On the Lebesgue measure of the attractor for piecewise continuous maps on compact manifolds.

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Este documento incluye:

1) Demostración inspirada en la de Pierre (con modificaciones) de condiciones suficientes para que el atractor tenga medida nula, en cualquier dimensión.

No incluye: 2) Ejemplo de atractor con medida positiva en dimensión 1, porque lo que había pensado estaba mal. Volví al principio, sigo sin saber si existe o no un tal ejemplo.

Comment about what the word "manifold" implicitly means (respuesta a una cuestión que planteó Pierre sobre cambiar la distancia en la variedad X):
X is a differentiable compact manifold (with or without boundary) of finite dimension, provided of a Riemannian metric i.e. a inner product in the tangent bundle, which defines:
• the distance dist between points in the manifold: for instance if $X \subset \mathbb{R}^d$ is a compact manifold with dimension d (or more generally if locally, the manifold is identified with its tangent space which is $\mathbb{R}^d$), and if the inner product is denoted by $u \cdot v$, then $\text{dist}(x, y) = \|x - y\|$, where the norm $\|v\| = \sqrt{v \cdot v}$. It is well known that any other distance dist^* in a Riemannian manifold X (provided that dist^* endows the topology of the manifold and no other topology, namely: the open balls according to the new distance dist^* are open sets in the manifold, and any open set in the manifold is the countable union of -not necessarily pairwise disjoint - balls according to the new distance dist^*), then the new distance dist^* is equivalent to the distance dist which is endowed by the Riemannian metric. La equivalencia entre distancias significa que todas las conclusiones topológicas con una se obtienen también con la otra: conexión, fronteras, clausuras, etc. - y además que existen constantes positivas k_1 y k_2 tales que $k_1 \cdot \text{dist} \leq \text{dist}^* \leq k_2 \cdot \text{dist}$. De esto último se desprende también (dicho groseramente) que las desigualdades que valgan con una distancia, valen también automáticamente - a menos de la introducción de un factor constante - para la otra distancia)
• the Lebesgue measure l for the Lebesgue-measurable sets of X: The Lebesgue measure of a set is defined by integrating a the volume form (i.e. the determinant of the local charts) which is obtained from the Riemannian structure of the manifold. Esto significa cubrir la

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variedad de entornos chiquitos, cada uno difeomorfo por una carta local a una bola o cubito de $\mathbb{R}^d$ (espacio tangente), y de manera que en la intersección de dos o más de esos entornos, los cambios de coordenadas sean ortogonales - la ortoganalidad no se puede definir si uno no tiene previamente el producto interno, es decir la métrica Riemanniana.

Es una definición absolutamente rígida que cuando se habla de "manifold", la distancia y la medida de Lebesgue no son cualquier cosa, sino las respectivas que provienen de una métrica Riemanniana. Esto es debido a lo siguiente: por un lado, si uno quisiera cambiar la distancia por otra que respete la misma topología, las conclusiones topológicas serían las mismas, y las desigualdades que cumpla la distancia nueva, son a menos de cambio de las constantes, las mismas que las que cumplía la distancia vieja que proviene de la métrica Riemanniana. Entonces, dicho groseramente, es lo mismo usar una que otra distancia, y usamos solo la mejor representante de todas las distancias, que es la que proviene de la misma métrica Riemanniana que la medida de Lebesgue. Por otro lado, si uno quisiera cambiar la distancia por otra que no respete la misma topología, entonces X no es una variedad, por convención. Será un espacio métrico, pero este espacio métrico NO tiene estructura de variedad, si la distancia no respeta la misma topología que le da estructura de variedad a X. De todas formas, si uno usara una tal distancia, sucederían cosas espantosas como que no todos los abiertos con la nueva topología del espacio métrico X son Lebesgue medibles con la vieja topología intrínseca de X que la da la estructura de variedad. Entonces no todos los compactos son medibles. Como nosotros no ponemos ninguna restricción a la elección de las piezas abiertas que son piezas de contractividad, si tomamos un piecewise contracting con una distancia en X que induce una topología en X que no es la de estructura de variedad (suponiendo que X tuviera por separado, una estructura de variedad) el atractor podría no ser Lebesgue-medible. Entonces no podríamos demostrar que $l(\Lambda) = 0$ en general, pues Λ podría ser no medible. Además no convendría llamar a X variedad, pues esto iría en contra de la definición de variedad universalmente aceptada.

NOTATION: All along this document I use the following notation:
• $f : X \mapsto X$ is a piecewise contracting map according to our definition in the paper.
• $\{X_i\}_{1 \leq i \leq N}$ is the set of open contracting pieces of f in X. Recall that $\bigcup_{i=1}^{N} X_i$ is dense in X.
• $\lambda \in [0, 1)$ is a contracting coefficient of f in any of its open pieces X_i. Namely

$$\forall 1 \leq i \leq N, \ \forall \ (x, y) \in X_i : \quad \text{dist}(f(x), f(y)) \leq \lambda \text{dist}(x, y).$$

• $\Delta \subset X$, which I call "discontinuity set", even if not all points of Δ were discontinuity points of f, is defined by

$$\Delta = \bigcup_{i=1} \partial X_i = X \setminus (\cup_{i=1}^{N} X_i).$$

• Λ is the atractor of f, according to our definition, which takes into account only the values of $f|_{X \setminus \Delta}$. Precisely

$$\Lambda = \bigcap_{n=1}^{+\infty} \Lambda_n,$$

where Λ_n is the union of atoms of generation n, and can be inductively defined by

$$\Lambda_1 = \bigcup_{i=1}^{N} \overline{f(X_i)}, \qquad \Lambda_{n+1} = \bigcup_{i=1}^{N} \overline{f(\Lambda_n \cap X_i)}. \tag{1}$$

• l is the Lebesgue probability measure on the COMPACT manifold X: Namely, l is defined from a volume form on X, after a rescaling such that $l(X) = 1$.
• For each $1 \leq i \leq N$ the map $f_i : \overline{X_i} \mapsto X$ is the continuous and contracting map, which I call continuous extension to $\overline{X_i}$, of $f|_{X_i}$, defined as follows:

$$f_i(x) := f(x) \text{ if } x \in X_i$$

$$\text{If } x \in \partial X_i : \quad f_i(x) := \lim_{k \to +\infty} f(x_k) \;\; \forall \; \{x_k\}_{k \in \mathbb{N}} \text{ such that } \lim_{k \to +\infty} x_k = x. \tag{2}$$

Due to the Lipschitz continuity of $f|_{X_i}$, it is standard to prove that $f_i(x)$, defined by Equality (2), does not depend on the chosen sequence $\{x_k\}_{k \in \mathbb{N}}$ that converges to x. Also due to the Lipschitz continuity of $f|_{X_i}$ it is immediate to deduce that $f_i : \overline{X_i} \mapsto X$ is Lipschitz continuous with the same Lipszhitz constant. Since this constant is $\lambda \in [0, 1)$, then f_i is a contraction with contracting coefficient equal to λ. Namely:

$$\text{dist}(f_i(x), f_i(y)) \leq \lambda \text{dist}(x, y) \;\; \forall \; (x, y) \in \overline{X}_i. \tag{3}$$

1 Demostración inspirada en la de Pierre (con modificaciones) de condición suficiente para que $l(\Lambda) = 0$

Theorem 1.1 *If $l(\Delta) = 0$ then $l(\Lambda) = 0$*

Corollary 1.2 *In dimension one, if Δ is finite or countably infinite, then Λ is totally disconnected.*

The proof of this corollary is immediate after Theorem 1.1: If Δ is countable, then $l(\Delta) = 0$ and if so $l(\Lambda) = 0$ so Λ can not contain open sets. In dimension one, any set with empty interior is totally disconnected.

Before proving Theorem 1.1 let us state the three lemmas that we use in the proof. [1]

The following lemma is indeed a reformulation of a well known result of Measure Theory on Riemannian Manifolds:

[1] Si llegaran a estar bien estos lemas y sus demostraciones, y si estuviera bien la prueba del Teorema 1.1, y si fuéramos a incluir este Teorema en el paper, opino que no hay que incluir las demostraciones de los lemas 1.3 y 1.4, sino solo sus enunciados, como afirmaciones, no como lemas, y diciendo que son “standard ”to prove. Quizás convenga incluir solo la demostración del lema 1.5, pues es el único lugar donde se usa la hipótesis $l(\Delta) = 0$ del Teorema 1.1.

Lemma 1.3 *There exists a constant $K > 0$ (which depends only on the manifold X and on its Riemannian structure) such that, for any nonempty open set $Z \subset X$, for any Lipschitz function $g : \overline{Z} \mapsto X$ with Lipschitz constant $\rho > 0$, and for any Lebesgue-measurable set $S \subset X$, the following assertions hold:*

$$g(S \cap \overline{Z}) \textit{ is Lebesgue-measurable },$$

$$l(g(S \cap \overline{Z})) \leq K \rho^d \, l(S \cap \overline{Z}),$$

where $d = dim(X)$.

Lemma 1.4 *Let $f : X \mapsto X$ be piecewise contracting map (according to our definition) and denote by $\{X_i\}_{1 \leq i \leq N}$ the set of its open contracting pieces, by $\Delta = \bigcup_{i=1}^{N} \partial X_i$ its discontinuity set, and by $0 \leq \lambda < 1$ the contracting Lipschitz constant of $f|_{X_i}$ for all $1 \leq i \leq N$.*

Fix $n \geq 1$. Define

$$Y_n := \bigcap_{s=0}^{n} f^{-s}(X \setminus \Delta)$$

where $f^{-s}(B)$ is the preimage of B by f^s (and f^0 is the identity map). Define

$$g_n : Y_n \mapsto X \setminus \Delta, \quad \textit{by} \quad g_n(x) := f^n(x) \ \forall \ x \in Y_n.$$

Then, there exists a finite family $\{Z_j\}_{1 \leq j \leq N_n}$ of pairwise disjoints open sets Z_j such that $\bigcup_{j=1}^{N_n} Z_j = Y_n$ and $g_n|_{Z_j}$ is a contraction with Lipschitz contracting rate λ^n.

Lemma 1.5 *Fix $n \geq 1$ and consider f, Y_n, g_n as in Lemma 1.4. Denote by Λ_n the union of the atoms of generation n of f defined by equalities* (1). Define:

$$S_n := g_n(Y_n).$$

Then

$$S_n \subset X \setminus \Delta, \quad S_n \subset \Lambda_n.$$

If besides $l(\Delta) = 0$ then

$$l(\Lambda_n) = l(S_n).$$

We will prove Lemmas 1.3, 1.4 and 1.5 at the end. First, assuming true these three lemmas, we will prove Theorem 1.1:

Proof of Theorem 1.1

Proof:

For each fixed $n \geq 1$ take g_n, Y_n and $\{Z_j\}_{1 \leq j \leq N_n}$ according to Lemma 1.4. Denote by $g_{n,j} : \overline{Z}_j \mapsto Y_n$ to the continuous extension of $g_n|_{Z_j}$ to $\overline{Z_j}$. Since $g_n|_{Z_j}$ is a contraction with contracting rate λ^n, the extended mapping $g_{n,j}$ is also a contraction with contracting rate λ^n. Namely, $g_{n,j}$ is Lipschitz with constant $\rho = \lambda^n$. Applying Lemma 1.3 (using $g = g_{n,j}$ and $S = Z = Z_j$), we have:

$$g_{n,j}(Z_j) = g_n(Z_j) \text{ is Lebesgue-measurable}$$

and

$$l(g_n(Z_j)) \leq K\,\lambda^{nd}\,l(Z_j),$$

where $d = \dim(X)$. Thus

$$g_n(Y_n) = g_n(\cup_{j=1}^{N_n} g(Z_j)) = \bigcup_{j=1}^{N_n} g(Z_j) \text{ is Lebesgue-measurable}$$

and

$$l(g_n(Y_n)) \leq \sum_{j=1}^{N_n} l(g(Z_j)) \leq K\,\lambda^{nd} l(Z_j) = K\,\lambda^{nd} l(Y_n).$$

The last equality is due that $\{Z_j\}_{1 \leq j \leq N_n}$ is a family of pairwise disjoint open sets whose union is Y_n (Lemma 1.4).

We apply now Lemma 9, using the hypothesis $l(\Delta) = 0$ of Theorem 1.1:

$$l(\Lambda_n) = l(S_n) = l(g_n(Y_n)) \leq K\,\lambda^{nd} l(Y_n) \leq K\,\lambda^{nd} \quad \forall\; n \geq 1$$

Since $\Lambda = \bigcap_{n=1}^{+\infty} \Lambda_n$ we conclude

$$l(\Lambda) \leq K\,\lambda^{nd} \quad \forall\; n \geq 1,$$

which, taking into account that $0 \leq \lambda < 1$ implies $l(\Lambda) = 0$, as wanted. □

Proof of Lemma 1.3[2]

Proof: Denote

$$A := S \cap \overline{Z}.$$

The Lebesgue probability measure l is regular in the compact manifold X (see for instance [1, Teorema 2.3.1 c), page 35]). Thus, for any $\epsilon > 0$ there exists an open set $V \subset X$ such that

$$A \subset V, \quad l(V) \leq l(A) + \epsilon. \tag{4}$$

V is open in the compact d-dimensional manifold X , and the manifold is locally diffeomorphic to an open and bounded set of $\mathbb{R}^d$. Thus, V inherits the topological and measurable properties of the open and bounded sets of $\mathbb{R}^d$.

We call a set $\overline{B} \subset X$ a compact cube, it is the image by a local diffeomorphic chart of X of a compact cube in $\mathbb{R}^d$. So, for any open set $V \subset X$ there exists countably many (non

[2]Creo que la demostración de este lema, si estuviera bien, no sería necesario incluirla en el paper. Me parece que es un resultado bastante estándar (si es que no metí la pata en algo). No lo busqué mucho en la bibliografía, pero debería estar en algún lado. Es este enunciado: la imagen por una función lipschitz-continua con constante ρ de un conjunto Lebesgue medible, es medible, y su medida de Lebesgue es menor que una constante por ρ^d por la medida de Lebesgue del conjunto. La parte que es más complicadita, es la de demostrar la medibilidad de la imagen, porque en general, la imagen continua de un medible puede ser no medible. La hipótesis de lipschitz continuidad es esencial. Lo que es clásico y está en todos lados, es que la imagen por un mapa de clase C^1 de un medible es medible y que la medida de Lebesgue de la imagen del conjunto es menor o igual que el máximo del determinante de la parte lineal en valor absoluto por la medida del conjunto

necessarily pairwise disjoint) compact *d-dimensional cubes* $\{\overline{V_j}\}_{j\geq 1}$, all with diameter small enough and such that

$$V \subset \bigcup_{j=0}^{+\infty} \overline{V_j} \quad \text{but} \quad \sum_{j=0}^{+\infty} l(\overline{V_j}) \leq l(V) + \epsilon. \tag{5}$$

From the definition of the Lebesgue measure on X, there exists constants $K_1 > 0$ and $K_2 > 0$ (which depend only on the manifold X and on its Riemannian structure - recall that this structure defines the distance on X and the volume form from which the Lebesgue measure l is constructed), such that:

$$k_1 \operatorname{diam}^d(\overline{B}) \leq l(\overline{B}) \leq k_2 \operatorname{diam}^d(B) \quad \text{for any } \textbf{compact cube } \overline{B} \subset X. \tag{6}$$

(Respecto a las desigualdades de arriba ver nota al pie [3] .)

Denote

$$K_j := \overline{V_j} \cap \overline{Z}.$$

(The set $K_{i,j}$ is compact, so it is Lebesgue-measurable).

Due to the Lipschitz continuity of g, we have the following inequality for any two points $x, y \in K_j \subset \overline{Z}$:

$$\operatorname{dist}(g(x), g(y)) \leq \rho \operatorname{dist}(x, y) \leq \ \lambda \operatorname{diam}(\overline{V_j})$$

Therefore

$$\operatorname{diam}(g(K_j)) \leq \rho \operatorname{diam}(\overline{V_j}),$$

and so, there exists a compact cube $\overline{B_j}$ such that

$$g(K_j) \subset \overline{B_j}, \quad \operatorname{diam}(\overline{B_j}) \leq 2\, d \operatorname{diam}(g(K_j) \leq 2\, d\, \rho \operatorname{diam}(\overline{V_j}). \tag{7}$$

Note that K_j is compact and g is continuous, so $g(K_j)$ is compact, hence Lebesgue measurable. Also $\overline{B_j}$ and $\overline{V_j}$ are compact, hence Lebesgue measurable. Thus

$$l(g(K_j)) \leq l(\overline{B_j}), \quad \operatorname{diam}(\overline{B_j}) \leq 2\, d \operatorname{diam}(g(K_j) \leq 2\, d\, \rho \operatorname{diam}(\overline{V_j}). \tag{8}$$

Taking into account that $\overline{V_j}$ and $\overline{B_j}$ are compact cubes in the manifold X, we can apply inequalities (6). So, from inequality (8), we deduce:

$$l(g(K_j)) \leq l(\overline{B_j}) \leq k_2 \, [\operatorname{diam}(\overline{B_j})]^d \leq 2^d\, d^d\, k_2\, \lambda^d \, [\operatorname{diam}(\overline{V_j})]^d \leq 2^d\, d^d\, \frac{k_2}{k_1}\, \rho^d\, l(\overline{V_j}). \tag{9}$$

[3] Por ejemplo, si la variedad es la superficie de una esfera, los cubitos Q 2-dimensionales en la variedad, son pedazos compactos difeormorfos a cuadrados en el espacio tangente. Tomando coordenadas esféricas con los polos que no pertenezcan a ese cuadradito Q, el área del cuadradito sobre la superficie esférica $l(Q)$ **no** es igual a la longitud del lado del cuadradito elevada a la 2 (como sería si la superficie esférica fuera isométrica a un pedazo de $\mathbb{R}^2$). O sea difiere de ser proporcional al cuadrado de la diagonal (diámetro) de Q. Pero de todas formas, haciendo cuentas en coordenadas esféricas por ejemplo, uno podría probar que $l(Q)$ está acotada por arriba y por abajo por el cuadrado del diámetro de Q multiplicado por sendas constantes.

Since $A \subset V \cap \overline{Z}$, we have

$$g(A) \subset g\Big(V \cap \overline{Z}\Big) \subset g\Big(\big(\cup_{j=1}^{+\infty} \overline{V_j}\big) \bigcap \overline{Z}\Big) \subset \bigcup_{j=1}^{+\infty} g(\overline{V_j} \cap \overline{Z}) = \bigcup_{j=1}^{+\infty} g(K_j). \tag{10}$$

First, assume that $g(A)$ is Lebesgue-measurable [4]. Combining assertions (9) and (10), we obtain:

$$l(g(A)) \leq \sum_{j=1}^{+\infty} l(g(K_j) \leq 2^d\, d^d\, \frac{k_2}{k_1}\, \rho^d \sum_{j=1}^{+\infty} l(\overline{V_j}).$$

Now, applying inequalities (4) and (5), we deduce:

$$l(g(A)) \leq 2^d\, d^d\, \frac{k_2}{k_1}\, \rho^d \big(l(A) + 2\,\epsilon\big).$$

Since the latter inequality holds for all $\epsilon > 0$, we conclude

$$l(g(A)) \leq 2^d\, d^d\, \frac{k_2}{k_1}\, \rho^d\, l(A) = k\, \rho^d\, l(A), \tag{11}$$

if $g(A)$ were Lebesgue-measurable and

$$k = 2^d\, d^d\, k_2/k_1.$$

So, in particular, inequality (11) holds if A is a compact set (because the continuous image $g(A)$ of a compact set is compact, hence measurable).

Second, let us prove that $g(A)$ is Lebesgue-measurable for all Lebesgue-measurable set $A \subset \overline{Z}$ (recall that $A = S \cap \overline{Z}$ is measurable because, by hypothesis, S so is, and $\overline{Z}$ is compact, hence measurable).

Due to the assertions (4), (5), (8), (10) and (11), and recalling that $g(K_j)$ is measurable (because K_j is compact), we have already proved that:

$$g(A) \subset g\Big(\bigcup_{j=1}^{+\infty} K_j\Big) = \bigcup_{j=1}^{+\infty} g(K_j), \qquad \sum_{j=1}^{+\infty} l(K_j) \leq l(A) + 2\epsilon, \qquad l\Big(g(K_j)\Big) \leq k\, \rho^d\, l(K_j). \tag{12}$$

The set defined by

$$T_\epsilon := \bigcup_{j=1}^{+\infty} K_j \tag{13}$$

is measurable because it is the countable union of compact sets, and analogously, the set

$$g(T_\epsilon) = g\Big(\bigcup_{j=1}^{+\infty} K_j\Big) = \bigcup_{j=1}^{+\infty} g(K_j) \quad \text{is measurable.} \tag{14}$$

[4] El conjunto A es medible porque es la intersección de S medible con $\overline{Z}$ que es medible porque es compacto. Pero la imagen continua de un conjunto medible puede no ser medible. En este caso queremos demostrar que la imagen por una función Lipschitz continua de un medible es medible.

So, we have proved that for all $\epsilon > 0$, the set $g(A)$ is contained in the measurable set $g(T_\epsilon)$, the set T_ϵ contains A,

$$l(T_\epsilon) \leq l(A) + 2\epsilon$$

and

$$l(g(T_\epsilon)) \leq k\,\rho^d\,(l(A) + 2\epsilon). \tag{15}$$

Taking $\epsilon = 1/r$ for $r \in \mathbb{N}$, $r \geq 1$, we deduce:

$$A \subset T := \bigcap_{r=1}^{\infty} T_{1/r}, \quad T \text{ is measurable }, \qquad l(T) \leq l(A) + (2/r) \forall\ r \geq 1.$$

(Note that T is measurable because it is the countable intersection of measurable sets.)

$$g(A) \subset g(T), \quad T \text{ is measurable }, \qquad l(T) = l(A). \tag{16}$$

(Note that $A \subset T$ and $l(T) \leq l(A)$ implies $l(T) = l(A)$.) Since

$$g(A) = g(T) \setminus (g(T) \setminus g(A)), \tag{17}$$

to end the proof that $g(A)$ is measurable, it is enough to prove that $g(T)$ and $g(T) \setminus g(A)$ are measurable. (Recall that $B \setminus C = B \cap (M \setminus C)$, if C is measurable then its complement $M \setminus C$ also is, and that the intersection of measurable sets is measurable).

So, let us prove that $g(T)$ is measurable:

$$g(T) = g\Big(\bigcap_{r=1}^{+\infty} T_{1/r}\Big) = g\Big(\bigcap_{n=1}^{+\infty} \bigcap_{r=1}^{n} T_{1/r}\Big) = \bigcap_{n=1}^{+\infty} g\Big(\bigcap_{r=1}^{n} T_{1/r}\Big). \tag{18}$$

(Recall that, for a sequence $\{A_n\}_{n\geq 1}$ of sets in the domain of any map g the image $g(\bigcap_r A_n)$ contains the intersection $\bigcap_r g(A_n)$ of the images, but the equality may not hold if g is not one to one. Nevertheless if $A_{n+1} \subset A_n$ for all $n \geq 1$, even if g is not one to one, it is easy to check that $g(\bigcap_n A_n) = \bigcap_n g(A_n)$. So Equality (18) follows because the sequence of sets $A_n := \bigcap_{r=1}^n T_{1/r}$, to which we apply the map g, is decreasing with n.)

Using the equality at right of assertion (18), to prove that $g(T)$ is measurable it is enough to prove that $g(\bigcap_{r=1}^n T_{1/r}\Big)$ is measurable. In fact, $T_{1/r}$ is the countable union of the compact sets K_j that were chosen at the beginning for a fixed value of $\epsilon = 1/r$. So $\bigcap_{r=1}^2 T_{1/r}$ is the intersection of two countable unions of compact sets. So, it is also the countable union of compact sets. Repeating this argument by induction on n, we deduce that for any natural number $n \geq 1$, the set $\bigcap_{r=1}^n T_{1/r}$ is the countable union of compact sets. Since the image by any map g of a union of sets, is the union of the images, we deduce that $g\Big(\bigcap_{r=1}^n T_{1/r}\Big)$ is the countable union of the image by g of compact sets. But g is continuous, so the image by g of each compact set is compact. So $g\Big(\bigcap_{r=1}^n T_{1/r}\Big)$ is the countable union of compact sets, hence it is measurable. This ends the proof that $g(T)$ is measurable.

Finally it is left to prove that $g(T) \setminus g(A)$ is measurable. For any map g we have:

$$g(T) \setminus g(A) \subset g(T \setminus A)$$

But, we have already proved that $A \subset T$ and $l(T) = l(A)$. So

$$l(T \setminus A) = l(T) - l(A) = 0$$

No, let us put $T \setminus A$ in the role of A, and H_ϵ in the role of T_ϵ, and apply the left assertion in (12), and also assertions (13), (14) and (15). We deduce that for all $\epsilon > 0$ the image $g(T \setminus A)$ is contained in a measurable set $g(H_\epsilon)$ such that $l(g(H_\epsilon) \leq k\,\rho^d \big(l(T \setminus A) + 2\epsilon\big)$. But $l(T \setminus A) = 0$. So, for all $\epsilon > 0$ the set $g(T \setminus A)$ is contained in a measurable set $g(H_\epsilon)$ of Lebesgue measure smaller than $2\,k\,\rho^d\,\epsilon$. Taking $\epsilon = 1/r$, for all natural values of $r \geq 1$, and the countable intersection of $g(H_{1/r})$ we deduce that $g(T \setminus A)$ is contained in a measurable set of zero Lebesgue measure.

The Lebesgue measure is complete (see for instance [1, pag. 23 and Theorem 2.5.2, pag. 44]). Namely, any set contained in a Lebesgue measurable set with zero Lebesgue measure, is Lebesgue measurable (and has zero Lebesgue measure). This implies that that $g(T \setminus A)$ is measurable and has zero Lebesgue measure. So, $g(T) \setminus g(A) \subset g(T \setminus A)$ is also measurable, as wanted. $\square$

Proof of Lemma 1.4 [5]

Proof: For each word $\mathbf{w} = (i_0, i_1, i_2, \ldots, i_n) \in \{1, \ldots, N\}^{n+1}$ of indexes, with length n, define the open set

$$Z_{\mathbf{w}} := X_{i_0} \cap f^{-1}X_{i_2} \cap \ldots \cap f^{-n}X_{i_n}.$$

Select the subfamily of all those open nonempty sets and denote it by $\{Z_1, \ldots, Z_{N_n}\}$. By construction $\bigcup_{j=1}^{N_n} Z_j = Y_n$. Also, by construction, for any two points $x, y \in Z_j$ we have that $f^s(x)$ and $f^s(y)$ belong to the same open contracting piece X_{i_s} for all $0 \leq s \leq n$. So,

$$\text{dist}(g_n(x), g_n(y)) = \text{dist}(f^n(x), f^n(y)) \leq \lambda \text{dist}(f^{n-1}(x), f^{n-1}(y)) \leq$$

$$\leq \lambda^2 \text{dist}(f^{n-2}(x), f^{n-2}(y)) \leq \ldots \leq \lambda^n \text{dist}(x, y).$$

$\square$

Proof of Lemma 1.5 [6]

Proof: By definition, $Y_n \subset f^{-n}(X \setminus \Delta)$, $g_n = f^n|_{Y_n}$ and $S_n = g_n(Y_n)$. So $S_n = f^n(Y_n) \subset f^n(f^{-n}(X \setminus \Delta) \subset X \setminus \Delta$.

Now let us prove that $S_n \subset \Lambda_n$ and that, if $l(\Delta) = 0$ then $l(\Lambda_n) = l(S_n)$. Let us argue by induction on the natural value of $n \geq 1$.

[5] Opino que la demostración del lema 1.4, si estuviera bien, no sería conveniente incluirla en el paper. Da la impresión que su enunciado es casi obvio, y muy fácil de demostrar.

[6] Opino que la demostración del lema 1.5, si estuviera bien, habría que incluirla en el paper, pues es el único lugar donde se usa la hipótesis del Teorema 1.1. La demostración de este lema copia las ideas de Pierre.

If $n=1$, then

$$S_1 = f((X\setminus\Delta)\cap f^{-1}(X\setminus\Delta)) \subset \bigcup_{i=1}^{N} f(X_i) \subset \bigcup_{i=1}^{N} \overline{f(X_i)} = \Lambda_1.$$

Notice that, for any set S:

$$\overline{f(S\cap X_i)} \subset f_i(\overline{S}\cap\overline{X_i}), \tag{19}$$

where f_i is the continuous extension of $f|_{X_i}$ to $\overline{X_i}$. This latter assertion holds due to the following argument: On the one hand, by definition of closure of a set, $\overline{f(S\cap X_i)}$ is the minimum closed set that contains $f(S\cap X_i)$. On the other hand $f(\overline{S}\cap\overline{X_i})$ is compact because it is the continuous image of a compact set; hence it is closed.

Let us assume that $l(\Delta)=0$. Then:

$$\Lambda_1 = \bigcup_{i=1}^{N}\overline{f(X_i)} \subset \bigcup_{i=1}^{N} f_i(\overline{X_i}) = \Big(\bigcup_{i=1}^{N} f(X_i)\Big) \bigcup \Big(\bigcup_{i=1}^{N} f_i(\partial X_i)\Big).$$

Since $\partial X_i\subset\Delta$, we have $l(\partial X_i)$. Applying Lemma 1.3 (putting ∂X_i in the role of S and f_i in the role of g) we deduce that $l(f_i(\partial X_i))=0$. So

$$l(\Lambda_1) \leq l\Big(\bigcup_{i=1}^{N} f(X_i)\Big) = l(f(X\setminus\Delta)).$$

Now

$$f(X\setminus\Delta) = \Big(f(X\setminus\Delta)\cap(X\setminus\Delta)\Big)\bigcup\Big(f(X\setminus\Delta)\cap\Delta\Big).$$

Since $l(\Delta)=0$ and $f(Y_1) = f\Big((X\setminus\Delta)\cap f^{-1}(X\setminus\Delta)\Big) \subset f(X\setminus\Delta)\cap(X\setminus\Delta)$, we deduce:

$$l(\Lambda_1) \leq l(f(X\setminus\Delta)) = l(f(Y_1)) = l(g_1(Y_1)) = l(S_1).$$

But since $S_1\subset\Lambda_1$ we also have the opposite inequality. So $l(\Lambda_1)=l(S_1)$, as wanted. We have proved the initial step of the induction. Let us prove the general step. Assume the inductive hipothesis:

$$S_n\subset\Lambda_n, \quad l(\Lambda_n)=l(S_n)$$

for some natural number $n\geq 1$. Let us prove the last two assertions for $n+1$ in the place of n:

$$S_{n+1} = f^{n+1}(Y_{n+1}) = f(f^n(Y_{n+1}) \subset f(S_n) = f(S_n\cap(X\setminus\Delta)),$$

because $S_n\subset X\setminus\Delta$. Then

$$S_{n+1} \subset \bigcup_{i=1}^{N} f(S\cap X_i) \subset \bigcup_{i=1}^{N} f(\Lambda_n\cap X_i) \subset \bigcup_{i=1}^{N}\overline{f(\Lambda_n\cap X_i)} = \Lambda_{n+1}.$$

We have proved that $S_{n+1}\subset\Lambda_{n+1}$. Now, it is left to prove that $l(\Lambda_{n+1})=l(S_{n+1})$. In fact:

$$\Lambda_{n+1} \subset \bigcup_{i=1}^{N} f_i(\Lambda_n\cap\overline{X_i}) = \Big(\bigcup_{i=1}^{N} f(\Lambda_n\cap X_i)\Big)\bigcup\Big(\bigcup_{i=1}^{N} f_i(\Lambda_n\cap\partial X_i)\Big).$$

Since $\partial X_i \subset \Delta$ and $l(\Delta) = 0$ we deduce:

$$l(\Lambda_{n+1}) \leq l\Big(\bigcup_{i=1}^{N} f(\Lambda_n \cap X_i)\Big).$$

But

$$\bigcup_{i=1}^{N} f(\Lambda_n \cap X_i) = \Big(\bigcup_{i=1}^{N} f(S_n \cap X_i)\Big) \bigcup \Big(\bigcup_{i=1}^{N} f((\Lambda_n \setminus S_n) \cap X_i)\Big).$$

From the inductive hypothesis $l(\Lambda_n \setminus X_n) = 0$. So, applying Lemma 1.3 we have $l(f(\Lambda_n \setminus X_n)) = 0$. Thus, we deduce

$$l(\Lambda_{n+1}) \leq l\Big(\bigcup_{i=1}^{N} f(S_n \cap X_i)\Big) = l\Big(f\big(\cup_{i=1}^{N} S_n \cap X_i\big)\Big) = l(f(S_n)).$$

So, to end the proof that $l(\Lambda_{n+1}) = l(S_{n+1})$ it is enough to show that $l(f(S_n)) = l(S_{n+1})$. Indeed, from the definition of Y_n and of $S_n = f^n(Y_n)$ we have that $y \in f(S_n)$ if and only if $y = f(f^n(z)$, where $f^s(z) \in X \setminus \Delta$ for all $0 \leq s \leq n$. We discuss two cases: either $f^{n+1}(z) \in X \setminus \Delta$ or $f^{n+1}(z) \in \Delta$. The first case is satisfied if and only if $z \in Y_{n+1}$, i.e. $y = f^{n+1}(z) \in Y_{n+1}$. The second case is satisfied if and only if $y = f^{n+1}(z) \in \Delta$. So, we have proved that $f(S_n) = S_{n+1} \cup \Delta$. Since $l(\Delta) = 0$ we conclude that $l(f(S_n)) = l(S_{n+1})$, ending the proof of Lemma 1.5. $\square$

References

[1] W. Rudin: *Análisis Real y Complejo* (translation to Spanish of the book *Real and Complex Analysis*), ISBN 84-205-0651-6, Editorial Alhambra, Madrid, 1979,