Expansive and topologically stable geodesic flows: from dynamics to global geometry and rigidity

Rafael Oswaldo Ruggiero

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Introduction

The works of Jorge Lewowicz about expansive homeomorphisms of compact surfaces had remarkable impact in the theory of geodesic flows without conjugate points. We shall make a survey of results about expansive and weakly stable geodesic flows in compact manifolds without conjugate points, starting from Lewowiczs results and views about expansive dynamics in low dimensions.

Geodesic flows

Let (M, g) C^∞ be a compact n-dimensional Riemannian manifold, $\tilde{M}$ universal covering of M, $p : \tilde{M} \longrightarrow M$ covering projection, $(\tilde{M}, \tilde{g})$ the pullback of g by the map p, TM the tangent bundle of M, T_1M the unit tangent bundle, $\pi : TM \longrightarrow M$ the canonical projection $\pi(p, v) = p$.

The geodesic flow of M is $\phi_t : T_1M \longrightarrow T_1M$, $\phi_t(p, v) = (\gamma_{(p,v)}(t), \gamma'_{(p,v)}(t))$, where $\gamma_\theta(t)$ is the geodesic having initial conditions $\gamma'_\theta(0) = v$, $\gamma_\theta(0) = p$. All the geodesics will be parametrized by arc length.

Let $T_\theta T_1 M = H_\theta \oplus V_\theta \oplus X(\theta)$ be the horizontal-vertical splitting of $T_\theta T_1 M$ (orthogonal in the Sasaki metric) where $X(\theta)$ is the geodesic vector field. The subspace $N_\theta = H_\theta \oplus V_\theta$ is invariant by the differential of the geodesic flow, whose action in N_θ is given by

$$D_\theta \phi_t(W) = D_\theta \phi_t(W_H, W_V) = (J_W(t), J'_W(t))$$

where $J_W(t)$ is a Jacobi field of the geodesic γ_θ defined by the initial conditions

$$J_W(0) = W_H,\ J'_W(0) = W_V.$$

Perpendicular Jacobi fields are solutions of the differential equation $J''(t) + K_\theta(t)J(t) = 0$, where $K_\theta(t)$ is the matrix of sectional curvatures of planes containing $\gamma'_\theta(t)$. The Liouville 1-form will be denoted by α, $d\alpha$ is invariant and symplectic in $T(TM)$.

Expansive flow

A non-singular smooth flow $\phi_t : \Sigma \longrightarrow \Sigma$ acting on a complete Riemannian manifold Σ is ϵ-expansive if given $x \in \Sigma$ we have that for each $y \in \Sigma$ such that there exists a continuous surjective function $\rho : \mathbb{R} \longrightarrow \mathbb{R}$ with $\rho(0) = 0$ satisfying

$$d(\phi_t(x), \phi_{\rho(t)}(y)) \leq \epsilon,$$

for every $t \in \mathbb{R}$ then y is actually in the orbit of x. A smooth non-singular flow is called expansive if it is expansive for some $\epsilon > 0$.

Topological stability

A non-singular smooth flow $\phi_t : \Sigma \longrightarrow \Sigma$ acting on a complete Riemannian manifold Σ is C^k topologically stable if there exists an open neighborhood U of ϕ_t in the C^k topology such that for each flow ψ_t in the neighborhood there exists a continuous surjective map $h : \Sigma \longrightarrow \Sigma$ such that for every $x \in \Sigma$ there exists $r : \mathbb{R} \longrightarrow \mathbb{R}$ continuous and surjective, $r(0) = 0$, with

$$h(\psi_t(x)) = \phi_{r(t)}(h(x)).$$

When h is a homeomorphism the flow is C^k structurally stable.

Stability in dynamics

Main subjects:

1. Lyapunov and topological stability.
2. Models (classification) of expansive and topologically stable dynamics in surfaces.
3. Persistent dynamics.

Main tools:

1. Analytic: Lyapunov quadratic forms and functions.
2. Topological: "Hyperbolic" topological dynamics of expansive and topologically stable systems (invariant set theory for expansive systems, shadowing property).

Lyapunov forms

Given a C^∞ manifold Σ and a smooth, non-singular vector field Y in $T\Sigma$, a C^k Lyapunov quadratic form $Q : T\Sigma \times T\Sigma \longrightarrow \mathbb{R}$ for the flow of Y is given by the following properties:

1. Q is C^k.
2. The Lie derivative $\mathcal{L}_Y Q$ of the form Q is positive.

Lyapunov quadratic forms are useful to find invariant cones of the dynamics of the differential of the flow of Y.

Lyapunov functions

Given a smooth manifold Σ and a neighborhood U of the diagonal of $\Sigma \times \Sigma$, A C^k Lyapunov function of two variables $f : U \longrightarrow \mathbb{R}$ for the flow of Y is a C^k non-negative function such that

1. $f(x,x) = 0$ for every $x \in \Sigma$,
2. The derivative of $f(x,y)$ with respect to the flow is positive for every $(x,y) \in \Sigma$.

Lyapunov functions can be obtained "integrating" Lyapunov forms. The so-called non-degenerate Lyapunov functions imply topological stability. Every expansive homeomorphism has a Lyapunov function (Lewowicz), not necessarily non-degenerate.

Hyperbolic dynamics and Lyapunov forms and functions

Theorem: Let Σ be compact, then the flow Y is Anosov if and only if there exists a non-degenerate Lyapunov quadratic form.

For diffeomorphisms, J. Lewowicz: Lyapunov functions and topological stability, J. of Diff. Equations (38), 1980. For flows:

1. Lewowicz, J.:Invariant manifolds for regular points. Pacific J. of Math. 1981.
2. Lewowicz, J., Lima de Sá, E., Tolosa, J. : Lyapunov functions of two variables and a conjugacy theorem for dynamical systems. Acta Científica Venezolana, 1981.
3. Lewowicz, J., Tolosa, J.: Local conjugacy of quasihyperbolic systems. Diff. equations, Qualitative theory I, II, 1984.

For geodesic flows

Theorem (J. Lewowicz): Let (M, g) be a compact surface with non-positive curvature. The geodesic flow is Anosov if and only if there exists an invariant sub-bundle W of $T(T_1M)$, and a non-degenerate quadratic form $Q : W \times W \longrightarrow \mathbb{R}$ such that the Lie derivative is positive definite. If the geodesic flow is not Anosov but the interior of the set of points with zero curvature is empty, then the flow has a Lyapunov function of two variables and is topologically stable.

This result is published in the article "Lyapunov functions and stability of geodesic flows", Geometric Dynamics (Rio de Janeiro) Lecture Notes in Math. 1007, 1983.

A closer look at the paper

Let α be the Liouville 1-form dual to the geodesic flow, ω the connection 1-form (dual to the vertical bundle in $T_1 M$), $\omega^{\perp}$ the 1-form such that $\alpha, \omega, \omega^{\perp}$ are the Cartan forms. Then:

1. If the curvature is negative the Lie derivative of the two form $Q(v,v) = \omega^{\perp} \times \omega$ is is positive (Cartan's formulae).
2. If the geodesic flow is Anosov there exists $T > 0$ such that the Lie derivative of $\bar{Q}(v,v) = \int_0^T \phi_t^* Q(v,v) dt$ is positive restricted to the unstable sub-bundle.

Remark: The characterization of Anosov by the form $\bar{Q}$ and the topological stability of expansive geodesic flows in manifolds without conjugate points extend to any dimension, regardless of the sign of the curvature.

Expansive homeos and Hyperbolic topological dynamics

Theorem: Expansive homeomorphisms of compact surfaces have local invariant sets with "product structure" in all but a finite number of periodic orbits, and are conjugate to pseudo-Anosov maps. In particular, there are no expansive homeomorphisms in the two sphere.

This result was proved by J. Lewowicz (Expansive homeomorphisms of surfaces, Bol. Soc. Bras. Math. 1989). Result obtained independently by Hiraide (Expansive homeomorphisms of compact surfaces are pseudo-Anosov, Osaka Math. Journal. 1990).
In my opinion, the most important work of J. Lewowicz. We shall comment its impact in the theory of geodesic flows.

Theorem: (R.–: Persistently expansive geodesic flows, Comm. Math. Physics, 1991)
Let (M, g) be a compact Riemannian manifold. If the geodesic flow is C^1 persistently expansive then the closure Ω of the set of periodic orbits is a hyperbolic set. If M is a surface, the geodesic flow is Anosov.

Proof:
First step.
C^1 persistent expansiveness
$\Rightarrow$ existence of a continuous Lagrangian invariant dominated splitting in Ω
$\Rightarrow$ hyperbolic splitting.

Second step.
If dim $(M) = 2$, dim $(T_1 M) = 3$, and product structure extends to expansive flows without singularities (M. Paternain, PhD thesis, Inaba-Matsumoto, Japan J. Math. 1990).
The product structure of invariant sets and Poincaré 's recurrence lemma imply density of periodic orbits.

Remark: The statement can be improved by assuming C^1 persistent expansiveness in the set of conformal perturbations of the metric (Mañé's genericity after a result by Rifford-R.– IMRN 2011.)

Theorem: (M. Paternain, Ergodic theory and dyn, sys. 1994)
If the geodesic flow of a compact Riemannian surface is expansive then the surface has no conjugate points.

Three-dimensional extensions of Lewowicz results about local product structure for expansive homeomorphisms play a crucial role in the proof.
There is a Hamiltonian version of the theorem by G. Paternain and M. Paternain (Comptes Rendu de l' Academie de Sciences, Paris, 1993). The n-dimensional version of the theorem is still an open problem for $n > 2$.

Local product structure and stability

Theorem: (R.–, Ergodic theory and dyn. sys. 1996, 1997)
Let (M, g) be a compact Riemannian manifold without conjugate points. If the geodesic flow is expansive. Then,

1. The flow has a local product structure, the pseudo-orbit tracing property, it is topologically stable, transitive and periodic orbits are dense.
2. There exists a C^1 neighborhood of the flow such that any expansive geodesic flow in the neighborhood with the same expansivity constant has no conjugate points.
3. If the geodesic flow of (M, g) is C^1-persistently expansive (in the above sense) then it is Anosov.

Coarse Hyperbolic geometry of the universal covering

Theorem: (R.–, Bol. Soc. Bras. Mat. 1994)
The fundamental group of a compact Riemannian manifold without conjugate points and expansive geodesic flow is Gromov hyperbolic.

By the solution of the Poincaré conjecture, a "prime" compact manifold admitting a Riemannian metric without conjugate points and expansive geodesic flow admits a hyperbolic metric.
M. Paternain proved for expansive geodesic flows of compact surfaces that the growth of the fundamental group is exponential (Bol. Soc. Bras. Mat. 1993).

Surfaces with non-positive curvature and finite area ideal triangles

Theorem: (With G. Contreras, Bol. Soc. Bras. Mat. 1997)
There exist expansive, non-Anosov, C^2 geodesic flows in compact surfaces with non-positive curvature with the following property: there exists a constant $C > 0$ such that every geodesic ideal triangle in the universal covering has area bounded above by C.

Theorem: (R.–. Bol. Soc. Bras. Mat. 1997)
If the geodesic flow of a compact surface (M, g) is C^3 and every geodesic ideal triangle has finite area (in the above sense) then the geodesic flow is Anosov.

Accessibility

Theorem: (R.–. Ergodic theory and dyn. sys. 2008)
Expansive geodesic flows in compact manifolds without conjugate points have the accessibility property (à la Brin).

Namely, for each point θ in the unit tangent bundle there exists an open neighborhood $V(\theta)$ such that every point $z \in V(\theta)$ can be joined to θ by a continuous path formed by (up to 4) continuous curves which are either in a stable leaf or in an unstable leaf (a sort of continuous version of non-joint-integrability of stable and unstable foliations of Anosov geodesic flows).

Rigidity in Finsler surfaces

Theorem: (With J. Barbosa Gomes, Houston J. of Math. 2011) Every k-basic Finsler metric in a compact surface with expansive geodesic flow is Riemannian.

A C^k **Finsler metric** in a smooth manifold M is a function $F : TM \longrightarrow [0, +\infty)$ such that

- $F(p, tv) = |t|F(p, v)$ for every $t \in \mathbb{R}$, and $(p, v) \in TM$.
- F is C^k in $TM - (M, 0)$.
- The Hessian of F^2 in the vertical variables is positive definite.

A Finsler metric is k-basic when the flag curvature $K(p, v)$ does not depend on the vertical variable v ((p, v) are canonical coordinates in the tangent bundle).

C-shadowing property

A flow $\phi_t : \Sigma \longrightarrow \Sigma$ acting on a complete Riemannian manifold Σ satisfies the C^k-C-shadowing property if there exists a C^k neighborhood of the flow such that the orbits of each flow ψ_t in the neighborhood can be C-shadowed by orbits of ϕ_t.
Namely, given $x \in \Sigma$, there exist $y \in \Sigma$, and a continuous surjective function $\rho_x : \mathbb{R} \longrightarrow \mathbb{R}$ with $\rho_x(0) = 0$ such that

$$d(\phi_t(y), \psi_{\rho_x(t)}(x)) \leq C,$$

for every $t \in \mathbb{R}$.

The flow ϕ_t satisfies the **lifted** C^k-C-shadowing property if the lift in $\tilde{\Sigma}$ of each orbit of the lifted C^k-close flow $\tilde{\psi}_t$ can be C-shadowed by an orbit of the lifted flow $\tilde{\phi}_t$ in the above sense.

The C-shadowing property implies the lifted C-shadowing property if C is small enough.

Shadowing and Preissmann's property

Theorem: (R.–. Ergodic Theory and dyn. sys. 1999, 2000, Bol. Soc. Bras. Mat. 1999, DCDS 2006)
Let (M, g) be a compact manifold without conjugate points,let $r(M)$ be the injectivity radius. Then we have:

1) If (M, g) has non-positive curvature, the lifted C^{∞}-C-shadowing property for the geodesic flow implies that every abelian subgroup of the fundamental group is infinite cyclic.

2) If (M, g) has non-positive curvature and is analytic, then the lifted C^{∞}-C-shadowing property implies that the fundamental group is Gromov hyperbolic.

3) If $(\tilde{M}, \tilde{g})$ is a quasi-convex space, and $C \leq \frac{1}{5} r(M)$, the C^{∞}-C-shadowing property implies that every abelian subgroup of the fundamental group is infinite cyclic.

Question (conjecture?): Does the lifted C-shadowing property imply Gromov hyperbolicity?

Clue: Gromov hyperbolic spaces are characterized by the shadowing of quasi-geodesics by geodesics. However, the C-shadowing is much weaker than the shadowing of quasi-geodesics.

Ricci flow

Lemma: Compact surfaces of non-positive curvature are in the closure of surfaces of negative curvature.

Proof: Maximum principle for parabolic equations applied to the Ricci flow.

Theorem: (With D. Jane) Compact surfaces without focal points such that the region of positive curvature consists of a finite number of "isolated" bubbles are in the closure of Anosov metrics.

Proof: Ricci-Yang-Mills flow and comparison theory for Jacobi fields.

The article by J. Lewowicz and R. Ures, On Smale diffeomorphisms close to pseudo-Anosov maps, Comput. Appl. Math. 2001, considers quotient spaces of maps which are close to pseudo-Anosov as expansive models of such systems up to semi-conjugacy. Quotients have positive Hausdorff dimension. (Mañé: positive topological dimension of compact metric spaces with an expansive homeo.) and the quotient dynamics inherits the topology of the initial dynamics (namely, isotopy to a Smale diffeo. implies transitivity, density of periodic orbits, etc, of the quotient dynamics).

Work in progress with A. De Carvalho

If (M, g) is a compact surface with non-positive curvature, the quotient Σ of T_1M by flats is a 3-dimensional manifold and the quotient $\bar{\phi}_t : \Sigma \longrightarrow \Sigma$ of the geodesic flow ϕ_t is semi-conjugate to ϕ_t by a time-preserving map.

Existence of non-time-preserving semi-conjugacies is well known (M. Gromov, E. Ghys).

AND IT WILL CONTINUE ...

THANK YOU VERY MUCH JORGE