On the work of Jorge Lewowicz on expansive systems

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La valeur de l'homme réside dans le meilleur de ses actes (The value of a person resides in his major contribution).

Photo of the entrance of the Institute du Monde Arab in Paris, France.

Expansive homeomorphisms

Definition (Expansive homeomorphism)

M compact metric space. $f : M \to M$ homeomorphism is *expansive* if $\exists\alpha$ such that

$$d(f^n(x), f^n(y)) \leq \alpha \quad \forall n \in \mathbb{Z} \Rightarrow x = y$$

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Theorem (Lewowicz)

The sphere S^2 does not admit expansive homeomorphisms.

$$S^2 = \{x \in \mathbb{R}^3 \ : \ \|x\| = 1\}$$

Structure of the talk

- Lyapunov functions.
- Persistence.
- Statement and sketch of the proof of classification theorem.

Lyapunov functions

Definition (Lyapunov function)

$V : U \subset M \times M \to \mathbb{R}$ (U neighborhood of diagonal) is a *Lyapunov function* if:

- $V(f(x), f(y)) - V(x, y) > 0$ for $x \neq y$
- $V(x, x) = 0$ for every x.

Theorem (Lewowicz)

A homeomorphism $f : M \to M$ is expansive if and only if it admits a Lyapunov function.

Lyapunov functions and topological stability

Quote from Lewowicz, J.Diff.Equations (1980): "*This paper contains some results on topological stability (see [2,3]) that generalize those obtained in [2] much in the same way as Lyapunov's direct theorem generalizes the asymptotic stability results of the hyperbolic case: if at a critical point, the linear part of a vector field has proper values with negative real parts, the point is asymptotically stable and the vector field has a quadratic Lyapunov function; however, asymptotic stability may also be proved for vector fields with non-hyperbolic linear approximations, provided they have a Lyapunov function. In a way this is what we do here, letting Anosov diffeomorphisms play the role of the hyperbolic critical point and replacing stability by topological stability; we get this time a class of topological stable diffeomorphisms wider than the class of Anosov diffeomorphisms.*"

Definition (Topological stability)

A homeomorphism $f : M \to M$ is *topologically stable* if there exists a C^0-neighborhood $\mathcal{U}$ of f such that for $g \in \mathcal{U}$ there exists a continuous and surjective $h : M \to M$ such that:

$$h \circ g = f \circ h$$

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Theorem (Lewowicz)

If f admits a non-degenerate Lyapunov function, then f is topologically stable.

Lewowicz's example:

$$F_c(x, y) = \left(2x - \frac{c}{2\pi}\sin(2\pi x) + y, x - \frac{c}{2\pi}\sin(2\pi x) + y\right)$$

Other related contributions

- New proofs of stability of Anosov and other classical results.
- Quadratic forms (we will return to this).
- (with Tolosa) Expansive homeomorphisms in the C^0-boundary of (codimension one) Anosov are conjugated.
- Coexistence in F_c for $c > 1$????

Question (Lewowicz)

Does the C^1-closure of Anosov diffeomorphisms contains expansive ones?

Persistence

Definition (Persistence)

$f : M \to M$ is *persistent* if for every $\varepsilon > 0$ $\exists$ a C^0-neighborhood $\mathcal{U}$ of f such that for every $g \in \mathcal{U}$ and $x \in M$ there exists $y \in M$ such that

$$d(f^n(x), g^n(y)) \leq \varepsilon \qquad \forall n \in \mathbb{Z}$$

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Quoting Lewowicz in ETDS (1983)"*(...)roughly, the dynamics of f may be found in each g close to f in the C^0-topology; however, these g may present dynamical features with no counterpart in f.*"

Question (Lewowicz)

Are all expansive homeomorphisms persistent (or semi-persistent)?

Selected Contributions

- Persistence of several types of expansive homeomorphisms. Including pseudo-Anosov maps. This was extended afterwards by Handel to the whole isotopy class.

- (with Lima de Sa) Analytic models of pseudo-Anosov maps.

- To prove that those models are non-uniformly hyperbolic (and in fact Bernoulli) he introduced *infinitesimal Lyapunov functions* or *quadratic forms*. Related to cone fields of Wojtkowski, then used and improved by Markarian and Katok-Burns.

Relationship between tangent bundle dynamics and underlying dynamics: Does an Anosov homeomorphism admit a smooth model? If yes, is it possible to construct a smooth model which is Anosov?

Expansive homeomorphisms of surfaces

Theorem (Lewowicz Bull.Bras. Mat. Soc (1989))

Let $f : S \to S$ be an expansive homeomorphism of a (compact orientable) surface S. Then, $S \neq S^2$ and:

- *If $S = \mathbb{T}^2$ then f is conjugated to Anosov.*
- *If $S = S_g$ then f is conjugated to pseudo-Anosov.*

Proved independently by Hiraide (Osaka J. of Math (1990)).

Stable and unstable sets

$$S_\varepsilon(x) = \{y \in S \ : \ d(f^n(x), f^n(y)) \leq \varepsilon \ \forall n \geq 0\}$$

$$U_\varepsilon(x) = \{y \in S \ : \ d(f^{-n}(x), f^{-n}(y)) \leq \varepsilon \ \forall n \geq 0\}$$

Reformulation of expansivity: $\exists \alpha > 0$ such that $\forall x \in M$

$$S_\alpha(x) \cap U_\alpha(x) = \{x\}$$

No stable points

Theorem (Lewowicz 1983)

If M is compact connected and locally connected and $f : M \to M$ is expansive, then $S_\varepsilon(x)$ has empty interior for every $x \in M$.

Quotation from Lewowicz Bull.Braz.Mat.Soc (1989): "*(...)expansivity means, from the topological point of view, that any point of the space M has a distinctive dynamical behavior. Therefore, a stronger interaction between the topology of M and the dynamics could be expected*".

Consequence of non-existence of stable points

Proposition

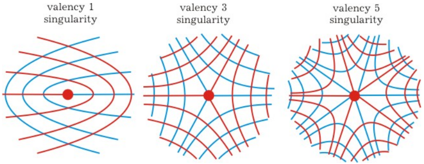
The sets $S_\varepsilon(x)$ and $U_\varepsilon(x)$ have connected components containing x with diameter bounded from bellow.

As a consequence: No expansive homeomorphisms on S^1.

Expansive homeomorphisms of surface: Local Classification

Theorem (Lewowicz)

The stable and unstable sets form transverse singular foliation of S with (finite) singularities of prong type and at least 3-legs.



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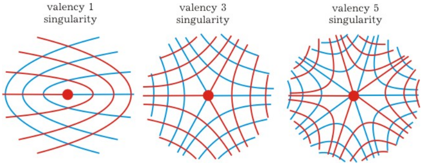

$\Rightarrow$ S^2 does not admit expansive homeomorphisms (an index argument).

Idea of the proof

Blackboard.

Thanks Jorge

(...) (El profesor de matemática) tiene que sentir un placer inmenso por lo que enseña, él tiene que estar seducido por la matemática (...) Estudiar matemática es igual que estudiar violín. Quien aprende a tocar ese instrumento se pasa años escuchando ruidos que ni siquiera los vecinos pueden tolerar. Los alumnos en el liceo estudian esos cinco años de sufrimiento, al concierto no lo llegan a conocer. Divulgar más la matemática, hacerla llegar y disfrutar como un concierto, ese es un desafío.

Translation: "(A math professor) has to feel an immense pleasure for what he teaches, he must be seduced by math (...) Learning math is like studying violin: Who learns to play the instrument must support years listening to noises which not even the neighbours can tolerate. Students in high-school have 5 years of suffering, they don't reach the concert. To divulgate math, make people enjoy it like a concert, that is a challenge" From an Interview by Diario "La República" December 2005.