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Quantum suppression of chaos in the Fermi accelerator

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Abstract

We address the question of how a quantum particle exchanges energy with a moving wall with an imposed smooth, periodic motion law. In the semiclassical region of parameter space, the energy gain of the particle is initially consistent with the classical result. This correspondence is maintained for some time, until quantum localization effects suppress the classical diffusion. We discuss similarities with the well studied case of the Quantum Kicked Rotor. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

Over the last years we have witnessed an impressive advance in the field of dynamical systems. This progress has not been matched by a corresponding understanding at the quantum mechanical level. This fact has motivated a renewed interest in the study of quantum dynamics in the semiclassical region. In particular, the correspondence that one would expect in this region between classical phase-space trajectories and quantal wave functions is far from being clear when the system is not integrable. Much insight into the classical-quantum correspondence can be gained from the study of simple model systems. For instance, the striking phenomena of quantum suppression of classical diffusion (quantum localization) was first found in the time evolution of the

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energy of a Kicked Rotator (KR) [1]. This system is described by a one-dimensional, impulsive, time periodic Hamiltonian. It has recently been shown [2] that other one-dimensional, impulsive, time periodic systems, such as the Quantum Bouncer [3] or the Fermi accelerator with discontinuous velocity of the moving wall can be formally reduced to a generalized Kicked Rotor form after a suitable transformation is applied. The time evolution of such quantum systems and their classical chaotic counterparts are different, even in the semiclassical region, due to the quantum suppression of classical diffusion resulting from localization of the wave function in energy space. Quantum localization has been understood [4,5] in terms of a formal analogy with the Anderson theory for disordered one-dimensional lattices [6]. The impulsive term in the Hamiltonian plays a central role in this analogy.

In this paper, we investigate numerically the time evolution of the energy observable for the Fermi accelerator with a non-impulsive, time periodic Hamiltonian. The quantum evolution is recurrent even at resonant points in parameter space, as expected for systems characterized by a discrete quasienergy spectrum. In the semiclassical region of parameter space, we find a quantum localization effect similar to the one observed in the KR. First, we briefly review the diffusion in phase space that occurs at low energies in the classical Fermi accelerator. We then discuss the quantum version of the problem and present the numerical results for the time evolution of the energy of the particle. Finally we discuss the nature of the quantum localization encountered and compare it to the Anderson localization mechanism observed in the Kicked Rotor.

2. Dynamical evolution

The Fermi accelerator consists of a particle confined in an infinite well with one periodically moving wall. The motion of the wall can be parametrized as $L(t) = L_0 [1 + \delta f(t)]$, where δ is the dimensionless amplitude of the wall motion and $f(t)$ is an analytic, periodic function of time scaled so that $|f(t)| \leqslant 1$. In the following, we assume that $f(t) = \sin(\omega t)$.

The classical Fermi accelerator has been thoroughly studied [7], since Ulam's pioneering work on the subject [8]. Let m be the mass of the particle, L_0 be the average width of the well and T be the period of the function $f(t)$. We choose units such that $m = 1$, $L_0 = 1$ and $T = 1$. The reduced phase-space variables are the particle's momentum after the nth collision with the wall, p_n, and the phase of the wall at that instant, $\theta_n = 2\pi t_n$. If elastic collisions with the wall are assumed and the displacement of the wall is neglected, the simplified Ulam map is obtained

$$\begin{aligned} p_{n+1} &= |p_n - 2\pi\delta\cos(\theta_n)|, \\ \theta_{n+1} &= \theta_n + \frac{4\pi}{p_{n+1}}, \qquad \bmod 2\pi. \end{aligned} \tag{1}$$

This map relates the phase-space variables after two consecutive collisions with the moving wall. For low velocities, the phase change of the wall between successive col-

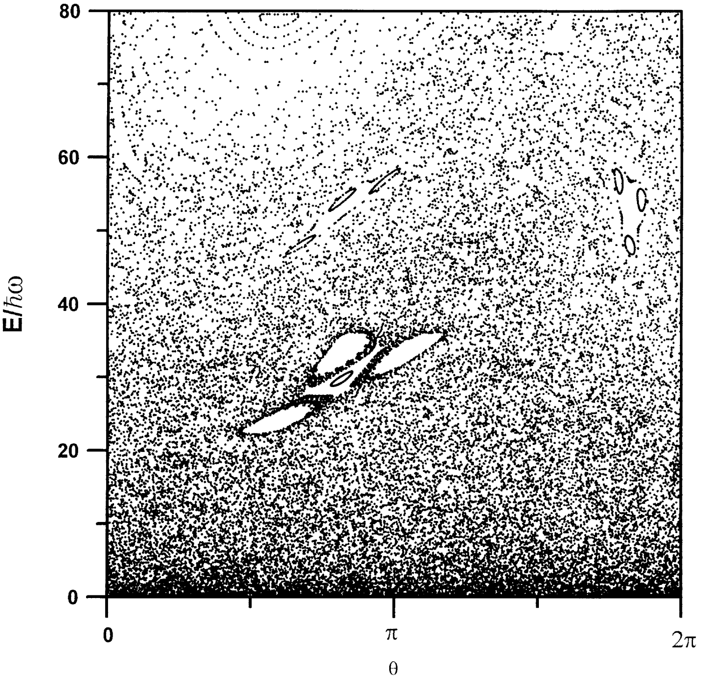

Fig. 1. Low-energy region of the phase space for Ulam's map (Eq. (1)). The energy of the particle after the nth collision is plotted vs. the phase of the moving wall for 20 000 collision events with several initial conditions. The energy is normalized by $\hbar\omega$ in order to compare with the quantum mechanical results. The parameters are $\varepsilon_0 = 0.0021$ and $\delta = 0.020$ (this corresponds to $\nu \approx 10$). The low-energy region is fully chaotic. Surviving KAM orbits emerge for energies above $20\hbar\omega$.

lisions is large compared with 2π and successive values of θ_n become uncorrelated. As a result, the particle executes a random walk in momentum space. The low energy region of phase space is shown in Fig. 1. In this region, an ensemble of initial conditions will spread out with a linear diffusion coefficient $D \equiv \frac{1}{2}\left\langle \Delta p^2 \right\rangle = \pi^2\delta^2$ and as a consequence the average kinetic energy will grow linearly in time with slope D. At higher energies, the existence of surviving KAM orbits will limit the energy growth, at least for analytic functions $L(t)$. At low energies, the situation is similar to that of the Kicked Rotor (represented by the Standard Map) which for kick strengths above a certain threshold presents a completely chaotic region in phase space in which diffusion takes place [9].

The quantum evolution is governed by the time periodic Hamiltonian operator $\mathbf{H} = -(\hbar^2/2)\partial^2/\partial x^2, +\mathbf{V}(x,t)$, where $\mathbf{V}(x,t) = 0$ if $x \in [0, L(t)]$ and $\mathbf{V}(x,t) = \infty$ otherwise. No exact solution is known for this problem, except for the trivial unperturbed case $\delta = 0$. The problem can be restated in terms of time-independent boundary conditions, by introducing an effective Hamiltonian which is the sum of an integrable term plus a time-dependent potential proportional to the perturbation strength δ [10]. As mentioned before, the classical Fermi accelerator is strongly chaotic in the low-velocity limit and thus it is an example of a non-adiabatic system. At the quantum level, except for particular regions of parameter space, the adiabatic approximation leads to a divergent series expansion of the evolution operator, as three of us have discussed in greater detail elsewhere [11]. For impulsive systems, such as the Kicked Rotor, the evolution operator can be obtained by direct time integration of the Hamiltonian and a quantum map can be generated. However, in the important case of analytic, periodic forcing functions $L(t)$, the evolution operator has not been obtained in closed form. In fact, the task of obtaining a closed expression for the evolution operator for the important case of sinusoidal wall motion has proved to be a very elusive one [10,12]. In order to obtain some information on this problem, one must resort to numerical methods.

The time evolution of the wave function $\Psi(x,t)$ is governed by the Schrödinger equation $i\hbar(\partial/\partial t)\Psi(x,t) = \mathbf{H}\Psi(x,t)$. We choose the instantaneous basis, $\Phi_n(x,t) = \sqrt{2/L(t)}\,\sin(n\pi x/L(t))$ which satisfies the time-dependent boundary conditions $\Phi_n(0,t) = \Phi_n(L(t),t) = 0$ of the problem. The wave function is expanded as

$$\Psi(x,t) = \sum_{n=1}^{\infty} a_n(t) e^{-i\varepsilon_0 n^2 \tau(t)} \Phi_n(x,t)\,.$$

We introduced the dimensionless scale parameter $\varepsilon_0 \equiv \hbar\pi^2/2m\omega L_0^2 = (\pi/4)\hbar$ (the unperturbed ground-state energy of the particle in units of $\hbar\omega$) and the Liouville time $\tau(t) = \int_0^t ds/L^2(s)$. In this parametrization, the classical limit corresponds to $\varepsilon_0 \to 0$ and the semiclassical region to the case $\varepsilon_0 \ll 1$. After substitution of this expansion in the Schrödinger equation, the coefficients $a_n(t)$ satisfy

$$a_n = \frac{2L}{L} \sum_{m \neq n} \mu_{mn} a_m(t) e^{-i\varepsilon_0 \left(m^2 - n^2\right)\tau(t)}, \tag{2}$$

where the coupling elements $\mu_{mn} \equiv (-1)^{m-n} mn/(m^2 - n^2)$ are defined for $m \neq n$. This evolution equation was integrated numerically, for given initial conditions, using a standard fourth order Runge–Kutta algorithm. The normalized average energy of the particle is given by

$$\varepsilon(t) = \frac{1}{\hbar\omega}\left\langle \Psi \left| -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} \right| \Psi \right\rangle = \frac{\varepsilon_0}{L^2(t)} \sum_{n=1}^{\infty} n^2 |a_n(t)|^2 .$$

It is convenient to describe the results in terms of the coupling strength parameter $v \equiv \delta/\varepsilon_0$. The system presents three qualitatively different time evolutions according to the value of v:

1. A perturbative regime for $v \ll 1$, in which the energy oscillates about its initial value with the period of the wall and an amplitude of order δ^2. If initialized in the nth basis state, the evolution of the energy is essentially given by $\varepsilon_0 n^2/L^2(t)$. This corresponds to a localized wave function with small localization length.
2. A resonant regime, for special values of v, in which the energy of the particle periodically increases to several times its initial value as shown in Fig. 2. The resonance widths in parameter space can be extremely small and there are smaller

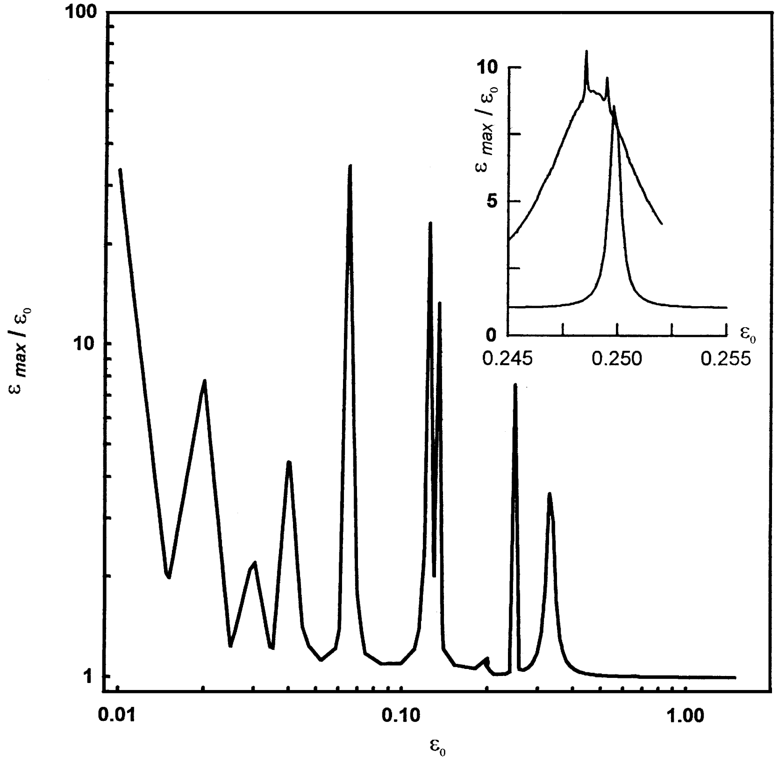

Fig. 2. Energy resonances in the quantum mechanical time evolution in a log–log scale. The maximum energy over several hundred periods of the wall oscillation (normalized to ε_0) is plotted as a function of the scale parameter ε_0. The resonances become more frequent and their characteristic widths increase as the semiclassical region $\varepsilon_0 \ll 1$ is approached. The inset shows the dependence of the resonances on the strength δ, illustrated for the case $\varepsilon_0 \approx 0.250$. For different values of the strength parameter $\delta = 0.020$ and $\delta = 0.060$, the width increases as δ^2 and smaller peaks appear in the main resonance implying some kind of fractal structure.

resonances within a given resonance, as the inset in Fig. 2 shows. The recurrence time can be orders of magnitude larger than the wall period as seen in Fig. 3. Similar quantum resonances have previously been observed in the KR model, where an analytical analysis is possible and results in a quadratic increase of the energy with time [13]. In the case of the Fermi accelerator, since an analytical form for the evolution operator is not known, a complete understanding of these resonant effects is still missing. It is clear, however, that the resonant behaviour in our model is bounded and recurrent, unlike the case of the KR. We observe that the resonant points are related to rational ratios of the particle's Bohr frequency to the moving wall frequency, that is, to rational values of ε_0. A similar (more strict) consideration applies for the Kicked Rotor where resonances occur exactly at rational values of the parameter. In this resonant regime, there is a specially strong coupling between a few channels and the probability flux os-

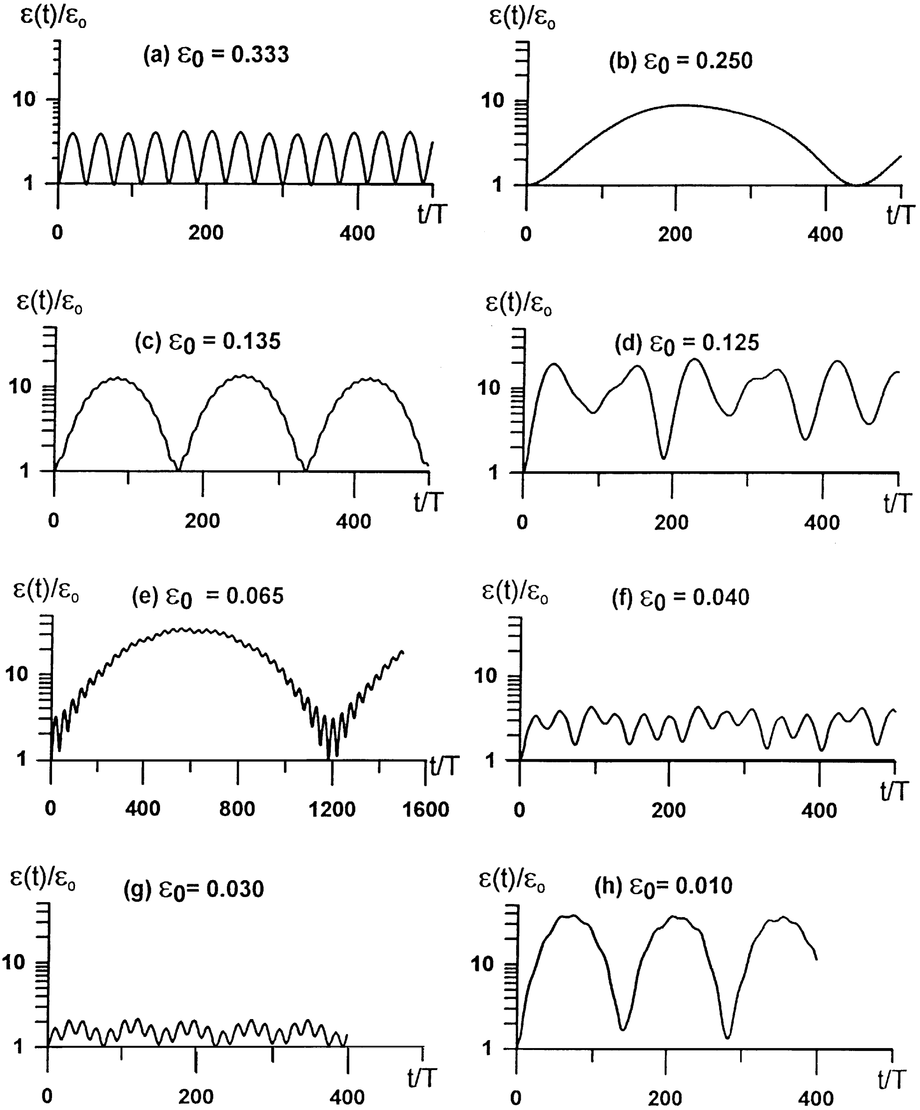

Fig. 3. The quantum mechanical evolution for the ground-state energy of the particle (normalized to ε_0) for several resonant cases. Notice the log scale in the energy axis. Time is in units of the period T of the wall and the figures are ordered sequentially in terms of increasing ν. The recurrence times can be of the order of thousands of periods as case (e) shows.

cillates between a reduced set of modes, causing the energy to oscillate correspondingly with a characteristic period. In some cases, the periodicity is almost destroyed by the complexity introduced by this resonant dynamics as is apparent in Fig. 2, cases (d) and (f). These resonances are dependent on the initial conditions.

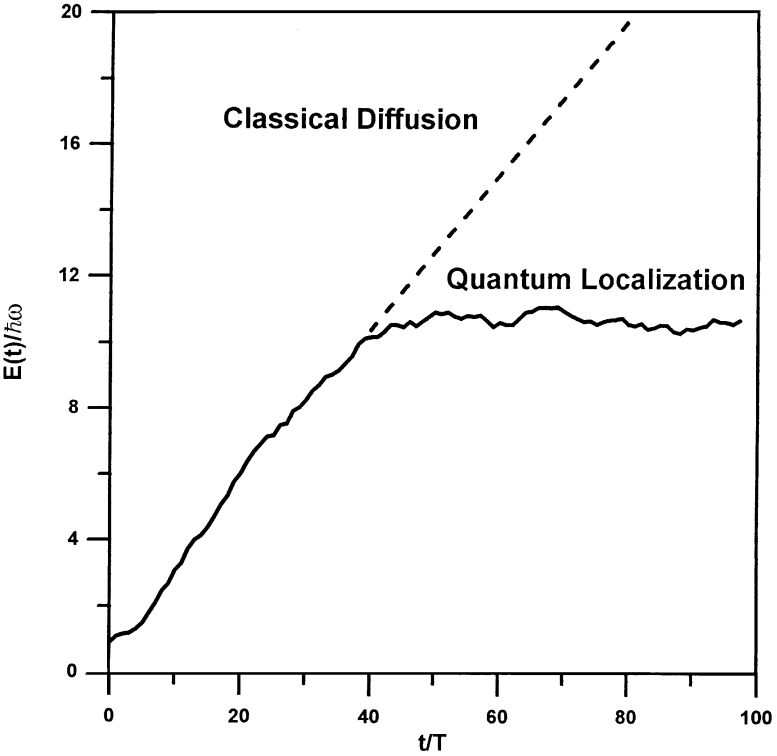

Fig. 4. Quantum mechanical evolution of the energy of a wavepacket in the semiclassical region $\nu \gg 1$. The initial state is a Gaussian wavepacket in the energy representation, centered at $n_0 = 20$ with a spread $\Delta_0 = 10$ modes. The parameters are, as in Fig. 1, $\varepsilon_0 = 0.0021$ and $\delta = 0.020$ so that $\nu = 9.524$. The dotted line shows the classical slope given by the linear approximation to the diffusion coefficient $D/\hbar\omega = \pi^2\delta^2/8\varepsilon_0$, as discussed in the text. Time is in units of the period T of the wall. The localization time is $t^* \approx 40$ in this case. It was numerically observed that $t^* \propto \nu$, so that $t^* \to \infty$ in the classical limit.

3. A semiclassical regime for $\nu \gg 1$, in which the energy increases linearly with a slope given by the classical diffusion coefficient D for some characteristic time t^*, after which the energy growth ceases due to quantum localization effects, as shown in Fig. 4. The localization time increases as ν when the classical limit ($\nu \to \infty$) is approached, thus preventing any conflict at finite times between the quantum and classical descriptions of the problem. In the classical limit, the quantum wave function becomes completely delocalized and extended over all Hilbert space. In this limit, the energy grows indefinitely and the system mimics classical diffusion for all finite times. Thus, the characteristic classical chaotic behavior emerges naturally from the quantum description in the classical limit. A similar situation is encountered in the KR, except that there the localization time increases quadratically as the classical limit is approached.

Periodically driven systems with smooth forcing functions have been shown to have a pure point quasienergy spectrum (for almost all values of the parameters) and the time evolution of the wave function is therefore recurrent [12]. The recurrence time depends on the number of states N that are significantly excited by the dynamical coupling, which in turn is very sensitive to the values of the parameters due to the resonant behavior described above. As Fig. 3 shows, a variety of recurrence times is present in a relatively small region of parameter space. It is clear however, that as the classical limit is approached and the number of excited states grows without bound, the recurrence time becomes infinite. This is consistent with the fact that, from a classical perspective, the particle wanders erratically in phase space.

The quantum localization observed is of special interest. As mentioned in the introduction, a similar localization effect has been obtained for the Kicked Rotor with a localization time and localization length that tends to infinity in the classical limit ($\hbar \rightarrow 0$). The formal analogy established for the KR with a one-dimensional, disordered Anderson lattice cannot be extended to the case of smooth, periodic forcing functions, since in this case there is no impulsive term in the Hamiltonian. Nevertheless, our numerical results imply that the dynamical coupling between the particle's wave function and the moving wall is sufficient to introduce the required decoherence in the wave function and cause it to mimic classical behaviour for some time.

The quasienergy spectrum for the Fermi accelerator shows a transition from a Poisson-like statistics to a Gaussian Orthogonal Ensemble (GOE) statistics as $\hbar \rightarrow 0$, as shown in [14]. There, it has been suggested that a Poisson statistics is associated to a localized state and a GOE statistics to delocalized state in the energy representation. We find that quantum localization effects are manifest in the semiclassical region (see Fig. 4) and (according to [14]) they coexist with a mixed (Poisson–GOE) distribution for the quasienergies.

3. Conclusions

The quantum Fermi accelerator with an analytical forcing function $L(t)$ has been numerically investigated. We have presented direct evidence of quantum suppression of classical chaos in a non-impulsive system. For special values of the parameters, we find a complex hierarchy of resonances in the time evolution of the energy observable. These energy resonances are bounded and recurrent unlike those of the KR. As the semiclassical region is approached, the dynamical evolution becomes increasingly complex and different resonances begin to overlap. The complexity of the quantum state becomes apparent in the increase of the localization length and the recurrence time as the classical limit is approached. It is a very remarkable result that the classical diffusion coefficient D emerges cleanly from this fully quantum picture. Our results for the time evolution of the average energy of the particle are similar to those obtained for the Kicked Rotor implying that strong similarities exist between these systems as other authors [2,15] have pointed out. However, we find some significant differences, as for example in the nature of the resonant behaviour. Further research is under way in order to clarify this and other important issues.

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