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Ergodic measures with infinite entropy

by

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Abstract. We construct ergodic probability measures with infinite metric entropy for generic continuous maps and homeomorphisms on compact manifolds. We also construct sequences of such measures that converge to a zero-entropy measure.

1. Introduction. Let M be a C^1 compact manifold of finite dimension $m \geq 1$, equipped with a Riemannian metric dist. The manifold M may or may not have boundary. Let $C^0(M)$ be the space of continuous maps $f : M \to M$ with the metric

$$\|f - g\|_{C^0} := \max_{x \in M} \operatorname{dist}(f(x), g(x)) \quad \forall f, g \in C^0(M).$$

We denote by $\operatorname{Hom}(M)$ the space of homeomorphisms $f : M \to M$ with the metric

$$\|f - g\|_{\operatorname{Hom}} := \max\{\|f - g\|_{C^0}, \|f^{-1} - g^{-1}\|_{C^0}\} \quad \forall f, g \in \operatorname{Hom}(M).$$

We note that the topology induced in $\operatorname{Hom}(M)$ by the above metric is the subspace topology induced by $C^0(M)$. Nevertheless, the metrics are different.

Since the metric spaces $C^0(M)$ and $\operatorname{Hom}(M)$ are complete, the Baire category theorem holds, namely a countable intersection of open dense sets is dense. A subset $\mathcal{S} \subset C^0(M)$ (or $\mathcal{S} \subset \operatorname{Hom}(M)$) is called a *$G_\delta$-set* if it is a countable intersection of open subsets of $C^0(M)$ (resp. $\operatorname{Hom}(M)$). We say that a property P of maps $f \in C^0(M)$ (or $f \in \operatorname{Hom}(M)$) is *generic,* or that generic maps satisfy P, if the set of maps that satisfy P contains a dense G_δ-set in $C^0(M)$ (resp. $\operatorname{Hom}(M)$).

The main result of this article is the following theorem.

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Theorem 1.1. *A generic map $f \in C^0(M)$ has an ergodic Borel probability measure μ such that $h_\mu(f) = +\infty$ and there exists $p \geq 1$ such that μ is mixing for the map f^p.*

Remark. In the case where M is a compact interval, Theorem 1.1 was proved in [CT, Theorem 39, and p. 33, second paragraph]. The statements and proofs of [CT] also hold for continuous maps of the circle S^1. In fact, each $f \in C(S^1)$ can be represented by a continuous map f in $[0,1]$ such that $f(0) = f(1)$. In the proof of genericity of the properties studied in [CT], no restrictions on the images of the endpoints 0 and 1 are imposed. In particular, the proof of the denseness condition was obtained by perturbing the map only in the interior of a finite number of compact subintervals contained in $[0,1]$. Finally, if a one-dimensional compact manifold M is not connected, the arguments of [CT] applied to a recurrent connected component of M extend the results to $C(M)$. This is why in this paper we will prove Theorem 1.1 only for m-dimensional manifolds with $m \geq 2$.

Yano proved that generic continuous maps of compact manifolds with or without boundary have infinite topological entropy [Ya]. Therefore, from the variational principle, there exist invariant measures with metric entropies as large as required. Nevertheless, this property alone does not imply the existence of invariant measures with infinite metric entropy. In fact, it is well known that the metric entropy function $\mu \mapsto h_\mu(f)$ is not upper semicontinuous for C^0-generic systems. Moreover, we prove that it is *strongly* non-upper-semicontinuous in the following sense:

Theorem 1.2. *For a generic map $f \in C^0(M)$ there exists a sequence of ergodic measures μ_n such that for all $n \geq 1$ we have $h_{\mu_n}(f) = +\infty$ and*

$$\lim_{n\to\infty}{}^{*} \mu_n = \mu \quad \textit{with} \quad h_\mu(f) = 0,$$

where $\lim^*$ *denotes the limit in the space of probability measures endowed with the weak* topology.*

Theorem 1.2 holds for any m-dimensional manifold, including $m = 1$. In this paper we will prove it for $m \geq 2$, but the proof for $m = 1$ is easily obtained by repeating our proof after some trivial substitutions that are explained at the beginning of Section 5.

Even if we had a priori some f-invariant measure μ with infinite metric entropy, we do not know if this property alone implies the existence of ergodic measures with infinite metric entropy as Theorems 1.1 and 1.2 state. Actually, if μ had infinitely many ergodic components, the proof that the metric entropy of at least one of those components must be larger than or equal to the entropy of μ uses the upper semicontinuity of the metric entropy function (see for instance [Ke, Theorem 4.3.7, p. 75]).

Yano also proved that generic homeomorphisms on manifolds of dimension 2 or larger have infinite topological entropy [Ya]. Thus one wonders if Theorems 1.1 and 1.2 hold also for homeomorphisms. We give a positive answer to this question for $m \geq 2$. If M is one-dimensional then a homeomorphism of M has zero topological entropy, so the following two theorems do not hold for one-dimensional manifolds.

THEOREM 1.3. *If* $\dim(M) \geq 2$, *then a generic homeomorphism* $f \in \mathrm{Hom}(M)$ *has an ergodic Borel probability measure* μ *satisfying* $h_\mu(f) = +\infty$ *and there exists* $p \geq 1$ *such that* μ *is mixing for the map* f^p.

THEOREM 1.4. *If* $\dim(M) \geq 2$, *then for a generic homeomorphism* $f \in \mathrm{Hom}(M)$ *there exists a sequence of ergodic measures* μ_n *such that for all* $n \geq 1$ *we have* $h_{\mu_n}(f) = +\infty$ *and*

$$\lim_{n\to\infty}^* \mu_n = \mu \quad \textit{with} \quad h_\mu(f) = 0.$$

To prove Theorems 1.1, 1.3 and 1.4 in dimension 2 or larger, we construct a family $\mathcal{H}$ of continuous maps, called models, on the cube $[0,1]^m$, including some homeomorphisms of the cube onto itself, which have a complicated behavior on a Cantor set (Definition 2.5). In the proof of Theorem 1.2 in dimension 1, the family $\mathcal{H}$ of model maps in M we use is the set of continuous maps that have an “atom doubling cascade”, according to [CT, Definition 35].

In any dimension $m \geq 1$, a *periodic shrinking box* is a compact set $K \subset M$ that is homeomorphic to the cube $[0,1]^m$ and such that for some $p \geq 1$, the sets $K, f(K), \dots, f^{p-1}(K)$ are pairwise disjoint and $f^p(K) \subset \mathrm{int}(K)$ (Definition 4.1).

The main steps of the proofs of Theorems 1.1 and 1.3 are the following results.

LEMMA 3.1. *For* $m \geq 1$, *any model* $\Phi \in \mathcal{H}$ *on the cube* $[0,1]^m$ *has a* Φ*-invariant mixing measure* ν *such that* $h_\nu(\Phi) = +\infty$.

LEMMAS 4.2 and 4.5. *For* $m \geq 1$, *generic maps in* $C^0(M)$, *and generic homeomorphisms of* M, *have a periodic shrinking box.*

LEMMAS 4.7 and 4.8. *If* $m \geq 1$ *then generic maps* $f \in C^0(M)$, *and if* $m \geq 2$ *also generic homeomorphisms of* M, *have a periodic shrinking box* K *such that the return map* $f^p|_K$ *is topologically conjugate to a model* $\Phi \in \mathcal{H}$.

We prove and use Lemma 3.1 only for $m \geq 2$ since the case $m = 1$ was proven in [CT, Theorem 38]. The other results in the above list will be fully proven here even in the case $m = 1$, independently of [CT].

A *good sequence of periodic shrinking boxes* is a sequence $\{K_n\}_{n\geq 1}$ of periodic shrinking boxes which accumulate (with the Hausdorff distance) on a periodic point x_0, and moreover their iterates $f^j(K_n)$ accumulate on the periodic orbit of x_0, uniformly for $j \geq 0$ (see Definition 5.1). The main tools used in the proofs of Theorems 1.2 and 1.4 are Theorems 1.1 (for $m \geq 1$) and 1.3 and Lemmas 4.2, 4.5, 4.7 and 4.8, together with

LEMMA 5.7. *If $m \geq 1$ then a generic map $f \in C^0(M)$, and if $m \geq 2$ also a generic homeomorphism f, has a good sequence $\{K_n\}$ of boxes such that the return map $f^{p_n}|_{K_n}$ is topologically conjugate to a model $\Phi_n \in \mathcal{H}$.*

2. Construction of the family of models. We call a compact set $K \subset D^m := [0,1]^m$ or more generally $K \subset M$ (where M is an m-dimensional manifold with $m \geq 1$) a *box* if it is homeomorphic to D^m. Models are certain continuous maps of D^m that we will define in this section.

We denote by $\mathrm{Emb}(D^m)$ the space of embeddings $\Phi : D^m \to D^m$ (i.e., Φ is a homeomorphism onto its image), with the topology of a subspace of $C^0(D^m)$.

DEFINITION 2.1. For $m = 1$ a *model* is a map that has an "atom doubling cascade" according to [CT, Definition 35]. Let $\mathcal{H}$ be the set of all model maps.

In the rest of this section we construct the family $\mathcal{H}$ of model maps for $m \geq 2$.

DEFINITION 2.2 (Φ-relation from a box to another). Let $\Phi \in C^0(D^m)$. Let $B, C \subset \mathrm{int}(D^m)$ be two boxes. We write

$$B \xrightarrow{\Phi} C \quad \text{if} \quad \Phi(B) \cap \mathrm{int}(C) \neq \emptyset.$$

Observe that this condition is open in $C^0(D^m)$. Let $\mathcal{A}$ be a finite family of boxes. Denote

$$\mathcal{A}^{2*} := \{(B,C) \in \mathcal{A}^2 : B \xrightarrow{\Phi} C\},$$
$$\mathcal{A}^{3*} := \{(D,B,C) \in \mathcal{A}^3 : D \xrightarrow{\Phi} B,\ B \xrightarrow{\Phi} C\}.$$

For all $n \geq 0$ we now define atoms of generation n for a map $\Phi \in C^0(D^m)$.

DEFINITION 2.3 (Atoms of generations $0 \leq n \leq k$; see Figure 1). Fix $\Phi \in C^0(D^m)$ and collections $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_k$ of boxes contained in the interior of D^m. For $n \geq 1$ and for $(D,B,C) \in \mathcal{A}^{3*}_{n-1}$ we define

$$\Omega_n(B) := \{G \in \mathcal{A}_n : G \subset \mathrm{int}(B)\},$$
$$\Omega_n(D,B) := \{G \in \Omega_n(B) : D \xrightarrow{\Phi} G\},$$
$$\Gamma_n(D,B,C) := \{G \in \Omega_n(D,B) : G \xrightarrow{\Phi} C\}.$$

Suppose that the following conditions hold for all $1 \leq n \leq k$:

(i) The family $\mathcal{A}_n$ consists of 2^{n^2} pairwise disjoint boxes.
(ii) For all $B \in \mathcal{A}_n$,

$$\#\{C \in \mathcal{A}_n : B \xrightarrow{\Phi} C\} = 2^n, \qquad \#\{D \in \mathcal{A}_n : D \xrightarrow{\Phi} B\} = 2^n.$$

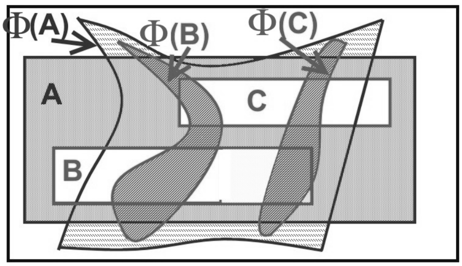

Fig. 1. An atom A of generation 0 and two atoms B, C of generation 1 for a map Φ of D^2.

Suppose furthermore:

(a) $\#\Omega_n(B) = \#\Omega_n(B')$ for all $B, B' \in \mathcal{A}_{n-1}$, and $\mathcal{A}_n = \bigcup_{B\in\mathcal{A}_{n-1}} \Omega_n(B)$.

(b) For all $(D, B) \neq (D', B') \in \mathcal{A}^{2*}_{n-1}$,
$$\#\Omega_n(D, B) = \#\Omega_n(D', B') \quad \text{and} \quad \Omega_n(D, B) \cap \Omega_n(D', B') = \emptyset.$$
Moreover, $\Omega_n(B) = \bigcup_{D:\,(D,B)\in\mathcal{A}^{2*}_{n-1}} \Omega_n(D, B)$ for all $B \in \mathcal{A}_{n-1}$.

(c) For all $(D, B, C) \neq (D', B', C') \in \mathcal{A}^{3*}_{n-1}$,
$$\#\Gamma_n(D, B, C) = \#\Gamma_n(D', B', C') \quad \text{and} \quad \Gamma_n(D, B, C)\cap\Gamma_n(D', B', C') = \emptyset,$$
and for all $(D, B) \in \mathcal{A}^{2*}_{n-1}$,
$$\Omega_n(D, B) = \bigcup_{C:\,(B,C)\in\mathcal{A}^{2*}_{n-1}} \Gamma_n(D, B, C), \tag{2.1}$$

(d) For each $(D, B, C) \in \mathcal{A}^{3*}_{n-1}$ and each $G \in \Gamma_n(D, B, C)$,
$$\Phi(G) \cap E = \emptyset \quad \forall E \in \mathcal{A}_n \setminus \Omega_n(B, C).$$

Then we call the members of $\mathcal{A}_n$ *atoms of generation* n or n*-atoms.*

Remark 2.4. From conditions (i), (ii) and (a)–(d) of Definition 2.3 we deduce the following properties of the families of atoms for $\Phi \in C^0(D^m)$:

• $\#\Omega_n(B) = 2^{2n-1}$ for all $B \in \mathcal{A}_{n-1}$. In fact, the families $\Omega_n(B)$ are pairwise disjoint because any two different atoms of generation n are disjoint. Therefore, from condition (a), we obtain
$$\#\mathcal{A}_n = \#\mathcal{A}_{n-1} \cdot \#\Omega_n(B) = 2^{(n-1)^2} \cdot \#\Omega_n(B) = 2^{n^2},$$
hence $\#\Omega_n(B) = 2^{2n-1}$.

• $\#\Omega_n(D, B) = 2^n$ for all $(D, B) \in \mathcal{A}^{2*}_{n-1}$. In fact, from condition (b),
$$\Omega_n(B) = \bigcup_{D:\,(D,B)\in\mathcal{A}^{2*}_{n-1}} \Omega_n(D, B) \quad \forall B \in \mathcal{A}_{n-1},$$

where the families of atoms in the above union are pairwise disjoint. Thus, for any $B \in \mathcal{A}_{n-1}$ we have

$$\begin{aligned}\#\Omega_n(B) &= \#\{D \in \mathcal{A}_{n-1} : D \xrightarrow{\Phi} B\} \cdot \#\Omega_n(D,B) \\ &= 2^{n-1} \cdot \#\Omega_n(D,B) = 2^{2n-1},\end{aligned}$$

hence $\#\Omega_n(D,B) = 2^n$.

• $\#\Gamma_n(D,B,C) = 2$ for all $(D,B,C) \in \mathcal{A}^{3*}_{n-1}$. In fact, from conditions (ii) and (c), for each 2-tuple $(D,B) \in \mathcal{A}^{2*}_{n-1}$ the collection $\Omega_n(D,B)$ is partitioned into 2^{n-1} pairwise disjoint subcollections $\Gamma_n(D,B,C)$, where the atoms $C \in \mathcal{A}_{n-1}$ are such that $B \xrightarrow{\Phi} C$. Since $\#\Omega_n(D,B) = 2^n$ (proved above), we deduce that $\#\Gamma_n(D,B,C) = 2$. For example, in Figure 2 we have $\Gamma_2(C,B,C) = \{F,G\}$.

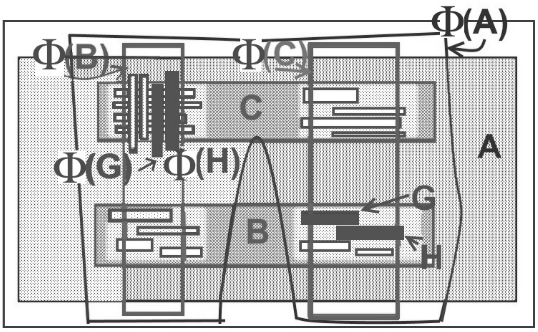

Fig. 2. An atom A of generation 0, two atoms B, C of generation 1, and 16 atoms of generation 2. In particular, the two atoms G, H of generation 2 satisfy $\Gamma_2(C,B,C) = \{G,H\}$.

• As a straightforward consequence of conditions (a)–(c) we obtain

$$\mathcal{A}_n = \bigcup_{(D,B,C)\in\mathcal{A}^{3*}_{n-1}} \Gamma_n(D,B,C), \tag{2.2}$$

where the families of atoms in the union are pairwise disjoint.

• For each $(D,B,C) \in \mathcal{A}^{3*}_{n-1}$, each $G \in \Gamma_n(D,B,C)$ and all $E \in \mathcal{A}_n$, either $G \xrightarrow{\Phi} E$, and this occurs if and only if $E \in \Omega_n(B,C)$, or $\Phi(G) \cap E = \emptyset$, and this occurs if and only if $E \notin \Omega_n(B,C)$. In fact, from condition (d), if $\Phi(G) \cap E \neq \emptyset$ then $E \in \Omega_n(B,C)$. So, recalling condition (ii), we obtain

$$\begin{aligned}2^n &= \#\{E \in \mathcal{A}_n : G \xrightarrow{\Phi} E\} \\ &\le \#\{E \in \mathcal{A}_n : \Phi(G) \cap E \neq \emptyset\} \le \#\Omega_n(B,C) = 2^n.\end{aligned}$$

Hence, all the above inequalities are equalities and the assertion is proved.

• $\#\mathcal{A}_n^{3*} = 2^{n^2+2n}$. In fact, all the 3-tuples $(D, B, C) \in \mathcal{A}_n^{3*}$ can be constructed by choosing freely $D \in \mathcal{A}_n$, later choosing $B \in \mathcal{A}_n$ such that $D \xrightarrow{\Phi_n} B$, and finally choosing $C \in \mathcal{A}_n$ such that $B \xrightarrow{\Phi_n} C$. Taking into account the equalities in (i) and (ii) we deduce

$$\begin{aligned}\#\mathcal{A}_n^{3*} &= \#\{(D,B,C) \in \mathcal{A}_n^3 : D \xrightarrow{\Phi_n} B,\ B \xrightarrow{\Phi_n} C\} \\ &= \#\mathcal{A}_n \cdot \#\{B \in \mathcal{A}_n : D \xrightarrow{\Phi_n} B\} \cdot \#\{C \in \mathcal{A}_n : B \xrightarrow{\Phi_n} C\} \\ &= 2^{n^2} 2^n 2^n = 2^{n^2+2n}.\end{aligned}$$

NOTATION. In certain statements we refer to families of atoms for several different maps. When necessary we will write $A \in \mathcal{A}_n(\Phi)$ or $(B, D) \in \mathcal{A}_n^{2*}(\Phi)$, etc., where Φ is the map to which the family of atoms is associated.

DEFINITION 2.5 (Models). We call $\Phi \in C^0(D^m)$ *a model* if $\Phi(D^m) \subset \mathrm{int}(D^m)$ and there exists a sequence $\{\mathcal{A}_n\}_{n \geq 0}$ of finite families of pairwise disjoint boxes contained in $\mathrm{int}(D^m)$ that are atoms of generations $n \geq 0$ for Φ such that

$$\lim_{n\to\infty} \max_{A \in \mathcal{A}_n} \mathrm{diam}(A) = 0. \tag{2.3}$$

Denote by $\mathcal{H}$ the family of all models in $C^0(D^m)$.

DEFINITION 2.6. For any $\Phi \in \mathcal{H}$, we denote by $\mathcal{H}_\Phi$ the family of maps in $C^0(D^m)$ that have the same atoms of all generations as Φ. Note that $\mathcal{H}_\Phi \subset \mathcal{H}$.

Construction of models. The rest of this section is dedicated to the proof of the following lemma.

LEMMA 2.7. *The family $\mathcal{H} \cap \mathrm{Emb}(D^m)$ is nonempty. As a consequence, we can choose a map Φ in this family such that the subfamilies $\mathcal{H}_\Phi$ and $\mathcal{H}_\Phi \cap \mathrm{Emb}(D^m)$ are nonempty G_δ-sets in $C^0(D^m)$ and $\mathrm{Emb}(D^m)$ respectively.*

Proof. For each fixed $n \geq 1$ the conditions (a)–(d) of Definition 2.3 are open conditions. So, for fixed $n \geq 0$, and for any given map Φ having families $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_n$ of atoms of generations $0, 1, \ldots, n$, the set of maps that have the same families of atoms of generations up to n as Φ (for that fixed n and not necessarily for all n) is an open set. Moreover, the condition $\Phi(D^m) \subset \mathrm{int}(D^m)$ is open.

From Definition 2.6 we deduce that if there existed some $\Phi \in \mathcal{H}$, the family $\mathcal{H}_\Phi \subset \mathcal{H}$ would be a nonempty G_δ-set in $C^0(M)$. In other words, $\mathcal{H}$ would contain a nonempty G_δ-set if $\mathcal{H}$ were nonempty.

Analogously, if there existed $\Phi \in \mathcal{H} \cap \mathrm{Emb}(D^m)$ (note that $\mathcal{H}_\Phi$ is not necessarily composed of embeddings of D^m), it would contain $\mathcal{H}_\Phi \cap \mathrm{Emb}(D^m)$, which would be a nonempty G_δ-set in $\mathrm{Emb}(D^m)$.

The difficult part is to prove the first assertion: $\mathcal{H} \cap \mathrm{Emb}(D^m) \neq \emptyset$. This will be proved in Lemma 2.8 below, in which we give a procedure to construct a map in that family. ∎

Note that the nonempty G_δ-set $\mathcal{H}_\Phi$ is not necessarily dense in $C^0(D^m)$!

LEMMA 2.8. *For all $f \in \mathrm{Emb}(D^m)$ such that $f(D^m) \subset \mathrm{int}(D^m)$, there exists $\psi \in \mathrm{Hom}(D^m)$ such that*

$$\psi|_{\partial D^m} = \mathrm{id}|_{\partial D^m} \quad \textit{and} \quad \Phi := \psi \circ f \in \mathcal{H} \cap \mathrm{Emb}(D^m).$$

We now outline the strategy of the proof of Lemma 2.8. The homeomorphism ψ is constructed as the limit of a convergent sequence $\psi_n \in \mathrm{Hom}(D^m)$ such that $\Phi_n := \psi_n \circ f \in \mathrm{Emb}(D^m)$ is convergent. The embedding Φ_n, by an inductive hypothesis, has atoms of generations $0, 1, \ldots, n$. Further, Φ_{n+1} will be constructed so that it coincides with Φ_n outside the interiors of all atoms of generation n for Φ_n. Hence, the collections of atoms of generations $0, 1, \ldots, n$ for Φ_n is also a collection of atoms of the same generations for Φ_{n+1} (see the proof of Lemma 2.11(a)).

To change Φ_n in the interior of each atom A of generation n for Φ_n, we will change ψ_n only inside some properly defined boxes $f(R)$ such that $R \subset \mathrm{int}(A)$ is a box (recall that f is an embedding). We will construct $\psi_{n+1}|_{f(R)}$ so that $\psi_{n+1}|_{\partial f(R)} = \psi_n|_{\partial f(R)}$, and finally extend $\psi_{n+1}(x) := \psi_n(x)$ for all x in the complement of the union of all boxes $f(R)$.

The new homeomorphism ψ_{n+1}, if properly constructed inside the boxes $f(R)$, will allow us to define the atoms of generation $n+1$ for $\Phi_{n+1} = \psi_{n+1} \circ f$. These atoms will be many little boxes in the interior of each box $f^{-1}(f(R)) = R \subset A$, where A is an atom of generation n for both Φ_n and Φ_{n+1}.

Lemma 2.8 will be proved by induction via several technical lemmas. One inductive hypothesis in the proof is that for a fixed $n \geq 0$ we have constructed an embedding Φ_n along with associated atoms of generations $0, 1, \ldots, n$. For each $(P, Q) \in \mathcal{A}_n^{2*}$, we will choose a connected component $S(P, Q)$ of $\Phi_n(P) \cap Q$. For each $(D, B, C) \in \mathcal{A}_n^{3*}$ we choose two disjoint boxes $G_0(D, B, C)$, $G_1(D, B, C)$ contained in $\mathrm{int}(S(D, B) \cap \Phi_n^{-1} S(B, C))$. By an additional inductive hypothesis on Φ_n a choice of the connected components $S(D, B)$ and $S(B, C)$ is assumed to exist such that the interior of this intersection is nonempty.

REMARK 2.9. We provisionally adopt an abusive notation for the families of such boxes $G_\cdot(\cdot, \cdot, \cdot)$. Even if they are not atoms of generation $n+1$ for Φ_n, we use the notation as if they were. We use this notation since in the proof of Lemma 2.8 we will modify Φ_n to construct a new embedding Φ_{n+1} for which the same atoms up to generation n for Φ_n are also atoms up to generation n for Φ_{n+1}, and moreover the boxes $G_\cdot(\cdot, \cdot, \cdot)$ are the atoms of generation $n+1$

for Φ_{n+1}. In brief, we first choose the boxes, candidates to be the atoms of generation $n+1$ for a new embedding Φ_{n+1}, and later we construct Φ_{n+1}. Let

$$\mathcal{A}_{n+1} := \{G_j(D,B,C) : j \in \{0,1\},\ (D,B,C) \in \mathcal{A}_n^{3*}(\Phi_n)\};$$
$$\Omega_{n+1}(B) := \{G_j(D,B,C) : j \in \{0,1\},\ (D,B,C) \in \mathcal{A}_n^{3*}(\Phi_n)\}$$

for each fixed $B \in \mathcal{A}_n(\Phi_n)$;

$$\Omega_{n+1}(D,B) := \{G_j(D,B,C) : j \in \{0,1\},\ B \xrightarrow{\Phi_n} C\}$$

for each fixed $(D,B) \in \mathcal{A}_n^{2^*}(\Phi_n)$; and

$$\Gamma_{n+1}(D,B,C) := \{G_j(D,B,C) : j \in \{0,1\}\} \tag{2.4}$$

for each fixed $(D,B,C) \in \mathcal{A}_n^{3^*}(\Phi_n)$. We will apply this abusive notation in Lemmas 2.10 and 2.11 and in Remark 2.12.

Lemma 2.10. *For all $(B,C) \in \mathcal{A}_n^{2*}(\Phi_n)$ and all $E \in \mathcal{A}_{n+1}$ we have $E \subset \Phi_n^{-1}(S(B,C))$ if and only if $E \in \Gamma_{n+1}(D,B,C)$ for some $D \in \mathcal{A}_n$ such that $D \xrightarrow{\Phi_n} B$.*

Proof. By the construction in Remark 2.9, for all $E \in \mathcal{A}_{n+1}$ we have $E \in \Gamma_{n+1}(D,B,C) = \{\Gamma_0(D,B,C), \Gamma_1(D,B,C)\}$ for some $(D,B,C) \in \mathcal{A}_n^{3^*}$. This means that $E = G_j(D,B,C) \subset \operatorname{int}(S(D,B) \cap \Phi_n^{-1}(S(B,C)))$ for some $j = 0,1$. Therefore, $E \subset \Phi_n^{-1}(S(B,C))$ if and only if there exists $D \in \mathcal{A}_n$ such that $D \xrightarrow{\Phi_n} B$ and $E \in \Gamma_{n+1}(D,B,C)$. ∎

Lemma 2.11. *Suppose that*

$$\operatorname{int}(S(D,B) \cap \Phi_n^{-1} S(B,C)) \neq \emptyset \quad \forall (D,B,C) \in \mathcal{A}_n^{3*}(\Phi_n).$$

Consider any $\Phi_{n+1} \in \operatorname{Emb}(D^m)$ which satisfies $\Phi_{n+1}(x) = \Phi_n(x)$ for each $x \notin \bigcup_{(B,C)\in\mathcal{A}_n^{2*}(\Phi_n)} \operatorname{int}(\Phi_n^{-1} S(B,C))$. Then:

(a) *For all $0 \le j \le n$ and any atoms $B, C \in \mathcal{A}_j^2(\Phi_n)$ we have $\Phi_n(B) = \Phi_{n+1}(B)$; hence $B \xrightarrow{\Phi_n} C$ if and only if $B \xrightarrow{\Phi_{n+1}} C$.*
(b) *$\#\mathcal{A}_{n+1} = 2^{(n+1)^2}$ and $E \cap F = \emptyset$ for all $E, F \in \mathcal{A}_{n+1}$ such that $E \neq F$.*
(c) *The family $\mathcal{A}_{n+1}$ is partitioned into the pairwise disjoint subfamilies $\Omega_{n+1}(B)$ where $B \in \mathcal{A}_n$. Moreover, $\#\Omega_{n+1}(B) = 2^{2n+1}$ and $\Omega_{n+1}(B) = \{G \in \mathcal{A}_{n+1} : G \subset \operatorname{int}(B)\}$ for all $B \in \mathcal{A}_n$.*
(d) *For all $B \in \mathcal{A}_n$ the family of boxes $\Omega_{n+1}(B)$ is partitioned into the pairwise disjoint subfamilies $\Omega_{n+1}(D,B)$ where $D \in \mathcal{A}_n$ is such that $D \xrightarrow{\Phi_{n+1}} B$. Moreover, for all $(D,B) \in \mathcal{A}_n^{2*}(\Phi_{n+1})$ we have $\#\Omega_{n+1}(D,B) = 2^{n+1}$ and $\Omega_{n+1}(D,B) = \{G \in \Omega_{n+1}(B) : D \xrightarrow{\Phi_{n+1}} G\}$.*

(e) *For all* $(D,B) \in \mathcal{A}_n^{2*}(\Phi_{n+1})$ *the family of boxes* $\Omega_{n+1}(D,B)$ *is partitioned into the pairwise disjoint subfamilies* $\Gamma_{n+1}(D,B,C)$, *where* $C \in \mathcal{A}_n$ *is such that* $B \xrightarrow{\Phi_{n+1}} C$. *Moreover, for all* $(D,B,C) \in \mathcal{A}_n^{3*}(\Phi_{n+1})$ *we have* $\#\Gamma_{n+1}(D,B,C) = 2$ *and* $\Gamma_{n+1}(D,B,C) = \{G \in \Omega_{n+1}(D,B) : G \xrightarrow{\Phi_{n+1}} C\}$.

(f) *For all* $(D,B,C) \in \mathcal{A}_n^{3*}(\Phi_{n+1})$, *all* $G \in \Gamma_{n+1}(D,B,C)$, *and all* $E \in \mathcal{A}_{n+1}$,

$$\Phi_{n+1}(G) \cap E \neq \emptyset \quad \textit{only if} \quad E \in \Omega_{n+1}(B,C).$$

Proof. (a) We prove (a) under the more general hypothesis $\Phi_{n+1}(x) = \Phi_n(x)$ for all $x \notin \bigcup_{B \in \mathcal{A}_n} \operatorname{int}(B)$. (Note that $x \notin \operatorname{int}(S(D,B) \cap \Phi_n^{-1} S(B,C))$ implies $x \notin B$.)

By hypothesis, $\Phi_n, \Phi_{n+1} \in \operatorname{End}(D^m)$ and $\Phi_n|_{\partial A} = \Phi_{n+1}|_{\partial A}$ for the boxes $A \in \mathcal{A}_j$ for all $0 \leq j \leq n$ (recall that, from condition (a) of Definition 2.3, each atom of generation n for Φ_n is contained in the interior of an atom of generation $0 \leq j \leq n$). Then $\Phi_{n+1}(A) = \Phi_n(A)$ for all $B \in \bigcup_{0 \leq j \leq n} \mathcal{A}_j$. Part (a) follows immediately.

(b) By construction, $E = G_j(D,C,B)$, $F = G_{j'}(D',B',C')$. If $E \neq F$, then either $(D,C,B) = (D',C',B')$ and $j \neq j'$, or $(D,C,B) \neq (D',C',B')$. In the former case, by construction,

$$G_0(D,C,B) \cap G_1(D,C,B) = \emptyset,$$

in other words $E \cap F = \emptyset$. In the latter case, either $D \neq D'$ or $B \neq B'$ or $C \neq C'$. By construction, $G_j(D,B,C) \subset \Phi_n(D) \cap B \cap \Phi_n^{-1}(C)$ and $G_{j'}(D',B',C') \subset \Phi_n(D') \cap B' \cap \Phi_n^{-1}(C')$. Since members of $\mathcal{A}_n$ are pairwise disjoint, and $\Phi_n \in \operatorname{Emb}(D^m)$, we deduce that $G_j(D,B,C) \cap G_{j'}(D',B',C') = \emptyset$, hence $E \cap F = \emptyset$ as required.

By the construction in Remark 2.9,

$$\mathcal{A}_{n+1} = \bigcup_{(D,C,B) \in \mathcal{A}_n^{3*}} \Gamma_{n+1}(D,B,C),$$

where the families in the union are pairwise disjoint and each one has two different boxes of $\mathcal{A}_{n+1}$. Therefore, taking into account the last assertion of Remark 2.4, we deduce that

$$\#\mathcal{A}_{n+1} = 2 \cdot \#\mathcal{A}_n^{3*} = 2 \cdot 2^{n^2+2n} = 2^{(n+1)^2}.$$

(c) Using the notation of the end of Remark 2.9, we have

$$\mathcal{A}_{n+1} = \bigcup_{B \in \mathcal{A}_n} \Omega_{n+1}(B).$$

Moreover, for all $G \in \mathcal{A}_{n+1}$ we have $G \subset \operatorname{int}(B)$ if and only if $G \in \Omega_{n+1}(B)$, because by construction, $G \subset \operatorname{int}(S(D,B)) \subset \operatorname{int}(B)$ for some $B \in \mathcal{A}_n$. Since members of $\mathcal{A}_n$ are pairwise disjoint, we deduce that $\Omega_{n+1}(B) \cap \Omega_{n+1}(B') = \emptyset$

if $B \neq B'$. We conclude that the above union of different subfamilies $\Omega_{n+1}(B)$ is a partition of $\mathcal{A}_{n+1}$, as required.

Note that

$$\Omega_{n+1}(B) = \bigcup_{D\in\mathcal{A}_n,\, D\xrightarrow{\Phi_n}B} \ \bigcup_{C\in\mathcal{A}_n,\, B\xrightarrow{\Phi_n}C} \Gamma_{n+1}(D,B,C),$$

where the families in the union are pairwise disjoint and each of them has two different boxes. Therefore, taking into account that $\mathcal{A}_n$ is a family of atoms for Φ_n (by hypothesis), equality (ii) of Definition 2.3 implies

$$\begin{aligned}\#\Omega_{n+1}(B) &= 2\cdot\#\{D\in\mathcal{A}_n : D \xrightarrow{\Phi_n} B\}\cdot\#\{C\in\mathcal{A}_n : B \xrightarrow{\Phi_n} C\}\\ &= 2\cdot 2^n\cdot 2^n = 2^{2n+1}.\end{aligned}$$

(d) By the construction at the end of Remark 2.9,

$$\Omega_{n+1}(B) = \bigcup_{D\in\mathcal{A}_n,\, D\xrightarrow{\Phi_n}B} \Omega_{n+1}(D,B).$$

Moreover, $\Omega_{n+1}(D,B)\cap\Omega_{n+1}(D',B)=\emptyset$ if $D\neq D'$ in $\mathcal{A}_n$, since different atoms of generation n are pairwise disjoint, and $G\in\Omega_{n+1}(D,B)$ implies $G\subset\Phi_n(D)$, which is disjoint from $\Phi_n(D')$ since Φ_n is an embedding.

By the construction in Remark 2.9,

$$\Gamma_{n+1}(D,C,B) = \{G_0(D,C,B), G_1(D,C,B)\},$$

where G_0 and G_1 are disjoint, hence different. Thus the cardinality of $\Gamma_{n+1}(D,C,B)$ is 2.

Also, $\Omega_{n+1}(D,B) = \bigcup_{C\in\mathcal{A}_n:\, B\xrightarrow{\Phi_n}C}\Gamma_{n+1}(D,B,C)$. Moreover,

$$\Gamma_{n+1}(D,B,C)\cap\Gamma_{n+1}(D,B,C') = \emptyset$$

if $C\neq C'$ in $\mathcal{A}_n$, because any two different atoms of generation n are disjoint and $G\in\Gamma_{n+1}(D,B,C)$ implies $G\subset\Phi_n^{-1}(C)$.

From the above assertions and from the equalities in (ii) of the definition of atoms of generation n, we deduce that

$$\#\Omega_{n+1}(D,B) = 2\cdot\#\{C\in\mathcal{A}_n : B\xrightarrow{\Phi_n} C\} = 2\cdot 2^n = 2^{n+1}.$$

Finally, for all $G\in\Omega_{n+1}(B)$ there exists a (unique) $D\in\mathcal{A}_n$ such that $G\subset S(D,B)\subset\Phi_n(D)=\Phi_{n+1}(D)$. Hence $D\xrightarrow{\Phi_{n+1}} G$ if and only if $G\in\Omega(D,B)$.

(e) Above we proved that $\Omega_{n+1}(D,B)$ is partitioned into the pairwise disjoint subfamilies $\Gamma_{n+1}(D,B,C)$, where C is such that $(B,C)\in\mathcal{A}_n^{2*}$.

We have also noticed that $\#\Gamma_{n+1}(D,B,C) = 2$. Finally, by the construction of Remark 2.9, for all $G\in\Omega_{n+1}(D,B)$ there exists $C\in\mathcal{A}_n$ such that $G\in S(D,B)\cap\Phi_n^{-1}(S(B,C)$. Therefore $\Phi_{n+1}(G)\subset\Phi_{n+1}(\Phi_n^{-1}(S(B,C))$. This latst set coincides with $\Phi_n(\Phi_n^{-1}(S(B,C))$ because, by hypothesis, Φ_n

and Φ_{n+1} are embeddings and coincide outside the interiors of all the sets $\Phi_n^{-1}(S(B,C))$. We deduce that

$$\begin{aligned}\Phi_{n+1}(G) \subset \Phi_n^{-1}(S(B,C)) &\subset \Phi_n\big(\Phi_n^{-1}(S(B,C))\big)\\ &\subset S(B,C) \subset \Phi_n(B)\cap C \subset C.\end{aligned}$$

Thus, the interior of $\Phi_{n+1}(G)$, which is nonempty because G is a box and Φ_{n+1} is an embedding, is contained in the interior of $C \in \mathcal{A}_n$. Since members of $\mathcal{A}_n$ are pairwise disjoint, we conclude that, for all $G \in \Omega(D,B)$, $G \in \Gamma_{n+1}(D,B,C)$ if and only if $G \xrightarrow{\Phi_{n+1}} C$, as required.

(f) If $G \in \Gamma_{n+1}(D,B,C)$ then $G \subset S(D,B) \subset \Phi_n(D)\cap B$. Therefore

$$\Phi_{n+1}(G) \subset \Phi_{n+1}(B) = \Phi_n(B).$$

Moreover, we have proved above that

$$\Phi_{n+1}(G) \subset C.$$

Assume that $\Phi_{n+1}(G)\cap E \neq \emptyset$ for some $E \in \mathcal{A}_{n+1}$. Since $E \in \Omega_{n+1}(B',C')$ for some $(B',C') \in \mathcal{A}_n^{2*}$, we have

$$E \subset S(B',C') \subset \Phi_n(B')\cap C'.$$

Since $\Phi_{n+1}(G)\cap E \neq \emptyset$, we deduce that $\Phi_n(B)\cap\Phi_n(B')\cap C\cap C' \neq \emptyset$. Since distinct atoms of generation n are disjoint and Φ_n is one-to-one, we conclude that $B = B'$, $C = C'$ and $E \in \Omega_{n+1}(B,C)$. ∎

Remark 2.12. Lemma 2.11(a) immediately implies that for $0 \le j \le n$:

- the families $\mathcal{A}_j^{2*}$ and $\mathcal{A}_j^{3*}$ for Φ_n and for Φ_{n+1} coincide,
- the members of the same families $\mathcal{A}_j$ are also atoms of the respective generations $0,1,\dots,n$ for Φ_{n+1}.

Parts (b)–(e) of Lemma 2.11 ensure that the family $\mathcal{A}_{n+1}$ of boxes constructed in Remark 2.9 satisfies conditions (i) and (a)–(d) of Definition 2.3 for Φ_{n+1}. Thus, the members of $\mathcal{A}_{n+1}$ are good candidates to be atoms of generation $n+1$ for Φ_{n+1}.

To actually obtain atoms of generation $n+1$ for Φ_{n+1} we will further modify the map in the interior of $S(D,B)\cap\Phi_n^{-1}S(B,C)$ for all $(D,B,C) \in \mathcal{A}_n^{3*}(\Phi_n)$ in such a way that for the new embedding Φ_{n+1} the boxes of $\mathcal{A}_{n+1}$ also satisfy condition (ii) of Definition 2.3.

Lemma 2.13. *Keeping the notation of Remark 2.9, let $\widetilde{L}_{n+1} \subset D^m$ be a finite set of cardinality $2^{(n+1)^2}2^{n+1}$, with a unique point $\widetilde{e}_i(E) \in \widetilde{L}_{n+1}$ for each $(i,E) \in \{1,\dots,2^{n+1}\}\times\mathcal{A}_{n+1}$. Assume that*

$$\widetilde{e}_i(E) \in \mathrm{int}(E) \quad \forall (i,E) \in \{1,\dots,2^{n+1}\}\times\mathcal{A}_{n+1}.$$

Then there exists a permutation $\theta : \widetilde{L}_{n+1} \to \widetilde{L}_{n+1}$ such that:

(a) *For all* $(i, E) \in \{1, \ldots, 2^{n+1}\} \times \mathit{\Gamma}_{n+1}(D, B, C)$ *for some* $(D, B, C) \in \mathcal{A}_n^{3*}(\mathit{\Phi}_n)$,

$$\theta(\widetilde{e}_i(E)) = \widetilde{e}_{i'}(E')$$

for a unique $i' \in \{1, \ldots, 2^{n+1}\}$ *and a unique* $E' \in \mathit{\Omega}_{n+1}(B, C)$.

(b) *For all* $(D, B, C) \in \mathcal{A}_n^{3*}(\mathit{\Phi}_n)$, *all* $E \in \mathit{\Gamma}_{n+1}(D, B, C)$ *and all* $F \in \mathit{\Omega}_{n+1}(B, C)$ *there exists a unique*

$$(i, i') \in \{1, \ldots, 2^{n+1}\}^2$$

such that $\theta(\widetilde{e}_i(E)) = \widetilde{e}_{i'}(F)$.

(c) *For all* $(B, C) \in \mathcal{A}_n^{2*}(\mathit{\Phi}_n)$,

$$\theta\Big(\Big\{\widetilde{e}_i(E) : E \in \bigcup_{D \in \mathcal{A}_n :\, (D,B) \in \mathcal{A}_n^{2*}} \mathit{\Gamma}_{n+1}(D, B, C),\ i \in \{1, \ldots, 2^{n+1}\}\Big\}\Big)$$
$$= \{\widetilde{e}_{i'}(F) : F \in \mathit{\Omega}_{n+1}(B, C),\ i' \in \{1, \ldots, 2^{n+1}\}\} = \widetilde{L}_{n+1} \cap S(B, C).$$

Proof. From the construction of $\mathcal{A}_{n+1}$ (see Remark 2.9), we deduce that for all $E \in \mathcal{A}_{n+1}$ there exist a unique $j \in \{0, 1\}$ and a unique $(D, B, C) \in \mathcal{A}_n^{3*}$ such that

$$E = G_j(D, B, C) \in \mathit{\Gamma}_{n+1}(D, B, C)$$

(recall (2.4)) and thus we will write

$$\widetilde{e}_i(G_j(D, B, C)) = \widetilde{e}_i(E)$$

for all $(i, E) \in \{1, \ldots, 2^{n+1}\} \times \mathcal{A}_{n+1}$.

By hypothesis, $\mathcal{A}_n$ is the family of atoms of generation n for $\mathit{\Phi}_n$, thus we can apply the equalities in (ii) of Definition 2.3. So, for each $B \in \mathcal{A}_n$, we can index the different atoms $D \in \mathcal{A}_n$ such that $D \xrightarrow{\mathit{\Phi}_n} B$ as follows:

$$\{D \in \mathcal{A}_n : D \xrightarrow{\mathit{\Phi}_n} B\} = \{D_1^-(B), \ldots, D_{2^n}^-(B)\}, \tag{2.5}$$

where $D_{k_1}^-(B) \neq D_{k_2}^-(B)$ if $k_1 \neq k_2$.

Analogously,

$$\{C \in \mathcal{A}_n : B \xrightarrow{\mathit{\Phi}_n} C\} = \{C_1^+(B), \ldots, C_{2^n}^+(B)\}, \tag{2.6}$$

where $C_{l_1}^+(B) \neq C_{l_2}^+(B)$ if $l_1 \neq l_2$.

Now, we index the distinct points of $\widetilde{L}_{n+1}$ as follows:

$$\widehat{e}_{i,j}(k, B, l) := \widetilde{e}_i(G_j(D, B, C)) = \widetilde{e}_i\big(G_j(D_k^-(B), B, C_l^+(B))\big)$$

for all $(i, j, B, k, l) \in \{1, \ldots, 2^{n+1}\} \times \{0, 1\} \times \mathcal{A}_n \times \{1, \ldots, 2^n\}^2$.

Define a correspondence $\theta : \widetilde{L}_{n+1} \to \widetilde{L}_{n+1}$ by

$$\theta(\widehat{e}_{i,j}(k, B, l)) = \widehat{e}_{i',j'}(k', B', l'),$$

where

- $B' := C_l^+(B)$,
- k' is such that $B = D_{k'}^-(C)$ (exists and is unique because $B \xrightarrow{\Phi_n} C$, by (2.5)),
- $l' = i \pmod{2^n}$,
- $j' = 0$ if $i \leq 2^n$ and $j' = 1$ if $i > 2^n$,
- $i' = k + j \cdot 2^n$.

Let us prove that θ is surjective; hence it is a permutation of the finite set $\widetilde{L}_{n+1}$.

Let $\widehat{e}_{i',j'}(k', B', l') \in \widetilde{L}_{n+1}$ be given, where

$$(i', j', B', k', l') \in \{1, \dots, 2^{n+1}\} \times \{0, 1\} \times \mathcal{A}_n \times \{1, \dots, 2^n\}^2.$$

Construct:

- $i := l' + j' \cdot 2^n$. Then $l' = i \pmod{2^n}$, $j' = 0$ if $i \leq 2^n$ and $j' = 1$ if $i > 2^n$.
- $B := D_{k'}^-(B')$. Then $B \xrightarrow{\Phi_n} B'$. So, there exists l such that $B' = C_l^+(B)$.
- $k := i' \pmod{2^n}$, $j := 0$ if $i' \leq 2^n$ and $j := 1$ if $i' > 2^n$. Therefore $i' = k + 2^n j$.

Thus we have constructed some θ^{-1} such that $\theta \circ \theta^{-1}$ is the identity map. So, θ is surjective, hence also one-to-one in the finite set $\widetilde{L}_{n+1}$, as required.

Now, let us prove that θ satisfies the assertions of Lemma 2.13.

(a) Fix $\widetilde{e}_i(E) \in \widetilde{L}_{n+1}$. By construction, $\theta(\widetilde{e}_i(E)) = \widetilde{e}_{i'}(E') \subset \operatorname{int}(E')$ for some $(i, E) \in \{1, \dots, 2^{n+1}\} \times \mathcal{A}_{n+1}$. Since members of $\mathcal{A}_{n+1}$ are pairwise disjoint (recall Lemma 2.11(b)), the box E' is unique. Moreover, by hypothesis, $\widetilde{e}_{i'}(E') \neq \widetilde{e}_{j'}(E')$ if $i' \neq j'$. So, the index i' is also unique. Therefore, to finish the proof of (a), it is enough to check that $E' \in \Omega_{n+1}(B, C)$ if $E \in \Gamma_{n+1}(D, B, C)$.

By the definition of the family $\Gamma_{n+1}(D, B, C)$ in Remark 2.9, if $E \in \Gamma_{n+1}(D, B, C)$, there exists $j \in \{0, 1\}$ such that $E = G_j(D, B, C)$. Thus, using the notation at the beginning, $\widetilde{e}_i(E) = \widetilde{e}_i(G_j(D, B, C)) = \widehat{e}_{i,j}(k, B, l)$, where $D = D_k^-(B)$ and $C = C_l^+(B)$. Then, using the definition of the permutation θ and the computation of its inverse θ^{-1}, we obtain $\widetilde{e}_{i'}(E) = \theta(\widetilde{e}_i(E)) = \widehat{e}_{i',j'}(k', B', l')$, where

$$B' = C_l^+(B) = C, \qquad D' = D_{k'}^-(B') = B.$$

We have proved that $\widetilde{e}_{i'}(E') = \widetilde{e}_{i'}(G_{j'}(B, C, C'))$. Finally, from the definition of the family $\Omega_{n+1}(B, C)$ in Remark 2.9 we conclude that $E' \in \Omega_{n+1}(B, C)$, as asserted in part (a).

(b) Fix $(D, B, C) \in \mathcal{A}_n^{3*}$ and $E \subset \Gamma_{n+1}(D, B, C)$. Then, using the definition of $\Gamma_{n+1}(D, B, C)$, we have a unique $(j, k, l) \in \{0, 1\} \times \{1, \dots, 2^n\}^2$ such that $E = G_j(D, B, C)$, $D = D_k^-(B)$, $C = C_l^+(B)$. Consider the finite set Z of 2^{n+1} distinct points $\widetilde{e}_i(E) = \widehat{e}_{i,j}(k, B, l)$, with j, k, B, l fixed as above and $i \in \{1, \dots, 2^{n+1}\}$. Let $i' := k + 2^n j$; then the image of each point in Z by the permutation θ is $\theta(\widetilde{e}_i(E)) = \widetilde{e}_{i'}(G_{j'}(B,C,C')$ (here we use assertion (a)). Since k, j are fixed, we deduce that there exists a unique i' such that all the points of $\theta(Z)$ are of the form $\widetilde{e}_{i'}(F)$, $F = G_{j'}(B,C,C')$ with $j' \in \{0,1\}$, $C' = C^+_{k'}(C)$, $k' \in \{1,\dots,2^{n+1}\}$. We have proved that the permutation $\theta|_Z$ is equivalent to

$$\{1,\dots,2^{n+1}\} \ni i \mapsto (j',k') \in \{0,1\} \times \{1,\dots,2^n\}$$

such that $\theta(\widetilde{e}_i(E)) = \widetilde{e}_{i'}(G_{j'}(B,C,C^+_{k'}(C)))$ with i' fixed.

Since $\#\{1,\dots,2^{n+1}\} = \#(\{0,1\} \times \{1,\dots,2^n\})$, from the injectivity of θ we deduce that $\theta(Z) = \{0,1\} \times \{1,\dots,2^n\}$. Thus for every $F \in \Omega(B,C)$ there exists a unique i such that $\theta(\widetilde{e}_i(E)) = \widetilde{e}_{i'}(F)$ (where i' is uniquely defined given E). This ends the proof of (b).

(c) For fixed $(B,C) \in \mathcal{A}^{2*}_n$, denote

$$P := \Big\{\widetilde{e}_i(E) : E \in \bigcup_{D\in\mathcal{A}_n,\, D \xrightarrow{\Phi_n} B} \Gamma_{n+1}(D,B,C),\ i \in \{1,\dots,2^{n+1}\}\Big\},$$

$$Q := \{\widetilde{e}_{i'}(F) : F \in \Omega_{n+1}(B,C),\ i' \in \{1,\dots,2^{n+1}\}\} \subset \widetilde{L}_{n+1}.$$

Applying (a) we deduce that $\theta(P) \subset Q$. So, to prove that $\theta(P) = Q$ it is enough to prove that $\#P = \#Q$. Applying Lemma 2.11 for the family of boxes $\mathcal{A}_{n+1}$ for the family of atoms $\mathcal{A}_n$, we obtain

$$\#P = 2^{n+1} \cdot \#\Gamma_{n+1}(D,B,C) \cdot \#\{D \in \mathcal{A}_n : D \xrightarrow{\Phi_n} B\} = 2^{n+1} \cdot 2 \cdot 2^n,$$
$$\#Q = 2^{n+1} \cdot \#\Omega_{n+1}(B,C) = 2^{n+1} \cdot 2^{n+1},$$

which proves that $\#P = \#Q$ and thus $\theta(P) = Q$.

Finally, let us prove that $Q = \widetilde{L}_{n+1} \cap S(B,C)$. On the one hand, if $F \in \Omega_{n+1}(B,C)$, then $F = G_j(B,C,C')$ for some (j,C'). Applying the construction of the boxes of $\mathcal{A}_{n+1}$ in Remark 2.9, we obtain $F \subset S(B,C)$, hence $\widetilde{e}_{i'}(F) \in \widetilde{L}_{n+1} \cap \operatorname{int}(F) \subset \widetilde{L}_{n+1} \cap S(B,C)$. This proves that $Q \subset \widetilde{L}_{n+1} \cap S(B,C)$.

On the other hand, if $\widetilde{e}_{i'}(F) \in \widetilde{L}_{n+1} \cup S(B,C)$, then $F \in \mathcal{A}_{n+1}$. We obtain $F = G_j(D',B',C') \subset S(D',B')$ for some $(D',B',C') \in \mathcal{A}^{3*}_n$. Since $S(D',B') \subset \Phi_n(D') \cap B'$ and $S(B,C) \subset \Phi_n(B) \cap C$, we deduce that $S(D',B') \cap S(B,C) = \emptyset$ if $(D',B') \neq (B,C)$. But $\widetilde{e}_{i'}(F) \in \operatorname{int}(F) \cap S(B,C) \subset S(D',B') \cap S(B,C)$. Consequently, $(D',B') = (B,C)$, thus $F = G_j(B,C,C') \in \Omega_{n+1}(B,C)$, hence $\widetilde{e}_{i'}(F) \in Q$. We have proved that $\widetilde{L}_{n+1} \cap S(B,C) \subset Q$. ∎

LEMMA 2.14. *Assume the hypotheses of Lemmas* 2.11 *and* 2.13. *Let* $\Phi_{n+1} \in \operatorname{Emb}(D^m)$ *moreover satisfy*

$$\Phi_{n+1}(\widetilde{e}) = \theta(\widetilde{e}) \quad \forall \widetilde{e} \in \widetilde{L}_{n+1},$$

where θ *is the permutation of* $\widetilde{L}_{n+1}$ *constructed in Lemma* 2.13. *Then:*

(a) $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_{n+1}$ *are collections of atoms up to generation* $n+1$ *for* Φ_{n+1}.

(b) *For each* $(E,F) \in \mathcal{A}^2_{n+1}$ *such that* $E \xrightarrow{\Phi_{n+1}} F$, *there exists exactly one point* $\widetilde{e}_i(E) \in \widetilde{L}_{n+1} \cap \operatorname{int}(E)$, *and exactly one point* $\widetilde{e}_{i'}(F) \in \widetilde{L}_{n+1} \cap \operatorname{int}(F)$, *such that*

$$\Phi_{n+1}(\widetilde{e}_i(E)) = \widetilde{e}_{i'}(F).$$

Proof. (a) By Remark 2.12, it is enough to establish the truth of condition (ii) of Definition 2.3 with $n+1$ instead of n.

Take $E \in \mathcal{A}_{n+1}$. There exists $(D,B,C) \in \mathcal{A}^{3^*}_n$ with $E \in \Gamma_{n+1}(D,B,C)$. Take $F \in \Omega_{n+1}(B,C)$. By Lemma 2.13(b), there exists a unique (i,i') such that $\theta(\widetilde{e}_i(E)) = \widetilde{e}_{i'}(F)$. Therefore

$$\Phi_{n+1}(\widetilde{e}_i(E)) = \widetilde{e}_{i'}(F).$$

As $\widetilde{e}_i(E) \in \operatorname{int}(E)$ and $\widetilde{e}_{i'}(F) \in \operatorname{int}(F)$, we conclude that $\Phi_{n+1}(E) \cap \operatorname{int}(F) \neq \emptyset$, that is, $E \xrightarrow{\Phi_{n+1}} F$. We have proved that

$$E \xrightarrow{\Phi_{n+1}} F \quad \forall E \in \Gamma_{n+1}(D,B,C),\ \forall F \in \Omega_{n+1}(B,C).$$

Combining this with Lemma 2.11(f), we deduce that for all $(D,B,C) \in \mathcal{A}^{3*}_n$, all $E \in \Gamma_{n+1}(D,B,C)$, and all $F \in \mathcal{A}_{n+1}$,

$$E \xrightarrow{\Phi_{n+1}} F \quad \text{if and only if} \quad F \in \Omega_{n+1}(B,C). \tag{2.7}$$

Given $E \in \mathcal{A}_n$, let us count how many $F \in \mathcal{A}_n$ satisfy $E \xrightarrow{\Phi_{n+1}} F$. Given E, there exists a unique $(D,B,C) \in \mathcal{A}^{3*}_n$ with $E \in \Gamma_{n+1}(D,B,C)$. Applying (2.7) and Lemma 2.11(d), we deduce

$$\#\{F \in \mathcal{A}_n : E \xrightarrow{\Phi_{n+1}} F\} = \Omega_{n+1}(B,C) = 2^{n+1}.$$

Finally, given $F \in \mathcal{A}_n$, let us count how many $E \in \mathcal{A}_n$ satisfy $E \xrightarrow{\Phi_{n+1}} F$. Given F, there exists a unique $(B,C) \in \mathcal{A}^{2*}_n$ such that $F \in \Omega_{n+1}(B,C)$. Applying (2.7), Lemma 2.11(e), and Definition 2.3(ii) for the atoms of generation n (for Φ_n and for Φ_{n+1}), we obtain

$$\begin{aligned}
&\#\{E \in \mathcal{A}_n : E \xrightarrow{\Phi_{n+1}} F\} \\
&\qquad = \#\{E \in \mathcal{A}_{n+1} : \exists D \in \mathcal{A}_n \text{ such that } D \xrightarrow{\Phi_{n+1}} B, E \in \Gamma_{n+1}(D,B,C)\} \\
&\qquad = \#\{D \in \mathcal{A}_n : D \xrightarrow{\Phi_{n+1}} B\} \cdot \#\Gamma_{n+1}(D,B,C) = 2^n \cdot 2 = 2^{n+1}.
\end{aligned}$$

We have proved that the boxes of $\mathcal{A}_{n+1}$ satisfy the equalities in (ii) of Definition 2.3 for Φ_{n+1}. The proof of (a) is complete.

(b) Take $(D,B,C) \in \mathcal{A}^{3*}_n$ and $E \in \Gamma_{n+1}(D,B,C)$. Take $F \in \mathcal{A}_{n+1}$. From Remark 2.4 (putting $n+1$ instead of n), we know that $\Phi_{n+1}(E) \cap F \neq \emptyset$ if and only if $F \in \Omega_{n+1}(B,C)$. Applying Lemma 2.13(b) we find a unique

$(i,i') \in \{1,\dots,2^{n+1}\}^2$ such that $\Phi_{n+1}(\widetilde{e}_i(E)) = \theta(\widetilde{e}_i(E)) = \widetilde{e}_{i'}(F)$. The proof of part (b) is complete. ■

LEMMA 2.15. *Let $\psi \in \mathrm{Emb}(D^m)$, $r \ge 1$, $P_1,\dots,P_r \subset D^m$ be pairwise disjoint boxes, and $Q_j := \psi(P_j)$ for all $j \in \{1,\dots,r\}$. For $k \ge 1$ and $j \in \{1,\dots,r\}$, let $p_{1,j},\dots,p_{k,j} \in \mathrm{int}(P_j)$ be distinct points and $q_{1,j},\dots,q_{k,j} \in \mathrm{int}(Q_j)$ also be distinct points. Then there exists a $\psi^* \in \mathrm{Emb}(D^m)$ such that*

$$\psi^*(x) = \psi(x) \qquad \forall x \notin \bigcup_{j=1}^{r} \mathrm{int}(P_j),$$
$$\psi^*(p_{i,j}) = q_{i,j} \qquad \forall (i,j) \in \{1,\dots,k\} \times \{1,\dots,r\}.$$

Proof. This is straightforward. ■

Proof of Lemma 2.8. We divide the construction of ψ and $\Phi \in \mathcal{H}$ into several steps:

STEP 1: *Construction of the atom of generation* 0. Since $f(D^m) \subset \mathrm{int}(D^m)$, there exists a box $A_0 \subset \mathrm{int}(D^m)$ such that $f(D^m) \subset \mathrm{int}(A_0)$. The box A_0 is the *atom of generation* 0 for the embedding $\Phi_0 := f$ which satisfies $\Phi_0 = \psi_0 \circ f$, where ψ_0 is the identity map. By the Brouwer Fixed Point Theorem, there exists a point $e_0 \in \mathrm{int}(\Phi_0(A_0))$ such that $\Phi_0(e_0) = e_0$. Define $S(A_0, A_0)$ to be the connected component of $A_0 \cap \Phi_0(A_0)$ containing e_0.

Note that $A_0 \cap \Phi_0(A_0) = \Phi(A_0)$ is connected. We introduce the notation $S(A_0, A_0)$ to stress that the inductive hypothesis is satisfied for $n = 0$.

STEP 2: *Construction of the atoms of generation* $n+1$. Inductively assume that we have constructed the families $\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_n$ of atoms up to generation n for $\Phi_n = \psi_n \circ f$, where $\psi_n \in \mathrm{Hom}(D^m)$, satisfying:

(I) $\psi_n|_{\partial D^m}$ is the identity map.
(II) $\max_{B \in \mathcal{A}_i} \max\{\mathrm{diam}(B), \mathrm{diam}(f(B))\} < 1/2^i$ for $i \in \{0,1,\dots,n\}$.
(III) $\Phi_i(x) = \Phi_{i-1}(x)$ for $x \in D^m \setminus \bigcup_{B \in \mathcal{A}_{i-1}} B$ and $i \in \{1,\dots,n\}$.
(IV) For all $(D,B,C) \in \mathcal{A}_n^{3*}(\Phi_n)$ there exists a point $e(D,B,C)$ such that

$$L_n := \{e(D,B,C) : (D,B,C) \in \mathcal{A}_n^{3*}\}$$

is Φ_n-invariant, and

$$e(D,B,C) \in \mathrm{int}\big(S(D,B) \cap \Phi_n^{-1}(S(B,C))\big), \tag{2.8}$$

where $S(D,B)$ and $S(B,C)$ are (properly chosen) connected components of $B \cap \Phi_n(D)$ and of $C \cap \Phi_n(B)$ respectively. (Recall the notation in Remark 2.9). Note that the sets $S(B,C)$ and $S(B',C')$ are disjoint if $(B,C) \neq (B',C')$, because two different atoms of generation n for Φ_n are disjoint (recall Definition 2.3) and Φ_n is one-to-one.

Next we will construct the family $\mathcal{A}_{n+1}$ of boxes, candidates to be atoms of generation $n+1$ for a new embedding Φ_{n+1} (as in Remark 2.9), and ψ_{n+1} such that $\Phi_{n+1} = \psi_{n+1} \circ f$.

First, for each $(B,C) \in \mathcal{A}_n^{2^*}(\Phi_n)$, we choose a box $R(D,B)$ such that

$$e(D,B,C) \in \operatorname{int}(R(B,C)), \qquad R(B,C) \subset \operatorname{int}(\Phi_n^{-1}(S(B,C)) \qquad \forall D \in \mathcal{A}_n \text{ such that } D \xrightarrow{\Phi_n} B. \tag{2.9}$$

Note that such boxes $R(\cdot,\cdot)$ are pairwise disjoint, because they are contained in pairwise disjoint sets.

Recall that $e(D,B,C) \in L_n$ and the set L_n is Φ_n-invariant. Consider assertions (2.8) and (2.9). Thus,

$$e(D,B,C) \in \operatorname{int}\big(R(B,C) \cap \Phi_n(R(D,B))\big) \neq \emptyset.$$

Next, for each $(D,B,C) \in \mathcal{A}_n^{3*}$ we choose two disjoint boxes, $G_0(D,B,C)$ and $G_1(D,B,C)$, contained in the interior of $R(B,C) \cap \Phi_n(R(D,B))$ that satisfy

$$\max\{\operatorname{diam}(G_i(D,B,C)), \operatorname{diam}(f(G_i(D,B,C)))\} < \frac{1}{2^{n+1}} \tag{2.10}$$

for $i = 0,1$. Now, we use the notation of Remark 2.9 to construct the family $\mathcal{A}_{n+1}$ of all the boxes $G_i(D,B,C)$. These boxes will be the $(n+1)$-atoms of two new embeddings $\widetilde{\Phi}_{n+1}$ and Φ_{n+1} that we will construct as follows.

First, in the interior of each box $E \in \mathcal{A}_{n+1}$ we choose 2^{n+1} distinct points $\widetilde{e}_i(E)$, $i = 1, \ldots, 2^{n+1}$, and denote

$$\widetilde{L}_{n+1} := \{\widetilde{e}_i(E) : E \in \mathcal{A}_{n+1},\ 1 \leq i \leq 2^{n+1}\}.$$

Second, we build a permutation $\widetilde{\theta}$ of $\widetilde{L}_{n+1}$ satisfying the properties of Lemma 2.13.

Third, we would like to apply Lemma 2.15 to construct $\widetilde{\psi}_{n+1} \in \operatorname{Hom}(D^m)$ satisfying the following constraints:

(a) For all $(B,C) \in \mathcal{A}_n^{2^*}(\Phi_n)$,

$$\widetilde{\psi}_{n+1}|_{f(R(B,C))} : f(R(B,C)) \to \psi_n \circ f(R(B,C)) = \Phi_n(R(B,C)),$$

(b) For all $x \notin \bigcup_{(B,C)\in\mathcal{A}_n^{2*} \text{ for } \Phi_n} f(R(B,C))$,

$$\widetilde{\psi}_{n+1}(x) = \psi_n(x).$$

(c) For all $\widetilde{e} \in \widetilde{L}_{n+1}$,

$$\widetilde{\psi}_{n+1}(f(\widetilde{e})) = \widetilde{\theta}(\widetilde{e}).$$

We turn to the verification of the hypotheses of Lemma 2.15. The boxes $R(B,C)$ where $(B,C) \in \mathcal{A}_n^{2*}$ are pairwise disjoint, so their images by f are also pairwise disjoint boxes. Furthermore, for each $(B,C) \in \mathcal{A}_n^{2*}(\Phi_n)$, the finite set

$$\{f(\widetilde{e}) : \widetilde{e} \in \widetilde{L}_{n+1} \cap \operatorname{int}(R(B,C))\} = \{f(\widetilde{e}) : \widetilde{e} \in \widetilde{L}_{n+1} \cap \operatorname{int}(\varPhi_n^{-1}(S(B,C)))\}$$

is contained in the interior of $f(R(B,C))$. Moreover, it coincides with

$$\{f(\widetilde{e}_i(E)) : E \in \varGamma_{n+1}(D,B,C) \text{ for some } D \in \mathcal{A}_n,\ i = 1,\dots,2^{n+1}\}$$

(recall Lemma 2.10). So, the image by $\widetilde{\theta}$ of such points $\widetilde{e}(\cdot)$ is

$$\{\widetilde{\theta}(\widetilde{e}_i(E)) : E \in \varGamma_{n+1}(D,B,C) \text{ for some } D \in \mathcal{A}_n,\ i = 1,\dots,2^{n+1}\}.$$

By Lemma 2.13(c), this set is

$$\{\widetilde{e}_k(F) : F \in \varOmega_{n+1}(B,C),\ k = 1,\dots,2^{n+1}\} = \widetilde{L}_{n+1} \cap S(B,C),$$

which is contained in the interior of $\varPhi_n(R(B,C)) = \widetilde{\psi}_n(f(R(B,C)))$.

We have proved that the points $f(\widetilde{e}(\cdot))$ are in the interior of the boxes $f(R(\cdot,\cdot))$, and their images $\theta(\widetilde{e}(\cdot))$ by $\widetilde{\psi}_{n+1}$ (to be constructed) are in the interiors of the images by $\widetilde{\psi}_n$ of those boxes. So, the hypothesis of Lemma 2.15 is satisfied.

Let

$$\widetilde{\varPhi}_{n+1} := \widetilde{\psi}_{n+1} \circ f.$$

Since

$$\widetilde{\varPhi}_{n+1}(x) = \widetilde{\psi}_{n+1} \circ f = \widetilde{\psi}_n \circ f = \varPhi_n(x)$$
$$\forall x \notin \bigcup_{(B,C)\in\mathcal{A}_n^{2*} \text{ for } \varPhi_n} \operatorname{int}(R(B,C)) \subset \bigcup_{(B,C)\in\mathcal{A}_n^{2*} \text{ for } \varPhi_n} \operatorname{int}(\varPhi_n^{-1}(S(B,C))),$$

the hypothesis of Lemma 2.11 is satisfied. Therefore the same atoms up to generation n for $\varPhi_n$ are still atoms up to generation n for $\widetilde{\varPhi}_{n+1}$. But moreover, by Lemma 2.14(a), the boxes of the new family $\mathcal{A}_{n+1}$ are now $(n+1)$-atoms for $\widetilde{\varPhi}_{n+1}$.

Step 3: *Construction of $\varPhi_{n+1}$ and ψ_{n+1}.* To argue by induction, we will not use $\widetilde{\varPhi}_{n+1}$ and $\widetilde{\psi}_{n+1}$, even if $\widetilde{\varPhi}_{n+1} = \widetilde{\psi}_{n+1} \circ f$ already has families $\mathcal{A}_0,\dots,\mathcal{A}_n,\mathcal{A}_{n+1}$ of atoms up to generation $n+1$, as required. Rather, we will modify them to obtain a new $\varPhi_{n+1}$ and a new ψ_{n+1} such that the inductive hypothesis (IV) and (2.8) also hold for $n+1$ instead of n. We will modify $\widetilde{\psi}_{n+1}$ only in the interiors of the boxes $f(G)$ for all the atoms $G \in \mathcal{A}_{n+1}$ for $\widetilde{\varPhi}_{n+1}$, and we will construct a new homeomorphism ψ_{n+1} such that $\varPhi_{n+1} := \psi_{n+1} \circ f$ has the same atoms up to generation $n+1$ for $\widetilde{\varPhi}_{n+1}$ (see the proof of Lemma 2.11(a)), and moreover satisfies (IV) with $n+1$ instead of n.

From the above construction of $\widetilde{\psi}_{n+1}$ and $\widetilde{\Phi}_{n+1}$, and from Lemma 2.14(b), we know that for each $(G,E) \in \mathcal{A}^{2*}_{n+1}$ for $\widetilde{\Phi}_{n+1}$ there exists a unique point $\widetilde{e}_i(G) \in \operatorname{int}(G)$, and a unique point $\widetilde{e}_k(E)$, such that

$$\widetilde{\Phi}_{n+1}(\widetilde{e}_i(G)) = \widetilde{\psi}_{n+1} \circ f(\widetilde{e}_i(G)) = \widetilde{\theta}(\widetilde{e}_i(G)) = \widetilde{e}_k(E) \in \operatorname{int}(E).$$

Therefore

$$\widetilde{e}_k(E) \in \operatorname{int}(E \cap \widetilde{\Phi}_{n+1}(G)).$$

Denote by $S(G,E)$ the connected component of $E \cap \widetilde{\Phi}_{n+1}(G)$ that contains $\widetilde{e}_k(E)$. Choose 2^{n+1} distinct points

$$e_i(G,E) \in \operatorname{int}(S(G,E)), \qquad i = 1, \dots, 2^{n+1},$$

and a permutation θ of the finite set

$$L_{n+1} := \{e_i(G,E) : (G,E) \in \mathcal{A}^{2*}_{n+1} \text{ for } \widetilde{\Phi}_{n+1},\ i = 1, \dots, 2^{n+1}\} \tag{2.11}$$

such that for each fixed $(G,E,F) \in \mathcal{A}^{3*}_{n+1}$ for $\widetilde{\Phi}_{n+1}$, there exists a unique point $e_i(G,E)$, and a unique point $e_k(E,F)$, satisfying

$$\theta(e_i(G,E)) = e_k(E,F). \tag{2.12}$$

The proof of the existence of such a permutation is similar to (but simpler than) the proof of Lemma 2.13.

Applying Lemma 2.15, construct $\psi_{n+1} \in \operatorname{Hom}(D^m)$ such that

$$\begin{aligned}
&\psi_{n+1}|_{f(G)} : f(G) \to \widetilde{\psi}_{n+1}(f(G)) = \widetilde{\Phi}_{n+1}(G) \quad \forall G \in \mathcal{A}_{n+1} \text{ for } \widetilde{\Phi}_{n+1},\\
&\psi_{n+1}(x) = \widetilde{\psi}_{n+1}(x) \quad \forall x \notin \bigcup_{G \in \mathcal{A}_{n+1}} f(G),\\
&\psi_{n+1}(f(e_i(G,E)) = \theta(e_i(G,E))\\
&\qquad\qquad \forall (E,G) \in \mathcal{A}^2_{n+1} \text{ such that } G \xrightarrow{\widetilde{\Phi}_{n+1}} E,\ \forall i = 1, \dots, 2^{n+1},
\end{aligned} \tag{2.13}$$

and extend ψ_{n+1} to the whole box D^m by defining $\psi_{n+1}(x) = \widetilde{\psi}_{n+1}(x)$ for $x \in D^m \setminus \bigcup_{G \in \mathcal{A}_{n+1}} f(G)$. In particular,

$$\psi_{n+1}|_{\partial D^m} = \widetilde{\psi}_{n+1}|_{\partial D^m} = \operatorname{id}|_{\partial D^m}.$$

Define

$$\Phi_{n+1} := \psi_{n+1} \circ f. \tag{2.14}$$

As said above, the property that Φ_{n+1} coincides with $\widetilde{\Phi}_{n+1}$ outside all the atoms of $\mathcal{A}_{n+1}$ for $\widetilde{\Phi}_{n+1}$ implies that the boxes of the families $\mathcal{A}_0, \dots, \mathcal{A}_{n+1}$, which are the family of atoms up to generation n for $\widetilde{\Phi}_{n+1}$, are also atoms up to generation $n+1$ for Φ_{n+1}. But now, due to (2.12)–(2.14), they have the following additional property: there exists a one-to-one correspondence between the triples $(G,E,F) \in \mathcal{A}^{3*}_{n+1}$ (for $\widetilde{\Phi}_{n+1}$ and also for Φ_{n+1}) and the points of the set L_{n+1} of (2.11), such that

$$e(G,E,F) := e_i(G,E) \in \operatorname{int}\big(S(G,E) \cap \Phi_{n+1}^{-1}(S(E,F))\big). \tag{2.15}$$

Recall that $S(G,E)$ and $S(E,F)$ are the connected components of $E \cap \Phi_n(G)$ and of $F \cap \Phi_n(E)$ respectively that were chosen after $\widetilde{\Phi}_{n+1}$ was constructed.

By construction, the finite set L_{n+1} satisfies $\Phi_{n+1}(L_{n+1}) = \psi_{n+1}(f(L_{n+1})) = \theta(L_{n+1}) = L_{n+1}$. Therefore, (I)–(IV) hold for $n+1$ and the inductive construction is complete.

STEP 4: *The limit homeomorphisms.* From the above construction we have

$$\psi_{n+1}(x) = \widetilde{\psi}_{n+1}(x) = \psi_n(x) \quad \text{ if } x \notin \bigcup_{B,C} \psi_n^{-1}(R(B,C)) \subset \bigcup_{B} f(B)$$

and

$$\begin{aligned}\psi_{n+1} \circ \psi_n^{-1}(R(B,C)) &= \widetilde{\psi}_{n+1} \circ \psi_n^{-1}(R(B,C)) \\ &= \psi_n \circ \psi_n^{-1}(R(B,C)) = R(B,C) \subset C.\end{aligned}$$

Therefore,

$$\operatorname{dist}(\psi_{n+1}^{-1}(x), \psi_n^{-1}(x)) \le \max_{B\in\mathcal{A}_n} \operatorname{diam}(f(B)) < \frac{1}{2^n} \quad \forall x \in D^m,$$

$$\operatorname{dist}(\psi_{n+1}(x), \psi_n(x)) \le \max_{C\in\mathcal{A}_n} \operatorname{diam}(C) < \frac{1}{2^n} \quad \forall x \in D^m,$$

$$\|\psi_{n+1} - \psi_n\|_{\mathrm{Hom}} < \frac{1}{2^n}.$$

From these inequalities we deduce that the sequence ψ_n is Cauchy in $\mathrm{Hom}(D^m)$. Therefore, it converges to a homeomorphism ψ. Moreover, by construction, $\psi_n|_{\partial D^m} = \mathrm{id}|_{\partial D^m}$ for all $n \ge 1$. Then $\psi|_{\partial D^m} = \mathrm{id}|_{\partial D^m}$.

The convergence of ψ_n to ψ in $\mathrm{Hom}(D^m)$ implies that $\Phi_n = \psi_n \circ f \in \mathrm{Emb}(D^m)$ converges to $\Phi = \psi \circ f \in \mathrm{Emb}(D^m)$ as $n \to \infty$. Since $f(D^m) \subset \operatorname{int}(D^m)$ and $\psi \in \mathrm{Hom}(D^m)$, we deduce that $\Phi(D^m) \subset \operatorname{int}(D^m)$. Moreover, by construction, $\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_n$ are families of atoms up to generation n for Φ_n, and $\Phi_j(x) = \Phi_n(x)$ for all $x \in D^m \setminus \bigcup_{B\in\mathcal{A}_n} B$ and all $j \ge n$. Since $\lim_j \Phi_j = \Phi$, the boxes of the family $\mathcal{A}_n$ are n-atoms for Φ for all $n \ge 0$. From (II) the diameters of the n-atoms converge uniformly to zero as $n \to \infty$. Thus Φ is a model. This completes the proof of Lemma 2.8. ∎

3. Infinite metric entropy and mixing property of the models. The purpose of this section is to prove the following lemma.

LEMMA 3.1. *Let $\mathcal{H} \subset C^0(D^m)$ be a family of models with $m \ge 2$. For each $\Phi \in \mathcal{H}$ there exists a Φ-invariant mixing (hence ergodic) measure ν supported on a Φ-invariant Cantor set $\Lambda \subset D^m$ such that $h_\nu(\Phi) = +\infty$.*

Throughout this section we assume $m \geq 2$ and we suppose there is a given $\Phi \in \mathcal{H}$ with a given sequence of families $\mathcal{A}_n$ $(n \geq 0)$ of atoms of generations $n \geq 0$ for Φ. When we refer to the atoms of generation n, we omit writing Φ and the families of atoms of previous generations.

Remark 3.2. Lemma 3.1 holds, in particular, for $\mathcal{H} \cap \mathrm{Emb}(D^m)$.

To prove Lemma 3.1 we need to define the paths of atoms and to discuss their properties. We also need to define the invariant Cantor set Λ that will support the measure ν and prove some of its topological dynamical properties.

Definition 3.3 (Paths of atoms). Let $\Phi \in \mathcal{H} \subset C^0(D^m)$, $l \geq 2$ and let $(A_1, \ldots, A_l)$ be an l-tuple of atoms for Φ of the same generation n such that

$$A_i \xrightarrow{\Phi} A_{i+1} \quad \forall i \in \{1, \ldots, l-1\}.$$

We call $(A_1, \ldots, A_l)$ an *l-path of n-atoms from A_1 to A_l*. Let $\mathcal{A}_n^{l*}$ denote the family of all the l-paths of atoms of generation l.

Lemma 3.4. *For all $n \geq 1$, all $l \geq 2n$, and all $A_1, A_2 \in \mathcal{A}_n$ there exists an l-path of n-atoms from A_1 to A_2.*

Proof. For $n = 1$, the result is trivial. Let us assume by induction that the result holds for some $n - 1 \geq 1$ and let us prove it for n.

Let $E, F \in \mathcal{A}_n$. From equality (2.2) of Remark 2.4, there exist unique atoms $B_{-1}, B_0, B_1 \in \mathcal{A}_{n-1}$ such that $E \in \Gamma_n(B_{-1}, B_0, B_1)$. Then $B_{-1} \xrightarrow{\Phi} B_0$, $E \subset B_0$ and, by Remark 2.4,

$$E \xrightarrow{\Phi} E_1 \quad \forall E_1 \in \Omega_n(B_0, B_1). \tag{3.1}$$

Analogously, there exist unique atoms $B_*, B_{*+1} \in \mathcal{A}_{n-1}$ such that $F \in \Omega_n(B_*, B_{*+1})$. Then $B_* \xrightarrow{\Phi} B_{*+1}$, $F \subset B_{*+1}$ and

$$E_* \xrightarrow{\Phi} F \quad \forall E_* \in \bigcup_{\substack{B_{*-1} \in \mathcal{A}_{n-1} \\ B_{*-1} \xrightarrow{\Phi} B_*}} \Gamma_n(B_{*-1}, B_*, B_{*+1}). \tag{3.2}$$

Since $B_1, B_* \in \mathcal{A}_{n-1}$, the induction hypothesis ensures that for all $l \geq 2n-2$ there exists an l-path $(B_1, \ldots, B_l)$ from B_1 to $B_l = B_*$. We write $B_{*-1} = B_{l-1}$, $B_* = B_l$, $B_{*+1} = B_{l+1}$. So (3.2) becomes

$$E_l \xrightarrow{\Phi} F \quad \forall E_l \in \Gamma_n(B_{l-1}, B_l, B_{l+1}). \tag{3.3}$$

Taking into account that $B_{i-1} \xrightarrow{\Phi} B_i$ for $1 < i \leq l$, and applying Remark 2.4, we deduce that if $E_{i-1} \in \Gamma_n(B_{i-2}, B_{i-1}, B_i) \subset \mathcal{A}_n$, then

$$E_{i-1} \xrightarrow{\Phi} E_i \quad \forall E_i \in \Omega_n(B_{i-1}, B_i), \forall 1 < i \leq l. \tag{3.4}$$

Combining (3.1), (3.3) and (3.4) yields an $(l+2)$-path $(E, E_1, \dots, E_l, F)$ of atoms of generation n from E to F, as required. ∎

LEMMA 3.5. *Let $n, l \geq 2$. For each l-path $(B_1, \dots, B_l)$ of $(n-1)$-atoms there exists an l-path $(E_1, \dots, E_l)$ of n-atoms such that $E_i \subset \operatorname{int}(B_i)$ for all $i = 1, \dots, l$.*

Proof. In the proof of Lemma 3.4, for each l-path $(B_1, \dots, B_l)$ of $(n-1)$-atoms we have constructed an l-path $(E_1, \dots, E_l)$ of n-atoms as required. ∎

DEFINITION 3.6 (The Λ-set). Let $\Phi \in \mathcal{H} \subset C^0(D^m)$ be a model map. Let $\mathcal{A}_0, \mathcal{A}_1, \dots$ be its sequence of families of atoms. The subset

$$\Lambda := \bigcap_{n \geq 0} \bigcup_{A \in \mathcal{A}_n} A$$

of $\operatorname{int}(D^m)$ is called the *Λ-set* of the map Φ.

From Definition 2.3, we know that, for each fixed $n \geq 0$, the set $\Lambda_n := \bigcup_{A \in \mathcal{A}_n} A$ is nonempty, compact, and $\operatorname{int}(\Lambda_n) \supset \Lambda_{n+1}$. Therefore, Λ is also nonempty and compact. Moreover, Λ_n is composed of a finite number of connected components $A \in \mathcal{A}_n$, which satisfy $\lim_{n\to\infty} \max_{A \in \mathcal{A}_n} \operatorname{diam}(A) = 0$ by Definition 2.5. Since $\Lambda := \bigcap_{n \geq 0} \Lambda_n$, we deduce that the Λ-set is a *Cantor set* contained in $\operatorname{int}(D^m)$.

LEMMA 3.7. *Let $n, l \geq 1$ and $A_1, A_2 \in \mathcal{A}_n$. If there exists an $(l+1)$-path from A_1 to A_2, then $\Phi^l(A_1 \cap \Lambda) \cap (A_2 \cap \Lambda) \neq \emptyset$.*

Proof. Assume that there exists an $(l+1)$-path from A_1 to A_2. So, from Lemma 3.5, for all $j \geq n$ there exist atoms $B_{j,1}, B_{j,2} \in \mathcal{A}_j$ and an $(l+1)$-path from $B_{j,1}$ to $B_{j,2}$ (with constant length $l+1$) such that

$$B_{n,i} = A_i, \quad B_{j+1,i} \subset B_{j,i} \quad \forall j \geq n, \forall i = 1, 2.$$

Construct the following two points x_1 and x_2:

$$\{x_i\} = \bigcap_{j \geq n_0} B_{j,i}, \quad i = 1, 2.$$

By Definition 3.6, $x_i \in A_i \cap \Lambda$. So, to finish the proof it is enough to prove that $\Phi^l(x_1) = x_2$.

Recall that l is fixed. Since Φ is uniformly continuous, for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $(y_0, y_1, \dots, y_l) \in (D^m)^l$ satisfies $d(\Phi(y_i), y_{i+1}) < \delta$ for $0 \leq i \leq l-1$, then the points y_0 and y_l satisfy $d(\Phi^l(y_0), y_l) < \varepsilon$. We choose δ small enough that additionally $d(\Phi^l(x), \Phi^l(y)) < \varepsilon$ if $d(x, y) < \delta$.

From (2.3), there exists $j \geq n$ such that $\operatorname{diam}(B_{j,i}) < \delta$. Since there exists an $(l+1)$-path from $B_{j,1}$ to $B_{j,2}$, there exists a $(y_0, \dots, y_l)$ as in the previous paragraph with $y_0 \in B_{j,1}$ and $y_l \in B_{j,2}$. Thus

$$\begin{aligned} d(\Phi^l(x_1), x_2) &\le d(\Phi^l(x_1), \Phi^l(y_0)) + d(\Phi^l(y_0), y_l) + d(y_l, x_1) \\ &< \operatorname{diam}(\Phi^l(B_{j,1})) + \varepsilon + \operatorname{diam}(B_{j,2}) < 3\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we obtain $\Phi^l(x_1) = x_2$, as required. ■

Lemma 3.8 (Topological dynamical properties of Λ).

(a) *The Λ-set of a model map $\Phi \in \mathcal{H}$ is Φ-invariant, i.e., $\Phi(\Lambda) = \Lambda$.*
(b) *The map Φ restricted to the Λ-set is topologically mixing.*
(c) *In particular, $\Phi^l(A_1 \cap \Lambda) \cap (A_2 \cap \Lambda) \neq \emptyset$ for any $n \ge 1$, any atoms $A_1, A_2 \in \mathcal{A}_n$ and $l \ge 2n - 1$.*

Proof. (a) Let $x \in \Lambda$ and let $\{A_n(x)\}_{n \ge 0}$ be the unique sequence of atoms such that $x \in A_n(x)$ and $A_n(x) \in \mathcal{A}_n$ for all $n \ge 0$. Then $\Phi(x) \in \Phi(A_n(x))$ for all $n \ge 0$. From Definition 2.3, for all $n \ge 0$ there exists an atom $B_n \in \mathcal{A}_n$ such that $A_n(x) \xrightarrow{\Phi} B_n$. Therefore $\Phi(A_n(x)) \cap B_n \neq \emptyset$. Let d denote the Hausdorff distance between subsets of D^m. We deduce

$$d(\Phi(x), B_n) \le \operatorname{diam}(\Phi(A_n(x))) + \operatorname{diam}(B_n).$$

Moreover, equality (2.3) and the continuity of Φ imply

$$\lim_{n \to +\infty} \max \{\operatorname{diam}(\Phi(A_n(x))), \operatorname{diam}(B_n)\} = 0.$$

Then for all $\varepsilon > 0$ there exists $n_0 \ge 0$ such that $d(\Phi(x), B_n) < \varepsilon$ for some atom $B_n \in \mathcal{A}_n$ for all $n \ge n_0$. Since any atom of any generation intersects Λ, we deduce that $d(\Phi(x), \Lambda) < \varepsilon$ for all $\varepsilon > 0$. Since Λ is compact, this implies $\Phi(x) \in \Lambda$. We have proved that $\Phi(\Lambda) \subset \Lambda$.

Now, let us prove the other inclusion. Let $y \in \Lambda$ and let $\{B_n(y)\}_{n \ge 0}$ be the unique sequence of atoms such that $y \in B_n(y)$ and $B_n(y) \in \mathcal{A}_n$ for all $n \ge 0$. From Definition 2.3, for all $n \ge 0$ there exists an atom $A_n \in \mathcal{A}_n$ such that $A_n \xrightarrow{\Phi} B_n(y)$. Therefore $\Phi(A_n) \cap B_n(y) \neq \emptyset$. We deduce that, for all $n \ge 0$, there exists a point $x_n \in A_n \in \mathcal{A}_n$ such that $\Phi(x_n) \in B_n(y)$. Since any atom A_n contains points of Λ, we obtain

$$d(x_n, \Lambda) \le \operatorname{diam}(A_n) \quad \text{and} \quad d(\Phi(x_n), y) \le \operatorname{diam}(B_n(y)) \quad \forall n \ge 0.$$

Let x be the limit of a convergent subsequence of $\{x_n\}_{n \ge 0}$. Applying (2.3) and the continuity of Φ, we deduce that $d(x, \Lambda) = 0$ and $d(\Phi(x), y) = 0$. This means that $y = \Phi(x)$ and $x \in \Lambda$. We have proved that $y \in \Phi(\Lambda)$ for all $y \in \Lambda$; that is, $\Lambda = \Phi(\Lambda)$, as required.

(c) We will prove a stronger assertion: for any two atoms, even of different generations, there exists $l_0 \ge 1$ such that

$$\Phi^l(A_1 \cap \Lambda) \cap (A_2 \cap \Lambda) \neq \emptyset \quad \forall l \ge l_0. \tag{3.5}$$

It is not restrictive to assume that A_1 and A_2 are atoms of the same generation n_0 (if not, take n_0 equal to the larger generation and substitute A_i by an atom of generation n_0 contained in A_i). By Lemma 3.4, for all $l \geq 2n_0 - 1$ there exists an $(l+1)$-path from A_1 to A_2. So, from Lemma 3.7, $\Phi^l(A_1 \cap \Lambda) \cap (A_2 \cap \Lambda) \neq \emptyset$, as required.

(b) The intersection of Λ with the atoms of all the generations generates its topology, thus (3.5) implies that Λ is topologically mixing. ∎

For fixed $(A_0, A_l) \in \mathcal{A}_n^2$ we set

$$\mathcal{A}_n^{l+1\,*}(A_0, A_l) := \{(A_0, A_1, \dots, A_{l-1}, A_l) \in \mathcal{A}_n^{l+1\,*}\}.$$

Lemma 3.9. *Let $l, n \geq 1$. Then:*

(a) $\#\mathcal{A}_n^{l+1\,*} = 2^{nl} \cdot \#\mathcal{A}_n$.
(b) $\#\mathcal{A}_n^{l+1\,*}(A_0, A_l) = 2^{nl}/\#\mathcal{A}_n$ *for all* $(A_0, A_l) \in \mathcal{A}_n^2$ *and all* $l \geq 2n - 1$.

Proof. (a) Each $(l+1)$-path $(A_0, A_1, \dots, A_l)$ of n-atoms is determined by a free choice of the atom $A_0 \in \mathcal{A}_n$, followed by the choice of the atoms $A_j \in \mathcal{A}_n$ such that $A_j \xrightarrow{\Phi} A_{j-1}$ for all $j = 1, \dots, l$. From the equalities in (ii) of Definition 2.3, we know that for any fixed $A \in \mathcal{A}_n$ the number of atoms $B \in \mathcal{A}_n$ such that $B \xrightarrow{\Phi} A$ is 2^n. This implies (a), as required.

(b) We argue by induction on n. Fix $n = 1$ and $l \geq 1$. Since any two atoms $A_j, A_{j+1} \in \mathcal{A}_1$ satisfy $A_j \xrightarrow{\Phi} A_{j+1}$, the number of $(l+1)$-paths

$$(A_0, A_1, \dots, A_j, A_{j+1}, \dots A_{l-1}, A_l)$$

of 1-atoms with (A_0, A_l) fixed equals $\#(\mathcal{A}_1)^{l-1} = 2^{l-1} = 2^l/2 = 2^{nl}/\#\mathcal{A}_n$ with $n = 1$.

Now, let us assume that (b) holds for some $n \geq 1$ and let us prove it for $n+1$. Let $l \geq 2(n+1) - 1 = 2n + 1 \geq 3$ and let $(B_0, B_l) \in \mathcal{A}_{n+1}^2$. From equality (2.2) and conditions (a) and (b) of Definition 2.3, there exists a unique $(A_{-1}, A_0, A_1) \in \mathcal{A}_n^{3*}$ and a unique $(A_{l-1}, A_l) \in \mathcal{A}_n^{2*}$ such that

$$B_0 \in \Gamma_{n+1}(A_{-1}, A_0, A_1), \qquad B_l \in \Omega_{n+1}(A_{l-1}, A_l).$$

As $(A_1, A_{l-1}) \in \mathcal{A}_n^2$ and $l - 2 \geq 2n - 1$, the induction hypothesis ensures that the number of $(l-1)$-paths $(A_1, \dots, A_{l-1})$ from A_1 to A_{l-1} is

$$\#\mathcal{A}_n^{l-1\,*}(A_1, A_{l-1}) = \frac{2^{n(l-2)}}{\#\mathcal{A}_n} = \frac{2^{n(l-2)}}{2^{n^2}} = 2^{nl-2n-n^2}. \tag{3.6}$$

Let

$$\mathcal{C}(B_0, B_l) := \bigcup_{(A_1,\dots,A_{l-1}) \in \mathcal{A}_n^{l-1\,*}(A_1, A_{l-1})} \{(B_0, B_1, \dots, B_l) \in \mathcal{A}_{n+1}^{l+1} : B_j \in \Gamma_{n+1}(A_{j-1}, A_j, A_{j+1})\ \forall j\}.$$

The families in the union $\mathcal{C}(B_0, B_l)$ are pairwise disjoint, because for $A \neq \widetilde{A}$ in $\mathcal{A}_n$ the families $\Gamma_{n+1}(\cdot, A, \cdot)$ and $\Gamma_{n+1}(\cdot, \widetilde{A}, \cdot)$ are disjoint.

A straightforward verification shows that

$$\mathcal{A}_{n+1}^{l+1\,*}(B_0, B_l) = \mathcal{C}(B_0, B_l). \tag{3.7}$$

Now, applying (3.6) and (3.7), we obtain

$$\begin{aligned}
\#&\mathcal{A}_{n+1}^{l+1\,*}(B_0, B_l) \\
&= \sum_{(A_1,\dots,A_{l-1})\in\mathcal{A}_n^{l-1\,*}(A_1,A_{l-1})} \#\{(B_0, B_1, \dots, B_l) \in \mathcal{A}_{n+1}^{l+1} : \\
&\qquad\qquad\qquad B_j \in \Gamma_{n+1}(A_{j-1}, A_j, A_{j+1})\ \forall j\} \\
&= \sum_{(A_1,\dots,A_{l-1})\in\mathcal{A}_n^{l-1\,*}(A_1,A_{l-1})} \prod_{j=1}^{l-1} \#\Gamma_{n+1}(A_{j-1}, A_j, A_{j+1}) \\
&= \#\mathcal{A}_n^{l-1\,*}(A_1, A_{l-1}) \cdot 2^{l-1} = 2^{nl-2n-n^2+l-1} = 2^{(n+1)l-(n+1)^2} = \frac{2^{(n+1)l}}{\#\mathcal{A}_{n+1}},
\end{aligned}$$

as required. ∎

Let $\vec{A}_n^l := (A_0, A_1, \dots, A_l)$ be an $(l+1)$-path of n-atoms, and $\mathcal{F}_{n,l}(\vec{A}_n^l) := \{G \in \mathcal{A}_{n+l} : G \cap \Lambda \subset \bigcap_{j=0}^{l} \Phi^{-j}(A_j)\}$.

LEMMA 3.10 (Intersection of Λ with l-paths). *Fix $l, n \geq 1$. Then:*

(a) *For any $G \in \mathcal{A}_{n+l}$, there exists a unique $(l+1)$-path $(A_0, A_1, \dots, A_l)$ of n-atoms such that $G \cap \Lambda \subset \bigcap_{j=0}^{l} \Phi^{-j}(A_j)$.*

(b) *For any atoms $G \in \mathcal{A}_{n+l}$, $A \in \mathcal{A}_n$ and $j \in \{0, 1, \dots, l\}$,*

$$(G \cap \Lambda) \cap \Phi^{-j}(A) \neq \emptyset \iff G \cap \Lambda \subset \Phi^{-j}(A).$$

(c) *For any $(l+1)$-path $\vec{A}_n^l = (A_0, A_1, \dots, A_l)$ of n-atoms,*

$$\Lambda \cap \bigcap_{j=0}^{l} \Phi^{-j}(A_j) = \bigcup_{G\in\mathcal{F}_{n,l}(\vec{A}_n^l)} G \cap \Lambda, \tag{3.8}$$

(d) *For any atom $G \in \mathcal{A}_{n+l}$ and any path $\vec{A}_n^l \in \mathcal{A}_n^{l+1\,*}$, we have $G \in \mathcal{F}_{n,l}(\vec{A}_n^l)$ if and only if there exists $(G_0, G_1, \dots, G_l) \in \mathcal{A}_{n+l}^{l+1\,*}$ such that $G_0 = G$ and $G_j \subset A_j$ for all $j = 0, 1, \dots, l$.*

(e) *For any $(l+1)$-path $(A_0, A_1, \dots, A_l)$ of n-atoms,*

$$\#\mathcal{F}_{n,l}(\vec{A}_n^l) = \frac{1}{2^{nl}} \cdot \frac{\#\mathcal{A}_{n+l}}{\#\mathcal{A}_n}.$$

Proof. (a) From (2.1) and (2.2), for any atom G of generation $n+l$ there exist two unique atoms B, C of generation $n+l-1$ such that $B \xrightarrow{\Phi} C$, $G \subset B$

and $G \xrightarrow{\Phi} E$ for all $E \in \Omega_{n+l}(B,C)$. Moreover, from Remark 2.9, we have

$$\Phi(G) \cap F \neq \emptyset \quad \text{if and only if} \quad F \in \Omega_{n+l}(B,C). \tag{3.9}$$

We claim that

$$\Phi(G \cap \Lambda) \subset \operatorname{int}(C). \tag{3.10}$$

Since Λ is Φ-invariant, for any $x \in G \cap \Lambda$, we have $\Phi(x) \in \Phi(G) \cap \Lambda$. Therefore $\Phi(x)$ is in the interior of some atom $E(x)$ of generation $n+l$ (see Definition 3.6). From (3.9), $E(x) \in \Omega_{n+l}(B,C)$. Thus $E(x) \subset \operatorname{int}(C)$ and $\Phi(x) \in \operatorname{int}(C)$ for all $x \in G \cap \Lambda$, proving (3.10).

So, there exists $C_1 \in \mathcal{A}_{n+l-1}$ such that $\Phi(G \cap \Lambda) \subset \operatorname{int}(C_1) \cap \Lambda$. Applying the same assertion to C_1 instead of G, we deduce that there exists $C_2 \in \mathcal{A}_{n+l-2}$ such that $\Phi(C_1 \cap \Lambda) \subset \operatorname{int}(C_2) \cap \Lambda$. So, by induction, we construct atoms $C_1, \ldots, C_l$ such that

$$C_j \in \mathcal{A}_{n+l-j} \quad \text{and} \quad \Phi^j(G \cap \Lambda) \subset \operatorname{int}(C_j) \cap \Lambda, \qquad \forall j = 1, \ldots, l.$$

Since any atom of generation larger than n is contained in a unique atom of generation n, there exist $A_0, A_1, \ldots, A_l \in \mathcal{A}_n$ such that $A_0 \supset G$ and $A_i \supset C_i$ for all $i = 1, \ldots, l$. We obtain

$$\Phi^j(G \cap \Lambda) \subset \operatorname{int}(A_j) \quad \forall j = 0, 1, \ldots, l.$$

Moreover, $(A_0, A_1, \ldots, A_l)$ is an $(l+1)$-path since $\emptyset \neq \Phi^j(G \cap \Lambda) \subset \Phi(A_{j-1}) \cap \operatorname{int}(A_j)$; hence $A_{j-1} \xrightarrow{\Phi} A_j$ for all $j = 1, \ldots, l$. Then $G \cap \Lambda \subset \Phi^{-j}(A_j)$ for all $j = 0, 1, \ldots, l$, proving the existence statement in (a).

To prove uniqueness assume that $(A_0, A_1, \ldots, A_l)$ and $(A'_0, A'_1, \ldots, A'_l)$ are paths of n-atoms such that

$$G \cap \Lambda \subset \Phi^{-j}(A_j) \cap \Phi^{-j}(A'_j) \quad \forall j \in \{0, 1, \ldots, l\}.$$

Then $A_j \cap A'_j \neq \emptyset$ for all $j \in \{0, 1, \ldots, l\}$. Since two different atoms of the same generation are pairwise disjoint, we deduce that $A_j = A'_j$ for all $j \in \{0, 1, \ldots, l\}$, as required.

(b) Trivially, if $G \cap \Lambda \subset \Phi^{-j}(A)$, then $(G \cap \Lambda) \cap \Phi^{-j}(A) \neq \emptyset$. Now, let us prove the converse assertion. Fix $G \in \mathcal{A}_{n+l}$ and $A \in \mathcal{A}_n$ satisfying $(G \cap \Lambda) \cap \Phi^{-j}(A) \neq \emptyset$. Applying part (a) we find $\widetilde{A} \in \mathcal{A}_n$ such that $G \cap \Lambda \subset \Phi^{-j}(\widetilde{A})$. Therefore $G \cap \Lambda \cap \Phi^{-j}(A) \subset \Phi^{-j}(\widetilde{A} \cap A) \neq \emptyset$. Since A and $\widetilde{A}$ are atoms of generation n, and two different atoms of the same generation are disjoint, we conclude that $\widetilde{A} = A$, hence $G \cap \Lambda \subset \Phi^{-j}(A)$, as required.

(c) For the $(l+1)$-path $\vec{A}^l_n = (A_0, A_1, \ldots, A_l)$ of n-atoms, construct

$$\widetilde{\mathcal{F}}_{n,l}(\vec{A}^l_n) := \{G \in \mathcal{A}_{n+l} : G \cap \Lambda \cap \Phi^{-j}(A_j) \neq \emptyset \ \forall j \in \{0, 1, \ldots, l\}\}. \tag{3.11}$$

From the definitions of the families $\mathcal{F}_{n,l}$ and $\widetilde{\mathcal{F}}_{n,l}$, and taking into account that $\varLambda$ is contained in the union of $(n+l)$-atoms, we obtain

$$\bigcup_{G\in\mathcal{F}_{n,l}(\vec{A}^l_n)} G\cap\varLambda \subset \varLambda\cap\bigcap_{j=0}^{l}\varPhi^{-j}(A_j) \subset \bigcup_{G\in\widetilde{\mathcal{F}}_{n,l}(\vec{A}^l_n)} G\cap\varLambda.$$

Therefore, to prove (3.8) it is enough to show that

$$\widetilde{\mathcal{F}}_{n,l}(\vec{A}^l_n) = \mathcal{F}_{n,l}(\vec{A}^l_n); \tag{3.12}$$

but this equality immediately follows from the construction of the families $\mathcal{F}_{n,l}(\vec{A}^l_n)$ and $\widetilde{\mathcal{F}}_{n,l}(\vec{A}^l_n)$ by assertion (b).

(d) For each $(l+1)$-path $\vec{A}^l_n = (A_0, A_1, \dots, A_l)$ of n-atoms construct the family

$$\mathcal{G}_{n,l}(\vec{A}^l_n) := \{G_0 \in \mathcal{A}_{n+l} : \exists (G_0, G_1, \dots, G_l) \in \mathcal{A}^{l+1\,*}_{n+l} \text{ such that } G_j \subset A_j\ \forall j\}.$$

We will first prove that $\mathcal{G}_{n,l}(\vec{A}^l_n) \supset \mathcal{F}_{n,l}(\vec{A}^l_n)$. In fact, take $G \in \mathcal{F}_{n,l}(\vec{A}^l_n)$, and any $x \in G\cap\varLambda$. We have $\varPhi^j(x) \in A_j\cap\varLambda$ for all $j \in \{0, 1, \dots, l\}$ (recall that $\varLambda$ is $\varPhi$-invariant). Since any point in $\varLambda$ is contained in the interior of some atom of any generation, there exists an atom G_j of generation $n+l$ such that $\varPhi^j(x) \in \operatorname{int}(G_j)$. Recall that each atom of generation $n+l$ is contained in a unique atom of generation n. As $\varPhi^j(x) \in G_j \cap A_j \neq \emptyset$, and different atoms of the same generation are disjoint, we conclude that $G_j \subset A_j$. Moreover, $G_0 = G$ because $x \in G \cap G_0$. Finally, $(G_0, G_1, \dots, G_l)$ is an $(l+1)$-path because $\varPhi^{j+1}(x) = \varPhi(\varPhi^j(x)) \in \varPhi(G_j) \cap \operatorname{int}(G_{j+1})$ for all $j \in \{0, 1, \dots, l-1\}$; namely $G_j \xrightarrow{\varPhi} G_{j+1}$. We have proved that $G \in \mathcal{G}_{n,l}(\vec{A}^l_n)$, as required.

Now, let us prove that $\mathcal{G}_{n,l}(\vec{A}^l_n) \subset \mathcal{F}_{n,l}(\vec{A}^l_n)$. Assume that $G_0 \in \mathcal{A}_{n+l}$ and $(G_0, G_1, \dots, G_l) \in \mathcal{A}^{l+1\,*}_{n+l}$ satisfies $G_j \subset A_j$ for all $j \in \{0, 1, \dots, l\}$. Therefore $(G_0, G_1, \dots, G_j)$ is a $(j+1)$-path of $(n+l)$-atoms for all $j \in \{1, \dots, l\}$. Applying Lemma 3.7, we obtain $G_0 \cap \varLambda \cap \varPhi^{-j}(G_j) \neq \emptyset$. Therefore, taking into account $G_j \subset A_j$, we deduce that

$$G_0 \cap \varLambda \cap \varPhi^{-j}(A_j) \neq \emptyset \quad \forall j \in \{0, 1, \dots, l\}.$$

Consequently, $G_0 \in \widetilde{\mathcal{F}}_{n,l}(\vec{A}^l_n) = \mathcal{F}_{n,l}(\vec{A}^l_n)$ (recall (3.11) and (3.12)). This holds for any $G_0 \in \mathcal{G}_{n,l}(\vec{A}^l_n)$, thus $\mathcal{G}_{n,l}(\vec{A}^l_n) \subset \mathcal{F}_{n,l}(\vec{A}^l_n)$, as required.

(e) From assertion (a) we obtain

$$\mathcal{A}_{n+l} = \bigcup_{\vec{A}^l_n \in \mathcal{A}^{l+1\,*}_n} \mathcal{F}_{n,l}(\vec{A}^l_n), \tag{3.13}$$

where the families in the above union are pairwise disjoint, due to the uniqueness property (a).

Recall the characterization of the family $\mathcal{F}_{n,1}(\vec{A}^l_n)$ given by assertion (d). From conditions (a) and (ii) of Definition 2.3, the number of atoms of each generation larger than n that are contained in each $A_j \in \mathcal{A}_n$, and also the number of atoms $G_j \in \mathcal{A}_{n+1}$ such that $G_j \xrightarrow{\Phi} G_{j+1}$, are constants that depend only on the generations but not on the chosen atom. Therefore, there exists a constant $k_{n,l}$ such that $\#\mathcal{F}_{n,l}(\vec{A}_n^l) = \#\mathcal{G}_{n,l}(\vec{A}_n^l) = k_{n,l}$ for all the $(l+1)$-paths of n-atoms. So, from (3.13) we obtain

$$\#\mathcal{A}_{n+l} = \#\mathcal{A}_n^{l+1\,*} \cdot \#\mathcal{F}_{n,l}(\{A_j\}),$$

and applying Lemma 3.9 we conclude that

$$\#\mathcal{A}_{n+l} = 2^{nl} \cdot \#\mathcal{A}_n \cdot \#\mathcal{F}_{n,l}(\{A_j\}),$$

as required. ■

We turn to the proof of Lemma 3.1. We will first construct the measure ν and then prove that it has the required properties.

We start by defining an additive premeasure on the Λ-set of Φ by

$$\nu^*(A \cap \Lambda) := \frac{1}{\#\mathcal{A}_n} \quad \forall A \in \mathcal{A}_n,\ \forall n \geq 0.$$

Since ν^* is a premeasure defined on a family of sets that generates the Borel σ-algebra of Λ, there exists a unique Borel probability measure ν supported on Λ such that

$$\nu(A \cap \Lambda) := \frac{1}{\#\mathcal{A}_n} \quad \forall A \in \mathcal{A}_n,\ \forall\, n \geq 0. \tag{3.14}$$

In the following lemmas we will prove that ν is Φ-invariant, mixing, and that the metric entropy $h_\nu(\Phi)$ is infinite. This will yield Lemma 3.1.

Lemma 3.11. *ν is invariant by Φ.*

Proof. Since the atoms of all generations intersected with Λ generate the Borel σ-algebra of Λ, it is enough to prove that

$$\nu(C \cap \Lambda) = \nu(\Phi^{-1}(C \cap \Lambda)) \quad \forall C \in \mathcal{A}_n,\ \forall n \geq 0. \tag{3.15}$$

From (2.2), taking into account that Λ is invariant and that any point in Λ belongs to an atom of generation $n+1$, we obtain

$$\Phi^{-1}(C \cap \Lambda) = \bigcup_{\substack{B \in \mathcal{A}_n \\ B \xrightarrow{\Phi} C}} \bigcup_{\substack{D \in \mathcal{A}_n \\ D \xrightarrow{\Phi} B}} \bigcup_{G \in \Gamma_{n+1}(D,B,C)} (G \cap \Lambda),$$

where both unions are of pairwise disjoint sets. Using the equalities in (ii) of Definition 2.3, we obtain

$$\begin{aligned} \nu(\Phi^{-1}(C \cap \Lambda)) &= \sum_{\substack{B \in \mathcal{A}_n \\ B \xrightarrow{\Phi} C}} \sum_{\substack{D \in \mathcal{A}_n \\ D \xrightarrow{\Phi} B}} \sum_{G \in \Gamma_{n+1}(B,C,D)} \nu(G \cap \Lambda) \\ &= N_C \cdot N_B \cdot \#\Gamma_{n+1}(B,C,D) \cdot \frac{1}{\#\mathcal{A}_{n+1}}, \end{aligned} \tag{3.16}$$

where $N_X := \#\{Y \in \mathcal{A}_n : Y \xrightarrow{\Phi} X\} = 2^n$ for all $X \in \mathcal{A}_n$. Since

$$\#\Gamma_{n+1}(B, C, D)) = 2$$

(see Remark 2.4) and $\#\mathcal{A}_{n+1} = 2^{(n+1)^2}$, we conclude

$$\nu(\Phi^{-1}(C \cap \Lambda)) = 2^n \cdot 2^n \cdot 2 \cdot \frac{1}{2^{(n+1)^2}} = \frac{1}{2^{n^2}} = \frac{1}{\#\mathcal{A}_n} = \nu(C \cap \Lambda),$$

proving (3.15), as required. ∎

LEMMA 3.12. *ν is mixing.*

Proof. The family of atoms of all generations intersected with Λ generates the Borel σ-algebra of Λ, thus it is enough to prove that for any pair (C_0, D_0) of atoms (of equal or different generations) there exists $l_0 \geq 1$ such that

$$\nu(\Phi^{-l}(D_0 \cap \Lambda) \cap (C_0 \cap \Lambda)) = \nu(C_0 \cap \Lambda) \cdot \nu(D_0 \cap \Lambda) \quad \forall l \geq l_0. \tag{3.17}$$

Let us first prove this in the case where C_0 and D_0 are atoms of the same generation n. Take $l \geq 2n - 1$. Applying Lemma 3.8(c), we have $\Phi^{-l}(D_0 \cap \Lambda) \cap (C_0 \cap \Lambda) \neq \emptyset$ for all $l \geq 2n - 1$.

Fix $l \geq 2n - 1$. We will use the notation

$$\vec{A}_n^l := (C_0, A_1, \ldots, A_{l-1}, D_0) \in \mathcal{A}_n^{l+1\,*}(C_0, D_0)$$

for any one of the $2^{nl}/\#\mathcal{A}_n$ different $(l+1)$-paths of n-atoms from C_0 to D_0 (see Lemma 3.9(b)).

We assert that

$$\Phi^{-l}(D_0 \cap \Lambda) \cap (C_0 \cap \Lambda) := \bigcup_{\vec{A}_n^l \in \mathcal{A}_n^{l+1\,*}(C_0, D_0)} \; \bigcup_{B \in \mathcal{F}_{n,l}(\vec{A}_n^l)} (B \cap \Lambda) =: T, \tag{3.18}$$

where the family $\mathcal{F}_{n,l}(\vec{A}_n^l)$ of $(n+l)$-atoms is defined in Lemma 3.10(c).

First, let us prove $\Phi^{-l}(D_0 \cap \Lambda) \cap (C_0 \cap \Lambda) \subset T$. Fix $x \in (D_0 \cap \Lambda) \cap (C_0 \cap \Lambda)$. Then C_0, D_0 are the unique atoms of generation n that contain x and $\Phi^l(x) \in \Phi^l(\Lambda) = \Lambda$ respectively. Since $x \in \Lambda$, there exists a unique atom B of generation $n + l$ that contains x. By Lemma 3.10(a) there exists a unique $(A_0, A_1, \ldots, A_l) \in \mathcal{A}_n^{l+1\,*}$ such that $B \cap \Lambda \subset \Phi^{-j}(A_j)$ for all $j \in \{0, 1, \ldots, l\}$. Since the n-atom that contains x is C_0, and two different n-atoms are disjoint, we deduce that $A_0 = C_0$. Analogously, since the n-atom that contains $\Phi^l(x)$ is D_0 and the preimages of two different n-atoms are disjoint, we deduce that $A_l = D_0$. Thus we have found $\vec{A}_n^l = (C_0, A_1, \ldots, A_{l-1}, D_0)$ and $B \in \mathcal{F}_{n,l}(\vec{A}_n^l)$ such that $x \in B \cap \Lambda$. In other words, $x \in T$, as required.

Next, let us prove that $\Phi^{-l}(D_0 \cap \Lambda) \cap (C_0 \cap \Lambda) \supset T$. Take $B \in \mathcal{F}_{n,l}(\vec{A}_n^l)$ for some $\vec{A}_n^l = (C_0, A_1, \ldots, A_{l-1}, D_0)$. From the definition of the family $\mathcal{F}_{n,l}(\vec{A}_n^l)$ in Lemma 3.10(c), we have $B \cap \Lambda \subset (C_0 \cap \Lambda) \cap \Phi^{-l}(D_0)$. Moreover, $B \cap \Lambda \in \Phi^l(\Lambda)$ because $\Phi^l(\Lambda) = \Lambda$. We conclude that $B \cap \Lambda \subset (C_0 \cap \Lambda) \cap \Phi^{-l}(D_0 \cap \Lambda)$, proving that $T \subset \Phi^{-l}(D_0 \cap \Lambda) \cap (C_0 \cap \Lambda)$, as required. This ends the proof of (3.18).

By definition, n-atoms are pairwise disjoint, thus the sets in the union forming T are pairwise disjoint. Therefore, from (3.18), and applying Lemmas 3.9(b) and 3.10(e), we deduce

$$\begin{aligned}\nu((C_0 \cap \Lambda) \cap \Phi^{-l}(D_0 \cap \Lambda)) &= \sum_{\vec{A}_n^l \in \mathcal{A}_n^{l+1\,*}(C_0,D_0)} \ \sum_{B \in \mathcal{F}_{n,l}(\vec{A}_n^l)} \nu(B \cap \Lambda) \\ &= \#\mathcal{A}_n^{l+1\,*}(C_0, D_0) \cdot \#\mathcal{F}_{n,l}(\vec{A}_n^l) \cdot \frac{1}{\#\mathcal{A}_{n+l}} \\ &= \frac{2^{nl}}{\#\mathcal{A}_n} \cdot \frac{1}{2^{nl}} \cdot \frac{\#\mathcal{A}_{n+l}}{\#\mathcal{A}_n} \cdot \frac{1}{\#\mathcal{A}_{n+l}} = \frac{1}{\#\mathcal{A}_n} \cdot \frac{1}{\#\mathcal{A}_n} \\ &= \nu(C_0 \cap \Lambda) \cdot \nu(D_0 \cap \Lambda).\end{aligned}$$

This ends the proof of (3.17) when C_0 and D_0 are atoms of the same generation n, taking $l_0 = 2n - 1$.

Now, let us prove (3.17) when C_0 and D_0 are atoms of different generations. Let n equal the maximum of the two generations. Take $l \geq 2n - 1$. Since Λ is contained in the union of the atoms of any generation, we have

$$C_0 \cap \Lambda = \bigcup_{C \in \mathcal{A}_n,\, C \subset C_0} C \cap \Lambda,$$

where the sets in the union are pairwise disjoint. Analogously,

$$\Phi^{-l}(D_0 \cap \Lambda) = \bigcup_{D \in \mathcal{A}_n,\, D \subset D_0} \Phi^{-l}(D \cap \Lambda),$$

where also the sets in the union are pairwise disjoint. So,

$$(C_0 \cap \Lambda) \cap \Phi^{-l}(D_0 \cap \Lambda) = \bigcup_{C \in \mathcal{A}_n,\, C \subset C_0} \ \bigcup_{D \in \mathcal{A}_n,\, C \subset D_0} (C \cap \Lambda) \cap \Phi^{-l}(D \cap \Lambda).$$

Since the sets in the union are pairwise disjoint, we deduce

$$\nu((C_0 \cap \Lambda) \cap \Phi^{-l}(D_0 \cap \Lambda)) = \sum_{C \in \mathcal{A}_n,\, C \subset C_0} \ \sum_{D \in \mathcal{A}_n, C \subset D_0} \nu((C \cap \Lambda) \cap \Phi^{-l}(D \cap \Lambda)).$$

As C, D are atoms of the same generation n, and $l \geq 2n - 1$, we can apply the first case proved above to deduce that

$$\begin{aligned}&\nu((C_0 \cap \Lambda) \cap \Phi^{-l}(D_0 \cap \Lambda)) \\ &\qquad = \#\{C \in \mathcal{A}_n : C \subset C_0\} \cdot \#\{D \in \mathcal{A}_n : C \subset D_0\} \cdot \frac{1}{(\#\mathcal{A}_n)^2}.\end{aligned} \tag{3.19}$$

The number of atoms of generation n contained in an atom C_0 of generation

$n_1 \geq n$ does not depend on the chosen atom C_0. Therefore,

$$\#\{C \in \mathcal{A}_n : C \subset C_0\} = \frac{\#\mathcal{A}_n}{\#\mathcal{A}_{n_1}} = \#\mathcal{A}_n \cdot \nu(C_0 \cap \Lambda).$$

Analogously

$$\#\{D \in \mathcal{A}_n : D \subset D_0\} = \#\mathcal{A}_n \cdot \nu(D_0 \cap \Lambda).$$

Finally, substituting this in (3.19) we conclude that

$$\nu(\Phi^{-l}(D_0 \cap \Lambda) \cap (C_0 \cap \Lambda)) = \nu(C_0 \cap \Lambda) \cdot \nu(D_0 \cap \Lambda) \quad \forall l \geq 2n-1. \blacksquare$$

LEMMA 3.13. $h_\nu(\Phi) = +\infty$.

Proof. For $n \geq 1$ we consider the partition $\mathcal{A}_n$ of Λ consisting of all the n-atoms intersected with Λ. By the definition of metric entropy,

$$h_\nu(\Phi) := \sup_{\mathcal{P}} h(\mathcal{P}, \nu) \geq h(\mathcal{A}_n, \nu), \tag{3.20}$$

where

$$h(\mathcal{A}_n, \nu) := \lim_{l\to\infty} \frac{1}{l} H\Big(\bigvee_{j=0}^{l} (\Phi^{-j}\mathcal{A}_n), \nu\Big) \tag{3.21}$$

with

$$H(\mathcal{Q}_l, \nu) := -\sum_{X \in \mathcal{Q}_l} \nu(X) \log \nu(X) \tag{3.22}$$

and

$$\mathcal{Q}_l := \bigvee_{j=0}^{l} \Phi^{-j}\mathcal{A}_n := \Big\{\bigcap_{j=0}^{l} \Phi^{-j} A_j \cap \Lambda \neq \emptyset : A_j \in \mathcal{A}_n\Big\}.$$

For any nonempty $X := \Lambda \cap \bigcap_{j=0}^{l} \Phi^{-j} A_j \in \mathcal{Q}_l$, Lemma 3.10(c) yields

$$\nu(X) = \nu\Big(\bigcap_{j=0}^{l} \Phi^{-j} A_j \cap \Lambda\Big) = \sum_{G \in \mathcal{F}_{n,l}(\vec{A}_j^l)} \nu(G \cap \Lambda).$$

Since G is an atom of generation $n+l$, we have $\nu(G \cap \Lambda) = 1/\#\mathcal{A}_{n+l}$, thus applying Lemma 3.10(e) yields

$$\nu(X) = \frac{\#\mathcal{F}_{n,l}(\{A_j\})}{\#\mathcal{A}_{n+l}} = \frac{1}{2^{nl} \cdot \#\mathcal{A}_n}.$$

Combining this with (3.22) yields $H(\mathcal{Q}_l) = \log \#\mathcal{A}_n + nl \log 2$. Finally, substituting this in (3.21), we obtain

$$h(\mathcal{A}_n, \nu) := \lim_{l\to\infty} \frac{1}{l} H(\mathcal{Q}_l, \nu) = n \log 2.$$

Combining this with (3.20) yields $h_\nu(\Phi) \geq n \log 2$ for all $n \geq 1$; hence $h_\nu(\Phi) = +\infty$. $\blacksquare$

4. Periodic shrinking boxes. In this section we will prove Theorems 1.1 and 1.3 for $m \geq 2$. The argument is based on the properties of models proved in the previous sections, and on the existence of periodic shrinking boxes which we construct here.

Throughout this section we consider $m \geq 1$, unless the condition $m \geq 2$ is explicitly stated.

DEFINITION 4.1 (Periodic shrinking box). Let $f \in C^0(M)$ and $K \subset M$ be a box. Then we call K *periodic shrinking with period* $p \geq 1$ for f if $K, f(K), f^2(K), \ldots, f^{p-1}(K)$ are pairwise disjoint, and $f^p(K) \subset \operatorname{int}(K)$. In that case, we call $f^p|_K : K \to \operatorname{int}(K)$ the *return map.*

Recall that the manifold M is compact. This assumption is important to obtain Lemmas 4.2 and 4.3 below. We will construct periodic shrinking boxes whose return maps are homeomorphisms onto their images. Although this last condition is unnecessary for the construction of the periodic shrinking boxes, it will be used later in the proofs of Lemmas 4.7 and 4.8, where the return maps must be topologically conjugate to model maps.

LEMMA 4.2. *For any $\delta > 0$, there exists an open and dense set of maps $f \in C^0(M)$ that have a periodic shrinking box K_f with* $\operatorname{diam}(K_f) < \delta$. *For a dense set of $f \in C^0(M)$ the return map to K_f is one-to-one.*

The proof of this lemma uses the following technical result.

LEMMA 4.3. *Let $f \in C^0(M)$ and $x_0 \in M$. For all $\varepsilon > 0$, there exists $g \in C^0(M)$ and a neighborhood H of x_0 such that $\|g - f\|_{C^0} < \varepsilon$, $g|_H$ is a homeomorphism onto its image and coincides with f off a neighborhood of x_0.*

Proof. Since the assertion is of local character, we may assume $M = \mathbb{R}^n$. Composing with a translation we may also assume that $x_0 = f(x_0) = 0$. Let $0 < \delta < \varepsilon$ be so small that the ball $\|x\| < \delta$ is mapped under f to a set of diameter smaller than ε. Let $\lambda : \mathbb{R}^n \to [0,1]$ be a continuous function such that $\lambda(x) = 0$ if $\|x\| \leq \delta/2$ and $\lambda(x) = 1$ if $\|x\| \geq \delta$. We define g by the formula $g(x) := \lambda(x)f(x) + (1 - \lambda(x))x$ if $\|x\| \leq \delta$ and $g(x) = f(x)$ if $\|x\| \geq \delta$. ∎

Proof of Lemma 4.2. According to Definition 4.1, the periodic shrinking box K_f for f is also a periodic shrinking box with the same period for all $g \in C^0(M)$ close enough to f, proving the openness assertion.

We turn to the denseness assertion. Let $f \in C^0(M)$ and $\varepsilon > 0$. We will construct $g \in C^0(M)$ and a periodic shrinking box K_g for g with $\operatorname{diam}(K)$ $< \delta$ such that $\|g - f\|_{C^0} < \varepsilon$. We suppose $\delta > 0$ to be smaller than the ε-modulus of continuity of f.

By the Krylov–Bogolyubov theorem, invariant measures exist (recall that the manifold M is compact), and thus by the Poincaré Lemma, there exists a recurrent point $x_0 \in M$ for f. First assume that $x_0 \notin \partial M$. So, there exists a box $B \subset M$ with $\operatorname{diam}(B) < \delta$ such that $x_0 \in \operatorname{int}(B)$. Since x_0 is a recurrent point, there exists a smallest $p \in \mathbb{N}$ such that $f^p(x_0) \in \operatorname{int}(B)$. Taking B slightly smaller if necessary, we can assume that $f^j(x_0) \notin B$ for all $j = 1, \dots, p-1$. So, there exists a small compact box $U \subset \operatorname{int}(B)$ as in Figure 3 such that $x_0 \in \operatorname{int}(U)$, the sets $U, f(U), \dots, f^{p-1}(U)$ are pairwise disjoint, and $f^p(U) \subset \operatorname{int}(B)$.

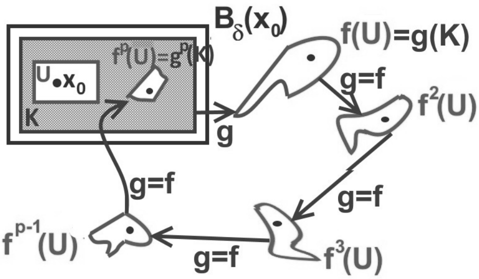

Fig. 3. Construction of g near f with a periodic shrinking box K for g

Since $U, f^p(U) \subset \operatorname{int}(B)$, there exists a box K such that $U, f^p(U) \subset \operatorname{int}(K) \subset K \subset \operatorname{int}(B)$, and there exists a homeomorphism $\psi : B \to B$ such that $\psi(x) = x$ for all $x \in \partial B$, and $\psi(K) = U$.

Finally, we construct $g \in C^0(M)$ as follows:

$$g(x) := \begin{cases} f(x) & \forall x \notin B, \\ f \circ \psi(x) & \forall x \in B. \end{cases}$$

By construction, K is a periodic shrinking box for g, say $K = K_g$; by the choice of δ we have $\|g - f\| < \varepsilon$.

Now, let us study the case for which M is a compact manifold with boundary and all the recurrent points of f belong to ∂M. Choose one such recurrent point $x_0 \in \partial M$. For any $\delta > 0$, there exists a compact box $B \subset M$ with $\operatorname{diam}(B) \le \delta$ such that $x_0 \in \partial M \cap B$. Since x_0 is recurrent, there exists a smallest natural number $p \ge 1$ such that $f^p(x_0) \in B$. But $f^p(x_0)$ is also recurrent. So, $f^p(x_0) \in \partial M \cap B$. The previous proof does not work as is. To overcome this problem, we choose a new point $\widetilde{x}_0 \neq x_0$, close enough to x_0, such that $\widetilde{x}_0 \in \operatorname{int}(B) \setminus \partial M$ and $f^p(\widetilde{x}_0) \in B$. By applying Lemma 4.3 and slightly perturbing f if necessary, we can assume that the restriction of f to a small neighborhood of $\widetilde{x}_0$ is a local homeomorphism onto its image. Hence, $f^p(\widetilde{x}_0) \in \operatorname{int}(B) \setminus \partial M$. To conclude, we repeat the construction of g and K_g above, replacing the recurrent point x_0 by $\widetilde{x}_0$.

Now, let us show that we can construct, for a dense set of $g \in C^0(M)$, a periodic shrinking box K_g such that the return map $g^p|_{K_g}$ is a homeomorphism onto its image. We repeat the beginning of the proof, up to the construction of the points $x_0, f(x_0), \ldots, f^p(x_0)$ such that $x_0, f^p(x_0) \in \operatorname{int}(B)$ and $f^j(x_0) \notin B$. Apply Lemma 4.3, and slightly perturb f if necessary inside small open neighborhoods $W_0, W_1, \ldots, W_{p-1}$ of the points $x_0, f(x_0), \ldots, f^{p-1}(x_0)$ respectively, so that $f|_{\overline{W}_i}$ is a homeomorphism onto its image for all $i = 0, 1, \ldots, p-1$. Finally, construct the box U (Figure 3), but small enough so $f^j(U) \subset W_j$ for all $j = 0, 1, \ldots, p-1$, and repeat the construction of $K = K_g$ and g as above. ∎

Remark 4.4. Note that to obtain the denseness property in the proof of the first sentence of Lemma 4.2, we only need to perturb the map f in the interior of the initial box B with diameter smaller than δ.

The following lemma is the homeomorphism version of Lemma 4.2.

Lemma 4.5. *For any $\delta > 0$, there exists an open and dense set of maps $f \in \operatorname{Hom}(M)$ that each have a periodic shrinking box K with $\operatorname{diam}(K) < \delta$.*

Proof. The proof of Lemma 4.2 also works when $f \in \operatorname{Hom}(M)$: in fact, the ε-perturbed map g constructed there is a homeomorphism, and to obtain $\|g - f\|_{\operatorname{Hom}(M)} < \varepsilon$ it is enough to take $\delta > 0$ smaller than the ε-modulus of continuity of f and f^{-1}. ∎

Remark 4.6. In the proof of Lemmas 4.2 and 4.5, if the starting recurrent point x_0 were a periodic point of period p, then the periodic shrinking box K so constructed would contain x_0 in its interior and have the same period p.

Lemma 4.7. *Assume $m \geq 2$. Fix $\delta > 0$ and $\Phi \in \mathcal{H} \cap \operatorname{Emb}(D^m)$ (recall Definition 2.5). Each generic map $f \in C^0(M)$ has a periodic shrinking box K with $\operatorname{diam}(K) < \delta$ such that the return map $f^p|_K$ is topologically conjugate to a model map in $\mathcal{H}_\Phi$ (recall Definition 2.6).*

Proof. Let $K \subset M$ be a periodic shrinking box for f. Fix a homeomorphism $\phi : K \to D^m$.

To prove the G_δ-set property, assume that $f \in C^0(M)$ has a periodic shrinking box K with $\operatorname{diam}(K) < \delta$ such that $\phi \circ f^p|_K \circ \phi^{-1} \in \mathcal{H}_\Phi$ (recall Definition 2.6 and Lemma 2.7). From Definition 4.1, the same box K is also periodic shrinking with period p for all $g \in \mathcal{N}$, where $\mathcal{N} \subset C^0(M)$ is an open neighborhood of f. From Lemma 2.7, $\mathcal{H}_\Phi$ is a nonempty G_δ-set in $C^0(D^m)$, i.e., it is the nonempty countable intersection of open families $\mathcal{H}_n \subset C^0(D^m)$. We define

$$\mathcal{V}_n := \{g \in \mathcal{N} : \phi \circ g^p|_K \circ \phi^{-1} \in \mathcal{H}_n\}.$$

Since the restriction to K of a continuous map g, and the composition of continuous maps, are continuous operations in $C^0(M)$, we deduce that $\mathcal{V}_n$ is an open family in $C^0(M)$. Moreover,

$$\phi \circ g^p|_K \circ \phi^{-1} \in \mathcal{H} = \bigcap_{n\geq 1} \mathcal{H}_n \quad \text{if } g \in \bigcap_{n\geq 1} \mathcal{V}_n \subset C^0(M). \tag{4.1}$$

In other words, the set of maps $g \in C^0(M)$ that have a periodic shrinking box K with $\operatorname{diam}(K) < \delta$, such that the return map $g^p|_K$ coincides, up to a conjugation, with a model map in $\mathcal{H}_\Phi$, is a G_δ-set in $C^0(M)$.

To show the denseness, fix $f \in C^0(M)$ and $\varepsilon > 0$. Applying Lemma 4.2, it is not restrictive to assume that f has a periodic shrinking box K with $\operatorname{diam}(K) < \min\{\delta, \varepsilon\}$ such that $f^p|_K$ is a homeomorphism onto its image. We will construct $g \in C^0(M)$ ε-near f and such that $\phi \circ g^p|_K \circ \phi^{-1} \in \mathcal{H}$.

Choose a box W such that $f^{p-1}(K) \subset \operatorname{int}(W)$. If $p \geq 2$, take W disjoint from $f^j(K)$ for all $j \in \{0, 1, \ldots, p-2\}$ (Figure 4). Let us see that we can assume that W has an arbitrarily small diameter. It is enough to prove that f can be chosen such that $f^{p-1}(K)$ has an arbitrarily small diameter. In fact, in the construction of f in the proof of Lemma 4.2, we can choose the box U (see Figure 3), after choosing K, as small as needed. So, we choose U small enough such that the $(p-1)$th image of U by the map before the perturbation has a small diameter. (Note that we do not change p.) After that, we construct the perturbed map, which we call f again, as in the proof of Lemma 4.2: the image $f^{p-1}(K)$ of the new map f coincides with the $(p-1)$th image of U by the map before the perturbation (Figure 3). So, it has an arbitrarily small diameter, as required.

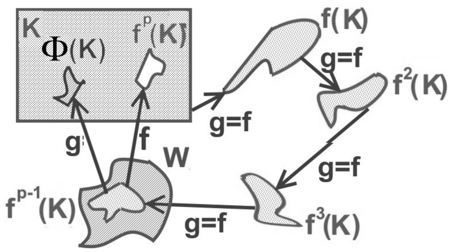

Fig. 4. Perturbation g of f such that $g^p|_K = \Phi$

To construct $g \in C^0(M)$ (see Figure 4) we consider the $\Phi \in \mathcal{H}$ chosen in the hypothesis, and let $g(x) := f(x)$ if $x \notin W$ and

$$g(x) := \phi^{-1} \circ \Phi \circ \phi \circ (f^p|_K)^{-1} \circ f(x) \quad \forall x \in f^{p-1}(K).$$

This defines a continuous map $g : f^{p-1}(K) \cup (M \setminus W) \to M$ such that $|g(x) - f(x)| < \operatorname{diam}(K) < \varepsilon$ for all $x \in f^{p-1}(K) \subset W$ and $g(x) = f(x)$ for all $x \in M \setminus W$. By the Tietze Extension Theorem, there exists a continuous extension of g to the whole compact box W, hence to M, such that

$\|g-f\|_{C^0}<\varepsilon$. Finally, by construction we obtain

$$\begin{aligned} g^p|_K &= g|_{f^{p-1}(K)} \circ f^{p-1}|_K \\ &= \phi^{-1}\circ \Phi \circ \phi \circ (f^p|_K)^{-1} \circ f \circ f^{p-1}|_K = \phi^{-1}\circ\Phi\circ\phi. \ \blacksquare \end{aligned}$$

LEMMA 4.8. *Let $\delta>0$. Fix $\Phi\in\mathcal{H}\cap\mathrm{Emb}(D^m)$. A generic homeomorphism $f\in\mathrm{Hom}(M)$ has a periodic shrinking box K with $\mathrm{diam}(K)<\delta$ such that the return map $f^p|_K$ is topologically conjugate to a model embedding in $\mathcal{H}_\Phi$.*

Proof. We repeat the proof of the G_δ-set property of Lemma 4.7, using $\mathcal{H}\cap\mathrm{Emb}(D^m)$ instead of $\mathcal{H}$, and $\mathrm{Hom}(M)$ instead of $C^0(M)$.

To show the denseness, fix $f\in\mathrm{Hom}(M)$ and $\varepsilon>0$. Applying Lemma 4.2, it is not restrictive to assume that f has periodic shrinking boxes of arbitrarily small diameters. Let $\delta\in(0,\varepsilon)$ be smaller than the ε-moduli of continuity of f and f^{-1}. Consider a periodic shrinking box K with $\mathrm{diam}(K)<\delta$ (Lemma 4.5). Fix a homeomorphism $\phi:K\to D^m$. We will construct $g\in\mathrm{Hom}(M)$ ε-near f in $\mathrm{Hom}(M)$ with $\phi\circ g^p|_K\circ\phi^{-1}=\Phi\in\mathcal{H}\cap\mathrm{Emb}(D^m)$.

From Definition 4.1 we know that the boxes $K, f(K), f^2(K),\dots,f^{p-1}(K)$ are pairwise disjoint and that $f^p(K)\subset\mathrm{int}(K)$. Denote $W:=f^{-1}(K)$. Since f is a homeomorphism, we deduce that W is a box as in Figure 4 such that $W\cap f^j(K)=\emptyset$ for all $j=0,1,\dots,p-2$ if $p\ge 2$, and $f^{p-1}(K)\subset\mathrm{int}(W)$. Since $\mathrm{diam}(K)<\delta$, we have $\mathrm{diam}(W)<\varepsilon$.

Consider $\phi\circ f^p|_K\circ\phi^{-1}\in\mathrm{Emb}(D^m)$. By Lemma 2.8, there exists a homeomorphism $\psi:D^m\to D^m$ such that

$$\psi|_{\partial D^m}=\mathrm{id}|_{\partial D^m},\qquad \psi\circ\phi\circ f^p|_K\circ\phi^{-1}=\Phi\in\mathcal{H}\cap\mathrm{Emb}(D^m).$$

So, we can construct $g\in\mathrm{Hom}(M)$ such that $g(x):=f(x)$ for all $x\notin W$, and $g(x):=\phi^{-1}\circ\psi\circ\phi\circ f(x)$ for all $x\in W$. Since $\psi|_{\partial D^m}$ is the identity map, we obtain $g|_{\partial W}=f|_{\partial W}$. Thus, the above equalities define a continuous map $g:M\to M$. Moreover, g is invertible because $g|_W:W\to K$ is a composition of homeomorphisms, and $g|_{M\setminus W}=f|_{M\setminus W}:M\setminus W\to M\setminus K$ is also a homeomorphism. So, $g\in\mathrm{Hom}(M)$. Moreover, by construction we have $|g(x)-f(x)|<\mathrm{diam}(K)<\varepsilon$ for all $x\in W$, and $g(x)=f(x)$ for all $x\notin W$. Also, the inverse maps satisfy $|g^{-1}(x)-f^{-1}(x)|<\mathrm{diam}(f^{-1}(K))=\mathrm{diam}(W)<\varepsilon$ for all $x\in K$, and $g^{-1}(x)=f^{-1}(x)$ for all $x\notin K$. Therefore $\|g-f\|_{\mathrm{Hom}}<\varepsilon$.

Finally, let us check that $g^p|_K$ is topologically conjugate to Φ:

$$\begin{aligned} g^p|_K &= g|_{f^{p-1}(K)}\circ f^{p-1}|_K = g|_W\circ f^{p-1}|_K = \phi^{-1}\circ\psi\circ\phi\circ f\circ f^{p-1}|_K \\ &= \phi^{-1}\circ(\psi\circ\phi\circ f^p|_K\circ\phi^{-1})\circ\phi = \phi^{-1}\circ\Phi\circ\phi. \ \blacksquare \end{aligned}$$

REMARK 4.9. In the proof of the denseness property in Lemmas 4.7 and 4.8, once a periodic shrinking box K is constructed with period $p\ge 1$, we only need to perturb the map f inside $W\cup\bigcup_{j=0}^{p-1}f^j(K)$, where $W=f^{-1}(K)$

if f is a homeomorphism, and $\mathrm{int}(W) \supset f^{p-1}(K)$ otherwise. In both cases, by reducing the set U of Figure 3 from the very beginning, we can construct W such that $\mathrm{diam}(W) < \varepsilon$ for any given small $\varepsilon > 0$.

Proof of Theorems 1.1 and 1.3. From Lemmas 4.7 and 4.8, a generic map $f \in C^0(M)$, and also a generic $f \in \mathrm{Hom}(M)$, has a periodic shrinking box K such that the return map $f^p|_K : K \to \mathrm{int}(K)$ is conjugate to a model map $\Phi \in \mathcal{H}$. We consider the homeomorphism $\phi^{-1} : K \to D^m$ such that $\phi^{-1} \circ f^p \circ \phi = \Phi \in \mathcal{H}$. Lemma 3.1 states that every map $\Phi \in \mathcal{H}$ has a Φ-invariant mixing measure ν with infinite metric entropy for Φ. Consider the push-forward measure $\phi_*\nu$, defined by $(\phi_*\nu)(B) := \nu(\phi^{-1}(B \cap K))$ for all the Borel sets $B \subset M$. By construction, $\phi_*\nu$ is supported on $K \subset M$. Since ϕ is a conjugation between Φ and $f^p|_K$, the push-forward measure $\phi_*\nu$ is f^p-invariant and mixing for f^p and moreover $h_{\phi_*\nu}(f^p) = +\infty$.

From $\phi_*\nu$, we will construct an f-invariant and f-ergodic measure μ supported on $\bigcup_{j=0}^{p-1} f^j(K)$, with infinite metric entropy for f. More precisely, for each Borel set $B \subset M$, define

$$\mu(B) := \frac{1}{p}\sum_{j=0}^{p-1}(f^j)_*(\phi_*\nu)(B \cap f^j(K)).$$

By this equality, and the fact that $\phi_*\nu$ is f^p-invariant and f^p-mixing, it is standard to check that μ is f-invariant and f-ergodic. From the convexity of the metric entropy function, we deduce that

$$h_\mu(f^p) = \frac{1}{p}\sum_{j=0}^{p-1} h_{(f^j)_*(\phi_*\nu)}(f^p) = +\infty.$$

Finally, recalling that $h_\mu(f^p) \le p h_\mu(f)$ for any f-invariant measure μ and any natural number $p \ge 1$, we conclude that $h_\mu(f) = +\infty$. ∎

5. Good sequences of periodic shrinking boxes. We now prove Theorems 1.2 and 1.4. Throughout this section we assume that $\dim(M) \ge 2$. When M is a one-dimensional manifold, Theorem 1.2 can be proved by repeating the proof of the 2-dimensional case after replacing Definition 2.5 by Definition 2.1.

Definition 5.1. Let $f \in C^0(M)$ and let $K_1, K_2, \dots$ be a sequence of periodic shrinking boxes for f. We call $\{K_n\}_{n\ge 1}$ *good* if it has the following properties (see Figure 5):

- $\{K_n\}_{n\ge 1}$ is composed of pairwise disjoint boxes.
- There exists a natural number $p \ge 1$, independent of n, such that K_n is a periodic shrinking box for f whose period p_n is a multiple of p.

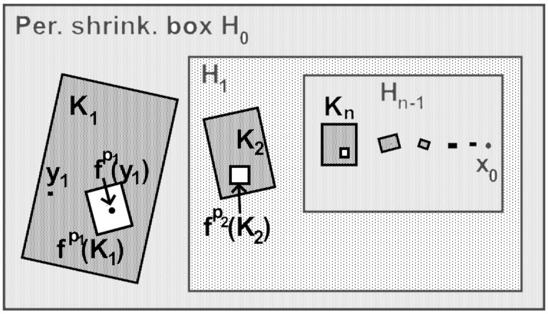

Fig. 5. Construction of a good sequence of periodic shrinking boxes

- There exists a sequence $\{H_n\}_{n\geq 0}$ of periodic shrinking boxes, all with period p, such that $K_n \cup H_n \subset H_{n-1}$, $K_n \cap H_n = \emptyset$ for all $n \geq 1$, and $\operatorname{diam}(H_n) \to 0$ as $n \to \infty$.

REMARK. Definition 5.1 implies that $\bigcap_{n\geq 1} H_n = \{x_0\}$, where x_0 is periodic with period p. Furthermore, for any $j \geq 0$ we have

$$d(f^j(K_n), f^j(x_0)) \leq \operatorname{diam}(f^j(H_{n-1})) \leq \max_{0\leq k\leq p-1} \operatorname{diam}(f^k(H_{n-1})) \xrightarrow{n\to\infty} 0,$$

and thus

$$\lim_{n\to\infty} \sup_{j\geq 0} d(f^j(K_n), f^j(x_0)) = 0. \tag{5.1}$$

We will construct a good sequence of periodic shrinking boxes for maps that are arbitrarily close to a given f. We start by constructing the zeroth level boxes:

LEMMA 5.2. *Let $f \in C^0(M)$ (resp. $f \in \operatorname{Hom}(M)$) and $\varepsilon, \delta > 0$. Then there exist $g_1 \in C^0(M)$ (resp. $g_1 \in \operatorname{Hom}(M)$), periodic shrinking boxes H_0 and K_1 for g_1 with periods p and p_1 respectively, where p_1 is multiple of p, and a periodic point $x_0 \in \operatorname{int}(H_0)$ for g_1 such that $K_1 \subset H_0 \setminus \{x_0\}$, and*

$$g_1^{p_1}|_{K_1} \text{ is topologically conjugate to } \Phi_1 \in \mathcal{H},$$
$$\operatorname{diam}(H_0) < \delta, \qquad \|g_1 - f\| < \varepsilon/2.$$

Proof. A generic map $\widetilde{f} \in C^0(M)$ (resp. $\widetilde{f} \in \operatorname{Hom}(M)$) has a periodic shrinking box H_0 with period $p \geq 1$ such that $\operatorname{diam}(H_0) < \delta$ and $\widetilde{f}^p|_{H_0}$ is conjugate to a model map $\Phi \in \mathcal{H}$ (Lemma 4.7, resp. 4.8). Fix such an $\widetilde{f}$ in the $(\varepsilon/6)$-neighborhood of f. The same box H_0 will be a shrinking periodic box for the map g_1 to be constructed.

Since $\widetilde{f}^p : H_0 \to \operatorname{int}(H_0) \subset H_0$ is continuous, by the Brouwer Fixed Point Theorem there exists a periodic point $x_0 \in \operatorname{int}(H_0)$ of period p. Lemma 3.1 and the argument at the end of the proof of Theorems 1.1 and 1.3 show that the map $\widetilde{f}$ has an ergodic measure μ supported on $\bigcup_{j=0}^{p-1} \widetilde{f}^j(H_0)$ such that $h_\mu(\widetilde{f}) = +\infty$. Therefore, by the Poincaré Recurrence Lemma, there exists some recurrent point $y_1 \in \operatorname{int}(H_0)$ for $\widetilde{f}$. We can choose $y_1 \neq x_0$ (see Figure 5) because μ is not supported on the orbit of the periodic point x_0 (recall that μ has infinite entropy and by construction its support is a perfect set).

Choose $\delta_1 > 0$ small enough and construct a box B_1 such that $y_1 \in \operatorname{int}(B_1)$, $\operatorname{diam}(B_1) < \delta_1$, the $\widetilde{f}$-orbit of x_0 (which is finite) does not intersect the finite piece of the $\widetilde{f}$-orbit of B_1 (until the first iterate of y_1 is in H_0) and $B_1 \subset \operatorname{int}(H_0)$. We repeat the proofs of the denseness property of Lemmas 4.2 and 4.5, using the recurrent point y_1 instead of x_0, and the box B_1 instead of B (see Figure 3). We deduce that there exist an $(\varepsilon/6)$-perturbation $\hat{f}$ of $\widetilde{f}$ and a periodic shrinking box $K_1 \subset B_1$ for $\hat{f}$ with some period $p_1 \geq p$ (see Figure 5). Moreover, $\hat{f}$ coincides with $\widetilde{f}$ in $M \setminus \operatorname{int}(B_1)$ (recall Remark 4.4). Therefore, the same periodic point x_0 of $\widetilde{f}$ survives for $\hat{f}$. Moreover, by the openness of the existence of the periodic shrinking box H_0, the same initial box H_0 is still periodic shrinking with period p for $\hat{f}$, provided that $\hat{f}$ is close enough to $\widetilde{f}$. So, the compact sets of the family $\{\hat{f}^j(H_0)\}_{j=0,1,\ldots,p-1}$ are pairwise disjoint, and $\hat{f}^p(H_0) \subset \operatorname{int}(H_0)$. This implies that the period p_1 of the new periodic shrinking box K_1 for $\hat{f}$ is a multiple of p.

Now, we apply the proofs of the denseness property of Lemmas 4.7 and 4.8, using the shrinking box K_1 instead of K (see Figure 4). We deduce that there exists an $(\varepsilon/6)$-perturbation g_1 of $\hat{f}$ such that K_1 is still a periodic shrinking box for g_1 with the same period p_1, but moreover the return map $g_1^{p_1}|_{K_1}$ is now topologically conjugate to $\Phi_1 \in \mathcal{H}$.

Consider a box W_1 satisfying $\hat{f}^{p_1-1}(K_1) \subset W_1 \subset K_1$, small enough so its $\hat{f}$-orbit is disjoint from the $\hat{f}$-orbit of the periodic point x_0. Taking into account Remark 4.9, we can construct g_1 to coincide with $\hat{f}$ in the complement of $W_1 \cup \bigcup_{j=0}^{p_1-1} \hat{f}^j(K_1)$. If g_1 is sufficiently close to $\hat{f}$, the point x_0 is still periodic of period p for g_1, and moreover H_0 is still a periodic shrinking box of period p for g_1 (recall that such a property is open). Finally,

$$\|g_1 - f\| < \|g_1 - \hat{f}\| + \|\hat{f} - \widetilde{f}\| + \|\widetilde{f} - f\| < \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{2}. \blacksquare$$

Assume that we have constructed the jth level of periodic shrinking boxes for all $0 \leq j \leq n-1$ of a good sequence. We will construct the periodic shrinking boxes of the nth level by perturbing the given map once more. Let us first define the following family of maps.

Definition 5.3. Fix $\delta > 0$, and let p, n be natural numbers such that $p, n \geq 1$. We denote by $\mathcal{G}_{p,n,\delta} \subset C^0(M)$ the family of all the maps $g \in C^0(M)$ such that there exist n boxes $K_1, \ldots, K_n$ satisfying the following properties:

- $\{K_j\}_{1\le j\le n}$ is composed of pairwise disjoint boxes.
- For all $1 \le j \le n$ the box K_j is periodic shrinking for g with period p_j that is a multiple of p, and
$$g^{p_j}|_{K_j} \text{ is topologically conjugate to } \Phi_j \in \mathcal{H}.$$
- There exists a sequence $\{H_j\}_{0\le j\le n-1}$ of periodic shrinking boxes for g, all of period p, and a periodic point $x_{n-1} \in \operatorname{int}(H_{n-1})$ of period p such that $K_j \cup H_j \subset H_{j-1}$, $K_j \cap H_j = \emptyset$ for all $1 \le j \le n-1$, $K_n \subset H_{n-1} \setminus \{x_{n-1}\}$ and $\operatorname{diam}(H_j) < \delta/2^j$ for all $0 \le j \le n-1$ (see Figure 5).

LEMMA 5.4. *Fix $\varepsilon, \delta > 0$ and natural numbers $n, p \ge 1$. Assume that $g_n \in \mathcal{G}_{p,n,\delta}$ or $g_n \in \mathcal{G}_{p,n,\delta} \cap \operatorname{Hom}(M)$. Then there exists an $(\varepsilon/2^{n+1})$-perturbation g_{n+1} of g_n such that $g_{n+1} \in \mathcal{G}_{p,n+1,\delta}$ or $g_{n+1} \in \mathcal{G}_{p,n+1,\delta} \cap \operatorname{Hom}(M)$, respectively. Moreover, for all $j = 1, \ldots, n$ the same boxes $K_1, \ldots, K_n$ and $H_0, \ldots, H_{n-1}$ are shrinking periodic for the new map g_{n+1} and for the given map g_n, with the same periods, and*
$$g_n^i|_{K_j} = g_{n+1}^i|_{K_j} \quad \forall i = 1, \ldots, p_j.$$

Proof. All the perturbations of g_n that we will construct are sufficiently close to g_n so that the same boxes $H_0, \ldots, H_{n-1}$ and $K_1, \ldots, K_n$ that are periodic shrinking for g_n are still periodic shrinking with the same periods for the perturbed maps. This is possible because the periodic shrinking property of a box and its period are open conditions. Moreover, we will only consider perturbations of g_n that coincide with g_n except in the interior of a finite number of boxes B, W, etc. whose g_n-iterates, up to the $(\max_{1\le j\le n} p_j)$th iterate, are disjoint from all the boxes of the family $\{g_n^i(K_j) : 1 \le j \le n, 0 \le i \le p_j - 1\}$. Therefore, if such a perturbation $\widetilde{g}$ of g_n is close enough to g_n, then the iterates by $\widetilde{g}$ of the boxes B, W, etc. (where $\widetilde{g}$ differs from g_n) are still disjoint from the g_n-iterates of K_j. This implies that for all $1 \le j \le n$,
$$g_n^i|_{K_j} = \widetilde{g}^i|_{K_j} \quad \forall i = 1, \ldots, p_j$$
and hence
$$\widetilde{g}^{p_j}|_{K_j} = g_n^{p_j}|_{K_j} \text{ is topologically conjugate to } \Phi_j \in \mathcal{H}.$$

Now let us perturb g_n as above, in several steps, to construct the boxes H_n and K_{n+1}.

By hypothesis, g_n has a periodic shrinking box H_{n-1} of period p, a periodic point $x_{n-1} \in \operatorname{int}(H_{n-1})$ of period p, and a periodic shrinking box $K_n \subset H_{n-1} \setminus \{x_{n-1}\}$ of period p_n, a multiple of p. It also has periodic shrinking boxes $K_1, \ldots, K_{n-1}, K_n$ whose g_n-orbits are disjoint from the periodic orbit of x_{n+1}. So, we can construct a box $\widetilde{B}_n \subset H_{n-1}$ containing the periodic point x_{n-1} in its interior, whose g_n-orbit up to the $(\max_{1\le j\le n} p_j)$th iterate is disjoint from all the sets of the family $\{f^i(K_j) : 1 \le j \le n, 0 \le i \le p_j - 1\}$. Moreover, we construct $\widetilde{B}_n$ such that $\operatorname{diam}(\widetilde{B}_n) < \widetilde{\delta}/2^n$. Repeating the proof of the density properties in Lemmas 4.2 and 4.5 (putting x_{n-1} instead of x_0), we construct an $(\varepsilon/(3\cdot 2^{n+1}))$-perturbation $\widetilde{g}_n$ of g_n, close enough to g_n, and a periodic shrinking box $H_n \subset \operatorname{int}(\widetilde{B}_n)$ for $\widetilde{g}_n$. Moreover, since x_{n-1} is a periodic point with period p for g_n, the period of H_n for $\widetilde{g}_n$ can be made equal to p (see Remark 4.6). By construction, $H_n \subset \widetilde{B}_n \subset H_{n-1}$ is disjoint from K_n. To construct $\widetilde{g}_n$ we only needed to modify g_n inside $\widetilde{B}_n$ (recall Remark 4.4). Therefore, if $\widetilde{g}_n$ is close enough to g_n, as observed at the beginning, the same periodic shrinking boxes $H_0, H_1, \dots, H_{n-1}$ and $K_1, \dots, K_n$ of g_n are preserved for $\widetilde{g}_n$ with the same periods, and $\widetilde{g}_n$ coincides with g_n on the g_n-orbits of the boxes $K_1, \dots, K_n$.

Now, as in the proofs of Lemmas 4.7 and 4.8, we will construct a new $(\varepsilon/(3\cdot 2^{n+1}))$-perturbation $\hat{g}_n$ of $\widetilde{g}_n$ such that $\hat{g}_n^p|_{H_n}$ is conjugate to a map in $\mathcal{H}$. To construct $\hat{g}_n$ we only need to modify $\widetilde{g}_n$ in $\widetilde{W}_n \cup \bigcup_{j=0}^{p-1} \widetilde{g}_n^j(H_n)$, where $\widetilde{W}_n$ is a small neighborhood of $\widetilde{g}_n^{p-1}(H_n)$ (see Remark 4.9). Since the $\widetilde{g}_n$-orbit of H_n is disjoint from the $\widetilde{g}_n$-orbits of K_j for all $1 \le j \le n$ (because H_n and K_j are disjoint periodic shrinking boxes for $\widetilde{g}_j$), we can choose W_n close enough to $\widetilde{g}_n^{p-1}(H_n)$ and $\hat{g}_n$ close enough to $\widetilde{g}_n$ so $\hat{g}_n$ coincides with $\widetilde{g}_n$ on the orbits of the boxes K_j, as observed at the beginning.

We conclude that the same shrinking boxes $K_1, \dots, K_n; H_0, \dots, H_{n-1}$ for $\widetilde{g}_n$ and g_n are still periodic shrinking for $\hat{g}_n$, with the same periods, and that $\hat{g}_n^{p_j}|_{K_j} = \widetilde{g}_n^{p_j}|_{K_j}$, which is conjugate to $\Phi_j \in \mathcal{H}$ for all $j = 1, \dots, n$.

When modifying g_n to obtain $\widetilde{g}_n$ and $\hat{g}_n$, the periodic point $x_{n-1} \in \operatorname{int}(H_{n-1})$ of period p for g_n may not be preserved as periodic for $\hat{g}_n$. But since $H_n \subset H_{n-1} \setminus K_n$ is a periodic shrinking box with period p for $\hat{g}_n$, by the Brouwer Fixed Point Theorem, there exists a periodic point $x_n \in \operatorname{int}(H_n) \setminus K_n$ for $\hat{g}_n$, with the same period p.

Since the return map $\hat{g}_n^p|_{H_n}$ is conjugate to a model map, there exists an ergodic measure μ with infinite entropy for $\hat{g}_n$ (see Lemma 3.1), supported on the $\hat{g}_n$-orbit of H_n. Therefore, there exists a recurrent point $y_n \in \operatorname{int}(H_n)$. We can choose $y_n \neq x_n$, because μ is not supported on the periodic orbit of x_n (in fact, μ has infinite entropy).

We now argue as in the proof of Lemma 5.2 (using $\hat{g}_n$, H_n and x_n in the role of $\tilde{f}$, H_0 and x_0) to construct an $\varepsilon/(3\cdot 2^n)$-perturbation g_{n+1} of $\hat{g}_n$, and a box $K_{n+1} \subset H_n \setminus \{x_n\}$ that is periodic shrinking for g_{n+1} of period p_{n+1} which is a multiple of p, and such that $g_{n+1}^{p_{n+1}}|_{K_{n+1}}$ is topologically conjugate to a model map.

As observed at the beginning, if we choose g_{n+1} close enough to $\hat{g}_n$, the boxes $H_0, \dots, H_n$ and $K_1, \dots, K_n$ are still periodic shrinking for g_{n+1} with the same periods, and

$$g_{n+1}^{p_j}|_{K_j} = \hat{g}_n^{p_j}|_{K_j} = g_n^{p_j}|_{K_j}$$

is topologically conjugate to a model map for all $1 \le j \le n$.

By construction we have $g_{n+1} \in \mathcal{G}_{p,n,\delta}$ and

$$\|g_{n+1} - g_n\| \le \|g_{n+1} - \hat{g}_n\| + \|\hat{g}_n - \widetilde{g}_n\| + \|\widetilde{g}_n - g_n\| < 3 \cdot \frac{\varepsilon}{3 \cdot 2^{n+1}} = \frac{\varepsilon}{2^{n+1}},$$

as required. ∎

DEFINITION 5.5. Fix $\delta > 0$. We denote by $\mathcal{G}_\delta \subset C^0(M)$ the family of all maps $g \in \bigcup_{p\ge 1} \bigcap_{n\ge 1} \mathcal{G}_{p,n,\delta}$ such that, for all $n \ge 1$, the boxes $H_0, \dots, H_{n-1}$ and $K_1, \dots, K_n$ of Definition 5.3 for g as belonging to $\mathcal{G}_{p,n,\delta}$ coincide with the boxes for g as belonging to $\mathcal{G}_{p,n+1,\delta}$.

LEMMA 5.6. *Fix $\delta > 0$. The family $\mathcal{G}_\delta$ is dense in $C^0(M)$ and its intersection with* $\mathrm{Hom}(M)$ *is dense in* $\mathrm{Hom}(M)$.

Proof. Let $f \in C^0(M)$ or $f \in \mathrm{Hom}(M)$, and $\varepsilon > 0$. We will construct $g \in \mathcal{G}_\delta$ such that $\mathrm{dist}(g, f) \le \varepsilon$.

By Lemma 5.2, there exist $p \ge 1$ and $g_1 \in \mathcal{G}_{p,1,\delta}$ such that $\mathrm{dist}(g_1, f) \le \varepsilon/2$. Denote by $H_0, K_1 \subset H_0$ the periodic shrinking boxes for g_1 as a map of $\mathcal{G}_{p,1,\delta}$ (recall Definition 5.3 for $n = 1$). By continuity, there exists $0 < \varepsilon_1 < \varepsilon$ such that for all g in the ε_1-neighborhood of g_1, H_0 is still a periodic shrinking box of period p for g.

By induction on $n \ge 1$ (Lemma 5.4 provides the inductive step), there is a sequence of maps $g_1, g_2, \dots$ and a strictly decreasing sequence of positive real numbers $\varepsilon > \varepsilon_1 > \varepsilon_2 > \cdots$ such that, for all $n \ge 1$, $g_n \in \mathcal{G}_{p,n,\delta}$, $\mathrm{dist}(g_{n+1}, g_n) \le \varepsilon_n/2^n$, the boxes $H_0, H_1, \dots, H_{n-1}$ and $K_1, \dots, K_n$ are still periodic shrinking for g_{n+1} with the same periods $p, p_1, \dots, p_n$ as for g_n, and $g_{n+1} = g_n$ when restricted to the g_n-orbits of the boxes K_j for $j = 1, \dots, n$. Moreover, for all g in the ε_n-neighborhood of g_n, H_{n-1} is still a periodic shrinking box of period p for g.

Since $\|g_{n+1} - g_n\| \le \varepsilon/2^{n+1}$ for all $n \ge 1$, the sequence $\{g_n\}_{n\ge 1}$ is Cauchy in $C^0(M)$ or $\mathrm{Hom}(M)$; let g be the limit map. Since g_n is an ε-perturbation of f for all $n \ge 1$, the limit map g satisfies $\mathrm{dist}(g, f) \le \varepsilon$.

Moreover, by construction, $g_k(x) = g_n(x)$ for all $x \in \bigcup_{j=0}^{p_n} g_n^j(K_n)$, for all $k \ge n \ge 1$. So $g_k^{p_n}|_{K_n} = g_n^{p_n}|_{K_n}$ is topologically conjugate to $\Phi_n \in \mathcal{H}$ for all $n \ge 1$ and all $k \ge n$ (recall $g_n \in \mathcal{G}_{p,n,\delta}$ and Definition 5.3). Thus K_n is still a periodic shrinking box for g of period p_n, and $g^{p_n}|_{K_n} = g_n^{p_n}|_{K_n}$ is topologically conjugate to a model map for all $n \ge 1$. Finally, for all $k > n \ge 1$ we have, by construction, $\mathrm{dist}(g_k, g_n) < \varepsilon_n(1/2^{n+1} + 1/2^{n+2} + \cdots + 1/2^k) \le \varepsilon_n$. So, taking the limit as $k \to \infty$, we obtain $\mathrm{dist}(g, g_n) \le \varepsilon_n$. This implies that H_{n-1} is still a periodic shrinking box of period p for g as it was for g_n. We have proved that $g \in \mathcal{G}_\delta$, as required. ∎

LEMMA 5.7. *A generic map $f \in C^0(M)$ for $m \ge 1$, and a generic homeomorphism f for $m \ge 2$, has a good sequence $\{K_n\}$ of boxes such that the return map $f^{p_n}|_{K_n}$ is topologically conjugate to a model $\Phi_n \in \mathcal{H}$.*

Proof. To see the G_δ property assume that f has a good sequence $\{K_n\}_n$ of periodic shrinking boxes. For each fixed n, the boxes K_n and H_n are also periodic shrinking with periods p_n and p respectively, for all g in an open set in $C^0(M)$ or in $\mathrm{Hom}(M)$ (see Definition 4.1). Taking the intersection of such open sets for all $n \geq 1$, we deduce that the same sequence $\{K_n\}$ is also a good sequence of periodic shrinking boxes for all g in a G_δ-set. Now, assume that moreover $f^{p_n}|_{K_n}$ is topologically conjugate to a model map for all $n \geq 1$. From Lemmas 4.7 and 4.8, for each fixed $n \geq 1$, the family of continuous maps g such that the return map $g^{p_n}|_{K_n}$ is topologically conjugate to a model, is a G_δ-set in $C^0(M)$ or in $\mathrm{Hom}(M)$. The (countable) intersection of these G_δ-sets produces a G_δ-set, as required.

To prove denseness, recall Definitions 5.3 and 5.5. Observe that the family of continuous maps or homeomorphisms that have a good sequence $\{K_n\}_{n\geq 1}$ of periodic shrinking boxes such that the return map to each K_n is topologically conjugate to a model map, contains the family $\mathcal{G}_\delta$ (or the intersection of this family with $\mathrm{Hom}(M)$) for any value of $\delta > 0$. Applying Lemma 5.6 we see that this last family is dense. ∎

Remark 5.8. As a consequence of Lemmas 5.7 and 3.1 (after applying the same arguments as at the end of the proof of Theorems 1.1 and 1.3), generic continuous maps and homeomorphisms f have a sequence of ergodic measures μ_n, each supported on the f-orbit of a box K_n of a good sequence $\{K_n\}_{n\geq 1}$ of periodic shrinking boxes for f, satisfying $h_{\mu_n}(f) = +\infty$ for all $n \geq 1$.

Let $\mathcal{M}$ denote the metrizable space of Borel probability measures on a compact metric space M, endowed with the weak* topology. Fix a metric dist^* in $\mathcal{M}$.

Lemma 5.9. *For all $\varepsilon > 0$ there exists $\delta > 0$ with the following property: if $\mu, \nu \in \mathcal{M}$ and $\{B_1, \ldots, B_r\}$ is a finite family of pairwise disjoint compact balls $B_i \subset M$, and if* $\mathrm{supp}(\mu) \cup \mathrm{supp}(\nu) \subset \bigcup_{i=1}^r B_i$, *and* $\mu(B_i) = \nu(B_i)$, $\mathrm{diam}(B_i) < \delta$ *for all* $i = 1, \ldots, r$, *then* $\mathrm{dist}^*(\mu, \nu) < \varepsilon$.

Proof. If $M = [0, 1]$, the proof is in [CT, Lemma 4]. If M is any other compact manifold of finite dimension $m \geq 1$, with or without boundary, just copy the proof of [CT, Lemma 4], substituting the pairwise disjoint compact intervals $I_1, \ldots, I_r \subset [0, 1]$ in that proof by the family of pairwise disjoint compact boxes $B_1, \ldots, B_r \subset M$. ∎

Proof of Theorems 1.2 and 1.4. Fix $\varepsilon > 0$, and let $\delta > 0$ be as in Lemma 5.9. By Lemma 5.7, generic continuous maps or homeomorphisms f have a good sequence $\{K_n\}_{n\geq 1}$ of periodic shrinking boxes, and a sequence $\{\mu_n\}$ of ergodic f-invariant measures such that $h_{\mu_n}(f) = +\infty$ (see Remark 5.8) and $\mathrm{supp}(\mu_n) \subset \bigcup_{j=0}^{p_n-1} f^j(K_n)$, where $p_n = l_n p$ is the period of the shrinking box K_n. Taking into account that $\{f^j(K_n)\}_{0\le j\le p_n-1}$ is a family of pairwise disjoint compact sets, and $f^{p_n}(K_n)\subset \operatorname{int}(K_n)$, we obtain, for each $j\in\{0,1,\dots,p_n\}$,

$$\mu_n(f^j(K_n)) = \mu_n(f^{-j}(f^j(K_n))) = \mu_n\big(f^{-j}(f^j(K_n))\cap \operatorname{supp}(\mu_n)\big) = \mu_n(K_n).$$

Since $1=\sum_{j=0}^{p_n-1}\mu_n(f^j(K_n)) = p_n\cdot\mu_n(K_n)$, we get

$$\mu_n(f^j(K_n)) = \mu_n(K_n) = \frac{1}{p_n} = \frac{1}{l_n\, p} \qquad \forall j=0,1,\dots,p_n.$$

From Definition 5.1, there exists a periodic point x_0 of period p such that $\lim_{n\to\infty}\sup_{j\ge 0}\operatorname{Hdist}(f^j(K_n), f^j(x_0)) = 0$, where Hdist denotes the Hausdorff distance. Therefore, there exists $n_0\ge 1$ such that $d(f^j(K_n), f^j(x_0))<\delta'$ for all $j\ge 0$ and all $n\ge n_0$, where $\delta'<\delta/2$ is chosen such that the balls $B_0, B_1,\dots,B_{p-1}$ centered at $f^j(x_0)$ and with radius δ' are pairwise disjoint. We obtain $f^j(K_n)\subset B_{j\,(\mathrm{mod}\,p)}$ for all $j\ge 0$ and all $n\ge 0$. Therefore,

$$\mu_n(B_j) = \frac{1}{p} \qquad \forall j=0,1,\dots,p-1,\ \forall n\ge n_0.$$

Finally, applying Lemma 5.9, we conclude that $\operatorname{dist}^*(\mu_n,\mu_0)<\varepsilon$ for all $n\ge n_0$, where $\mu_0 := (1/p)\sum_{j=0}^{p-1}\delta_{f^j(p)}$ is the f-invariant probability measure supported on the periodic orbit of x_0, which has zero entropy. ■

6. Open questions. Lipschitz maps have finite topological entropy and thus cannot have infinite entropy invariant measures. The following question arises: do Theorems 1.1 and 1.3 hold also for maps with more regularity than continuity but lower regularity than Lipschitz? For instance, do they hold for Hölder-continuous maps?

A priori there is a chance to answer this question positively in situations where the topological entropy is generically infinite, for example for one-dimensional Hölder-continuous endomorphisms and also for bi-Hölder homeomorphisms on manifolds of dimension 2 or larger. In both cases generic infinite entropy is known [FHT1, FHT2]. This is a good question for further research.

Theorems 1.1 and 1.3 are proved for compact manifolds; we wonder if some of the results also hold in other compact metric spaces that are not manifolds. Do they hold if the space is a Cantor set K?

If the aim were just to construct $f\in\operatorname{Hom}(K)$ with ergodic measures with infinite metric entropy, the answer is positive. Theorem 1.3 holds for the 2-dimensional square $D^2 := [0,1]^2$. One of the steps of the proof consists in constructing a Cantor set $\Lambda\subset D^2$, and a homeomorphism Φ on M that leaves Λ invariant, and possesses an Φ-invariant ergodic measure supported on Λ with infinite metric entropy (see Lemma 3.1 and Remark 3.2). Since any two Cantor sets K and Λ are homeomorphic, we deduce that any Cantor set K supports a homeomorphism f and an f-ergodic measure with infinite metric entropy.

If the purpose were to prove that such homeomorphisms are generic in $\mathrm{Hom}(K)$, the answer is negative. On the one hand, there also exist homeomorphisms on K with finite, and even zero, topological entropy, for example $f \in \mathrm{Hom}(K)$ conjugate to the homeomorphism on the attractor of a Smale horseshoe, or to the attractor of the C^1-Denjoy example on the circle. On the other hand, it is known that each homeomorphism on a Cantor set K is topologically locally unique, i.e., it is conjugate to any of its small perturbations [AGW]. Therefore, the topological entropy is locally constant in $\mathrm{Hom}(K)$. We conclude that the homeomorphisms on the Cantor set K with infinite metric entropy, which do exist, are not dense in $\mathrm{Hom}(K)$; hence they are not generic.

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