

A SURVEY OF PARTIALLY HYPERBOLIC DYNAMICS.

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ABSTRACT. Some of the guiding problems in partially hyperbolic systems are the following: (1) Examples, (2) Properties of invariant foliations, (3) Accessibility, (4) Ergodicity, (5) Lyapunov exponents, (6) Integrability of central foliations, (7) Transitivity and (8) Classification. Here we will survey the state of the art on these subjects, and propose related problems.

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1. INTRODUCTION

Here we will survey the state of the art in the area of partially hyperbolic dynamics, that is, diffeomorphisms that display some hyperbolic behavior ruling an intermediate one.

Hyperbolic behavior has proved to be a powerful tool to get different types of chaotic properties from the ergodic and topological viewpoints. As early as the late 60's or early 70's the need of relaxing the full hyperbolicity hypothesis appeared. Indeed, C. Pugh and M. Shub [PuSh1] in their study of the ergodicity of Anosov actions and with M. Hirsch [HiPuSh1], [HiPuSh2] in their study of invariant manifolds proposed the notion of normal hyperbolicity, the intermediate part being played by a foliation transversal to the hyperbolic part; M. Brin and Y. Pesin [BrPe1], [BrPe2] studying ergodicity of skew products and frame flows, proposed the notion of partial hyperbolicity where the intermediate part is assumed to be tangent to a bundle. Both approaches are obviously quite related and they essentially contain the same known examples. Here we will follow this partially hyperbolic approach, that is we will be assuming that $f : M \rightarrow M$ leaves a splitting $TM = E^s \oplus E^c \oplus E^u$ invariant, where vectors in E^s are exponentially contracted in the future and vectors in E^u are exponentially contracted in the past. The dynamics in the intermediate bundle E^c has its point-wise spectrum between the ones of E^s and E^u .

In the last decade the area became quite active, here we will focus on the following problems (1) Examples, (2) Properties of invariant foliations, (3) Accessibility, (4) Ergodicity, (5) Lyapunov exponents, (6) Integrability of central foliations, (7) Transitivity and (8) Classification. We think that the theory will still be growing, and that the formulation of problems, the basic and even the simple ones, should be one of the main tasks.

Finally, we were not be able to cover all branches of study and some of them were treated only laterally. Examples of these are Pesin theory, the search of SRB measures or Gibbs states, the dominated splitting approach proposed by Mañé, partially hyperbolic actions by more general groups, partially hyperbolic maps that are not diffeomorphisms, etc. We encourage the reader also to read the works [BuPuShWi], [PuSh4], [Pe2], [BoDíVi] and [HaPe].

This work will be organized as follows. In section 2, we enumerate the known examples of partially hyperbolic diffeomorphisms. It is worth noting that this theory grows mainly from examples. We propose different viewpoints to treat them. First, in section 3, we present the invariant foliations viewpoint. There we explore different kinds of regularities to be found (or not) in the invariant foliations. In section 4, we focus on accessibility, that is, on the ability to access from a point x to another one y , by moving only along lines piecewise tangent to E^s and E^u , the bundles composing the ruling hyperbolic part. Some results on accessibility will be commented, and we shall compute accessibility classes on three-dimensional nil-manifolds, and the general affine case. Accessibility classes of stably ergodic toral automorphisms are analyzed, and a short sketch of the proof that ergodic toral automorphisms are stably ergodic is presented.

There is evidence that accessibility is abundant, and indeed there is a known conjecture of C. Pugh and M. Shub that accessibility would be open and dense; and also, that accessibility implies ergodicity among partially hyperbolic diffeomorphisms. In sections 4 and 5, we talk about advances there have been in that sense, under certain restrictions on the center bundle E^c .

In section 6, the Lyapunov exponents viewpoint, and other kind of volume growth rates is analyzed for partially hyperbolic systems.

A completely different scope is the study of the integrability of the center distribution. This area seems quite open, and problems are posed. This is reviewed in section 7. Section 8 is devoted to robust transitivity, a point of view related to ergodicity. Finally, in section 9, the classification problem is reviewed.

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2. EXAMPLES

Before going into the examples, let us define some terms more precisely. A diffeomorphism $f : M \rightarrow M$ is *partially hyperbolic* if there is an invariant splitting of the tangent bundle $TM = E^s \oplus E^c \oplus E^u$ such that for all $x \in M$ and all unitary vectors $v^\sigma \in E^\sigma$, with $\sigma = s, c, u$, the following inequalities apply:

$$|D_x f v^s| < |D_x f v^c| < |D_x f v^u|$$

It is also required that $|Df|_{E^s} < 1$ and $|Df|_{E^u} > 1$.

As it will be mentioned below, there are two invariant foliations \mathcal{F}^s and \mathcal{F}^u , the *stable* and *unstable foliations*, that are tangent, respectively, to E^s and E^u . These are the only foliations with these features. But, in general, there is no invariant foliation tangent to E^c ; and, in case there were, it is not known if it must be unique.

In studying partially hyperbolic systems, one of the problems is that it is not clear if the amount of existing examples is small, or if it essentially includes all the examples. Thus we get two parallel problems: the search of examples and the classification problem. We would like to split the examples into two categories in nature, a grosser or topological one and another finer or geometric one; or even a measure theoretic one.

For the topological type we would be interested in knowing in which manifolds and in which homotopy classes the partially hyperbolic dynamics can occur. For example, we say that two partially hyperbolic systems $f : M \rightarrow M$ and $g : N \rightarrow N$, both having a central foliation \mathcal{F} are *centrally conjugated* or conjugated modulo the central direction [HiPuSh2] if there is a homeomorphism $h : M \rightarrow N$ such that

i) $h(\mathcal{F}_f(x)) = \mathcal{F}_g(h(x))$

ii) $h(f(\mathcal{F}_f(x))) = g(h(\mathcal{F}_f(x)))$ or, which is equivalent, $\mathcal{F}_g(h(f(x))) = \mathcal{F}_g(g(h(x)))$

It would be useful to classify partially hyperbolic systems modulo central conjugacy. It would be interesting also to have an analogous concept when the central distribution is not integrable.

Bellow we give a list of some of the existing examples. We hope that we had put there most of them.

The second type of examples typically live within the first type and will be appearing along the survey.

2.1. Anosov Diffeomorphisms. [An1] A diffeomorphism $f : M \rightarrow M$ is an *Anosov diffeomorphism* if its derivative Df leaves the splitting $TM = E^s \oplus E^u$ invariant, where Df contracts vectors in E^s exponentially fast, and Df expands vectors in E^u exponentially fast.

Anosov systems are the hallmark of hyperbolic and chaotic behaviors. Nevertheless, they are far from being completely understood. For example, the following problem is still open.

PROBLEM 1. [Sm] *Is every Anosov diffeomorphism conjugated to an infra-nil-manifold automorphism?*

When the manifold underlying the dynamics is a nil-manifold or if the unstable foliation has codimension one the answer is yes, [Fr], [Man], [Ne1]. For expanding maps (when every vector is expanded by the derivative) the answer also is yes, they are always conjugated to infra-nil-manifold endomorphisms, [Sh1], [Gro]. It would be interesting to get analogous results for partially hyperbolic diffeomorphisms, or at least to have an answer to the following:

PROBLEM 2. [BrBuIv], *Section 9. Let f be a partially hyperbolic diffeomorphism on \mathbb{T}^3 . Is it true that its action in homology is partially hyperbolic?*

From the ergodic point of view, Anosov diffeomorphisms are very much better understood.

Theorem 2.1. [An2] *Volume preserving Anosov diffeomorphisms are ergodic.*

The partially hyperbolic systems share lots of their properties with the Anosov systems. Let us describe some of those properties. There are two invariant foliations \mathcal{F}^s and \mathcal{F}^u tangent to E^s and E^u . Both foliations have smooth leaves (as smooth as the diffeomorphism), but the foliations themselves are not smooth a priori. In fact, although there are some interesting cases where the invariant foliations are smooth, the general case is that they are rarely smooth [An4], [HaWi]. Thus, it became an interesting problem to study the transversal regularity of these foliations. For

example, it turned out that the holonomies of these foliations are *absolutely continuous* [An2], [An3], [AnSi], [Si] i.e. we say that a map $h : \Sigma_1 \rightarrow \Sigma_2$ is absolutely continuous if it sends zero measure sets into zero measure sets, see subsection 3.2 for more details. The importance of absolute continuity of the holonomies is that it implies that Fubini's theorem is true for these foliations, that is, a measurable set A has zero measure if and only if for a.e. point x in M the intersection of A with the leaf through x has zero leaf-wise measure. It is worth mentioning that in smooth ergodic theory, when dealing with any kind of hyperbolicity, the smooth regularity of the system is typically required to be at least $C^{1+\text{H\"older}}$. In fact, in the C^1 category the following is still unknown:

PROBLEM 3. *Are there examples of non ergodic volume preserving Anosov diffeomorphisms?*

2.2. Geodesic Flows. Let V be an n -dimensional manifold and let g be a metric on V . On TV it is defined the *geodesic flow* as follows. Given a point $x \in V$ and a vector $v \in T_x V$ there is a unique geodesic γ with $\gamma(0) = x$ and $\dot{\gamma}(0) = v$. For $t \in \mathbb{R}$ we define $\phi_t(x, v) = (\gamma(t), \dot{\gamma}(t))$. It follows that $|\dot{\gamma}(t)| = |v|$ for every $t \in \mathbb{R}$ or more precisely, $g_{\gamma(t)}(\dot{\gamma}(t)) = g_x(v)$ for every $t \in \mathbb{R}$. Thus the geodesic flow preserves the vectors of a given magnitude. Let $M = T_1 V$ be the bundle of unit vectors tangent to V and let us restrict the geodesic flow to M . It turns out that if the sectional curvature is negative then the geodesic flow is in fact an Anosov flow [An1]. Indeed, for every unit vector v in M , $T_v M$ may be identified with the orthogonal Jacobi fields. Thus, if we call E^s the set of orthogonal Jacobi fields that are bounded for the future and E^u the set of orthogonal Jacobi fields that are bounded for the past, then negative sectional curvature implies that $TM = E^s \oplus E^0 \oplus E^u$ and the vectors in E^s are exponentially contracted in the future, E^0 is the one dimensional space spanned by the vector-field defining the geodesic flow and the vectors in E^u are exponentially contracted in the past.

The geodesic flow preserves a natural measure defined on M , the Liouville measure *Liou*. Let us first define a one-form η over TV as follows: if $\omega \in TV$ and $\chi \in T_\omega TV$ then we define $\eta_\omega(\chi)$ as being $\omega \cdot d_\omega p(\chi)$ where $x = p(\omega)$, $p : TV \rightarrow V$ is the canonical projection. It turns out that $d\eta$ is a symplectic 2-form on TV and that the geodesic flow preserves this symplectic form. Thus, $L = d\eta \wedge \dots \wedge d\eta$ (n -times) is a $2n$ -form. The restriction of L to M is the $(2n - 1)$ -form defining *Liou*.

It was for the geodesic flows on surfaces of negative curvature that E. Hopf [Ho] developed the machinery now called the Hopf argument to prove ergodicity w.r.t. *Liou* and the antecedent for D. Anosov work. For general manifolds of negative sectional curvature D. Anosov proved that the geodesic flow is ergodic w.r.t. *Liou*. In fact, he proved more generally that C^2 volume preserving Anosov systems are ergodic, thus, since being an Anosov flow is an open condition, they form an open set of ergodic flows [An1], [An2], [An3], [AnSi].

The time-one map of the geodesic flow on negative curvature, i.e. ϕ_1 , is naturally a partially hyperbolic diffeomorphism. It was not until 1992 that M. Grayson, C. Pugh and M. Shub, [GrPuSh] proved that the time-one map of the geodesic flow on a surface of constant negative curvature is a *stably ergodic* diffeomorphism, that is, as

in the Anosov case, their perturbations remain ergodic. Later, A. Wilkinson proved the same result but for variable curvature [Wi].

There is also a the related topological question about robust transitivity for partially hyperbolic systems that remains widely open. In [BoDí] it is proven that close to the time-one map of the geodesic flow on a negatively curved surface there are whole open sets of transitive diffeomorphisms. But the following is still open:

PROBLEM 4. *Is the time-one map of the geodesic flow on a negatively curved surface robustly transitive?*

As it will be seen in section 8 it is enough to prove that, for perturbations, the non-wandering set is still the whole manifold.

2.3. Anosov Flows. We say that a flow ϕ_t on a manifold M is an *Anosov flow* if admits an invariant splitting $TM = E^s \oplus E^0 \oplus E^u$, where, as usual, vectors in E^s and E^u are respectively contracted and expanded, and E^0 is the space spanned by the vector-field. One of the main difference between Anosov flows and Anosov diffeomorphisms is that there are known examples of Anosov flows where the non-wandering set is not the whole manifold, [FrWi]. In fact, this changes completely the hope of finding a complete classification of Anosov flows like that stated in Problem 1. On the other hand, when dealing with transitive Anosov flows, there is a dichotomy, either they are mixing or else the bundle $E^s \oplus E^u$ is jointly integrable. In fact, in [Pl], it is proven that either the strong unstable manifold is minimal or $E^s \oplus E^u$ is integrable. In this second case J. Plante also proved that the flow is conjugated to a suspension but possibly changing the time. In fact it is still an open problem to know if the su -foliation is by compact leafs, and this is closely related to the following long-standing problem

PROBLEM 5. *Is the action in homology of an Anosov diffeomorphism hyperbolic?*

As already mentioned, volume preserving Anosov flows are ergodic. Moreover, the following is proven in [BuPuWi]:

Theorem 2.2. *Let ϕ_1 be the time-one map of a volume preserving Anosov flow ϕ . If ϕ is mixing then it is stably ergodic.*

Of course, the time-one map of the suspension of an Anosov diffeomorphism by a constant roof function is not stably ergodic.

2.4. Frame Flows. [Br1], [BrGr], [BrKa], [BrPe1], [BuPo]. The *frame flow* on a riemannian manifold (V, g) fibers over its geodesic flow. Let \hat{M} be the space of positively oriented orthonormal n -frames in TV . Thus \hat{M} naturally fibers over $M = T_1V$, where the projection takes a frame to its first vector. The associated structure group $SO(n-1)$ acts on fibres by rotating the frames keeping the first vector fixed. In particular, we can identify each fiber with $SO(n-1)$. Let $\hat{\phi}_t : \hat{M} \rightarrow \hat{M}$ denote the frame flow, which acts on frames by moving their first vectors according to the geodesic flow and moving the other vectors by parallel transport along the geodesic defined by the first vector. The projection is a semi-conjugacy from $\hat{\phi}_t$ to ϕ_t . In particular, $\hat{\phi}_t$ is an $SO(n-1)$ -group extension of ϕ_t . The frame flow preserves the

measure $\mu = \text{Liou} \times \nu_{SO(n-1)}$, where $\nu_{SO(n-1)}$ is the (normalized) Haar measure on $SO(n-1)$. It turns out that the time- t map of the frame flow is a partially hyperbolic diffeomorphism [BrPe2]. The neutral direction has dimension $1 + \dim SO(n-1)$ and is spanned by the flow direction and the fibre direction.

The frame flow on manifolds of negative sectional curvature is known to be ergodic in lots of cases. The study of the ergodicity of the frame flow restricts to the study of its accessibility classes (see section 4 for the notion of accessibility) and is a very interesting example to begin with, in order to learn how to manage them. Finally the frame flow is stably ergodic in the cases it is known to be ergodic. But it is not always ergodic, Kähler manifolds with negative curvature and real dimension at least 4 have non-ergodic frame flows because the complex structure is invariant under parallel translation. We suggest the reader to see [BuPo] for a good account of the existing results, problems and conjectures.

2.5. Affine diffeomorphisms. Let G be a Lie group and $B \subset G$ a subgroup. Given a one parameter subgroup of G it defines an *homogeneous flow* on G/B . Examples of homogeneous flows are geodesic flows of hyperbolic surfaces. There are lots of interplays between the dynamics of homogeneous flows and the algebraic properties of the groups involving it, see for example [St] for an account.

The time- t map of an homogeneous flow is a particular case of an *affine diffeomorphism*. In fact affine diffeomorphisms and homogeneous flows are typically treated in a similar way. Let G be a connected Lie group, $A : G \rightarrow G$ an automorphism, B a closed subgroup of G with $A(B) = B$, and $g \in G$. Then we define the affine diffeomorphism $f : G/B \rightarrow G/B$ as $f(xB) = gA(x)B$. We shall assume that G/B supports a finite left G -invariant measure and call, in this case, G/B a finite volume homogeneous space. If G/B is compact and B is discrete the existence of such a measure is immediate, but if B is not discrete the assumption is nontrivial.

The affine diffeomorphism f is covered by the diffeomorphism $\bar{f} = L_g \circ A : G \rightarrow G$; where $L_g : G \rightarrow G$ the left multiplication by g . If \mathfrak{g} is the Lie algebra of G , we may identify $T_e G = \mathfrak{g}$ where e is the identity map. Let us fix a right invariant metric on G , i.e. R_g is an isometry for every g where R_g is right multiplication by g . Let us define the naturally associated automorphism $\mathfrak{a}(f) : \mathfrak{g} \rightarrow \mathfrak{g}$ by $\mathfrak{a}(f) = Ad(g) \circ D_e A$ where $Ad(g)$ is the adjoint automorphism of g , that is the derivative at e of $x \rightarrow gxg^{-1}$. In other words, $\mathfrak{a}(f)$ is essentially the derivative of \bar{f} , but after right multiplication by g^{-1} (which is an isometry) in order to send $T_g G$ to $T_e G$. So we have the splitting $\mathfrak{g} = \mathfrak{g}^s \oplus \mathfrak{g}^c \oplus \mathfrak{g}^u$ w.r.t the eigenvalues of $\mathfrak{a}(f)$ being of modulus less than one, one, or bigger than one respectively and similarly, \mathfrak{g}^s is formed by the vectors going exponentially to 0 in the future, \mathfrak{g}^u is formed by the vectors going exponentially to 0 in the past and \mathfrak{g}^c is formed by the vectors that grow at most polynomially for the future and the past. Observe that if v_λ and v_σ are eigenvectors for $\mathfrak{a}(f)$ w.r.t. λ and σ respectively then we have that

$$\mathfrak{a}(f)([v_\lambda, v_\sigma]) = [\mathfrak{a}(f)(v_\lambda), \mathfrak{a}(f)(v_\sigma)] = \lambda\sigma[v_\lambda, v_\sigma]$$

and hence if $[v_\lambda, v_\sigma] \neq 0$ then it is an eigenvector for $\lambda\sigma$. As a consequence we get that $\mathfrak{g}^s, \mathfrak{g}^u, \mathfrak{g}^c, \mathfrak{g}^{cs} = \mathfrak{g}^c \oplus \mathfrak{g}^s$ and $\mathfrak{g}^{cu} = \mathfrak{g}^c \oplus \mathfrak{g}^u$ are subalgebras tangent to connected

subgroups G^s, G^u, G^c, G^{cs} and G^{cu} of G and their translates will define the stable, unstable, center, center-stable and center-unstable foliations respectively.

Let \mathfrak{h} denote the smallest Lie subalgebra of \mathfrak{g} containing \mathfrak{g}^s and \mathfrak{g}^u . Using Jacobi identity it is not hard to see that it is an ideal, \mathfrak{h} , called the *hyperbolic subalgebra* of \bar{f} . Moreover, let us denote $H \subset G$ the connected subgroup tangent to \mathfrak{h} and call it the hyperbolic subgroup of \bar{f} . As \mathfrak{h} is an ideal in \mathfrak{g} , H is a normal subgroup of G . Finally let us denote with $\mathfrak{b} \subset \mathfrak{g}$ the Lie algebra of $B \subset G$. Then we have the following:

Theorem 2.3. [PuSh4] *Let $f : G/B \rightarrow G/B$ be an affine diffeomorphism as above, then f is partially hyperbolic if and only if $\mathfrak{h} \not\subset \mathfrak{b}$. Moreover, if f is partially hyperbolic then the left action of G^σ , $\sigma = s, u, c, cs, cu$ on G/B foliates G/B into the stable, unstable, center, center-stable and center-unstable foliations respectively.*

PROBLEM 6. *Is there an example of a non-Anosov affine diffeomorphism that is robustly transitive? Are they exactly the same as the stably ergodic ones?*

2.6. Linear Automorphisms on Tori. A special case of affine diffeomorphisms are the affine automorphisms on tori. In fact, the torus \mathbb{T}^N may be seen as the quotient $\mathbb{R}^N/\mathbb{Z}^N$. Integer entry $N \times N$ matrices with determinant ± 1 define what we shall call linear automorphisms of tori simply via matrix multiplication. Thus, given such a matrix A and a vector $v \in \mathbb{R}^N$, it is defined an affine diffeomorphism of the torus f by $f(x) = Ax + v$. It is quite easy to see that, conjugating by a translation, it is enough to study the case where v belongs to the eigenspace corresponding to the eigenvalue 1, E_1 . Observe also that E_1 is a rational space, that is, it has a basis formed by vectors of rational coordinates.

The corresponding splitting of the tangent bundle here, is the splitting given by the eigenspaces of A . Thus, a not quite involved argument proves that f is partially hyperbolic unless all the eigenvalues of A are roots of unity. Moreover, using a little bit of harmonic analysis (Fourier series) it is seen [Hal] that f is ergodic if and only if A has no eigenvalues that are roots of the identity other than one itself and v has irrational slope inside E_1 . Finally, notice that if E_1 is not trivial, we may always perturb in order to make v of rational slope, thus in order to get that perturbations remain ergodic it is necessary that also 1 be not in the spectrum of A . Thus we reach to the following problem:

PROBLEM 7. [HiPuSh2], [FRH1], *Subsection 4.4.* *Are the ergodic linear automorphisms stably ergodic?*

Of course, an analogous problem may be posed in the topological category, that is, are their perturbations also transitive? [HiPuSh2].

2.7. Direct Products. Given a partially hyperbolic diffeomorphism $f : M \rightarrow M$ and $g : N \rightarrow N$ a diffeomorphism, the product $f \times g : M \times N \rightarrow M \times N$ is partially hyperbolic if the dynamics of g is less expanding and contracting, respectively, than the expansions and contractions of f . This is essentially the most trivial way a partially hyperbolic dynamics appears, Anosov \times identity. Besides, we can also make the product of two partially hyperbolic diffeomorphisms.

It is quite interesting that by making perturbations of this product dynamics, lots of nontrivial examples arise. For instance, the first example of a robustly transitive non-Anosov diffeomorphism constructed by M. Shub [Sh2], see section 8, although not a product, is a large perturbation of a product. In fact direct products as well as the construction of M. Shub are part of a more general type of construction, the partially hyperbolic systems that fiber over other partially hyperbolic systems.

2.8. Fiberings over partially hyperbolic diffeomorphisms. Let $f : B \rightarrow B$ be a partially hyperbolic diffeomorphism with splitting $TM = E_f^s \oplus E_f^c \oplus E_f^u$. Let $p : N \rightarrow B$ be a fibration with fiber F , let us call $F(x)$ the fiber through x . Then any lift $g : N \rightarrow N$ of f is a partially hyperbolic diffeomorphism if

$$|D_{p(x)}f|E_f^s| < m(D_xg|T_xF(x)) \leq |D_xg|T_xF(x)| < m(D_{p(x)}f|E_f^s).$$

where $m(A) = |A^{-1}|^{-1}$. As we said, M. Shub's example of a robustly transitive diffeomorphism is of this kind, and, in fact, many of the existing examples are of this kind. It would be interesting to find the minimal pieces over which partially hyperbolic systems are built. For example:

PROBLEM 8. *Find the partially hyperbolic diffeomorphisms f such that no partially hyperbolic diffeomorphism g homotopic to f^n , $n > 0$, fibers over a lower dimensional partially hyperbolic diffeomorphism. The geodesic flow on negative curvature as well as the ergodic automorphisms of tori defined in [FRH1] are examples of that building blocks. Find other types of gluing technics to generate new partially hyperbolic systems.*

2.9. Skew products. Another type of systems that fiber over lower dimensional partially hyperbolic diffeomorphisms are the *skew products*: Let $f : M \rightarrow M$ be a partially hyperbolic diffeomorphism, G a Lie group and $\theta : M \rightarrow G$ a function. Define the skew product $f_\theta : M \times G \rightarrow M \times G$ by $f_\theta(x, g) = (f(x), \theta(x)g)$. Skew products were extensively studied in the context of partially hyperbolic diffeomorphisms, see for example [AdKiSh], [Br1], [Br2], [BrPe2], [BuWi1], [FiPa].

3. PROPERTIES OF THE INVARIANT FOLIATIONS

By a foliation we shall mean a topological foliation. In the dynamical framework it is useful to treat the regularity of the leaves of a foliation and the regularity of their holonomies separately. In fact, typically the foliations we deal with have smooth leaves but the holonomies are only Hölder continuous.

Given a continuous plane field $E \subset TM$, we say that E is *integrable* if there is a foliation \mathcal{F} tangent to E , i.e. each leaf of the foliation is a C^1 manifold everywhere tangent to E .

Given a partially hyperbolic diffeomorphism $f : M \rightarrow M$, with splitting $TM = E^s \oplus E^c \oplus E^u$, we are mainly interested in studying if the plane fields are integrable, and in case they are, how good their leaf and holonomy regularity are.

The study of the regularity of the invariant bundles and the invariant foliations is in the core of the theory. There is a vast bibliography on the subject and we will not be able to mention every reference here. Nevertheless, the book of M. Hirsch, C.

Pugh and M. Shub [HiPuSh2] is still one of the more inspiring ones in the literature. So that this subject is quite well developed although it is still very active and there continue to appear new types of regularity to deal with. In fact one of the themes here is what we will mean by regularity.

3.1. Existence and regularity of the strong foliations. First we shall deal with the case of the strong bundles, that is, the stable and unstable bundles. Let us first go into the integrability problem:

Theorem 3.1. [BrPe2], [HiPuSh2] *Given a partially hyperbolic diffeomorphism the stable and unstable bundles are always integrable to foliations \mathcal{F}^s and \mathcal{F}^u whose leaves are as differentiable as the diffeomorphism.*

From this, we have that the integrability of the strong bundles is quite well solved and understood.

There are mainly three different ways to study the regularity of a foliation: regularity of the tangent plane field, of the foliation charts and of the holonomy maps. See Section 6 of [PuShWi1] for an enlightening discussion about the differences and relationships between these approaches. Frobenius Theorem implies that regularity of the tangent bundle of the foliation is the strongest of the regularities mentioned above and this kind of regularity is what is obtained in many cases like, for instance, in Theorem 3.2 below. However, in the setting of this survey (and in most dynamical systems so far), what is in fact needed is the regularity of the holonomies.

So, let us go into the problem of the regularity of the holonomies. Let us split this problem. We shall treat the regularity of the holonomies from a purely differentiable viewpoint. In the next subsection we shall deal with the absolute continuity problem. Given $r \in \mathbb{Z}$, $r \geq 0$ and $0 < \theta \leq 1$, we say that a map between smooth manifolds is $C^{r+\theta}$ if it is C^r and its r^{th} derivative is Hölder continuous with Hölder exponent θ . Of course, in order to say that a map is $C^{r+\theta}$ we need first that the manifolds where it is defined are at least $C^{r+\theta}$. This last observation, though it is an obvious remark, becomes a real problem when trying to prove regularity of the holonomies.

The regularity of the holonomies is much related to what one could call the spectrum of the differential of the partially hyperbolic diffeomorphism. Instead of going into the definition of the Mather spectrum (the reader may found a good exposition in [HaPe]) we shall go into its point-wise version.

For a linear transformation A between Banach spaces, we define $m(A) = \inf_{|v|=1} |Av|$. Observe that if A is invertible then $m(A) = |A^{-1}|^{-1}$.

The following theorem is essentially proved in [PuShWi1], we thank K. Burns for pointing out this statement.

Theorem 3.2. *Let M be a complete riemannian manifold and let $f : M \rightarrow M$ be a C^k diffeomorphism with an invariant splitting $TM = E_1 \oplus E_2$ satisfying*

$$\sup_x \frac{|D_x f|_{E_1}|}{m(D_x f|_{E_2})} < 1.$$

Let us assume that

$$\sup_x |D_x f|_{E_1}| \frac{|D_x f|_{E_2}|^r}{m(D_x f|_{E_2})} < 1$$

Then there is a foliation tangent to E_1 whose leaves are as smooth as f and whose holonomies are C^l where $l = \min\{k - 1, r\}$. Here r and k are allowed to be any positive real numbers bigger than or equal to 1.

For example, if f is a partially hyperbolic with E^{cs} integrable and

$$(3.1) \quad \sup_x |D_x f|_{E^s} \frac{|D_x f|_{E^c}|^r}{m(D_x f|_{E^c})} < 1$$

then we have that the stable foliation is C^l smooth when restricted to each center-stable leaf, where l depends on the differentiability of f , the differentiability of the leaves of the center-stable foliation, and r . But typically the leaves of the center-stable foliation are not better than $C^{1+\text{Hölder}}$. Nevertheless, if f is C^2 , the stable foliation is still C^1 when restricted to each center-stable leaf if equation 3.1 holds with $r = 1$, [PuShWi1]. Moreover, it can be seen that if the center dimension is one, then the stable foliation is still C^1 when restricted to any $c + s$ dimensional manifold everywhere tangent to E^{cs} [RHRHU1]. Notice that when the center dimension is one, equation 3.1 with $r = 1$ is trivially satisfied.

PROBLEM 9. *Prove that the stable foliation is still C^1 when restricted to any $(c + s)$ -dimensional manifold everywhere tangent to E^{cs} whenever equation (3.1) is satisfied with $r = 1$.*

Another interesting case is in dimension 3, if f is volume preserving, then equation (3.1) is satisfied with $r = 2$. But the problem is that a priori it is not known if there is a C^2 center-stable manifold and hence theorem 3.2 cannot be straightforwardly applied. It would be interesting to know if the results in [PuShWi1], [PuShWi2] can be adapted to solve the following:

PROBLEM 10. *If $f : M \rightarrow M$ is a volume preserving, partially hyperbolic C^r diffeomorphism, r big enough, with $\dim M = 3$, prove that the stable and unstable holonomies are C^2 when restricted to some suitable 2 dimensional manifolds. At least prove that if $E^s \oplus E^u$ is integrable then it integrates to a C^2 foliation.*

Related to this we want to ask the following:

PROBLEM 11. *What is the relation between bunching and the smoothness of $E^s \oplus E^u$ if any?*

There are results when f is only $C^{1+\text{Hölder}}$, see [BuWi4]. Also, when there is no integrability of the center-stable distribution at all, we still have the differentiability of the stable foliation restricted to some “fake” foliations, if equation (3.1) holds with $r = 1$, see for instance [BuWi3]. The reader may find problems related to this in subsection 7.1.

Another related problem about the holonomies is how they vary as we perturb the dynamics. In [FRH1] it is proven that their variation is quite good in the particular case analyzed there. It would be interesting to have more general results with explicit constants.

There is another form of regularity that is between the smooth category and the absolute continuity category. It is the *quasi-conformality* and it proved to be a very

powerful notion at the time of establishing ergodicity. We shall return to this subject in subsection 5.4.

3.2. Absolute continuity of the strong invariant foliations. Let us enter now into absolute continuity. To begin with, there are several notions of absolute continuity of foliations. It seems that the strong foliations satisfy the strongest one. But let us begin with the weakest definition.

Given \mathcal{F} a foliation let us denote with $W(x)$ the leaf of \mathcal{F} through x . Given a measure μ we call $\mu_{W(x)}$ the conditional measures along $W(x)$. We say that \mathcal{F} is *absolutely continuous with respect to μ* if given a measurable set A we have that $\mu(A) = 0$ if and only if there is a set B such that $\mu(B) = 1$ and such that $\mu_{W(x)}(A \cap W(x)) = 0$ for every $x \in B$. Typically, the absolute continuity problem is analyzed w.r.t. Lebesgue measure, but it would be interesting to know the absolute continuity of the strong foliations with respect to other measures. For example, it seems likely that the unstable foliation is absolutely continuous with respect to any u -Gibbs state.

We say that the foliation \mathcal{F} has *absolutely continuous holonomies* if given $x, y \in W(x)$ and two transversals Σ_x and Σ_y then the holonomy map $h : \Sigma_x \rightarrow \Sigma_y$ sends zero sets into zero sets. Again here typically we analyze the case when the measure on the sections are Lebesgue, but one may ask about other transversal measures. Moreover, if \mathcal{F} has absolutely continuous holonomies, then given an holonomy $h : \Sigma_x \rightarrow \Sigma_y$ one may talk about the Radon-Nikodym derivative of the pull back measure on Σ_y over the measure on Σ_x , and call it the *Jacobian* of h , that is, if we denote with μ_{Σ_x} and μ_{Σ_y} the measures on the transversals, then the jacobian will be the map $Jh : \Sigma_x \rightarrow \mathbb{R}$ satisfying

$$\mu_{\Sigma_y}(A) = \int_A Jh(t) d\mu_{\Sigma_x}(t)$$

for every measurable set $A \subset \Sigma_x$. So the absolute continuity theorem is:

Theorem 3.3. [An1], [BrPe2], [PuSh1] *The stable and unstable foliation for a partially hyperbolic diffeomorphism have absolutely continuous holonomies. Moreover, the holonomies are uniformly Hölder continuous, that is, the exponent and constant may be taken uniformly on the whole manifold if the transversals are taken good enough (uniformly smooth, bounded and reasonable angle with the foliation) and close enough (distance between the points not going to infinity along the leaf).*

3.3. Non-absolute continuity of central foliations. In this section we shall see how absolute continuity fails completely for the central foliation. In fact, there are examples with a full measure set that intersects each leaf only in a finite number of points. This phenomenon is sometimes known as Fubini's nightmare.

In [Mi] the reader may find the first example, built by A. Katok, of such phenomenon of a non-absolutely continuous foliation. The idea is essentially as follows, take a smooth path f_t , $t \in [0, 1]$ of area preserving Anosov diffeomorphisms on \mathbb{T}^2 beginning with a linear one. Then one can define a partially hyperbolic diffeomorphism $F : \mathbb{T}^2 \times [0, 1] \rightarrow \mathbb{T}^2 \times [0, 1]$ by $F(x, t) = (f_t(x), t)$. It follows that F leaves a center foliation invariant, and that the leaves through $(x, 0)$ can be parameterized by

$(h_t(x), t)$ where $h_t : \mathbb{T}^2 \rightarrow \mathbb{T}^2$, $t \in [0, 1]$ is a conjugacy homotopic to identity between f_0 and f_t , i.e. $f_t \circ h_t = h_t \circ f$. Hence we have that h_t is the central holonomy between the transversal $\mathbb{T}^2 \times \{0\}$ and $\mathbb{T}^2 \times \{t\}$. Now, if we define $\mu_t(A) = Leb(h_t^{-1}(A))$, then μ_t is the entropy maximizing measure for f_t . On the other hand it is known that, for an Anosov diffeomorphism of \mathbb{T}^2 , if Leb is the entropy maximizing measure then the eigenvalues at periodic points for the diffeomorphism coincide with the ones of the linear model, moreover, it is smoothly conjugated to the linear model. So, if we take the path f_t in such a way that the eigenvalues at the fixed point are different from the linear one for every t , then μ_t is not Lebesgue measure and hence it must be singular. Hence, for every $t > 0$, there is a set A_t of full Lebesgue measure on \mathbb{T}^2 whose image under the center holonomy $h_t(A_t)$ has zero Lebesgue measure. In fact, if the eigenvalue at the fixed point of any two f_t are different, then there is a full measure set $B \subset \mathbb{T}^2 \times [0, 1]$ that cuts each central leaf at exactly one point. Take B to be the set of points (x, t) such that its time average converges to its space average w.r.t. Leb_t , the Lebesgue measure on $\mathbb{T}^2 \times \{t\}$. Then B has full measure because it has full measure when restricted to each torus $\mathbb{T}^2 \times \{t\}$. Now, given $t \neq s$, we can send Leb_t to $\mathbb{T}^2 \times \{s\}$ by central holonomy (that correspond to conjugating f_t with f_s) and get a measure that should be singular w.r.t. Leb_s (this follows from a theorem of R. de la Llave [deL]). Hence, if $(x, t) \in B$, the corresponding point (y, s) by central holonomy would be typical for the measure singular to Leb_s and hence $(y, s) \notin B$, in other words, B intersects each central leaf at exactly one point.

In [ShWi2], M. Shub and A. Wilkinson found the same phenomenon with a different approach, thus finding an open set where the center foliation is not absolutely continuous. In this case, F is a diffeomorphism of $\mathbb{T}^3 = \mathbb{T}^2 \times \mathbb{S}^1$ close to a skew product over an Anosov diffeomorphism of \mathbb{T}^2 . They built an ergodic volume preserving diffeomorphism with nonzero central exponents, see section 6. Since all central curves are compact it follows from an argument by R. Mañé that the central foliation cannot be absolutely continuous. Indeed the argument is as follows: assume that the central exponent is positive. If the foliation were absolutely continuous then, using Pesin theory, one can find a positive leaf measure set where there is actual central expansion, i.e. there is a set $A \subset W^c$ of positive length measure such that $|D_x F^n|_{E^c}| > \sigma^n$, $\sigma > 1$ for $x \in A$, and every $n \geq n_0$. But then

$$C \geq length(f^n(W^c)) \geq length(f^n(A)) \geq \sigma^n length(A)$$

a contradiction. Here it is also possible to find a full measure set $B \subset \mathbb{T}^3$ that cuts each central curve in a finite set, see [RuWi].

In [HiPe], M. Hirayama and Y. Pesin generalize R. Mañé's argument and prove the following general theorem.

Theorem 3.4. *Let f be a C^2 diffeomorphism of a compact smooth Riemannian manifold M preserving a smooth measure μ . Let also \mathcal{W} be an f -invariant foliation of M with smooth leaves. Assume that \mathcal{W} has finite volume leaves almost everywhere. If f is \mathcal{W} -dissipative then the foliation \mathcal{W} is not absolutely continuous.*

Here, f is said to be \mathcal{W} -dissipative if

$$\int_M \log \text{Jac}(Df|_{\mathcal{W}}) d\mu \neq 0.$$

Another type of example is as follows, take a linear Anosov diffeomorphism A of \mathbb{T}^3 with three one-dimensional invariant bundles, $E^{uu} \oplus E^u \oplus E^{ss}$. Let us call χ^{uu} , χ^u and χ^{ss} the corresponding Lyapunov exponents of A and χ_f^{uu} , χ_f^u and χ_f^{ss} the ones w.r.t. Lebesgue measure of a volume preserving perturbation f . The first observation is that if f is a volume preserving perturbation of A , since the strong foliations are absolutely continuous, it follows that the strong stable exponent cannot decrease, and the strong unstable Lyapunov exponent cannot increase, i.e. $-\chi_f^{ss} \leq -\chi^{ss}$ and $\chi_f^{uu} \leq \chi^{uu}$. On the other hand, if we make the perturbation in such a way that it preserves the $E^{uu} \oplus E^u$ foliation then the strong stable exponent will be preserved and hence the entropy (w.r.t. Lebesgue measure) will not change, notice that the entropy also equals $\chi_f^{uu} + \chi_f^u$. Similarly to the strong case, if the central foliation were absolutely continuous then the central exponent will not increase either, i.e. $\chi_f^u \leq \chi^u$. Then the final stage to get the non absolutely continuous central foliation will be to make the perturbation so that $\chi_f^{uu} < \chi^{uu}$. One way to make this perturbation is along the lines of [ShWi2]. Another way is just to notice that if $\chi^{uu} = \chi_f^{uu}$, then the strong unstable eigenvalues at any periodic point coincides with the linear ones, so brake any of them and we are done.

We found this example in a conversation with A. Katok, A. Gogoleg, a student of A. Katok, and the first author. One week later, in the International Workshop on Global Dynamics Beyond Uniform Hyperbolicity at Chicago, in his talk, R. Saghin exposed his work with Z. Xia about the construction of similar examples within the study of different kinds of dynamical growths (see section 6).

It is generally believed that the failure of absolute continuity of the central foliation is a generic phenomenon.

PROBLEM 12. *Prove that there is a C^1 open, C^∞ dense, set of diffeomorphism whose central foliation fails to be absolutely continuous. Moreover, try to characterize the case where the central foliation is absolutely continuous, at least when the central dimension is one.*

We finish with another problem.

PROBLEM 13. *Analyze regularity of the central foliation, maybe when restricted to the center-stable or center-unstable. For example, if the dynamics on the central direction is an isometry, is there some type of regularity of the the central foliation?. What about the case when there are no periodic points or when all the central Lyapunov exponents are 0.*

Observe that in A. Katok's original example, the dynamics on the central direction is an isometry for some suitable metric. In fact, if we take the sup norm $|v| = \max\{|v_{\mathbb{T}_2}|, |v_{\mathbb{S}^1}|\}$ then it follows that $|D_x F|_{E^c} = 1$. Hence absolute continuity of the central foliation seems to imply some kind of strong rigidity phenomenon, not only on the growth of the central direction but also on the strong directions. It would

be interesting to find out what can be said when the central foliation is absolutely continuous. For example, is there some kind of converse to A. Katok's original example when the diffeomorphism is a perturbation of Anosov times identity? and finally,

PROBLEM 14. *Can we have zero-central exponent, accessibility and non-absolutely continuous central foliation?, what about for perturbations of Anosov times identity on $\mathbb{T}^2 \times \mathbb{S}^1$?*

4. ACCESSIBILITY

Given two uniquely integrable sub-bundles $E, F \subset TM$, an equivalence relation may be defined by saying that y is (E, F) -*accessible* from x if there is a piecewise smooth path, piecewise tangent to either E or F beginning at x and ending at y . Another way to define it is to say that the accessibility class of x is the minimal set containing x , saturated by leaves of the E -foliation and the F -foliation. When $(E, F) = (E^s, E^u)$ corresponding to a partially hyperbolic system, we call it the *su-accessibility relation* and we denote the *su*-accessibility class of x by $C(x)$. The study of the accessibility classes is quite well developed in control theory, but typically, in control theory the bundles are assumed to be smooth. In dynamics, as we already have said, the bundles are rarely smooth, so that much of the work already done in control theory should be redone in this new setting. Some properties follow straightforwardly, but others become quite difficult.

To the best of our knowledge, the use of the accessibility property to prove ergodicity was first used by R. Sacksteder in [Sa]. He essentially proved that accessibility implies ergodicity when the strong foliations are smooth. Later, M. Brin and Y. Pesin in [BrPe2] used it again in the context of frame flows and skew products where they also look at its relation with transitivity, see subsection 5.7. Finally, C. Pugh and M. Shub, beginning with [GrPuSh], used it systematically within a plan to prove ergodicity for partially hyperbolic systems. Indeed, C. Pugh and M. Shub asked the following:

Conjecture 1. *Stable ergodicity is C^r -dense among partially hyperbolic systems, $r \geq 2$.*

Their plan is to split it into two conjectures related to the accessibility.

Conjecture 2. *Essential accessibility implies ergodicity*

Here *essential accessibility* means that any measurable *su*-saturated set has either full or null measure.

Conjecture 3. *Stable accessibility is C^r -dense among partially hyperbolic systems, volume preserving or not, $r \geq 2$.*

When $\dim E^c = 1$ Conjecture 3 is proven in its full strength in [RHRHUr1] for the volume preserving case and in [BuRHRHTaUr] for the non preserving case, see also subsection 4.2 for an account on this. Moreover, in [RHRHUr1], the authors also prove Conjecture 2 when $\dim E^c = 1$, thus proving the Main Conjecture 1 when $\dim E^c = 1$. We removed the dynamical coherence hypothesis of the paper [BuWi2], when $\dim E^c = 1$, essentially using the notion of weak-integrability in [BrBuIv]. At the same time, K. Burns and A. Wilkinson also removed the dynamical coherence hypothesis for any central dimension (under a bunching assumption that is trivially satisfied if $\dim E^c = 1$) but using the idea of fake foliations, [BuWi3], see subsection 5 where Conjecture 2 will be treated.

Conjecture 3 is also known to be true in full generality but only for $r = 1$ by [DoWi]. There are also lot of special cases where Conjecture 3 holds, that is, systems that are known to be stably accessible or that can be approached by stably accessible ones. Historically, the first non Anosov examples having the accessibility property were the ones in R. Sacksteder work, [Sa], for some affine diffeomorphisms. Then M. Brin [Br1], [Br2] in his work on skew products proved that most skew products over Anosov systems have the accessibility property, and also M. Brin with M. Gromov and H. Karcher [BrGr], [BrKa] proved the accessibility property for some frame flows. But by that time none of those works guaranteed that this accessibility was stable. The first case where the accessibility property was guaranteed to be stable was for the geodesic flows on surfaces of constant negative curvature, [GrPuSh], after this comes the variable curvature, [Wi], the contact Anosov flows, [KaKo], and then in [PuSh2] it was proven that if E^s and E^u are C^1 then accessibility implies stable accessibility which implies that lots of examples are stably accessible, for example, all the affine diffeomorphisms having the accessibility property. Then Conjecture 3 was proven in the context of skew products, [BuWi1], then in the context of Anosov flows by [BuPuWi], using a result of [Pl]. In [ShWi1] it was proven that some examples can be approximated by stable accessible ones. Then in [NiTö] it was proven when $\dim E^c = 1$ and some more technical requirements that involve the existence of two nearby closed invariant central leaves and that any two central leaves can be joined by an su -path. P. Didier, in [Di] proved that accessibility implies stable accessibility if $\dim(E^c) = 1$. On the other hand, in [FRH1] it is proven that some ergodic linear automorphisms on tori are stably essentially accessible, but as they are not accessible then they will be not stably accessible.

4.1. The differentiable case. In the case when E^s and E^u are differentiable the behavior of the accessibility classes are well understood. Essentially the accessibility classes form a stratification of the manifold. Indeed the following is true:

Theorem 4.1. [Su1], [Ste], [PuSh2] *If E and F are differentiable then the accessibility classes have a natural smooth manifold structure such that the inclusion is a differentiable map. Moreover the accessibility classes and their dimension vary lower semi-continuously w.r.t. points in the manifold.*

The semi-continuous variation we are talking about above is that if $x_n \rightarrow x$ then the limit of the accessibility classes through x_n contains the accessibility class of x

and the limit of the dimension of the accessibility classes of x_n is larger than the dimension of the accessibility class of x .

PROBLEM 15. *Prove that if $E^s \oplus E^u$ is differentiable then the accessibility classes behave as when E^s and E^u are differentiable.*

PROBLEM 16. *Prove that, in the general case, the accessibility classes are topological manifolds that vary semi-continuously as well as their dimension. Prove that, with bunching, they are indeed smooth manifolds.*

When the central dimension is 1 we give a solution to this problem in [RHRHUr1], see subsection 4.2. When the central direction is two dimensional we think that a proof in the lines of [FRH1] should be possible. For the general case we think that the approach in [FRH1] should be useful, but more machinery would be needed.

More generally, we say that a topological space is *topologically locally homogeneous* if given two points, there is a local homeomorphism from a neighborhood of one onto a neighborhood of the other. There are lots of unsolved problems on this subject, see for example [Mo], [MoZi], [HuWa], but we want to address the following:

PROBLEM 17. *Let X be a subset of \mathbb{R}^n and assume it is topologically locally homogeneous. Assume also that the local homeomorphisms extend to local diffeomorphisms of \mathbb{R}^n and that they are diffeotopic to the identity through local diffeomorphisms preserving X . Prove that X is indeed a differentiable manifold.*

With respect to perturbations we have the following,

Theorem 4.2. [Gra], [PuSh2] *Let E and F be integrable C^1 bundles with the accessibility property. If E' and F' are uniquely integrable C^0 bundles, C^0 close to E and F then (E', F') have the accessibility property.*

Thus, accessibility is stable in this setting. Moreover, having the accessibility property in control theory is a generic property,

Theorem 4.3. [Lo] *The C^r pairs (E, F) , $r \geq 1$, of uniquely integrable plane fields that have the accessibility property form an open and dense set.*

Conjecture 3 looks for the analogous of theorem 4.3 in the setting of partially hyperbolic systems. Observe that the main difficulty in the partially hyperbolic setting is that we perturb f instead of the bundles themselves. Maybe the following is not that hard to prove:

PROBLEM 18. *If $E^s \oplus E^u$ is smooth then f can be approximated by a stably accessible diffeomorphism. Or maybe assuming that E^s and E^u are smooth.*

Related to this problem we can mention the result of M. Shub and A. Wilkinson [ShWi1] that solves problem 18 in two types of cases. In both cases it is required the integrability of $E^s \oplus E^u$ and some kind of global product structure of the center manifolds times the su -manifolds.

4.1.1. *C^1 density of stable accessibility.* As we have already said Conjecture 3 was completely solved in the C^1 category by D. Dolgopyat and A. Wilkinson.

Theorem 4.4. [DoWi] *Stable accessibility is C^1 dense among the partially hyperbolic diffeomorphisms, volume preserving or not.*

Their strategy is the following: Let f be a partially hyperbolic diffeomorphism. First of all, they find a finite collection of small disks, $\{D_j\}$, approximately tangent to the center direction in such a way that f is accessible modulo these disks. More precisely, for $x, y \in M$ there is a finite sequence of su -paths such that the first path begins at x , the last path ends at y and all the other starting and ending points of the sequence of paths belong to a disk in $\{D_j\}$, each path beginning in the disk where the previous path ends. It is not difficult to show, perhaps enlarging the size of the disks a little bit, that accessibility modulo $\{D_j\}$ is an open property. After that they perturb f in a small neighborhood of $\cup D_j$ in such a way that they obtain (stably) that any pair of points of any disk D_j of the collection can be joined by an su -path. These local accessibility together the accessibility modulo $\{D_j\}$ give the accessibility for f . To prove the local accessibility in a disk some control on the effect of the perturbation of f on the strong bundles is needed. They show that the contribution of a C^1 perturbation in a small neighborhood of a point x to $E^s(x)$ and $E^u(x)$ is larger than the contribution of the rest of the orbit. This seems to be false for a C^2 perturbation.

4.2. **The case $\dim E^c = 1$.** In [RHRHUr1] it was proven Conjecture 3 for the volume preserving case when $\dim E^c = 1$. In [BuRHRHTaUr] the volume preserving condition is removed. Let us see how it works.

The idea is first to understand how do the accessibility classes behave. In fact the first lemma is very simple, completely general and it does not need the $\dim E^c = 1$ assumption.

Lemma 4.5. *The following properties are equivalent:*

- i) $C(x)$ is open,
- ii) $C(x)$ has nonempty interior,
- iii) the intersection of $C(x)$ with some c -dimensional manifold transversal to $E^s \oplus E^u$ has nonempty interior.

Then, we define the sets $U(f) = \{x; C(x) \text{ is open}\}$ and $\Gamma(f)$ the complement of $U(f)$. $U(f)$ is clearly open and invariant and hence $\Gamma(f)$ is closed and invariant. By connectedness of M , accessibility means that $U(f) = M$. When a point is in $\Gamma(f)$ we say that $E^s \oplus E^u$ is integrable. This is motivated by the following:

Theorem 4.6. *$\Gamma(f)$ is laminated by the accessibility classes, that is, the partition by accessibility classes restricted to $\Gamma(f)$ form a lamination. Moreover the map $f \rightarrow \Gamma(f)$ varies upper semi-continuously w.r.t. partially hyperbolic f , i.e. if $f_n \rightarrow_{C^1} f$ and f is partially hyperbolic then $\limsup \Gamma(f_n) \subset \Gamma(f)$.*

Observe that the semi-continuity of Γ automatically guarantees that the accessibility property is stable, see also [Di].

So that the accessibility classes are extremely well behaved. We think that an analogous result should be true in quite full generality, for example something like:

PROBLEM 19. Define $\Gamma_k(f)$, $k = u + s, \dots, N$, to be the set of points X where the dimension of $C(x)$ is k and $\Lambda_k(f) = \bigcup_{j=u+s}^k \Gamma_j(f)$. Then, $\Lambda_k(f)$ is closed for every k , Γ_j is laminated, for every j and they fit to build a stratification on Λ_k . Moreover, $f \rightarrow \Lambda_k(f)$ varies semi-continuously.

The next stage is to know how to break the integrability. To this end we prove the following:

Proposition 4.7. Let \mathcal{R} be the set of partially hyperbolic diffeomorphism such that $Per(f) \subset U(f)$, that is the diffeomorphism whose accessibility classes at periodic points are open. Then \mathcal{R} is C^∞ dense.

The proof follows the lines of Kupka-Smale theorem once we know how to make a perturbation at a periodic point to open its accessibility class. To make the perturbation smooth, the idea is to find where to make the perturbation. So we have the following lemma that is quite general and that guarantees that there are lots of points that do not return to themselves. Let us denote with $\mathcal{F}_\varepsilon(x)$ the ε ball of x in the leaf $\mathcal{F}(x)$. Given $x \in M$ let Σ be a transversal to \mathcal{F} at some fixed angle with \mathcal{F} . Let us define $A_\varepsilon(x) = \mathcal{F}_{4\varepsilon}(\Sigma) \setminus \mathcal{F}_\varepsilon(\Sigma)$.

Lemma 4.8 (Keepaway lemma). Let f be a diffeomorphism preserving a foliation \mathcal{F} . Assume that $m(Df|_{\mathcal{F}}) \geq \lambda > 1$ and let $N > 0$ be such that $\lambda^N \geq 4$. Then for every $\varepsilon > 0$ and every $x \in M$, if $f^j(A_\varepsilon(x)) \cap B_\varepsilon(x) = \emptyset$ for $j = 1, \dots, N$, then there is $y \in \mathcal{F}_\varepsilon(x)$ such that $d(f^n(y), x) \geq \varepsilon$ for every $n \geq 0$.

The idea is then to take a periodic point p , and a non-returning point $z \in W^u(p)$ close to p . If p is in $\Gamma(f)$ we can take a three legged su -path beginning at z and ending at p . We can suitably choose it in such a way that the breaking points of the su -path are also non-returning, for the past and/or for the future. Then, essentially any push supported in a small ball around z , transversal to the $E^s \oplus E^u(z)$ direction, will not change the three legged su -path, but will change the unstable leaf at z and hence it will break the integrability at p .

So the idea is to find periodic points to prove the following:

Proposition 4.9. $\mathcal{R} = \mathcal{A} \cup \mathcal{B}$ where \mathcal{A} is the set of diffeomorphism having the accessibility property and \mathcal{B} is the set of diffeomorphism with $E^s \oplus E^u$ integrable, and having no periodic points, i.e. $\Gamma(f) = M$ and $Per(f) = \emptyset$.

The end of the proof of the denseness of the accessibility property is just to notice that the set \mathcal{B} is closed by the semi-continuity of $\Gamma(f)$ and is nowhere dense by the same type of perturbation carried on periodic points, but now on any other point.

To prove proposition 4.9 it is used the following proposition which will be quite useful in the description of the accessibility partition that is carried out in [RHRHUr3], see also next subsection. Given a compact invariant, su -saturated set $K \subset \Gamma(f)$ let us define the central boundary of K , $\partial^c K$ as the set of points $x \in K$ such that for any central curve W , x is in the boundary of an interval of $K^c \cap W$.

Proposition 4.10. *Let us assume that $\Omega(f) = M$. If $K \subset \Gamma(f)$ is a compact invariant, su -saturated set then $\partial^c K$ is su -saturated and the set of periodic points in $\partial^c K$ is dense in $\partial^c K$.*

The proof of this proposition uses heavily that $\Omega(f) = M$. The case $\Omega(f) \neq M$ needs another type of argument, exploiting in a more subtle way the semi-continuity of $f \rightarrow \Gamma(f)$, see [BuRHRHTaUr] for more details.

4.3. Some special cases. In this subsection we shall exploit proposition 4.10 and find some interesting description of the partition into accessibility classes when the unstable (or the stable) manifold has also dimension 1.

The main theorem here is an application of proposition 4.10 and J. Franks work on codimension one Anosov diffeomorphisms. To this end let us give first some example. Let A be a codimension one Anosov linear automorphism on \mathbb{T}^N and let B be another linear automorphism commuting with A . Let M_B be the manifold that is the quotient $\mathbb{T}^N \times \mathbb{R} / \sim$ where $(x, t) \sim (y, s)$ if and only if $B^n x = y$ and $t = s + n$. Then on M_B can be defined lots of partially hyperbolic systems related to A , for instance, any diffeomorphism $F : \mathbb{T}^N \times [0, 1] \rightarrow \mathbb{T}^N \times [0, 1]$ such that $F|_{\mathbb{T}^N \times \{0\}} = F|_{\mathbb{T}^N \times \{1\}}$ is homotopic to A defines one of them. Then we have the following:

Theorem 4.11. *Let $F : M \rightarrow M$ be a partially hyperbolic diffeomorphism on a compact manifold M . Assume that $\Omega(F) = M$, $\dim E^u = 1$ and $\dim E^c = 1$. Then either $M = M_B$ and F is as one of the examples above, or $E^s \oplus E^u$ is integrable, or F has the accessibility property, or M decompose as the union $M = \bigcup_{i=1}^n M_i$ where M_i is a compact manifold whose boundary is a finite union of tori, each of which injects in homotopy, and $M_i \cap M_j \subset \partial M_i \cap \partial M_j$ for $i \neq j$. M_i is su -saturated, F^{k_i} invariant for some $k_i > 0$ and its boundary are accessibility classes. For each i , either the interior of M_i is itself an accessibility class or M_i is homeomorphic to a torus times an interval. Finally, if for each i we take the M_i that are homeomorphic to torus times interval in a maximal fashion then n , the number of elements in the decomposition is less than the first Betti number of M , i.e. $n \leq \dim H_1(M, \mathbb{R})$.*

When M is foliated by the accessibility classes, it can be shown that this su foliation has the following minimality property, the unique nonempty open F -invariant and su -saturated set is the whole M .

Notice that in order to have a nontrivial decomposition one needs to have at least more than one invariant torus injecting in $\pi_1(M)$ and with the dynamics on it being Anosov. This is a highly nontrivial topological restriction. On the other hand, if all the components are of the form torus times interval then the dynamics of F restricted to any of the boundary components are all conjugated to the same linear map A and $M = M_B$ as one of the examples above.

Finally, observe that when the decomposition is trivial, then either the system has the accessibility property or $E^s \oplus E^u$ integrates to a foliation that jointly with F have a minimal property. Aside from some linear Anosov diffeomorphisms on torus, we do not know of any example of this last case. In the volume preserving case, we

believe that this last case is already enough to guaranty ergodicity. We want to put the following related problem:

PROBLEM 20. *Given a C^2 codimension one minimal foliations on a compact manifold M , there is essentially a finite number of measurable saturated subsets of M , i.e. the σ -algebra of measurable saturated sets is finite mod 0 w.r.t. Lebesgue.*

We thought that in fact the foliation should be ergodic w.r.t. Lebesgue measure, but it seems that some construction beginning with some minimal but non-ergodic flow on a surface could be carried out to build some non-ergodic minimal foliation. In any case, we think that in the particular case that the codimension one foliation comes from this partially hyperbolic setting, then the σ -algebra should be trivial. But it is still not clear in this case why the su -foliation is C^2 , see problem 10.

We finish with another problems:

PROBLEM 21. *Prove that on most three dimensional manifolds, partial hyperbolicity already implies ergodicity.*

PROBLEM 22. *If $f : M \rightarrow M$ is an Anosov diffeomorphism on a complete riemannian manifold M . Is it true that if $\Omega(f) = M$ then M is compact?*

When f is a codimension one Anosov diffeomorphism the answer is yes and that is one of the ingredients to the proof of theorem 4.11.

4.3.1. Three dimensional nil-manifolds. Here we shall see what happens in one of the first algebraic, non-trivial examples. It appears in R. Sacksteder work [Sa] where he proves its ergodicity using the accessibility property.

Let \mathcal{H} be the Heisenberg group of upper triangular 3×3 matrices with ones in the diagonal. This is the non-abelian nilpotent simply connected three dimensional Lie group. We may identify \mathcal{H} with the pairs (\mathbf{x}, y) where $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, $y \in \mathbb{R}$, $(\mathbf{x}, y) \cdot (\mathbf{a}, b) = (\mathbf{x} + \mathbf{a}, y + b + x_1 a_2)$ and $(\mathbf{x}, y)^{-1} = (-\mathbf{x}, x_1 x_2 - y)$. For (\mathbf{x}, y) and (\mathbf{a}, b) in \mathcal{H} , their commutator is $[(\mathbf{x}, y), (\mathbf{a}, b)] = (0, x_1 a_2 - a_1 x_2)$. Hence (\mathbf{x}, y) commutes with (\mathbf{a}, b) if and only if \mathbf{x} and \mathbf{a} are colinear. We have also the projection $p : \mathcal{H} \rightarrow \mathbb{R}^2$, $p(\mathbf{x}, y) = \mathbf{x}$ which is also an homomorphism.

If we denote with \mathfrak{h} the Lie algebra of \mathcal{H} , then \mathfrak{h} corresponds with the upper triangular matrices with zeros in the diagonal. We may also identify \mathfrak{h} with the pairs (\mathbf{x}, y) where $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, $y \in \mathbb{R}$. We have the exponential map $\exp : \mathfrak{h} \rightarrow \mathcal{H}$ given by $\exp(\mathbf{x}, y) = (\mathbf{x}, y + \frac{1}{2}x_1 x_2)$, \exp is one to one and onto. Its inverse, the logarithm, $\log : \mathcal{H} \rightarrow \mathfrak{h}$ is given by $\log(\mathbf{x}, y) = (\mathbf{x}, y - \frac{1}{2}x_1 x_2)$ and the Lie bracket is given also by $[(\mathbf{x}, y), (\mathbf{a}, b)] = (0, x_1 a_2 - a_1 x_2)$.

The homomorphisms from \mathcal{H} to \mathcal{H} are of the form $L(\mathbf{x}, y) = (A\mathbf{x}, l(\mathbf{x}, y))$, where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad l(\mathbf{x}, y) = \alpha x_1 + \beta x_2 + \det(A)y + \frac{ac}{2}x_1^2 + \frac{bd}{2}x_2^2 + bcx_1x_2.$$

If we denote with $\hat{L} : \mathfrak{h} \rightarrow \mathfrak{h}$, $\hat{L} = D_0L$, it is induced by the matrix

$$\hat{L} = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ \alpha & \beta & \det(A) \end{pmatrix} = \begin{pmatrix} A & 0 \\ \mathbf{v} & \det(A) \end{pmatrix}$$

where $\mathbf{v} = (\alpha, \beta)$ and it follows that $\exp(\hat{L}(\mathbf{x}, y)) = L(\exp(\mathbf{x}, y))$.

The centralizer of \mathcal{H} coincides with its first commutator, i.e. $Z(\mathcal{H}) = [\mathcal{H}, \mathcal{H}] = \mathcal{H}_1$ which consists of the elements of the form $(0, y)$. Any homomorphism from \mathcal{H} to \mathcal{H} must leave \mathcal{H}_1 invariant. Similarly $\mathfrak{h}_1 = [\mathfrak{h}, \mathfrak{h}]$ also consists of the elements of the form $(0, y)$. The automorphisms of \mathcal{H} are exactly the ones with $\det(A) \neq 0$.

Any lattice in \mathcal{H} is isomorphic to $\Gamma_k = \{(\mathbf{x}, y) : \mathbf{x} \in \mathbb{Z}^2, y \in \frac{1}{k}\mathbb{Z}\}$, for k a positive integer and the automorphisms leaving Γ_k invariant are the ones with $A \in GL(2, \mathbb{Z})$ (the matrices with integral entries and determinant ± 1) and $\alpha, \beta \in \frac{1}{k}\mathbb{Z}$. On the other hand, every automorphisms of Γ_k extends to an automorphism of \mathcal{H} .

Lemma 4.12. *If T is a subgroup of Γ_k isomorphic to \mathbb{Z}^2 , $T \cap \mathcal{H}_1 \neq \{(0, 0)\}$.*

We define the quotient compact nil-manifold $N_k = \mathcal{H}/\Gamma_k$ by the relation $(\mathbf{x}, y) \sim (a, b)$ if and only if $(\mathbf{x}, y)^{-1} \cdot (a, b) \in \Gamma_k$. The first homotopy group is $\pi_1(N_k) = \Gamma_k$ and two maps of N_k to itself are homotopic if and only if their action on Γ_k coincide. Moreover, any map from N_k to itself is homotopic to an automorphism as the described above leaving Γ_k invariant. The projection $p : \mathcal{H} \rightarrow \mathbb{R}^2$ descends to a projection $p : N_k \rightarrow \mathbb{T}^2$ and also p serves as semiconjugacy between any automorphism L and its corresponding matrix $A \in GL(2, \mathbb{Z})$.

Notice that lemma 4.12 plus theorem 4.11 and its comments imply that any partially hyperbolic diffeomorphism on N_k either has the accessibility property or $E^s \oplus E^u$ integrates to a minimal foliation. We think that this second possibility does not exist.

Given an automorphism L of N_k , it is partially hyperbolic if and only if the associated matrix $A \in GL(2, \mathbb{Z})$ is hyperbolic. In this case, by taking some finite covering it is possibly to make $\mathbf{v} = 0$, but we will not use this fact here.

Let us denote the invariant subspaces for A with $E_A^s, E_A^u \subset \mathbb{R}^2$ with corresponding eigenvalues λ^s and λ^u . Then the invariant subspaces for $D_0L = \hat{L}$ are

$$E_L^s = \left\{ \left(\mathbf{w}, \frac{\mathbf{v} \cdot \mathbf{w}}{\lambda^s - \det(A)} \right) : \mathbf{w} \in E_A^s \right\},$$

and similarly

$$E_L^u = \left\{ \left(\mathbf{w}, \frac{\mathbf{v} \cdot \mathbf{w}}{\lambda^u - \det(A)} \right) : \mathbf{w} \in E_A^u \right\}.$$

Finally, $E_L^c = \mathfrak{h}_1$ and the invariant bundles are formed by the translates of the invariant spaces at 0. Similarly, the translates of E_L^σ , $\sigma = s, u, c$ project onto the invariant foliations in N_k . Observe that E_L^c projects onto a circle and hence the projection of the translates of E_L^c is a foliation by circles. Moreover, these circles are collapsed by the projection p and hence the central foliation is a nontrivial fibration with base \mathbb{T}^2 and fiber \mathbb{S}^1 .

Finally, the accessibility class of 0 will be the smallest subgroup containing E_L^s, E_L^u and being closed under the bracket operation. Thus, its lift to the Lie algebra will be the smallest Lie sub-algebra containing E_L^s and E_L^u that also must contain $\mathfrak{h}_1 = [E_L^s, E_L^u]$ and hence equals \mathfrak{h} . So that any such L has the accessibility property.

4.3.2. *The general affine case.* Let $f : G/B \rightarrow G/B$ be a partially hyperbolic affine diffeomorphism. Recall, subsection 2.5, that this is equivalent to say that $\mathfrak{h} \not\subset \mathfrak{b}$ where \mathfrak{h} is the hyperbolic subalgebra of f . Let $\bar{f} : G \rightarrow G$, $\bar{f} = L_g \circ A$ be the affine diffeomorphism covering f and $\mathfrak{a}(f) : \mathfrak{g} \rightarrow \mathfrak{g}$ the corresponding automorphism of its Lie algebra. Observe that the partition into accessibility classes for \bar{f} corresponds exactly to the translates of H , the hyperbolic subgroup of \bar{f} and hence it is a foliation, moreover it comes from the left action of H on G . The following is also true:

Theorem 4.13. [BreSh], [PuSh4] *If $f : G/B \rightarrow G/B$ is a partially hyperbolic affine diffeomorphism then its accessibility classes are the orbits of the left action of H on G/B . Hence f has the accessibility property if and only if $\mathfrak{h} + \mathfrak{b} = \mathfrak{g}$. Moreover, f has the essential accessibility property if and only if $\overline{HB} = G$.*

With respect to the way in which the accessibility classes behave after perturbations, the following is proven by A. Starkov in the appendix of [PuSh4],

Proposition 4.14. *There exists a neighborhood $O(g) \subset G$ such that H is contained in the hyperbolic subgroup H_x of $f_x = L_x \circ A$ for every $x \in O(g)$.*

So that after perturbation by left translation the accessibility classes can only grow. It would be interesting to know in which cases it is possible to get accessibility, or essential accessibility only by applying left translation, i.e.

PROBLEM 23. *Given an automorphism of a finite volume homogeneous space, $A : G/B \rightarrow G/B$ let us define $G_i \subset G$, $i = \dim G^s + \dim G^u + \dim B, \dots, d = \dim G$ by $G_i = \{g \in G \text{ such that } \dim \overline{H_g B} = i\}$ where H_g is the hyperbolic subgroup of $L_g \circ A$. This gives a partition of G . How is the structure of this partition?, How do G_i look like, are they manifolds?, For which automorphisms is $G_d \neq \emptyset$? For which automorphisms is $G_i \neq \emptyset$ for every i ? Is this possible? What is their relation with the algebraic properties of G ? Analyze also the analogous subsets but for $\mathfrak{h}_g + \mathfrak{b}$ instead of $\overline{H_g B}$, do they coincide?*

Hence in the affine case, the accessibility classes behave the best possible, also under perturbations. As we said, we think that this should be true in the general context.

PROBLEM 24. *Study the affine diffeomorphism f on G/B having $\mathfrak{h} + \mathfrak{b}$ of codimension one and their perturbations. It seems that some description of all possible cases should be plausible, at least with low dimensional center space. What about codimension 2? What about one and two dimensional center space?*

4.4. Stable ergodicity of toral automorphisms. In [FRH1] it is given a partial answer to problem 7. It is proven the stable ergodicity for some toral automorphism with two dimensional center bundle and some extra assumption. This extra assumption essentially asks for the irreducibility of the matrix A defining the linear automorphisms. In fact, the assumption is that all the powers of A have irreducible characteristic polynomial. This irreducibility condition is not very much restrictive because every toral automorphisms can be essentially decomposed by blocks satisfying this irreducibility condition. Thus, if one understands what the behavior is

for this type of linear automorphisms and its perturbations, it seems likely that one can go through to the general case. The restrictive assumption here is the two dimensionality of the center bundle. Let us formulate the results:

Theorem 4.15. *Every linear automorphism of \mathbb{T}^N with the hypothesis listed above is C^5 -stably ergodic if $N \geq 6$.*

When $N = 4$ we have the following:

Theorem 4.16. *Every linear automorphisms of \mathbb{T}^4 with the hypothesis listed above is C^{22} -stably ergodic.*

Here C^r -stable ergodicity means that the perturbations are made in the C^r topology.

Moreover, all ergodic linear automorphisms acting on \mathbb{T}^4 are either Anosov or satisfy the extra assumptions; and all ergodic linear automorphisms acting on \mathbb{T}^2 , \mathbb{T}^3 and \mathbb{T}^5 are Anosov. Hence, we have the obvious corollary:

Theorem 4.17. *Every ergodic linear automorphism of \mathbb{T}^N is stably ergodic for $N \leq 5$.*

Before giving a rough idea of how the proof goes, let us spend a few words about the high differentiability used. In fact most of the proof follows only with a C^1 assumption, but at some places it is used KAM linearizing theorems that make use of high differentiability assumption. When $N \geq 6$ it is used the standard linearizing theorem of V. Arnold and J. Moser but for $N = 4$ things look a little bit different so it is needed to adapt a linearizing theorem of J. Moser on commuting diffeomorphisms of the circle to this setting but then the differentiability increases to 22. We think that the differentiability could be improved without new techniques to allow C^1 perturbations but by C^r diffeomorphism.

PROBLEM 25. *Are the ergodic automorphisms of tori stably Bernoulli? Maybe a little simpler, in dimension 4 when the $E^s \oplus E^u$ bundle is integrable, is f Bernoulli?*

Let us go now into the proof. As one can imagine, the idea is to prove the essential accessibility of the perturbations. Observe that for the linear map, the bundle $E^s \oplus E^u$ integrates to a foliation by planes that coincide with the partition by accessibility classes. So there is no hope of getting stable accessibility, and one needs to find a way of distinguishing when there is accessibility.

The first step is to prove that the perturbation, in some sense looks much like the linear case. To this end, it is much more useful to work in the universal covering \mathbb{R}^N . One of the most useful properties in the linear case is that all the invariant foliations are foliations by planes. So any two leaves of the same foliation are parallel and two leaves of different foliations always intersect. For the perturbation we have that all the distributions but the su are integrable. Moreover the center, center-stable and center-unstable leaves for f stay in a bounded tubular neighborhood of the corresponding for the linear. But for the strong foliations this is no longer the case, typically the leaves will not stay in a bounded tubular neighborhood of the linear ones. Nevertheless, if one works with logarithmic type tubular neighborhoods, then

one gets that the leaves of the strong foliations stay in that logarithmic type tubular neighborhoods of the linear ones. This is enough to guarantee that any leaf of the stable foliation intersects any leaf of the center-unstable at exactly one point, and the same holds for the unstable foliation. Moreover, this also allows us to define the asymptotic direction for the strong foliations and to see that they coincide with the linear case. So one can define global stable holonomies between two center-unstable leaves and global unstable holonomies between two center-stable leaves.

The second step is to study the stable and unstable holonomies. It turns out that when one restricts the stable holonomy to center leaves (whenever it makes sense) then they are differentiable. Moreover, one sees that if the center leaves are not much far away then the stable holonomy between center leaves are close to the ones of the linear in the C^r uniform topology in the whole center leaf if the perturbation is C^r small. Besides, when the center leaves are far, one can still measure the Lipschitz constant of the stable holonomy and see that is no worse than a small power of the distance between the leaves. With this control on the growth of the Lipschitz constant, one can measure some type of growth of the volume of accessibility classes. For example, if one takes a central ball of radius ε and then take all the unstable balls of radius $1/\varepsilon^\beta$ and on this set, the stable balls of radius ε , it follows that this set is open, and for some choice of β , its volume is more or less $1/\varepsilon^\gamma$ for some $\gamma > 0$. This gives some type of recurrence for the accessibility classes.

In the third step it is shown, using the listed properties and that the central direction is two dimensional, that the partition by accessibility classes is essentially minimal. In fact it is proven that the only open, f -invariant and su -saturated sets are the whole manifold and the empty set. The idea of the proof is to recover some flavor of the Denjoy argument for the rotations on the circle. The information on the growth of the volume will take the place, in some sense, of the Denjoy-Koksma inequality. One also make use of the diophantine property of the asymptotic directions of the strong foliations (which are the same as the linear one). Finally, the criterion used to prove that such a non-empty open set is the whole manifold is to have the same homotopy type of the torus, and as the torus is a $K(\mathbb{Z}^N, 1)$ Eilenberg-MacLane space this is done by proving that all the homotopy groups π_k , $k \geq 2$ are trivial. Here is, maybe, the crucial step where the two dimensionality of the center direction appears. Indeed, to make all the homotopy groups trivial, it is seen that the homotopy groups of an su -saturated set are the same as the ones of its intersection with a central leaf and hence, as the π_k of any subset of the plane is trivial if $k \geq 2$, one only has to deal with the π_0 and π_1 . Observe that this fact is no longer true in higher dimensions ($\pi_3(S^2) \neq 0$). The π_0 is treated easily, it is the π_1 that consumes the biggest effort.

Observe that by the above minimality property it is quite easy to make the accessibility property appear, for example, if there is an accessibility class with nonempty interior then, as the system is volume preserving, it is not hard to see that it should be open and essentially invariant and hence we get the accessibility. So, to get accessibility it is enough to get some open accessibility class somewhere. But as we have said, there is no hope to get always accessibility, so it is needed to know what happens when there is no accessibility, and that is step 4.

In the fourth step, it is shown that the accessibility classes are (topological) manifolds and moreover, their dimension vary semi-continuously. Although at this stage it is used the fact that the central dimension is two, we believe that this is true in full generality. Here one mostly works with the intersection of an accessibility class with a center leaf which we call *central accessibility class* and call the dimension of this intersection the *central dimension*. We get for example that the set of points whose accessibility class has zero central dimension is a closed, f -invariant, su -saturated set and hence, by the minimality, is either the whole manifold or the empty set. If it were empty then either would we have that there is an accessibility class that has central dimension two and hence should be open and hence we get the accessibility property, or else all accessibility classes would have central dimension one and hence the central accessibility classes would be curves, in particular this would be the case for a fix point (there is always a fixed point) that needs to have complex eigenvalues if the perturbation is small. So we would have an invariant curve through a fixed point with a complex eigenvalue, thus this curve should spiral. But then we know that the accessibility classes are homogeneous (a neighborhood of each point is homeomorphic to some neighborhood of any other point) and moreover the homeomorphisms that make it homogeneous are holonomies between central leaves and hence they are diffeomorphisms. So we get that the curve should spiral at all its points and this is impossible. Thus, we get that if we do not have the accessibility property then the central accessibility classes are zero dimensional, that is, they are points. Recall that we are working in the universal covering, thus if we fix a central leaf, we get that the intersection of an accessibility class with this central leaf is just a point. So we finally get what we were looking for, what happens when we do not have the accessibility property.

In the final step we have to deal with the case where we do not have the accessibility property and we have to prove the essential accessibility property. At this stage the two dimensionality of the central foliation is not needed. The step above allows us to define the su -holonomy between center leaves simply as the intersection of the accessibility class with the leaf. By the properties we got in the second step, this holonomies are C^r and C^r -close to the linear ones. This property and the fact that the asymptotic directions are diophantine allow us to use the KAM scheme to get a smooth conjugacy between the perturbed partition by accessibility classes and the linear one, thus getting the essential accessibility for the perturbed system and hence the stable ergodicity. The way we use the KAM scheme is by taking a global transversal to the partition by accessibility classes and hence to get a \mathbb{Z}^N acting on it. Thus we take 2 elements of the action and build a two-dimensional torus on it and we still have an action of \mathbb{Z}^{N-2} on this two dimensional torus. At this point, the case of dimension 4 and dimension bigger than 4 are quite different. In dimension bigger than four, any element of this action on the torus has irrational (Diophantine) translation vector with irrational (Diophantine) slopes and we can find an element close to the corresponding translation and hence the usual linearizing theorem of V. Arnold and J. Moser applies, see [He], and hence the differentiability required is 5, in fact $4 + \alpha$, any $\alpha > 0$ will be enough. But when $N = 4$, this is no longer the case, in fact, already for the linear case any element of the \mathbb{Z}^2 action on the two torus

have translation vectors that have rational slopes, we find this an interesting feature that we were not able to see reflected in the su -foliation. In this case we need to use the \mathbb{Z}^2 action in its full strength and we have to prove a theorem on linearization of commuting diffeomorphism on torus analogous to the one of J. Moser in [Mos] and to make the algorithm converge we need the differentiability of at least $21 + \alpha$ for some definite $\alpha > 0$ that can be computed. Nevertheless this differentiability is required for the convergence of this specific algorithm and it seems to be far from optimal. Another approach that may improve the required differentiability is the result of R. Hamilton [Ham] on the stability of some foliations.

5. ERGODIC PROPERTIES OF PARTIALLY HYPERBOLIC SYSTEMS

Given a diffeomorphism $g : N \rightarrow N$, we say that g is *ergodic with respect to Lebesgue measure* if any invariant set has either full or null measure; we do not assume a priori that g preserves Lebesgue measure. More generally, given a partition of a manifold, we say that *the partition is ergodic* w.r.t. Lebesgue measure if any measurable saturated set has either full or null measure. We say that a partially hyperbolic diffeomorphism has the *essential accessibility property* if its partition into accessibility classes is ergodic.

5.1. The differentiable case. [Ho],[Sa],[PuShSt] We shall first present a sketch of the proof that some partially hyperbolic systems are ergodic when the stable and unstable foliations are smooth. This sketch, popularly known as the Hopf argument, is essentially the one used in the more general case.

Theorem 5.1. *Let f be a partially hyperbolic system and assume that E^s and E^u are differentiable. Then if f has the essential accessibility property, f is ergodic.*

This theorem is a consequence of a more general one that may be found in [Sa]. In [PuShSt] there is also a proof for the homogeneous case. This gives essentially the ergodic decomposition of Lebesgue measure.

Theorem 5.2. *Let f be a diffeomorphism and assume it preserves two smooth foliations, \mathcal{F}^s and \mathcal{F}^u . Assume that vectors tangent to \mathcal{F}^s are exponentially contracted and vectors tangent to \mathcal{F}^u are exponentially expanded. Then, for each invariant continuous function ϕ , ϕ is essentially constant along accessibility classes, that is, there is a full measure set R such that if $x, y \in R$ and x is in the accessibility class of y then $\phi(x) = \phi(y)$. In other words, the accessibility classes determine the ergodic decomposition of Lebesgue measure.*

The property described in this last theorem is sometimes called the Mautner's phenomenon.

Observe that in this theorem it is not assumed that f is partially hyperbolic. Nevertheless, it would be interesting to have an example of a non partially hyperbolic system where the theorem applies.

Instead of giving a proof of theorem 5.2 we shall go directly to the proof of theorem 5.1.

Proof. By the Birkhoff ergodic theorem we have that for every L^1 function ϕ , the times or Birkhoff averages

$$(5.1) \quad \frac{1}{n} \sum_{k=0}^n \phi(f^k(x)) \quad \text{and} \quad \frac{1}{n} \sum_{k=0}^n \phi(f^{-k}(x))$$

converge a.e., as $n \rightarrow +\infty$, to measurable invariant functions $\tilde{\phi}^+$ and $\tilde{\phi}^-$ respectively. Moreover, $\tilde{\phi}^- = \tilde{\phi}^+$ a.e. Finally, f is ergodic if and only if $\tilde{\phi}^+$ is constant a.e. for every continuous ϕ . Thus we wish to prove that $\tilde{\phi}^+$ is constant a.e.

Let us call R^+ the set where the forward Birkhoff averages are convergent, and R^- the set where backward Birkhoff averages are convergent. Observe that by uniform continuity of ϕ we have that R^+ is saturated by stable leaves, and R^- is saturated by unstable leaves. Moreover, if y is in the stable leaf of $x \in R^+$ then $\tilde{\phi}^+(x) = \tilde{\phi}^+(y)$, similarly if z is in the unstable leaf of $x \in R^-$ then $\tilde{\phi}^-(x) = \tilde{\phi}^-(z)$. In other words, $\tilde{\phi}^+$ is constant along stable leaves and $\tilde{\phi}^-$ is constant along unstable leaves. Thus, if either the stable or the unstable foliations were ergodic w.r.t. Lebesgue measure, then we would get ergodicity of f . But a priori, we do not know if this is the case (see Problem 26). Let us see how we will overcome this difficulty.

As Lebesgue measure is invariant, we have that the integrals of $\tilde{\phi}^+$, $\tilde{\phi}^-$ and ϕ are equal and let us call them I . Call A^+ the set of x such that $\tilde{\phi}^+(x)$ is less than I , A^- the set of x such that $\tilde{\phi}^-(x)$ is less than I , and observe that A^+ and A^- differ in a null measure set. We need to prove that A^+ has null measure too. So, proving Theorem 5.1 is reduced to proving:

Proposition 5.3. *The set of Lebesgue density points of an s -saturated set, is also s -saturated.*

This finishes the proof of the theorem because of the following: Lebesgue density points of sets that differ in a null measure set are the same, so, if we denote by $D(X)$ the set of Lebesgue density points of the set X , we have that $D(A^+) = D(A^-)$. Now, the proposition says that $D(A^+)$ is s -saturated, and that $D(A^-)$ is u -saturated. Hence, $D(A^+) = D(A^-)$ is s and u -saturated. By the essential accessibility property, $D(A^+)$ has either full or null measure. Hence A^+ has either full or null measure, it clearly cannot have full measure, so we are done. \square

Let us go into the proof of proposition 5.3

Proof. of proposition 5.3 The proof of the proposition uses the following general lemma:

Lemma 5.4. *Let \mathcal{F} be an absolutely continuous foliation and let Σ be a transversal to \mathcal{F} . There is a constant $C > 0$ such that if A is an \mathcal{F} -saturated set then*

$$\frac{1}{C}m(A) \leq m_{\Sigma}(A \cap \Sigma) \leq Cm(A)$$

where m_{Σ} is Lebesgue measure on Σ .

In fact this is essentially the definition of absolute continuity for a foliation we use. As a consequence of the lemma we have that x is a Lebesgue density point for a

saturated set A if and only if x is a Lebesgue density point of $A \cap \Sigma$ for the measure m_Σ . Thus, the proposition follows from this observation and the fact that the holonomies are C^1 , and C^1 diffeomorphisms preserve Lebesgue density points. \square

Let us observe that in the proof of Theorem 5.1 the C^1 hypothesis was only used in the proof of proposition 5.3. In fact, this is the proposition that is being generalized for more general settings in [GrPuSh], [PuSh2], [PuSh3], [BuWi2], [BuWi3], [RHRHUr1]. Nevertheless, it is not clear that proposition 5.3 remains valid in the general case. In some cases, instead of using proposition 5.3, it is only used the absolute continuity as in the Anosov case, but in this case typically it is needed some extra hypothesis in the Lyapunov exponents, see [BuDoPe]; we shall review this item in subsection 5.8.

Observe that the proof of Theorem 5.1 works without change if one only requires, instead of partial hyperbolicity, that the diffeomorphisms leave invariant two C^1 foliations \mathcal{F}^s and \mathcal{F}^u with the property that points in an s -leaf are forward asymptotic and points in an s -leaf are backward asymptotic (not necessarily exponentially fast).

Let us put some problems related to the proof above.

PROBLEM 26. *If f is partially hyperbolic and has the essential accessibility property, are the stable and/or the unstable foliations ergodic w.r.t. Lebesgue measure? In the homogeneous case it is true, [PuShSt]. What about the general C^1 case (the stable and unstable foliations are smooth), or just putting some bunching condition?*

Let us remark that there is an affirmative answer for the topological analogous problem, $\dim M = 3$: for an open and dense set of partially hyperbolic systems f , if f is robustly transitive, then the stable or the unstable foliation is minimal [BoDiUr]. In [RHRHUr2], the authors extend this result to $\dim M \geq 3$ and $\dim E^c = 1$. Also, note that Problem 26 has an affirmative answer in the case of Anosov diffeomorphisms.

PROBLEM 27. *Prove the analogous of theorem 5.2 for general partially hyperbolic systems.*

When f has some center bunching then problem 27 has a positive answer. Let us call $GC(x) = \bigcup_{n \in \mathbb{Z}} C(f^n(x))$ the f -saturation of the accessibility classes, then

Theorem 5.5. *Let f be a center bunched partially hyperbolic diffeomorphism. Then the measurable hull of the partition into $GC(x)$ generalized accessibility classes coincides mod 0 with the partition into ergodic components.*

In other words we will be proving that any f -invariant measurable set coincides mod 0 with an f -invariant su -saturated set.

Proof. By proposition 5.7 we know that the set of density points of every essentially s -saturated and essentially u -saturated set is su -saturated. So we only need to prove that any invariant set A coincides mod 0 with an s -saturated set A^s and also with an u -saturated set A^u . Take a sequence of continuous functions φ_n converging a.e. to χ_A the characteristic function of A . By Birkhoff ergodic

theorem we know that $\lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{k=0}^{N-1} \varphi_n(f^k(x)) \rightarrow \tilde{\varphi}_n^+(x)$ for Lebesgue almost every x , where $\tilde{\varphi}_n^+$ is a measurable invariant function. Let us define the sets $V_n = \{x \in M \text{ such that } \tilde{\varphi}_n^+(x) > \frac{1}{2}\}$. As in the proof of theorem 5.1, since φ_n is continuous, if $y \in W^s(x)$ then $\tilde{\varphi}_n^+(x)$ exists if and only if $\tilde{\varphi}_n^+(y)$ exists and in the case they exist they are equal. Also, by Birkhoff ergodic theorem we know that $\tilde{\varphi}_n^+ \rightarrow \tilde{\chi}_A = \chi_A$ a.e. Let $A^s = \bigcup_{N \geq 0} \bigcap_{n \geq N} V_n$ and observe that it is s -saturated. We claim that $\text{Leb}(A \triangle A^s) = 0$. In fact, let Z be the set of points x such that $\tilde{\varphi}_n(x)$ exists for every n and such that $\tilde{\varphi}_n(x) \rightarrow \chi_A(x)$ and let us see first that $A \cap Z \subset A^s$. If $x \in A \cap Z$ then we have that $\tilde{\varphi}_n(x) \rightarrow \chi_A(x)$ since $x \in Z$ but as x is also in A then $\chi_A(x) = 1$ and hence there is N_x such that $\tilde{\varphi}_n(x) > \frac{1}{2}$ for every $n \geq N_x$ hence $x \in V_n$ for every $n \geq N_x$ and so $x \in A^s$.

On the other hand, if $x \in A^s \cap Z$ then we have that $\tilde{\varphi}_n(x) \rightarrow \chi_A(x)$ since $x \in Z$ and as x is also in A^s there is N_x such that $x \in V_n$ for every $n \geq N_x$. Hence $\tilde{\varphi}_n(x) > \frac{1}{2}$ for every $n \geq N_x$. So we get that $\chi_A(x) = \lim \tilde{\varphi}_n(x) \geq \frac{1}{2}$ but as $\chi_A(x)$ can only be 0 or 1 we get that $\chi_A(x) = 1$ and hence $x \in A$. So A coincides mod 0 with A^s and hence it is essentially s -saturated.

The proof that it is essentially u -saturated is exactly the same but putting $N \rightarrow -\infty$ in Birkhoff theorem. \square

Let us follow with some other problems,

PROBLEM 28. *Prove ergodicity only assuming that $E^s \oplus E^u$ is differentiable.*

PROBLEM 29. *Prove that if E^s and E^u are differentiable then the diffeomorphism is approached by an ergodic one, or even by a stably ergodic one.*

PROBLEM 30. *Prove that if E^s and E^u are differentiable and have the accessibility property then the diffeomorphism is stably ergodic.*

5.2. Accessibility implies ergodicity. In this section we shall give an idea of how the proof of the following theorem goes.

Theorem 5.6. [BuWi3] *Let f be a C^2 partially hyperbolic diffeomorphism, and let all unit vectors $v^\sigma \in E^\sigma$, with $\sigma = s, c, u$ satisfy the following inequalities*

$$|D_p f v^s| < \nu(p) < \gamma(p) < |D_p f v^c| < \hat{\gamma}^{-1}(p) < \hat{\nu}^{-1}(p) < |D_p f v^u|$$

where $\nu, \hat{\nu} < 1$. Let f satisfy the following center bunching conditions

$$(5.2) \quad \nu < \gamma \hat{\gamma} \quad \text{and} \quad \hat{\nu} < \gamma \hat{\gamma}$$

Then, if f has the essential accessibility property, it is ergodic.

The main ingredients of the proof of this theorem may be found already in [GrPuSh]. They were subsequently improved reaching this final, quite general form. It seems likely that in order to go further, some new techniques should appear. For example, one of the main uses of the center bunching condition is to prove differentiability of the strong foliations when restricted to some weak “fake” foliations and the bunching condition is sharp for this purpose. The proof, a priori, relies heavily on this differentiability. With this idea we shall put the following problem:

PROBLEM 31. *As a step to remove the center bunching assumption, maybe it is useful to reduce it only to one of the inequalities, for example, assume only that $\nu < \gamma\hat{\gamma}$. It would be interesting also to understand what happens in the limit case, for example, assume that $\nu(p), \hat{\nu}(p) \leq \gamma(p)\hat{\gamma}(p)$ for every point and equality only holds for a fixed point and at most at that point the corresponding holonomies are not differentiable. How should the proof work then? What about assuming only that the stable foliation is smooth?*

PROBLEM 32. *Is it true that essential accessibility implies ergodicity only assuming that the stable and unstable holonomies are C^1 when restricted to center-stables and center-unstables?*

5.3. A sketch of the proof when $\dim E^c = 1$. Let us give an idea of how we prove, in [RHRHUr1], that accessibility implies ergodicity when $\dim E^c = 1$. As it was said in p. 28, we are reduced to proving the following:

Proposition 5.3. *The set of Lebesgue density points of an s -saturated set, is also s -saturated.*

In fact, we shall prove a weaker result, that will be enough for our purpose. Let us recall that, for $\sigma = s, u$, an *essentially σ -saturated* set is one that differs from an σ -saturated set only in a set of null measure.

Proposition 5.7. *The set of Lebesgue density points of an essentially s -saturated and essentially u -saturated set is su -saturated.*

That is, Lebesgue density points of essentially s - and essentially u -saturated sets flow through stable and unstable leaves. In [GrPuSh], it was suggested that certain shapes called *juliennes* would be more natural, rather than merely riemannian balls, in order to treat preservation of density points. We will follow this line and use certain solid *juliennes* instead of balls. Of course, these new neighborhood bases will define different sets of density points. Let us say that a point x is a C_n -density point of a set X if $\{C_n(x)\}_n$ is a local neighborhood basis of x , and

$$\lim_{n \rightarrow \infty} \frac{m(X \cap C_n(x))}{m(C_n(x))} = 1$$

Recall that *Lebesgue density points* are B_{r^n} -density point, where $B_{r^n}(x)$ are riemannian balls of radii $0 < r^n < 1$. We will be particularly interested in a dynamically defined neighborhood basis $\{J_n\}_n$, the *juliennes*, that consists of certain local stable and unstable saturations of a small center arc. For this new neighborhood basis we obtain:

Proposition 5.8. *The set of J_n -density points of an essentially s -saturated set is s -saturated.*

By changing the neighborhood basis, we have solved the problem of preserving density points, that is we have established proposition 5.3 but for *julienne* density points. However, we need to know now what the relationship is between the *julienne* density points, and Lebesgue density points. Given a family \mathcal{M} of measurable sets,

let us say that two systems $\{C_n\}_n$ and $\{E_n\}_n$ are *Vitali equivalent* over \mathcal{M} , if the set of C_n -density points of X equals (pointwise) the set of E_n -density points of X for all $X \in \mathcal{M}$. That is, if $D_{C_n}(X) = D_{E_n}(X)$ for all sets $X \in \mathcal{M}$.

We obtain

Proposition 5.9. *$\{J_n\}$ is Vitali equivalent to Lebesgue over essentially u -saturated sets.*

Hence, if X is an essentially s - and essentially u -saturated set, we have that $D(X)$, the set of Lebesgue density points of X , is an s - and u -saturated set. As we have said, this is enough to prove Theorem 5.1 for this setting. Indeed, let ϕ be a continuous function, and let A^+ the set of points x such that

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi \circ f^k(x) < \int \phi$$

Also let A^- be the set of points such that $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi \circ f^{-k}(x) < \int \phi$. Then A^+ and A^- differ in a set of null measure, and their density points set is essentially s - and essentially u -saturated, so the proof follows as in the differentiable case to prove that A^+ has null measure, see below Proposition 5.3.

5.4. Juliennes. Let us briefly mention the construction of the juliennes. As one can infer from the previous paragraph, juliennes are local bases whose main features are: (1) their density points are equivalent to Lebesgue over essentially u -saturated sets (2) their density points are s -saturated if the set is essentially s -saturated. Let us just try to construct them to fulfill these conditions.

As it was said above, juliennes are dynamically defined balls obtained by locally su -saturating center arcs in a certain way.

Now, in order that condition (2) be fulfilled, we need that the julienne density points can flow through stable leaves. If we restrict ourselves to a center-stable leaf, this is indeed the case, because the stable holonomy is C^1 if the bunching conditions (5.2) are satisfied, and, in particular, if $\dim E^c = 1$. So, if we merely su -saturate center arcs of a certain length $\sigma^n \in (0, 1)$, by arcs of the same length, we would obtain a ‘‘cubic system’’ that is easily seen to be equivalent to Lebesgue, and to preserve density points when restricted to the center-stable leaf of the center point. This is not enough yet.

Indeed, since the global stable holonomy is not C^1 in general, the unstable saturation of the center arc could be very much distorted by the stable holonomy, thus possibly producing the appearance of new density points or the disappearance of old ones. However, if the saturation is dynamically defined, one can bound this distortion:

Let us assume for simplicity that $\nu, \hat{\nu}, \gamma, \hat{\gamma}$ in Theorem 5.6 are constant, and choose σ so that $\nu/\gamma < \sigma < \min(1, \hat{\gamma})$. We define the *center-unstable juliennes* as

$$(5.3) \quad J_n^{cu}(x) = \bigcup_{y \in W_{\sigma^n}^c(x)} f^{-n}(W_{\nu^n}^u(f^n(y))) = \bigcup_{y \in W_{\sigma^n}^c(x)} J_n^u(y)$$

Note that these are not solid juliennes, but laminae. However, stable holonomy takes these laminae $J_n^{cu}(x)$ into $J_{n-k}^{cu}(x')$ for all n , and some fixed k , so we have attained a bounded distortion. If we saturate these sets by stable leaves of length σ^n , we obtain a local base made of *juliennes* $J_n(x)$, whose density points are preserved under stable holonomy. This fulfills condition (2).

We must check now condition (1). As it was mentioned, it is enough to see that the cubic system formed by the σ^n s - and u -saturation of a center arc of length σ^n is Vitali equivalent to J_n over u -saturated sets. We will see that both systems are Vitali equivalent to a third one over essentially u -saturated sets. Indeed, due to Lemma 5.4, the cubic system is easily seen to be equivalent to:

$$\bigcup_{y \in W_{\sigma^n}^{cs}(x)} J_n^u(y)$$

the rest of the proof consists in seeing that these new juliennes constructed by first σ^n s -saturating $W_{\sigma^n}^c(x)$, and then saturating by $J_n^u(y)$, are Vitali equivalent to the ones obtained above. This is not too difficult and can be found in full detail in [RHRHUr1], Proposition B. 9. With this, we have proved Proposition 5.9. With an analogous procedure, we can obtain

Proposition 5.10. *The set of Lebesgue density points of an essentially s - and essentially u -saturated set is u -saturated.*

With this, we get that the set of Lebesgue density points of an essentially s - and essentially u -saturated set is s - and u -saturated. The procedure consists in building another family of juliennes, which are also equivalent to Lebesgue, but on essentially s -saturated, sets. This can be attained by taking $\hat{\sigma}$ such that $\hat{\nu}/\hat{\gamma} < \hat{\sigma} < \min(1, \gamma)$, and

$$J_n^{cs}(x) = \bigcup_{y \in W_{\hat{\sigma}^n}^c(x)} f^{-n}(W_{\hat{\nu}^n}^s(f^n(y))) = \bigcup_{y \in W_{\hat{\sigma}^n}^c(x)} J_n^s(y)$$

If we saturate by unstable leaves of length $\hat{\sigma}^n$, we obtain a family of juliennes J_n' , which is analogously seen to be equivalent to Lebesgue over essentially s -saturated sets and u -saturated over essentially u -saturated sets.

5.5. Some interesting corollaries.

Proposition 5.11. *Let f be a partially hyperbolic system. Given Σ a transversal to $E^s \oplus E^u$ there is a constant $C > 0$ such that if A is an su -saturated set then*

$$\frac{1}{C}m(A) \leq m_{\Sigma}(A \cap \Sigma) \leq Cm(A)$$

where m_{Σ} is Lebesgue measure on Σ .

As a corollary of the above theorem we have:

Corollary 5.12. *Let f be a bunched partially hyperbolic diffeomorphism. Let P^c be a closed manifold everywhere tangent to E^c such that $f(P^c) = P^c$ and that every point can be joined to P^c by an su -path. If $f|_{P^c}$ is ergodic then f is ergodic. If $f|_{P^c}$ is stably ergodic then f is stably ergodic.*

Applying this corollary we get:

Corollary 5.13. *Let ϕ be a Lebesgue measure preserving Anosov flow. Then either ϕ_t is ergodic for every $t \in \mathbb{R}$ or ϕ is flow equivalent to the suspension of an Anosov diffeomorphism by a constant function ω and t/ω is rational.*

Another corollary is:

Corollary 5.14. *Let $f : M \rightarrow M$ be a stably ergodic diffeomorphism and let $g : N \rightarrow N$ be a volume preserving Anosov diffeomorphism, then, if $f \times g$ is bunched, using $M \times \{y\}$ as central foliations then it is stably ergodic. Also, if $g : N \rightarrow N$ is partially hyperbolic with the stable accessibility property, and $f \times g$ is bunched, using $T_x M \times E_y^c$ as central space, then $f \times g$ is stably ergodic.*

What is interesting here is that one do not needs the accessibility property for $f \times g$ to guaranty its stable ergodicity.

PROBLEM 33. *Are products of stably ergodic systems stably ergodic?*

PROBLEM 34. *Find an example of a volume preserving stably ergodic diffeomorphism that is not robustly transitive, or either prove that there is no such example.*

5.6. The affine case. Affine diffeomorphisms are always center bunched w.r.t. the splitting $\mathfrak{g} = \mathfrak{g}^s \oplus \mathfrak{g}^c \oplus \mathfrak{g}^u$. Hence theorem 5.6 always applies and hence $\overline{HB} = G$ implies ergodicity. Of course, for an affine diffeomorphism, one does not need theorem 5.6 to prove ergodicity, in fact it was already known by S. Dani, [Da1] [Da2], that $\overline{HB} = G$ implies the Kolmogorov property. But for perturbations it is needed and hence, as a consequence of theorems 4.2 and 5.6 the following is also true

Theorem 5.15. [PuSh3] *If an affine diffeomorphism has the accessibility property then it is stably ergodic.*

We say that an affine diffeomorphism is *stably ergodic among left translations* if its perturbations by left translations are also ergodic.

Theorem 5.16. [BreSh], [PuSh4] *If an affine diffeomorphism is stably ergodic among left translations then it must be partially hyperbolic and $\overline{HB} = G$.*

J. Brezin and M. Shub have proven theorem 5.16 in the semisimple and solvable cases and then A. Starkov proved it in full generality. This motivated C. Pugh and M. Shub to formulate

PROBLEM 35. *Does $\overline{HB} = M$ implies stable ergodicity?, in other words, is stable ergodicity among left translations enough for stable ergodicity?*

So far, the only known examples of affine diffeomorphism being stably ergodic but not having the accessibility property are the automorphism of torus in [FRH1]. On the other hand, A. Tahzibi has asked

PROBLEM 36. *Does stable ergodicity for partially hyperbolic systems imply essential accessibility when the central dimension is one? i.e. does theorem 5.16 hold in this general context?*

PROBLEM 37. *Are the affine diffeomorphisms Bernoulli whenever $\overline{HB} = G$? What happens with their perturbations?*

5.7. Weak ergodicity. In the general case, the accessibility property gives a weak form of ergodicity that in some cases it is used indeed to prove ergodicity. We say that f is *weakly ergodic* if almost every orbit is dense. Obviously ergodicity implies weak ergodicity and in some cases, weak ergodicity is enough to prove ergodicity.

Recall that $GC(x) = \bigcup_{n \in \mathbb{Z}} C(f^n(x))$.

Proposition 5.17. [Br1], [BrPe2], [BuDoPe], [DoPe] *Let f be a partially hyperbolic system and assume that for Lebesgue almost every point x , $GC(x)$ is ε -dense. Then Lebesgue almost every orbit is ε -dense. In other words, a.e. ε -accessibility implies that a.e. orbit is ε -transitive.*

We essentially take the proof from [BuDoPe].

Proof. Let B be a ball of radius ε . Let us say that a point p is good if there is a neighborhood of p such that a.e. point in this neighborhood enters B . Then if we prove that a.e. point is good we are done. Take p such that $GC(p)$ is ε -dense and let us see that p is good. By definition we have there is a su-path $[z_0, \dots, z_k]$ with $z_0 \in B$ and $z_k = f^N(p)$ for some $N \in \mathbb{Z}$. It is clear that if z_k is good then, as f is a diffeomorphism, p will be good. We have obviously that z_0 is good since it is itself in B . Let us see by induction that all the z_j are good. Assume z_i is good and let us go to z_{i+1} . By assumption z_i has a neighborhood N such that the orbit of a.e. point in this neighborhood enters B . Take $S \subset N$ the whose orbits enter B and are forward and backward recurrent. By Poincarè recurrence theorem we have that S has full measure in N . If $x \in S$ then the orbit of any point $y \in W^s(x) \cup W^u(x)$ enters B . The absolute continuity of the foliations W^s and W^u means that the set

$$\bigcup_{x \in S} W^s(x) \cup W^u(x)$$

has full measure in the set

$$\bigcup_{x \in N} W^s(x) \cup W^u(x)$$

The latter is a neighborhood of z_{i+1} . Hence z_{i+1} is good. \square

As a corollary we get:

Theorem 5.18. *If for Lebesgue a.e. x $GC(x)$ is dense, f is weakly ergodic.*

Observe that the proof of proposition 5.17 only works for Lebesgue measure. Indeed, the main tools used are Poincarè recurrence and absolute continuity of the strong foliations. It would be interesting to have an analogous theorem for other measures. See Subsection 3.2.

5.8. Ergodicity via Lyapunov exponents. A first corollary of theorem 5.17 is that accessibility plus local ergodicity somewhere implies ergodicity. In [BuDoPe] it is proven that negative central Lyapunov exponents implies local ergodicity. But they also prove that all this situation is stable under perturbations, in fact, they prove the following:

Theorem 5.19. [BuDoPe] *Let f be a volume preserving partially hyperbolic diffeomorphism and assume that almost every accessibility class is dense. If the exponents corresponding to the central bundle are all negative on a positive measure set then f is stably ergodic.*

In this paper, K. Burns, D. Dolgopyat and Y. Pesin pose the following problem:

PROBLEM 38. *Is accessibility plus a.e. non-zero Lyapunov exponents enough for ergodicity?*

In [BuDoPe] the authors also used ideas from the papers [AlBoVi], [BoVi] where it is proven that with a dominated splitting, see section 8 for a definition, and some assumptions on the appearance of nonzero Lyapunov exponents one obtains the existence of SRB measures. Also using this type of technics, A. Tahzibi [Ta] found the following:

Theorem 5.20. [Ta] *On \mathbb{T}^n , there is a stably ergodic diffeomorphism homotopic to an Anosov diffeomorphism that admits a dominated splitting but has no invariant hyperbolic subbundle.*

Moreover he also proved the uniqueness of the SRB measure, that exists by [AlBoVi], for non-conservative perturbations. Hence he proves in particular that partially hyperbolicity is not necessary for proving stable ergodicity, see section 8 for some related problems.

6. LYAPUNOV EXPONENTS

In this section we review some results relating Lyapunov exponents and some other types of growth rates, with partially hyperbolic systems.

Given a C^1 diffeomorphism $f : M \rightarrow M$, Oseledec's theorem asserts the existence of some asymptotic directions with some asymptotic growth rates corresponding to vectors in this directions for a.e. point with respect to any invariant measure. Indeed, he proves the following:

Theorem 6.1. *Given an invariant measure μ , for μ -a.e. point x there is an invariant splitting $T_x M = E_x^1 \oplus \dots \oplus E_x^{k(x)}$, where $k(f(x)) = k(x)$ and $Df_x(E_x^j) = E_{f(x)}^j$. There are also invariant functions $\lambda_1(x) > \dots > \lambda_{k(x)}(x)$ such that*

$$\lim_{n \rightarrow \pm\infty} \frac{1}{n} \log |Df_x^n(v)| = \lambda_j(x)$$

for $v \in E_x^j \setminus \{0\}$ and $j = 1, \dots, k(x)$. Moreover, if $1 \leq j \neq l \leq k(x)$, then

$$\lim_{n \rightarrow \pm\infty} \frac{1}{n} \log \angle \left(E_{f^n(x)}^j, E_{f^n(x)}^l \right) = 0.$$

The numbers λ_i are called the *Lyapunov exponents* and the splitting is called the *Oseledec's splitting*. Notice that if the measure is ergodic then the Lyapunov exponents and $k(x)$ are constant a.e.

Observe that in the partially hyperbolic case, the Oseledec's splitting refines the partially hyperbolic splitting. We call *central Lyapunov exponents* (or central exponents) the Lyapunov exponents corresponding to vectors in E^c , similarly for the

strong stable and strong unstable Lyapunov exponents. Observe that the sum of the central Lyapunov exponents, counted with multiplicities equals:

$$\sum_j \lambda_j(x) \dim E_x^j = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \log \text{Jac} \left(D_{f^k(x)} f | E^c \right)$$

where the sum ranges over all j such that $E_x^j \subset E_x^c$. We will be interested in the integrated sum of central Lyapunov exponents:

$$\int \sum_j \lambda_j(x) \dim E_x^j d\mu$$

which equals

$$\int \log \text{Jac} (D_x f | E^c) d\mu$$

and, as always, in most cases the invariant measure is Lebesgue measure. Notice that when the measure is ergodic the integrated sum of central Lyapunov exponents equals a.e. the sum of central Lyapunov exponents.

6.1. Removing zero central exponents. Let us begin with M. Shub and A. Wilkinson result about the approximation by non-zero central Lyapunov exponents (see also, [Do, Ru2] for related results).

Theorem 6.2. [ShWi2] *There is an open set of partially hyperbolic volume preserving Bernoulli diffeomorphism with non-zero Lyapunov exponents on \mathbb{T}^3 .*

The idea is to get stably ergodicity and non-zero exponents, then Y. Pesin's results give the Bernoulli property [Pe1]. They begin with a linear map f of $\mathbb{T}^3 = \mathbb{T}^2 \times \mathbb{T}$ associated with the matrix

$$B = \begin{pmatrix} A & 0 \\ \mathbf{w}_0 & 1 \end{pmatrix},$$

where $\mathbf{w}_0 \in \mathbb{Z}^2 \setminus \{0\}$ and $A \in GL(2, \mathbb{Z})$, for example,

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

and $\mathbf{w}_0 = (1, 1)$. They embed f into a two parameter family $f_{a,b}$ such that $f_{0,0} = f$. With the parameter b they will guarantee accessibility and hence ergodicity and with the parameter a they will get non-zero exponents. It is important in order to get non-zero exponents in their construction that $\mathbf{w}_0 \neq 0$. Although a posteriori one can arrange the example to get one close to Anosov times identity.

Let us see how it works. Observe that for a partially hyperbolic diffeomorphism with central dimension 1 the sum of the central exponents is just the central exponent. They look for a family $f_{a,b} = g_a \circ h_b$ where h_b is a skew product over A having the accessibility property and g_a is a perturbation along the unstable direction that will make the strong unstable exponent decrease without touching the strong stable exponent. Let us denote $\mathbf{x} = (x, y)$, then

$$h_b(\mathbf{x}, z) = (A\mathbf{x}, z + \mathbf{w}_0 \cdot \mathbf{x} + b\varphi(\mathbf{x}))$$

where $\varphi : \mathbb{T}^2 \rightarrow \mathbb{R}$ is suitably chosen, for example $\varphi(x, y) = \sin(2\pi y)$. We shall see later what is meant by suitable in general. And

$$g_a(\mathbf{x}, z) = (\mathbf{x} + a\psi(z)\mathbf{v}_0, z)$$

where $\psi : \mathbb{T} \rightarrow \mathbb{R}$ is also suitably chosen, for example $\psi(z) = \sin(2\pi z)$, and \mathbf{v}_0 is a unit vector in the unstable direction, for instance $\mathbf{v}_0 = ((1 + \sqrt{5})/2, 1)$. Observe that h_b and g_a are volume preserving for every a, b .

Let us see what is suitable for φ . The idea here is to apply the following criterion to guarantee accessibility for skew products, see [BrPe2, BuWil]. Let $\theta : \mathbb{T}^2 \rightarrow \mathbb{T}$ be a smooth map and define the skew product $h_\theta : \mathbb{T}^2 \times \mathbb{T} \rightarrow \mathbb{T}^2 \times \mathbb{T}$ by $h(\mathbf{x}, z) = (A\mathbf{x}, z + \theta(\mathbf{x}))$ where $A \in GL(2, \mathbb{Z})$ is a hyperbolic matrix. Then h is partially hyperbolic and has the accessibility property if and only if there is no solution to the cohomological equation

$$(6.1) \quad k\theta(\mathbf{x}) = \Phi(A\mathbf{x}) - \Phi(\mathbf{x}) + c$$

for $\Phi \in C^0(\mathbb{T}^2, \mathbb{T})$, $c \in \mathbb{T}$, where $k = \det(A - Id)$.

Observe that θ may be written as $\theta(\mathbf{x}) = \mathbf{w}_0 \cdot \mathbf{x} + \varphi(\mathbf{x})$ where $\mathbf{w}_0 \in \mathbb{Z}^2$ and φ is homotopic to constant and hence may be seen as a function $\varphi : \mathbb{T}^2 \rightarrow \mathbb{R}$. Similarly, the unknown Φ may be written as $\Phi(\mathbf{x}) = \mathbf{w}_1 \cdot \mathbf{x} + k\eta(\mathbf{x})$ where $\mathbf{w}_1 \in \mathbb{Z}^2$ and η is homotopic to constant and hence also $\eta : \mathbb{T}^2 \rightarrow \mathbb{R}$. Observe that in order that the cohomological equation 6.1 have a solution, it is necessary that $(A^t - Id)\mathbf{w}_1 = k\mathbf{w}_0$. And this has solution $\mathbf{w}_1 \in \mathbb{Z}^2$ since k is exactly $\det(A^t - Id) = \det(A - Id)$. Hence taking \mathbf{w}_1 this solution, the cohomological equation 6.1 transforms into

$$(6.2) \quad \varphi(\mathbf{x}) = \eta(A\mathbf{x}) - \eta(\mathbf{x}) + c$$

for $\eta \in C^0(\mathbb{T}^2, \mathbb{R})$ and $c \in \mathbb{R}$. Finally, notice that for the cohomological equation 6.2 to have solution it is necessary that the average of φ over any invariant measure be c . So that any function φ whose average at two different periodic points differs will be good to guarantee accessibility and hence φ will be suitable.

So, for any $b \neq 0$, $f_{0,b} = h_b$ has the accessibility property and hence it belongs to an open set of ergodic diffeomorphisms. So, for any fixed b one can move the parameter a a little still having ergodicity. Let us see what happens when we move a , let us first look at the invariant foliations for $a = 0$. It follows that the central, the center-stable and center-unstable foliations are still the same as for $b = 0$. For the stable foliation we have that each leaf is the graph of a function from the stable manifold of A to the circle and it is invariant under translations of the form $(\mathbf{x}, z) \rightarrow (\mathbf{x}, z + z_0)$ for every $z_0 \in \mathbb{T}$. Now, for any a , g_a preserves the center-unstable foliation by planes. And hence $f_{a,b}$ also preserves the center-unstable foliation by planes.

Let us see that the strong stable Lyapunov exponent for $f_{a,b}$ is the same as for $f_{0,0}$. To that end one can work in the universal covering and hence there is a well defined linear projection $\pi^s : \mathbb{R}^3 \rightarrow E_B^s$ along the center-unstable planes into the stable manifold of the linear one such that $\pi^s \circ f = B \circ \pi^s$. This is because f preserves the center-unstable foliation by planes. Hence $\pi^s(D_x f(v)) = \sigma^{-1}\pi^s(v)$ for any vector v , where σ is the unstable eigenvalue. So, if we put the sup-norm $|v| = \max\{|v^s|, |v^{cu}|\}$ where $v = v^s + v^{cu}$ w.r.t. the splitting for the linear, we get

that for any vector V in the stable direction of $f_{a,b}$, $|V| = |V^s|$ since the invariant spaces for the perturbation are close to the linear ones. Hence it follows that

$$|D_x f^{-n}(V)| = |\pi^s(D_x f^{-n}(V))| = \sigma^n |V^s| = \sigma^n |V|$$

and hence

$$\frac{1}{n} \log |D_x f^{-n}|_{E_{a,b}^s} = \log \sigma$$

for any n which implies that the strong stable Lyapunov exponents coincide. Now, as $f_{a,b}$ is volume preserving it follows that the sum of the strong unstable exponent and the center exponent should equal $\log \sigma$. Hence, if the strong unstable exponent decrease, the central exponent should be positive and that is what should be computed.

The idea again is to use that $f_{a,b}$ is a perturbation so that we may write a vector-field $V^u(\mathbf{x}, z) = (\mathbf{v}_0, u_{a,b}(\mathbf{x}, z))$ where \mathbf{v}_0 is the vector in the unstable direction, and $u_{a,b}$ is uniquely defined by this requirement. Observe that $u_{0,0} = \frac{\mathbf{v}_0 \cdot \mathbf{w}_0}{\sigma - 1} \neq 0$. Then after some computation they obtain that the strong unstable exponent is

$$\log \sigma - \int_{\mathbb{T}^3} \log[1 - a\psi'(w)u_{a,b}(w)]dw$$

Now, they see how this number varies and they get that it decreases. That is made by some heavy computation calculating its derivatives.

The idea is somehow that when a is non zero, the unstable distribution will have some component on the center bundle and hence it will force the dynamics on the strong unstable to slow down.

In the C^1 setting, A. Baraviera and C. Bonatti were able to push this technic to a more general context proving:

Theorem 6.3. [BaBo] *For a C^1 open and dense set of volume preserving diffeomorphism $f : M \rightarrow M$ admitting a dominated splitting $TM = E_1 \oplus \dots \oplus E_k$, $k > 1$ the integrated sum of Lyapunov exponents on E_i are non zero for $i = 1, \dots, k$, i.e. $\int_M \log \text{Jac}(D_x f|_{E_i}) dx \neq 0$.*

For the definition of dominated splitting see section 8.

6.2. C^1 genericity and Lyapunov exponents. In [Ma3] it appeared a sketch of a proof of the following:

Theorem 6.4. [Boc, Ma3] *For a C^1 -generic area preserving diffeomorphism of surfaces f , either f has a.e. zero exponents or f is Anosov*

In particular if the surface is not a torus then the generic f has zero exponent. In [Boc], J. Bochi fixed and completed the proof in [Ma3] and then in [BocVi], jointly with M. Viana, they generalized the result to any dimension but losing a little of strength. In section 8 the reader may find a definition of dominated splitting. We say that the Oseledec's splitting is *dominated at x* if it extend to a dominated splitting on the closure of the orbit of x .

Theorem 6.5. *For a C^1 -generic volume preserving diffeomorphism f of a manifold M , for a.e. $x \in M$ the Oseledec's splitting of f is either trivial or dominated at x .*

The techniques here involve the use of some tower constructions and some perturbations that resemble the Shub-Wilkinson case in order to slow down an exponent when there is no domination, but here of course the slowdown will be not continuous. In fact, they deduce theorem 6.5 from the following

Theorem 6.6. *Let f_0 be a C^1 volume preserving diffeomorphism such that the map*

$$f \rightarrow (LE_1(f), \dots, LE_{d-1}(f))$$

is continuous at $f = f_0$. Then for almost every $x \in M$, the Oseledec's splitting of f is either dominated or trivial at x .

Here M is d -dimensional and $LE_i(f)$ is the integrated sum of the first i Lyapunov exponents.

Of course it will be interesting to get a dominated splitting on the whole manifold. Moreover, maybe the following is true:

PROBLEM 39. *C^1 generically among volume preserving diffeomorphisms, either f has a.e zero Lyapunov exponents or f is ergodic.*

They also have some counterpart for the symplectic case stating that a C^1 generic symplectic diffeomorphism is either Anosov or has at least one (necessarily double) zero Lyapunov exponent. But a complete counterpart is still open, that is, they ask if one can get in fact partial hyperbolicity along the orbit of x . They put the following problem:

PROBLEM 40. *Is it true that the Oseledec's splitting of generic symplectic C^1 diffeomorphisms is either trivial or partially hyperbolic at almost every point?*

Indeed one can ask to get either zero exponents or partial hyperbolicity. On the other hand, in the partially hyperbolic setting, one can put together theorems 6.5, 6.3, 4.4 and 5.18 plus some trick to bypass some absolute continuity and get that a C^1 generic volume preserving, partially hyperbolic system has a globally defined dominated splitting that coincides a.e. with the Oseledec's splitting and also has non-zero integrated Lyapunov exponents. So, on each bundle, the dynamics is asymptotically conformal for a.e. point.

6.3. Dynamical growth rates. The Lyapunov exponents, as we have seen, measure the growth rate of the derivative along some directions. There are others types of asymptotic growth rates that can be defined, and they are all typically related in some way. Let $f : M \rightarrow M$ be a diffeomorphism and assume it leaves a foliation \mathcal{F} invariant, a priori, as always, with smooth leaves of dimension d and tangent to a continuous sub-bundle $E \subset TM$. We shall also assume that f expands \mathcal{F} . Let us define the *dynamical volume growth* of f on \mathcal{F} as

$$vg_r(x) = \liminf_{n \rightarrow \infty} \frac{1}{n} \log v(f^n(\mathcal{F}_r(x)))$$

Where $v(A)$ is the volume of A on the corresponding leaf volume. This dynamical growth of volume was treated in the general case by S. Newhouse [Ne2] and Y. Yomdin [Yo], in this last paper is also proven the entropy conjecture for C^∞ maps,

that is, the hyperbolicity of the action of f in homology is a lower bound for the entropy of f .

Then we define the *dynamical homological growth* of f on \mathcal{F} as the current defined in the following way, [RuSu], given a d -form ω :

$$C_{r,x}(\omega) = \lim_{n \rightarrow \infty} \frac{1}{\nu(f^n(\mathcal{F}_r(x)))} \int_{\mathcal{F}_r(x)} f_*^n \omega$$

Recall that a *current* is an element of the dual of the differential forms. It turns out that C is a closed current (it vanishes on exact forms) and hence it defines an homology class. In [PuSh3] C. Pugh and M. Shub asked:

PROBLEM 41. *Do the strong stable and unstable manifolds represent non-trivial homology classes in the homology of M ?*

In the Anosov case the answer is positive, see [RuSu]. In [Ru1], D. Ruelle also defines some transverse measure to \mathcal{F} , ρ , associated to some cocycles and such that $f_*\rho = \lambda\rho$ for some $\lambda > 1$. In particular when the cocycle is trivial he gets a transverse invariant measure ρ_0 with $f_*\rho_0 = \lambda_0\rho_0$.

All this quantities typically exists for almost every point and do not depend on r . They are all related when the foliation is absolutely continuous. This is the subject of a work of R. Saghin and Z. Xia that uses this relations to build as a corollary some non-absolutely continuous invariant foliations.

6.4. Lyapunov exponents and uniform hyperbolicity. It is known that if a continuous map $f : M \rightarrow M$ is uniquely ergodic then the Birkhoff averages converge uniformly. With essentially the same proof one can prove that for a function φ , if the integral of φ w.r.t. any ergodic invariant measure is less than a constant C , then the Birkhoff averages should be less than C , i.e. $1/n \sum_{k=0}^{n-1} \varphi(f^k(x)) < C$ for every x and $n \geq N_0$. In particular, if the integral w.r.t. any ergodic invariant measure is always the same, then the Birkhoff averages converge uniformly. The results in this subsection can be found in [FRH2]. The reader may found results related to the ones in this subsection in [AlArSa, Ca1, Ca2, CaLuRi, Sc], we would like to thank Y. Cao for putting these references into our attention.

The next proposition says that for the multiplicative case we have essentially the same phenomenon.

Proposition 6.7. *Let $f : X \rightarrow X$ be a continuous map of a compact metric space. Let $a_n : X \rightarrow \mathbb{R}$, $n \geq 0$ be a sequence of continuous functions such that $a_{n+k}(x) \leq a_n(f^k(x)) + a_k(x)$ for every $x \in X$, $n, k \geq 0$ and such that there is a sequence of continuous functions b_n , $n \geq 0$ satisfying $a_n(x) \leq a_n(f^k(x)) + a_k(x) + b_k(f^n(x))$ for every $x \in X$, $n, k \geq 0$. If*

$$\inf_n \frac{1}{n} \int_X a_n d\mu < 0$$

for every ergodic f -invariant measure, then there is $N \geq 0$ such that $a_N(x) < 0$ for every $x \in X$.

An interesting case where we will apply the proposition is the case when $a_n(x) = \log |D_x f^n|E|$ and $b_n(x) = \log m(D_x f^n|E)$ getting the following corollary. A regular C^1 map is a map whose derivative is invertible at each point.

Corollary 6.8. *Let $f : M \rightarrow M$ be a regular C^1 map and Λ a compact invariant set. Assume f leaves a continuous bundle E over Λ invariant. If the Lyapunov exponents of the restriction of Df to E are all negative (positive) for every ergodic invariant measure, then Df contracts (expands) E uniformly.*

Also, using the fact that hyperbolic measures (measures with nonzero Lyapunov exponents) are sent to hyperbolic measures by Hölder continuous conjugacies, we have the following:

Corollary 6.9. *Let $f : M \rightarrow M$ be a diffeomorphism and $g : N \rightarrow N$ be a $C^{1+\text{Hölder}}$ diffeomorphism. Let Λ be a transitive hyperbolic set for f and assume there is a Hölder continuous homeomorphism $h : U \rightarrow V$ from a neighborhood U of Λ onto $V \subset N$ such that $h \circ f = g \circ h$. Let us assume that g leaves a continuous splitting $TM = E_1 \oplus E_2$ over $h(\Lambda) = \Lambda_g$ invariant, and that it coincides with the Lyapunov (stable \oplus unstable) splitting for some (necessarily hyperbolic) g -invariant measure. Then Λ_g is a hyperbolic set for g .*

Also related to regularity of the invariant distributions we have the following:

Corollary 6.10. *Let g be a C^k Anosov diffeomorphism, and assume it preserves a continuous splitting $TM = E_1 \oplus E_2$ (not necessarily the hyperbolic splitting). Given a periodic point p , let us call $\chi_1^+(p)$ the biggest Lyapunov exponent of the restriction of Df to E_1 , $\chi_2^+(p)$ the biggest Lyapunov exponent of the restriction of Df to E_2 and $\chi_2^-(p)$ the smallest Lyapunov exponent of the restriction of Df to E_2 . If there is a constant $c < 0$ such that $\chi_1^+(p) - \chi_2^-(p) < c < 0$ and $\chi_1^+(p) + r\chi_2^+(p) - \chi_2^-(p) < c < 0$, where $r \geq 1$, for every periodic point p then there is a C^s foliation tangent to E_1 where $s = \min\{k - 1, r\}$.*

7. INTEGRABILITY OF THE CENTRAL DISTRIBUTION

The integrability of the central distribution is one of the more striking problems in the study of partial hyperbolic systems. Indeed, there are quite few new results towards such integrability. In general, given a plane field $E \subset TM$, there are two possible obstructions to the integrability of E .

- i) One obstruction is that E does not satisfy the Frobenius bracket condition.
- ii) The other one is the lack of differentiability of the bundle itself.

In the partially hyperbolic setting, let us mention that although there are examples of non integrable central distributions, in this examples the problem is Frobenius bracket condition and not the differentiability, see subsection 7.1.1. Moreover, subsection 7.1 suggest that the Frobenius part of the problem is intimately related to bunching, that is, if f satisfies some bunching then E^{cs} and E^{cu} should be “involutive”.

7.1. The smooth case. In this first section we shall deal with the first reason of non integrability mentioned above. First let us see a positive result and then an example of non-integrability.

In [BuWi2] it is proven the integrability of the center distribution when some bunching condition is available.

Theorem 7.1. [BuWi2] *If E^{cu} is smooth and the bunching condition $\hat{\nu} < \hat{\gamma}^2$ holds then E^{cu} is integrable, analogously, if E^{cs} is smooth then it is integrable whenever $\nu < \gamma^2$. Finally, if both E^{cs} and E^{cu} are smooth and both bunching conditions hold, E^c is integrable.*

In [BuWi2] the authors give a geometric proof. Their proof essentially follows the lines of the proof of Froebenius' theorem. We shall present two proofs of the theorem. First a wrong proof that uses Froebenius' theorem itself. Second, a correction of the first proof.

Proof. First (wrong) proof. Let us see the E^{cs} case, the other case is analogous. By Froebenius' theorem, it has to be proven that whenever X and Y are two vector-fields tangent to E^{cs} their Lie bracket $[X, Y]$ is also tangent to E^{cs} . If we denote the norm by $|\cdot|$, we have that

$$|D_p f^n([X, Y])| = |[D_p f^n(X), D_p f^n(Y)]| \leq C|D_p f^n(X)||D_p f^n(Y)| \leq C\hat{\gamma}_n(p)^{-2}|X||Y|.$$

since $X, Y \in E^{cs}$. On the other hand, recall that E^{cs} may be characterized as the vectors where $|D_p f^n(Y)|\hat{\nu}_n(p) \rightarrow 0$. Finally, as $\hat{\nu} < \hat{\gamma}^2$ we get that $C\hat{\gamma}_n(p)^{-2}\hat{\nu}_n(p)|X||Y| \rightarrow 0$ and we are done.

What is wrong in this proof is that we are tacitly using that $|[X, Y]| \leq C|X||Y|$ for some constant C . That is false, what you have is that $|[X, Y]| \leq C|X|_{C^1}|Y|_{C^1}$, but this inequality is much more complicated to deal with. \square

Proof. Second (corrected) proof. We use again Froebenius' theorem, let X and Y be two vector-fields tangent to E^{cs} and let us see that their Lie bracket $[X, Y]$ is in E^{cs} . Given $p \in M$ take $z \in \omega(p)$, and η^1, \dots, η^u , u linearly independent 1-forms defining $E^s \oplus E^c$ in a neighborhood of z , that is, $E^s \oplus E^c$ is the intersection of the kernels of η^i . Take $n_j \rightarrow \infty$ such that $p_j = f^{n_j}(p) \rightarrow z$ and call

$$v_j = \frac{D_p f^{n_j}[X, Y](p)}{|D_p f^{n_j}[X, Y](p)|}$$

and we may assume that $v_j \rightarrow v$. Now, if $[X, Y](p) \notin E_p^s \oplus E_p^c$, then, v_j loses its center-stable component and its unstable component persists, so that $v \in E_z^u$ and $v \neq 0$. Thus we get that there is i such that $\eta_z^i(v) \neq 0$ and thus $|\eta_{p_j}^i(v_j)| = |\eta_{p_j}^i(v_j^u)| > c > 0$, where $v_j = v_j^{cs} + v_j^u$. Let us call $X_{n_j}(x) = D_x f^{n_j} X(x)$ and $Y_{n_j}(x) = D_x f^{n_j} Y(x)$. We may assume that p_j are in the neighborhood of z so that, on one hand we have

$$\begin{aligned}
|\eta_{p_j}^i(D_p f^{n_j}[X, Y](p))| &= |\partial_{X_{n_j}} \eta_{p_j}^i(Y_{n_j}) - \partial_{Y_{n_j}} \eta_{p_j}^i(X_{n_j}) - d\eta_{p_j}^i(X_{n_j}(p), Y_{n_j}(p))| \\
&= |d\eta_{p_j}^i(D_p f^{n_j} X(p), D_p f^{n_j} Y(p))| \\
&\leq C |D_p f^{n_j} X(p)| |D_p f^{n_j} Y(p)| \\
&\leq C \hat{\gamma}_{n_j}(p)^{-2} |X| |Y|
\end{aligned}$$

and on the other hand we have that

$$\begin{aligned}
|\eta_{p_j}^i(D_p f^{n_j}[X, Y](p))| &= |D_p f^{n_j}[X, Y](p)| |\eta_{p_j}^i(v_j)| \\
&\geq |D_p f^{n_j}[X, Y]^u(p)| |\eta_{p_j}^i(v_j^u)| \geq c \hat{\nu}_{n_j}(p)^{-1} |[X, Y]^u|
\end{aligned}$$

where $[X, Y] = [X, Y]^{cs} + [X, Y]^u$. Thus, since $[X, Y]^u \neq 0$, we get that $\hat{\nu}_{n_j}(p)^{-1} \leq C \hat{\gamma}_{n_j}(p)^{-2}$ which contradicts the bunching condition. \square

It would be interesting to prove the integrability of the central distribution but only assuming the differentiability of E^c and bunching. In fact, this result lead K. Burns and A. Wilkinson to the formulation of the following problem:

PROBLEM 42. *If f is center-bunched then the center bundle is integrable.*

Related to this problem we want to mention the weak-integrability notion defined in [BrBuIv]. A bundle $E \subset TM$ is said to be weakly integrable if for every point $x \in M$ there is a complete manifold $x \in W(x)$ tangent to E . In [BrBuIv] it is proven that if the center bundle is one dimensional then E^{cs} , E^{cu} and E^c are weakly integrable. Also they prove that if two partially hyperbolic diffeomorphisms are homotopic through a path of partially hyperbolic diffeomorphisms and one of them has E^{cs} weakly integrable then so does the other. Let us put a problem weaker than problem 42:

PROBLEM 43. *If f is center-bunched, is the center bundle weakly-integrable?.*

Observe that if the bundle E is smooth then weak-integrability and integrability coincide.

7.1.1. An example of non-integrability. Let \mathcal{H} be the Heisenberg group as in subsection 4.3.1. Then f will be essentially A. Borel's construction of an Anosov diffeomorphism on a quotient of $\mathcal{H}^2 = \mathcal{H} \times \mathcal{H}$ that appeared in S. Smale's paper [Sm]. The Lie algebra of \mathcal{H}^2 is $\mathfrak{h} \oplus \mathfrak{h}$, where the bracket is defined componentwise. Let $\lambda = 2 + \sqrt{3}$, then $\lambda^{-1} = 2 - \sqrt{3}$. Given $\alpha, \beta \in \mathbb{Z}$ let us define the automorphism of $\mathfrak{h} \oplus \mathfrak{h}$

$$f(x_1, x_2, y, a_1, a_2, b) = (\lambda^\alpha x_1, \lambda^\beta x_2, \lambda^{\alpha+\beta} y, \lambda^{-\alpha} a_1, \lambda^{-\beta} a_2, \lambda^{-\alpha-\beta} b).$$

It is well defined and hence it defines an automorphism F of \mathcal{H}^2 . It is partially hyperbolic if α or β are not zero and hyperbolic if $\alpha, \beta, \alpha + \beta$ are non-zero. Now the whole theme is to find a lattice Γ of \mathcal{H}^2 that is F -invariant and cocompact, i.e with \mathcal{H}^2/Γ compact. But before doing this let us identify the invariant bundles, and the invariant foliations already in $\mathfrak{h} \oplus \mathfrak{h}$, recall that $\mathfrak{h}_1 = [\mathfrak{h}, \mathfrak{h}]$ and that $[\mathfrak{h}_1, \mathfrak{h}] = 0$. Let

us assume that $\alpha + \beta > \beta \geq \alpha$. Then we can take $E^u = \mathfrak{h}_1 \oplus \{0\}$ to be the strong unstable direction and $E^s = \{0\} \oplus \mathfrak{h}_1$ to be the strong stable one. Then take E^c to be the space spanned by $X_1 = (1, 0, 0, 0, 0, 0)$, $X_2 = (0, 1, 0, 0, 0, 0)$, $A_1 = (0, 0, 0, 1, 0, 0)$ and $A_2 = (0, 0, 0, 0, 1, 0)$, that is, the orthogonal complement to $\mathfrak{h}_1 \oplus \mathfrak{h}_1 = E^u \oplus E^s$. Then observe that E^c is not integrable since $[E^c, E^c] = \mathfrak{h} \oplus \mathfrak{h}$ and hence it is not involutive. In fact $0 \neq Y = [X_1, X_2] \in \mathfrak{h}_1 \oplus \{0\}$ and $0 \neq B = [A_1, A_2] \in \{0\} \oplus \mathfrak{h}_1$. Observe that also we could also take $E^s = \{0\} \oplus \mathfrak{h}$ and then E^c the orthogonal complement of $\mathfrak{h}_1 \oplus \mathfrak{h}$ and then $E^c \subset \mathfrak{h} \oplus \{0\}$ and $[E^c, E^c] = \mathfrak{h} \oplus \{0\}$ and it is still not integrable.

Let us see now how to build an F -invariant, cocompact lattice. We take this part from [AuSch]. Take, $a \in \mathbb{Z}$ such that λ be a root of $x^2 + 2ax + 1$ and define the following basis for $\mathcal{H} \times \mathcal{H}$,

$$\left\{ X_1 + A_1, \sqrt{a^2 - 1}(X_1 - A_1), X_2 + A_2, \sqrt{a^2 - 1}(X_2 - A_2), Y + B, \sqrt{a^2 - 1}(Y - B) \right\}$$

and call the vectors E_1, \dots, E_6 respectively, then it follows that $[E_1, E_3] = E_5$, $[E_1, E_4] = E_6$, $[E_2, E_3] = E_6$ and $[E_2, E_4] = (a^2 - 1)E_5$ and all other possible combinations are either zero or the opposite of the existing ones. So taking the integer lattice $L \subset \mathfrak{h} \oplus \mathfrak{h}$ formed by the integer combination of E_1, \dots, E_6 , and $\Gamma = \exp(L)$ we get that Γ is a discrete subgroup of \mathcal{H}^2 and that \mathcal{H}^2/Γ is a compact nilmanifold. Let us see that F defines an automorphism of \mathcal{H}/Γ , for this we need to see that $f(L) = L$ or, what is the same, that the matrix associated to f in the basis $\{E_i\}$ is an integer matrix. Then, if we take the two by two matrix

$$C = \begin{pmatrix} -a & a^2 - 1 \\ 1 & -a \end{pmatrix}$$

then the associated matrix to f is:

$$\begin{pmatrix} C^\alpha & 0 & 0 \\ 0 & C^\beta & 0 \\ 0 & 0 & C^{\alpha+\beta} \end{pmatrix}$$

In the case $\lambda = 2 + \sqrt{3}$, $a = -2$ and if we take $\alpha = 1$ and $\beta = 2$ we obtain

$$\begin{pmatrix} 2 & 3 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 7 & 12 & 0 & 0 \\ 0 & 0 & 4 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 26 & 45 \\ 0 & 0 & 0 & 0 & 15 & 26 \end{pmatrix}$$

Observe that using some variation of theorem 4.2 it can be proven that for any perturbation of F , the center-stable and the center-unstable bundles are not integrable. In fact, one can prove that any two points can be joined by central curves. Also it follows the stability of the non integrability using the notion of weak-integrability in [BrBuIv]. In fact, neither E^{cs} , nor E^{cu} are weakly-integrable and this is a widely stable property (valid in the whole connected component of the set of partially hyperbolic diffeomorphisms containing F).

7.2. Some special cases. In [RHRHUr2] we prove the unique integrability of the central distribution when the diffeomorphism f is transitive, the manifold is of dimension three and there are no periodic points at all. When the center-stable or the center-unstable is integrable, if the fundamental group of M is abelian, the isomorphism induced by f in the homology is partially hyperbolic, see [BrBuIv] and subsection 9.2. This implies that, in particular, if $M = \mathbb{S}^3$ then $Per(f) \neq \emptyset$. However, it is not known whether \mathbb{S}^3 supports a partially hyperbolic diffeomorphism.

PROBLEM 44. *Does \mathbb{S}^3 support a partially hyperbolic diffeomorphism?*

We think the answer is negative. For such a diffeomorphism, neither E^{cu} nor E^{cs} could be integrable (see [DiPuUr, BrBuIv]).

7.2.1. Quasi-isometric foliations and integrability. A foliation \mathcal{W} of a simply connected riemannian manifold is said to be quasi-isometric if there are constants a and b such that whenever x and y are in the same leaf $d_{\mathcal{W}}(x, y) \leq ad(x, y) + b$, where $d_{\mathcal{W}}(x, y)$ denotes the distance on the leaf. In [Br3] M. Brin prove the following:

Theorem 7.2. *Let f be a partially hyperbolic diffeomorphism of a compact riemannian manifold M . Suppose the unstable foliation \mathcal{W}^u of f is quasi-isometric in the universal cover \tilde{M} . Then the distribution E^{cs} is locally uniquely integrable.*

Of course if \mathcal{W}^s is quasi-isometric in \tilde{M} then E^{cu} is locally uniquely integrable, and if both are quasi-isometric in \tilde{M} then E^c is locally uniquely integrable. Although the unstable foliation do not need to be quasi-isometric in the universal covering, for example for the geodesic flow on a hyperbolic surface, it is quasi-isometric in some interesting examples.

Proposition 7.3. [Br3] *Let \mathcal{W} be a k -dimensional foliation on \mathbb{T}^m . Suppose there is a codimension k plane A such that $T\mathcal{W} \cap A = \{0\}$. Then the lift of \mathcal{W} is quasi-isometric in the universal cover \mathbb{R}^m .*

It would be interesting to weaken somehow the hypothesis of the proposition, at least to a more topological type of hypothesis. In [Br3] M. Brin ask the following:

PROBLEM 45. *If f is a partially hyperbolic diffeomorphism of \mathbb{T}^3 , are the stable and unstable foliations necessarily quasi-isometric?*

Of course we may ask the same for \mathbb{T}^n .

For other cases of unique integrability of the center bundle see subsection 9.1.

7.3. Plaque expansive. Given a partially hyperbolic diffeomorphism $f : M \rightarrow M$ having a center foliation \mathcal{F}^c , we define a δ -pseudo-orbit respecting the central plaques to be a sequence x_n , $n \in \mathbb{Z}$, such that $f(x_n) \in \mathcal{F}_\varepsilon^c(x_{n+1})$. We say that f is plaque expansive at \mathcal{F}^c if there is an $\varepsilon > 0$ such that if x_n and y_n are ε -pseudo-orbits preserving the central plaques and $d(x_n, y_n) < \varepsilon$ for every $n \in \mathbb{Z}$ then $x_0 \in \mathcal{F}_\varepsilon^c(y_0)$. The main reference for plaque expansivity is still [HiPuSh2].

If \mathcal{F}^c is a C^1 foliation it is plaque expansive, [HiPuSh2] otherwise it is only known in some cases.

PROBLEM 46. *Are the central foliations always plaque expansive? What about when the strong foliations are quasi-isometric?*

One of the main consequences of plaque expansiveness is the following:

Theorem 7.4. [HiPuSh2] *Let $f : M \rightarrow M$ be a plaque expansive partially hyperbolic diffeomorphism then there is a neighborhood of f , U , such that if $g \in U$ then g leave invariant a plaque expansive center foliation \mathcal{F}_g^c and there is an homeomorphism $h : M \rightarrow M$ such that $h(\mathcal{F}_f(x)) = \mathcal{F}_g(h(x))$ and $h(f(\mathcal{F}_f(x))) = g(h(\mathcal{F}_f(x)))$*

In [HiPuSh2] they put the following:

PROBLEM 47. *If f is partially hyperbolic and plaque expansive at \mathcal{F}^c , is \mathcal{F}^c the unique f -invariant foliation tangent to E^c ?*

We heard the following problem from C. Pugh:

PROBLEM 48. *Let $f : M \rightarrow M$ be a partially hyperbolic diffeomorphism and assume that it has a central foliation by compact leaves. Is it true that the volume of central leaves is uniformly bounded?, is it true that the central foliation is plaque expansive?, is it true that there is a fibration $p : M \rightarrow N$ whose fibers are the central leaves and an Anosov diffeomorphism $g : N \rightarrow N$ such that p is a semiconjugacy between f and g ?*

Maybe the last question goes a little beyond of what C. Pugh was asking, and maybe it goes a little beyond the reality.

7.3.1. *Lyapunov Stable.* Let $f : M \rightarrow M$ leave invariant a bundle $E \subset TM$. We say that f is Lyapunov stable in the direction E if for any $\varepsilon > 0$ there is $\delta > 0$ such that for any C^1 path γ tangent to E , $length(\gamma) < \delta$ implies $length(f^n \gamma) < \varepsilon$ for every $n \geq 0$.

Theorem 7.5. *Let $f : M \rightarrow M$ admit a splitting $TM = E^{cs} \oplus E^u$, where $Df|_{E^u}$ is expanding. Let us assume that f is Lyapunov stable in the direction E^{cs} . Then E^{cs} is tangent to a unique lamination W^{cs} . Moreover, W^{cs} is plaque expansive.*

Proof. Let us assume for simplicity that $|Df^{-1}|_{E^u}| \leq \mu^{-1}$, $\mu > 1$. The existence of the lamination tangent to E^{cs} follows directly from the technics in Theorem (7.5) of [HiPuSh2]. Let us see that W^{cs} is plaque expansive. Let \mathcal{P} be a plaquation of W^{cs} and let $\{x_n\}_n$ and $\{y_n\}_n$ be two ν -pseudo orbits respecting \mathcal{P} . Let us assume that $d(x_n, y_n) < \nu$ for every $n \in \mathbb{Z}$. We want to show that if ν is small enough, then x_0 and y_0 are in the same plaque. Let $z_n = W_{loc}^{cs}(x_n) \cap W_{loc}^u(y_n)$. Then z_n is an δ -pseudo orbit respecting \mathcal{P} and $d(z_n, y_n) < \delta$ for every $n \in \mathbb{Z}$ where δ goes to zero with ν . If z_0 and y_0 lie in a common plaque then x_0 and y_0 are in the same plaque and we are done. Let us call $\sigma_k = \sup_{n \in \mathbb{Z}} d(f^k(y_n), f^k(z_n))$. There is ε_0 independent of δ and ν such that if $\sigma_j \leq \varepsilon_0$ for $0 \leq j \leq k-1$ then $\sigma_k \geq \mu^k \sigma_0$. Take $k_0 + 1$ the first time $\sigma_k > \varepsilon_0$, which exists since we assume by contradiction that $\sigma_0 \neq 0$. Then $\varepsilon_0 \geq \sigma_{k_0} \geq \frac{\varepsilon_0}{|Df|_{E^u}^{k_0}} = \varepsilon_1$.

Let us fix a small ε and let us take δ from the definition of Lyapunov stable. Given $k \geq 0$, we have that $\{f^k(y_n)\}_n$ and $\{f^k(z_n)\}_n$ are ε -pseudo orbits respecting \mathcal{P} . In

fact $d(f(f^k(y_n)), f^k(y_{n+1})) = d(f^k(f(y_n)), f^k(y_{n+1})) < \varepsilon$ since $d(f(y_n), y_{n+1}) < \delta$. Let us call $p_n = f^{k_0}(y_n)$ and $q_n = f^{k_0}(z_n)$. We have that

$$\begin{aligned} \mu d(p_n, q_n) &\leq d(f(p_n), f(q_n)) \\ &\leq d(f(p_n), p_{n+1}) + d(p_{n+1}, q_{n+1}) + d(f(q_n), q_{n+1}) \\ &\leq 2\varepsilon + \sigma_{k_0} \end{aligned}$$

Taking n such that $d(p_n, q_n) \geq \sigma_{k_0} - \varepsilon$, we get that $\varepsilon_1 \leq \sigma_{k_0} \leq \left(\frac{2+\mu}{\mu-1}\right)\varepsilon$. So, taking ε small, since ε_1 is fixed we get a contradiction and thus we get the theorem. \square

Corollary 7.6. *Let f be a partially hyperbolic diffeomorphism. If f and f^{-1} are Lyapunov stable in the direction E^c then E^c is tangent to a unique lamination W^c . Moreover, W^c is plaque expansive. The same holds if $1/C \leq m(Df^n|_{E^c}) \leq |Df^n|_{E^c} \leq C$ for every $n \geq 0$ and some constant $C > 0$ which is the case when $Df|_{E^c}$ is an isometry.*

This last case appeared in [HiPuSh2], but no proof were available since then.

Related with Liapunov stability in the center direction, A. Wilkinson proposed us the following problem:

PROBLEM 49. *Does an accessible, center bunched system with Lyapunov stable center would have to be Bernoulli?*

8. ROBUST TRANSITIVITY

A diffeomorphism f of a closed manifold is C^r robustly (stably) transitive if it belongs to the C^r interior of the transitive diffeomorphisms.

Transitive Anosov diffeomorphisms are examples of such diffeomorphisms (it is well known that the transitivity of any Anosov diffeomorphism is an open question). The first nonhyperbolic examples were given by M. Shub ([Sh2]) in \mathbb{T}^4 . Later R. Mañé gave an example on \mathbb{T}^3 ([Ma1]), in dimension 2 robust transitivity implies Anosov [Ma2]. The list of examples that are known to be robustly transitive is very small:

- Transitive Anosov diffeomorphisms.
- Some derived from Anosov (see [Ma1, Sh2]).
- Perturbations of $f \times Id$ where f is a transitive Anosov diffeomorphism Id is the identity map of any closed manifold (see [BoDí]).
- Perturbations of the time-one map of a transitive Anosov flow (see [BoDí]).
- Examples that are not partially hyperbolic but presenting some form of weak hyperbolicity (see [BoVi]).

An important tool in order to construct examples are the center-stable blenders first introduced in [BoDí]. The blenders resemble a high dimensional skew horse-shoe and their more important property is that in their presence a quasi-transversal intersection between stable and unstable manifolds of hyperbolic sets of different indices turns out to be persistent under perturbations. For instance, this enables C. Bonatti and L. Díaz to prove that the closure of the stable manifold of certain

periodic point contains a stable manifold of greater dimension. This property, combined with some global property of the original diffeomorphism (the presence of two periodic points that persistently have dense stable or unstable manifold), implies that the transverse homoclinic points of these periodic points are dense giving the desired transitivity.

Although there exists this abstract construction of robustly transitive diffeomorphisms, the first problem is to enlarge the known set of examples. It seems that many of the examples that are known to be stably ergodic should be robustly transitive. For instance:

PROBLEM 50. *Is the time-one map of the geodesic flow of a surface of negative curvature robustly transitive? And the partially hyperbolic automorphisms of three dimensional nilmanifolds of section 4.3.1?*

Moreover, there are no satisfactory sufficient conditions implying robust transitivity. There are some theorems with necessary conditions in the C^1 category: some weak form of hyperbolicity is needed (we shall explain it better below) and if f is partially hyperbolic and the center bundle is one-dimensional, generically, at least one of the strong foliations must be minimal (see [BoDíUr, RHRHUr2]).

PROBLEM 51. *Let f be a partially hyperbolic diffeomorphism. Does minimality of the strong unstable foliation imply that f is robustly transitive? And if, in addition, we demand f to have the accessibility property?*

In [DíPuUr] and [BoDíPu] it is proved that some amount of hyperbolicity is needed in order to obtain robust transitivity (at least in the C^1 topology). In fact, for surface diffeomorphisms R. Mañé results (see [Ma2]) implies that C^1 robustly transitive diffeomorphisms are Anosov and, of course, there are not robustly transitive diffeomorphisms of \mathbb{S}^1 .

Let us explain the results of [BoDíPu] that generalize to any dimension those of [DíPuUr] for dimension 3. A continuous invariant splitting $TM = E \oplus F$ is called dominated if there exists $n \in \mathbb{N}$ such that

$$\|D_x f^n|_E\| \cdot \|D_{f^n(x)} f^{-n}|_F\| < 1/2.$$

Moreover, we will say that a continuous invariant splitting $TM = E_1 \oplus \dots \oplus E_k$ is dominated if, for each $i = 1, \dots, k$, the bundles $E = E_1 \oplus \dots \oplus E_i$ and $F = E_i \oplus \dots \oplus E_k$ define a dominated splitting in the latter sense.

Theorem 8.1. [DíPuUr, BoDíPu] *Let f be a C^1 robustly transitive diffeomorphism. Then, f admits a dominated splitting $TM = E_1 \oplus \dots \oplus E_k$ such that the Jacobian of $Df^n|_{E_1}$ and $Df^{-n}|_{E_k}$ decrease exponentially with n .*

This theorem says that robustly transitive diffeomorphisms are partially “volume” hyperbolic.

PROBLEM 52. *Is it possible to build a theory of partially volume hyperbolic diffeomorphism? Observe that, in the easier case of dimension 3, at least one of the bundles is hyperbolic (it is one dimensional) but the other could be only volume hyperbolic and, then, it is not known if it is integrable. Are these strong bundles uniquely integrable as in the partially hyperbolic case?*

This motivates also the following problem: Give sufficient conditions (optimal) in such a way that a volume preserving diffeomorphism on a 3 dimensional manifold admitting a splitting of the form $TM = E^{cs} \oplus E^u$ be ergodic. Here, E^{cs} is volume contracting and E^u expands vectors. What about the following?

PROBLEM 53. *Is it true that a C^1 open and C^∞ dense set of volume preserving diffeomorphism admitting a splitting $TM = E^{cs} \oplus E^u$ is ergodic? Here again E^{cs} is volume contracting and E^u expands vectors. What about in dimension 3?*

Aside from the case some hypothesis is made on Lyapunov exponents, very little is known to guaranty ergodicity. Maybe in dimension 3, if E^{cs} cannot be split then its Lyapunov exponents are C^r typically negative, $r \geq 2$.

Let us also mention that V. Horita and A. Tahzibi [HorTa] and independently R. Saghin have proven that stable ergodicity among C^1 symplectic diffeomorphisms implies partial hyperbolicity.

Related to transitivity, there is the following counterpart of theorem 5.18:

Theorem 8.2. [Br1] *If $\Omega(f) = M$ and $GC(x)$ is dense for some point x then f is transitive.*

The reader may find also a proof of this in [RHRHUr2]. So we have the following counterpart of Corollary 5.12.

PROBLEM 54. *Let $f : M \rightarrow M$ be a partially hyperbolic diffeomorphism. If P^c is a compact f -invariant manifold tangent to the central direction, $f|_{P^c}$ is robustly transitive and f has the stable accessibility property, is f robustly transitive?*

We expect that the answer to this problem is no. In fact if we take the non transitive Anosov flow ϕ constructed in [FrWi], it has the stable accessibility property and we can take a time t in such a way that for some closed orbit $\phi_t|_{P^c}$ is an irrational rotation. Of course $\phi_t|_{P^c}$ is not robustly transitive, but maybe if we multiply by an Anosov diffeomorphism A on \mathbb{T}^2 and make a perturbation g to get that $g|_{P^c \times \mathbb{T}^2}$ is robustly transitive... Compare with proposition (8.4) of [HiPuSh2].

9. CLASSIFICATION

As we have already said, the problem of the classification of partially hyperbolic systems is widely open. In [PuSh3] C. Pugh and M. Shub posed the following problem related to the topology of the manifold supporting a partially hyperbolic system:

PROBLEM 55. *If a manifold M supports a partially hyperbolic diffeomorphism, does M fibers over a lower, positive dimensional manifold?*

9.1. Transitive systems in dimension 3. In [BoWi] C. Bonatti and A. Wilkinson have done some substantial advances in an attempt of classification of transitive partially hyperbolic diffeomorphisms on three manifolds. Their main hypothesis concern the existence of an invariant embedded circle γ (observe that γ will always be tangent to E^c) and the behavior of the invariant foliations around it. In their

own words, if a transitive partially hyperbolic diffeomorphism f looks like the perturbations of a skew products or of the time-1 map of an Anosov flow in just a small region of a 3-manifold, then f is the perturbation of a skew product or the time-1 map of an Anosov flow.

$W_\delta^s(\gamma)$ and $W_\delta^u(\gamma)$ will denote the union of the strong stable and strong unstable segments, respectively, of length δ through the points of γ . Let us state the theorems:

Theorem 9.1. *Let f be a transitive partially hyperbolic diffeomorphism of a 3-manifold M . Assume that there is an embedded circle γ such that $f(\gamma) = \gamma$. Suppose there exists $\delta > 0$ such that $W_\delta^s(\gamma) \cap W_\delta^u(\gamma) \setminus \gamma$ contains a connected circle. Then:*

- (1) f is dynamically coherent.
- (2) Each center leaf is a circle and the center foliation is a Seifert bundle on M .
- (3) If the center-stable and the center-unstable foliations are transversely orientable, then M is a \mathbb{S}^1 -bundle over \mathbb{T}^2 , and f is conjugate to a (topological) skew product over a linear Anosov map of \mathbb{T}^2 .
- (4) If the center-stable or the center-unstable foliations are not orientable, then a covering of M corresponding to the possible transverse orientations is a \mathbb{S}^1 -bundle and the natural lift \tilde{f} of f is conjugate to a (topological) skew product over a linear Anosov map of \mathbb{T}^2 .

In this result they adopt a more general definition of a skew product. They say that a homeomorphism F of any circle bundle is a skew product over an Anosov map A if F preserves the fibration and projects to A . Observe that this definition includes the examples of R. Sacksteder on nilmanifolds (see [Sa] and subsection 4.3.1).

Theorem 9.2. *Let f be a partially hyperbolic dynamically coherent diffeomorphism on a compact 3-manifold M . Assume that there is a closed center leaf γ which is periodic under f and such that each center leaf in $W_{loc}^s(\gamma)$ is periodic for f .*

Then:

- (1) there is an $n \in \mathbb{N}$ such that f^n sends every center leaf to itself.
- (2) there is an $L > 0$ such that for any $x \in M$ the length of the smaller center segment joining x to $f^n(x)$ is bounded by L .
- (3) each center-unstable leaf is a cylinder or a plane (according it contains a closed center leaf or not) and is trivially bi-foliated by center and strong unstable leaves.
- (4) the center foliation supports a transitive expansive continuous flow.

9.2. Growth of curves in dimension 3. The results in this subsection follows essentially from [BrBuV]. Their idea is to analyze the action of f on homology and thus get some restrictions on the homotopy type. Here we push this argument a little more. The results here are for dimension 3. One of the main tools here is Novikov theorem on Reeb components.

Given a compact manifold M and $x \in \tilde{M}$, let us define $v_x(r) = \text{vol}(B(x, r))$. Notice that there is $C > 0$ such that $v_x(r) \leq C v_y(r)$ for any two point x and y . So let us fix $x_0 \in \tilde{M}$ and call $v(r) = v_{x_0}(r)$.

Proposition 9.3. *Let $f : M \rightarrow M$ be a partially hyperbolic diffeomorphism on a three dimensional manifold. Assume that either $E^s \oplus E^u$ or $E^c \oplus E^u$ is integrable. Then there is a constant $C > 0$ such that if $I \subset \tilde{M}$ is an unstable arc then $\text{length}(I) \leq C\nu(\text{diam}(I)) + C$. Moreover, if $x \in W^u(y)$ then $d^u(x, y) \leq C\nu(d(x, y)) + C$.*

Thus, proposition 9.3 gives something that recalls the quasi-isometric property used by M. Brin in [Br3]. In fact, in some cases we can get something closer, for example for nilmanifolds, as they have polynomial growth of volume we get:

Corollary 9.4. *In the setting of proposition 9.3, if M is a nil-manifold then there is $C > 0$ such that $\text{length}(I) \leq C(\text{diam}(I))^4 + C$, and in fact if $x \in W^u(y)$ then $d^u(x, y) \leq Cd(x, y)^4 + C$.*

But if the manifold is the unit tangent bundle of a hyperbolic surface, then the volume growth exponentially and hence proposition 9.3 only give that $d(f^n(x), f^n(y))$ growth linearly.

Another property that is useful is the following:

Proposition 9.5. *Let $f : M \rightarrow M$ be a diffeomorphism on a three dimensional manifold, and assume it leave invariant a codimension one torus T such that $f|_T$ leave invariant an expanding foliation, then T is not the boundary of a solid torus.*

This proposition together the following, that is also a consequence of Novikov's theorem, will give a good description of the homotopy type of some partially hyperbolic systems on some three dimensional manifolds.

Proposition 9.6. *Let $f : M \rightarrow M$ be a partially hyperbolic diffeomorphism with $\dim M = 3$. If either $E^s \oplus E^u$, E^{cs} or E^{cu} is integrable then M is a $K(\pi_1(M), 1)$ manifold and thus its universal covering is contractible.*

Using all this facts it is proven:

Theorem 9.7. [BrBuIv] *Let $f : M \rightarrow M$ be a partially hyperbolic diffeomorphism with $\dim M = 3$. If either $E^s \oplus E^u$, E^{cs} or E^{cu} is integrable and $\pi_1(M)$ is abelian then the action of f on its first homology group is partially hyperbolic.*

The idea is the following, take a point $x_0 \in \tilde{M}$, the universal covering and let $\Gamma = \pi_1(M)$ acts on \tilde{M} by the deck transformations. Then $\gamma : \Gamma \rightarrow \tilde{M}$, $\gamma(L) = L(x_0)$ is a quasi isometry, that is, there is $C > 0$, independent of x_0 such that $\frac{1}{C}d(\gamma(L_1), \gamma(L_2)) - C \leq d_\Gamma(L_1, L_2) \leq Cd(\gamma(L_1), \gamma(L_2)) + C$ for every L_1 and L_2 , where $d_\Gamma(L_1, L_2)$ is the word length. Then, if $\pi_1(M)$ is nilpotent, it can be proven that there are $L_1, L_2 \in \Gamma$ such that $d_\Gamma(f_*^n(L_1), f_*^n(L_2)) \geq \sigma^n$ for some $\sigma > 1$ for every $n \geq n_0$.

It would be interesting to go to higher dimensions, for example,

PROBLEM 56. *If f is a partially hyperbolic diffeomorphism on \mathbb{T}^n , $n \geq 3$, is it homotopic to a partially hyperbolic automorphism?*

Also here is quite evident the importance of the integrability to the classification problem. Let us finish with other type of problem.

PROBLEM 57. Let $E \subset TM$ be a plane field on a three dimensional manifold. Assume that E integrates to a lamination on a closed set $\Lambda \subset M$. Is it true Novikov's theorem in this setting?, if η is a homotopically trivial closed curve transversal to E , and $\eta \cap \Lambda \neq \emptyset$, does Λ contains a torus tangent to E ?

In [RHRHUr3] we recall all this results to find the ergodic partially hyperbolic diffeomorphisms on dimension 3.

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