-invariant measurable functions are a.e. constant on E.

An ergodic component E is *hyperbolic* if all Lyapunov exponents are different from zero on E, that is, if for almost all $x \in E$ and $v \in T_xM \setminus \{0\}$, we have:

$$\lambda(x, v) = \lim_{|n| \to \infty} \log \frac{1}{n} \|Df^n(x)v\| \neq 0 \tag{2.30}$$

The number $\lambda(x, v)$ defined in Equation (2.30) above is the *Lyapunov exponent* of x in the v-direction, that is, the exponential growth rate of Df along v. The Lyapunov exponents are well defined for every vector almost everywhere on M. Moreover, since they are measurable invariant functions, they are almost everywhere constant over ergodic components, see next subsection.

2.7.1. Pesin Theory. We review some facts about Pesin theory, a good summary of which may be found, for instance, in Pugh and Shub's paper[**102**], Ledrappier and Young's paper [**86**], Katok's paper [**80**] and the book by Barreira and Pesin [**12**].

Given $x \in M$, let $E_\lambda(x)$ be the subspace of T_xM consisting of all v such that the Lyapunov exponent of x in the v-direction is λ. Then we have the following:

THEOREM 2.7.1 (Osedelec [**95**]). *For any C^1 diffeomorphism $f : M \to M$ there is an f-invariant Borel set R of total probability, that is, $\mu(R) = 1$ for all invariant probability measures μ, with* *the following properties. For each* $\varepsilon > 0$ *there is a Borel function* $C_\varepsilon : R \to (1, \infty)$ *such that for all* $x \in R$*,* $v \in T_x M$ *and* $n \in \mathbb{Z}$

(1) $T_x M = \bigoplus_\lambda E_\lambda(x)$ *(Oseledec's splitting)*
(2) *For all* $v \in E_\lambda(x)$

$$C_\varepsilon(x)^{-1} exp[(\lambda - \varepsilon)n]|v| \leq |Df^n(x)v| \leq C_\varepsilon(x) exp[(\lambda + \varepsilon)n]|v|$$

(3) $\angle(E_\lambda(x), E_{\lambda'}(x)) \geq C_\varepsilon(x)^{-1}$ *if* $\lambda \neq \lambda'$
(4) $C_\varepsilon(f(x)) \leq \exp(\varepsilon) C_\varepsilon(x)$

The set R is called the set of *regular points*. For simplicity, we will assume that all points in R are Lebesgue density points. We also have that $Df(x)E_\lambda(x) = E_\lambda(f(x))$. If an f-invariant measure μ is ergodic then the Lyapunov exponents and $\dim E_\lambda(x)$ are constant μ-a.e.

In short, what Oseledets proved is that for almost every point in M, there exists a splitting of the tangent bundle

$$T_x M = E_1(x) \oplus \cdots \oplus E_{l(x)}(x)$$

and numbers $\hat{\lambda}_1(x) > \cdots > \hat{\lambda}_{l(x)}(x)$ describing all the exponential growth rates of $\|Df\|$ in $T_x M$, that is, all vectors in the subspace $E_i(x)$ have Lyapunov exponent $\hat{\lambda}_i(x)$. We can group this splitting as the *zipped Oseledets splitting*:

$$T_x M = \bigoplus_{\lambda < 0} E_\lambda(x) \oplus E^0(x) \bigoplus_{\lambda > 0} E_\lambda(x) = E^-(x) \oplus E^0(x) \oplus E^+(x) \tag{2.31}$$

where $E^0(x)$ is the subspace generated by the vectors having zero Lyapunov exponents. For every $f \in \mathrm{Diff}^1_m(M)$, the *Pesin region* of f is defined as the set

$$\mathrm{Nuh}(f) = \{x \in M : T_x M = E^-(x) \oplus E^+(x)\} \tag{2.32}$$

This is an invariant set, and all ergodic components contained in the Pesin region are hyperbolic ergodic components.

From now on, we assume that $f \in \mathrm{Diff}^{1+\alpha}_m(M)$. Given a regular point $x \in R$, we define its *stable Pesin manifold* by

$$W^s(x) = \left\{ y : \limsup_{n \to +\infty} \frac{1}{n} \log d(f^n(x), f^n(y)) < 0 \right\} \tag{2.33}$$

The *unstable Pesin manifold* of x, $W^u(x)$ is the stable Pesin manifold of x with respect to f^{-1}. Stable and unstable Pesin manifolds of points in R are immersed manifolds [**97**]. We stress that $C^{1+\alpha}$ regularity is crucial for this to happen. In this way we obtain a partition $x \mapsto W^s(x)$, which we call *stable partition.* The unstable partition is defined analogously. The stable and unstable partitions are invariant.

For each fixed $\varepsilon > 0$ and $l > 1$, we define the *Pesin blocks*:

$$\Lambda(\varepsilon, l) = \{x \in R : C_\varepsilon(x) \leq l\}.$$

Note that Pesin blocks are not necessarily invariant. However $f(\Lambda(\varepsilon, l)) \subset \Lambda(\varepsilon, \exp(\varepsilon) l)$. Also, for each $\varepsilon > 0$, we have

$$R = \bigcup_{l=1}^{\infty} \Lambda(\varepsilon, l) \tag{2.34}$$

We loose no generality in assuming that the $\Lambda(\varepsilon, l)$ are compact. On the Pesin blocks we have a continuous variation: Let us call $W^s_{loc}(x)$ the connected component of $W^s(x) \cap B_r(x)$ containing x, where $B_r(x)$ denotes the Riemannian ball of center x and radius $r > 0$, which is sufficiently small but fixed. Then

THEOREM 2.7.2 (Stable Pesin Manifold Theorem [**97**]). *Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism preserving a smooth measure m. Then, for each $l > 1$ and small $\varepsilon > 0$, if $x \in \Lambda(\varepsilon, l)$:*

(1) *$W^s_{loc}(x)$ is a disk such that $T_x W^s_{loc}(x) = \bigoplus_{\lambda < 0} E_\lambda(x)$*
(2) *$x \mapsto W^s_{loc}(x)$ is continuous over $\Lambda(\varepsilon, l)$ in the C^1 topology*

In particular, the dimension of the disk $W^s_{loc}(x)$ equals the number of negative Lyapunov exponents of x. An analogous statement holds for the unstable Pesin manifold.

2.7.2. The Pesin homoclinic classes. Let $f \in \mathrm{Diff}^{1+\alpha}_m(M)$. Given a hyperbolic periodic point p, we define the *s-Pesin homoclinic class* of p by:

$$\mathrm{Phc}^s(p) = \{x \in R : W^s(x) \pitchfork W^u(o(p))\} \tag{2.35}$$

where $o(p)$ denotes the orbit of p, and $\pitchfork$ denotes non-empty transverse intersection. Analogously we define the *u-Pesin homoclinic class* $\mathrm{Phc}^u(p)$. $\mathrm{Phc}^s(p)$ is an s-saturated set, and $\mathrm{Phc}^u(p)$ is a u-saturated set. Both sets are f-invariant. We define the *Pesin homoclinic class* of p by:

$$\mathrm{Phc}(p) = \mathrm{Phc}^s(p) \cap \mathrm{Phc}^u(p) \tag{2.36}$$

$\mathrm{Phc}(p)$ is f-invariant, but is not necessarily s- or u-saturated.. See Figure 9.

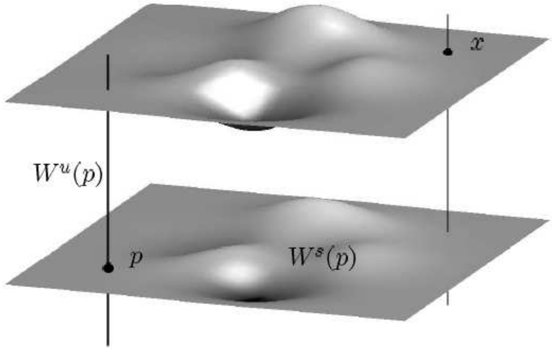

FIGURE 9. A point x in the Pesin homoclinic class of p

2.7.3. Criterion. We have the following criterion to recognize hyperbolic ergodic components of a smooth measure:

THEOREM 2.7.3. [**71**] *Let $f \in \mathrm{Diff}^{1+\alpha}_m(M)$. If p is a hyperbolic periodic point such that $m(\mathrm{Phc}^s(p)) > 0$ and $m(\mathrm{Phc}^u(p)) > 0$, then* $\mathrm{Phc}(p)$ *is a hyperbolic ergodic component of f and*

$$\mathrm{Phc}(p) \mathrel{\mathring{=}} \mathrm{Phc}^s(p) \mathrel{\mathring{=}} \mathrm{Phc}^u(p)$$

With this Criterion, and Katok's Closing lemma [**80**], we obtain the following version of Pesin's Ergodic Component Theorem:

THEOREM 2.7.4 (Pesin's Ergodic Component Theorem [**97**], [**80**], [**71**]). *Let $f \in \mathrm{Diff}^{1+\alpha}_m(M)$, then there exist hyperbolic periodic points p_n such that*

$$\mathrm{Nuh}(f) \mathrel{\mathring{=}} \mathrm{Phc}(p_1) \cup \mathrm{Phc}(p_2) \cup \dots \cup \mathrm{Phc}(p_n) \cup \dots$$

2.7.4. Proof. The proof of this criterion follows the line of the Hopf argument, and it is split in two parts. First, it is proved that if $\mathrm{Phc}(p)$ has positive measure, then it is a hyperbolic ergodic component of the measure. A more delicate proof is required to show that if

$\mathrm{Phc}^s(p)$ and $\mathrm{Phc}^u(p)$ have positive measure then they coincide modulo a zero set.

We prove that $\mathrm{Phc}(p)$ is a hyperbolic ergodic component by showing that all continuous functions $\varphi : M \to \mathbb{R}$ have almost constant forward Birkhoff limits φ^+ on $\mathrm{Phc}(p)$. To do this, we consider two typical points x and y in $\mathrm{Phc}(p)$ and try to see that $\varphi^+(x) = \varphi^+(y)$. Observe that φ^+ is constant on stable Pesin leaves, due to continuity of φ. Since x, y are typical, m_u^x-a.e. point in $W^u(x)$ takes the value $\varphi^+(x)$, and m_u^y-a.e. point in $W^u(y)$ takes the value $\varphi^+(y)$. We may consider iterates of x and y so large that they are very close to $W^u(p)$. We will therefore find two disks D_x and D_y, one contained in $W^u(f^k(x))$ and the other contained in $W^u(f^m(y))$, such that they are very close. The stable holonomy takes positive measure sets on D_x into positive measure sets in D_y. In particular it takes the set of points in D_x for which the value is $\varphi^+(x)$ into a set of positive measure in D_y. The fact that φ^+ is constant along stable Pesin leaves, together with the fact that m_u^y-a.e. point in D_y has the value $\varphi^+(y)$ prove that $\varphi^+(x) = \varphi^+(y)$.

The proof that $\mathrm{Phc}^s(p)$ coincides modulo zero with $\mathrm{Phc}^u(p)$ requires more delicate steps. Indeed, we want to prove that a typical point x in $\mathrm{Phc}^u(p)$ is contained in $\mathrm{Phc}^s(p)$. In order to do that, we take a typical point y in $\mathrm{Phc}^s(p)$. The fact that x is typical implies that x belongs to $\mathrm{Phc}^s(p)$ if and only if m_u^x-a.e. point in $W^u(x)$ belongs to $\mathrm{Phc}^s(p)$. Proceeding as in the previous proof, one takes suitable iterates of x and y so that they are very close to $W^u(p)$. We would like to follow as in the proof above, by taking holonomies between close unstable disks; however, the dimension of $W^u(x)$ might be less than $\dim(W^u(y))$. We shall therefore sub-foliate $W^u(y)$ with disks of dimension $\dim(W^u(x))$, and choose a disk $D_y \subset W^u(y)$ such that m_D-a.e. point in D_y belongs to $\mathrm{Phc}^s(p)$, where m_D is the Lebesgue measure induced on D_y. This is possible due to a Fubini argument, since y is a typical point in $\mathrm{Phc}^s(p)$.

Now, $\mathrm{Phc}^s(p)$, is an f-invariant and s-saturated set. This means that if $x \in \mathrm{Phc}^s(p)$, then $W^s(x) \subset \mathrm{Phc}^s(p)$. An analogous statement holds for $\mathrm{Phc}^u(p)$. See Figure 10. By Birkhoff Ergodic Theorem,

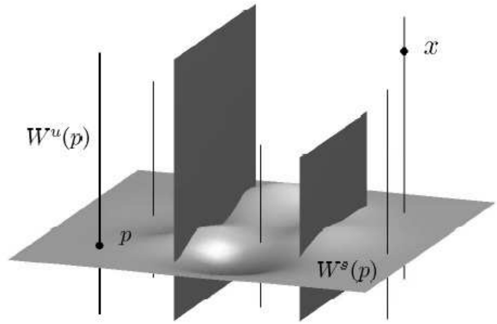

FIGURE 10. $\mathrm{Phc}^u(p)$ is u-saturated

the limit defined in Equation (2.1) exists and coincides almost everywhere with $\varphi^+(x)$ (defined in Equation (2.2)), and $\varphi^-(x)$, defined analogously. $\tilde{\varphi}$ and $\varphi^\pm$ are f-invariant.

LEMMA 2.7.5 (Typical points for continuous functions). *There exists an invariant set $\mathcal{T}_0$ of typical points with $m(\mathcal{T}_0) = 1$ such that for all $\varphi \in C^0(M)$ if $x \in \mathcal{T}_0$ then $\varphi^+(w) = \varphi^+(x)$ for all $w \in W^s(x)$ and m_u^x-a.e. $w \in W^u(x)$.*

PROOF. Let us consider the full measure set:

$$\mathcal{S}_0 = \{x \in M : \exists \varphi^+(x) = \varphi^-(x)\}$$

For almost all $x \in \mathcal{S}_0$, we have that m_u^x-a.e. $\xi \in W^u_{loc}(x)$, $\xi \in \mathcal{S}_0$. Otherwise, there would exist a positive measure set $A \subset M$ such that for all $x \in A$ there is a subset $B_x \subset W^u_{loc}(x) \setminus \mathcal{S}_0$ with $m_u^x(B_x) > 0$. Considering a density point y of A and integrating along a transverse small disk T, we would obtain a set $B \subset M \setminus \mathcal{S}_0$ such that

$$m(B) = \int_T m_u^x(B_x) dm_T(x) > 0$$

which is an absurd. As we have seen, the following is a full measure set:

$$\mathcal{S}_1 = \{x \in \mathcal{S}_0 : m_u^x\text{-a.e. } \xi \in W^u_{loc}(x), \xi \in \mathcal{S}_0\}$$

For all $x \in \mathcal{S}_1$ there exists ξ_x such that m_u^x-a.e. $\xi \in W^u_{loc}(x)$, $\varphi^+(\xi) = \varphi^-(\xi) = \varphi^-(\xi_x) = \varphi^+(\xi_x)$. But almost every $x \in \mathcal{S}_1$

satisfies $\varphi^+(x) = \varphi^+(\xi_x)$. Otherwise, we would obtain a positive measure set $C \subset \mathcal{S}_1$ such that $m_u^x(C \cap W^u_{loc}(x)) = 0$ for almost every x, which contradicts absolute continuity. The invariance of φ^+ yields a set $\mathcal{T}_0 \subset S_1$, with $m(\mathcal{T}_0) = 1$ and such that if $x \in \mathcal{T}_0$ then m_u^x-a.e. $\xi \in W^u(x)$ satisfies $\varphi^+(x) = \varphi^+(\xi)$. Since φ is continuous, we obviously have that φ^+ is constant on $W^s(x)$. $\square$

Assume for simplicity that p is a fixed point. Given a continuous function $\varphi : M \to \mathbb{R}$, let $\mathcal{T}_0$ be the set of typical points obtained in Lemma 2.7.5 and R be the set of regular points. We will see that φ^+ is constant on $\mathrm{Phc}(p) \cap \mathcal{T}_0 \cap R$, and hence almost everywhere constant on $\mathrm{Phc}(p)$. This will prove that $\mathrm{Phc}(p)$ is an ergodic component of f.

For any $\varepsilon > 0$ and $l > 1$ such that $m(\Lambda(p) \cap \Lambda(\varepsilon, l)) > 0$, let us call $\Lambda = \mathrm{Phc}(p) \cap \Lambda(\varepsilon, l) \cap \mathcal{T}_0$. Without loss of generality, we may assume that all points in Λ are Lebesgue density points of Λ, and return infinitely many times to Λ in the future and the past. Note that there exists $\delta > 0$ such that $W^s_{loc}(x)$ contains an s-disc of radius δ for all $x \in \Lambda$.

Take $x, y \in \Lambda$, and consider $n > 0$ such that $y_n = f^n(y) \in \Lambda$ and $d(y_n, W^u(p)) < \delta/2$. We have $W^s_{loc}(y_n) \pitchfork W^u(p)$.

As a consequence of the λ-lemma, there exists $k > 0$ such that $x_k = f^k(x) \in \Lambda$ and $W^u(x_k) \pitchfork W^s_{loc}(y_n)$. See Figure 11.

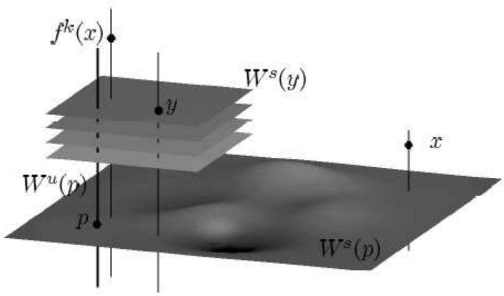

FIGURE 11. $\mathrm{Phc}(p)$ is an ergodic component of f

Since y_n is a typical point for φ, for $m_u^{y_n}$-a.e. w in $W^u(y_n)$ we have $\varphi^+(w) = \varphi^+(y_n)$. Also we have $\varphi^+(z) = \varphi^+(x_k)$. Since y_n is a Lebesgue density point of Λ, applying Fubini's theorem we get a point ξ near y_n such that the stable holonomy between $W^u_{loc}(\xi)$ and $W^u_{loc}(y_n)$ is defined for a set of $m_u^{y_n}$-positive measure in $W^u_{loc}(y_n)$. The λ-lemma implies that the stable holonomy between $W^u_{loc}(y_n)$ and $W^u_{loc}(z)$ is defined for a $m_u^{y_n}$-positive measure set, where z is a point in $W^u(x_k)$, see Figure 11.

Now, φ^+ is constant along stable Pesin manifolds. And due to absolute continuity, stable holonomy takes the set of points w in $W^u_{loc}(y_n)$ for which $\varphi^+(w) = \varphi^+(y_n)$ into a set of positive measure in $W^u_{loc}(z)$ for which the value of φ^+ will be $\varphi^+(y_n)$. The fact that x_k is a typical point of φ then implies that $\varphi^+(x) = \varphi^+(x_k) = \varphi^+(y_n) = \varphi^+(y)$. This proves that $\mathrm{Phc}(p)$ is an ergodic component of f.

Exercise 2.7.6. *Prove that* $\mathrm{Phc}(p)$ *is a hyperbolic ergodic component. Hint: use the ergodicity and prove that* $\dim W^s + \dim W^u = n$ *for almost every x in* $\mathrm{Phc}(p)$.

In order to prove that $\mathrm{Phc}^s(p)$ and $\mathrm{Phc}^u(p)$ coincide modulo a zero set, we shall need a refinement of Lemma 2.7.5:

Lemma 2.7.7 (Typical points for L^1 functions). *Given $\varphi \in L^1$ there exists an invariant set $\mathcal{T} \subset M$ of* typical points *of φ, with $m(\mathcal{T}) = 1$ such that if $x \in \mathcal{T}$ then $\varphi^+(z) = \varphi^+(x)$ for m_x^s-a.e. $z \in W^s(x)$ and m_x^u-a.e. $z \in W^u(x)$.*

Proof. Given $\varphi \in L^1$ take a sequence of continuous functions φ_n converging to φ in L^1. Now, φ_n^+ converges in L^1 to φ^+, so there exists a subsequence $\varphi_{n_k}^+$ converging a.e. to φ_+. Call $\mathcal{S}$ this set of a.e. convergence. Then the set $\mathcal{T}$ is the intersection of the set $\mathcal{T}_0$ obtained in Lemma 2.7.5 with $\mathcal{S}$. □

Let $\mathcal{T}$ be the set of typical points for the characteristic function $\mathbf{1}_{\mathrm{Phc}^s(p)}$ of the set $\mathrm{Phc}^s(p)$. Take $x \in \mathrm{Phc}^u(p) \cap \mathcal{T}$ such that all iterates of x are Lebesgue density points of $\mathrm{Phc}^s(p)$ and x returns infinitely many times to $\mathrm{Phc}^s(p)$. We shall see that $x \in \mathrm{Phc}^s(p)$. This will prove $\mathrm{Phc}^u(p) \overset{\circ}{\subset} \mathrm{Phc}^s(p)$. The converse inclusion follows analogously.

Let $\varepsilon > 0$, $l > 1$ be such that $m(\mathrm{Phc}^s(p) \cap \Lambda(\varepsilon, l)) > 0$, and let $\delta > 0$ be such that for all $z \in \Lambda^s(p) \cap \Lambda(\varepsilon, l)$, the set $W^s_{loc}(z)$ contains an s-disc of radius $\delta > 0$. Consider a Lebesgue density point y of $\Lambda^s = \mathrm{Phc}^s(p) \cap \Lambda(\varepsilon, l) \cap \mathcal{T}$ such that $d(y, W^u(p)) < \delta/2$.

As a consequence of the λ-lemma, there exists $k > 0$ such that $x_k = f^k(x) \in \mathrm{Phc}^u(p) \cap \mathcal{T}$ and $W^u(x_k) \pitchfork W^s_{loc}(y)$. Note that this intersection could a priori have positive dimension. See Figure 12.

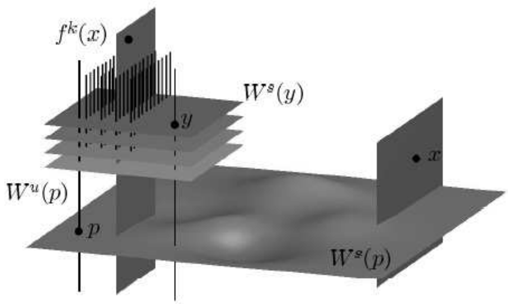

FIGURE 12. Proof of $\mathrm{Phc}^s(p) \mathrel{\mathring{=}} \mathrm{Phc}^u(p)$

Since y is a Lebesgue density point of Λ^s, we have $m(\Lambda^s \cap B_\delta(y)) > 0$. Take a smooth foliation $\mathcal{L}$ in $B_\delta(y)$ of dimension $u_y = n - \dim W^s_{loc}(y)$ and transverse to $W^s_{loc}(y)$. Note that $u_y \leq \dim W^u(p)$. We can also ask that the leaf L_w of $\mathcal{L}$ containing a point $w \in W^u(x_k)$ be contained in $W^u(x_k)$. See Figure 12.

By Fubini's theorem we have:

$$m(\Lambda^s \cap B_\delta(y)) = \int_{W^s_{loc}(y)} m^\xi_L(L_\xi \cap \Lambda^s) dm^s_y(\xi)$$

so $m^\xi_L(L_\xi \cap \Lambda^s) > 0$ for m^y_s-a.e. $\xi \in W^s_{loc}(y)$. Take $L \in \mathcal{L}$ such that $m^\xi_L(L \cap \Lambda^s) > 0$, this means that there is a m^ξ_L-positive measure set of points $w \in L_\xi$ such $w \in \Lambda^s(p)$. The stable holonomy takes this m^ξ_L-positive measure set into a m^w_L-positive measure set in L_w for all $w \in W^u(x_k) \cap B_\delta(y)$. But $\mathrm{Phc}^s(p)$ is a set saturated by stable leaves. This means that $m^w_L(L_w \cap \Lambda^s) > 0$ for all $w \in W^u(f^k(x)) \cap B_\delta(y)$.

If $W^u(x_k) \cap W^s_{loc}(y)$ is zero dimensional, this readily implies that $m_u^{x_k}(W^u(x_k) \cap \mathrm{Phc}^s(p)) > 0$, and hence, since x_k is a typical point, x_k and x belong to $\mathrm{Phc}^s(p)$.

Otherwise, take an open submanifold T of $W^u(x_k) \pitchfork W^s_{loc}(y)$. Then, by Fubini again:

$$m_u^{x_k}(\Lambda^s \cap W^u(x_k) \cap B_\delta(x)) \geq \int_T m_L^w(L_w \cap \Lambda^s) dm_T(w) > 0$$

We have that a $m^u_{x_k}$-positive measure set of $w \in W^u(x_k)$ satisfies $\mathbf{1}_{\mathrm{Phc}^s(p)}(w) = 1$. Since x_k is a typical point, this implies that x_k and hence x are in $\mathrm{Phc}^s(p)$. Therefore, $\mathrm{Phc}^u(p) \overset{\circ}{\subset} \mathrm{Phc}^s(p)$. The converse inclusion follows in an analogous way.

REMARK 2.7.8. *As an immediate consequence of the λ-lemma, we have that if p and q are two hyperbolic points such that $W^u(p) \pitchfork W^s(q)$ then $\mathrm{Phc}^u(p) \subset \mathrm{Phc}^u(q)$ and $\mathrm{Phc}^s(q) \subset \mathrm{Phc}^s(p)$.*

2.8. State of the art in 2011

Let us briefly describe the state of the art in these days with respect to partial hyperbolicity and stable ergodicity. The Pugh-Shub conjecture has been solved with the highest possible differentiability for one-dimensional center bundle:

THEOREM 2.8.1 (Rodriguez Hertz, Rodriguez Hertz, Ures [**67**]). *Stable ergodicity is C^∞-dense among partially hyperbolic diffeomorphisms with one-dimensional center bundle.*

For two-dimensional center bundle we have the following result:

THEOREM 2.8.2 (Rodriguez Hertz, Rodriguez Hertz, Tahzibi, Ures). *Stable ergodicity is C^1-dense among partially hyperbolic diffeomorphisms with two-dimensional center bundle.*

With respect to the Conjecture 2.2.1, we have the following results:

THEOREM 2.8.3 (Burns, Wilkinson [**33**]). *In $\mathrm{PH}^{1+\alpha}_m(M)$, essential accessibility and center bunching condition (2.11) imply ergodicity.*

A number λ is a center Lyapunov exponent of f if λ is a Lyapunov exponent in the direction of a vector in the center bundle, that is if $\lambda = \lambda(x, v)$ (as defined in Equation (2.30)) for some $v \in E^c$.

THEOREM 2.8.4 (Burns, Dolgopyat, Pesin [**28**]). *If $f \in \mathrm{PH}^{1+\alpha}_m(M)$ has the accessibility property and all the center Lyapunov exponents have the same sign on a set of positive measure, then f is stably ergodic.*

With respect to Conjecture 2.2.2 the advances are the following:

THEOREM 2.8.5 (Dolgopyat, Wilkinson [**46**]). *Stable accessibility is C^1-dense among partially hyperbolic diffeomorphisms, volume preserving or not, and symplectic.*

The following higher differentiability version holds for one-dimensional center bundle:

THEOREM 2.8.6 (Burns, Rodriguez Hertz, Rodriguez Hertz, Talitskaya, Ures [**67**], [**29**]). *Stable accessibility is C^∞-dense among partially hyperbolic diffeomorphisms, volume preserving or not.*

CHAPTER 3

Partial hyperbolicity and entropy

3.1. Introduction

In this chapter, our main issue will be the study of the properties of the entropy maximizing measures for partially hyperbolic systems. Recall that an **ergodic** invariant measure μ is *entropy maximizing* if its metric entropy coincides with the topological entropy of the system.

Entropies are quantities that measure the complexity of the orbits of a system. While the topological entropy "sees" the whole complexity of the orbits of a system, the metric entropy "sees" the complexity of the orbits that are relevant for a given measure. An entropy maximizing measure (or just a maximizing measure) is an ergodic measure such that its metric entropy equals the topological entropy of the system. In the previous informal words, the complexity of the orbits that are seen by a maximizing measure is the same as the complexity of the orbits of the whole system.

On the one hand, it is well-known that uniformly hyperbolic systems admit maximizing measures and their topological transitivity implies uniqueness. See the works of R. Bowen [**15**] and G. Margulis [**90**]. On the other hand, when the system is not hyperbolic the existence can fail if it is not smooth enough (see [**92**] and [**94**]). However, if we are in the setting of partially hyperbolic diffeomorphism, the results of W. Cowieson and L.-S. Young in [**41**] (see also [**42**]) imply that there are always entropy maximizing measures if the center bundle is one-dimensional even if the diffeomorphism is C^1.

Although existence is already provided by Cowieson-Young's results our method gives it immediately as a consequence of the existence of a (especial type of) semiconjugacy with a hyperbolic system.

Along these notes we are mostly focussed on three dimensional partially hyperbolic diffeomorphisms. In this chapter we will mainly study properties of entropy maximizing measures for two families of 3-dimensional partially hyperbolic diffeomorphisms. The first type of examples are the partially hyperbolic diffeomorphisms isotopic to an hyperbolic automorphism of $\mathbb{T}^3$. The second ones are the partially hyperbolic diffeomorphisms with compact center manifolds, that is, it is dynamically coherent and every center leaf is compact.

Let us state the results. For the diffeomorphisms isotopic to Anosov we obtain the following:

Theorem 3.1.1 ([**122**]). *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be an absolutely partially hyperbolic diffeomorphism homotopic to a hyperbolic linear automorphism A. Then, f has a unique (entropy) maximizing measure μ. Moreover, the pair (f, μ) is isomorphic to (A, m) where m is the volume measure on $\mathbb{T}^3$.*

A natural question arises:

Problem 3.1.2. *Even in dimension three, what is the situation for the other cases? In particular, what does it happen with the perturbations of time-one maps of (transitive) Anosov flows?*

This theorem contrast with the following result for the case having center compact leaves:

Theorem 3.1.3 ([**72**]). *Let $f : M \to M$ be a partially hyperbolic diffeomorphism, dynamically coherent with compact one dimensional central leaves and satisfying accessibility property. Then one and only one of the following occurs*

(1) *f admits a unique entropy maximizing measure μ and $\lambda_c(\mu) = 0$. Moreover (f, μ) is isomorphic to a Bernoulli shift,*
(2) *f has a finite number (strictly greater than one) of ergodic maximizing measure all of which with non vanishing central Lyapunov exponent. Moreover (f, μ) is a finite extension of Bernoulli shift for any entropy maximizing measure μ.*

Moreover, the diffeomorphisms fulfilling the conditions of the second item form a C^1-open and $C^\infty-$dense subset of the dynamically coherent partially hyperbolic diffeomorphism with compact one dimensional central leaves.

One of the main differences between Theorems 3.1.1 and 3.1.3 and the situation of Problem 3.1.2 is the presence of an especial semiconjugacy. In fact, in both theorems the partially hyperbolic diffeomorphisms are semiconjugate to a hyperbolic automorphism of a torus. Moreover, this semiconjugacies have the particular property of preserving the topological entropy, that is, the partially hyperbolic diffeomorphisms have the same entropy than the corresponding linear Anosov diffeomorphism of the torus. In particular this implies that the entropy of this systems is a locally constant function. In the case of the time-one map of an Anosov flow the situation is very different. On the one hand, there is no semicunjugacy to a simpler model and on the other hand, entropy is not locally constant.

3.2. The Ledrappier-Walters entropy formula

Leddrapier-Walters formula (LW formula, [**84**]) is an important generalization of the variational principle. The variational principle states that the topological entropy is the supremum of the metric entropies of the invariant measures of the system. Essentially states the following: suppose that our space is a fibration preserved by the dynamics (our diffeomorphisms sends fibers to fibers) Then, we have a dynamics induced on the base space of the fibration. Consider an invariant measure μ defined on the fibration. Then, Rokhlin disintegration theorem implies that μ projects onto ν, an invariant measure of the base dynamics. Then, the LW formula says that the entropy of mu (h_μ) is not greater that the entropy of ν plus the ν average of the topological entropy of he fibers. Moreover, this sum (h_ν plus average of entropy of fibers) is the supremum of the entropies of the invariant measures whose projection is ν.

Before stating precisely the theorem that gives the LW formula we need some previous definitions.

Consider $T : X \to X$, a continuous map on a compact metric space X. Recall that, given $K \subset X$, $C \subset K$ is an (n, ε)-spanning set for K if $\forall y \in K$ there is $x \in C$ such that for all $j \in \{0, \dots n\}$ we have $\operatorname{dist}(T^j(x), T^j(y)) < \varepsilon$. Another way of understanding what is the meaning of an (n, ε)-spanning set for K is to define, for each n a new distance $d_n(x, y) = \min_{i=0,\dots n} \operatorname{dist}(T^i(x), T^i(y))$. An (n, ε)-spanning set for K is a set such that its d_n-balls of radius ε cover K. Let

$Sp_n(K, \varepsilon)$ be the minimal cardinality of an (n, ε)-spanning set for K. Then, we define the *entropy of T in K* as

$$h(T, K) = \lim_{\varepsilon \to 0} \limsup \frac{1}{n} \log Sp_n(K, \varepsilon).$$

There is a dual definition using (n, ε)-separated sets and it is a classical result that both definitions coincide (see, for instance, [**81**]) Also observe that in case of having $K = X$ we obtain the classical definition of $h_{top}(T)$, the topological entropy of T. Denote $h_\mu(T)$ the metric entropy of a T-invariant probability μ and $\mathcal{M}_T$ the space of all T-invariant probabilities. A fundamental result in Ergodic Theory is the Variational Principle.

THEOREM 3.2.1. $h_{top}(T) = \sup_{\mathcal{M}_T} h_\mu(T)$.

As we have already said, LW formula is a nontrivial generalization of the Variational Principle. Suppose that we have $T : X \to X$ and $S : Y \to Y$ continuous maps of compact metric spaces X, Y. Additionally suppose that the map $\pi : X \to Y$ is a semiconjugacy between T and S, that is, π is a continuous surjective map such that $\pi \circ T = S \circ \pi$.

THEOREM 3.2.2 (LW-formula [**84**]).

$$\sup_{\{\mu : \mu\circ\pi^{-1} = \nu\}} h_\mu(T) = h_\nu(S) + \int_Y h(T, \pi^{-1}(y)) d\nu(y).$$

To understand Theorem 3.2.2 recall that Rokhlin disintegration theorem says that given a probability μ defined on X there exist a family of probabilities μ_y, $y \in Y$, supported on the corresponding fibers $\pi^{-1}(y)$ and a probability ν defined on Y such that $\mu = \int_Y \mu_y d\nu(Y)$. It is not difficult to see that ν is S-invariant if μ is T-invariant. Dually, if ν is an S-invariant probability it can be shown that there is at least one measure μ such that $\mu \circ \pi^{-1} = \nu$. The existence of such a measure μ can be shown by using Hahn-Banach Theorem. Although in the cases that we will need its existence , in the present notes, it can be obtained more directly.

Observe that Theorem 3.2.1 (Variational Principle) is a consequence of Theorem 3.2.2. Indeed, if Y is a singleton (i.e. consists of one point) we have that $h_\nu(S) = 0$, $\int_Y h(T, \pi^{-1}(y)) d\nu(y) = h_{top}(T)$ and $\{\mu : \mu \circ \pi^{-1} = \nu\} = \mathcal{M}_T$.

Another corollary that can be easily obtained from Theorem 3.2.2 is the following formula of Bowen.

COROLLARY 3.2.3 ([**16**]).

$$h_{top}(T) \leq h_{top}(S) + \sup_{y \in Y} h(T, \pi^{-1}(y)).$$

In fact we will apply the LW formula to the case when the entropy of each fiber is zero, that is $h(T, \pi^{-1}(y)) = 0 \ \forall y \in Y$. That means that the dynamics along the fibers does not contribute to the topological entropy. Then, we have that $h_{top}(T) = h_{top}(S)$. Moreover, we have the following corollary:

COROLLARY 3.2.4. *Suppose that $h(T, \pi^{-1}(y)) = 0 \ \forall y \in Y$. Then, μ is an entropy maximizing measure for T if and only if $\nu = \mu \circ \pi^{-1}$ is an entropy maximizing measure for S.*

A way to ensure that a set K has null entropy is to show that its dynamics is, in some sense, one-dimensional. The following lemma will be very useful.

LEMMA 3.2.5. *Suppose that C is an arc, for some $L > 0$,* length$(T^n(C)) < L$ *for all $n \geq 0$ and $K \subset C$. Then, $h(T, K) = 0$.*

PROOF. Of course, $K \subset C$ implies that $h(T, K) \leq h(T, C)$. Then, it is enough to prove the lemma for $K = C$.

For each $n \geq 0$ consider $H_n = \{x_1, \ldots x_{k_n}\} \subset T^n(C)$ in such a way that

- the length of all the subarcs of $T^n(C)$ determined by the points of H_n is less than ε,
- $\#H_n = k_n \leq \frac{L}{\varepsilon} + 1$.

Define the set $J_n = \bigcup_{j=0,\ldots,n} T^{-j}(H_n)$.

Observe that $\#J_n \leq (n+1)(\frac{L}{\varepsilon} + 1)$. We shall prove that J_n is an (n, ε)-spanning set. Let $x \in C$, $x_r \in J_n$ be the closest point to x on the right and x_l on the left. Take $j \in \{0, \ldots, n\}$. Then, there are $x_i, x_{i+1} \in J_j$ such that $T^{-j}(x_i) \leq x_l < x < x_r \leq T^{-j}(x_{i+1})$ (where the inequalities correspond to the natural order of C) So, $x_i \leq T^j(x_l) < T^j(x) < T^j(x_r) \leq x_{i+1}$ (or with the inverse inequalities in case T reverse orientation) Since the arc joining x_i and x_{i+1} has length less than ε we have that the length of the arc joining $T^j(x)$ and

$T^j(x_r)$ (or $T^j(x_l)$) is less than ε. Hence, $\mathrm{dist}(T^j(x), T^j(x_r)) < \varepsilon$ (and $\mathrm{dist}(T^j(x), T^j(x_r)) < \varepsilon$) for all $j \in \{0, \ldots, n\}$ implying, as claimed, that J_n is an (n, ε)-spanning set.

Since $\# J_n \leq (n+1)(\frac{L}{\varepsilon}+1)$ we have that $Sp_n(C, \varepsilon) \leq \# J_n \leq (n+1)(\frac{L}{\varepsilon}+1)$. Finally, we have that $h(T, C) = \lim_{\varepsilon \to 0} \limsup \frac{1}{n} \log Sp_n(C, \varepsilon) \leq \lim_{\varepsilon \to 0} \limsup \frac{1}{n} \log[(n+1)(\frac{L}{\varepsilon}+1)] = 0$. This proves the lemma. $\square$

3.3. Isotopic to Anosov

In this section we present an sketch of the proof of the main result of [**122**]. See also [**36**] for similar results.

THEOREM 3.3.1 (see Theorem 3.1.1). *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be an absolutely partially hyperbolic diffeomorphism homotopic to a hyperbolic linear automorphism A. Then, f has a unique (entropy) maximizing measure μ. Moreover, the pair (f, μ) is isomorphic to (A, m) where m is the volume measure on $\mathbb{T}^3$.*

The proof of this theorem depends on some new results about partially hyperbolic diffeomorphisms. The first one of these results is that an absolutely partially hyperbolic diffeomorphism of $\mathbb{T}^3$ has quasi-isometric strong foliations (see [**22**]). The second result is a byproduct of A. Hammerlindl's leaf conjugacy (see [**57**]). Hammerlindl showed that the quasi-isometry property for strong foliations implies the quasi-isometry property for the center foliation. We can talk about the center foliation because Brin [**20**] has shown that having quasi-isometric strong foliations implies dynamical coherence. Recall that we say that f is *dynamically coherent* if there exist invariant foliations $\mathcal{W}^{c\sigma}$ tangent to $E^{c\sigma} = E^c \oplus E^\sigma$ for $\sigma = s, u$. Note that by taking the intersection of these foliations we obtain an invariant foliation $\mathcal{W}^c$ tangent to E^c that subfoliates $\mathcal{W}^{c\sigma}$ for $\sigma = s, u$. In order to understand these results we need the definition of quasi-isometry.

DEFINITION 3.3.2. *A foliation $\mathcal{W}$ of a simply connected Riemannian manifold is quasi-isometric if there are $a, b \in \mathbb{R}$ so that $d_W(x, y) \leq a\, d(x, y) + b$ for any x, y in the same leaf W of $\mathcal{W}$. Here d_W stands for the distance induced by the restriction to W of the ambient Riemannian metric.*

When we say that the strong (or the center) foliations of a partially hyperbolic diffeomorphism are quasi-isometric we are meaning that their lifts to the universal cover are quasi-isometric.

As we have already said a result of Brin, Burago and Ivanov states that the strong foliations of an absolute partially hyperbolic diffeomorphism on $\mathbb{T}^3$ are quasi-isometric (in the universal cover) and this implies dynamical coherence.

THEOREM 3.3.3 ([**22, 20**]). *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be an absolutely partially hyperbolic diffeomorphism. Then, the strong foliations are quasi-isometric and therefore, it is dynamically coherent.*

We observe that this is not true for general partially hyperbolic diffeomorphisms. In fact, there are non-dynamically coherent partially hyperbolic diffeomorphisms on $\mathbb{T}^3$ (see [**69**]) Of course, the strong foliations of these diffeomorphisms are not quasi-isometric. However, the examples there in are not homotopic to a hyperbolic diffeomorphism. Then, we have the following questions.

PROBLEM 3.3.4. *Are there dynamically incoherent partially hyperbolic diffeomorphisms on $\mathbb{T}^3$? What about other manifolds (for instance nilmanifolds)? See* [**96, 58**].

Related with this problem, we observe that the construction of the examples in [**69**] heavily relies on the existence of an invariant 2-torus with hyperbolic dynamics. Obviously, this kind of torus cannot exist for diffeomorphisms homotopic to Anosov. The main result in [**70**] implies that they cannot exist in many manifolds too, in particular in nilmanifolds (see also Chapter 5)

The following lemma proved in [**57**] will also be useful.

LEMMA 3.3.5 (Hammerlindl, [**57**]). *$W^c(f)$ is quasi-isometric in the universal cover.*

3.3.1. Properties of the semiconjugacy. As we have previously said our strategy is to apply LW formula to a semiconjugacy, with a simpler model, such that its fibers have null entropy. In the case we are studying now there is a semiconjugacy between the diffeomorphism and the hyperbolic automorphism A. This is a well-known result of Franks [**48**].

THEOREM 3.3.6 ([**48**]). *Let $f : \mathbb{T}^n \to \mathbb{T}^n$ be a diffeomorphism homotopic to a hyperbolic automorphism A. Then, there exists a continuous surjection $h : \mathbb{T}^n \to \mathbb{T}^n$ homotopic to the identity such that $A \circ h = h \circ f$.*

Franks' Theorem is of great help in our strategy. On the one hand, m, the Lebesgue (volume) measure, is the unique entropy maximizing measure for A. Then, if we were able to prove that the fibers of h have null entropy we would have that the entropy maximizing measures of f would be the ones that projects via h onto m. On the other hand, one way of characterize h is the following: given lifts to the universal cover $\tilde{x}$ of x and $\tilde{h}$ of h, $\tilde{h}(\tilde{x})$ is the (unique) point of $\mathbb{R}^3$ such that there exists $K > 0$ with the property that $\operatorname{dist}(\tilde{f}^n(\tilde{x}), \tilde{f}^n(\tilde{h}(\tilde{x}))) < K$ for all $n \in \mathbb{Z}$ and $\tilde{f}$ a lift of f. Moreover, K can be chosen independently of x. This, in particular, implies that given to points x, y, they have the same h-image if they have lifts to universal cover $\tilde{x}, \tilde{y}$ such that its $\tilde{h}$-orbits remain a bounded distance and $K > 0$ can be chosen in such a way that two points $\tilde{x}_1, \tilde{x}_2 \in \mathbb{R}^3$ are in the same $\tilde{h}$-preimage of a point $\tilde{z} \in \mathbb{R}^3$ if and only if $\operatorname{dist}(\tilde{h}^n(\tilde{x}_1), \tilde{h}^n(\tilde{x}_2)) < K \; \forall n \in \mathbb{Z}$. In particular the diameter of $\tilde{h}^{-1}(\tilde{z})$ is smaller than K for all $\tilde{z} \in \mathbb{R}^3$. Also observe that $\tilde{f}(\tilde{h}^{-1}(\tilde{z})) = \tilde{h}^{-1}(\tilde{f}(\tilde{z}))$ and $p(\tilde{h}^{-1}(\tilde{z})) = h^{-1}(z)$ with $p : \mathbb{R}^3 \to \mathbb{T}^3$ is the covering projection.

Now, we want to show that for every $\tilde{z}$, $\tilde{h}^{-1}(\tilde{z})$ is a closed arc of center manifold (possible a singleton) If we are able to prove this then we are almost done. All the "fibers" of the semiconjugacy h are center arcs and its iterates have uniformly bounded diameter. Then, since the center foliation is quasi-isometric the lengths of these center arcs are uniformly bounded too. This implies that the entropy of the fibers is zero and so, the entropy maximizing measures are the ones that project onto the volume measure m. Let explain this with more detail.

Our first lemma states that if $h(x) = h(y)$ then, x and y are in the same center ($y \in W^c(x)$) Of course, it is enough to prove it in the universal cover.

LEMMA 3.3.7. *$\tilde{y} \in W^c(\tilde{x})$ if $\tilde{h}(\tilde{x}) = \tilde{h}(\tilde{y})$.*

PROOF. Suppose that $\tilde{y} \notin W^c(\tilde{x})$. In this case we can suppose that $\tilde{y} \notin W^{cs}(\tilde{x})$ (the case when $\tilde{y} \notin W^{cu}(\tilde{x})$ being analogous)

Let $\tilde{z} = W^u(\tilde{y}) \cap W^{cs}(\tilde{x})$ and call $D_{cs} = d_{cs}(\tilde{x}, \tilde{z})$ and $D_u = d_u(\tilde{y}, \tilde{z})$. With d_u and d_{cs} we are denoting the intrinsic distances in W^u and W^{cs}, that is, the distances induced by the restriction of the ambient Riemannian metric to the corresponding manifolds. The existence (and uniqueness) of $\tilde{z}$ was proved in [**22**] (see also Proposition 2.15 of Hammerlindl thesis [**57**]) Now, the absolute partial hyperbolicity implies the existence of constants $1 < \lambda_c < \lambda_u$ such that $\forall n > 0$ $d(\tilde{f}^n(\tilde{x}), \tilde{f}^n(\tilde{z})) \leq \lambda_c^n D_{cs}$ and $d_u(\tilde{f}^n(\tilde{y}), \tilde{f}^n(\tilde{z})) \geq \lambda_u^n D_u$. Since $\mathcal{W}^u$ is quasi isometric we have that $d(\tilde{f}^n(\tilde{y}), \tilde{f}^n(\tilde{z})) > \frac{1}{a}(\lambda_u^n D_u - b)$. Finally, $d(\tilde{f}^n(\tilde{x}), \tilde{f}^n(\tilde{y})) > \frac{1}{a}(\lambda_u^n D_u - b) - \lambda_c^n D_{cs}$. This quantity goes to infinity with n implying that $\tilde{h}(\tilde{x}) \neq \tilde{h}(\tilde{y})$ and finishing the proof of the lemma. $\square$

REMARK 3.3.8. *Similar arguments give that the constants of the partial hyperbolicity of f and A are comparable. Moreover, this implies that the center manifolds of f goes through h into center manifolds (lines) of A. See Lemma 3.4 and Remark 3.5 of* [**122**].

The next step is to show that when $\tilde{h}(\tilde{x}) = \tilde{h}(\tilde{y})$ the whole center arc determined by $\tilde{x}$ and $\tilde{y}$, call it $[\tilde{x}, \tilde{y}]_c$, is contained in $\tilde{h}^{-1}(\tilde{(}\tilde{x}))$, that is, $\tilde{h}(\tilde{x}) = \tilde{h}(\tilde{z})$ for all $\tilde{z} \in [\tilde{x}, \tilde{y}]_c$. But this is a consequence of the fact that the center foliation is quasi-isometric. Suppose that $\tilde{z} \in [\tilde{x}, \tilde{y}]_c$ and $\tilde{h}(\tilde{x}) \neq \tilde{h}(\tilde{z})$. This implies that the sequence $\operatorname{dist}(\tilde{f}^n(\tilde{x}), \tilde{f}^n(\tilde{z}))$ is unbounded. In particular, there is an $n_0 \in \mathbb{Z}$ for which $\operatorname{length}(\tilde{f}^{n_0}([\tilde{x}, \tilde{y}]_c))$ is larger than $aK + b$, where a, b are the constants given by the quasi-isometric property of the center foliation. Finally, the quasi-isometry property gives that

$$d_c(\tilde{f}^{n_0}(\tilde{x}), \tilde{f}^{n_0}(\tilde{y})) \leq a \operatorname{dist}(\tilde{f}^{n_0}(\tilde{x}), \tilde{f}^{n_0}(\tilde{y})) + b < aK + b$$

which leads to contradiction. Observe that this argument implies that the lengths of this center arcs are smaller than $aK + b$ because the ambient distance between any two points, in particular its extremes, is less than K. Then, we have proved the following

PROPOSITION 3.3.9. *The fibers of h are center arcs of uniformly bounded length. In particular, $h(f, h^{-1}(x)) = 0$ for all $x \in \mathbb{T}^3$.*

3.3.2. Uniqueness of the entropy maximizing measure. In the previous subsection we have arrived to the conclusion that the entropy maximizing measures of f projects onto the volume m measure that is the unique entropy maximizing measure of A. We also have proved that the fibers of the semiconjugacy h are uniformly bounded center arcs. In this subsection we will show that this is enough to guarantee the uniqueness of the entropy maximizing measure of f.

We can classify the points of $\mathbb{T}^3$ into two classes according to their h-preimage. Let $R = \{x \in \mathbb{T}^3 : \#h^{-1}(x) = \infty\}$. Recall that the alternatives are: either $\#h^{-1}(x) = 1$ or $\#h^{-1}(x) = \infty$, since in the second case it is a non-degenerate center arc.

On each center manifold there is at most a countable number of disjoint non-degenerate arcs. Since h sends f-center manifolds into A-center manifolds we have that the intersection of R with a center manifold of A is a countable set. The center foliation of A is formed by parallel lines then, using Fubini we obtain that $m(R) = 0$. Hence, to obtain the measures that project onto m it is enough to restrict h to $h^{-1}()\mathbb{T}^3 \setminus R)$, but on this set h is injective. This clearly gives the uniqueness of the entropy maximizing measure μ of f and h itself is the desired isomorphism between (f, μ) and (A, m).

3.4. Compact center leaves

In this section we present a sketch of the proof of the main result of [**72**].

THEOREM 3.4.1 (See Theorem 3.1.3). *Let $f : M \to M$ be a partially hyperbolic diffeomorphism, dynamically coherent with compact one dimensional central leaves and satisfying accessibility property. Then one and only one of the following occurs*

1. *f admits a unique entropy maximizing measure μ and $\lambda_c(\mu) = 0$. Moreover (f, μ) is isomorphic to a Bernoulli shift,*
2. *f has a finite number (strictly greater than one) of ergodic maximizing measure all of which with non vanishing central Lyapunov exponent. Moreover (f, μ) is a finite extension of Bernoulli shift for any entropy maximizing measure μ.*

Moreover, the diffeomorphisms fulfilling the conditions of the second item form a C^1-open and $C^\infty-$dense subset of the dynamically coherent partially hyperbolic diffeomorphism with compact one dimensional central leaves.

Recall that a partially hyperbolic diffeomorphism satisfies the accessibility property if any pair of points can be joined by a continuous curve that is formed by a finite number of arcs that are tangent either to E^s or to E^u. In other words, the only nonempty set that is simultaneously saturated by stable and unstable leaves is the whole manifold.

In this section we will present all the result for one type of examples. This examples are the perturbations of $A \times Id : \mathbb{T}^2 \times \mathbb{S}^1 = \mathbb{T}^3 \to \mathbb{T}^3$ where $A : \mathbb{T}^2 \to \mathbb{T}^2$ is a hyperbolic automorphism and $Id : \mathbb{S}^1 \to \mathbb{S}^1$ is the identity map. Certainly, $A \times Id$ is partially hyperbolic and then, its perturbations so are. Moreover, the results of the classical book by Hirsch, Pugh and Shub [**74**] imply that if f is close enough to $A \times Id$ it has a center foliations that is conjugate (in particular homeomorphic) to the center foliations of $A \times Id$ by a homeomorphism nearby the identity. That means that the center foliation of f is formed by circles, the space of leaves is homeomorphic to $\mathbb{T}^2$ and the dynamics induced by f on the space of the center leaves is conjugate to A.

These last comments can be translated in terms of the LW formula. Call M_C to the space of center leaves (recall that M_C is homeomorphic to $\mathbb{T}^2$) Then, we have the natural projection $p : \mathbb{T}^3 \to M_C$ that sends a point $x \in \mathbb{T}^3$ to its center manifolds $W^c(x) \cong \mathbb{S}^1$. Since f sends center leaves into center leaves we have a naturally defined map on M_C, that is, we have a homeomorphism $g : M_C \cong \mathbb{T}^2 \to M_C \cong \mathbb{T}^2$. Moreover, g is conjugate to A. Joining all these observations together it is not difficult to conclude that f is semiconjugate to A. We will call π to this semiconjugacy. The fibers of this semiconjugacy are the center manifolds of f that are homeomorphic to circles and therefore, have null entropy. Since A has a unique entropy maximizing measure (that is the area form on $\mathbb{T}^2$) we have the following proposition.

Proposition 3.4.2. *μ is an entropy maximizing measure for f if and only if its π-projection is the area measure of $\mathbb{T}^2$. Moreover, $h_{top}(f) = h_{top}(A \times Id) = h_{top}(A)$.*

3.4.1. Null center Lyapunov exponent. The proof of Theorem 3.1.3 naturally splits into two cases according with the center behavior. We will see that the sign of the center Lyapunov exponent is the key property that determines whether f has a unique entropy maximizing measure or not. In this subsection we will, most of the time, assume that the center Lyapunov exponent is zero and we will conclude that this implies that f is, in a certain sense, rigid and has a unique entropy maximizing measure.

The proofs in this subsection use an invariant principle proved by Ávila and Viana [**10**] (see also Chapter 4) Although this principle is more general we will state it in our context. With this aim we introduce some concepts and observations.

Recall that we are assuming that f is a perturbation of $A \times Id$. Take a point $x \in \mathbb{T}^3$, then its center manifold $W^c(x)$ is homeomorphic to $\mathbb{S}^1$. We have $W^{cs}(x) = W^s(W^c(x)) = \cup_{y \in W^c(x)} W^s(y)$. In fact, the first equality is given by the dynamical coherence. It is not difficult to see that $W^{cs}(x)$ is an injectively immersed cylinder for any x and is equal to $\pi^{-1}(W^s(\pi(x)))$ where $W^s(\pi(x)$ here stands for the A-stable manifold of the π-image of x. Since $W^s(\pi(x)$ is an injectively immersed line and the fibers are homeomorphic to $\mathbb{S}^1$ we obtain that $W^{cs}(x) = \pi^{-1}(W^s(\pi(x)))$ is an injectively immersed cylinder.

The cylinder $W^{cs}(x)$ is subfoliated by both center and stable manifolds. Moreover, the reader can show that these foliations are such that any center circle is intersected at exactly one point by every stable leaf. Then, if C_1 and C_2 are two center leaves in the same center stable leaf there exists a homeomorphism $h_s : C_1 \to C_2$ sending a point $x \in C_1$ to the intersection of $W^s(x)$ with C_2 ($h_s(x) = W^s(x) \cap C_2$) This homeomorphism is called the stable holonomy (inside the center stable manifold) In fact this homeomorphism is C^1 is f is C^2 (see, for instance, [**101**]) Obviously, there is an analogous definition of unstable holonomies h_u using the unstable and center unstable leaves.

Now, we can "travel" from one center leaf C' to another C'' using stable and unstable holonomies. For this we only need to choose a path of stable and unstable leaves (like in the definition of accessibility) joining C' and C'' and travel using stable or unstable holonomies as appropriate. Observe that on the one hand, there is not a unique way of traveling from C' to C'', it depends on the choose of the stable-unstable path. On the other hand, there is always a path joining C' and C'' (if you suppose that f is accessible it is obvious, if not there is always a path of this kind joining C' and C'' as points of M_C and it is sufficient to lift this path in an adequate way) We will call to all these homeomorphisms (diffeomorphisms) holonomies of f.

As we have already said, given a probability measure μ on $\mathbb{T}^3$ we have defined automatically a probability $\nu = \mu \circ \pi^{-1}$ on $\mathbb{T}^2$ (the codomain of π) and a family of probabilities $\mu_z,\ z \in \mathbb{T}^2$ that are defined on each center leaf $\pi^{-1}(z)$. Rokhlin Disintegration Theorem states that there exists a unique (ν-a.e. z) such decomposition such that $\mu = \int_{\mathbb{T}^2} \mu_z d\nu(z)$. The μ invariance by f implies the invariance of ν and the uniqueness of Rokhlin disintegration implies that $\nu_z = \nu_{g(z)}$ for ν-a.e. z. Ávila-Viana's invariance principle gives conditions under which the conditional measures μ_z are holonomy invariant. We state Avila-Viana result adapted to our setting.

THEOREM 3.4.3 (Theorem D, [**10**]). *Let f be a perturbation of $A \times Id$ and $(m_k)_k$ be a sequence of f-invariant probability measures whose projection ν is a probability measure that has local product structure. Assume the sequence converges to some probability measure μ in the weak* topology and $\int |\lambda_c(x)| dm_k(x) \to 0$ when $k \to \infty$. Then, μ admits a disintegration $\{\mu_z : z \in \mathbb{T}^2\}$ which is invariant by holonomies and whose conditional probabilities μ_z vary continuously with z on the support of ν.*

This statement needs more explanation. Firstly, in this subsection, we will apply it to the case when the sequence of probability measures is constant and the measure itself has center Lyapunov exponent equal to 0. Secondly, we have to explain the meaning of the local product structure of ν. Observe that on $\mathbb{T}^2$ we have defined the automorphism A that has local product structure. That means that there are local neighborhoods of the form (it is enough homeomorphic to) $W^s_{loc}(z) \times W^u_{loc}(z)$. ν has *local product structure* if its restriction to all these neighborhoods is a product of a measure on $W^s_{loc}(z)$ times a measure on $W^u_{loc}(z)$. In fact, it is enough that ν restricted to $W^s_{loc}(z) \times W^u_{loc}(z)$ be equivalent to a product measure.

Then, if we suppose that μ is an entropy maximizing measure for f, it projects onto m, the area measure, that clearly has local product structure. These observations imply:

PROPOSITION 3.4.4. *Let μ be an entropy maximizing measure for f with null center Lyapunov exponent. Then, the conditional measures μ_z can be chosen in such a way that they vary continuously with z in the weak* topology and are invariant by holonomies.*

Observe in both statements, the theorem of Ávila-Viana and the last proposition, there an election of the conditional measures. Of course, since Rokhlin disintegration is unique a.e. this freedom of election involves only a set of null measure but there is a unique election that makes the conditional measures vary continuously.

The next conclusion is a consequence of the accessibility and the holonomy invariance. Recall that the accessibility property holds for an open and dense set of partially hyperbolic diffeomorphisms with one-dimensional center [**29**].

PROPOSITION 3.4.5. *Let f be a perturbation of $A \times Id$ with the accessibility property and suppose that μ is an entropy maximizing measure for f with null center Lyapunov exponent. Then, the support of any conditional measure μ_z is the whole center manifold $\pi^{-1}(z)$. Moreover, μ_z is non-atomic.*

Indeed, let $x, y \in \pi^{-1}(z)$. Then, the accessibility property implies that there is an su-path joining x with y. This implies that there is a holonomy $\bar{h} : \pi^{-1}(z) = W^c(x) \to W^c(x)$ with $\bar{h}(x) = y$. If x were an atom for μ_z holonomy invariance would imply that y would also be an atom. But since $y \in W^c(x)$ is arbitrary this leads to a contradiction with the fact that μ_z is a probability measure (moreover, it contradicts the fact of being σ-finite, μ_z would be the counting measure!)

Now, an argument of Ávila, Viana and Wilkinson [**11**] gives that f is conjugate to an accessible rotation extension of A. What do we mean by a rotation extension of A? It is a homeomorphism (in fact, it is needed more regularity that is included in the definition) $F : \mathbb{T}^2 \times \mathbb{S}^1 \to \mathbb{T}^2 \times \mathbb{S}^1$ such that $F(x, \theta) = (Ax, \theta + \varphi(x))$ for some Hölder function $\varphi : \mathbb{T}^2 \to \mathbb{S}^1$. In other words, we have a rotation on each fiber while A acts on the "base" $\mathbb{T}^2$. For this kind of diffeomorphisms with the accessibility property it was already known by Brin (see, for instance, [**45**]) that the volume measure on $\mathbb{T}^3$ is the unique invariant probability that projects onto the area measure m of $\mathbb{T}^2$. Of course, this implies that F that the volume measure is the unique entropy maximizing measure for F and then, modulo proving that f is conjugate to F, we have that it also has a unique entropy maximizing measure (the measure μ with null center Lyapunov exponent) Observe that the accessibility of F is a consequence of the accessibility of f and the fact that the conjugacy sends stable and unstable manifolds of f to stable and unstable manifolds of F.

Then. we are done with the first part of Theorem 3.1.3 if we prove the following lemma.

LEMMA 3.4.6. *f is conjugate to a rotation extension.*

PROOF. Consider the 2-torus $T = \mathbb{T}^2 \times \{0\}$. Since f is a perturbation of $A \times Id$ and the center leaves vary continuously in the C^1-topology, T intersect transversely each center leaf in unique point. In other words, T is a transverse section of the center foliation. We want to define a homeomorphism $h : \mathbb{T}^3 \to \mathbb{T}^3 = \mathbb{T}^2 \times \mathbb{S}^1$ that conjugates f with a rotation extension. First of all, take $h|_T$ as the conjugacy between $z \mapsto \pi(f(\pi^{-1}(z) \cap T))$ (observe that this gives an homeomorphism of the base $\mathbb{T}^2$) and A. We will call h_T to this homeomorphism. Secondly, take an orientation of the center foliation. Let $x \in T$, $y \in W^c(x)$ and $[x, y]_c$ the center segment contained in $W^c(x)$ going from x to y in the positive direction. We will define $h(y)$ as the point such that $[h_T(x), h(y)] \subset \mathbb{S}^1$ has length equal to $\mu_{\pi(x)}$. It is left as an exercise for the reader to verify that this homeomorphism h conjugates f with a rotation extension F. $\square$

We would like to mention that in this case the entropy maximizing measure is Bernoulli. It is enough to show this for F and the volume measure. But, in this case, the volume measure is F^n-ergodic for all n and F is a rotation extension of A for which the area is a Bernoulli probability. Then, a result of Rudolph [**109**] implies the Bernoulli property for F. Then, finally we have:

PROPOSITION 3.4.7. *Let f be a perturbation of $A \times Id$ with the accessibility property. If f has an entropy maximizing measure μ with zero center exponent then,*

- *μ is the unique entropy maximizing measure for f*
- *f has no hyperbolic periodic points (it is conjugated to a rotation extension)*
- *μ is Bernoulli*

Moreover, the same argument leads to a slightly more general (and more useful) statement (see the Ávila-Viana Theorem 3.4.3)

PROPOSITION 3.4.8. *Let f be a perturbation of $A \times Id$ with the accessibility property. If f has a sequence of entropy maximizing measures μ_n with center exponents converging to 0 and μ is a limit of this sequence then,*

- *μ is the unique entropy maximizing measure for f*
- *f has no hyperbolic periodic points (it is conjugated to a rotation extension)*
- *μ is Bernoulli*

Finally, observe that the diffeomorphisms that satisfy the conditions of Proposition 3.4.8 form a meager set. Although the accessibility property is open and dense, it is not difficult to see that the perturbations of $A \times Id$ having a hyperbolic periodic point also form an open and dense set.

3.4.2. Non-vanishing center exponents. In this part we will show the second part of Theorem 3.1.3. On the one hand, we will prove that if there is one maximizing measure with nonzero, say negative, center exponent then, there at least another entropy maximizing measure with positive center exponent. On the other hand, we will show that the number of entropy maximizing measures is finite.

Firstly we claim that there exists $c > 0$ such $|\lambda_c(\mu)| > c$ for every entropy maximizing measure μ. Here $\lambda_c(\mu)$ stands for the center Lyapunov exponent of the measure μ. Indeed, if this claim were false it would exist a sequence of maximizing measures μ_n that we can suppose that converges to a measure μ. Observe that our previous considerations imply that, since the μ_n are entropy maximizing probability measures, all the μ_n project through π onto the same measure

m, namely the area measure of $\mathbb{T}^2$. Then, the same argument of the previous subsection gives that this limit measure μ is the unique entropy maximizing measure of f and it is conjugate to a rotation extension. This contradicts the fact that there is a sequence of entropy maximizing measures (with nonzero center exponent) converging to μ.

The second step is to show that the existence of an entropy maximizing measure μ^+ with $\lambda_c(\mu^+) > 0$ implies that existence of an entropy maximizing measure μ^- with $\lambda_c(\mu^-) < 0$. The proof of the existence of such a measure μ^- is based in the following lemma.

LEMMA 3.4.9. *There exists a set S and $k \in \mathbb{N}$ such that $\mu^+(S) = 1$ and for every $x \in S$, we have $\#S \cap \pi(\pi^{-1}(x)) = k$.*

In other words, the center conditional measures of μ^+, μ^+_z $z \in \mathbb{T}^2$, have finite support for m almost every point z. The fact that this number is the constant k is a consequence of the ergodicity of m.

This lemma is a corollary of the results of Ruelle and Wilkinson in [**110**]. Intuitively one has that in the unstable Pesin manifold of almost every point intersected with a center the conditional measure must have an atom (the manifolds are exponentially contacted for the past) Then, Lemma 3.4.9 means that almost every conditional measure is supported in k points with weight $\frac{1}{k}$ by ergodicity. We also can suppose, by intersecting with the Pesin regular points, that the points in S have an unstable manifold and we call $W^u_P(x)$ that is two dimensional. Call $W^c_{\lambda_c}(x) = W^u_P(x) \cap W^c(x)$. Since $W^c(x)$ is diffeomorphic to $\mathbb{S}^1$ we have that $W^c_{\lambda_c}(x)$ is an arc. We are supposing that $x \in S$ is an atom of the conditional measure $\mu^+_{\pi(x)}$. Now, call $\bar{x}$ to the boundary point of the arc $W^c_{\lambda_c}(x)$ that is in the positive direction with respect to x and we will proceed to define the measure μ^-. Since we want that μ^- be an entropy maximizing measure its projection must be m. So, it is enough to define the conditional measures for almost every fiber and we do this in the following way: given $x \in S$ we will assign weight $\frac{1}{k}$ to $\bar{x}$. This is definition of μ^-.

It is left to the reader the proofs of the invariance of μ^- and its ergodicity (recall that, after satisfying these properties it is entropy maximizing by definition, it projects onto m) Then, we will show that μ^- has negative center exponent.

LEMMA 3.4.10. *$\lambda_c(\mu^-) < 0$.*

PROOF. First of all observe that $\lambda_c(\mu^-) \neq 0$. Indeed, if it were not the case, the arguments in the preceding subsection would imply that μ^- is the unique entropy maximizing measure. Now, if $\lambda_c(\mu^-) > 0$ the Pesin unstable manifolds of $\bar{x}$ coincide intersects the Pesin unstable manifolds of x contradicting that $\bar{x}$ is in the boundary of $W_{\lambda_c}(x)$. This implies the lemma. $\square$

REMARK 3.4.11. *Observe that we have proved that all entropy maximizing measures with nonzero center Lyapunov exponent are equivalent to finite extensions of Bernoulli shifts. We can not expect more that this. $A \times Id$ can be approximated by an f that is $A \times R$ where R is a Morse-Smale diffeomorphism of $\mathbb{S}^1$ having two periodic (not fixed) orbits one attracting and the other one repelling. Then, $A \times R$ has exactly two entropy maximizing measures, i.e. the product of the area measure of the 2-torus times the Dirac measure defined on each periodic orbit of R, but these measures are not equivalent to a Bernoulli shift (they are not even mixing since there is an iterate such that are not ergodic)*

3.4.3. Finitely many maximizing measures. We have proved that we have more than one entropy maximizing measure if we have an entropy maximizing measure with nonzero center exponent. The only thing we have to prove in order to finish the proof of the second item of Theorem 3.1.3 is the finiteness of the entropy maximizing measures.

With this aim suppose that we have infinitely many entropy maximizing measures with nonzero center Lyapunov exponent.

LEMMA 3.4.12. *Suppose that there is a sequence of entropy maximizing measures $(\mu_n^+)_n$ with $\mu_i^+ \neq \mu_j^+$ for $i \neq j$. Moreover, suppose that $\lambda_c(\mu_n^+) > 0$ for all n and that the sequence converges in the weak* topology to a measure μ. Then, the sequence $(\mu_n^-)_n$ also converges to μ.*

PROOF. Since the μ_n^+, $n \in \mathbb{N}$, are infinitely many different measures we have that for almost every center manifold the lengths of the Pesin center manifolds of the points that supports the corresponding conditional measures $(\mu_n^+)_z$ go to 0. For this is important to observe that the Birkhoff Theorem implies that the support of these conditional measures do not intersect. Then, we have that given $\varepsilon > 0$ there exists an $N > 0$ such that for $n > N$ the length of the Pesin center manifold of x is less than ε for every x in a set of center manifold of quotient measure m greater than $1 - \varepsilon$. This implies that for any continuous $\phi : M \to \mathbb{R}$ we have that $|\int \phi \, d\mu_n^+ - \int \phi \, d\mu_n^-| < (1-\varepsilon)\varepsilon + 2\varepsilon \max|\phi| \to 0$ with $\varepsilon \to 0$. Then, we obtained that $\mu_n^- \to \mu$ if $\mu_n^+ \to \mu$. $\square$

Now let us show that there exists just a finite number of ergodic entropy maximizing measures, that is, the compact subset of measures that project onto m is a finite simplex.

Suppose by contradiction that we have infinitely many entropy maximizing measures. As we have already shown the center Lyapunov exponents of these measures are nonzero. Then, we can take an infinite sequence having the center exponent with the same sign. Suppose that for this sequence the center exponent is positive (if not take the inverse). Since the set of invariant probabilities is sequentially compact we obtain a sequence of measures $(\mu_n^+)_n$ converging to a measure μ and satisfying the hypothesis of Lemma 3.4.12. Then, $(\mu_n^-)_n$ also converges to μ. On the one hand,

$$\int \log ||Df|_{E^c}||\, d\mu = \lim \int \log ||Df|_{E^c}||\, d\mu_n^+ = \lim \lambda_c(\mu_n^+) \geq 0$$

and analogously we have that

$$\int \log ||Df|_{E^c}||\, d\mu = \lim \int \log ||Df|_{E^c}||\, d\mu_n^- = \lim \lambda_c(\mu_n^-) \leq 0.$$

These two in equalities clearly imply that

$$\int \log ||Df|_{E^c}||\, d\mu = 0.$$

Then, $\lim \lambda_c(\mu_n^+) = 0$ and this yields to a contradiction with the observation done at the beginning of this subsection that there is $c > 0$ such that $|\lambda_c(\mu_n^+)| > c$.

This finishes the proof of the second part of Theorem 3.1.3.

Remark 3.4.13. *Bonatti and Díaz have shown in* [**13**] *that there are perturbations of $A \times Id$ that are robustly transitive. In other words, there exists an open set of diffeomorphisms $\mathcal{U}$ such that $A \times Id \in \overline{\mathcal{U}}$*

and $f \in \mathcal{U}$ implies that f is transitive. Moreover, the results in [**14**] *(see also* [**65**] *in greater dimensions) imply that there is an open (and C^1 dense) set of topologically mixing diffeomorphisms in $\mathcal{U}$. Then, Theorem 3.1.3 imply the existence of an open set of topologically mixing diffeomorphisms with more than one entropy maximizing measure.*

This remark naturally leads to some questions. For instance:

PROBLEM 3.4.14. *There are topologically mixing perturbations of $A \times Id$ with more than two entropy maximizing measures?*

3.5. Miscellany of results on entropy and maximizing measures

In this section we will roughly present some results related with topological entropy and maximizing measures. We will concentrate our presentation on the known results in the three dimensional setting. Then, we will not describe the very recent results of [**35, 51**].

3.5.1. h-expansiveness and maximizing measures. A sufficient condition for the existence of entropy maximizing is that the system be h-expansive (in fact it is enough with asymptotically h-expansiveness)

Given $\varepsilon > 0$ and $x \in M$ call $\Phi(x, \varepsilon) = \{y \in M; \operatorname{dist}(f^n(x), f^n(y)) \leq \varepsilon\}$ for all $n \in \mathbb{Z}$. In other words $\Phi(x, \varepsilon)$ is the set of points belonging to the closed ε-ball of x whose iterates remain forever a distance less or equal than ε of the orbits for the future and the past. $\Phi(x, \varepsilon)$ is a compact set and then, we have that its entropy is well defined. Denote $\tilde{h}_f(x, \varepsilon) = h(\Phi(x, \varepsilon), f)$ and $\tilde{h}_f(\varepsilon) = \sup_{x \in M} \tilde{h}_f(x, \varepsilon)$.

Intuitively, $\tilde{h}_f(\varepsilon)$ measures how much of the entropy is generated locally. If this quantity does not go to 0 with ε then the system is producing a bounded from bellow amount of entropy at smaller and smaller scales. If we want certain simplicity of the system from this point of view we will need that this quantity be zero. Then, the following are natural definitions (see the Bowen and Misiurewicz papers [**17, 92**])

DEFINITION 3.5.1. *f is h-expansive (or entropy expansive)if there exists ε_0 such that $\tilde{h}_f(\varepsilon) = 0$ for any $0 < \varepsilon < \varepsilon_0$. It is asymptotically h-expansive if $\lim_{\varepsilon \to 0} \tilde{h}_f(\varepsilon)$.*

Asymptotic h-expansiveness is enough for the existence of entropy maximizing measures and the C^∞ diffeomorphisms are asymptotically h-expansive [**34**] (in fact, Buzzi result gives the existence of symbolic extensions that implies the existence of entropy maximizing measures, the existence of such a measure was already known, see Newhouse and Yomdin results [**94, 126**])

The natural questions is what happens when the diffeomorphism is less regular. Then, if the center dimension is one we have the following result of Cowieson and Young.

THEOREM 3.5.2 ([**41**]). *If f is a partially hyperbolic diffeomorphism with one dimensional center then it is $h-$expansive.*

The idea of the proof is quite simple. Take $y \in \Phi(x, \varepsilon)$ with ε small enough. If $y \notin W^{cs}_\varepsilon(x)$ then, iterating for the future it goes away from the ε-neighborhood. Then, $y \in W^{cs}_\varepsilon(x)$. Analogously, taking backward iterates we obtain $y \in W^{cu}_\varepsilon(x)$. Hence, $y \in W^c_\varepsilon(x)$ and therefore $\Phi(x, \varepsilon) \subset W^c_\varepsilon(x)$. Since the center is one dimensional this implies that $h(\Phi(x, \varepsilon), f) = 0$ for all x.

The argument is obviously false if the center has dimension greater or equal than 2. In fact, Buzzi and Jana Rodriguez Hertz have an example of a partially hyperbolic diffeomorphism with 4-dimensional center and having no entropy maximizing measure [**37**].

3.5.2. Center Lyapunov exponent of entropy maximizing measures. In this subsection we will describe an estimation of the center Lyapunov exponents given in [**76**]. In this work the authors give some topological conditions on the strong foliations in order that the topological entropy be locally constant for C^∞ partially hyperbolic diffeomorphisms with one-dimensional center and be a continuous function if the center dimension is two. We will only present, without proofs a refined Pesin-Ruelle-like inequality for the one-dimensional case.

We need to define the volume growth of f on a foliation $\mathcal{W}$. For our purposes W will be the strong unstable (stable) foliation.

DEFINITION 3.5.3.

$$\textit{Let } \chi_{\mathcal{W}}(x, r) = \limsup \frac{1}{n} \log \operatorname{Vol}(f^n(W(x, r)))$$

where $W(x,r)$ is the ball of center x and radius r of the leaf $\mathcal{W}$ through x. Define the volume growth of f on $\mathcal{W}$ as

$$\chi_{\mathcal{W}}(f) = \sup_{x \in M} \chi_{\mathcal{W}}(x,r).$$

it is not difficult to show that $\chi_{\mathcal{W}}(f)$ does not depend on r.

Suppose that f is a $C^{1+\alpha}$ partially hyperbolic diffeomorphism with one dimensional center Hua, Saghin and Xia proved the following.

THEOREM 3.5.4 ([**76**]). *Let ν be an ergodic f-invariant measure and $\lambda_c(\nu)$ its center Lyapunov exponent then, $h_\nu(f) \leq \lambda_c(\nu) + \chi_{\mathcal{W}^u}(f)$.*

We present here an example of how one can apply this result. We will suppose that f is as in Section 3.3. That is, f is an absolutely partially hyperbolic diffeomorphism isotopic to a hyperbolic automorphism A of $\mathbb{T}^3$. We will assume that the unstable dimension of A is two. If this were not the case then replace f with f^{-1} in the following considerations.

The first thing we have is that $h_{top}(f) = h_{top}(A) = \lambda_u(A) + \lambda_c(A)$ where $\lambda_u(A) > 0$ and $\lambda_c(A) > 0$ are the Lyapunov exponents of A corresponding to the strong unstable and center direction respectively. Observe that the Lyapunov exponents of A do not depend on the measure because A is linear. Then, if $\nu = \mu$ is the entropy maximizing measure of f (uniqueness was shown in Section 3.3) we want to apply Theorem 3.5.4 to estimate the center Lyapunov exponent of μ. With this aim we will estimate the volume (length) growth of f on its strong unstable foliation.

Let $\tilde{h}$ be a lift of the semi-conjugacy of f with A. Observe that the volume growth does not change if we calculate it in the universal cover.

Take an strong unstable arc γ. On the one hand, the diameter of $\tilde{f}^n(\gamma)$, $\operatorname{diam}(\tilde{f}^n(\gamma))$, can be estimated in function of the diameter of $\tilde{A}^n(h(\gamma))$ and a constant K that bounds the distance between $\tilde{h}$ and the identity. We have that $\operatorname{diam}(\tilde{f}^n(\gamma)) \leq \operatorname{diam}(\tilde{A}^n(h(\gamma))) + 2K \leq \exp(n\lambda_u(A)) \operatorname{diam}(h(\gamma)) + 2K$.

On the other hand, the strong unstable foliation is quasi-isometric that is, there are constants a, b such that $\operatorname{dist}_u(x,y) \leq a\operatorname{dist}(x,y) + b$ for x, y in the same strong unstable manifold.

Then, if we put all these considerations together we obtain

$$\begin{aligned}\operatorname{dist}_u(f^n(x), f^n(y)) &\leq a\,\operatorname{dist}(f^n(x), f^n(y)) + b \\ &\leq a\,\exp(n\lambda_u(A))\operatorname{dist}(\tilde{h}(x), \tilde{h}(y)) + b\end{aligned}$$

and this easily implies that

$$\chi_{\mathcal{W}^u}(f) \leq \lambda_u(A).$$

Now, we are in conditions to apply Theorem 3.5.4 and obtain that the center Lyapunov exponent of the entropy maximizing measure is positive (see [**122**]):

$$\lambda_u(A) + \lambda_c(A) = h_{top}(f) \leq \lambda_c(\mu) + \chi_{\mathcal{W}^u}(f) \leq \lambda_c(\mu) + \lambda_u(A)$$

Then, $\lambda_c(\mu) \geq \lambda_c(A) > 0$.

3.5.3. The Entropy Conjecture. The arguments of this subsection follow [**51**] for the case that the center is one-dimensional case. They obtain the Entropy Conjecture for diffeomorphisms C^1-far from tangencies. The Entropy Conjecture for the case of one-dimensional center was firstly proved by Saghin and Xia in [**112**] using the volume growth of the strong unstable foliation.

Firstly, let us explain what the entropy conjecture is. If one has a diffeomorphism $f : M^m \to M^m$ it induces, for any $k = \{0, \ldots, m\}$ a linear operator of the real homology group $H_k(M, \mathbb{R})$. The *spectral radius* of f is the maximum on the spectral radius of these induced operators. More precisely, the spectral radius is

$$\operatorname{sp}(f) = \max_{k=\{0,\ldots,m\}} \operatorname{sp}_k(f)$$

where $\operatorname{sp}_k(f)$ is the spectral radius of the linear operator of the k homology group induced by f. Shub has conjectured that the logarithm of the spectral radius of a C^1-diffeomorphism f is a lower bound of its topological entropy [**115**].

Conjecture 3.5.5 (Entropy Conjecture).

$$\log(\operatorname{sp}(f)) \leq h_{top}(f).$$

It is well-known that the Entropy Conjecture is true for C^∞ diffeomorphisms, see [**126**], and false for Lipschitz homeomorphisms (at least if the dimension is great enough), see [**100**]. It is also known to be true for 3-dimensional homeomorphisms [**89**].

Then, one of the ideas in [**51**] is to show that, in their setting, the topological entropy is an upper semi-continuous function of f. This implies the entropy conjecture. Indeed, as we have already mention, we know that the entropy conjecture is true for the C^∞ diffeomorphisms [**126**]. Then, take a sequence f_l of partially hyperbolic diffeomorphisms with one-dimensional center that converges to f. Of course, we have that $\mathrm{sp}(f_l) = \mathrm{sp}(f)$ and $h_{top}(f_l) \geq \mathrm{sp}(f_l)$ since the diffeomorphisms f_l are C^∞. Finally the upper semi-continuity of the entropy gives

$$\mathrm{sp}(f) = \mathrm{sp}(f_l) \leq \limsup h_{top}(f_l) \leq h_{top}(f).$$

Then, the proof of the entropy conjecture in this setting is reduced to the proof of the following proposition:

PROPOSITION 3.5.6. *$f \mapsto h_{top}(f)$ is upper semi-continuous in the set of C^1 diffeomorphisms with one-dimensional center.*

On the one hand, observe that the arguments we give in the proof of Theorem 3.5.2 showing the h-expansivity of f remain true in a whole neighborhood. More precisely, there is an ε such that, for any g close enough to f, $\tilde{h}(g, \varepsilon) = 0$ (i.e. the entropy of the set of points of whose g-orbits that ε-shadow the g-orbit of x has null entropy and this is true for any g close enough to f and for any $x \in M$)

On the other hand, Bowen proved [**16**, Theorem 2.4] that if we have an ε given by the h-expansiveness we obtain that $h_{top}(g) = \limsup \frac{1}{n} Sp_n(M, \varepsilon)$ i.e. we do not need to take the limit when ε goes to 0.

In order to relate this Bowen's result with the semi-continuous variation of the topological entropy we need an equivalent definition of it. Suppose that $\mathcal{U}$ is an open covering of M. Denote by $r(\mathcal{U})$ the minimum number of elements of $\mathcal{U}$ needed to cover M. Given two coverings $\mathcal{U}_1$ and $\mathcal{U}_2$ denote $\mathcal{U}_1 \vee \mathcal{U}_2$ the covering formed by the open sets $U \cap V$ with $U \in \mathcal{U}_1$ and $V \in \mathcal{U}_2$ and define

$$\begin{aligned} h(f, \mathcal{U}) &= \lim \frac{1}{n} \log r(\mathcal{U} \vee f^{-1}(\mathcal{U}) \vee \cdots \vee f^{-n}(\mathcal{U})) \\ &= \inf_{n \geq 1} \frac{1}{n} \log r(\mathcal{U} \vee f^{-1}(\mathcal{U}) \vee \cdots \vee f^{-n}(\mathcal{U})). \end{aligned}$$

Observe that $r(\mathcal{U} \vee f^{-1}(\mathcal{U}) \vee \cdots \vee f^{-n}(\mathcal{U})) \leq r(\mathcal{U})^{n+1}$.

Then, one can define $h_{top}(f) = \sup_{\mathcal{U}} h(f, \mathcal{U})$. In fact, it is known that this definition coincides with the one we have previously used and, moreover, it was the original definition given in [**2**]. For the equivalence between both definitions see [**87**, Page 315, Exercise 7.1].

It is not difficult to convince yourself that $Sp_n(M, \varepsilon) \leq r(\mathcal{U} \vee f^{-1}(\mathcal{U}) \vee \cdots \vee f^{-n}(\mathcal{U}))$ if the maximum diameter of a set belonging to $\mathcal{U}$ is less than ε. Then, $h_{top}(f) = \limsup \frac{1}{n} \log Sp_n(M, \varepsilon) \leq h(f, \mathcal{U}) \leq h_{top}(f)$ if ε is a constant of h-expansiveness. The first equality is given by the Bowen's result mentioned earlier and the last inequality is because the topological entropy is the supremum on the coverings $\mathcal{U}$ of $h(f, \mathcal{U})$. So, $h_{top}(f) = h(f, \mathcal{U})$ if the diameter of the covering $\mathcal{U}$ is less than ε.

Finally, we know that $h(f, \mathcal{U}) = \inf_{n \geq 1} \frac{1}{n} \log r(\mathcal{U} \vee f^{-1}(\mathcal{U}) \vee \cdots \vee f^{-n}(\mathcal{U}))$. The functions $r(\mathcal{U} \vee f^{-1}(\mathcal{U}) \vee \cdots \vee f^{-n}(\mathcal{U}))$ are upper semi-continuous functions of f for each n and then, so are $\frac{1}{n} \log r(\mathcal{U} \vee f^{-1}(\mathcal{U}) \vee \cdots \vee f^{-n}(\mathcal{U}))$. The infimum of upper semi-continuous is an upper semi-continuous function. This finishes the proof of the upper semi-continuity of the topological entropy and then, the proof of the Entropy Conjecture for partially hyperbolic diffeomorphisms with one dimensional center.

CHAPTER 4

Partial hyperbolicity and cocycles

4.1. Introduction

Much of the richness of partially hyperbolic dynamics appears already in the dynamics of cocycles over hyperbolic maps. Such cocycle maps may sometimes be seen as partially hyperbolic diffeomorphisms.

We will try to maintain the discussion in the lowest possible dimensional level and in the simplest models. We think that most of the complexity already appears in this rather simple setting.

Let us give some flavor of what will happen along this chapter. Let $f : M \to M$ be a diffeomorphism and let $g : M \to Diff(S^1)$ be a smooth map so that $F : M \times S^1 \to M \times S^1$ given by

$$F(x, \theta) = (f(x), g_x(\theta))$$

(here $g_x(\theta)$ stands for $g(x)(\theta)$) be a diffeomorphism (we shall discuss smoothness later). Let $\omega_{0,M}$ be a smooth volume form on M and $d\theta$ the standard arclength in S^1 and consider $\omega_0 = \omega_{0,M} \wedge d\theta$ on $M \times S^1$. So, a continuous volume form on $M \times S^1$ would be of the form $\omega = u\omega_0$ where $u : M \times S^1 \to (0, \infty)$ is a continuous map. Hence, in order that F leave invariant a continuous volume form ω it is necessary and sufficient that

$$\frac{u}{u \circ F} = JDF, \tag{4.1}$$

where JDF is the jacobian of F w.r.t. ω_0. Now

$$JDF(x, \theta) = JDf(x) g_x'(\theta), \tag{4.2}$$

where JDf is the jacobian of f w.r.t. $\omega_{0,M}$ and g_x' is derivative of the diffeomorphism g_x w.r.t. θ. So that putting 4.1 and 4.2 together we get

$$u(x, \theta) = JDf(x) u(f(x), g_x(\theta)) g_x'(\theta) \tag{4.3}$$

Defining

$$v(x) = \int_{S^1} u(x,\theta)d\theta,$$

and integrating 4.3 on both sides w.r.t. θ plus the chain rule we get that

$$\begin{aligned} v(x) &= \int_{S^1} u(x,\theta)d\theta = JDf(x)\int_{S^1} u(f(x), g_x(\theta))g_x'(\theta)d\theta \\ &= JDf(x)\int_{S^1} u(f(x),\theta)d\theta = JDf(x)v(f(x)) \end{aligned}$$

So we get that defining $\omega_M = v\omega_{0,M}$, f preserves the volume form ω_M. This a much more general results that basically says that if we have a smooth fibering dynamics, i.e. p below is a smooth onto submersion,

$$\begin{tikzcd} N \arrow[r, "F"] \arrow[d, "p"'] & N \arrow[d, "p"'] \\ M \arrow[r, "f"] & M \end{tikzcd}$$

and F leaves invariant a volume then f also leaves invariant a volume form and p essentially intertwine the volume forms.

EXERCISE 4.1.1. *Make precise this generalization. (You will need to consider corresponding volume forms on the fibers.)*

Let us follow with our analysis. So we have that f leaves invariant the form $\omega_M = v\omega_{0,M}$ and that $\omega = a\omega_M \wedge d\theta$ where $a = u/v$ (we are assuming all volume forms positive everywhere). Hence, equation 4.3 and our condition gives that

$$a(x,\theta) = a(f(x), g_x(\theta))g_x'(\theta). \tag{4.4}$$

Now, integrating w.r.t. θ we get a function

$$A(x) = \int_{S^1} a(x,\theta)d\theta$$

which by the change of variable formula satisfies that $A \circ f \equiv A$. A very mild condition (less than ergodicity since A is continuous) will warrant that $A \equiv const$ and we may assume (and do assume) that this constant is 1.

EXERCISE 4.1.2. *Show that without any assumption a can be modified to a continuous function $\tilde{a}$ satisfying equation 4.4 and whose corresponding integral over the circles $\tilde{A}$ is constant* 1.

Now, let us define a continuous Riemannian metric on $M \times S^1$. For a point $p = (x, \theta)$ we have that $T_{(x,\theta)}M \times S^1 = T_x M \times T_\theta S^1$, so we will ask that $T_x M$ be orthogonal to $T_\theta S^1$ at p, that the volume of an orthonormal basis in $T_x M$ be 1 w.r.t. ω_M and that for $v \in T_\theta S^1$ (i.e. with 0 coordinate in $T_x M$), $|v|_{(x,\theta)} = a(x,\theta)|v|$, where $|v|$ is standard length in $T_\theta S^1$.

Restricting this Riemannian metric to each circle gives us a notion of length and hence of distance. Let us denote with d_x the distance on the circle $\{x\} \times S^1$. Then equation 4.4 gives us that for θ and θ' in S^1,

$$d_f(x)(g_x(\theta), g_x(\theta')) = d_x(\theta, \theta').$$

In other words, F may be seen us an isometric extension of f. Indeed, identifying S^1 to $\mathbb{R}/\mathbb{Z}$ we can take $0 \in \mathbb{R}/\mathbb{Z}$ and define the map $h_x : S^1 \to S^1$ by $h_x(\theta) = \int_0^\theta a(x,u)du$. (Prove that this map is well defined, i.e. it does not depend on the particular lift of $a(x,\cdot)$ to a map from $\mathbb{R}$ to $\mathbb{R}$ after we project the integral to the circle.)

h_x is clearly a C^1 diffeomorphism varying continuously with x (prove). If we define $H : M \times S^1 \to M \times S^1$ by $H(x,\theta) = (x, h_x(\theta))$ then H is a homemorphism that is smooth along the circle coordinate and preserve the circles. Moreover, if we conjugate F by H we get that $G = H \circ F \circ H^{-1} = (f(x), k_x(\theta))$. Using the change rule and equation 4.4 we get that $k_x'(\theta) \equiv 1$ (prove, it will be easier to consider the second coordinate of $H \circ F$ vs the second coordinate of H). Then $k_x(\theta) \equiv \theta + \alpha(x)$ where $\alpha(x)$ is a continuous function (why?). So finally we got that F is conjugated to the map

$$G(x,\theta) = (f(x), \theta + \alpha(x))$$

which is a rotation extension of f. Moreover, the conjugacy H was smooth along the circles and is identity in the base M and its smoothness depends directly on the smoothness of a. Indeed, looking at the proof it follow that if the volume form preserved by F, ω, is smooth then a is smooth and hence H is smooth.

Observe that if f has a fixed point (i.e. a circle invariant by F) then the dynamics over this circles has to be equivalent to a rotation.

In this chapter we will see that when f presents some sort of hyperbolicity, and under very mild conditions on the foliation by circles, or maybe even more general foliations the same type of rigidity applies, i.e. the volume preserving property of F implies the dynamics is smoothly conjugated to some cocycle dynamics. We hope the reader recognize the above example in most of the discussions of this chapter.

Let now $F : N \to N$ be a diffeomorphism. Assume F preserves a foliation by circles $\mathcal{F}$, the foliation yet need not be smooth, but we will assume that the leafs, i.e. the circles are smooth (at least C^1). Let μ be a measure, invariant by F. Since the foliation by circles form a measurable partition, Rokhlin decomposition theorem will disintegrate μ over this foliation to give conditional probability measures $\mu_x^{\mathcal{F}}$ over the circle $\mathcal{F}(x)$ for μ a.e. x. Uniqueness of the Rokhlin decomposition gives that $F_*\mu_x^{\mathcal{F}} = \mu_{F(x)}^{\mathcal{F}}$. Define on the circle $\mathcal{F}(x)$, $d_x(a,b) = \inf \mu_x^{\mathcal{F}}(A)$ where the infimum is taken over all open arcs A containing a and b. If $\mu_x^{\mathcal{F}}$ is non atomic then d_x is a pseudo-distance, i.e. it is symmetric, satisfies triangular inequality, $d_x(a,b) \geq 0$ and $d_x(a,a) = 0$ but $d_x(a,b) = 0$ with $a \neq b$ may happens. If $\mu_x^{\mathcal{F}}$ is moreover fully supported then d_x is a distance. If μ_x has atoms, then one may take an orientation on the circle and work with semi-open arcs. Let us assume for a moment that the measure has no atoms and follow with the argument.

EXERCISE 4.1.3. *Prove the assertions above about the properties of d_x.*

The invariance property of $\mu_x^{\mathcal{F}}$ gives that F behaves as an isometry along the circles w.r.t. d_x, i.e.

$$d_{F(x)}(F(a), F(b)) = d_x(a,b)$$

for a.e. x and for every $a, b \in \mathcal{F}(x)$. Also it follows that $x \to d_x$ is a measurable map since $x \to \mu_x^{\mathcal{F}}$ is measurable. If $x \to \mu_x^{\mathcal{F}}$ were continuous, then $x \to d_x$ would be also continuous (prove this).

So, the outcome is that if $\mu_x^{\mathcal{F}}$ has no atoms and is fully supported on $\mathcal{F}(x)$ then F is an isometry w.r.t. d_x. So, considering the quotient $M = N/\mathcal{F}$ with the quotient measure μ_M it can be seeing that F is an isometric extension of the quotient dynamics $f : M \to M$, of course this is only from a measurable viewpoint. Since isometries on the circle are essentially rotations (some orientation should be assumed, for instance let us assume that the circles can be oriented and that F preserves this orientation) we can see that at a measurable level we get the same as before, i.e.

PROPOSITION 4.1.4. *If the conditional measures $\mu_x^{\mathcal{F}}$ have no atoms and are fully supported then $F : (N, \mu) \to (N, \mu)$ is measurably conjugated to $G : (M \times S^1, \mu_M \times \lambda) \to (M \times S^1, \mu_M \times \lambda)$ given by $G(x, \theta) = (f(x), \theta + c(x))$ where $c : M \to S^1$ is a measurable map. (Here λ is Lebesgue measure on the circle). Moreover, if $x \to \mu_x^{\mathcal{F}}$ is continuous and the foliation by circles is a trivial fibration then the conjugacy and c (and hence G) can be taken continuous. If moreover $\mu_x^{\mathcal{F}}$ is absolutely continuous with continuous Radon Nikodym derivative, $x \to \mu_x^{\mathcal{F}}$ is differentiable and N is a smooth trivial fibration then the conjugacy and G are smooth.*

EXERCISE 4.1.5. *Prove that if $\mu_x^{\mathcal{F}}$ have no atoms for μ a.e. x then the Proposition holds without the assumption that $\mu_x^{\mathcal{F}}$ be fully supported.*

To understand (or try to understand) the case that $\mu_x^{\mathcal{F}}$ have atoms, let us assume that μ is ergodic. Now, ergodicity implies the following:

PROPOSITION 4.1.6. *There is and integer $n \geq 1$ such that for μ a.e. x, there is a set of n points $F(x) \subset \mathcal{F}(x)$ and $\mu_x^{\mathcal{F}}$ is the measure supported in $F(x)$ with equal mass $\frac{1}{n}$ on each atom. Moreover, $F : (N, \mu) \to (N, \mu)$ is measurably conjugated to $G : (M \times \mathbb{Z}_n, \mu_M \times \lambda_n) \to (M \times \mathbb{Z}_n, \mu_M \times \lambda_n)$ given by $G(x, \theta) = (f(x), \theta + c(x) mod\ n)$ where $c : M \to \mathbb{Z}_n$ is a measurable map (here, λ_n is equally distributed measure in $\mathbb{Z}_n = \{0, 1, \dots, n-1\}$).*

Hence in the measurable category, F is a finite extension of f.

EXERCISE 4.1.7. *Prove the proposition.*

The type of phenomenon we described above holds in more general situations but as we said we only try to give a glimpse of this. In the measurable category, we saw that the properties of F are mostly governed by the properties of a cocycle map $G(x, \theta) = (f(x), \theta + c(x))$.

Let us try to translate the measurable properties of G in terms of that of F and some properties of c.

4.2. Ergodic properties of cocycles

Let $f: M \to M$ be a diffeomorphism preserving a measure μ and $c: M \to S^1$ be a continuous map. Define the $F: M \times S^1 \to M \times S^1$ by $F(x,\theta) = (f(x), \theta + c(x))$. Clearly G preserves the measure $\mu \times \lambda$. Let us conjugate F by a map $H: M \times S^1 \to M \times S^1$ given by $H(x,\theta) = (x, \theta + u(x))$ where $u: M \to S^1$. Then $G = H^{-1} \circ F \circ H$ has the form

$$G(x,\theta) = (f(x), \theta + c(x) + u(x) - u(f(x))).$$

Consider the equation

$$u(f(x))) - u(x) = c(x) - c_0, \tag{4.5}$$

where c_0 is a constant, here we are taking the equation mod 1. Equation 4.5 is known as a *cohomological equation* with values in S^1. Observe that even if u and c take values in S^1 it make sense to integrate them and the result will be an element in S^1. If we integrate both sides of equation 4.5 we get that to have a measurable solution u we need that $\int_{S^1} c(x) d\mu(x) = c_0$, this determines the value of c_0. Moreover we can see the following,

PROPOSITION 4.2.1. *If equation 4.5 has a measurable, continuous or smooth solution then F is measurably, topologically or smoothly conjugated to $G(x,\theta) = (f(x), \theta + c_0)$. Moreover, in this case, if c_0 is rational, then F is not ergodic and in any case, F is not weakmixing.*

What can be said if equation 4.5 has no solution? Assuming f is weakmixing, mixing, Kolomogorov, Bernoulli, is it true that F will be weakmixing, mixing, Kolmogorov, Bernoulli? Although we could follow this analysis in this general framework and we encourage the reader to do so, we want to concentrate now in the case f is a hyperbolic diffeomorphism. Indeed, in this case it follows that F is a partially hyperbolic diffeomorphism and hence the accessibility property for F would imply that F is Kolmogorov. Hence let us jump into the next section.

4.3. Accessibility

Let by now $f: M \to M$ be an Anosov diffeomorphism and consider $F: M \times S^1 \to M \times S^1$ by $F(x,\theta) = (f(x), \theta + c(x))$ where $c: M \to S^1$ is Hölder continuous function, here we are identifying $S^1 = \mathbb{R}/\mathbb{Z}$ with its inherited group structure. Let 0 be the neutral element of S^1. We are not making any assumption on the transitivity of f yet. Then the stable manifold of the point (x,θ) is given by:

$$W^s(x,\theta) = \{(y, \theta + \sum_{k\geq 0} c(f^n(x)) - c(f^n(y))) : y \in W^s(x)\}$$

Here $W^s(x)$ stands for the stable manifold of x w.r.t. f. Similarly

$$W^u(x,\theta) = \{(y, \theta + \sum_{k\leq -1} c(f^n(x)) - c(f^n(y))) : y \in W^u(x)\}.$$

Observe that

$$W^\sigma(x,\theta') = \theta' - \theta + W^\sigma(x,\theta).$$

for $\sigma = s, u$. This helps to compute the accessibility class of a point (x,θ) since we get a nice structure for the stable and unstable manifolds. Indeed, let us define for $y \in W^s(x)$,

$$P^s(x,y) = \sum_{k\geq 0} c(f^n(x)) - c(f^n(y))$$

and for $y \in W^u(x)$,

$$P^u(x,y) = \sum_{k\leq -1} c(f^n(x)) - c(f^n(y)).$$

Let $x \in M$ and $y \in W^s(x)$ then we can define the stable holonomy map

$$H^s_{x,y} : \{x\} \times S^1 \rightarrow \{y\} \times S^1$$

by

$$H^s_{x,y}(x,\theta) = W^s(x,\theta) \cap \{y\} \times S^1.$$

It follows that

$$H^s_{x,y}(x,\theta) = (y, \theta + P^s(x,y)).$$

So, given an su-path $\gamma = [x_0, x_1, \ldots, x_n]$ in M (remember the dynamics in M is Anosov) we can concatenate the holonomies and get a map

$$H_\gamma : \{x_0\} \times S^1 \rightarrow \{x_n\} \times S^1$$

by

$$H_\gamma(x_0,\theta) = (x_n, \theta + a(\gamma)).$$

If in particular we take the Γ_x the set of su-paths starting and ending in x then we a map $a : \Gamma_x \rightarrow S^1$. This map behaves very nicely under the operation of concatenating paths in Γ_x, i.e. $\gamma_1 * \gamma_2$ is the concatenation of the path γ_1 and γ_2. We can also define the inverse path, γ^{-1} is the same path traveled in opposite direction. Even though Γ_x is not a group under this operations, we have easily that $a(\gamma_1 * \gamma_2) = a(\gamma_1) + a(\gamma_2)$ and $a(\gamma^{-1}) = -a(\gamma)$. Hence $H(x) := \{a(\gamma) : \gamma \in \Gamma_x\}$ is a subgroup of S^1.

Proposition 4.3.1. *$H(x)$ does not depend on x.*

Proof. Indeed, since $f : M \to M$ is Anosov, any to points can be joined by an su-path (prove this). Then, if $a = a(\gamma) \in H(x)$ for some $\gamma \in \Gamma_x$, we can take $\gamma_{x,y}$ an su-path joining x and y and then we get that $\gamma_{x,y} * \gamma * \gamma_{x,y}^{-1} \in \Gamma_y$ and $a(\gamma_{x,y} * \gamma * \gamma_{x,y}^{-1}) = a(\gamma)$, hence $a = a(\gamma) \in H(y)$. the other inclusion follows as well. □

It is a very nice exercise to work also in the case M is only a (possibly totally disconnected) hyperbolic set.

So let us call $H(x)$ simply by H. Now let $\Gamma_x^0 \subset \Gamma_x$ be the set of su-paths in M that can be deformed by an homotopy inside Γ_x to the constant path, i.e. $\gamma \in \Gamma_x^0$ if there is a continuous map $H : I \times I \to M$ such that $H(t, \cdot) \in \Gamma_x$ for every $t \in I$, $H(0, \cdot) = \gamma$ and $H(1, \cdot) \equiv x$ (here $I = [0, 1]$). It is not hard to see that given an su-path $\gamma_{x,y}$ joining to points x and y, $\gamma_{x,y}^{-1} * \gamma_{x,y} \in \Gamma_x^0$.

Exercise 4.3.2. *Prove that Γ_x^0 is the same as the set of su-paths in Γ_x homotopic to constant. Prove this by showing that any path is homotopic to a an su path through su-paths (use uniform local product structure).*

So, we can define $H_0 = \{a(\gamma) : \gamma \in \Gamma_x^0\}$ and the same proof gives that H_0 is a group and H_0 does not depend on x (repeat the proof to make sure it follows as well). Clearly

$$\{0\} \subset H_0 \subset H \subset \bar{H} \subset S^1.$$

Now we have the following:

Theorem 4.3.3. *H_0 is connected and moreover it coincides with the connected component of the identity of H. F has the accessibility property if and only if $H_0 = S^1$ if and only if $H = S^1$. F has the essential accessibility property if and only if $\bar{H} = S^1$.*

Observe that so far everything works under the milder assumption that f be a hyperbolic homeomorphism plus some irreducibility.

In the proof of the theorem we will use that M is a manifold, although some more general result should follow. Also for more general groups than S^1 there is a counterpart. The interested reader may found more on this subject in [**19**], [**32**], both are very enjoyable papers.

Proof. Connectedness of H_0 follows from connectedness of Γ_x^0 plus continuous dependence of holonomy maps on base points. Let us skip the second assertion until the end of the proof. Now, if $H = S^1$ then any two points in the same circle can be joined by an su-path in $M \times S^1$. So, let (x, θ) and (y, θ') be two points in $M \times S^1$, take and su- path γ_1 joining (y, θ') with a point (x, θ'') this can be done by first chosing an su-path in M joining y with x and then taking the corresponding su-path in $M \times S^1$. Now, since $H = S^1$ there should be an su-path joining (x, θ'') with (x, θ) and we are done.

Let us see that if F has the accessibility property then $H_0 = S^1$. Consider small 4 legged su-paths in M contained in a local product structure neighborhood this paths are in Γ_0^x. If for such paths the corresponding elements in H_0 are always 0, i.e. the corresponding su-paths in $M \times S^1$ are closed, then it is not hard to see as in previous section that the accessibility class of x is a manifold, i.e. W^s and W^u are jointly integrable and hence F has not the accessibility property. Hence for one of such quadrilaterals the corresponding element in H_0 is non trivial. But there are not nontrivial connected subgroups of S^1, hence we get that $H_0 = S^1$. And we are done. Now the prove that H_0 coincides with the connected component of the identity of H is an exercise.

FInally, that F has the essential accessibility property means that any su-saturated set has either full or null Lebesgue measure. Assume first that $\bar{H} \neq S^1$, then $H = \bar{H}$ is discrete and hence finite, so there is a set $A \subset S^1$ with intermediate measure ($0 < |A| < 1$) such that $A + H = A$, i.e. $A + a = A$ for every $a \in H$. Take a point $x \in M$ consider the su-saturation of $\{x\} \times A$, i.e. all the points in $M \times S^1$ that can be joined with $\{x\} \times A$ by an su-path, call this set U. This set can be taken measurable since A can be taken an open set on S^1 and hence U will be open. Now, given any $y \in M$ we have that there is a map $H_{x,y} : \{x\} \times S^1 \to \{y\} \times S^1$ that is a translation on the circle (holonomy along an su-path joining x and y) such that $H_{x,y}(\{x\} \times A = U \cap \{y\} \times S^1$ hence, the Lebesgue measure λ_S^1 of

$U \cap \{y\} \times S^1$ is the same as $\lambda_{S^1}(A)$ which is positive and less than 1 for every $y \in M$. Then U canhave neither full measure nor null measure and hence F has not the essential accessibility property.

Assume now that F has not the essential accessibility property and take U an su-saturated set of intermediate measure. Then if we consider $U_x \subset S^1$ such that $U \cap \{x\} \times S^1 = \{x\} \times U_x$ then U_x is invariant by H. Moreover, for some x, U_x needs to have intermediate measure and hence H leaves invariant a set of intermediate measure, so H cannot be dense. $\square$

It is a nice exercise to analyze the accessibility classes for dynamics of the type $F(x,\theta) = (f(x), g(x,\theta))$, where $f : M \to M$ is an Anosov diffeomorphism and $g : M \times S \to S$ is a smooth map such that F is a partially hyperbolic diffeomorphism with center space $\{0\} \times TS$. Particularly interesting is the case when $S = S^1$ is a circle.

4.4. Holonomy invariance and continuity of conditional measures

In this section we follow the analysis of maps $F : M \times S \to M \times S$ of the form $F(x,\theta) = (f(x), g(x,\theta))$, let us denote $E = M \times S$ and $p : E \to M$ the projection. Although the general context works with different types of S we will focus in the case $S = S^1$ and comment about the generalizations. Everything we would say in this section essentially goes back to the work of F. Ledrappier [**83**] on random product of matrices that was later extended by A. Avila and M Viana [**9**] to a framework closer to what we will be working here and in [**8**] by A. Avila, J. Santamaria and M Viana [**9**] where the case f is partially hyperbolic was addressed.

Returning to the discussion of the introduction of the chapter, let us assume that f is hyperbolic (again a hyperbolic homeomorphism will be good enough, say a shift of finite type). Let us assume that for every $x \in M$ and $y \in W^s(x)$ there is a homeomorphism $H^s_{x,y} : S \to S$ that we shall call the stable holonomy such that if $y, z \in W^s(x)$ then

$$H^s_{x,z} = H^s_{x,y} \circ H^s_{y,z},$$

$$H^s_{x,y} = (H^s_{y,x})^{-1}$$

and

$$H^s_{f(x),f(y)} \circ F = F \circ H^s_{x,y}.$$

Moreover we want that $(x, y) \to H^s_{x,y}$ depends continuously in the appropriate topology (compact-open should be ok, uniform). Similarly we shall assume the existence of unstable holonomies. Also we will assume that the map $\theta \to g(x, \theta) =: g_x(\theta)$ is a smooth diffeomorphism. The dependence on x may be milder, measurable or continuous, depending on the result we are looking for.

Let ν be a probability measure invariant by F, let μ be the projected measure, i.e $p_*\nu = \mu$ or $\mu(A) = \nu(p^{-1}(A))$. The partition $\{x\} \times S$ is a measurable partition, hence we can apply Rokhlin decomposition and get a measurable map $M \to P(S)$, $x \to \nu_x$, where P is the set of probability measures on S with the weak* topology that disintegrates ν. We call ν_x the conditional measures. The map $x \to \nu_x$ is defined only a.e. We say that the Rokhlin decomposition is continuous if it coincides a.e. with a continuous map.

We want to understand how ν_x looks like and also the dependence on x. In particular, in the case $S = S^1$ we want to recover the work of the introduction. We want to prove for instance that if $y \in W^s(x)$ then $(H^s_{x,y})_*\nu_x = \nu_y$ for μ a.e. $x, y \in M$, in such a case we say that the disintegration is s-invariant, similarly for u-holonomies. Also we would like to know continuity of the disintegration. To this end, let us introduce some quantities.

First the extremal Lyapunov exponents along the fiber S. Let us denote with $g^n_x = g_{f^{n-1}(x)} \circ \ldots g_x$. Let us denote for an invertible linear map A, $m(A) = \|A^{-1}\|^{-1}$. Assuming that the maps $(x, \theta) \to \log \|D_\theta g_x\|$ and $(x, \theta) \to \log m(D_\theta g_x)$ are integrable let us define

$$\lambda_+(x, \theta) = \lim_{n \to +\infty} \frac{1}{n} \log \|D_\theta g^n_x\| \tag{4.6}$$

$$\lambda_-(x, \theta) = \lim_{n \to +\infty} \frac{1}{n} \log m(D_\theta g^n_x) \tag{4.7}$$

The limits exist ν a.e. by the multiplicative ergodic theorem and $\lambda_- \leq \lambda_+$ (exercise). Let us define the integrated extremal Lyapunov exponents

$$\lambda_\pm(\nu) = \int \lambda_\pm(x, \theta) d\nu \tag{4.8}$$

Observe that if ν is ergodic then $\lambda_\pm(\nu) = \lambda_\pm(x, \theta)$ for ν a.e. (x, θ).

We will make a first very important assumption on the measure μ, the invariant measure for the hyperbolic map. The assumption is that such a measure has *local product structure*, this means that Fubini holds for such a measure w.r.t. to stable and unstable foliations and hence the Hopf argument can be applied to this measure. To be more precise, let us denote with μ_x^s and μ_x^u the stable and unstable conditional measures. We say that μ has *local product structure* if stable holonomy map between unstable manifolds are absolutely continuous w.r.t. conditional measures, i.e. there is a set R of full μ measure such that if $x, y \in R$, $y \in W^s_{loc}(x)$ and

$$hol : W^u_{loc}(x) \to W^u_{loc}(y)$$

is stable holonomy ($hol(z) = W^s_{loc}(z) \cap W^u_{loc}(y)$), then for every set $A \subset W^u_{loc}(x)$, if $\mu_x^u(A) = 0$ then $\mu_y^u(hol(A)) = 0$.

THEOREM 4.4.1. *Let $F : M \times S \to M \times S$, $F(x, \theta) = (f(x), g_x(\theta))$ be as above with f hyperbolic and let μ be an ergodic measure invariant by f with local product structure. Let ν_k be a sequence of measures invariant by F projecting to μ and assume that*

$$\int |\lambda_\pm(x, \theta)| d\nu_k \to 0$$

as $k \to \infty$. If $\nu_k \to \nu$ then the disintegration ν_x is continuous and $s-$ and $u-$invariant.

The proof is in two steps, first one proves s- and $u-$invariance, this holds without the assumption of local product structure in μ. Then proves that s- and $u-$invariance implies continuity when μ has local product structure.

Let us begin with this second part.

PROOF. **s- and $u-$invariance implies continuity** Assume ν is $s-$ and $u-$ invariant. This means that there is a set R of full μ measure such that if $x, y \in R$ and $y \in W^s(x)$ then $(H^s_{x,y})_* \nu_x = \nu_y$ and if $x, y \in R$ and $y \in W^u(x)$ then $(H^u_{x,y})_* \nu_x = \nu_y$. Now continuity of holonomy maps plus holonomy invariance implies continuity of conditional measures when moving along stable and unstable manifolds, i.e. chose a metric defining the weak* topology on measures, then for any $\varepsilon > 0$ there is a $\delta > 0$ such that if $x, y \in R$, $y \in W^s_{loc}(x)$, or $y \in W^u_{loc}(x)$ and $d(x, y) < \delta$ then $d(\nu_x, \nu_y) < \varepsilon$.

Now we will apply Hopf argument to prove that there is a set R' of full measure such that $x \to \nu_x$ on R' is uniformly continuous. Let $R' \subset R$ be the set of points x such that $R \cap W^u(x)$ has full μ_x^u measure. Take $\varepsilon > 0$ and take $\delta > 0$ as above. Let $\delta' > 0$ be small to be determined later, it will depend on continuity of holonomy maps for f. Take $x, y \in R$ such that $d(x, y) < \delta'$, we assume δ' is small enough so that $hol : W^u_{loc}(x) \to W^u_{loc}(y)$ is well defined. Since hol is absolutely continuous and $R \cap W^u(x)$ has full μ_x^u and $R \cap W^u(y)$ has full μ_y^u there is a point $z \in R \cap W^u(x)$ such that $hol(z) \in R \cap W^u(y)$. We may assume, if δ' is small enough than $d(x, z)$, $d(z, hol(z))$, $d(hol(z), y)$ are all less than δ. We get that since $x, z \in R$, $z \in W^s_{loc}(x)$ and $d(x, z) < \delta$, hence $d(\mu_x, \mu_z) < \varepsilon$. Similarly with z and $hol(z)$ along unstables and $hol(z)$ and y again along stables so we get that

$$d(\mu_x, \mu_y) \leq d(\mu_x, \mu_z) + d(\mu_z, \mu_{hol(z)}) + d(\mu_{hol(z)}, \mu_y) < 3\varepsilon$$

and we get the uniform continuity. Hence it extends to a continuous map. □

Let us go into the prove of $u-$ invariance. Let us assume for simplicity that $\nu_k = \nu$ for every k, that ν is ergodic, that $\lambda_+(\nu) \leq 0$ but no assumption on the projected measure μ.

Let us see a somewhat more geometric proof that is inspired in a related theorem by F. Ledrappier and J-S. Xie, [**85**], see also [**82**].

Proof. $u-$**holonomy invariance**

Let ξ be an increasing measurable partition of M subordinated to W^u. Given a measurable set $A \subset \{x\} \times S$ let us define

$$\xi(A) = \{(y, H^u_{x,y}(\theta)) : (x, \theta) \in A \text{ and } y \in \xi(x)\}$$

a local unstable saturation of A in $M \times S$. Let us consider the partition of $M \times S$ given by $\xi^{cu}(x) = \xi(\{x\} \times S) = \xi(x) \times S$, observe that if $y \in \xi(x)$ then $\xi^{cu}(x) = \xi^{cu}(y)$. Recall that ν_x are the conditional measure w.r.t. the partition $\{x\} \times S$. Let $\nu_x^{\xi^{cu}}$ be the conditional measures w.r.t. the partition ξ^{cu} and observe that if $y \in \xi(y)$ then $\nu_x^{\xi^{cu}} = \nu_y^{\xi^{cu}}$. $u-$invariance is equivalent to prove that for x-a.e. point w.r.t. μ and for every measurable set $A \subset \{x\} \times S$,

$$\nu_x(A) = \nu_x^{\xi^{cu}}(\xi(A)).$$

We shall prove that for any such increasing partition ξ, for μ-a.e. x and for any measurable set $A \subset \{x\} \times S$,

$$\nu_x^{\xi^{cu}}(\xi(A)) = \nu_x^{(f^{-1}\xi)^{cu}}((f^{-1}\xi)(A)) \tag{4.9}$$

where the partition $f^{-1}\xi$ is given by $(f^{-1}\xi)(x) = f^{-1}(\xi(f(x)))$. Observe that $F^{-1}\xi^{cu} = (f^{-1}\xi)^{cu}$. Since ξ is increasing, $f^{-1}\xi > \xi$ and hence $F^{-1}\xi^{cu} > \xi^{cu}$. Invariance of ν gives that

$$\nu_x^{F^{-1}\xi^{cu}} = F_*^{-1}\nu_{f(x)}^{\xi^{cu}}$$

and hence, the right hand side of 4.9 is equal to

$$\nu_x^{(f^{-1}\xi)^{cu}}((f^{-1}\xi)(A)) = \nu_{f(x)}^{\xi^{cu}}(\xi(F(A))).$$

Let us fix by now an increasing partition ξ and let us put for $A \subset \{x\} \times S$, $\nu_x^0(A) = \nu_x^{\xi^{cu}}(\xi(A))$ and $\nu_x^1(A) = \nu_x^{(f^{-1}\xi)^{cu}}((f^{-1}\xi)(A))$. Using Lebesgue-Radon-Nycodim decomposition we get that $\nu_x^1 = \rho_x \nu_x^0 + \eta_x$. Let us define the entropy of ν_x^1 w.r.t. ν_x^0 as

$$H(\nu_x^1, \nu_x^0) = -\int \log \rho_x(\theta) d\nu_x^0(\theta)$$

and

$$h(F, \nu, \xi) = \int H(\nu_x^1, \nu_x^0) d\mu(x).$$

Jensen's inequality gives that $H(F, \nu, \xi) = 0$ if and only if $\nu_x^1 = \nu_x^0$ for μ a.e. x. So we have to prove that $H(F, \nu, \xi) = 0$, this is some kind of fiber entropy. The vanishing of the fiber entropy follows from a kind of Ruelle inequality, i.e.

PROPOSITION 4.4.2. *There is a constant* $C > 0$ *such that* $H(F, \nu, \xi) \leq C \max\{0, \lambda_+(\nu)\}$.

We omit here the proof of this lemma although as we said this is a kind of fiber Ruelle's inequality. Indeed, the following seems to be true: if f is a hyperbolic map then

$$h(F, \nu) = h(f, \mu) + H(F, \nu, \xi)$$

and hence Ledrappier-Young entropy formula essentially implies an improvement of the proposition. $\square$

Let us state now the partially hyperbolic case. We will not show a proof here, we only mention that the proof is somehow a mix of the above proof combined with the technics of jullienes. Let λ denote volume measure.

THEOREM 4.4.3. [**8**] *Let $f : M \to M$ be a partially hyperbolic, volume preserving diffeomorphism with the accessibility property and center-bunched. Let $F : M \times S \to M \times S$ such that $F(x, \theta) = (f(x), g_x(\theta))$ be as before. Let ν_k be a sequence of measures projecting into λ and such that $\int |\lambda_\pm(x,\theta)| d\nu_k \to 0$. Then, if $\nu_k \to \nu$ then the disintegration $x \to \nu_x$ is $s-$ and $u-$invariant and continuous.*

Again the proof splits in two steps. The first step is to prove $s-$ and $u-$invariance. This part works the same as the previous case, so essentially the same proof works. But the proof of continuity is more subtle since a priori the measure λ has not local product structure. The idea is to use Hopf-Pugh-Shub argument using Jullienes instead of the standard Hopd argument using local product structure.

4.5. Absolute continuity of center foliations and rigidity

4.5.1. Conservative systems. In the present section we want to discuss the following problem. Let $f : M \to M$ be a volume preserving partially hyperbolic diffeomorphism with the accessibility property and assume that the centre bundle integrates to a C^0 foliation is by circles. We would like to study the absolute continuity of this centre foliation. We addressed this problem in our survey [**66**]. In [**11**], based in the work of [**8**], A. Avila, M. Viana and A. Wilkinson solved this problem, let us see the outcome.

THEOREM 4.5.1. [**11**] *Let $F : N \to N$ be a volume preserving partially hyperbolic diffeomorphism with the accessibility property whose center bundle integrates to a foliation by circles. Assume that the centre foliation is absolutely continuous, then $F : N \to N$ is smoothly conjugated to an isometric (or rotation) extension of an Anosov diffeomorphism. In particular if the foliation is trivializable, e.g. N is a torus, then F is smoothly conjugated to a map $G : M \times S^1 \to M \times S^1$ of the form $G(x, \theta) = (f(x), \theta + c(x))$ where f is a volume preserving Anosov diffeomorphism and $x \to c(x)$ is a smooth map.*

Observe that this implies automatically that absolute continuity is a very rare property among such systems. The volume preserving hypothesis is very important since there are examples of partially hyperbolic, accessible diffeomorphisms on $\mathbb{T}^3$ whose central foliation by circles is robustly absolutely continuous. Of course this are non volume preserving systems, but they present an SRB measure. See Subsection 4.5.2.

Let us briefly explain the proof. First of all, let us see that the central Lyapunov exponents vanish. If this were not the case, then an argument like the ones in the entropy maximizing case proves that conditional measures along the central foliation are atomic contradicting absolute continuity.

Let us assume by now, for simplicity in the exposition, that $N = \mathbb{T}^3$. Then it turns out that the central foliation is trivial (see [**40**]). Let us consider the map of the 4 dimensional manifold $\hat{F} : N \times S^1 \to N \times S^1$ given by $\hat{F}(x, \theta) = (F(x), g_x(\theta))$ where $g_x(\theta)$ is the diffeomorphism (after a fixed parametrization of center leaves) between the circle through x and the circle through $F(x)$, i.e. if $\gamma : N \to Emb(S^1, N)$ is a parametrization of the circles then $g_x = \gamma(F(x))^{-1} \circ F \circ \gamma(x)$. Since γ is Hölder continuous g_x depends Hölder continuously on x. We shall consider the following invariant measure for $\hat{F}$. On the base N, ν equals λ =volume. On the fiber through x, $\nu_x = \gamma(x)_*^{-1}\lambda_x$ is the conditional measure which is absolutely continuous w.r.t. Lebesgue, i.e.

$$\int \phi(x, \theta) d\nu(x, \theta) = \int \int \phi(x, \gamma(x)^{-1}(t)) d\lambda_x(t) d\lambda(x).$$

ν is $\hat{F}$ invariant and projects to Lebesgue= λ (exercise), hence, since the central exponents vanishes, we can apply Theorem 4.4.3 and get that $x \to \nu_x$ varies continuously.

We have that $\lambda_x = \gamma(x)_* \nu_x$ for λ a.e. x and hence $x \to \lambda_x$ varies continuously, also, Theorem 4.4.3 gives that λ_x is s- and $u-$ holonomy invariant. Once we get this, we can follow the same proof of the introduction and get that F is topologically conjugated to $(x, \theta) \to (f(x), \theta + c(x))$.

In the case the foliation is not by compact leaves, also there are some cases that can be understood. In the same paper [**11**] the authors deal with perturbations of the geodesic flow on a negatively curved surface. Also, the work of A. Gogolev [**52**] for Anosov diffeomorphisms of three manifolds and its extension by F. Micena, [**91**] to partially hyperbolic diffeomorphisms in dimension 3 homotopic to Anosov diffeomorphisms prove the rigidity of the absolute continuity of the center foliation. Let A be an Anosov linear map on $\mathbb{T}^3$ and consider $f : \mathbb{T}^3 \to \mathbb{T}^3$ a volume preserving partially hyperbolic diffeomorphism homotopic to A with the accessibility property.

THEOREM 4.5.2. [**52, 91**] *Assume that the central Lyapunov exponent $\lambda^c > 0$ then there is a λ^u such that for every invariant measure μ the unstable Lyapunov exponent $\lambda^u(\mu) = \lambda^u$. In particular all periodic orbits have the same Lyapunov exponents.*

4.5.2. Dissipative systems. As we already mentioned, the volume preserving condition in the previous subsection is necessary. M. Viana and J. Yang [**123**] constructed an example of a partially hyperbolic diffeomorphism robustly presenting an absolutely continuous central foliation.

THEOREM 4.5.3. [**123**] *Let $g : M \to M$ be a transitive Anosov diffeomorphism. And let $F_0 : M \times S^1 \to M \times S^1$, $F_0(x,\theta) = (f(x), g_x(\theta))$ be a diffeomorphism such that for some fixed point p of f, g_p is north-south pole map. Then there is an open set $\mathcal{U}$ such that F_0 is in the closure of $\mathcal{U}$ and every F in $\mathcal{U}$ has absolutely continuous center-stable, center-unstable and central foliation.*

The idea is to perturb F to get an open set of diffeomorphisms wich are simultaneously mostly contracting for F and for F^{-1}. Here we say that a diffeomorphism is *mostly contracting* if for Lebesgue a.e. point x,

$$\limsup_{n \to \infty} \frac{1}{n} \log |D_x F^n | E^c| < -\lambda < 0.$$

Of course if F is Lebesgue preserving, it is impossible to be mostly contracting for both F and F^{-1} but since for dissipative systems, SRB measures for F and for F^{-1} are singular, this is not at all impossible to get such diffeomorphisms. Indeed under very mild assumptions, one gets that for a mostly contracting diffeomorphism there is only one SRB measure and this is a robust property. So, once there is an example the example is robust.

What happens when one get a diffeomorphism which is mostly contracting for F and for F^{-1}?. As we mentioned, under mild conditions there is an SRB measure μ^+ for F and an SRB measure μ^- for F^{-1} . This SRB measure has negative central exponents hence, Pesin theory gives that the Pesin stable manifolds of F form an absolutely continuous foliation and the Pesin stable manifolds of F^{-1} form an absolutely continuous foliation. Now, for μ^+ a.e. point and for Lebesgue a.e. point x, the Pesin stable manifold for F coincide with an open set of the center-stable manifold through x. Similarly, for μ^- a.e. point and for Lebesgue a.e. point x, the Pesin unstable manifold for F (which equals Pesin stable manifold for F^{-1}) coincide with an open set of the center-unstable manifold through x. The next step is to go from this a.e. absolute continuity to full absolute continuity, so, one needs to bypass some possible holes. In the case of the center-stable foliation, this can be done for instance if the unstable foliation is minimal, though much milder condition will suffices.

CHAPTER 5

Partial hyperbolicity in dimension 3

5.1. Introduction

The purpose of this chapter is to present the state of the art in the study of the ergodicity of conservative partially hyperbolic diffeomorphisms on three dimensional manifolds. In fact, we shall mainly describe the results contained in [**68, 70**]. The study of partial hyperbolicity has been one of the most active topics on dynamics over the last years and we do not pretend to describe all the related results, even for 3-manifolds.

A diffeomorphism $f : M \to M$ of a closed smooth manifold M is partially hyperbolic if TM splits into three invariant bundles such that one of them is contracting, the other is expanding, and the third, called the center bundle, has an intermediate behavior, that is, not as contracting as the first, nor as expanding as the second (see Subsection 5.2.3 for a precise definition). The first and second bundles are called strong bundles.

A central point in dynamics is to find conditions that guarantee ergodicity (see also Chapter 2 in this book) In 1994, the pioneer work of Grayson, Pugh and Shub [**53**] suggested that partial hyperbolicity could be "essentially" a sufficient condition for ergodicity. Indeed, soon afterwards, Pugh and Shub conjectured that stable ergodicity (open sets of ergodic diffeomorphisms) is dense among partially hyperbolic systems. They proposed as an important tool the accessibility property (see also the previous work by Brin and Pesin [**26**]): f is accessible if any two points of M can be joined by a curve that is a finite union of arcs tangent to the strong bundles. Essential accessibility is the weaker property that any two measurable sets of positive measure can be joined by such a curve. In fact, accessibility will play a key role in this chapter.

Pugh and Shub split their Conjecture into two sub-conjectures: (1) essential accessibility implies ergodicity, (2) the set of partially hyperbolic diffeomorphisms contains an open and dense set of accessible diffeomorphisms.

Many advances have been made since then in the ergodic theory of partially hyperbolic diffeomorphisms. In particular, there is a result by Burns and Wilkinson [**33**] (Theorem 2.8.3) proving that essential accessibility plus a bunching condition (trivially satisfied if the center bundle is one dimensional) implies ergodicity. There is also a result by the authors [**67**] obtaining the complete Pugh-Shub conjecture for one-dimensional center bundle (Theorem 2.8.1) See [**66**] for a survey on the subject.

We have therefore that almost all partially hyperbolic diffeomorphisms with one dimensional bundle are ergodic. This means that the *non-ergodic* partially hyperbolic systems are very few. Can we describe them? Concretely,

QUESTION 5.1.1. *Which manifolds support a non-ergodic partially hyperbolic diffeomorphism? How do they look like?*

In this chapter we give a description of what is known about this question for three dimensional manifolds. We study the sets of points that can be joined by paths everywhere tangent to the strong bundles (accessibility classes), and arrive, using tools of geometry of laminations and topology of 3-manifolds, to the somewhat surprising conclusion that there are strong obstructions to the non-ergodicity of a partially hyperbolic diffeomorphism. See Theorems 5.1.4, 5.1.6 and 5.1.7.

This gave us enough evidence to conjecture the following:

CONJECTURE 5.1.2 ([**68**]). *The only orientable manifolds supporting non-ergodic partially hyperbolic diffeomorphisms in dimension 3 are the mapping tori of diffeomorphisms of surfaces which commute with Anosov diffeomorphisms.*

Specifically, they are (1) the mapping tori of Anosov diffeomorphisms of $\mathbb{T}^2$*, (2)* $\mathbb{T}^3$*, and (3) the mapping torus of* $-id$ *where* $id : \mathbb{T}^2 \to \mathbb{T}^2$ *is the identity map on the 2-torus.*

Indeed, we believe that for 3-manifolds, all partially hyperbolic diffeomorphisms are ergodic, unless the manifold is one of the listed above.

In the case that $M = \mathbb{T}^3$ we can be more specific and we also conjecture that:

CONJECTURE 5.1.3. *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a conservative partially hyperbolic diffeomorphism homotopic to a hyperbolic automorphism. Then, f is ergodic.*

In [**68**] we proved Conjecture 5.1.2 when the fundamental group of the manifold is nilpotent:

THEOREM 5.1.4. *All the conservative C^2 partially hyperbolic diffeomorphisms of a compact orientable 3-manifold with nilpotent fundamental group are ergodic, unless the manifold is $\mathbb{T}^3$.*

A paradigmatic example is the following. Let M be the mapping torus of $A_k : \mathbb{T}^2 \to \mathbb{T}^2$, where A_k is the automorphism given by the matrix $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, k a non-zero integer. That is, M is the quotient of $\mathbb{T}^2 \times [0, 1]$ by the relation $\sim$, where $(x, 1) \sim (A_k x, 0)$. The manifold M has nilpotent fundamental group; in fact, it is a nilmanifold. Theorem 5.1.4 then implies that all conservative partially hyperbolic diffeomorphisms of M are ergodic.

In fact, the above case, namely the case of nilmanifolds, is the only one where Theorem 5.1.4 is non-void (see [**68**]). Moreover, the other cases of Theorem 5.1.4 are ruled out by a remarkable result by Burago and Ivanov [**27**]:

THEOREM 5.1.5 ([**27**]). *There are no partially hyperbolic diffeomorphisms in $\mathbb{S}^3$ or $\mathbb{S}^2 \times \mathbb{S}^1$.*

The proofs of most of the theorems of this chapter involve deep results of the geometry of codimension one foliations and the topology of 3-manifolds. In Subsection 5.2.1 we shall include, for completeness, the basic facts and definitions that we shall be using in this work. However, the interested reader is strongly encouraged to consult [**38**], [**39**], [**61**] and [**62**] for a well organized and complete introduction to the subject.

Let us explain a little bit our strategy. In the first place, it follows from the results in [**33, 67**] that accessibility implies ergodicity.

So, our strategy will be to prove that all partially hyperbolic diffeomorphisms of compact 3-manifolds except the ones of the manifolds listed in Conjecture 5.1.2 satisfy the (essential) accessibility property.

In dimension 3, and in fact, whenever the center bundle is 1-dimensional, the non-open accessibility classes are codimension one immersed manifolds (see [**67**] and Theorem 2.3.1, Chapter 2 in this book); the union of all non-open accessibility classes is a compact set *laminated* by the accessibility classes (see Subsection 5.2.1 for definitions). So, either f has the accessibility property or else there is a non-trivial lamination formed by non-open accessibility classes.

Let us first assume that the lamination is not a foliation (i.e. does not cover the whole manifold). Then in [**68**] it is showed that it either extends to a true foliation without compact leaves, or else it contains a leaf that is a periodic 2-torus with Anosov dynamics. In the first case, we have that the boundary leaves of the lamination contain a dense set of periodic points, see [**67**] and Theorem 2.5.10, Chapter 2 in this book. Moreover, the fundamental group of any boundary leaf injects in the fundamental group of the manifold. In the second case, let us call any embedded 2-torus admitting an Anosov dynamics extendable to the whole manifold, an *Anosov torus*. That is, $T \subset M$ is an Anosov torus if there exists a homeomorphism $h : M \to M$ such that $h|_T$ is homotopic to an Anosov diffeomorphism. In [**70**] we obtained that the manifold must be again one of the manifolds of Conjecture 5.1.2 if it has an Anosov torus.

THEOREM 5.1.6. *A closed oriented irreducible 3-manifold admits an Anosov torus if and only if it is one of the following:*

(1) *the 3-torus*
(2) *the mapping torus of $-id$*
(3) *the mapping torus of a hyperbolic automorphism*

Let us recall that a 3-manifold is *irreducible* if any embedded 2-sphere bounds a ball. After the proof of the Poincaré conjecture this is the same of having trivial second fundamental group. Three dimensional manifolds supporting a partially hyperbolic diffeomorphism are always irreducible thanks to Burago and Ivanov results in [**27**]. Indeed, the existence of a Reebless foliation implies that the manifold is irreducible or it is $\mathbb{S}^2 \times \mathbb{S}^1$.

Secondly suppose that there are no open accessibility classes. Then, accessibility classes must foliate the whole manifold. Let us see that this foliation can not have compact leaves. Observe that any such compact leaf must be a 2-torus. So, we have three possibilities: (1) there is an Anosov torus, (2) the set of compact leaves forms a strict non-trivial lamination, (3) the manifold is foliated by 2-tori. The first case has just been ruled out. In the second case, we would have that the boundary leaves contain a dense set of periodic points, as stated above, and hence they would be Anosov tori again, which is impossible. Finally, in the third case, we conclude that the manifold is a fibration of tori over $\mathbb{S}^1$. This can only occur, in our setting, only if the manifold is the mapping torus of a diffeomorphism which commutes with an Anosov diffeomorphism as in Conjecture 5.1.2.

The following theorem is the first step in proving Conjectures 5.1.2 and 5.1.3. See definitions in Subsection 5.2.1:

THEOREM 5.1.7. *Let $f : M \to M$ be a conservative partially hyperbolic diffeomorphism of an orientable 3-manifold M. Suppose that the bundles E^σ are also orientable, $\sigma = s, c, u$, and that f is not accessible . Then one of the following possibilities holds:*

(1) *M is the mapping torus of a diffeomorphism which commutes with an Anosov diffeomorphism as in Conjecture 5.1.2.*
(2) *there is an f-invariant lamination $\emptyset \neq \Gamma(f) \neq M$ tangent to $E^s \oplus E^u$ that trivially extends to a (not necessarily invariant) foliation without compact leaves of M. Moreover, the boundary leaves of $\Gamma(f)$ are periodic, have Anosov dynamics and dense periodic points.*
(3) *there is a minimal invariant foliation tangent to $E^s \oplus E^u$.*

The assumption on the orientability of the bundles and M is not essential, in fact, it can be achieved by a finite covering. The proof of Theorem 5.1.7 appears at the end of Section 5.5.

We do not know of any example satisfying (2) in the theorem above. We have the following question.

QUESTION 5.1.8. *Let $f : N \to N$ be an Anosov diffeomorphism on a complete Riemannian manifold N. Is it true that if $\Omega(f) = N$ then N is compact?*

5.2. Preliminaries

5.2.1. Geometric preliminaries. In this section we state several definitions and concepts that will be useful in the rest of this paper. From now on, M will be a compact connected Riemannian 3-manifold.

A *lamination* is a compact set $\Lambda \subset M$ that can be covered by open charts $U \subset \Lambda$ with a local product structure $\phi : U \to \mathbb{R}^n \times T$, where T is a locally compact subset of $\mathbb{R}^k$. On the overlaps $U_\alpha \cap U_\beta$, the transition functions $\phi_\beta \circ \phi_\alpha^{-1} : \mathbb{R}^n \times T \to \mathbb{R}^n \times T$ are homeomorphisms and take the form:

$$\phi_\beta \circ \phi_\alpha^{-1}(u, v) = (l_{\alpha\beta}(u, v), t_{\alpha\beta}(v)),$$

where $l_{\alpha\beta}$ are C^1 with respect to the u variable. No differentiability is required in the transverse direction T. The sets $\phi^{-1}(\mathbb{R}^n \times \{t\})$ are called *plaques*. Each point x of a lamination belongs to a maximal connected injectively immersed n-submanifold, called the *leaf* of x in L. The leaves are union of plaques. Observe that the leaves are C^1, but vary only continuously. The number n is the *dimension* of the lamination. If $n = \dim M - 1$, we say Λ is a *codimension-one* lamination. The set L is an *f-invariant lamination* if it is a lamination such that f takes leaves into leaves.

We call a lamination a *foliation* if $\Lambda = M$. In this case, we shall denote by $\mathcal{F}$ the set of leaves. In principle, we shall not assume any transverse differentiability. However, in case $l_{\alpha\beta}$ is C^r with respect to the v variable, we shall say that the foliation is C^r. Note that even purely C^0 codimension-one foliations admit a transverse 1-dimensional foliation (see Siebenmann [**116**],). In our case the existence of this 1-dimensional foliation is trivial thanks to the existence of the 1-dimensional center bundle E^c. These allows us to translate many local deformation arguments, usually given in the C^2 category, into the C^0 category as observed, for instance, by Solodov [**119**]. In particular, Theorems 5.2.1 and 5.2.3, which were originally formulated for C^2 foliations hold in the C^0 case. We shall say that a codimension-one foliation $\mathcal{F}$, is *transversely orientable* if the transverse 1-dimensional foliation mentioned above is orientable. An *invariant foliation* is a foliation that is an invariant lamination.

Let Λ be a codimension-one lamination that is not a foliation. A *complementary region* V is a component of $M \setminus \Lambda$. A *closed complementary region* $\hat{V}$ is the metric completion of a complementary region V with the path metric induced by the Riemannian metric, the distance between two points being the infimum of the lengths of paths in V connecting them. A closed complementary region is independent of the metric. Note that they are not necessarily compact. If Λ does not have compact leaves, then every closed complementary region decomposes into a compact *gut* piece and non-compact *interstitial regions* which are I-bundles over non-compact surfaces, and get thinner and thinner as they go away from the gut (see [**62**] or [**50**]). The interstitial regions meet the gut along annuli. The decomposition into interstitial regions and guts is unique up to isotopy. Moreover, one can take the interstitial regions as thin as one wishes.

A *boundary leaf* is a leaf corresponding to a component of ∂V, for V a closed complementary region. That is, a leaf is a non-boundary leaf if it is not contained in a closed complementary region.

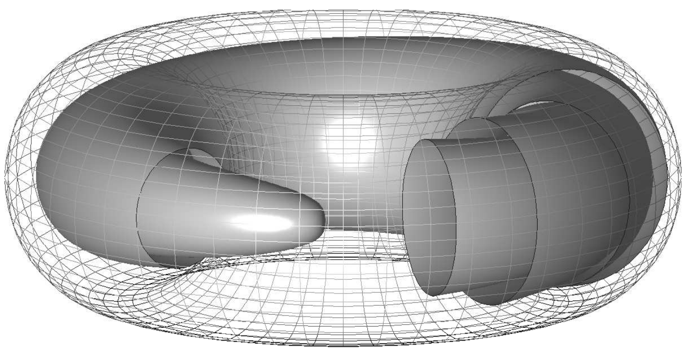

FIGURE 1. A Reeb component

The geometry of codimension-one foliations is deeply related to the topology of the manifold that supports them. The following subset of a foliation is important in their description. A *Reeb component* is a solid torus whose interior is foliated by planes transverse to the of core of the solid torus, such that each leaf limits on the boundary torus, which is also a leaf (see Figure 1). A foliation that has no Reeb components is called *Reebless*.

The following theorems show better the above mentioned relation:

THEOREM 5.2.1 (Novikov). *Let M be a compact orientable 3-manifold and $\mathcal{F}$ a transversely orientable codimension-one foliation. Then each of the following implies that $\mathcal{F}$ has a Reeb component:*

(1) *There is a closed, nullhomotopic transversal to $\mathcal{F}$*
(2) *There is a leaf L in $\mathcal{F}$ such that $\pi_1(L)$ does not inject in $\pi_1(M)$*

The statement of this theorem can be found, for instance, in [**39**, Theorems 9.1.3 & 9.1.4., p.288]. We shall also use the following theorem

THEOREM 5.2.2 (Haefliger). *Let Λ be a codimension one lamination in M. Then the set of points belonging to compact leaves is compact.*

This theorem was originally formulated for foliations [**55**]. However, it also holds for laminations, see for instance [**62**].

We have the following consequence of Novikov's Theorem about Reebless foliations. This theorem is stated in [**108**] as Corollary 2 on page 44.

THEOREM 5.2.3. *If M is a compact 3-manifold and $\mathcal{F}$ is a transversely orientable codimension-one Reebless foliation, then either $\mathcal{F}$ is the product foliation of $\mathbb{S}^2 \times \mathbb{S}^1$, or $\tilde{\mathcal{F}}$, the foliation induced by $\mathcal{F}$ on the universal cover $\tilde{M}$ of M, is a foliation by planes $\mathbb{R}^2$. In particular, if $M \neq \mathbb{S}^2 \times \mathbb{S}^1$ then M is irreducible.*

This theorem was originally stated for C^2 foliations, but it also holds for C^0 foliations, due to Siebenmann's theorem mentioned above.

5.2.2. Topologic preliminaries. Let M be a 3-dimensional manifold. A manifold M is *irreducible* if every 2-sphere $\mathbb{S}^2$ embedded in the manifold bounds a 3-ball. Recall that a 2-torus T embedded in M is an *Anosov torus* if there exists a diffeomorphism $f : M \to M$ such that $f(T) = T$ and the action induced by f on $\pi_1(T)$, that is,

$f_{\#}|_T : \pi_1(T) \to \pi_1(T)$, is a hyperbolic automorphism. Equivalently, f restricted to T is isotopic to a hyperbolic automorphism.

We shall assume from now on, that M is an irreducible 3-manifold since this is the case for 3-manifolds supporting partially hyperbolic diffeomorphisms. In this subsection, we will focus on what is called the JSJ-decomposition of M (see below). That is, we will cut M along certain kind of tori, called incompressible, and will obtain certain 3-manifolds with boundary that are easier to handle, which are, respectively, Seifert manifolds, and atoroidal and acylindrical manifolds. Let us introduce these definitions first.

An orientable surface S embedded in M is *incompressible* if the homomorphism induced by the inclusion map $i_{\#} : \pi_1(S) \hookrightarrow \pi_1(M)$ is injective; or, equivalently, if there is no embedded disc $D^2 \subset M$ such that $D \cap S = \partial D$ and $\partial D \nsim 0$ in S (see, for instance, [**61**, Page 10]). We also require that $S \neq \mathbb{S}^2$.

A manifold with or without boundary is *Seifert* , if it admits a one dimensional foliation by closed curves, called a Seifert fibration. The boundary of an orientable Seifert manifold with boundary consists of finite union of tori. There are many examples of Seifert manifolds, for instance $\mathbb{S}^3$, T_1S where S is a surface, etc.

The other type of manifold obtained in the JSJ-decomposition is atoroidal and acylindrical manifolds. A 3-manifold with boundary N is *atoroidal* if every incompressible torus is *∂-parallel*, that is, isotopic to a subsurface of ∂N. A 3-manifold with boundary N is *acylindrical* if every incompressible annulus A that is *properly embedded*, i.e. $\partial A \subset \partial N$, is ∂-parallel, by an isotopy fixing ∂A.

As we mentioned before, a closed irreducible 3-manifold admits a natural decomposition into Seifert pieces on one side, and atoroidal and acylindrical components on the other:

THEOREM 5.2.4 (JSJ-decomposition [**77**], [**78**]). *If M is an irreducible closed orientable 3-manifold, then there exists a collection of disjoint incompressible tori $\mathcal{T}$ such that each component of $M \setminus \mathcal{T}$ is either Seifert , or atoroidal and acylindrical. Any minimal such collection is unique up to isotopy. This means that if $\mathcal{T}$ is a collection as described above, it contains a minimal sub-collection $m(\mathcal{T})$ satisfying the same claim. All collections $m(\mathcal{T})$ are isotopic.*

Any minimal family of incompressible tori as described above is called a *JSJ-decomposition* of M. When it is clear from the context we shall also call JSJ-decomposition the set of pieces obtained by cutting the manifold along these tori. Note that if M is either atoroidal or Seifert, then, $\mathcal{T} = \emptyset$.

5.2.3. Dynamic preliminaries. Throughout this paper we shall work with a *partially hyperbolic diffeomorphism* f, that is, a diffeomorphism admitting a non-trivial Tf-invariant splitting of the tangent bundle $TM = E^s \oplus E^c \oplus E^u$, such that all unit vectors $v^\sigma \in E^\sigma_x$ ($\sigma = s, c, u$) with $x \in M$ verify:

$$\|T_x f v^s\| < \|T_x f v^c\| < \|T_x f v^u\|$$

for some suitable Riemannian metric. f also must satisfy that $\|Tf|_{E^s}\| < 1$ and $\|Tf^{-1}|_{E^u}\| < 1$. We shall say that a partially hyperbolic diffeomorphism f that satisfies

$$\|T_x f v^s\| < \|T_y f v^c\| < \|T_z f v^u\| \ \forall x, y, z \in M$$

is *absolutely partially hyperbolic.*

We shall also assume that f is *conservative*, i.e. it preserves Lebesgue measure associated to a smooth volume form.

It is a known fact that there are foliations $\mathcal{W}^\sigma$ tangent to the distributions E^σ for $\sigma = s, u$ (see for instance [**26**]). The leaf of $\mathcal{W}^\sigma$ containing x will be called $W^\sigma(x)$, for $\sigma = s, u$. The connected component of x in the intersection of $W^s(x)$ with a small ε-ball centered at x is the *ε-local stable manifold of* x, and is denoted by $W^s_\varepsilon(x)$.

In general it is not true that there is a foliation tangent to E^c. It is false even in case $\dim E^c = 1$ (see [**69**]). However, in Proposition 3.4 of [**21**] it is shown that if $\dim E^c = 1$, then f is *weakly dynamically coherent*. This means that for each $x \in M$ there are complete immersed C^1 manifolds which contain x and are everywhere tangent to E^c, E^{cs} and E^{cu}, respectively. We will call a *center curve* any curve which is everywhere tangent to E^c. Moreover, we will use the following fact:

PROPOSITION 5.2.5 ([**21**]). *If γ is a center curve through x, then*

$$W^s_\varepsilon(\gamma) = \bigcup_{y\in\gamma} W^s_\varepsilon(y) \qquad \textit{and} \qquad W^u_\varepsilon(\gamma) = \bigcup_{y\in\gamma} W^u_\varepsilon(y)$$

are C^1 immersed manifolds everywhere tangent to $E^s \oplus E^c$ and $E^c \oplus E^u$ respectively.

We shall say that a set X is *s-saturated* or *u-saturated* if it is a union of leaves of the strong foliations $\mathcal{W}^s$ or $\mathcal{W}^u$ respectively. We also say that X is su-saturated if it is both s- and u-saturated. The *accessibility class* $AC(x)$ of the point $x \in M$ is the minimal su-saturated set containing x. Note that the accessibility classes form a partition of M. If there is some $x \in M$ whose accessibility class is M, then the diffeomorphism f is said to have the *accessibility property*. This is equivalent to say that any two points of M can be joined by a path which is piecewise tangent to E^s or to E^u. A diffeomorphism is said to be *essentially accessible* if any su-saturated set has full or null measure.

The theorem below relates accessibility with ergodicity. In fact it is proven in a more general setting, but we shall use the following formulation:

THEOREM 5.2.6 ([**33**],[**67**]). *If f is a C^2 conservative partially hyperbolic diffeomorphism with the (essential) accessibility property and* $\dim E^c = 1$*, then f is ergodic.*

In [**68**] it is proved that there are manifolds whose topology implies the accessibility property holds for all partially hyperbolic diffeomorphisms. In these manifolds, all partially hyperbolic diffeomorphisms are ergodic.

Sometimes we will focus on the openness of the accessibility classes. Note that the accessibility classes form a partition of M. If all of them are open then, in fact, f has the accessibility property. We will call $U(f) = \{x \in M; AC(x) \text{ is open}\}$ and $\Gamma(f) = M \setminus U(f)$. Note that f has the accessibility property if and only if $\Gamma(f) = \emptyset$. We have the following property of non-open accessibility classes:

PROPOSITION 5.2.7 ([**67**]). *The set $\Gamma(f)$ is a codimension-one lamination, having the accessibility classes as leaves.*

In fact, any compact su-saturated subset of $\Gamma(f)$ is a lamination.

The above proposition is Proposition A.3. of [**67**]. The fact that the leaves of $\Gamma(f)$ are C^1 may be found in [**44**]. See also Chapter 2 in this book. The following proposition is Proposition A.5 of [**67**]:

PROPOSITION 5.2.8 ([**67**]). *If Λ is an invariant sub-lamination of $\Gamma(f)$, then each boundary leaf of Λ is periodic and the periodic points are dense in it (with the induced topology).*

Moreover, the stable and unstable manifolds of each periodic point are dense in each plaque of a boundary leaf of Λ.

Observe that the proof of Proposition A.5 of [**67**] shows in fact that periodic points are dense in the accessibility classes of the boundary leaves of V endowed with its intrinsic topology. In other words, periodic points are dense in each plaque of the boundary leaves of V.

We shall also use the following theorem by Brin, Burago and Ivanov, whose proof is in [**21**], after Proposition 2.1.

THEOREM 5.2.9 ([**21**]). *If $f : M^3 \to M^3$ is a partially hyperbolic diffeomorphism, and there is an open set V foliated by center-unstable leaves, then there cannot be a closed center-unstable leaf bounding a solid torus in V.*

5.3. Anosov tori

In this section we will say a few words about the proof of Theorem 5.1.6. The idea in its proof is that, given an Anosov torus T, we can "place" T so that either T belongs to the family $\mathcal{T}$ given by the JSJ-decomposition (Theorem 5.2.4), or else T is in a Seifert component, and it is either transverse to all fibers, or it is union of fibers of this Seifert component. See Proposition 5.3.3.

It is important to note the following property of Anosov tori:

THEOREM 5.3.1 ([**68**]). *Anosov tori are incompressible.*

An Anosov torus in an atoroidal component will then be ∂-parallel to a component of its boundary. In this case, we can assume $T \in \mathcal{T}$. On the other hand, the Theorem of Waldhausen below, guarantees that we can always place an incompressible torus in a Seifert manifold in a "standard" form; namely, the following: a surface is *horizontal* in a Seifert manifold if it is transverse to all fibers, and *vertical* if it is union of fibers:

THEOREM 5.3.2 (Waldhausen [**124**]). *Let M be a compact connected Seifert manifold, with or without boundary. Then any incompressible surface can be isotoped to be horizontal or vertical.*

The architecture of the proof of Theorem 5.1.6 is contained in the following proposition.

PROPOSITION 5.3.3. *Let T be an Anosov torus of a closed irreducible orientable manifold M. Then, there exists a diffeomorphism $f : M \to M$ and a JSJ-decomposition $\mathcal{T}$ such that*

(1) *$f|T$ is a hyperbolic toral automorphism,*
(2) *$f(\mathcal{T}) = \mathcal{T}$, and*
(3) *one of the following holds*
   (a) *$T \in \mathcal{T}$*
   (b) *T is a vertical torus in a Seifert component of $M \setminus \mathcal{T}$, and T is not ∂-parallel in this component.*
   (c) *M is a Seifert manifold ($\mathcal{T} = \emptyset$), and T is a horizontal torus,*

The proposition above allows us to split the proof of Theorem 5.1.6 into cases. Note that case (3b) includes the case in which M is a Seifert manifold and T is a vertical torus.

In the case that T is a vertical torus in a Seifert component we can cut this component along T. Then we can suppose that T is in the boundary. We take profit of the fact that in most manifolds the Seifert fibration is unique up to isotopy. Since the dynamics restricted to T is Anosov we have that the manifold has more than one Seifert fibration. This lead us to show that this Seifert component must be $\mathbb{T}^2 \times [0, 1]$. This gives that the whole manifold must be one of the manifolds of Theorem 5.1.6.

If T is horizontal torus then the manifold M is Seifert and T intersects all the fibers. This is discarded in a case by case study thanks to the fact that the Seifert manifolds having horizontal torus a finite.

The last and more difficult case is when T is part of the JSJ-decomposition but it is not the boundary of a Seifert component. The proof in this case is complicated but a very rough idea is to take a properly embedded surface S with an essential circle of T in its boundary. Taking a large iterate $f^n(S)$ and considering $S \cap f^n(S)$, it is possible to construct a non-parallel incompressible cylinder as a union of a band in S and a band in $f^n(S)$. This leads to contradiction because the component is not Seifert and then, it is acylindrical.

5.4. The su-lamination $\Gamma(f)$

Let f be a partially hyperbolic diffeomorphism of a compact 3-manifold M. From Subsection 5.2.3 it follows that we have three possibilities: (1) f has the accessibility property, (2) the union of all non-open accessibility classes is a strict lamination, $\emptyset \subsetneq \Gamma(f) \subsetneq M$ or (3) the union of all non-open accessibility classes foliates M: $\Gamma(f) = M$.

Now, we shall distinguish two possible cases in situations (2) and (3):

(a) the lamination $\Gamma(f)$ does not contain compact leaves
(b) the lamination $\Gamma(f)$ contains compact leaves

In this section we deal with the case (2a). In fact, for our purposes it will be sufficient to assume that there exists an f-invariant sub-lamination Λ of $\Gamma(f)$ without compact leaves. Section 5.5 treats the cases (2b) and (3b). Section 5.6 treats the case (3a).

In this section, we will prove that the complement of Λ consists of I-bundles. To this end, we shall assume that the bundles E^σ ($\sigma = s, c, u$) and the manifold M are orientable (we can achieve this by considering a finite covering).

Theorem 5.4.1 ([**68**], Theorem 4.1). *If $\emptyset \subsetneq \Lambda \subset \Gamma(f)$ is an orientable and transversely orientable f-invariant sub-lamination without compact leaves such that $\Lambda \neq M$, then all closed complementary regions of Λ are I-bundles.*

Theorem 5.4.1 was proved by showing:

Proposition 5.4.2. *Let $\Lambda \subset \Gamma(f)$ be a nonempty f-invariant sub-lamination without compact leaves. Then E^c is uniquely integrable in the closed complementary regions of Λ.*

The proof of this proposition is rather technical. The interested reader may found a proof in [**68**].

Let us consider $\hat{V}$ a closed complementary region of Λ, and call $\mathcal{I}(V)$ the union of all interstitial regions of V and $\mathcal{G}(V)$ the gut of $\hat{V}$ (see Subsection 5.2.1), so that

$$\hat{V} = \mathcal{I}(V) \cup \mathcal{G}(V).$$

The following statement is rather standard:

LEMMA 5.4.3. *Let $f : M \to M$ be a partially hyperbolic diffeomorphism. If U is an open invariant set such that $U \subset \Omega(f)$, then the closure of U is su-saturated.*

Let us observe that if $\hat{V}$ is connected then there are only two boundary leaves of $\hat{V}$. Indeed, as we mentioned before periodic points are dense in boundary leaves. This fact jointly with the local product structure imply, using standard arguments, that the stable and unstable leaves of periodic points are dense too. Take a periodic point p in a boundary leaf and in the intersticial region. There are center curves joining the points in the local stable manifold of p with other boundary curve L_1 of $\hat{V}$ (the same property holds for the local unstable manifold). Invariance of the stable manifold of p and boundary leaves give that the center curve of any point of the stable manifold joins the boundary leaf L_0 containing p with L_1. Denseness of the stable and unstable manifolds of p implies that the complement of the set of points such that their center manifold join L_0 with L_1 is totally disconnected. Then, it is not difficult to see that L_0 and L_1 are the unique boundary leaves of $\hat{V}$.

Also, since periodic points are dense in the boundary leaves due to Proposition 5.2.8, there is an iterate of f that fixes all connected components of $\hat{V}$, so we will assume when proving Theorem 5.4.1 that $\hat{V}$ is connected and has two boundary leaves L_0 and L_1.

PROOF OF THEOREM 5.4.1. We will present a sketch of a different approach to a proof than the one given in [**68**]. The strategy will be to show that all center leaves in $\hat{V}$ meet both L_0 and L_1. Let p be a periodic point in $L_0 \cap \mathcal{I}(V)$. As we mentioned before its center leaf meets L_1, and the same happens for all points in its stable and unstable manifolds. Now stable and unstable manifolds of a periodic point are dense in each plaque of L_0 (Proposition 5.2.8). So the set of points in L_0 whose center leaf does not reach L_1 is contained in a totally disconnected set.

Let us suppose that x_0 is a point in L_0 whose center leaf does not reach L_1. Then, since center curves of points of the intersticial region clearly reach the boundary, $W^c(x_0)$ is contained in $\mathcal{G}(V)$. Take a small rectangle R in L_0 around x_0 formed by arcs of stable and unstable manifolds of a periodic point. Moreover, we can assume that the center curves of the points of R_0 reach L_1. Of course, the image is another rectangle R_1 formed by stable and unstable arcs. Then, the center arcs of the points of R_0 and the interiors of R_0 and R_1 form a 2-sphere S. Since Rosenberg's theorem [**107**] remains valid in this setting and $\hat{V}$ is foliated by $\mathcal{W}^{cs}$ that is Reebless and transverse to the boundary, we have that $\hat{V}$ is irreducible. Then, S bounds a ball B. Now, since $W^c(x_0)$ does not reach L_1 and is contained in B, it accumulates in B but Novikov's Theorem implies the existence of a Reeb component, a contradiction. □

Theorem 5.4.1 implies that any non trivial invariant sub-lamination $\Lambda \subset \Gamma(f)$ without compact leaves can be extended to a foliation of M without compact leaves. Indeed, any complementary region V is an I-bundle, and hence it is diffeomorphic to the product of a boundary leaf times the open interval: $L_0 \times (0,1)$. The foliation $F_t = L_0 \times \{t\}$ induces a foliation of V.

This has the following consequence in case the fundamental group of M is nilpotent:

Proposition 5.4.4. *If M is a compact 3-manifold with nilpotent fundamental group, and $\emptyset \subsetneq \Lambda \subsetneq M$, is an invariant sub-lamination of $\Gamma(f)$, then there exists a leaf of Λ that is a periodic 2-torus with Anosov dynamics.*

Proof. If Λ has a compact leaf, let us consider the set Λ_c of all compact leaves of Λ. Λ_c is in fact an invariant sub-lamination, due to Theorem 5.2.2. Hence Proposition 5.2.8 implies that the boundary leaves of Λ_c are periodic 2-tori with Anosov dynamics, and we obtain the claim.

If, on the contrary, Λ does not have compact leaves, then due to Theorem 5.4.1 above, we can extend Λ to a foliation $\mathcal{F}$ of M without compact leaves. In particular, $\mathcal{F}$ is a Reebless foliation. Item (2) of Theorem 5.2.1 implies that for all boundary leaves L of Λ, $\pi_1(L)$ injects in $\pi_1(M)$, and is therefore nilpotent.

Now, this implies that the boundary leaves can only be planes or cylinders. Theorem 5.2.8 implies that stable and unstable leaves of periodic points are dense in those leaves, which is impossible for the case of the plane or the cylinder. Therefore, Λ must contain a compact leaf, and due to what was shown above, it must contain a periodic 2-torus with Anosov dynamics. □

In fact, Theorem 5.1.6 implies that periodic 2-tori with Anosov dynamics are not possible in 3-manifolds with nilpotent fundamental group, unless the manifold is $\mathbb{T}^3$. Hence the hypotheses of Proposition 5.4.4 are not fulfilled, unless the manifold is $\mathbb{T}^3$. This will eliminate case (2) mentioned at the beginning of this section.

5.5. A trichotomy for non-accessible diffeomorphisms

In this section we will prove Theorem 5.1.7. This theorem and the results in this section are valid for any 3-manifold M, and do not require that its fundamental group be nilpotent. Moreover, Theorem 5.3.1 does not even require the existence of a partially hyperbolic diffeomorphism.

Let T be an embedded 2-torus in M. We shall call T an *Anosov torus* if there exists a homeomorphism $g : M \to M$ such that T is g-invariant, and $g|_T$ is homotopic to an Anosov diffeomorphism.

Also, let S be a two-sided embedded closed surface of M^3 other than the sphere. S is *incompressible* if and only if the homomorphism induced by the inclusion map $i_\# : \pi_1(S) \hookrightarrow \pi_1(M)$ is injective; or, equivalently, after the Loop Theorem, if there is no embedded disc $D^2 \subset M$ such that $D \cap S = \partial D$ and $\partial D \nsim 0$ in S (see, for instance, [**61**]).

Recall that Theorem 5.3.1 says that Anosov tori are incompressible. We insist that this theorem is general, and does not depend on the existence of a partially hyperbolic dynamics in the manifold.

We also need the following fact about codimension one laminations.

THEOREM 5.5.1. *Let $\mathcal{F}$ be a codimension one C^0-foliation without compact leaves of a three dimensional compact manifold M. Then, $\mathcal{F}$ has a finite number of minimal sets.*

We are now in position to prove Theorem 5.1.7 of Page 107:

PROOF OF THEOREM 5.1.7. If $\Gamma(f) = M$ then there are no Reeb components. Indeed, since f is conservative, if there were a Reeb component, then its boundary torus should be periodic. We get a contradiction from Theorem 5.3.1. This gives case (3) except the minimality.

Let us assume that $\Gamma(f) \neq M$. If $\Gamma(f)$ contains a compact leaf then the set of compact leaves is a sub-lamination Λ of $\Gamma(f)$ by Theorem 5.2.2. Proposition 5.2.8 implies that the boundary leaves of Λ are Anosov tori, and we obtain case (1) as a consequence of Theorem 5.1.6.

If $\Gamma(f) \neq M$ and contains no compact leaves, then Theorem 5.4.1 and Proposition 5.2.8 give us case (2).

Finally we show minimality in case (3). On the one hand, if $\Gamma(f) = M$ and has a compact leaf we have two possibilities: either all leaves are compact or not. If not then, the previous argument implies the existence of an Anosov torus and we are in case (1). If all leaves are compact, as we mentioned before, the manifold is a torus bundle and the hyperbolic dynamics on fibers implies that we are again in case (1). On the other hand, if $\Gamma(f)$ has no compact leaves and has a minimal sub-lamination $\mathcal{L}$, we have that $\mathcal{L}$ is periodic (recall that minimal sub-laminations of a codimension one foliation are finite, Theorem 5.5.1). Then, we are again in case (2). $\square$

5.6. Nilmanifolds

This section deals with the proof of Theorem 5.1.4. Let $f : M \to M$ be a conservative partially hyperbolic diffeomorphism of a compact orientable three dimensional nilmanifold $M \neq \mathbb{T}^3$. As consequence of Proposition 5.4.4 and Theorems 5.1.6 and 5.1.7 we have that $E^s \oplus E^u$ integrates to a minimal foliation $\mathcal{F}^{su}$ if f does not have the accessibility property. Indeed the only possibilities in the trichotomy of Theorem 5.1.7 are (2) and (3) and Proposition 5.4.4 says that there is an Anosov torus if we are in case (2). But this last case is impossible for a nilmanifold $M \neq \mathbb{T}^3$. In this section we shall give some arguments showing that the existence of a minimal foliation tangent to $E^s \oplus E^u$ leads us to a contradiction. In [**68**] the reader can find a different proof of the same fact. Without loss of generality we may assume, by taking a double covering if necessary, that $\mathcal{F}^{su}$ is transversely orientable. Observe that the double covering of a nilmanifold is again a nilmanifold.

The first step is that Parwani [**96**] proved (following Burago-Ivanov [**27**] arguments) that the action induced by f in the first homology group of M is hyperbolic. By duality the same is true for the first cohomology group.

The second step is given by Plante results in [**99**] (see also [**62**]). Since $\mathcal{F}^{su}$ is a minimal foliation of a manifold whose fundamental group has non-exponential growth there exists a transverse holonomy invariant measure μ of full support. This measure is unique up to multiplication by a constant and represents an element of the first cohomology group of M. The action of f leaves $\mathcal{F}^{su}$ invariant and induces a new transverse measure ν, an image of former one. The uniqueness implies that $\nu = \lambda\mu$ for some $\lambda > 0$. Since of the action of f on $H^1(M)$ is hyperbolic, then $\lambda \neq 1$. Suppose that $\lambda > 1$ (if the contrary is true take f^{-1}).

The third step is to observe that $\lambda > 1$ implies that f is expanding the μ measure of center curves. Since μ has full support and the $su-$bundle is hyperbolic we would obtain that f is conjugated to Anosov leading to contradiction with the fact that $M \neq \mathbb{T}^3$.

5.7. Homotopic to Anosov on $\mathbb{T}^3$

In this section we present the results announced by Hammerlindl and Ures on Conjecture 5.1.3, that the nonexistence of nonergodic partially hyperbolic diffeomorphisms homotopic to Anosov in dimension 3. They are able to prove the following result.

THEOREM 5.7.1 ([**59**]). *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a $C^{1+\alpha}$ conservative partially hyperbolic diffeomorphism homotopic to a hyperbolic automorphism A. Suppose that f is not ergodic. Then,*

1. *$E^s \times E^u$ integrates to a minimal foliation.*
2. *f is topologically conjugated to A and the conjugacy sends strong leaves of f into the corresponding strong leaves of A.*
3. *The center Lyapunov exponent is* 0 *a.e.*

We remark that it is not known if there exists a diffeomorphism satisfying the conditions of the theorem above.

Now, in order to prove Conjecture 5.1.3 we have two possibilities: either we prove that a diffeomorphism satisfying the conditions of Theorem 5.7.1 is ergodic or we prove that such a diffeomorphism cannot exist. Hammerlindl and Ures announced that if f is C^2 and the center stable and center unstable leaves of a periodic point are C^2 then, f is ergodic.

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