# Entropy formula for $C^1$ expanding maps

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**Abstract**

We prove that the (necessarily existing) pseudo-physical or SRB-like measures of $C^1$ expanding dynamical systems on a compact Riemannian manifold satisfy Pesin's entropy formula.

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# 1 Introduction

We consider the dynamical system by iteration of a map $f : M \to M$ of $C^1$ class on a compact Riemannian manifold $M$ of finite dimension. The search for "natural" invariant measures that describe the statistical behavior of an observable set of orbits of the system is a key point in the ergodic theory. Usually, the concept of observability of a set of orbits is associated to its volume, namely, a set of orbits is observable if it has positive Lebesgue measure. This is particularly restrictive if the map is not Lebesgue preserving. The ergodicity of an invariant measure, if the system is not volume preserving, does not ensure that the measure is "natural". In fact, an ergodic measure $\mu$ describes the statistical behavior of $\mu$-almost all the orbits, but if $\mu$ is mutually singular with the Lebesgue measure, the set of such orbits may zero volume.

## 1.1 Physical and pseudo-physical measures.

We denote by $\mathcal{M}$ the set of Borel probability measures on $M$, endowed with the weak*-topology (see for instance [34]). The weak* topology in $\mathcal{M}$ is defined by the following equality:

$$\lim_{n\to+\infty}{}^{*}\ \mu_n = \mu \text{ for } \mu_n, \mu \in \mathcal{M} \quad \text{if and only if}$$

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$$\lim_{n\to+\infty}\int \phi\, d\mu_n = \int \phi\, d\mu \text{ for all the continuous functions } \phi: M \to \mathbb{R}.$$

We denote by $\mathcal{M}_f \subset \mathcal{M}$ the subset of measures $\mu$ that are invariant by $f$, i.e. $f^*\mu = \mu$, where $f^*: \mathcal{M} \to \mathcal{M}$ is the pull-back of $f$, defined by $f^*\mu(B) = \mu(f^{-1}(B))$ for any Borel-measurable set $B$.

In [21] a measure $\mu \in \mathcal{M}_f$ is called *natural* if it satisfies

$$\mu = \lim_{n\to+\infty}{}^{*}\frac{1}{n}\sum_{j=0}^{n-1}(f^*)^j\nu,$$

for some (not necessarily invariant) Borel probability measure $\nu$ that is absolutely continuous with respect to the Lebesgue measure. The problem in this definition of natural measures, is that they do not necessarily exist.

One of the most used concept of relevance of an invariant measure, from the statistical viewpoint for a positive volume set of orbits, is the property of being "physical", that we will define below:

**Definition 1.1. (Empiric probabilities and basin of statistical attraction.)**

For any initial point $x \in M$ the *empiric probability measure* $\sigma_n(x) \in \mathcal{M}$ up to time $n$ of the orbit of $x$ is

$$\sigma_n(x) := \frac{1}{n}\sum_{j=0}^{n-1}\delta_{f^j(x)},$$

where $\delta_y$ denotes the Dirac delta probability measure supported on $y$. In other words, the empiric probability is equally supported on the points of the finite piece of orbit from $x$ up to $f^{n-1}(x)$.

For any point $x \in M$ the sequence $\{\sigma_n(x)\}_{n\geq 1} \subset \mathcal{M}$ of empiric probability has convergent subsequences, because $\mathcal{M}$ is a compact space with the weak$^*$-topology. We call the set of probabilities measures that are the limits$^*$ of the convergent subsequences of $\{\sigma_n(x)\}_{n\geq 1}$, the *p-omega limit* of the orbit of $x$ (i.e. the omega limit in the space of probabilities), and denote it by $p\omega(x)$. Precisely,

$$p\omega(x) = \{\nu \in \mathcal{M} : \nu = \lim^*_{j\to+\infty}\sigma_{n_j}(x)$$

$$\text{for some convergent subsequence } \{\sigma_{n_j}(x)\}_{j\geq 1}\}.$$

For any invariant measure $\mu$, the *basin $B(\mu)$ of statistical attraction* of $\mu$ is the set

$$B(\mu) := \{x \in M : \mu = \lim^*_{n\to+\infty}\sigma_n(x)\} = \{x \in M : p\omega(x) = \{\mu\}\}.$$

**Definition 1.2. (Physical measures.)** We call an invariant probability measure $\mu$ *physical* if its basin of statistical attraction has positive Lebesgue measure, i.e.

$$m(B(\mu)) > 0,$$

where $m$ denotes the Lebesgue measure.

The physical measures are also called Sinai-Ruelle-Bowen (SRB) measures, due to the early works in the decade of 1970's of Ya. Sinai [29], D. Ruelle and R. Bowen [11], [10], [27], introducing the physical measures for smooth dynamical systems with uniform hyperbolicity.

When working in the $C^1$ topology, we prefer to call them physical measures, instead of SRB-measures, to avoid confusions: In fact, there is abundant literature studying the physical or SRB-measure for smooth systems or at least $C^{1+\alpha}$ systems with $\alpha > 0$. With such regularity, relevant properties appear: the conditional measures of the physical probabilities along the unstable submanifolds are absolutely continuous with respect to the Lebesgue measures of these submanifolds [24] [19]. In the literature, these properties are required or proved, before calling the invariant measures SRB [35]. But in our context, where regularity is only $C^1$ (but not necessarily $C^{1+\alpha}$), the existence of unstable submanifolds fails [25]. Besides, also in the particular cases for which the unstable manifolds exist, the properties of absolute continuity do not hold [7].

One of the most relevant problems in the ergodic theory, is to prove the existence of physical measures, since a priori the sequence of empiric probabilities may be non convergent for a set of orbits with positive volume. The existence of physical measures, is mainly obtained in a scenario of some kind of uniform or non-uniform hyperbolicity or, at least, domination of the expanding directions. The existence of physical measures was proved for $C^1$-generic expanding map of the circle in [12], for $C^1$ generic diffeomorphisms having an hyperbolic attractor in [26], and for $C^{1+\alpha}$ diffeomorphisms with dominated splitting in [6]. More recently, a characterization in the $C^{1+\alpha}$ scenario of the existence of SRB measures on surfaces was given in [16].

Besides the existence, the problem of uniqueness or, at least finitude, of the physical measures is the object of research mainly for partially hyperbolic systems. In [22] it is proved the existence and uniqueness of SRB measures for certain class of $C^2$ partially hyperbolic systems. In [2] the authors prove the existence of at most a finite number of SRB-measures for a class of $C^{1+\alpha}$ partially hyperbolic dynamical systems.

As said above, the existence of physical or SRB measures was mainly proved for systems that are $C^{1+\alpha}$ regular (and with some kind of hyperbolicity or expanding properties), except some few articles that explore their existence in the $C^1$ topology. In an intermediate situation, in [9] the author finds SRB measures for hyperbolic systems that are more regular than $C^1$ but with weaker regularity than $C^{1+\alpha}$.

To overcome the problem of nonexistence of physical measures, a generalization of such measures, was introduced in [14]: the concept of pseudo-physical or SRB-like measure, which we define in the following paragraphs.

Recall Definition 1.1 of the $p\omega$-limit set of an orbit in the space of probabilities and of basin of statistical attraction of an invariant measure.

**Definition 1.3. (Epsilon-weak basin of statistical attraction.)** Choose a distance $\mathrm{dist}^*$ in $\mathcal{M}$ that endows the weak$^*$ topology.

For any $f$-invariant probability measure $\mu$, and for $\epsilon > 0$, we call the following set $B_\epsilon(\mu) \subset M$ the *$\epsilon$-weak basin of statistical attraction of $\mu$:*

$$B_\epsilon(\mu) := \{x \in M : \mathrm{dist}^*(p\omega(x), \mu) < \epsilon\}.$$

We note that the basin of statistical attraction defined in 1.1, may not coincide with the zero-weak basin of statistical attraction. In fact, the weak*-distance between $p\omega(x)$ and $\mu$ may be zero, but the sequence of empiric probabilities may not converge, and have convergent subsequences whose limits are different from $\mu$.

**Definition 1.4. (Pseudo-physical measures.)**

We call an invariant probability measure $\mu$ *pseudo-physical or SRB-like,* if its $\epsilon$-weak basin of statistical attraction has positive Lebesgue measure for all $\epsilon > 0$. In brief

$$m(B_\epsilon(\mu)) > 0 \text{ for all } \epsilon > 0.$$

The following properties were proved in [14]:

The pseudo-physical measures do always exist for any continuous $f$, and do not depend of the chosen dist$^*$ in the space of probability measures (provided it endows the weak*-topology). Besides for Lebesgue-almost all the orbits, any convergent subsequence of empiric probabilities converges to a pseudo-physical measure. In other words, the set of all the pseudo-physical measures completely describes the observable statistical behavior of the system if the criteria of observability is that of the orbits with positive volume. Any physical measure is pseudo-physical, so pseudo-physicality is a generalization of physicality. If the set of all the pseudo-physical measures is finite, then all the pseudo-physical measures are physical.

## 1.2 Equilibrium states and Pesin's entropy formula.

Other important definition in the study of statistical properties of a dynamical system, coming from the statistical mechanics, is the concept of equilibrium states of a variational principle (see for instance [10], [18]), and in particular the set of measures that satisfy Pesin's entropy formula. We will state the definitions of those concepts in the following paragraphs.

To define the equilibrium states we will use the *metric entropy* of the map $f$ with respect to an $f$-invariant probability measure $\mu$. For a definition of the metric entropy, see Section 3 of this article. For a more detailed exposition of the properties of the entropy see also for instance the book [34].

**Definition 1.5. (Equilibrium States.)**

Let $\psi : M \mapsto \mathbb{R}$ be a continuous function called *potential.* .

We call the following supremum $P(f, \psi)$ the *pressure* of $f$ with respect to the potential $\psi$:

$$P(f, \psi) := \sup_{\nu \in \mathcal{M}_f} \left\{ h_\nu(f) + \int \psi \, d\nu \right\},$$

where $h_\nu(f)$ is the metric entropy of $f$ with respect to the $f$-invariant measure $\nu$.

An $f$-invariant measure $\mu$ is *an equilibrium state* of $f$ with respect to the potential $\psi$ if

$$h_\mu(f) + \int \psi \, d\mu = P(f, \psi).$$

In the decade of 1970', Sinai, Ruelle and Bowen proved important relations between the equilibrium states and the physical measures for smooth hyperbolic systems [29], [11], [10], [27]. More recently, [1] proves the existence of equilibrium states for certain type of partially hyperbolic endomorphisms of $C^1$ class, and for certain type of potentials. In [17], the existence and uniqueness of equilibrium states are proved for non-uniformly expanding skew products and for Hölder continuous potentials. Also the uniqueness of equilibrium states, besides their existence, is proved in [23] for a class of flows satisfying a version of the specification property among other conditions. In [4] the existence of finitely meany ergodic equilibrium states for a type of non-uniformly expanding maps, with respect to Hölder continuous potentials.

The equilibrium states are mainly applied when the potential is related with the positive Liapunov exponents which translate to the tangent space the chaotic behavior of the dynamics by iterations of $f$.

Oseledet's Theorem (see for instance [8]) states that for any $f$-invariant measure $\mu$, at almost all the points $x$ with respect to $\mu$ there exists a splitting of the tangent space

$$T_xM = \oplus_{i=1}^{k(x)} E_x^i$$

into $Df$-invariant measurable subspaces $E_x^i$ along which the Liapunov exponents exist according with the following definition:

**Definition 1.6. (Liapunov exponents.)** The *Liapunov exponent* $\chi_i(x)$ of $f$ at the point $x \in M$ along the measurable $Df$-invariant tangent subspace $E_x^i$ is:

$$\chi_i(x) := \lim_{n\to\pm\infty} \frac{\|Df_x^n v\|}{n} \quad \text{for all } 0 \neq v \in E_x^i.$$

The Liapunov exponents are the exponential rate of increasing (if positive) or decreasing (if negative) of the vectors of the tangent space, when iterating $Df : TM \mapsto TM$.

We denote by

$$\sum_{i=1}^{m(x)} \chi_i^+(x)$$

the sum of the Liapunov exponents that are strictly larger than zero at the point $x$, counting each one as many times as its multiplicity. If all the Liapunov exponents are smaller or equal than zero, that sum is null. The following Theorem is due to Margulis [20] and Ruelle [28], and states an upper bound for the metric entropy, related with the positive Liapunov exponents:

**Theorem 1.7. (Margulis-Ruelle inequality)**

*For any $f$-invariant measure $\mu$,*

$$h_\mu(f) \leq \int \sum_{i=1}^{m(x)} \chi_i^+(x)\, d\mu$$

For a proof, see for instance [34].

**Definition 1.8. (Pesin's entropy formula)** An $f$-invariant measure $\mu$ satisfies *Pesin's entropy formula* if

$$h_\mu(f) = \int \sum_{i=1}^{m(x)} \chi_i^+(x)\, d\mu$$

Measures satisfying Pesin's entropy formula may not exist. But if someone exists, its metric entropy is the maximum possible with respect to the chaotic behavior of $f$ that is expressed by the positive Liapunov exponents.

When the system has a continuous $Df$-invariant unstable sub-bundle $U \subset TM$, the integral of the sum of the positive Liapunov exponents equal the integral of

$$\phi := \log |\det Df|_U|.$$

If this latter function is continuous, its opposite $\psi = -\phi$ can be used as the potential to study the equilibrium states. In this case the pressure $P(f, \psi) \leq 0$, due to Margulis-Ruelle inequality. So the measures satisfying Pesin's entropy formula, if someone exists, are the equilibrium states of $f$ with respect to the potential $\psi$, and the pressure is zero.

Ya. B. Pesin [24] early initiated the so called Pesin's Theory, proving important relations between the Liapunov exponents and the existence of measures satisfying Pesin's entropy formula for some smooth systems, is a scenario for which there exists invariant measures that have properties of absolute continuity with respect to the Lebesgue measure along the unstable submanifolds.

Later, Ledrappier and Young [19] proved that the condition of absolute continuity used in Pesin's Theory is indeed a characterization of the measures (if they exist) that satisfy Pesin's entropy formula, provided the system is of $C^{1+\alpha}$ $(\alpha > 0)$ class. This characterization is relevant: it is the key point in the later research proving the existence of measures satisfying Pesin's entropy formula. For instance in [6] the existence of a SRB measure that satisfies Pesin's entropy formula is proved for $C^{1+\alpha}$ diffeomorphisms with dominated splitting. In [3] it is proved the existence and uniqueness of SRB measure satisfying Pesin' entropy formula for Gibbs-Markov induced maps, that translate to a piecewise $C^{1+\alpha}$ dynamics.

The characterization of Ledrappier and Young of measures satisfying Pesin's entropy formula via the properties of absolute continuity with respect to the Lebesgue measure along the unstable submanifolds, do not hold for $C^1$ systems if they are not $C^{1+\alpha}$. In fact, generic $C^1$ systems do not have measures with that property of absolute continuity [25], [7]. Nevertheless, under some kind of hyperbolicity, $C^1$ systems still have measures satisfying Pesin's entropy formula: In [31], A. Tahzibi proved that generic $C^1$ systems of dimension two have an invariant measure satisfying Pesin's entropy formula. Later, in [30], Sun and Tian extended Tahzibi's result to $C^1$- generic volume-preserving diffeomorphisms in any dimension with a dominated splitting. In [13] it is proved that the necessarily existing pseudo-physical measures satisfy Pesin's entropy formula for all the $C^1$ systems with dominated splitting in any dimension. In [15] it was proved the same result but for $C^1$ expanding maps in dimension one. And in [5] it is proved the result for $C^1$ nonuniform expanding maps in any dimension.

## 1.3 Statement of the result for expanding maps.

**Definition 1.9. (Expanding maps.)**

The $C^1$ map $f : M \to M$ is (uniformly) *expanding* if there exists a constant $\lambda > 1$ such that

$$\|Df_x(v)\| \geq \lambda \|v\| \quad \forall\ (x,v) \in TM.$$

Recall Definition 1.6 of the Liapunov exponents. Since for an expanding map, the norm of all the vectors in the tangent space grow more than $\lambda > 1$ at each iterate, the exponential rate of growing for the vectors of any direction, is larger than $\log \lambda > 0$. So all the Liapunov exponents are positive.

**Theorem 1.10. (Liouville formula)** *For any $C^1$ map $f$ and for any $f$-invariant measure $\mu$*

$$\int \sum_{i=1}^{k(x)} \chi_i(x)\, d\mu = \int \log|\det(Df)|\, d\mu,$$

*where $\sum_{i=1}^{k(x)} \chi_i(x)$ is the sum of all the Liapunov exponents at the point $x$, counting each one as many times as its multiplicity.*

For a proof of Liouville formula, see for instance [32].

**Corollary 1.11.** *If the $C^1$ map $f$ is expanding then, for any $f$-invariant measure $\mu$*

$$\int \sum_{i=1}^{m(x)} \chi_i^+(x)\, d\mu = \int \log|\det(Df)|\, d\mu,$$

*where $\sum_{i=1}^{m(x)} \chi_i^+(x)$ is the sum of all the positive Liapounov exponents at the point $x$, counting each one as many times as its multiplicity.*

*Proof.* Since the map is expanding, all the Liapunov exponents are positive. Therefore, this corollary is a restatement of Liouville formula. $\square$

**Proposition 1.12.** *For a $C^1$ expanding map $f$, an invariant probability measure $\mu$ satisfies Pesin's entropy formula if and only if it is an equilibrium state for the potential $\psi = -\log|\det(Df)|$ and the pressure $P(f,\psi) = 0$.*

*Proof.* Due to Theorem 1.7 and Corollary 1.11, we have

$$P(f,\psi) = \sup_{\mu \in \mathbb{M}_f} (h_\mu(f) + \int \psi\, d\mu) =$$

$$= \sup_{\mu \in \mathbb{M}_f} (h_\mu(f) - \int \log|\det(Df)|\, d\mu) \leq 0.$$

So the pressure $P(f,\psi)$ is not positive.

Now, recalling Definition 1.8 of Pesin's entropy formula, and using again Corollary 1.11, an $f$-invariant measure $\mu$ satisfies Pesin's formula if and only if

$$0 = h_\mu(f) - \int \log|\det(Df)|\,d\mu = h_\mu(f) + \int \psi\,d\mu = \sup_{\mu\in\mathbb{M}_f}(h_\mu(f) + \int \psi\,d\mu).$$

□

The main result to be proved along this paper is the following:

**Theorem 1.** *Let $f : M \mapsto M$ be an expanding $C^1$ map on a compact Riemannian manifold $M$ of finite dimension.*

*Then, any (necessarily existing) pseudo-physical measure $\mu$ for $f$ satisfies Pesin's entropy formula. Namely,*

$$h_\mu(f) = \int \sum_{i=1}^{m(x)} \chi_i^+(x)\,d\mu = \int \log|\det(Df)|\,d\mu.$$

*Equivalently, $\mu$ is an equilibrium state for the potential*

$$\psi = -\log|\det(Df)|,$$

*and the pressure $P(f,\psi) = 0$.*

Theorem 1 is a generalization to any finite dimension of the result previously proved in [15] in dimension one. The proof of Theorem 1, which we will expose along this paper, was presented by F. Valenzuela is his unpublished thesis in 2017 [33]. In 2019, and previously in 2017 as a preprint, Araujo and Santos [5] proved a more general result that holds not only for $C^1$ (uniformly) expanding maps of Theorem 1 (according to Definition 1.9), but also for maps that are non-uniformly topologically expanding.

# 2 Expanding maps are expansive.

To prove Theorem 1 we need an important topological property defined for continuous maps, called expansiveness. We need to show that, in particular, the $C^1$ expanding maps, according to Definition 1.9, are expansive.

**Definition 2.1. (Expansive maps.)**

A continuous map $f : M \mapsto M$ is *expansive in the future* , if there exists a constant $\alpha > 0$, called the *expansivity constant,* such that if $x, y \in M$ satisfy

$$\text{dist}(f^n(x), f^n(y)) \leq \alpha \ \ \forall\, n \geq 0,$$

then $x = y$.

The expansiveness is understood as the sensitivity to the initial condition. In fact, the two orbits with two different initial states, even if these initial states are arbitrarily near, they separate more than $\alpha$ for some iterate in the future.

**Proposition 2.2.** *If the $C^1$ map $f : M \mapsto M$ on the compact Riemannian manifold $M$ is expanding, then it is expansive in the future.*

*Proof.* Let $x \in M$, and $\delta > 0$. Denote by $B_\delta(0) \subset T_xM$ the open ball in the tangent space at $x$, centered at 0 with radius $\delta$, i.e. the set of vectors in $T_xM$ with norm smaller than $\delta$. Choose $\delta$ small enough such that the exponential map $\exp_x : B_\delta(0) \subset T_xM \to M$ is a diffeomorphism onto its image. Explicitly, for any $y \in M$ such that $\text{dist}(x, y) < \delta$, there exists a unique vector $v = \exp_x^{-1}(y) \in T_xM$, and this vector satisfies $\|v\| = \text{dist}(x, y) < \delta_1$. Since $M$ is compact, we can choose a uniform $\delta > 0$, namely $\delta$ does not depends on $x$.

In the sequel we will denote $y - x$ to refer to the vector $v = \exp_x^{-1}(y) \in B_\delta(0) \subset T_xM$.

Since $f$ is of $C^1$ class

$$\|f(y) - f(x)\| = \|Df_z(y - x)\|,$$

for some point $z$ in the (convex) ball centered at $x$ of radius $\delta$.

Then

$$\|f(y) - f(x)\| \geq \lambda\|y - x\|,$$

where $\lambda > 1$ is the constant in Definition 2.1 of the expanding map $f$.

It is enough to prove that $\delta > 0$ is a constant of expansivity for $f$, as in Definition 2.1. Assume that

$$\text{dist}(f^j(x), f^j(y)) = \|f^j(y) - f^j(x)\| \leq \delta \;\; \forall\, j \geq 0.$$

Then

$$\|f^{j+1}(y) - f^{j+1}(x)\| \geq \lambda\|f^j(y) - f^j(x)\| \;\; \forall\, j \geq 0,$$

and therefore

$$\delta \geq \text{dist}(f^n(x), f^n(y)) = \|f^n(y) - f^n(x)\| \geq \lambda^n\|y - x\| \;\; \forall\, n \geq 0.$$

Since $\lambda^n \to +\infty$ with $n$, while $\lambda^n\|y - x\| \leq \alpha$ for all $n \geq 0$, we deduce that $x = y$. □

# 3 The metric entropy.

In this section we review the definition of metric entropy of $f$ with respect to an invariant measure $\mu$ and state some of its properties that we will use in the proof of Theorem 1.

A *finite measurable partition* $\mathcal{P}$ is a finite family of measurable sets $P \subset M$ that are pairwise disjoint and whose union is $M$. The sets $P \in \mathcal{P}$ are *the pieces of* $\mathcal{P}$. We agree to simply name *partition* to refer to a finite measurable partition.

The *boundary* $\partial\mathcal{P}$ of a partition is the union of the topological boundaries of its pieces. Namely,

$$\partial\mathcal{P} := \bigcup\{\partial P : \; P \in \mathcal{P}\}.$$

The *diameter* $\text{diam}(\mathcal{P})$ of a partition is the maximum diameter of its pieces. Namely,

$$\text{diam}(\mathcal{P}) = \max_{P \in \mathcal{P}} \text{diam}(P).$$

The *product* $\mathcal{P} \vee \mathcal{Q}$ of two partitions is the new partition whose pieces are $P \bigcap Q$, where $P \in \mathcal{P}$ and $Q \in \mathcal{Q}$.

More generally, for each natural number $N \geq 1$, if $\{\mathcal{P}_i\}_{1 \leq i \leq N}$ is a collection of $N$ partitions $\mathcal{P}_i$, we define their *product* as follows:

$$\bigvee_{i=1}^{N} \mathcal{P}_i = \left\{ \bigcap_{i=1}^{N} P_i : \quad P_i \in \mathcal{P}_i \ \ \forall\, 1 \leq i \leq N \right\}.$$

**Definition 3.1.** The *entropy* $H(\mathcal{P}, \mu)$ of the partition $\mathcal{P}$ with respect to a probability measure $\mu$ is

$$H(\mathcal{P}, \mu) := -\sum_{P \in \mathcal{P}} \mu(P) \log \mu(P),$$

where we agree to take $0 \cdot \log 0 = 0$.

Now, let us introduce the dynamics of $f : M \mapsto M$ to the study of the entropy of the partitions with respect to a probability measure.

For any partition $\mathcal{P}$, we consider the following product partition:

$$\mathcal{P}_f^n := \bigvee_{i=0}^{n-1} f^{-i}\mathcal{P} = \mathcal{P} \vee f^{-1}\mathcal{P} \vee \ldots \vee f^{-(n-1)}\mathcal{P}, \tag{1}$$

where $f^{-i}\mathcal{P} := \{f^{-i}(P) : P \in \mathcal{P}\}$.

**Proposition 3.2.** *If $\mu$ is $f$-invariant, then the following limit exists and satisfies the equality and inequality at right:*

$$\lim_{n\to\infty} \frac{H(\mathcal{P}_f^n, \mu)}{n} = \inf_{n \geq 1} \frac{H(\mathcal{P}_f^n, \mu)}{n} \leq H(\mathcal{P}, \mu).$$

*Proof.* See for instance [34], Lemma 9.1.12. □

**Definition 3.3.** Let $\mathcal{P}$ be a partition, and $\mu$ be an $f$-invariant measure, We call the following expression $h_\mu(f, \mathcal{P})$ *the entropy of $f$ with respect to the partition $\mathcal{P}$ and to the measure $\mu$:*

$$h_\mu(f, \mathcal{P}) := \lim_{n\to\infty} \frac{H(\mathcal{P}_f^n, \mu)}{n} = \inf_{n \geq 1} \frac{H(\mathcal{P}_f^n, \mu)}{n}.$$

**Definition 3.4. (The metric entropy.)**

Let $\mu$ be an $f$-invariant probability measure. We call the following expression $h_\mu(f)$ *the metric entropy of $f$ with respect to the measure $\mu$:*

$$h_\mu(f) \ := \ \sup \{h(f, \mathcal{P}) : \ \mathcal{P} \textit{ finite measurable partition}\}.$$

## 3.1 Properties of the entropy of partitions.

In this subsection we state some properties of the entropy of partitions that will be used in the proof of Theorem 1. For more properties of the entropy, see for instance the books [18] and [34].

**Proposition 3.5.** *For any (finite measurable) partition $\mathcal{P}$ and any probability measure $\mu$*

$$H(\mathcal{P},\mu) \leq \log p,$$

*where $p$ is the number of pieces of $\mathcal{P}$.*

*Proof.* See for instance [34], Lemma 9.1.3. □

**Proposition 3.6.** . *Let $\mathcal{P}, Q$ be two partitions and $\mu$ any probability measure. Then,*

$$H(\mathcal{P},\mu) \leq H(\mathcal{P}\vee Q,\mu) \leq H(\mathcal{P},\mu) + H(\mathcal{Q},\mu).$$

*Proof.* See for instance [34], Section 9.1. □

**Proposition 3.7.** *For any partition $\mathcal{P}$ and any (not necessarily $f$-invariant) probability measure $\mu$:*

$$H(\mathcal{P}_f^n,\mu) \leq \sum_{i=0}^{n-1} H(f^{-i}\mathcal{P},\mu) = \sum_{i=0}^{n-1} H(\mathcal{P}, f^{*i}\mu).$$

*Proof.* Applying Equality (1) and Proposition 3.6 we have

$$H(\mathcal{P}_f^n,\mu) = H\left(\bigvee_{i=0}^{n-1} f^{-i}\mathcal{P},\mu\right) \leq \sum_{i=0}^{n-1} H(f^{-i}\mathcal{P},\mu). \tag{2}$$

Besides, from Definition 3.1:

$$H(f^{-i}\mathcal{P},\mu) = -\sum_{P\in\mathcal{P}} \mu(f^{-i}P)\log\mu(f^{-i}P),$$

and from the definition of the pull-back $f^*$, we have $\mu(f^{-i}P) = f^{*i}\mu(P)$. So,

$$H(f^{-i}\mathcal{P},\mu) = H(\mathcal{P}, f^{*i}\mu).$$

Finally, substituting this last equality in (2), we deduce

$$H(\mathcal{P}_f^n,\mu) \leq \sum_{i=0}^{n-1} H(f^{-i}\mathcal{P},\mu) = \sum_{i=0}^{n-1} H(\mathcal{P}, f^{*i}\mu),$$

as wanted. □

**Proposition 3.8.** *For any partition $\mathcal{P}$ and any (not necessarily $f$-invariant) probability measure $\mu$,*

$$\frac{H(\mathcal{P}_f^n,\mu)}{n} \leq \frac{1}{n}\sum_{i=0}^{n-1} H(\mathcal{P}, f^{*i}\mu) \leq H\left(\mathcal{P}, \frac{1}{n}\sum_{n=0}^{n-1} f^{*i}\mu\right).$$

*Proof.* Consider the following continuous real function $\phi : [0,1]\mathbb{R}$:

$$\phi(u) = -u \log u \text{ if } u \in (0,1], \quad \phi(0) = 0.$$

It is easy to check that $\phi$ is $C^\infty$ in $(0,1)$, and that $\phi''(u) < 0$ for all $u \in (0,1)$. Therefore, the graph of $\phi$ is above the secant line. Thus, the value of $\phi$ at the convex combination of $n$ values $u_0, \dots, u_{n-1} \in [0,1]$ (which is the ordinate of a point in the graph of $\phi$), is larger or equal than the convex combination of $\phi(u_0), \dots, \phi(u_{n-1})$ (which is the ordinate of a point in the secant line). Precisely, if $0 < \lambda_i < 1$ for all $0 \le i \le n-1$ and $\sum_{i=0}^{n-1} \lambda_i = 1$, then

$$\phi\left(\sum_{i=0}^{n-1} \lambda_i u_i\right) \ge \sum_{i=0}^{n-1} \lambda_i \phi(u_i).$$

Therefore

$$H\left(\mathcal{P}, \frac{1}{n}\sum_{n=0}^{n-1} f^{*i}\mu\right) = \sum_{P\in\mathcal{P}} \phi\left(\frac{1}{n}\sum_{i=0}^{n-1} f^{*i}\mu(P)\right) \ge$$

$$\sum_{P\in\mathcal{P}} \frac{1}{n}\sum_{i=0}^{n-1} \phi(f^{*i}\mu(P)) =$$

$$\frac{1}{n}\sum_{i=0}^{n-1}\sum_{P\in\mathcal{P}} \phi(f^{*i}\mu(P)) = \frac{1}{n}\sum_{i=0}^{n-1} H(\mathcal{P}, f^{*i}\mu).$$

Finally, using Proposition 3.7 the last expression is greater or equal than $(1/n)H(\mathcal{P}_f^n, \mu)$, as wanted. ☐

**Proposition 3.9.** *Let $\mathcal{P}$ be a (finite measurable) partition and $\mu$ a probability measure such that*

$$\mu(\partial P) = 0.$$

*If $\{\mu_n\}_{n\ge 0}$ is a sequence of probabilities measures such that*

$$\lim^*_{n\to+\infty} \mu_n = \mu,$$

*then*

$$\lim_{n\to+\infty} H(\mathcal{P}, \mu_n) = H(\mathcal{P}, \mu).$$

In other words, this proposition states the continuity at $\mu$ of the entropy of a partition $\mathcal{P}$ as a function of the measure $\mu$, if the boundary of the partition has zero $\mu$-measure.

To prove Proposition 3.9, we will use the following lemma:

**Lemma 3.10.** *Let $\mu_n$, $\mu$ be probability measures such that*

$$\lim^*_{n\to+\infty} \mu_n = \mu$$

(1) *If $K \subset M$ is compact, then $\limsup_{n\to+\infty} \mu_n(K) \le \mu(K)$.*

(2) *If $V \subset M$ is open, then $\liminf_{n\to+\infty} \mu_n(V) \ge \mu(V)$.*

(3) *If $A$ is a Borel set such that $\mu(\partial A) = 0$, then*

$$\lim_{n\to+\infty} \mu_n(A) = \mu(A).$$

*Proof.* (1) Let $\epsilon > 0$ and $V \subset M$ such that $K \subset V$ and $\mu(V \setminus K) < \epsilon$. Let $\phi : M \mapsto [0,1]$ be a continuous function such that $\phi|_K = 1$ and $\phi|_{M\setminus V} = 0$. Then

$$\mu_n(K) \leq \int \phi d\mu_n \quad \forall\, n \geq 1.$$

From the continuity of $\phi$, and the convergence in the weak* topology of $\mu_n$ to $\mu$, we obtain

$$\int \phi d\mu_n \to \int \phi d\mu = \int_{\phi\neq 0} \phi\, d\mu \leq \int_V \phi\, d\mu \leq \int_V 1\, d\mu =$$

$$= \mu(V) = \mu(K) + \mu(V \setminus K) < \mu(K) + \epsilon.$$

Luego

$$\mu_n(K) \leq \int \phi\, d\mu_n < \mu(K) + \epsilon \quad \forall\, n \geq 1.$$

Now, taking lim sup we obtain

$$\limsup_{n\to+\infty} \mu_n(K) \leq \mu(K) + \epsilon.$$

Since the above inequality holds for all $\epsilon > 0$, we conclude

$$\limsup_{n\to+\infty} \mu_n(K) \leq \mu(K),$$

as wanted.

(2) Let $K = M \setminus V$. We have

$$\mu_n(V) = 1 - \mu_n(K) \quad \forall\, n \geq 1.$$

Taking lim inf, we obtain

$$\liminf_{n\to+\infty} \mu_n(V) = 1 - \limsup_{n\to+\infty} \mu_n(K).$$

Since $K$ is a closed set in the compact metric space $M$, it is compact. Applying property (1), we conclude

$$\liminf_{n\to+\infty} \mu_n(V) = 1 - \limsup \mu_n(K) \geq 1 - \mu(K) = \mu(V),$$

as wanted.

(3) We consider the interior $\mathrm{int}(A)$ of $A$ and its closure $\overline{A}$. Each one of these sets differs from $A$ i the boundary of $A$, which has zero $\mu$-measure. Thus

$$\mu(\mathrm{int}(A)) = \mu(A) = \mu(\overline{A}).$$

For $\mu_n$ we have

$$\mu_n(\mathrm{int}(A)) \leq \mu_n(A) \leq \mu_n(\overline{A}).$$

Applying properties (1) and (2), we obtain

$$\mu(\mathrm{int}(A)) \leq \liminf_{n\to+\infty} \mu_n(\mathrm{int}(A)) \leq \liminf_{n\to+\infty} \mu_n(A) \leq \limsup_{n\to+\infty} \mu_n(A)$$

$$\leq \limsup_{n\to+\infty} \mu_n(\overline{A}) \leq \mu(\overline{A}).$$

Since $\mu(\text{int}(A)) = \mu(\overline{A}) = \mu(A)$, the last chain of inequalities is a chain of equalities. Therefore, $\limsup$ and $\liminf$ of $\mu_n(A)$ coincide and are equal to $\mu(A)$. We conclude

$$\lim_{n\to+\infty} \mu_n(A) = \mu(A),$$

as wanted. □

*Proof. of Proposition 3.9*:

Consider the following continuous real function $\phi : [0,1] \mapsto \mathbb{R}$:

$$\phi(u) = -u \log u \text{ if } 0 < u \leq 1, \quad \phi(0) = 0.$$

From Definition 3.1 we have

$$H(\mathcal{P}, \mu) = \sum_{P\in\mathcal{P}} \phi(\mu(P)), \quad H(\mathcal{P}, \mu_n) = \sum_{P\in\mathcal{P}} \phi(\mu_n(P))$$

Since $\mu(\partial\mathcal{P}) = 0$ we can apply part (3) of Lemma 3.10 to each piece $P \in \mathcal{P}$. Using also the continuity of the function $\phi$, we obtain

$$\lim_{n\to+\infty} \phi(\mu_n(P)) = \phi(\mu(P)) \quad \forall\ P \in \mathcal{P}.$$

Finally we sum the above equality for all the finite number of pieces $P$ of the partition $\mathcal{P}$, concluding that

$$\lim_{n\to+\infty} H(\mathcal{P}, \mu_n) = \lim_{n\to+\infty} \sum_{P\in\mathcal{P}} \phi(\mu_n(P)) = \sum_{P\in\mathcal{P}} \lim_{n\to+\infty} \phi(\mu_n(P)) =$$

$$= \sum_{P\in\mathcal{P}} \phi(\mu(P)) = H(\mathcal{P}, \mu),$$

as wanted. □

## 3.2 The metric entropy for expansive maps.

For expansive maps, Kolmogorov-Sinai Theorem (Theorem 3.12 and Corollary 3.13 in this section) states there exists partitions that reach the supremum in Definition 3.4. In other words, the metric entropy of $f$ may be computed as the entropy of $f$ with respect to a concrete partition.

**Definition 3.11.** A partition $\mathcal{P}$ of $M$, is a *generator* (in the future) for $f : M \to M$ if the $\sigma$-algebra generated by $\left\{ \bigvee_{j=0}^{k} f^{-j}\mathcal{P} \right\}_{k\geq 0}$ is the Borel $\sigma$-algebra.

Recall Definitions 3.3 and 3.4 of $h_\mu(f, \mathcal{P})$ and $h_\mu(f)$.

**Theorem 3.12. (Kolmogorov-Sinai)**

*If the (finite measurable) partition $\mathcal{P}$ of $M$ is a generator for $f : M \to M$, then for any $f$-invariant probability measure $\mu$:*

$$h_\mu(f) = h_\mu(f, \mathcal{P}).$$

*Proof.* See for instance [18], Theorem 3.2.18. $\square$

Recall Definition 2.1 of expansiveness in the future of a map.

**Corollary 3.13.** *If $f : M \mapsto M$ is expansive in the future with expansivity constant $\alpha$, and $\mathcal{P}$ is a partition with diam($\mathcal{P}$) $< \alpha$ then for any $f$-invariant probability measure $\mu$:*

$$h_\mu(f) = h(f, \mathcal{P}) = \lim_{n \to +\infty} \frac{H(\mathcal{P}_f^n, \mu)}{n}.$$

*Proof.* Applying Theorem 3.12 it is enough to prove that $\mathcal{P}$ is a generator for $f$.

Let $k \geq 0$ and for each point $x \in M$ consider the piece $A_k(x)$ of the partition $\bigvee_{i=0}^{k} f^{-i}\mathcal{P}$ that contains $x$. We denote by $B_\delta(x)$ the ball centered at $x$ with radius $\delta < 0$. We will first prove the following statement:

**Assertion A.** *For all $0 < \delta < \alpha$ there exists $k = k(\delta)$ such that*

$$A_k(x) \subset B_\delta(x) \quad \forall\, x \in M. \tag{3}$$

Suppose for a contradiction that there is $\delta \in (0, \alpha)$ such that for all $k \geq 0$ there exist points $x_k, y_k \in M$ satisfying

$$y_k \in A_k(x_k) \setminus B_\delta(x_k) \ \ \forall\, k \geq 0.$$

Since $M$ is compact, there are subsequences $\{x_{k_j}\}_j$ and $\{y_{k_j}\}_j$ convergent to the points $x$ and $y$ respectively. On the one hand, as $y_k \notin B_\delta(x_k)$ we have

$$\operatorname{dist}(x, y) = \lim_{j \to +\infty} \operatorname{dist}(x_{k_j}, y_{k_j}) \geq \delta. \tag{4}$$

On the other hand, as $y_k \in A_k(x_k) \in \bigvee_{i=0}^{k} f^{-i}\mathcal{P}$ and diam($\mathcal{P}$) $< \alpha$, we obtain

$$\operatorname{dist}(f^i(x_{k_j}), f^i(y_{k_j})) < \alpha \ \ \forall\, 0 \leq i \leq k_j.$$

Taking the limit in the inequality above when $j \to +\infty$ with $i$ fixed, we deduce

$$\operatorname{dist}(f^i(x), f^i(y)) \leq \alpha \ \ \forall\, i \geq 0.$$

Due to the expansiveness in the future of $f$, the inequality above implies $x = y$ contradicting inequality (4), and ending the proof of Assertion A.

Any open set $V \subset M$ can be written as the union of open balls $B_{\delta(x)}(x) \subset V$ with $\delta(x) < \alpha$. Using Assertion A, each point $x$ of $V$ is inside a piece $A_{k(x)}(x) \subset B_\delta(x)$ for some $k(x) \geq 0$. Therefore $V$ is the union of a family of pieces

$$A_{k(x)}(x) \in \bigcup_{k \geq 0} \left\{ \bigvee_{i=0}^{k} f^{-i}\mathcal{P} : \ k \geq 0 \right\}.$$

Since the family of all such pieces is countable (because they are the pieces of a countable union of finite partitions), we deduce that the $\sigma$-algebra generated by them includes all the open subsets of $M$. Thus, it includes the Borel $\sigma$-algebra. Conversely, all the pieces $A_k(x)$ are Borel measurable sets. Therefore, the $\sigma$-algebra generated by them is included in the Borel $\sigma$-algebra. We conclude that both $\sigma$-algebras coincide, as wanted. $\square$

# 4 Proof of Theorem 1.

To prove Theorem 1, we will fix some notation:

For the $C^1$ expanding map $f : M \mapsto M$ denote

$$\psi(x) := -\log|\det Df_x| < 0, \quad \forall\, x \in M. \tag{5}$$

Recall that $\psi : M \to \mathbb{R}$ is the potential in the statement of Theorem 1.

In the space $\mathcal{M}$ of Borel probability measures on the manifold $M$, we fix the following weak* metric:

$$\text{dist}^*(\mu, \nu) := \sum_{i=0}^{+\infty} \frac{1}{2^i} \left| \int \phi_i \, d\mu - \int \phi_i \, d\nu \right|, \tag{6}$$

where $\phi_0 := \psi$ and $\{\phi_i\}_{i\geq 1}$ is a countable family of continuous functions that is dense in the space $C^0(M, [0,1])$.

**Lemma 4.1.** *For any $\mu \in \mathcal{M}$ and any $\epsilon > 0$ the ball $\mathcal{B} := \{\nu \in \mathcal{M} : \ dist^*(\mu, \nu) < \epsilon\}$ is convex.*

*Proof.* Let $\nu_1$, $\nu_2 \in \mathcal{B}$. We will prove that if $\lambda_1, \lambda_2 \in [0,1]$ are such that $\lambda_1 + \lambda_2 = 1$, then $\lambda_1\nu_1 + \lambda_2\nu_2 \in \mathcal{B}$.

Using the triangle inequality, we have

$$\text{dist}(\lambda_1\nu_1 + \lambda_2\nu_2, \mu) = \sum_{i=0}^{+\infty} \frac{1}{2^i} \left| \int \phi_i d\mu - \int \phi_i d(\lambda_1\nu_1 + \lambda_2\nu_2) \right|$$

$$\leq \sum_{i=0}^{+\infty} \frac{1}{2^i} \left| \lambda_1 \int \phi_i d\mu - \lambda_1 \int \phi_i d\nu_1 \right| + \sum_{i=0}^{+\infty} \frac{1}{2^i} \left| \lambda_2 \int \phi_i d\mu - \lambda_2 \int \phi_i d\nu_2 \right| =$$

$$= \lambda_1 \sum_{i=0}^{+\infty} \frac{1}{2^i} \left| \int \phi_i d\mu - \phi_i d\nu_1 \right| + \lambda_2 \sum_{i=0}^{+\infty} \frac{1}{2^i} \left| \int \phi_i d\mu - \phi_i d\nu_2 \right| <$$

$$(\lambda_1 + \lambda_2)\epsilon = \epsilon,$$

as wanted. $\square$

We state and prove now a series of lemmas that we will use in the proof of Theorem 1.

Recall Definition 3.1 of the entropy $H(\mathcal{P}, \mu)$ of a partition with respect to a probability measure $\mu$, and the equality (1) defining the product partition $\mathcal{P}_f^n$.

**Lemma 4.2.** *Let $\mu$ be an $f$-invariant probability measure. Let $\mathcal{P}$ be any finite partition of the manifold $M$ into measurable sets. If $\mu(\partial\mathcal{P}) = 0$, then, for all $\epsilon > 0$, and for all $q \geq 1$ there exists $\epsilon^* > 0$ such that*

$$\Big|\frac{H(\mathcal{P}_f^q, \rho)}{q} - \frac{H(\mathcal{P}_f^q, \mu)}{q}\Big| < \epsilon \tag{7}$$

*for any probability measure $\rho$ such that dist$(\rho, \mu) < \epsilon^*$.*

*Proof.* Assume by contradiction that for all $\epsilon^* > 0$ there exists a probability measure $\rho$, whose distance to $\mu$ is smaller than $\epsilon^*$, and that does not satisfy inequality (7). Thus, in particular for $\epsilon^* = 1/m$ where $m \in \mathbb{N}$, there exists $\rho_m$ such that

$$\text{dist}(\rho_m, \mu) < \frac{1}{m},$$

$$\Big|\frac{H(\mathcal{P}_f^q, \rho_m)}{q} - \frac{H(\mathcal{P}_f^q, \mu)}{q}\Big| \geq \epsilon \ \ \forall\ m \geq 1. \tag{8}$$

We have $\lim_{m\to+\infty} \rho_m = \mu$ in the weak$^*$ topology and $\mu(\partial\mathcal{P}) = 0$. Note that $\mu(\partial\mathcal{P}_f^q) = 0$ because $\mu$ is $f$-invariant. So, applying part (iii) of Lemma 3.10, $\lim_{m\to+\infty} \rho_m(Y) = \mu(Y)$ for any piece $Y \in \mathcal{P}_f^q$. And, from Proposition 3.9, for fixed $q \geq 1$, we have:

$$\lim_{m\to+\infty} \frac{H(\mathcal{P}_f^q, \rho_m)}{q} = \frac{H(\mathcal{P}_f^q, \mu)}{q},$$

contradicting inequality (8). □

Recall Definition 2.1 of expansiveness, and Proposition 2.2. Let $\alpha > 0$ be an expansivity constant for $f$. From Corollary 3.13 of Kolmogorov-Sinai Theorem for expansive maps, for any $f$-invariant probability measure $\mu$ we have:

$$h_\mu(f) = \lim_{q\to+\infty} \frac{H(\mathcal{P}_f^q, \mu)}{q} \quad \text{if} \quad \text{diam}(\mathcal{P}) < \alpha. \tag{9}$$

**Lemma 4.3.** *Let $f : M \mapsto M$ be a $C^1$ expanding map on the compact manifold $M$. Let $\alpha > 0$ be an expansivity constant for $f$.*

*For all $0 < \delta < \alpha$, for all $\epsilon > 0$, and for any $f$-invariant measure $\mu$, there exists a finite partition $\mathcal{P}$ of $M$, a real number $\epsilon^* > 0$, and a natural number $n_0 \geq 1$, such that:*

(i) $\text{diam}(\mathcal{P}) < \delta < \alpha$,

(ii) $\mu(\partial\mathcal{P}) = 0$,

(iii) *For any sequence of non necessarily invariant probabilities $\nu_n$, if $\mu_n := \frac{1}{n}\sum_{j=0}^{n-1}(f^j)^*\nu_n$, and if* $\text{dist}(\mu_n, \mu) < \epsilon^* \ \ \forall\ n \geq 1$, *then*

$$\frac{1}{n} H(\mathcal{P}_f^n, \nu_n) \leq h_\mu(f) + \epsilon \quad \forall\ n \geq n_0.$$

*Proof.* Recall Equality (1) defining the product partition $\mathcal{P}_f^n$ and Definition 3.4 of the metric entropy $h_\mu(f)$ of $f$. For simplicity along this proof, since we will not change the map $f$, we will denote $h_\mu$ instead of $h_\mu(f)$ and $\mathcal{P}^n$ instead of $\mathcal{P}_f^n$.

Take any finite covering $\mathcal{U} = \{Y_1, \dots, Y_p\}$ of $M$ with open balls with radia smaller than $\delta/2$. Denote $\partial\mathcal{U} := \bigcup_{i=1}^p \partial Y_i$. Since the family of boundaries of the balls with radius $r > 0$ is non countable when changing $r$, but these boundaries can have positive $\mu$-measure only for at most a countable subfamily, the radius of each ball $Y_i \in \mathcal{U}$ can be chosen such that $\mu(\partial Y_i) = 0$. Thus $\mu(\partial\mathcal{U}) = 0$. Therefore, the partition $\mathcal{P} = \{X_i\}_{1\leq i\leq p}$ defined by $X_1 := Y_1 \in \mathcal{U}, \;\; X_{i+1} := Y_{i+1} \setminus (\cup_{j=1}^i X_i)$, satisfies the assertions (i) and (ii).

To end the proof, for any given $\epsilon > 0$ let us find $\epsilon^* > 0$ and $n_0 \geq 1$ such that assertion (iii) holds.

Let us fix two integer numbers $q \geq 1$ and $n \geq q$. Write $n = Nq + j$ where $N, j$ are integer numbers such that $0 \leq j \leq q - 1$ Fix a (non necessarily invariant) probability $\nu$. Applying Propositions 3.6 and 3.7, we obtain:

$$H(\mathcal{P}^n, \nu) = H(\mathcal{P}^{Nq+j}, \nu) \leq$$
$$H(\vee_{i=0}^{j-1} f^{-(Nq+i)}\mathcal{P}, \nu) + H(\vee_{i=0}^{N-1} f^{-iq}\mathcal{P}^q, \nu) \leq$$
$$\sum_{i=0}^{j-1} H(f^{-(Nq+i)}\mathcal{P}, \nu) + \sum_{i=0}^{N-1} H(f^{-iq}\mathcal{P}^q, \nu) = \sum_{i=0}^{j-1} H(\mathcal{P}, (f^{Nq+i})^*\nu) + \sum_{i=0}^{N-1} H(\mathcal{P}^q, (f^{iq})^*\nu).$$

From the above inequality, using Proposition 3.5, and recalling that $j \leq q - 1 < q$, we obtain

$$H(\mathcal{P}^n, \nu) \leq q \log p + \sum_{i=0}^{N-1} H(\mathcal{P}^q, (f^{iq})^*\nu) \;\; \forall\, q \geq 1, \quad n \geq q,$$

where $p$ is the number of pieces of the partition $\mathcal{P}$.

The inequality above holds also for $f^{-l}\mathcal{P}$ instead of $\mathcal{P}$, for any $l \geq 0$, because it holds for any partition with exactly $p$ pieces. Thus:

$$H(f^{-l}\mathcal{P}^n, \nu) \leq q \log p + \sum_{i=0}^{N-1} H(f^{-l}\mathcal{P}^q, (f^{iq})^*\nu) =$$

$$q \log p + \sum_{i=0}^{N-1} H(\mathcal{P}^q, (f^{iq+l})^*\nu).$$

Adding the above inequalities for $0 \leq l \leq q - 1$, we obtain:

$$\sum_{l=0}^{q-1} H(f^{-l}\mathcal{P}^n, \nu) \leq q^2 \log p + \sum_{l=0}^{q-1} \sum_{i=0}^{N-1} H(\mathcal{P}^q, (f^{iq+l})^*\nu)$$

Therefore, on the one hand we have:

$$\sum_{l=0}^{q-1} H(f^{-l}\mathcal{P}^n, \nu) \leq q^2 \log p + \sum_{i=0}^{Nq-1} H(\mathcal{P}^q, (f^i)^*\nu). \tag{10}$$

On the other hand, applying Proposition 3.6, for all $0 \leq l \leq q-1$ we have

$$H(\mathcal{P}^n, \nu) \leq H(\mathcal{P}^n \vee f^{-n}\mathcal{P} \vee f^{-(n+1)}\mathcal{P} \vee \ldots \vee f^{-(n+l-1)}\mathcal{P}, \nu) = H(\mathcal{P}^{n+l}, \nu) =$$

$$H((\vee_{i=0}^{l-1} f^{-i}\mathcal{P}) \vee (f^{-l}\mathcal{P}^n)).$$

Therefore,

$$H(\mathcal{P}^n, \nu) \leq H(\mathcal{P}^{n+l}, \nu) \leq \Big(\sum_{i=0}^{l-1} H(f^{-i}\mathcal{P}, \nu)\Big) + H(f^{-l}\mathcal{P}^n, \nu).$$

So,

$$H(\mathcal{P}^n, \nu) \leq q \log p + H(f^{-l}\mathcal{P}^n, \nu).$$

Adding the above inequalities for $0 \leq l \leq q-1$ and joining with the inequality (10), we obtain:

$$qH(\mathcal{P}^n, \nu) \leq q^2 \log p + \sum_{i=0}^{q-1} H(f^{-l}\mathcal{P}^n, \nu) \leq 2q^2 \log p + \sum_{i=0}^{Nq-1} H(\mathcal{P}^q, (f^i)^*\nu).$$

Recall that $n = Nq + j$ with $0 \leq j \leq q-1$. So $n-1 = Nq + j - 1 \geq Nq - 1$ and then

$$qH(\mathcal{P}^n, \nu) \leq 2q^2 \log p \ + \sum_{i=0}^{n-1} H(\mathcal{P}^q, (f^i)^*\nu)$$

Now we put $\nu = \nu_n$ and divide by $n$. Using that $\mu_n = (1/n)\sum_{j=0}^{n-1} (f^j)^*\nu_n$ and applying Proposition 3.8, we obtain

$$\frac{q\, H(\mathcal{P}^n, \nu_n)}{n} \leq \frac{2q^2 \log p}{n} + \frac{1}{n}\sum_{i=0}^{n-1} H(\mathcal{P}^q, (f^i)^*\nu_n) \leq \frac{2q^2 \log p}{n} + H(\mathcal{P}^q, \mu_n).$$

For any fixed $\epsilon > 0$ (and the natural number $q \geq 1$ still fixed), take $n \geq n(q) := \max\{q, 6\, q \log p/\epsilon\}$ in the inequality above. We deduce:

$$\frac{q}{n} H(\mathcal{P}^n, \nu_n) \leq \frac{q\epsilon}{3} + H(\mathcal{P}^q, \mu_n) \quad \forall\ n \geq n(q) \quad \forall\ q \geq 1,$$

from where we obtain

$$\frac{1}{n} H(\mathcal{P}^n, \nu_n) \leq \frac{\epsilon}{3} + \frac{H(\mathcal{P}^q, \mu_n)}{q} \qquad \forall\ n \geq n(q) \quad \forall\ q \geq 1. \tag{11}$$

The inequality above holds for for any fixed $q \geq 1$ and for any $n$ large enough, depending on $q$.

By hypothesis, $\mu$ is $f$-invariant. So, after Equality (9), there exists $q \geq 1$ such that

$$\frac{H(\mathcal{P}^q, \mu)}{q} \leq h_\mu + \frac{\epsilon}{3}. \tag{12}$$

Fix such a value of $q$. Since $\mu(\partial(\mathcal{P})) = 0$ due to the construction of $\mathcal{P}$ (depending on the given measure $\mu$), we can apply Lemma 4.2 to find $\epsilon^* > 0$ such that

$$\frac{H(\mathcal{P}^q, \rho)}{q} \leq \frac{H(\mathcal{P}^q, \mu)}{q} + \frac{\epsilon}{3} \text{ if } \operatorname{dist}(\rho, \mu) < \epsilon^*.$$

To prove assertion (iii) we assume $\operatorname{dist}(\mu_n, \mu) < \epsilon^*$ for all $n \geq 1$. We deduce

$$\frac{H(\mathcal{P}^q, \mu_n)}{q} \leq \frac{H(\mathcal{P}^q, \mu)}{q} + \frac{\epsilon}{3} \quad \forall\ n \geq 1.$$

Joining this latter assertion with inequalities (11) and (12) we obtain

$$\frac{1}{n} H(\mathcal{P}^n, \nu_n) \leq h_\mu + \epsilon \quad \forall\ n \geq n(q).$$

Thus, after denoting $n_0 := n(q)$, assertion (iii) is proved. $\square$

**Notation.** Recall Equality (5) defining the continuous real function $\psi : M \to \mathbb{R}$, which is the potential in the statement of Theorem 1. For any real number $r \geq 0$ construct

$$\mathcal{K}_r := \{\nu \in \mathcal{M}_f : \int \psi \, d\nu + h_\nu \geq -r\}. \tag{13}$$

(We note that, a priori, the set $\mathcal{K}_r$ of $f$-invariant probabilities may be empty.)

For any integer $n \geq 1$ and for all $x \in M$ recall the Definition 1.1 of the empirical probability $\sigma_n(x)$), and of the p$\omega$-limit set $p\omega(x)$ in the set $\mathcal{M}$ of Borel probabilities. We also recall the weak$^*$ metric $\operatorname{dist}^*$ in the space $\mathcal{M}$ of probability measures constructed by equality (6).

**Lemma 4.4.** *Let $f$ be a $C^1$ expanding map on $M$. Let $m$ be the Lebesgue measure on $M$. Fix $r > 0$ and let $\mathcal{K}_r$ be defined by Equality (13). Then, for all $0 < \epsilon < r/2$, and for all $\mu \in \mathcal{M}_f$ such that $\mu \notin \mathcal{K}_r$, there exists $n_0 \geq 1$ and $0 < \epsilon^* \leq \epsilon/3$ such that*

$$m(\{x \in M : \operatorname{dist}(\sigma_n(x), \mu) < \epsilon^*\}) < e^{n(\epsilon - r)} < e^{-nr/2} \qquad \forall\ n \geq n_0. \tag{14}$$

*Proof.* As in the proof of Lemma 4.3, for simplicity along this proof we write $\mathcal{P}^n$ instead of $\mathcal{P}^n_f$, and $h_\mu$ instead of $h_\mu(f)$.

From Proposition 2.2, $f$ is expansive in the future. Let $\alpha > 0$ be a expansivity constant for $f$. For the given value of $\epsilon > 0$, fix a uniform continuity modulus $0 < \delta < \alpha$ for $\epsilon/3$ of the function $\psi = -\log|\det(Df)|$. Namely

$$|\psi(x) - \psi(y)| < \epsilon/3 \text{ if } \operatorname{dist}(x, y) < \delta. \tag{15}$$

For such a value of $\delta$, for the given measure $\mu \in \mathcal{M}_f$, and for $\epsilon/3$ instead of $\epsilon$, apply Lemma 4.3 to construct the partition $\mathcal{P}$ in $M$, and the numbers $\epsilon^* > 0$ and $n_0 \geq 1$, such that assertions (i), (ii) and (iii) hold. In particular assertion (iii) states that for any sequence of probability measures $\nu_n$, if $\mu_n := \frac{1}{n}\sum_{j=0}^{n-1}(f^j)^*\nu_n$, satisfies $\operatorname{dist}(\mu_n, \mu) < \epsilon^* \ \ \forall\ n \geq 1$, then

$$\frac{1}{n} H(\mathcal{P}^n, \nu_n) \leq h_\mu + \frac{\epsilon}{3} \quad \forall\ n \geq n_0. \tag{16}$$

It is not restrictive to assume that

$$\epsilon^* \leq \epsilon/3.$$

Denote, for all $n \geq 1$:

$$C_n := \{x \in M : \ \operatorname{dist}(\sigma_n(x), \mu) < \epsilon^*\}. \tag{17}$$

To prove this Lemma we must prove that

$$m(C_n) \leq e^{n(\epsilon - r)} \ \ \forall \ \ n \geq n_0 \quad \text{(to be proved)} \tag{18}$$

Since $f$ is $C^1$ expanding, its derivative $Df_x$ is invertible for all $x \in M$. Thus, by the local inverse map theorem, $f$ is a local diffeomorphism. The compactness of $M$ implies that there exists a uniform value $\delta_1 > 0$ such that $f$ restricted to any ball of radius $\delta_1$ is a diffeomorphism onto its image. Therefore, if the diameter of the partition $\mathcal{P}$ is chosen small enough, the restricted map $f^n|_X : X \mapsto f^n(X)$ is a diffeomorphism for all $X \in \mathcal{P}^n$ and for all $n \geq 1$. Thus, recalling that $\psi = -\log|\det Df|$, we deduce the following equality for all $X \in \mathcal{P}^n$:

$$m(X \cap C_n) = \int_{f^n(X \cap C_n)} |\det Df^{-n}| \, dm = \int_{f^n(X \cap C_n)} e^{\sum_{j=0}^{n-1} \psi \circ f^j} \, dm.$$

Therefore

$$m(C_n) = \sum_{X \in \mathcal{P}^n} \int_{f^n(X \cap C_n)} e^{\sum_{j=0}^{n-1} \psi \circ f^j} \, dm. \tag{19}$$

Either $C_n = \emptyset$, and Assertion (18) becomes trivially proved, or the finite family of pieces $\{X \in \mathcal{P}^n : \ X \cap C_n \neq \emptyset\} = \{X_1, \ldots, X_N\}$ has $N = N(n) \geq 1$ pieces. In this latter case, choose a single point $y_k \in X_k \cap C_n$ for each $k = 1, \ldots, N$. Denote by $Y(n) = \{y_1, \ldots, y_N\}$ the collection of such points. Due to the construction of $\delta > 0$ according to Equation (15), and since the partition $\mathcal{P}$ has diameter smaller than $\delta$ (because it satifies (i) of Lemma 4.3), we deduce:

$$\sum_{j=0}^{n-1} \psi(f^j(y)) \leq \sum_{j=0}^{n-1} (\psi(f^j(y_k)) + \epsilon/3) \ \ \forall \ y, y_k \in X_k, \ \forall \ k = 1, \ldots, N.$$

Therefore, substituting in Equality (19),

$$m(C_n) \leq e^{n\epsilon/3} \sum_{k=1}^{N} e^{\sum_{j=0}^{n-1} \psi(f^j(y_k))} \, m(f^n(X_k \cap C_n)).$$

Thus

$$m(C_n) \leq e^{n\epsilon/3} \sum_{k=1}^{N} e^{\sum_{j=0}^{n-1} \psi(f^j(y_k))}.$$

Define

$$L := \sum_{k=1}^{N} e^{\sum_{j=0}^{n-1} \psi(f^j(y_k))}, \qquad \lambda_k := \frac{1}{L} \, e^{\sum_{j=0}^{n-1} \psi(f^j(y_k))} \in (0, 1). \tag{20}$$

Then,

$$\sum_{k=1}^{N} \lambda_k = 1$$

and

$$m(C_n) \le e^{(n\epsilon/3)+\log L}, \tag{21}$$

where

$$\log L = \left(\sum_{k=1}^{N} \lambda_k \sum_{j=0}^{n-1} \psi(f^j(y_k))\right) - \left(\sum_{k=1}^{N} \lambda_k \log \lambda_k\right). \tag{22}$$

(To prove the equality above, take log in the equality at right in (20), multiply by $\lambda_k$ and take the sum for $k = 1, \dots, N$. )

Define the probability measures

$$\nu_n := \sum_{k=1}^{N} \lambda_k \delta_{y_k}, \tag{23}$$

$$\mu_n := \frac{1}{n} \sum_{j=0}^{n-1} (f^j)^*(\nu_n) = \sum_{k=1}^{N} \lambda_k \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(y_k)} = \sum_{k=1}^{N} \lambda_k \sigma_n(y_k). \tag{24}$$

(To prove the above equality at right recall Definition 1.1 of the empirical probability measures $\sigma_n(y_k)$.)

Then,

$$\sum_{k=1}^{N} \lambda_k \sum_{j=0}^{n-1} \psi(f^j(y_k)) = n \int \psi \, d\mu_n. \tag{25}$$

Recall that for any piece $X_k \in \mathcal{P}^n$ such that $C_n \cap X_k \neq \emptyset$ we have chosen a single point $y_k \in C_n \cap X_k$. Then $\nu_n(X_k) = \lambda_k \delta_{y_k}(X_k) = \lambda_k$, and we deduce that

$$-\sum_{k=1}^{N} \lambda_k \log \lambda_k = H(\mathcal{P}^n, \nu_n). \tag{26}$$

Therefore, combining Equations (21), (22), (25) and (26), we obtain

$$m(C_n) \le \exp\Big(\frac{n\epsilon}{3} + \log L\Big) = \exp\Big(n\Big(\frac{\epsilon}{3} + \int \psi \, d\mu_n + \frac{H(\mathcal{P}^n, \nu_n)}{n}\Big)\Big). \tag{27}$$

Now, we assert that

$$\operatorname{dist}(\mu_n, \mu) < \epsilon^* \le \frac{\epsilon}{3} \quad \forall \, n \ge 1. \tag{28}$$

In fact, by construction $y_k \in C_n$ for all $k = 1, \dots, N$, Thus, from Equality (17), we have

$$\operatorname{dist}(\sigma_n(y_k), \mu) < \epsilon^*.$$

Recalling Lemma 4.1, the ball in $\mathcal{M}$ of center $\mu$ and radius $\epsilon^*$ is convex. Thus, any convex combination of the measures $\sigma_n(y_k)$ belongs to that ball. From Equality (24) at right, $\mu_n$

is a convex combination of the measures $\sigma_n(y_k)$. We deduce that $\mu_n$ belongs to that ball. Hence, inequality (28) is proved. So equation (16) holds.

Combining Equations (16) and (27), we deduce that

$$m(C_n) \leq \exp\Big(n\Big(\frac{2\cdot\epsilon}{3} + \int \psi \, d\mu_n + h_\mu\Big)\Big) \;\; \forall \; n \geq n_0.$$

Besides, from inequality (28) and the construction of the weak*-metric dist* in $\mathcal{M}$ with $\phi_0 = \psi$ (recall Equality (6)), we deduce that

$$\Big|\int \psi \, d\mu_n - \int \psi \, d\mu\Big| < \frac{\epsilon}{3}, \quad \int \psi \, d\mu_n < \int \psi \, d\mu + \frac{\epsilon}{3}.$$

Therefore, we obtain

$$m(C_n) \leq \exp\Big(n\Big(\epsilon + \int \psi \, d\mu + h_\mu\Big)\Big) \;\; \forall \; n \geq n_0. \tag{29}$$

Finally, by hypothesis $\mu \notin \mathcal{K}_r$. Thus, $\int \psi \, d\mu + h_\mu < -r$. Substituting this latter inequality in (29), we conclude (18), ending the proof. □

The following lemma is a well-known elementary result in Probability Theory. We will apply it in the particular case for which $M$ is a compact Riemannian manifold and $\mathcal{B}$ is the Borel $\sigma$-algebra of subsets of $M$.

**Lemma 4.5. (Borel-Cantelli)** *Let $\mu$ be a probability measure on a measurable space $(M, \mathcal{B})$. Let $\{C_n\}_{n\geq 1}$ be a sequence of measurable subsets $C_n \subset M$ such that*

$$\sum_{n=1}^{+\infty} \mu(C_n) < +\infty.$$

*Then*

$$\mu\left(\bigcap_{N\geq 1} \bigcup_{n\geq N} C_n\right) = 0.$$

*Proof.* The sequence $\left\{\bigcup_{n\geq N} C_n\right\}_{N\geq 1}$ is (not necessarily strictly) decreasing with $N$. Then

$$\mu\left(\bigcap_{N\geq 1} \bigcup_{n\geq N} C_n\right) = \lim_{N\to+\infty} \mu\left(\bigcup_{n\geq N} C_n\right) \leq \lim_{N\to+\infty} \sum_{n=N}^{+\infty} \mu(C_n).$$

Finally, $\lim_{N\to+\infty} \sum_{n=N}^{+\infty} \mu(C_n) = 0$ because $\sum_{n=N}^{+\infty} \mu(C_n)$ is the tail of the convergent series $\sum_{n=1}^{+\infty} \mu(C_n)$. □

## 4.1 End of the proof of Theorem 1

*Proof.* We will prove that any pseudo-phisical measure $\mu$ satisfies Pesin Entropy Formula, namely, for the $C^1$- expanding map $f$, according to Proposition 1.12:

$$h_\mu(f) + \int \psi \, d\mu = 0 \qquad \text{(to be proved)},$$

where

$$\psi := -\log |\det Df|.$$

For any $r > 0$ consider the compact set $\mathcal{K}_r \subset \mathcal{M}$ defined by Equality (13). Since $\{\mathcal{K}_r\}_r$ is decreasing when decreasing $r$, we have

$$\mathcal{K}_0 = \bigcap_{r>0} \mathcal{K}_r, \quad \text{where}$$

$$\mathcal{K}_0 := \Big\{ \mu \in \mathcal{M}_f : \quad \int \psi \, d\mu + h_\mu(f) \geq 0 \Big\}.$$

By Margulis-Ruelle's inequality (see Theorem 1.7) and Corollary 1.11 applied to $C^1$ expanding maps, we have

$$h_\mu(f) \leq \int \log |\det Df| \, d\mu = - \int \psi \, d\mu \;\; \forall \; \mu \in \mathcal{M}_f. \tag{30}$$

Therefore, the (a-priori maybe empty) set $\mathcal{K}_0$ is composed by all the invariant measures $\mu$ such that

$$\int \psi \, d\mu + h_\mu(f) = 0,$$

or, in other words, $\mathcal{K}_0$ is the set of invariant measures $\mu$ that satisfy Pesin Entropy Formula. So, to prove that any pseudo-physical measure $\mu$ satisfies Pesin Entropy Formula, we must prove that $\mu \in \mathcal{K}_r$ for all $r > 0$.

Assume by contradiction that there exists $r > 0$ such that the pseudo-physical measure $\mu$ does not belong to $\mathcal{K}_r$. From Lemma 4.4, there exists $n_0 \geq 1$ and $\epsilon^* > 0$ such that,

$$m\big(\{x \in M : \text{dist}^*(\sigma_n(x), \mu) < \epsilon^*)\} \leq e^{-nr/2} \quad \forall \; n \geq n_0, \tag{31}$$

where $m$ denotes the Lebesgue measure.

From Definition 1.4 of pseudo-physical measure, for any $\epsilon^* > 0$ the set

$$A = \{x \in M : \quad \text{dist}(p\omega(x), \mu) < \epsilon^*\}$$

has positive Lebesgue measure: $m(A) > 0$. For each $n \geq 1$, denote

$$C_n := \{x \in M : \quad \text{dist}(\sigma_n(x), \mu) < \epsilon^*\}.$$

Apply Definition 1.1 of the set $p\omega(x)$ in $\mathcal{M}$ composed by the weak*-limits of all the convergent subsequences of $\{\sigma_n(x)\}$. Therefore,

$$A \subset \bigcap_{N \geq 1} \bigcup_{n \geq N} C_n,$$

So, we deduce the following inequality:

$$m\Big(\bigcap_{N\geq 1}\bigcup_{n\geq N} C_n\Big) \geq m(A) > 0. \tag{32}$$

But inequality (31) implies that $\sum_{n=1}^{\infty} m(C_n) < +\infty$; hence, applying Borel-Cantelli Lemma (see Lemma 4.5), it follows that

$$m\Big(\bigcap_{N\geq 1}\bigcup_{n\geq N} C_n\Big) = 0,$$

contradicting inequality (32). □

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