Series-Parallel Reductions in Monte Carlo Network Reliability Evaluation

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Summary & Conclusions — Series-parallel reductions allow to evaluate in linear time the $\mathcal{K}$*-terminal network (un)reliability* parameter for *sp-reducible* topologies. To deal with general networks, these reductions have been combined with Moskowitz' decomposition giving the well-known factoring exact methods. Since exact evaluation is an *NP*-hard problem, any exact algorithm will be prohibitive when the size of network is large. An alternative is the use of Monte Carlo simulation methods.

In this paper, we show that series-parallel reductions can be incorporated to the Recursive Variance Reduction simulation method, leading to a more efficient estimator.

Key words — Network reliability, Monte Carlo methods, variance reduction, series-parallel reductions.

Resumen y conclusiones: El uso de reducciones serie-paralelo permite evaluar en tiempo lineal el parámetro de $\mathcal{K}$*-confiabilidad* para topologías *sp-reductibles*. En el caso de redes genéricas, estas reducciones han sido combinadas con la descomposición de Moskowitz, resultando en los conocidos métodos exactos de factorización. Dado que la evaluación exacta es un problema *NP*-difícil, cualquier algoritmo exacto resultara prohibitivo cuando la red considerada es de gran tamaño. Una alternativa es el empleo de métodos de simulación Monte Carlo.

En este trabajo, mostramos que las reducciones serie-paralelo pueden ser incorporadas al método de reducción recursiva de la varianza, obteniendo un estimador más eficiente, con buenas performances computacionales, como lo ilustran resultados experimentales.

Palabras clave: Simulación, confiabilidad de redes de comunicaciones, métodos Monte Carlo, reducciones serie-paralelo.

1 Introduction

The system under study is an undirected connected communication network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{K})$ consisting of a set of nodes $\mathcal{V}$, a set of connecting links $\mathcal{E}$ and a set of terminals $\mathcal{K}$ (a fixed subset of the node-set). Nodes do not fail, but each link $l \in \mathcal{E}$ is assigned an s-independent probability of failure q_l (called link unreliability).

The success of communication between nodes in $\mathcal{K}$ is a random event, which has probability $R(\mathcal{G})$. The problem of evaluating $R(\mathcal{G})$ is usually called the $\mathcal{K}$-*terminal reliability* problem, and it has received considerable attention from the research community (see [8] and [17] for many references).

Series-parallel reductions are frequently used in the exact evaluation context [2]. They allow to deal in linear time with *sp-reducible* networks [25]. In the general case, they can be used as a precomputing task of any algorithm in order to reduce the size of the network without changing its reliability. Also, series-parallel reductions have been combined with a recursive application of Moskowitz' decomposition giving the well-known factoring methods [5], [6], [7], [9], [10], [20], [21], [25], [28]. Unfortunately, the exact evaluation of $R(\mathcal{G})$ was shown to be *NP*-hard [1], implying that no algorithm is known to exist for exactly solving this problem in time bounded by a polynomial function of the network size. This limitation leads to consider Monte Carlo alternatives, which permit to obtain an estimate and a confidence interval for the measure of interest [3], [4], [11], [12], [13], [14], [15], [16], [22], [23].

In this paper, we address the use of series-parallel reductions in the Monte Carlo simulation context, in particular in the case of the Recursive Variance Reduction (RVR) algorithm [3], which is an efficient variance-reduction Monte Carlo method for computing the $\mathcal{K}$-*terminal unreliability* parameter $Q(\mathcal{G})$ ($Q(\mathcal{G}) = 1 - R(\mathcal{G})$).

The paper is organized as follows. Series-parallel reductions and their use in exact evaluation are recalled in Section 2. Section 3 discusses how to incorporate series-parallel reductions to the RVR method, and shows some evidence as to the potential efficiency gains. Finally, Section 4 is devoted to numerical results and possible improvements.

Notation (General)

$\mathcal{G}$	$(\mathcal{V}, \mathcal{E}, \mathcal{K})$: an undirected network topology
$\mathcal{V}$	$\{1, \ldots, n\}$: the node-set of $\mathcal{G}$
$\mathcal{E}$	$\{l_1, \ldots, l_m\}$: the link-set of $\mathcal{G}$
$\mathcal{K}$	*target set* of $\mathcal{G}$, $\mathcal{K} \subseteq \mathcal{V}$; the nodes of $\mathcal{K}$ are the *terminals* of $\mathcal{G}$ [11]
m,n,k	the number of [links, nodes,terminals] of $\mathcal{G}$
$\|A\|$	cardinality of the set A
$\mathbf{1}_{(.)}$	indicator function: $\mathbf{1}_{True} = 1$, $\mathbf{1}_{False} = 0$
$\overline{E}$	complement of event E
$\overline{a}$	real $1 - a$
x_l	$\mathbf{1}_{(\text{link } l \text{ is up})}$: state of link l
r_l	$\Pr\{x_l = 1\}$: operating probability of link l
q_l	$\Pr\{x_l = 0\} = 1 - r_l$: failure probability of link l
$X_{\mathcal{G}}$	$(x_1, \ldots, x_m)$: random network-state vector
$\mathcal{G}_X$	network with node-set $\mathcal{E}$ and link-set derived from $\mathcal{E}$ by removing all failed links in X
$\Phi_{\mathcal{G}}$	structure function associated with the $\mathcal{K}$-*terminal reliability* measure : $\Phi_{\mathcal{G}}(X) = \mathbf{1}_{(\mathcal{G}_X \text{ is } \mathcal{K}\text{-connected})}$ for $X \in \{0,1\}^m$

$Y_{\mathcal{G}}$	$1-\Phi_{\mathcal{G}}(X_{\mathcal{G}})$: random state of the network $\mathcal{G}$
$R(\mathcal{G})$	$\Pr\{\mathcal{G}$ is $\mathcal{K}$-connected$\} = \mathrm{E}\{\Phi_{\mathcal{G}}(X_{\mathcal{G}})\}$: *$\mathcal{K}$-terminal reliability* of $\mathcal{G}$
$Q(\mathcal{G})$	$\Pr\{\mathcal{G}$ is not $\mathcal{K}$-connected$\} = \mathrm{E}\{Y_{\mathcal{G}}\} = 1 - R(\mathcal{G})$: *$\mathcal{K}$-terminal unreliability* parameter of $\mathcal{G}$
$\widehat{W}$	$(1/N)\sum_{i=1}^{N} W^{(i)}$: sample mean with fixed sample size N and i.i.d. $W^{(1)},\ldots,W^{(N)}$ with Cdf$\{W\}$
$\widehat{V}_W$	$(1/N(N-1))\sum_{i=1}^{N}(W^{(i)} - \mathrm{E}\{W\})^2$: unbiased estimator of $\mathrm{Var}\left\{\widehat{W}\right\}$.

Definitions & Nomenclature

- A network $\mathcal{G} = (\mathcal{V},\mathcal{E},\mathcal{K})$ is $\mathcal{K}$-connected if there is at least one path in $\mathcal{G}$ between every pair of nodes in $\mathcal{K}$ (all the nodes in $\mathcal{K}$ are connected in $\mathcal{G}$).
- A subset C of $\mathcal{E}$ is a $\mathcal{K}$-cutset of $\mathcal{G}$ if the subnetwork $\mathcal{G}' = (\mathcal{V},\mathcal{E}-C,\mathcal{K})$ of $\mathcal{G}$ is not $\mathcal{K}$-connected.
- A $\mathcal{K}$-tree of $\mathcal{G}$ is a $\mathcal{K}$-connected tree of $\mathcal{G}$ whose leaves belong to $\mathcal{K}$ [25].
- Two links l_i, l_j are in series if they are adjacent and they have only one common node with degree two and not belonging to $\mathcal{K}$.
- A series reduction consists of replacing two links in series by a single link between the non-shared extreme nodes. The operating probability of the new link is $r_{l_i}r_{l_j}$.
- Two links l_i, l_j are in parallel if they have the same extremities.
- A parallel reduction consists of replacing two parallel links by a single link between the same nodes. The operating probability of the new link is $r_{l_i} + r_{l_j} - r_{l_i}r_{l_j}$.
- A network $\mathcal{G}$ is *sp-reducible* if it is either a $\mathcal{K}$-tree or it can be reduced to a $\mathcal{K}$-tree by successive series and parallel reductions [25].
- $\tilde{\mathcal{G}}$ denotes the network resulting from applying all possible series and parallel reductions to the network $\mathcal{G}$.
- A network $\mathcal{G}$ is *complex* if it is not *sp-reducible* [25] ($\tilde{\mathcal{G}}$ is not a $\mathcal{K}$-tree).
- For a given link l in $\mathcal{G} = (\mathcal{V},\mathcal{E},\mathcal{K})$, $\mathcal{G} - l$ denotes the network with node-set $\mathcal{V}$ and link-set derived from $\mathcal{E}$ by removing link $l \in \mathcal{E}$. The target set of $\mathcal{G} - l$ is equal to $\mathcal{K}$.
- For a given link l in $\mathcal{G} = (\mathcal{V},\mathcal{E},\mathcal{K})$, $\mathcal{G} * l$ denotes the network derived from $\mathcal{G}$ by contracting link $l = \{u,v\} \in \mathcal{E}$ (eliminate l and merge its extremities u and v into a new node w). The target set of $\mathcal{G} * l$ is equal to $\mathcal{K} - \{u,v\} \cup \{w\}$ if u or v belongs to $\mathcal{K}$, and $\mathcal{K}$ otherwise.

Other, standard notation is given in "Information for Readers & Authors" at rear of each issue.

2 Series-parallel reductions and exact evaluation

In the evaluation of network (un)reliability, simplification techniques are frequently used to reduce the size of considered networks. A well known technique is the use of series–parallel reductions [2], [18] which can be performed in time linear on the network size and do not change the (un)reliability measure ($Q(\mathcal{G}) = Q(\tilde{\mathcal{G}})$) [8], [25]. In this section, we recall how series-parallel reductions can be applied in the exact evaluation context.

When the network studied is a $\mathcal{K}$-tree, its reliability $R(\mathcal{G})$ is the product of the individual link reliabilities, as all links must be operational to ensure the communication between the terminals.

If the network is *sp-reducible*, then its reliability can be computed in linear time by applying series-parallel reductions to transform it to a $\mathcal{K}$-tree.

When the network is complex, Moskowitz' decomposition [19] (based on conditioning on the state of a given link l):

$$Q(\mathcal{G}) = Q(\tilde{\mathcal{G}}) = r_l Q(\tilde{\mathcal{G}} * l) + q_l Q(\tilde{\mathcal{G}} - l) \tag{1}$$

allows to express the unreliability of $\tilde{\mathcal{G}}$ in terms of the unreliabilities of the network $\tilde{\mathcal{G}} * l$ with one node and one link fewer than $\tilde{\mathcal{G}}$ and the network $\tilde{\mathcal{G}} - l$ with one fewer link. The networks $\tilde{\mathcal{G}} * l$ and $\tilde{\mathcal{G}} - l$ may contain links in parallel and in series respectively, which can be reduced by using series-parallel reductions. Some *factoring* algorithms recursively apply these reductions and equality (1) to compute $Q(\mathcal{G})$. This recursive process can be summarized by

$$Q(\mathcal{G}) = \begin{cases} 1 & \text{if } \mathcal{G} \text{ is not } \mathcal{K}\text{–connected;} \\ 0 & \text{if } \mathcal{K} \text{ is a singleton set;} \\ 1 - \prod_{l \in \tilde{\mathcal{G}}} r_l & \text{if } \mathcal{G} \text{ is } \textit{sp-reducible}; \\ r_l Q(\tilde{\mathcal{G}} * l) + q_l Q(\tilde{\mathcal{G}} - l) & \text{otherwise, with } l \in \tilde{\mathcal{G}} \text{ an arbitrary link.} \end{cases} \tag{2}$$

The computations involved in this process can be represented by a binary computational structure. Each node of the structure corresponds to a call to the recursive procedure and each leaf corresponds to one of the following types of networks:

- *sp-reducible* network;
- $\mathcal{K}$-unconnected network;
- network with only one terminal.

The size of this binary structure depends on the strategy used to select the link for applying Moskowitz' decomposition. The optimal strategy is obtained from domination theory [7],[25].

Since exact evaluation is an *NP*-hard problem, any exact algorithm will be prohibitive when the size of network is large. An alternative is the use of Monte Carlo simulation algorithms. In the next section, we address the use of series-parallel reductions in Monte Carlo context.

3 Series-parallel reductions and simulation

An unbiased estimator $\widehat{U}$ is more accurate than another unbiased estimator $\widehat{W}$ iff the variance of $\widehat{U}$ is smaller than $\widehat{W}$ [26]. If $\widehat{U}$ and $\widehat{W}$ are sample mean estimators for a sample size N we obtain

$$\mathrm{Var}\left\{\widehat{U}\right\} = \mathrm{Var}\left\{U\right\}/N \text{ and } \mathrm{Var}\left\{\widehat{W}\right\} = \mathrm{Var}\left\{W\right\}/N.$$

Consequently, it suffices to look at the variances of r.v. U and W to compare $\widehat{U}$ and $\widehat{W}$.

The aim of variance-reduction methods is to construct random variables having same s-expectation as $Y_{\mathcal{G}} = 1 - \Phi_{\mathcal{G}}(X_{\mathcal{G}})$ and smaller variance, since a sample mean based on such a r.v. is a more accurate estimator than the crude one based on $Y_{\mathcal{G}}$. References in this topic are [3], [4], [11], [12], [13], [14], [15], [16], [22], [23].

The main objective of our work is to show that series-parallel reductions can be incorporated to the Recursive Variance Reduction Monte Carlo method [3], leading to a more efficient method. We begin by showing that the random variable $Y'_{\mathcal{G}}$ that is equal to the exact unreliability of $\mathcal{G}$ in the trivial cases and to the random network-state of $\tilde{\mathcal{G}}$ otherwise ($\tilde{\mathcal{G}}$ is the network resulting from $\mathcal{G}$ by successive series and parallel reductions) may be more interesting to use than $Y_{\mathcal{G}}$.

Proposition 3.1 *For any network $\mathcal{G}$, let $Y'_{\mathcal{G}}$ the r.v. defined by*

$$Y'_{\mathcal{G}} = \begin{cases} 1 & \text{if } \mathcal{G} \text{ is not } \mathcal{K}\text{-connected;} \\ 0 & \text{if } \mathcal{K} \text{ is a singleton set;} \\ 1 - \prod_{l \in \tilde{\mathcal{G}}} r_l & \text{if } \mathcal{G} \text{ is sp-reducible;} \\ Y_{\tilde{\mathcal{G}}} = 1 - \Phi_{\tilde{\mathcal{G}}}(X_{\tilde{\mathcal{G}}}) & \text{otherwise.} \end{cases} \tag{3}$$

Then $Y'_{\mathcal{G}}$ has same s-expectation as $Y_{\mathcal{G}}$ and smaller or equal variance.

Proof . In the first three cases, $Y'_{\mathcal{G}}$ is constant and equal to the exact unreliability of $\mathcal{G}$ that is the value of the s-expectation of $Y_{\mathcal{G}}$. Moreover, $\mathrm{Var}\left\{Y'_{\mathcal{G}}\right\} = 0$ and consequently it is smaller than the variance of $Y_{\mathcal{G}}$.

For the remaining case, since series-parallel reductions do not modify network unreliability ($Q(\mathcal{G}) = Q(\tilde{\mathcal{G}})$) [7], we obtain $\mathrm{E}\left\{Y_{\mathcal{G}}\right\} = Q(\mathcal{G}) = Q(\tilde{\mathcal{G}}) = \mathrm{E}\left\{Y'_{\mathcal{G}}\right\}$. Moreover, as $Y_{\mathcal{G}}$ and $Y_{\tilde{\mathcal{G}}}$ are Bernoulli r.v. with the same s-expectation, they have the same variance.

It results that we always have

$$\mathrm{Var}\left\{Y'_{\mathcal{G}}\right\} \leq \mathrm{Var}\left\{Y_{\mathcal{G}}\right\}$$

and

$$\mathrm{E}\left\{Y'_{\mathcal{G}}\right\} = \mathrm{E}\left\{Y_{\mathcal{G}}\right\}.$$

□

Remark 3.2 *When $\mathcal{G}$ is a complex network, the r.v. $Y'_{\mathcal{G}}$ has the same variance as $Y_{\mathcal{G}}$. Then, for a fixed sample size, the sample mean based on $Y'_{\mathcal{G}}$ has the same accuracy than the one based on $Y_{\mathcal{G}}$. The interest of $Y'_{\mathcal{G}}$ comes from the fact that $\tilde{\mathcal{G}}$ may be smaller than $\mathcal{G}$ leading to a smaller simulation time.*

In a previous paper [3], we have shown that the sample mean $\widehat{Z_{\mathcal{G}}}$ based on $Z_{\mathcal{G}}$ defined in the following proposition is more accurate than the one based on $Y_{\mathcal{G}}$.

The r.v. $Z_{\mathcal{G}}$ is expressed in terms of $|C|$ network-state random variables $Y_{\mathcal{G}_i}$, $1 \leq i \leq |C|$. Since each $\mathcal{G}_i$ is deduced from $\mathcal{G}$ by $(i-1)$ deletions and one contraction, then the opportunity for applying series and parallel reductions may appear. Based on the results of Proposition 3.1, we propose to replace each $Y_{\mathcal{G}_i}$ with the r.v. $Y'_{\mathcal{G}_i}$. In the following proposition, we show that this replacement leads to a more accurate estimator.

Proposition 3.3 *For any network $\mathcal{G}$, let*

C	$\{l_1, l_2, \ldots, l_{\lvert C\rvert}\}$: *a fixed $\mathcal{K}$-cutset in $\mathcal{G}$*
A_C	*the event "all components in C are in failed state"*
Q_C	$\Pr\{A_C\} = \prod_{i=1}^{\lvert C\rvert} q_{l_i}$
B_i	*the event "all components in $\{l_1, \ldots, l_{i-1}\}$ are failed and l_i is up"*
$\mathcal{G}_i$	$(\mathcal{G} - l_1 - \ldots - l_{i-1}) * l_i$
V	*discrete r.v. s-independent from r.v. $Y_{\mathcal{G}_i}$'s, with*

$$pmf\{v\} = \Pr\{V = v\} = \Pr\{B_v\}/\overline{Q_C} = ((1 - q_{l_v})\prod_{j=1}^{v-1} q_{l_j})/\overline{Q_C}, \ 1 \leq v \leq |C|,$$

then the random variable

$$Z_{\mathcal{G}} = Q_C + \overline{Q_C}\sum_{i=1}^{|C|} \mathbf{1}_{(V=i)} Y_{\mathcal{G}_i} \tag{4}$$

verifies

$$\mathrm{E}\{Z_{\mathcal{G}}\} = Q(\mathcal{G}) \tag{5}$$

and

$$\mathrm{Var}\{Z_{\mathcal{G}}\} = (Q(\mathcal{G}) - Q_C)R(\mathcal{G}) \leq Q(\mathcal{G})R(\mathcal{G}) = \mathrm{Var}\{Y_{\mathcal{G}}\}. \tag{6}$$

Moreover, the random variable

$$Z'_{\mathcal{G}} = Q_C + \overline{Q_C}\sum_{i=1}^{|C|} \mathbf{1}_{(V=i)} Y'_{\mathcal{G}_i} \tag{7}$$

where each $Y'_{\mathcal{G}_i}$, $1 \leq i \leq |C|$, is defined by

$$Y'_{\mathcal{G}_i} = \begin{cases} 1 & \text{if } \mathcal{G}_i \text{ is not } \mathcal{K}_i\text{-connected;} \\ 0 & \text{if } \mathcal{K}_i \text{ is a singleton set;} \\ 1 - \prod_{l \in \tilde{\mathcal{G}}_i} r_l & \text{if } \mathcal{G}_i \text{ is sp-reducible;} \\ Y_{\tilde{\mathcal{G}}_i} & \text{otherwise;} \end{cases}$$

verifies

$$\mathrm{E}\left\{Z'_{\mathcal{G}}\right\} = \mathrm{E}\{Z_{\mathcal{G}}\} \tag{8}$$

and

$$\mathrm{Var}\left\{Z'_{\mathcal{G}}\right\} \leq \mathrm{Var}\{Z_{\mathcal{G}}\}. \tag{9}$$

Proof . The proof of results (5) and (6) is given in [3].

By Proposition 3.1, $Y'_{\mathcal{G}_i}$ has same s-expectation as $Y_{\mathcal{G}_i}$ and smaller variance for each i, $1 \leq i \leq |C|$. Using these results and the lemma given in the appendix, we deduce that the r.v.

$$\sum_{i=1}^{|C|} \mathbf{1}_{(V=i)} Y'_{\mathcal{G}_i}$$

has same s-expectation as the r.v.

$$\sum_{i=1}^{|C|} \mathbf{1}_{(V=i)} Y_{\mathcal{G}_i}$$

and smaller variance.

From these comparisons and expressions (4) and (7) of $Z_{\mathcal{G}}$ and $Z'_{\mathcal{G}}$ respectively, we obtain the results (8) and (9). □

In [3], we have shown that the construction process of $Z_{\mathcal{G}}$ can be recursively used to construct a random variable $F(\mathcal{G})$ giving a more accurate estimator than the one based on $Z_{\mathcal{G}}$. Using the same notation as in the previous proposition 3.3, $F(\mathcal{G})$ has been expressed as follows:

$$F(\mathcal{G}) = \begin{cases} 1 & \text{if } \mathcal{G} \text{ is not } \mathcal{K}\text{-connected;} \\ 0 & \text{if } \mathcal{K} \text{ is a singleton set;} \\ Q_C + \overline{Q_C} \sum_{i=1}^{|C|} \mathbf{1}_{(V=i)} F(\mathcal{G}_i) & \text{otherwise.} \end{cases} \qquad (10)$$

The results (8) and (9) show that applying series-parallel reductions in the definition of $Z_{\mathcal{G}}$ leads to a more accurate sample mean estimator based on $Z'_{\mathcal{G}}$.

By combining these two ideas, (recursive application of the procedure used to construct the r.v. $Z_{\mathcal{G}}$ (4) and series-parallel reductions at each step), we obtain the following r.v.:

$$F'(\mathcal{G}) = \begin{cases} 1 & \text{if } \mathcal{G} \text{ is not } \mathcal{K}\text{-connected;} \\ 0 & \text{if } \mathcal{K} \text{ is a singleton set;} \\ Q(\mathcal{G}) & \text{if } \mathcal{G} \text{ is } \textit{sp-reducible}; \\ Q_C + \overline{Q_C} \sum_{i=1}^{|C|} \mathbf{1}_{(V=i)} F'(\tilde{\mathcal{G}}_i) & \text{otherwise.} \end{cases} \qquad (11)$$

The r.v. $F'(\mathcal{G})$ can be seen as a recursive application of the construction process of $Z'_{\mathcal{G}}$ in Proposition 3.3. Using the lemma given in the appendix, we can show by induction on the number of links that $F'(\mathcal{G})$ has same s-expectation than $Z'_{\mathcal{G}}$ and smaller variance (the same result was proven in [3] for $Z_{\mathcal{G}}$ and $F(\mathcal{G})$). Unfortunately, it is not easy to analytically compare the variances of $F(\mathcal{G})$ and $F'(\mathcal{G})$, even if we expect the last one to have better behavior, by analogy with what happens with $Z_{\mathcal{G}}$ and $Z'_{\mathcal{G}}$. This conjecture is supported by numerical comparisons given in next section.

A procedure to obtain a trial of $F'(\mathcal{G})$ can be summarized as follows:

Procedure RVR–SP($\mathcal{G}$)

Input: network $\mathcal{G}$
Output: a random sample of r.v. $F'(\mathcal{G})$

1. Check end recursion condition:
   1.1. Check if $\mathcal{G}$ is always $\mathcal{K}$-connected: If $|\mathcal{K}| = 1$ return(0).
   1.2. Check if the network is not $\mathcal{K}$-connected: If $\mathcal{G}$ is not $\mathcal{K}$-connected return(1).
   1.3. Check if the network is *sp-reducible*: If $\mathcal{G}$ is *sp-reducible* return($1 - \prod_{l \in \tilde{\mathcal{G}}} r_l$).
2. Find a K-cutset C of $\mathcal{G}$: $C = \{l_1, \ldots, l_{|C|}\}$ the set of all links adjacent to s, where $s \in K$ arbitrary.
3. Compute the probability that all components in C are failed: $Q_C = \prod_{i=1}^{|C|} q_{l_i}$.
4. Generate a trial v of V (with distribution $\Pr\{V = i\} = \Pr\{B_i\} / \overline{Q_C}$, $1 \le i \le |C|$).
5. Construct the corresponding network : $\mathcal{G}_v = (\mathcal{G} - l_1 - l_2 - \ldots - l_{v-1}) * l_v$.
6. Apply series-parallel reductions to $\mathcal{G}_v$ to compute $\tilde{\mathcal{G}}_v$.
7. Recursive step: return($Q_C + \overline{Q_C} \times$ RVR–SP($\tilde{\mathcal{G}}_v$)).

When computing a trial for $F'(\mathcal{G})$, we generate a trial v of the r.v. V (task 4). Since $\mathbf{1}_{(V=i)}$ is equal to 0 for all $i \neq v$, then only the term $\mathbf{1}_{(V=v)}$ survives in the sum of expression (11). It results that the computations involved in the recursion process can be represented by a linear computational structure. Its root corresponds to the network $\mathcal{G}$ under study, each internal node corresponds to a recursive call (task 7) and the last one presents a network that can be exactly evaluated (conditions 1.1, 1.2 and 1.3 of the above procedure). Because at each recursive step the number of links of the network resulting from tasks 5 and 6 is diminished by at least 1, the size of the related linear structure is bounded by $|\mathcal{E}|$.

In [3], we have shown that each recursive step of the RVR procedure has linear time complexity (in the number of links $|\mathcal{E}|$). The RVR–SP procedure has only tasks 1.3 and 6 in addition of those of RVR. Since these supplementary tasks can be done in linear time, then each RVR–SP recursive step also has linear time complexity. The total complexity to generate a trial of $F'(\mathcal{G})$ is then quadratic, as in the RVR case.

Remark 3.4 *In RVR–SP procedure, the recursion depth will be smaller than in the RVR procedure. This comes from two reasons:*

- *the RVR–SP procedure uses series-parallel reductions in order to diminish the number of links of the networks involved in the recursion process;*
- *the recursion process of the RVR–SP procedure stops when the network is sp-reducible. This is not true for the RVR procedure.*

4 Numerical Results and Possible Improvements

In this section, we give numerical results comparing the performance of the sample means $\widehat{F}'(\mathcal{G})$ and $\widehat{F}(\mathcal{G})$ where $\mathcal{G}$ is the network of 20 nodes and 30 arcs in Figure 1 with $\mathcal{K} = \{s, t\}$, and whose arcs are assigned equal but s-independent unreliabilities $q_l = 0.50$, 0.10, 0.05 and 0.02 (this topology is often used to compare the performances of different Monte Carlo methods).

In Table 1 we give the estimations of $Q(\mathcal{G})$ obtained with $\widehat{F}'(\mathcal{G})$ and $\widehat{F}(\mathcal{G})$ at column 2 and column 3 respectively. We also tabulate the ratio between the estimates of the variances of these estimators and the ratio between their execution times.

If $\widehat{V}_F / \widehat{V}_{F'} = v$, this implies that $\widehat{F}'(\mathcal{G})$ has variance v times smaller than $\widehat{F}(\mathcal{G})$. We observe that this ratio is always greater than 1, meaning that $\widehat{F}'(\mathcal{G})$ gives more accurate estimates.

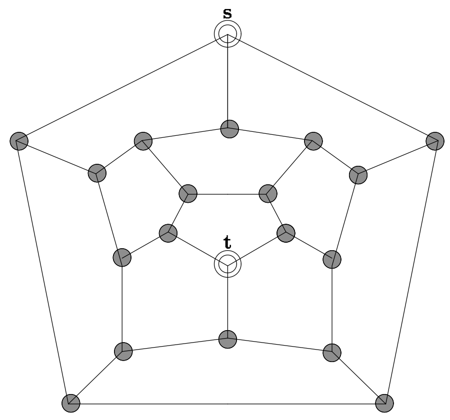

Figure 1: The "Dodecahedron" Network Topology

q_l	$\widehat{F}(\mathcal{G})$ [3]	$\widehat{F}'(\mathcal{G})$	$\widehat{V}_F/\widehat{V}_{F'}$	$T_F/T_{F'}$	W_{RVR}/W_{RVR-SP}
.50	0.7099	0.7099	1.37	1.04	1.42
.10	0.002880	0.002886	3.28	1.20	3.92
.05	0.0002946	0.0002954	4.74	1.22	5.79
.02	0.00001702	0.00001702	2.91	1.21	3.53

Table 1: Numerical comparisons of RVR and RVR-SP simulation methods

If $T_F/T_{F'} = t$, this implies that $\widehat{F}'(\mathcal{G})$ is t times faster than $\widehat{F}(\mathcal{G})$. In our example, this ratio is always greater than 1. The overhead (in terms of execution time) resulting from the series–parallel reductions is less important than the gain resulting from working with smaller networks.

To account for both variance ratio and execution times ratio, we tabulate at column 6 their product $W_F/W_{F'} = (\widehat{V}_F/\widehat{V}_{F'}) \times (T_F/T_{F'})$. This measure is called the speedup of $\widehat{F}'(\mathcal{G})$ with respect to $\widehat{F}(\mathcal{G})$. It is easily seen that $\widehat{F}'(\mathcal{G})$ has a better overall performance.

Table 2 presents the speedups of the methods RVR and RVR-SP with respect to the CMC method (Crude Monte Carlo method based on $Y_{\mathcal{G}} = 1 - \Phi_{\mathcal{G}}(X_{\mathcal{G}})$). Speedup results of the RVR method are from [3]. By these results and the values at column 6 of Table 1, we obtain the speedup results of RVR-SP method.

These comparisons show that the series and parallel reductions can be incorporated to the Recursive Variance Reduction method to increase its efficiency.

There are other reductions which have been proposed in the literature, as for instance the polygon-to-chain reductions, which have been used to accelerate factoring methods [27] and which allow to evaluate in linear time a wider class of networks which includes the *sp-reducible* topologies [24]. These techniques could be used in combination with series-parallel reductions to construct a more efficient estimator.

q_l	W_{CMC}/W_{RVR} [3]	W_{CMC}/W_{RVR-SP}
.50	6.56	9.32
.10	85.5	335.16
.05	681	3942.99
.02	22000	77660

Table 2: Speedups of RVR and RVR-SP methods with respect to crude Monte Carlo method

Acknowledgment

This research was supported by the ECOS French-Uruguayan Scientific Cooperation Program (Action U93E03) and the PEDECIBA Program. The work of H. Cancela was also supported by grants from BID/CONICYT (Uruguay).

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Appendix

Lemma .1 *Let Z and Z' be a couple of r.v defined by*

$$Z = \sum_{i=1}^{I} \mathbf{1}_{(V=i)} A_i \tag{12}$$

$$Z^{'} = \sum_{i=1}^{I} \mathbf{1}_{(V=i)} A_i^{'} \tag{13}$$

where

I	*is any integer greater than* 1,
A_i	*is a random variable,* $1 \leq i \leq I$,
$A_i^{'}$	*is a random variable,* $1 \leq i \leq I$,
V	*is a discrete random variable taking values in* $\{1, \ldots, I\}$.

If V is s-independent from A_i and $A_i^{'}$, $\mathrm{E}\left\{A_i^{'}\right\} = \mathrm{E}\{A_i\}$ and $\mathrm{Var}\left\{A_i^{'}\right\} \leq \mathrm{Var}\{A_i\}$, then Z' has same s-expectation as Z and smaller or equal variance.

Proof .

(a) By $\mathrm{E}\left\{A_i^{'}\right\} = \mathrm{E}\{A_i\}$ and the s-independence of V from A_i and $A_i^{'}$ for all i, $1 \leq i \leq I$, we obtain $\mathrm{E}\left\{Z^{'}\right\} = \mathrm{E}\{Z\}$.

(b) $\mathrm{Var}\{Z\} = \mathrm{E}\{Z^2\} - \mathrm{E}\{Z\}^2$, and $\mathrm{Var}\{Z'\} = \mathrm{E}\{Z'^2\} - \mathrm{E}\{Z'\}^2$. As for (a) $\mathrm{E}\left\{Z^{'}\right\} = \mathrm{E}\{Z\}$, to prove $\mathrm{Var}\{Z'\} \leq \mathrm{Var}\{Z\}$ is enough to show that $\mathrm{E}\{Z'^2\} \leq \mathrm{E}\{Z^2\}$. Developing $\mathrm{E}\{Z^2\}$, we obtain

$$\begin{aligned} \mathrm{E}\left\{Z^2\right\} &= \mathrm{E}\left\{(\sum_{i=1}^{I} \mathbf{1}_{(V=i)} A_i)(\sum_{j=1}^{I} \mathbf{1}_{(V=j)} A_j)\right\} \\ &= \sum_{i=1}^{I}\sum_{j=1}^{I} \mathrm{E}\left\{\mathbf{1}_{(V=i)} \mathbf{1}_{(V=j)} A_i A_j\right\}. \end{aligned}$$

By the s-independence between r.v. V and A_i's we obtain

$$\mathrm{E}\left\{Z^2\right\} = \sum_{i=1}^{I}\sum_{j=1}^{I} \mathrm{E}\left\{\mathbf{1}_{(V=i)} \mathbf{1}_{(V=j)}\right\} \mathrm{E}\{A_i A_j\}.$$

Since $\mathrm{E}\left\{\mathbf{1}_{(V=i)} \mathbf{1}_{(V=i)}\right\} = \mathrm{E}\left\{\mathbf{1}_{(V=i)}\right\}$ and for $i \neq j$ $\mathrm{E}\left\{\mathbf{1}_{(V=i)} \mathbf{1}_{(V=j)}\right\} = 0$, then

$$\mathrm{E}\left\{Z^2\right\} = \sum_{i=1}^{I} \mathrm{E}\left\{\mathbf{1}_{(V=i)}\right\} \mathrm{E}\left\{A_i^2\right\}.$$

By hypothesis $\mathrm{E}\{A_i\} = \mathrm{E}\left\{A_i^{'}\right\}$ and $\mathrm{Var}\{A_i\} \geq \mathrm{Var}\left\{A_i^{'}\right\}$, then $\mathrm{E}\{A_i^2\} \geq \mathrm{E}\left\{A_i^{'2}\right\}$ and

$$\mathrm{E}\left\{Z^2\right\} \geq \sum_{i=1}^{I} \mathrm{E}\left\{\mathbf{1}_{(V=i)}\right\} \mathrm{E}\left\{A_i^{'2}\right\} = \mathrm{E}\left\{Z'^2\right\}.$$

□