Topological dynamics

Basic concepts

Jana Rodriguez Hertz

Universidad de la República - Uruguay

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dynamical system

$$f : X \to X$$

dynamical system

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orbit of $x \in X$:

dynamical system

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orbit of $x \in X$:

$$x,$$

dynamical system

$f : X \to X$

orbit of $x \in X$:

$$x, \quad f(x),$$

dynamical system

$f : X \to X$

orbit of $x \in X$:

$$x, \quad f(x), \quad f^2(x),$$

dynamical system

$f : X \to X$

orbit of $x \in X$:

$$x, \quad f(x), \quad f^2(x), \quad \ldots$$

dynamical system

$f : X \to X$

orbit of $x \in X$:

$$f^{-1}(x), \quad x, \quad f(x), \quad f^2(x), \quad \ldots$$

dynamical system

$f : X \to X$

orbit of $x \in X$:

$$f^{-2}(x), \quad f^{-1}(x), \quad x, \quad f(x), \quad f^2(x), \quad \ldots$$

dynamical system

$f : X \to X$

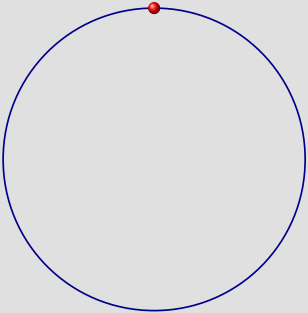
orbit of $x \in X$:

$$\ldots \quad f^{-2}(x), \quad f^{-1}(x), \quad x, \quad f(x), \quad f^2(x), \quad \ldots$$

periodic orbit

Definition (periodic orbit)

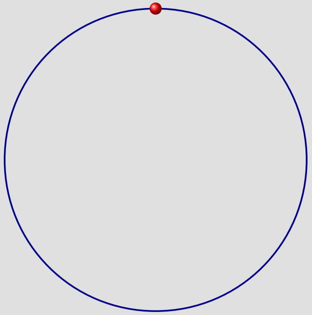
- x is a periodic point



periodic orbit

Definition (periodic orbit)

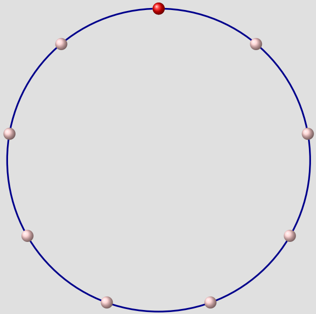
- x is a periodic point
- if the orbit of x is finite:



periodic orbit

Definition (periodic orbit)

- x is a periodic point
- if the orbit of x is finite: $\exists n$
- $x, f(x), \ldots, f^n(x) = n$



example 1

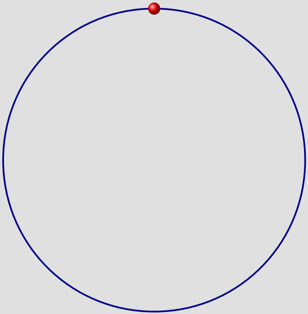
example 1

a dynamical system with all its orbits periodic

dense orbit

Definition (dense orbit)

- x has a dense orbit



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- x has a dense orbit
- if for each $\varepsilon > 0$ and $y \in X$

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- there is $n > 0$ such that
- $d(f^n(x), y) < \varepsilon$

example 2

example 2

a dynamical system with all its orbits dense

example 3

example 3

a dynamical system with periodic orbits and dense orbits

transitive and minimal dynamical systems

Definition (transitivity and minimality)

- a system with a dense orbit is transitive

transitive and minimal dynamical systems

Definition (transitivity and minimality)

- a system with a dense orbit is transitive
- a system with all its orbits dense is minimal

the circle

Definition (the circle)

$$\mathbb{S}^1 = \{(\cos 2\pi\theta, \sin 2\pi\theta) : \theta \in \mathbb{R}\}$$

the circle

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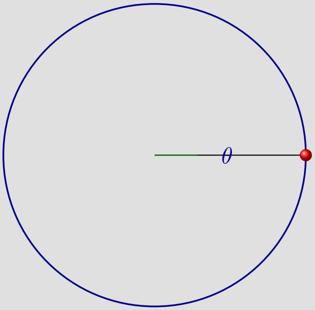
$$\mathbb{S}^1 = \{(\cos 2\pi\theta, \sin 2\pi\theta) : \theta \in \mathbb{R}\}$$



the circle

Definition (the circle)

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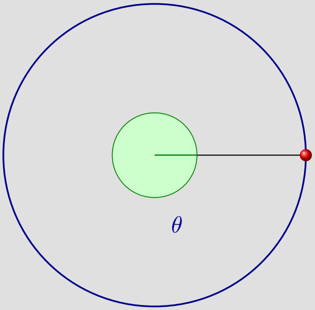

$$\theta \sim \theta + 1$$

the circle

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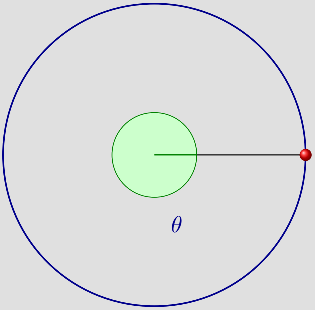

$$\theta \sim \theta + 1 \sim \theta + 2 \ldots$$

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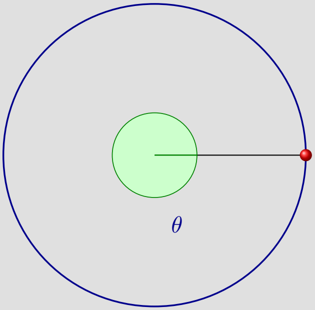
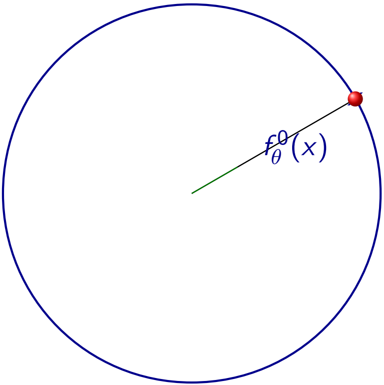


$$\theta \sim \theta + 1 \sim \theta + 2 \ldots$$

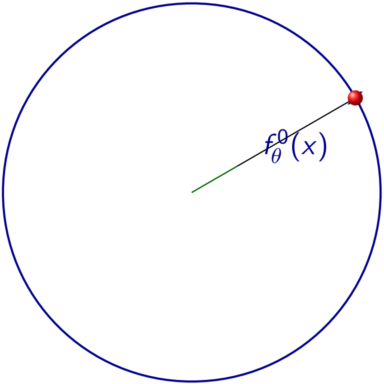
$$\mathbb{S}^1 \leftrightarrow \mathbb{R}/\mathbb{Z}$$

rotations of the circle



$f_\theta : \mathbb{S}^1 \to \mathbb{S}^1$

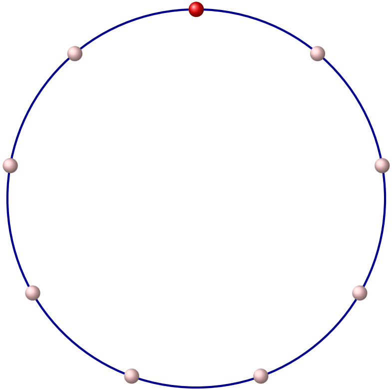
rotations of the circle



$$f_\theta : \mathbb{S}^1 \to \mathbb{S}^1$$

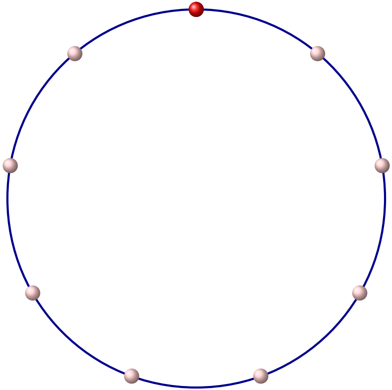
$$f_\theta(x) = x + \theta \pmod 1$$

rational rotations



if $\theta = {}^{p}/_{q} \in \mathbb{Q}$

rational rotations



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$$f_{p/q}(x) = x + {}^{p}/_{q}$$

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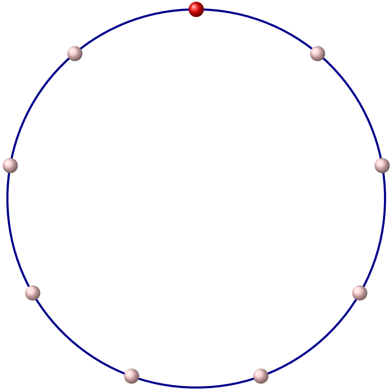

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rational **rotation**

proposition

Proposition (rational rotations)

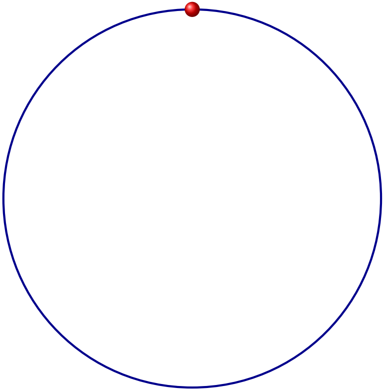
- $f_{p/q} : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$

proposition

Proposition (rational rotations)

- $f_{p/q} : \mathbb{S}^1 \to \mathbb{S}^1$
- $\Rightarrow$ all orbits are periodic

irrational rotation



if $\theta \notin \mathbb{Q}$

irrational rotation

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irrational rotation

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irrational **rotation**

proposition

Proposition (irrational rotations)

- $f_\theta : \mathbb{S}^1 \to \mathbb{S}^1$ irrational rotation

proposition

Proposition (irrational rotations)

- $f_\theta : \mathbb{S}^1 \to \mathbb{S}^1$ irrational rotation
- $\Rightarrow$ all orbits are dense

proposition

Proposition (irrational rotations)

- $f_\theta : \mathbb{S}^1 \to \mathbb{S}^1$ irrational rotation
- $\Rightarrow$ all orbits are dense
- f_θ is minimal

limit sets

Definition (ω-limit set)

- $f : X \to X$ dynamical system

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limit sets

Definition (ω-limit set)

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- $y \in \omega(x)$ if
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- $\omega(x)$ is called the ω-limit set of x

limit sets

Definition (α-limit set)

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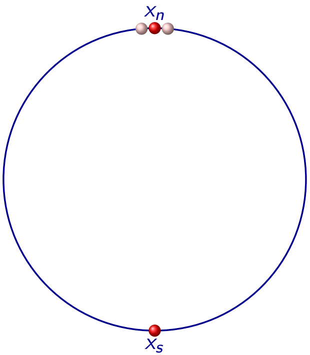
example

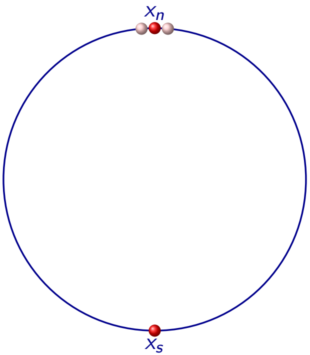
north pole - south pole



- $\forall x \neq x_n$

example

north pole - south pole



- $\forall x \neq x_n$
- $\omega(x) = \{x_s\}$

example

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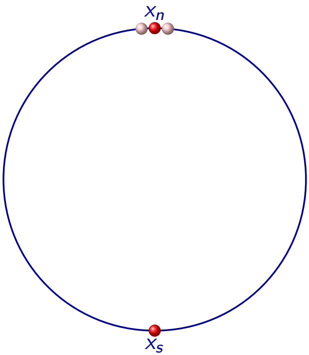
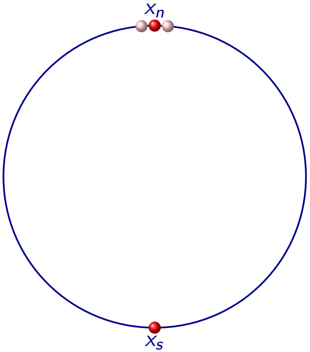


- $\forall x \neq x_n$
- $\omega(x) = \{x_s\}$
- x_s attracting fixed point

example

north pole - south pole



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example

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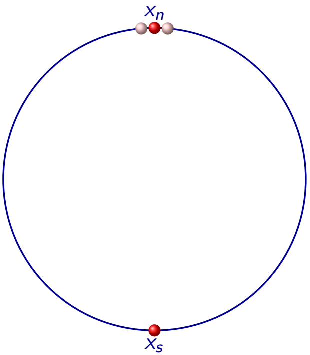

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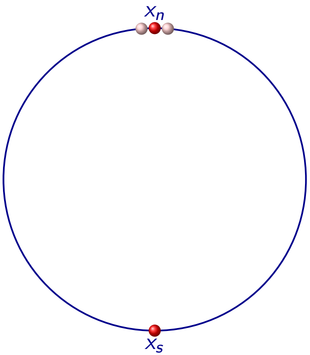

- $\forall x \neq x_s$
- $\alpha(x) = \{x_n\}$
- x_n repelling fixed point

example - saddle node

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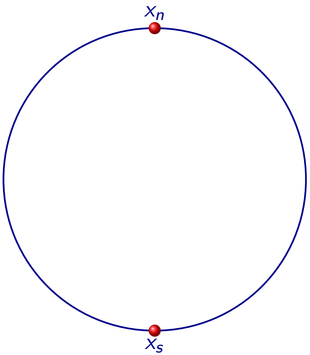

- $x \in (1/2, 3/2)$

example - saddle node

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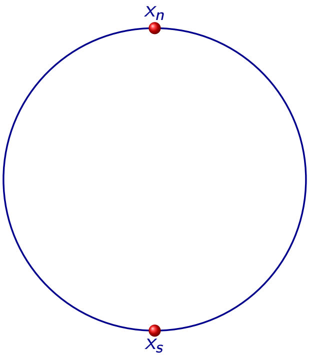

- $x \in \left({}^{1}\!/_{2}, {}^{3}\!/_{2}\right)$
- $\Rightarrow \alpha(x) = x_s$ and $\omega(x) = x_n$

example - saddle node

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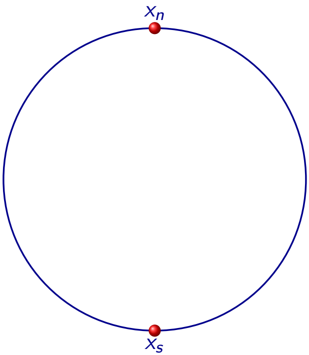

- $x \in (^1/_2, ^3/_2)$
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- $x \in (^3/_2, ^1/_2)$

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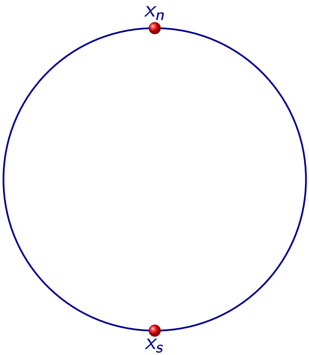

- $x \in (1/2, 3/2)$
- $\Rightarrow \alpha(x) = x_s$ and $\omega(x) = x_n$
- $x \in (3/2, 1/2)$
- $\Rightarrow \alpha(x) = x_n$ and $\omega(x) = x_s$

example - saddle node

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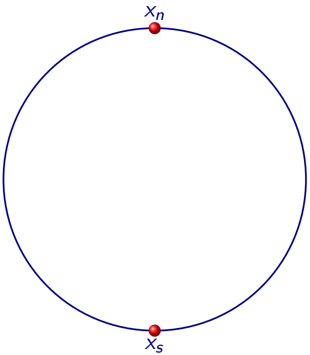

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- $x \in \left({}^3\!/_2, {}^1\!/_2\right)$
- $\Rightarrow \alpha(x) = x_n$ and $\omega(x) = x_s$
- x_n and x_s are saddle nodes

example - circle rotations

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- $f_{p/q}$ is a rational rotation

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- $f_{p/q}$ is a rational rotation
- $\Rightarrow \alpha(x) = \omega(x) = o(x) \qquad \forall x \in \mathbb{S}^1$

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- f_θ is an irrational rotation

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- f_θ is an irrational rotation
- $\Rightarrow \alpha(x) = \omega(x) = \mathbb{S}^1 \qquad \forall x \in \mathbb{S}^1$

the map $2x \mod 1$

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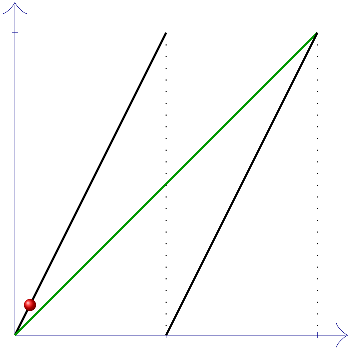

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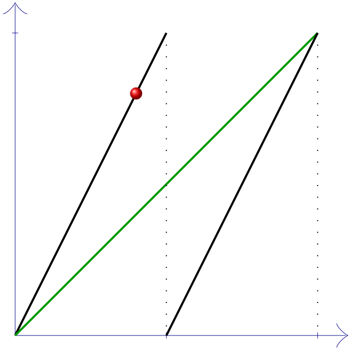

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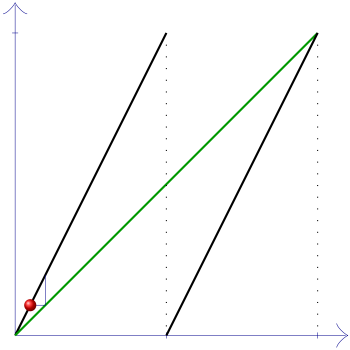

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