

Observable invariant measures.

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Abstract

For continuous maps on a compact manifold M , particularly for those that do not preserve the Lebesgue measure m , we define the *observable invariant probability measures* as a generalization of the physical measures. We prove that any continuous map has observable measures, and characterize those that are physical in terms of the observability. We prove that there exist physical measures whose basins cover Lebesgue a.e, if and only if the set of all observable measures is finite or infinite numerable. We define for any continuous map, its *generalized attractors* using the set of observable invariant measures where there is no physical measure, and prove that any continuous map defines a decomposition of the space in up to infinitely many generalized attractors whose basins cover Lebesgue a.e. We apply the results to the C^1 expanding maps f in the circle, proving that the set of all observable measures (even if f is not $C^{1+\alpha}$) is a subset of the set of all the equilibrium states of $-\log |f'|$.

1 Introduction

It is an old problem to find “good” probability measures for a map $f: M \mapsto M$, meaning for that, an invariant probability that resume in some sense, the dynamics by iterations of the map. Sometimes, the map is born with a good measure, as in the case of maps preserving a Lebesgue ergodic measure. But this is not true in general, and it is not an easy question to determine, in most examples, a single or a few probability measures representing the dynamics of the map.

There have been proposed several ideas to define a “good” invariant probability measure μ :

1. Lebesgue a.e. point in a set is generic with respect μ , that is, $\mu = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_f^j(x)$,

where the convergence is in the weak* topology of the space $\mathcal{P}$ of probabilities on M .

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2. The conditional measures of μ on unstable manifolds are absolutely continuous respect to the Lebesgue measure along those manifolds.
3. μ verifies the Pesin-Ledrappier-Young (PLY) equality:

$$h_\mu(f) = \int \sum_i \lambda_i^+(x) \dim E_i(x) d\mu(x)$$

where $h_\mu(f)$ is the entropy of μ , and $\dim E_i(x)$ is the multiplicity of the positive Lyapunov exponent $\lambda^+(x)$ in the Oseledec's decomposition.

4. The measure is the limit of measures which are invariant under stochastic perturbations.

It is a remarkable property that the four definitions above are equivalent for Axiom A attractors. Moreover, Ledrappier and Young (see [LY85a] and [LY85b]), under suitable hypothesis of differentiability, proved that a measure verifies property 2 if it verifies property 3 while the converse result is the well known result of Pesin's entropy formula. Ergodic measures verifying 2 (or, equivalently, 3) with no zero Lyapunov exponents describe chaotic behavior, and are accompanied by rich geometric and dynamical structures (see [Y02]). Nevertheless, the other measures listed above are also interesting, because reveal statistical aspects of the behavior of the future iterations of the map. For instance, a measure verifying definitions 1 or 4 is concentrated in the part of the space which is statistically more visited.

We will call physical measure, a probability measure verifying 1, and stochastically stable, a probability measure verifying 4. We will call SRB (Sinai-Ruelle-Bowen) measure a probability verifying 2, and a PLY measure, a probability verifying 3. In this work, we propose another concept of "good" probabilities, which we call observable measures. The following question was the motivation of this work: Is it possible to describe probabilistically in the space, in some *minimal way, and in a very general regular or irregular setting*, the asymptotic behavior of the time averages of Lebesgue a.e. orbit? We answer this question in Theorem 1.5.

Generalized ergodic attractors and observable measures, that we define and theoretically develop along this work, do always exist for any continuous map (Theorems 1.3 and 2.5). On the other hand, physical measures and ergodic attractors do not necessarily exist (see examples 4.8 and 4.9).

It is largely known the difficulties to characterize, or just find, non hyperbolic or non $C^{1+\alpha}$ maps that do have physical measures. This is a hard problem even in some systems whose iterated topological behavior is known ([C93], [E98], [H00], [HY95]). The difficulties appear when trying to apply to a non hyperbolic setting, or to a non $C^{1+\alpha}$ map, the known techniques for constructing the physical measures of hyperbolic $C^{1+\alpha}$ maps ([P77], [S72], [A67]). The $C^{1+\alpha}$ hypothesis allows the existence of SRB measures ([BR75], [R76], [S72]) and are relevant and widely studied occupying an important focus of interest in the ergodic differentiable theory of dynamical systems ([A67], [PS82], [PS04], [V98], [BDV00]). But if having a weak or non uniform hyperbolic setting, the obstruction usually resides in the

irregularity of the invariant manifolds, which technically translate the non trivial relations between topologic, measurable and differentiable properties of the system ([PS04]). The difficulties arise even for maps posed in a very regular setting as Lewowicz diffeomorphisms in the two-torus (see [L80]). For them, the differentiable regularity of the given transformation (they are analytic maps) and the topological known behavior of the iterated system (they are conjugated to Anosov maps), were not enough, up to the moment, to prove the existence of physical measures and ergodic attractors, except in some meager set of examples [CE01]. Before stating the results we need to formalize some definitions that we will use all along this work: Let $f: M \mapsto M$ be a continuous map with M a compact, finite-dimensional manifold. Let m be a probability Lebesgue measure, and not necessarily f -invariant. We denote $\mathcal{P}$ the set of all Borel probability measures in M , provided with the weak* topology, and a metric structure inducing this topology.

For any point $x \in M$ we denote $p\omega(x)$ to the set of the Borel probabilities in M that are the partial limits of the (not necessarily convergent) sequence

$$\left\{ \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)} \right\}_{n \in \mathbb{N}} \quad (1)$$

where δ_y is the Dirac delta probability measure supported in $y \in M$.

The set $p\omega(x) \subset \mathcal{P}$ is the collection of the spatial probability measures describing the asymptotic time average (given by (1)) of the system states, provided the initial state is x . If the sequence (1) converges then we denote $p\omega(x) = \{\mu_x\}$. To include also those cases for which the sequence (1) is not convergent (for a set of orbits with positive Lebesgue measure) we consider, for a given measure μ , the set of points $x \in M$ such that the minimum distance between μ and the set of partial limits of the sequence (1) is small. We define:

Definition 1.1 (Observable probability measures.) A probability measure $\mu \in \mathcal{P}$ is *observable* if for all $\varepsilon > 0$ the set $A_\varepsilon = \{x \in M : \text{dist}^*(p\omega(x), \mu) < \varepsilon\}$ has positive Lebesgue measure. The set $A_\varepsilon = A_\varepsilon(\mu) \subset M$ is called *the ε -basin of partial attraction* of the probability μ .

We note that the definition above is independent of the choice of the distance in $\mathcal{P}$, provided that the metric structure induces its weak* topology. Observable measures are f -invariant, and usually at most a few part of the space of invariant measures for f are observable measures (see the examples in Section 4). We remark that for observable measures, the condition $m(A_\varepsilon) > 0$ must be verified not only *for some* but *for all* $\varepsilon > 0$.

Definition 1.2 (Physical probability measures.) A probability measure $\mu \in \mathcal{P}$ is *physical* (even if it is not ergodic), if the set $B = \{x \in M : p\omega(x) = \{\mu\}\}$ has positive Lebesgue measure. The set $B = B(\mu) \subset M$ is called *the basin of attraction* of μ .

Therefore, all physical measures are observable, but not all observable measures are physical (Examples 4.8 and 4.9).

We state the following starting results:

Theorem 1.3 (Existence of observable measures and physical measures.)

a) *For any continuous map f , the set $\mathcal{O}$ of all observable probability measures for f is non-empty and weak*-compact.*

b) *$\mathcal{O}$ is finite or countably infinite if and only if there exist (resp. finitely or countably infinitely many) physical measures of f such that the union of their basins of attraction cover Lebesgue a.e.*

The first statement of this theorem is proved in paragraph 3.4 and the second one in paragraph 6.2.

The $p\omega(x)$ limit set of convergent subsequences of (1) may have many different partial limit measures. Nevertheless, we prove that $p\omega(x)$ is formed only with observable measures, for Lebesgue almost all $x \in M$, as stated in the following Theorem 1.5.

Definition 1.4 (Basin of attraction.)

The *basin of attraction* $B(\mathcal{K})$ of a compact subset $\mathcal{K}$ of the space $\mathcal{P}$ of all the Borel probability measures in M , is the (maybe empty) subset of M defined as:

$$B(\mathcal{K}) = \{x \in M : p\omega(x) \subset \mathcal{K}\}$$

If the purpose is to study the asymptotic to the future time average behaviors of Lebesgue almost all points in M , then the *consideration of the set $\mathcal{O}$ of all the observable measures for f , is the necessary and sufficient condition.* In fact we have the following:

Theorem 1.5 (Attracting minimality property of the set of observable measures.) *The set $\mathcal{O}$ of all observable measures for f is the minimal compact subset of the space $\mathcal{P}$ whose basin of attraction has total Lebesgue measure.*

We prove this theorem in paragraph 3.12.

The theory about the observable measures describe the statistical asymptotic behavior of time averages, instead of the theory of physical measures when these last probabilities do not exist. In particular it is suitable to study the statistics of the future iterations of maps, disregarding their regularity, that do not preserve a Lebesgue equivalent measure, or that do preserve it but are not ergodic. In other words, the results about the observable measures, independently if they are or not useful to find physical measures in some concrete examples, substitute the physical measures, and do that in a wide setting of dynamical systems (all the continuous maps), without losing their statistical meaning (see Proposition 7.2.)

In Section 2 we define *attractor* A , to the support in M of a physical, non necessarily ergodic measure ν , similarly as done by Pugh and Shub in [PS89]. Analogously, we call *generalized attractor*, to the union A of the supports of an adequately reduced weak*-compact family $\mathcal{O}_1$ of observable measures. We will not require strict topological attraction to A , but weak topological: in the average the orbit lays in an arbitrarily small neighborhood of A as much time as wanted. (Proposition 7.2.) In Theorem 2.5 we prove that there always

exist up to countable many generalized attractors whose basins cover Lebesgue a.e. The following open question refers to the existence and finiteness of physical measures and to the convergence of the sequence (1) of time averages of the system for a set of initial states with total Lebesgue measure. It is posed in [P99]:

1.6 Palis Conjecture *Most dynamical systems have up to finitely many physical measures (or ergodic attractors) such that their basins of attraction cover Lebesgue almost all points.*

This conjecture admits the following equivalent statement, that seems weaker. (In fact, the definition 1.1 of observability is certainly weaker than the definition 1.2 of physical measures.)

1.7 Equivalent formulation of Palis Conjecture:

For most dynamical systems the set of observable measures is finite.

Note: To prove the equivalence of statements 1.6 and 1.7 it is enough to join the results of Theorems 1.3.b and 1.5.

For systems preserving the Lebesgue measure the main question is their ergodicity. It is immediate the following result:

Remark 1.8 (Observability and ergodicity.)

If f preserves the Lebesgue measure m then the following assertions are equivalent:

- (i) f is ergodic respect to m .
- (ii) There exists a unique observable measure μ for f .
- (iii) There exists a unique physical measure ν for f attracting Lebesgue a.e.

Besides, if the assertions above are verified, then $m = \mu = \nu$

Given a Lebesgue measure preserving map f , the question if f has a physical measure is mostly open, for differentiable maps that do not have some kind of uniform total or partial hyperbolicity [V98]. The existence of a physical measure attracting Lebesgue a.e. is equivalent to the ergodicity of the map, and is also an open question if most of conservative maps are ergodic ([PS04], [BMVW03]). The key difficult point resides in those maps that do not have any kind of uniform total or partial C^1 hyperbolicity in the space, as in the inspiring Lewowicz examples in the two torus [L80]. Due to Remark 1.8 those open questions are equivalent to the *uniqueness of the observable measure*.

2 Attractors

Due to the conjecture in 1.6 and Theorem 1.3, we are interested in partitioning the set $\mathcal{O}$ of observable measures, or to reduce it as much as possible, into different compact subsets whose basins of attractions have positive Lebesgue measure. Due to Theorem 1.5, no proper compact part of $\mathcal{O}$ has a total Lebesgue basin. We define:

Definition 2.1 (Generalized Attractors - Reductions of the space $\mathcal{O}$.) A *generalized attractor* $(A, \mathcal{A}) \subset \mathcal{M} \times \mathcal{O}$, (or a *reduction* $\mathcal{A}$ of the space $\mathcal{O}$ of all observable measures for f), is a compact subset $(A, \mathcal{A})$ such that the basin of attraction $B(\mathcal{A}) = \{x \in M : p\omega(x) \subset \mathcal{A}\}$ has positive Lebesgue measure in M , and A is the (minimal) compact support in M of all the probability measures in $\mathcal{A}$. We call $(A, \{\mu\})$ an *attractor* if it is a generalized attractor with a single invariant probability μ , i.e. μ is a physical measure.

Remark 2.2 Sometimes we refer to a generalized attractor only to A or only to $\mathcal{A}$. The *irreducible* generalized attractors (if they exist), attract the time averages distributions and are minimal in some sense, but are not formed necessarily with ergodic measures (Example 4.8). In spite a system could not exhibit a physical measure, still the reductions of the space of observable measures divide the manifold in the basins of *generalized attractors*. Each reduction $\mathcal{A}$ has a basin $C = C(\mathcal{A})$ with positive Lebesgue measure and is minimal respect to C . We state this result as follows:

Theorem 2.3 (Minimality of generalized attractors.) *Any generalized attractor $\mathcal{A}$ is the minimal compact set of observable measures attracting its basin $C(\mathcal{A})$. More precisely*

$$m(C(\mathcal{A}) \setminus C(\mathcal{K})) > 0 \quad \text{for all compact subset } \mathcal{K} \subset \mathcal{A}.$$

We prove Theorem 2.3 in paragraph 3.12.

Definition 2.4 Irreducibility

A generalized attractor $\mathcal{A} \subset \mathcal{P}$ is *irreducible* if it does not contain proper compact subsets that are also generalized attractors.

It is *trivial* or *trivially irreducible* if its diameter in $\mathcal{P}$ is zero, or in other words, if $\mathcal{A}$ has a unique observable measure μ .

In other words: physical measures are trivially irreducible and conversely.

The following result is much weaker than, but related with, the Palis' conjecture stated in paragraph 1.6:

Theorem 2.5 (Decomposition Theorem)

For any continuous map $f: M \mapsto M$ there exist a collection of (up to countable infinitely many) generalized attractors whose basins of attraction are pairwise Lebesgue-almost disjoint and cover Lebesgue-almost all M .

The space of all continuous maps divide in two disjoint classes:

- *The generalized attractors of the decomposition are all irreducible and then the decomposition is unique.*
- *For all $\varepsilon > 0$ there exists a decomposition for which the reducible generalized attractors have all diameter (in the weak* space of probabilities), smaller than ε*

We prove this theorem in paragraph 6.10. Note that, in the second class of systems, as the reducible generalized attractors have in the space of probabilities a small diameter, for a rough observer, each of those attractors acts as a physical measure.

In Theorem 6.4 and Corollary 6.9 we characterize those maps whose generalized attractors are the support of physical measures, as asked in the statement of Palis conjecture.

Definition 2.6 Milnor attractor. [M85] A Milnor attractor is a compact set $A \subset M$ such that its topological basin of attraction

$$B(A) = \{x \in M : \omega(x) \subset A\}$$

has positive Lebesgue measure, and for any compact proper subset $K \subset A$ the set

$$B(A) \setminus B(K) = \{x \in M : \omega(x) \subset A, \omega(x) \not\subset K\}$$

also has positive Lebesgue measure.

Note: Here $\omega(x)$ is the ω -limit set in M of the orbit of x , i.e. the set of limit points in M of the orbit with initial state x .

The generalized attractors $(A, \mathcal{A})$ where $\mathcal{A} \subset \mathcal{P}$, were inspired in the definition 2.6. They play in $\mathcal{P}$ a similar topological role that Milnor attractors play in M . In particular, the physical measures in $\mathcal{P}$ play the role that sinks do in M , those first considered as punctual attractors of time averages probabilities, and these last considered as punctual attractors of the points in M , along the future orbits.

3 Proofs of Theorems 1.3 (a) and 1.5.

Definition 3.1 (Weak* topology in the space of probability measures.)

The weak* topologic structure in the space $\mathcal{P}$ is defined as:

$$\mu_n \rightarrow \mu \text{ in } \mathcal{P} \quad \text{iff} \quad \lim \int \phi d\mu_n = \int \phi d\mu \quad \text{for all } \phi \in C^0(M, \mathbb{R})$$

where $C^0(M, \mathbb{R})$ denotes the space of the continuous real functions in M .

A classic basic theorem on Topology states that the space $\mathcal{P}$ is compact and metrizable when endowed with the weak* topology [M89]. Let us denote as $\mathcal{P}_f \subset \mathcal{P}$ the set of the Borel probability measures in M that are f -invariant, that is $\mu(f^{-1}(B)) = \mu(B)$ for all Borel set $B \subset M$. Note that the Lebesgue measure m does not necessarily belong to $\mathcal{P}_f$. Fix any metric in $\mathcal{P}$ giving its weak* topology structure. We denote as $\mathcal{B}_\varepsilon(\mu)$ to the open ball in $\mathcal{P}$ centered in $\mu \in \mathcal{P}$ and with radius $\varepsilon > 0$.

3.2 The $p\omega$ - limit sets.

At the beginning of this paper we defined, for each initial state $x \in M$, the set $p\omega(x)$ in the space $\mathcal{P}$ as the partial limits, in the weak* topology in $\mathcal{P}$, of the sequence (1) of time averages. In other words:

$$p\omega(x) = \left\{ \mu \in \mathcal{P} : \lim_{n_i} \frac{1}{n_i} \sum_{j=0}^{n_i-1} \phi(f^j(x)) = \int \phi d\mu \forall \phi \in C^0(M, \mathbb{R}) \text{ for some } n_i \rightarrow +\infty \right\}$$

For further uses we state here the following property for the $p\omega$ -limit sets:

Theorem 3.3 Convex-like property.

For any point $x \in M$

i) If $\mu, \nu \in p\omega(x)$ then for each real number $0 \leq \lambda \leq 1$ there exists a measure $\mu_\lambda \in p\omega(x)$ such that

$$\text{dist}(\mu_\lambda, \mu) = \lambda \text{dist}(\mu, \nu)$$

ii) The set $p\omega(x)$ either has a single element or non-countable infinitely many.

Proof: See 7.3 in the appendix.

3.4 Proof of Theorem 1.3 (a): (The proof of the assertion (b) of Theorem 1.3 is delayed until paragraph 6.2.)

Let us prove that, given any continuous map f , the set $\mathcal{O}_f$ of the probability measures that are ε -observable for all $\varepsilon > 0$ is a non-empty compact subset of the (weak* topologic) space $\mathcal{P}_f$ of the f -invariant measures.

The key question is that $\mathcal{O}_f$ is not empty, which we prove at the end.

Let us first prove that $\mathcal{O}_f \subset \mathcal{P}_f$. In fact, given $\mu \in \mathcal{O}_f$ then, for any $\varepsilon = 1/n > 0$ there exists some $\mu_n \in \mathcal{B}_\varepsilon(\mu)$ which is the limit of a convergent subsequence of (1) for some $x \in M$. As the limits of all convergent subsequences of (1) are f -invariant, we have that $\mu_n \in \mathcal{P}_f \subset \mathcal{P}$ for all natural number n , and $\mu_n \rightarrow \mu$ with the weak* topologic structure of $\mathcal{P}$. The space $\mathcal{P}_f$ is a compact subspace of $\mathcal{P}$ with the weak* topologic structure, so $\mu \in \mathcal{P}_f$ as wanted.

Second, let us prove that $\mathcal{O} = \mathcal{O}_f$ is compact. The complement $\mathcal{O}^c$ of $\mathcal{O}$ in $\mathcal{P}$ is the set of all probability measures μ (not necessarily f -invariant) such that for some $\varepsilon = \varepsilon(\mu) > 0$ the set $\{x \in M : p\omega(x) \cap \mathcal{B}_\varepsilon(\mu) \neq \emptyset\}$ has zero Lebesgue measure. Therefore $\mathcal{O}^c$ is open in $\mathcal{P}$, and $\mathcal{O}$ is a closed subspace of $\mathcal{P}$. As $\mathcal{P}$ is compact we deduce that $\mathcal{O}$ is compact as wanted.

Third, let us prove that $\mathcal{O}$ is not empty. Suppose by contradiction that it is empty. Then $\mathcal{O}^c = \mathcal{P}$, and for every $\mu \in \mathcal{P}$ there exists some $\varepsilon = \varepsilon(\mu) > 0$ such that the set $A = \{x \in M : p\omega(x) \subset (\mathcal{B}_\varepsilon(\mu))^c\}$ has total Lebesgue probability.

As $\mathcal{P}$ is compact, let us consider a finite covering of $\mathcal{P}$ with such open balls $\mathcal{B}_\varepsilon(\mu)$, say $B_1, B_2, \dots, B_k$, and their respective sets $A_1, A_2, \dots, A_k$ defined as above. As $m(A_i) = 1$ for all $i = 1, 2, \dots, k$ we have that the intersection $B = \bigcap_{i=1}^k A_i$ is not empty. By construction, for all $x \in B$ the $p\omega$ -limit of x is contained in the complement of B_i for all $i = 1, 2, \dots, k$, and so it would not be contained in $\mathcal{P}$, that is the contradiction ending the proof. $\square$

Note that the proof of Theorem 1.3 (a) does not use any property of the Lebesgue measure on M different from those that any Borel probability m on M also has. The same result works (but maybe defining a different subset of observable measures) if any other, non necessarily invariant probability measure m in $\mathcal{P}$ is used, as the reference probability distribution for the choice of the initial state x , instead of the Lebesgue measure. If so, the concept of *physical* measure also changes accordingly. Nevertheless, along this work, we are calling m to the Lebesgue measure, i.e. the volume form measure, given by the Riemannian metric on the manifold M , adequately rescaled to be a probability: $m(M) = 1$.

Definition 3.5 Observability size. If $\mu \in \mathcal{P}$ is a (non necessarily invariant) probability measure (in particular if it is an observable measure, see Definition 1.1), we call *observability size* of μ to the non negative real function $o = o_\mu: \overline{\mathbb{R}^+} \mapsto \overline{\mathbb{R}^+}$ defined as

$$o_\mu(\varepsilon) = m(A(\varepsilon, \mu))$$

where m is the Lebesgue measure in M and $A(\varepsilon, \mu)$ is the set

$$A(\varepsilon, \mu) = \{x \in M : p\omega(x) \cap \mathcal{B}_\varepsilon(\mu) \neq \emptyset\} \quad \text{being} \quad \mathcal{B}_\varepsilon(\mu) = \{\nu \in \mathcal{P} : \text{dist}^*(\nu, \mu) < \varepsilon\}$$

For some fixed $\varepsilon > 0$, we say that $\mu \in \mathcal{P}$ is ε -observable if $o_\mu(\varepsilon) > 0$.

Remark 3.6 For any probability measure μ , its observability size function $o(\varepsilon)$ is *positive and decreasing* with $\varepsilon > 0$. Then $o(\varepsilon)$ has always a non-negative limit value when $\varepsilon \rightarrow 0^+$. We reformulate Definition 1.1 of observability of measures, in the following equivalent terms:

3.7 Remark: (Observability revisited.) $\mu \in \mathcal{P}$ is *observable* for f , if and only if it is ε -observable (see 1.1) for all $\varepsilon > 0$. In particular, μ is *physical* if and only if $\lim_{\varepsilon \rightarrow 0} o_\mu(\varepsilon) > 0$.

The characterization of those continuous maps having physical-measures as those whose sets of observable measures, or some reductions of them, are finite or countable infinite (Theorem 6.2), derives the attention to try to define and find sufficient conditions to reduce as much as possible the set of observable measures.

Besides, the *reductions* of the space of observable measures will work as *Generalized Ergodic Attractors*, even in the case that this reduction can not be done as much as to obtain physical measures. We first prove that the reducibility of the set $\mathcal{O}$ of observable measures for f must be defined carefully, because in the following sense, this set $\mathcal{O}$ is minimal.

Theorem 3.8 (Reformulation of Theorem 1.5)

Let $f: M \mapsto M$ be any given continuous map in the compact manifold M .

The set $\mathcal{O}_f$ of all its observable measures belongs to the family

$$\mathfrak{N} = \{\mathcal{K} \subset \mathcal{P} : \mathcal{K} \text{ is compact and } p\omega(x) \subset \mathcal{K} \text{ for Lebesgue almost every point } x \in M\}$$

Moreover

$$\mathcal{O}_f = \bigcap_{\mathcal{K} \in \mathfrak{N}} \mathcal{K}$$

and thus $\mathcal{O}_f$ is the unique minimal set in $\mathfrak{N}$.

Proof: For simplicity let us denote $\mathcal{O} = \mathcal{O}_f$. Given any subset $\mathcal{K} \subset \mathcal{P}$ (this $\mathcal{K}$ is neither necessarily in $\aleph$ nor necessarily compact), let us consider:

$$A(\mathcal{K}) = \{x \in M : p\omega(x) \cap \mathcal{K} \neq \emptyset\}, \quad C(\mathcal{K}) = \{x \in M : p\omega(x) \subset \mathcal{K}\} \quad (2)$$

It is enough to prove that $m(C(\mathcal{O})) = 1$ and that $\mathcal{K} \supset \mathcal{O}$ for all $\mathcal{K} \in \aleph$. Let us first prove the second assertion.

To prove that $\mathcal{K} \supset \mathcal{O}$ it is enough to show that $\mu \notin \mathcal{O}$ if $\mu \notin \mathcal{K} \in \aleph$.

If $\mu \notin \mathcal{K}$ take $\varepsilon = \text{dist}(\mu, \mathcal{K}) > 0$. For all $x \in C(\mathcal{K})$ the set $p\omega(x) \subset \mathcal{K}$ is disjoint with the ball $B_\varepsilon(\mu)$. But almost all Lebesgue point $x \in C(\mathcal{K})$, because $\mathcal{K} \in \aleph$. Therefore $p\omega(x) \cap B_\varepsilon(\mu) = \emptyset$ Lebesgue a.e. This last assertion and Definition 1.1 and paragraph 3.7 imply that $\mu \notin \mathcal{O}$, as wanted.

Now let us prove that $m(C(\mathcal{O})) = 1$, which is the key matter of this theorem. We know $\mathcal{O}$ is compact and not empty. So, for any $\mu \notin \mathcal{O}$ it is defined the distance $\text{dist}(\mu, \mathcal{O}) > 0$. Observe that the complement $\mathcal{O}^c$ of $\mathcal{O}$ in $\mathcal{P}$ can be written as the increasing union of compacts sets $\mathcal{K}_n$ (not in the family $\aleph$) as follows:

$$\mathcal{O}^c = \bigcup_{n=1}^{\infty} \mathcal{K}_n, \quad \mathcal{K}_n = \{\mu \in \mathcal{P} : \text{dist}(\mu, \mathcal{O}) \geq 1/n\} \supset \mathcal{K}_{n+1} \quad (3)$$

Let us take the sequence $A_n = A(\mathcal{K}_n)$ of sets in M defined in (2) at the beginning of this proof, and denote $A_\infty = A(\mathcal{O}^c)$. We deduce from (2) and (3) that:

$$A_\infty = \bigcup_{n=1}^{\infty} A_n, \quad m(A_n) \rightarrow m(A_\infty), \quad A_\infty = A(\mathcal{O}^c)$$

To finish the proof is thus enough to show that $m(A_n) = 0$ for all $n \in \mathbb{N}$.

In fact, $A_n = A(\mathcal{K}_n)$ and $\mathcal{K}_n$ is compact and contained in $\mathcal{O}^c$. By Definition 1.1 and paragraph 3.7 there exists a finite covering of $\mathcal{K}_n$ with open balls $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_k$ such that

$$m(A(\mathcal{B}_i)) = 0 \quad \text{for all } i = 1, 2, \dots, k \quad (4)$$

By (2) the finite collection of sets $A(\mathcal{B}_i)$; $i = 1, 2, \dots, k$ cover A_n and therefore (4) implies $m(A_n) = 0$ ending the proof. $\square$

As shown in the examples of Section 4, there exist maps whose spaces $\mathcal{O}$ of observable measures are irreducible and maps for which they are reducible. Also there exist maps that do not have irreducible subsets in $\mathcal{O}$. In section 6.3 we define chains and co-chains of reductions. Those systems having physical measures can be characterized also according to the existence of adequate decreasing sequences (chains) of generalized attractors.

For further uses we define:

Definition 3.9 (Diameter and Attracting Size.) Let $\mathcal{O}$ be the set of the observable measures for f . Let $\mathcal{O}_1$ be a reduction or generalized attractor of $\mathcal{O}$.

The diameter of $\mathcal{O}_1$ is $\max\{\text{dist}(\mu, \nu) : \mu, \nu \in \mathcal{O}_1\}$. The attracting size of $\mathcal{O}_1$ is $m(B(\mathcal{O}_1))$, where $B(\mathcal{O}_1)$ is the basin of attraction of $\mathcal{O}_1$ (see definition 1.4).

By definition of generalized attractor its attracting size is positive. If the basin of attraction B of some compact subset of $\mathcal{O}$ has positive Lebesgue measure, then there exists a compact set $K \subset B$ with positive Lebesgue measure. Thus we obtain the following characterization of all the reductions of $\mathcal{O}$, as a consequence of Egoroff Theorem:

Proposition 3.10 (Generalized attractors and uniform convergence.)

The subspace $\mathcal{O}_1$ is a reduction of the space $\mathcal{O}$ of the observable measures for f , (i.e. $\mathcal{O}_1$ is a generalized attractor for f), if and only if there exists a positive Lebesgue measure set $K \subset M$ such that

$$\text{dist} \left(\frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)}, \mathcal{O}_1 \right) \rightarrow 0 \text{ uniformly in } x \in K$$

Note: The set $K \subset M$ is not necessarily f -invariant.

Proof: Let us call B to the basin of attraction of $\mathcal{O}_1$ (see definition 1.4). We have $m(B) > 0$ and therefore, the sequence in 3.10 converges to 0 m -a.e. $x \in B$. The direct result is now a straightforward consequence of Egoroff Theorem and its converse is obvious. $\square$

The following is other characterization of the reductions of the space of observable measures for $f \in C^0(M)$, in terms of the invariant subsets in M that have positive Lebesgue measure:

Proposition 3.11 (Restricting the map to reduce the set of observable measures.)

The subspace $\mathcal{O}_1$ is a reduction of the space $\mathcal{O}_f$ of the observable measures for f (i.e. $\mathcal{O}_1$ is a generalized attractor for f) if and only if $\mathcal{O}_1 = \mathcal{O}_{f_1}$, where $\mathcal{O}_{f_1}$ is the set of all observable measures of the map $f_1 = f|_C$, obtained when f is restricted to some invariant set $C \subset M$ that has positive Lebesgue measure. Besides C can be chosen as the basin attraction $B(\mathcal{O}_1)$ of $\mathcal{O}_1$.

Proof: This Theorem is a corollary of Theorem 3.8. In fact, to prove the converse statement apply 3.8 to $f|_C$ instead of f , taking $C = B(\mathcal{O}_1)$ where $\mathcal{O}_1$ is the given reduction of $\mathcal{O}_f$. To prove the direct result apply also 3.8 to $f|_C$, but now taking C as the given invariant subset in M with positive Lebesgue measure. $\square$

3.12 - Proof of Theorem 1.5: By Proposition 3.11 and Theorem 3.8 applied to $f|_C$ where $C = B(\mathcal{O}_1)$, we have that Lebesgue almost all $x \in C$ verifies $p\omega(x) \subset \mathcal{O}_1$, and any $\mu \in \mathcal{O}_1$ is observable for $f|_C$. Take any $\mu \in \mathcal{O}_1 \setminus \mathcal{K}$. There exists an open ball $B_\varepsilon(\mu)$ that does not intersect $\mathcal{K}$. As μ is observable for $f|_C$, (see Definition 1.1), $p\omega(x)$ is not contained in $\mathcal{K}$ for a set of $x \in C$ with positive Lebesgue measure. Therefore $m(C \setminus B(\mathcal{K})) > 0$ as wanted. $\square$

4 Examples.

The examples in this section are well known or very simple, but give a scenario of the possible dynamics, in terms of the statistics given by the time mean sequence (1). In fact, they are paradigmatic of some different classes of continuous dynamical systems $f \in C^0(M)$. After Theorem 2.5), some of these examples may appear joint with the others, in each of the basins of the (up to countable infinitely many) generalized attractors, in which the complete set $\mathcal{O}_f$ of observable measures decomposes. In a general case, the complete topological dynamics may be much more complicated, since the basins of the infinitely many generalized attractors of f , may be topologically riddled in M , as explained in the discussion at the end of this section.

Example 4.1 For a map with a single periodic point x_0 , being a topological sink whose topological basin is M almost all point, the set $\mathcal{O}$ has a unique measure that is the δ -Dirac measure supported on x_0 .

Example 4.2 For any transitive Anosov $C^{1+\alpha}$ diffeomorphism the set $\mathcal{O}$ is irreducible containing uniquely the SRB measure μ . But there are also infinitely many other ergodic and non ergodic invariant probabilities, that are not observable (for instance the equally distributed Dirac delta measures combination supported on a periodic orbit). In particular, if f preserves a probability μ equivalent to the Lebesgue measure, then f has μ as the unique observable, and thus physical, probability. This result is generalized also for some subclass of C^3 diffeomorphisms on the two-torus, conjugated to transitive Anosov, with a non hyperbolic fixed point in which the derivative of f has double eigenvalue 1 and a single eigendirection ([CE01]). But is is mostly open for other conjugated to transitive Anosov, even if they are analytic.

Example 4.3 In [HY95] it is studied the class of C^2 diffeomorphisms f in the two-torus obtained from a transitive Anosov, and in the same conjugation class of the Anosov, when the unstable eigenvalue of a fixed point x_0 is weakened to be 1, maintaining its stable eigenvalue strictly smaller than 1 and maintaining also the uniform hyperbolicity in each iterate outside a neighborhood (non invariant) of the fixed point. It is proved that f has a single physical measure that is the Dirac delta supported on x_0 and that its basin has total Lebesgue measure. Therefore this is the single observable measure for f , although there are infinitely many other ergodic invariant measures. As the physical measure is supported in a fixed point x_0 , statistically x_0 acts as a sink, attracting the sequences of time averages of Lebesgue almost all orbit. Nevertheless, as f is conjugated to Anosov, it is topologically chaotic (i.e. expansive, or sensible to initial conditions).

Example 4.4 The diffeomorphism $f: [0, 1]^2 \mapsto [0, 1]^2$; $f(x, y) = (x/2, y)$ has the set $\mathcal{O}$ of observable measures as the set of Dirac delta measures $\delta_{(0,y)}$ for all $y \in [0, 1]$. In this case $\mathcal{O}$ coincides with the set of all ergodic invariant measures for f , it is infinitely reducible (i.e. $\mathcal{O}$ is reducible and any reduction of $\mathcal{O}$ is also reducible). Not all f -invariant measure μ for f is observable: for instance, the one-dimension Lebesgue measure on the interval $[0] \times [0, 1]$

is invariant and is not observable. This example shows that the set $\mathcal{O}$ is not necessarily convex.

Example 4.5 The maps exhibiting infinitely many simultaneous hyperbolic sinks, constructed from Newhouse's theorem ([N74]) has a space $\mathcal{O}$ of observable measures that is reducible. But it has infinitely many reductions (the Dirac delta supported on the hyperbolic sinks), each of them being irreducible. Also the maps exhibiting infinitely Hénon-like attractors, constructed by Colli in [C98], has a space of observable measures that is reducible, having infinitely many reductions (the physical measures supported on the Hénon-like attractors), each one that is irreducible.

Example 4.6 Consider the quadratic family $\{f_t\}_t$ in the interval I , and in this family, the map f where the first cascade or period doubling bifurcating maps converge. It has a single attractor A which is Feigenbaum-Coulet-Tresser. This attractor A is formed by a single orbit, whose closure K is a Cantor set having as extremes of its gaps, the future orbit of the critical point. For all $x \in A$, $f^n(x)$ moves quasi-periodically in a single orbit (with quasi-periods 2^n for all $n \geq 1$) and attracts topologically all the points of I , except those of countable many periodic hyperbolic repellers (with periods 2^n , for all $n \geq 0$). The map f is infinitely doubling renormalizable and has a single observable probability μ (and thus physical measure) supported on K . This physical measure is constructed as follows: For any fixed $n \geq 1$ call $\{K_{i,n} : 0 \leq i \leq 2^n - 1\}$ to the family of 2^n atoms of generation n , i.e. the sets $K_{i,n} : 0 \leq i \pmod{2^n} \leq 2^n - 1$ are the pairwise disjoint compact intervals such that $f(K_{i,n}) \subset K_{i+1,n}$, and $K = \bigcap_{n \geq 1} \bigcup_{i=0}^{2^n-1} K_{i,n}$. Then define the probability μ such that $\mu(K_{i,n}) = 1/2^n$ for all $n \geq 1$ and for all $0 \leq i \leq 2^n - 1$.

Example 4.7 (Example 1 in [A01]). Define the function $\phi : [-\frac{1}{\pi}, \frac{1}{\pi}] \mapsto \mathbb{R}$, $\phi(s) = s^4 \sin(\frac{1}{s})$ and identify the extremes of the interval to obtain $\varphi : S^1 \mapsto \mathbb{R}$. Consider the time one of the gradient flow given by $\dot{x} = \nabla \varphi(x)$. This map has infinitely many sources and sinks, which accumulate at 0. The physical measures are the infinitely many sinks. The Dirac delta on the accumulation point of the sinks and sources, is not a physical measure, but it is an observable measure. It is besides the unique stochastically stationary measure. We conclude that physical measures, even if their basins attract Lebesgue a.e., do not necessarily include the stochastically stationary probabilities but, at least in this example, observable measures include them.

Example 4.8 The following example, due to Bowen (see also example 2 in [A01]), shows that the space of observable measures may be formed by measures that are partial limits of the sequences of time averages of the system states and that this sequence may be not convergent for Lebesgue almost all points. Consider a diffeomorphism f in a ball of $\mathbb{R}^2$ with two hyperbolic saddle points A and B such that the unstable global manifold $W^u(A)$ of A is an embedded arc that coincides (except for A and B) with the stable global manifold $W^s(B)$ of B , and the unstable global manifold $W^u(B)$ of B is also an embedded arc that coincides with the stable manifold $W^s(A)$ of A . Let us take f such that there exists a

source C in the open ball U with boundary $W^u(A) \cup W^u(B)$, and all orbits in that ball U have α -limit set C and ω -limit set $W^u(A) \cup W^u(B)$. If the eigenvalues of the derivative of f at A and B are well chosen, then one can get that the time average sequences of the orbits in $U \setminus \{C\}$ are not convergent, have at least one subsequence convergent to the Dirac delta δ_A on A and have other subsequence convergent to the Dirac delta δ_B on B .

Due to Theorem 3.3, for each $x \in U \setminus \{C\}$ there are non countably many probability measures which are the limit measures of the time average sequence of the future orbit starting on x . All these measures are invariant under f and therefore, due to Poincaré Recurrence Theorem (see [M89]), all of them are supported on $\{A\} \cup \{B\}$. Due to this last observation and due to Theorem 3.3 all the convex combinations of δ_A and δ_B are limit measures of the sequence of time averages of any orbit starting at $U \setminus \{C\}$ and conversely.

Therefore the set $\mathcal{O}$ of observable measures for f coincides with the set of convex combinations of δ_A and δ_B . The set $\mathcal{O}$ is irreducible and formed by non-countable many probability measures. It is not the set of all invariant measures; in fact the measure δ_C is not observable.

This example also shows that the observable measures are not necessarily ergodic.

A different exact adjustment in the eigenvalues of the two saddles A and B allows a different example, for which all the convergent subsequences of (1) converge to the same exact previously chosen convex combination μ of δ_A and δ_B . So, there is a physical measure that attracts the time averages of the orbits of U and that is not ergodic. This proves that physical measures are not necessarily ergodic.

In [A01] it is shown that this physical measure, which is a convex combination μ of δ_A and δ_B , is stochastically stable, even being non ergodic.

Example 4.9 The $C^{1+\varepsilon}$ expanding maps $f: S^1 \mapsto S^1$ in the circle (i.e. $f'(x) > 1 \ \forall x \in S^1$), have been extensively studied, they present a unique SRB and physical measure attracting Lebesgue a.e. In [CQ01] it is shown that also in the C^1 topology, generically f has a unique physical measure. Nevertheless this physical measure is not SRB (it is not absolute continuous respect to Lebesgue) in those generic C^1 (not $C^{1+\varepsilon}$) examples. They show that, for C^1 uniformly hyperbolic maps, (that are besides topologically mixing), if there is a unique observable (and thus physical) probability, this measure is not necessarily SRB.

On the other hand, there exists C^1 examples of expanding maps for which a non ergodic invariant measure μ is equivalent to Lebesgue ([Q96]). Thus, there is not a unique observable measure. This shows that, out of the $C^{1+\varepsilon}$ case, when uniform hyperbolicity and topological mixing hold, the set of observable measures is not necessarily reduced to a single probability, even if the sequence (1) converges Lebesgue a.e.

The case of $f \in C^0$ stands in contrast with the former situation. In [M04] (see Theorem 3.4) is proved in particular that there exist maps $f: S^1 \rightarrow S^1$ topologically conjugate with $g_d: S^1 \rightarrow S^1$, $g_d(x) = dx$ such that for Lebesgue almost every point x in S^1 and every f -invariant measure μ , some subsequence of the sequence (1) converges to μ .

Discussion: When formulating the theorems about chains and co-chains of generalized attractors in Section 6, we have in mind the three paradigmatic different statistical

dynamical behaviors that C^0 systems may exhibit (in relation to the limit probabilities of the sequence (1)):

First, in Examples 4.4 Lebesgue a.e. orbit defines a convergent subsequence (1) of probabilities, but no physical measure exist (at least for a subset of initial states with positive Lebesgue measure).

Second, in Examples 4.8 Lebesgue a.e. orbit defines a non convergent subsequence (1).

Third, in Examples 4.9 (the C^2 or the C^1 generic expanding maps), 4.6 (the Feigenbaum attractor), 4.5 (infinitely many coexisting sinks or Hénon like attractors), 4.2 and 4.3 (C^2 conjugated to Anosov in the two-torus), Lebesgue almost all orbit defines a convergent subsequence (1) that converges to a physical measure, and there are at most countable many such physical measures.

To state the Theorems of Section 6, we are thinking in those different statistical dynamical behaviors, but not as dynamical systems that are topological isolated one from the others. Precisely, we are considering that the basins of the generalized attractors of all those examples may be topologically immersed in a larger dimension compact manifold $\widehat{M}$, in such a way that become mutually riddled (i.e. they are dense subsets of $\widehat{M}$).

5 Equilibrium states and observable measures for C^1 expanding maps.

We will develop the theorems in this section for order preserving expanding maps in the circle S^1 , but the proofs also work for order reversing expanding maps in S^1 . Some of the results also work for expanding maps, similarly defined, in appropriate manifolds of dimension larger than one.

Definition 5.1 A C^1 (order preserving) map $f : S^1 \mapsto S^1$ is expanding if $f'(x) > 1$ for all $x \in S^1$. (If f is order reversing then it is expanding if $-f'(x) > 1$.)

In particular, for all integer $d > 1$ we denote $g_d : S^1 \mapsto S^1$ to the linear expanding map

$$g_d(x) = dx \ \forall x \in S^1$$

If f is expanding, then there exist a unique integer $d > 1$, called the degree of f , such that f is topologically conjugated to g_d . Thus, all the topological properties of g_d are translated to f .

We denote $\mathcal{E}^1 \subset C^1(S^1)$ to the family of C^1 expanding maps in the circle S^1 . For all $r \geq 1$, we denote $\mathcal{E}^r \subset \mathcal{E}^1$ to the C^r expanding maps in S^1 . Denote $\text{Homeo}(S^1)$ to the set of homeomorphisms on the circle S^1 with the C^0 topology. Finally, for all $f \in \mathcal{E}^1$, denote $\text{Conj}(f) \subset \text{Homeo}(S^1)$ to the set of all the conjugacies between f and g_d .

For a seek of completeness we state here a Lemma from [CQ01]. For each $x \in S^1$ denote U_x to the C^1 open and dense set of all expanding maps f in S^1 such that $f(x) \neq x$.

Lemma 5.2 For each $x \in S^1$ there is a continuous map $\Pi_x: U_x \rightarrow \text{Homeo}(S^1)$ such that $\Pi_x(f) \in \text{Conj}(f)$ for each f . In particular, given $f \in \mathcal{E}^1$ of degree d , there is a neighborhood U of f on which there is a continuous choice of conjugacies to the map g_d .

Proof: See Lemma 1 of [CQ01] and Theorem 2.4.6. of [KH95].

Definition 5.3 Pressure and equilibrium states.

Let f a C^1 -expanding map in the circle. We denote $\psi = -\log f' \in C^0(S^1, \mathbb{R})$ and $\mathcal{P}_f$ to the set of f -invariant Borel probabilities in S^1 . For $\nu \in \mathcal{P}_f$ we denote $h_\nu(f)$ to the entropy of ν .

Let $\varphi \in C^0(S^1, \mathbb{R})$. The pressure of φ , is

$$P_f(\varphi) = \sup_{\nu \in \mathcal{P}_f} \left\{ h_\nu(f) + \int \varphi d\nu \right\}.$$

A measure $\mu \in \mathcal{P}_f$ is an *equilibrium state* for φ if

$$h_\mu(f) + \int \varphi d\mu = P_f(\varphi).$$

In particular $\mu \in \mathcal{P}_f$ is an *equilibrium state* for $\psi = -\log f'$ if

$$h_\mu(f) = \int \log f'(x) d\mu(x) + P_f(-\log f').$$

We denote as $ES_f \subset \mathcal{P}_f$ to the set of f -invariant probabilities that are equilibrium state for $\psi = -\log f'$.

For a seek of completeness we recall well known results in the following Theorem 5.4 and Corollary 5.5:

Theorem 5.4 Ruelle inequality and Pesin-Ledrappier-Young Equality for $C^{1+\alpha}$ expanding maps. *If $f \in \mathcal{E}^{1+\alpha}$ (i.e. f is a C^1 expanding map in the circle such that f' is $\alpha > 0$ -Hölder continuous), then there exists a unique f -invariant and ergodic measure μ that is the equilibrium state for $\psi = -\log f'$. Besides μ is the unique f -invariant measure that is absolute continuous respect to the Lebesgue measure m , and the pressure $P_f(\log f') = 0$. In other words, for all $\nu \in \mathcal{P}_f$ the inequality of Ruelle holds:*

$$h_\nu(f) \leq \int \log f'(x) d\nu.$$

Besides, this last is an equality if and only if $\nu \ll m$, and this holds if and only if $\nu = \mu$.

Proof: See Theorem 6.3.8 of [K98].

Corollary 5.5 Ruelle inequality for C^1 expanding maps. *If $f \in \mathcal{E}^1$ (i.e. f is a C^1 expanding map in the circle), then the pressure $P_f(\log f') = 0$. In other words, for all $\nu \in \mathcal{P}_f$ the inequality of Ruelle holds:*

$$h_\nu(f) \leq \int \log f'(x) d\nu.$$

Proof: We reproduce the proof of Lemma 2 in [CQ01]. After the PLY Equality of Theorem 5.4 it follows that

$$P_f(-\log f') = h_\mu(f) - \int \log f' d\mu = 0 \quad \text{if } f \in \mathcal{E}^2$$

for the unique f -invariant measure μ which is absolute continuous respect to the Lebesgue measure m .

If $f \in \mathcal{E}^1$ has degree d , after the lemma 5.2, there exists a neighborhood $U \subset \mathcal{E}^1$ of f , and for all $g \in U$, conjugacies $\gamma_g \in \text{Conj}(g)$, between $g \in U$ and the linear expanding map g_d of degree d , such that the application $g \rightarrow \gamma_g$ is continuous on U . Denote $\psi_g(x) = -\log g'(x)$ for all $g \in U$. Take $\{f_i\}_{i \in \mathbb{N}} \subset \mathcal{E}^2$ such that $f_i \rightarrow f \in \mathcal{E}^1$ (the convergence is in the C^1 topology). Then $\psi_{f_i} \circ \gamma_{f_i} \rightarrow \psi_f \circ \gamma_f \in C^0(S^1, \mathbb{R})$. Note that the pressure $p_g(\varphi)$, for any fixed $g \in \mathcal{E}^1$, depends continuously on $\varphi \in C^0(S^1, \mathbb{R})$. Then

$$P_{g_d}(\psi_{f_i} \circ \gamma_{f_i}) = P_{g_d}(\psi_f \circ \gamma_f).$$

Also note that, if $\gamma \in \text{Conj}(g)$ for some $g \in \mathbb{E}^1$ and if $\varphi \in C^0(S^1, \mathbb{R})$, then

$$P_{g_d}(\varphi \circ \gamma) = P_g(\varphi).$$

Indeed, γ induces a bijection γ^* between the g_d invariant measures $\nu \in \mathcal{P}_{g_d}$, and the g invariant measures $\gamma^*\nu \in \mathcal{P}_g$. Then $\int \varphi \circ \gamma d\nu = \int \varphi d\gamma^*\nu$ for all $\nu \in \mathcal{P}_{g_d}$. Besides, since γ^* is a measure-theoretic isomorphism, the entropies coincide $h_\nu(g_d) = h_{\gamma^*\nu}(g)$.

We conclude that

$$0 = P_{f_i}(\psi_{f_i}) = P_{g_d}(\psi_{f_i} \circ \gamma_{f_i}) = P_{g_d}(\psi_f \circ \gamma_f) = P_f(\psi_f). \quad \square$$

Remark 5.6 If f is a C^1 expanding map in the circle that is not $C^{1+\alpha}$, then (except the inequality of Ruelle), the thesis of Theorem 5.4 does not necessarily hold. In fact, the uniqueness of the equilibrium state of $-\log f'$, or its absolute continuity respect to Lebesgue, may fail, as in the following cases:

On one hand, the PLY equality may hold for a unique equilibrium state that is a physical measure but singular respect to Lebesgue ([CQ01]).

On the other hand, the PLY equality may hold for some invariant probability that is absolute continuous respect to Lebesgue, but not ergodic: see [Q96] and Theorem of Ledrappier (Theorem 2 in [W75]). Therefore its ergodic components are also equilibrium states of $-\log f'$.

The following theorem states that any observable measure is an equilibrium state. It is a stronger version of Theorem 6.1.8 of the book of Keller [K98]¹, but its proof is in essence the same.

Theorem 5.7 *Let f be a C^1 expanding map on the circle S^1 . Then, any observable measure of f and any convex combination of observable measures of f , is an equilibrium state for $\psi = -\log f'$, and its pressure is equal to 0, i.e. any observable measure μ of f satisfies the PLY equality for the entropy:*

$$h_\mu(f) = \int \log f' d\mu.$$

We prove Theorem 5.7 in the subsection 5.12. Let us now state its Corollaries. The first Corollary is a well known result. Nevertheless a new point of view for its proof rises from Theorems 5.7, 1.3 and 1.5.

Corollary 5.8 *A C^1 expanding map on the circle always has a non empty set ES_f of probability measures that verify the PLY equality of the entropy. Besides, f has a physical measure μ attracting Lebesgue a.e. if and only if the observable measure is unique, and this happens if there is a unique probability $\mu \in ES_f$.*

Proof: After Theorem 1.3: $\mathcal{O}_f \neq \emptyset$. After Theorem 5.7, the closed convex hull of $\mathcal{O}_f$ is contained in ES_f , so $ES_f \neq \emptyset$. Finally, from Theorem 1.5, f has a physical measure μ attracting Lebesgue a.e. if and only if $\mathcal{O}_f = \{\mu\}$ and this trivially holds if $ES_f = \{\mu\}$ $\square$

We say that a probability measure is atomic if it is supported on a finite set.

Corollary 5.9 *There is no atomic ergodic observable measure of a C^1 expanding map.*

Proof of the corollary 5.9: By contradiction, if μ is an atomic ergodic observable measure of an expanding map f , then $h_\mu(f) = 0$. As $\psi = -\log f' < 0$, and μ is an equilibrium state for ψ , then the pressure $P_f(\psi)$ is negative, contradicting Corollary 5.5. $\square$

In the following definitions and lemmas, we reproduce those of the book of Keller ([K98]) about the equilibrium states in its section §4.4, applied in particular to expansive C^1 maps f in the circle S^1 . Recall that if $f \in \mathcal{E}^1$, the set of equilibrium states of $\psi = -\log f'$ is $ES_f \subset \mathcal{P}_f$, i.e. $\mu \in ES_f$ if and only if the PLY equality of the entropy holds: $h_\mu(f) = \int \log f' d\mu$.

5.10 Notation: For all $\nu \in \mathcal{P}_f$ we denote

$$V_f(\nu) = h_\nu(f) - \int \log f'(x) d\nu(x)$$

Due to Ruelle inequality (Corollary 5.5):

$$V_f(\nu) \leq 0 \quad \forall \nu \in \mathcal{P}_f.$$

¹Our definition of observability is weaker than the definition in [K98]

For all $r \geq 0$ we denote

$$\mathcal{K}_r = \{\nu \in \mathcal{P}_f : V_f(\nu) \geq -r\}$$

In particular $\mathcal{K}_0 = ES_f$. For all $r \geq 0$ the set $\mathcal{K}_r$ is non empty, compact (in the weak* topology) and convex (join the proof of Theorem 4.2.3 of the book in [K98], with Theorem 4.2.4 and Remark 6.1.10 of the same book).

For all integer $n \geq 1$ and all $x \in S^1$ denote $\sigma_{n,x}$ to the (non necessarily f invariant) probability of the sequence (1), called the *empirical distribution of the future orbit of x up to time n* , i.e.:

$$\sigma_{n,x} = \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)}$$

where δ_x is the Dirac-delta probability supported on the point x .

As stated from the beginning of this paper, for $x \in S_1$ fixed, we denote

$$pw(x) \subset \mathcal{P}_f$$

to the set of all probabilities that are the weak*-partial limits (limits of the convergent subsequences) of the sequence $\{\sigma_{n,x}\}_{n \geq 1}$ of the empirical distributions of the future orbit of x .

Lemma 5.11 *Let f be a C^1 expanding map of the circle S^1 . Let $\mathcal{K}_r$ the compact set of f invariant probabilities defined in 5.10. For all $r \geq 0$ and for all open neighborhood $\mathcal{V}$ of $\mathcal{K}_r$ in the space $\mathcal{P}$ of all the (not necessarily f -invariant) probabilities, the following inequality holds:*

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \log m\{x \in S_1 : \sigma_{n,x} \notin \mathcal{V}\} \leq -r,$$

where m is the Lebesgue probability in the circle S^1 .

Proof: This Lemma is the Proposition 6.1.11 of the book of Keller [K98]. All the hypothesis of that proposition² hold in the case that f is a C^1 expanding map of the circle S^1 : see Remark 6.1.10 of [K98].

5.12 Proof of Theorem 5.7: As ES_f is convex, it is enough to prove that $\mathcal{O}_f \subset ES_f$. Consider, for any $r > 0$ the compact set $K_r \subset \mathcal{P}_f$ defined in 5.10. By definition, the intersection of the (decreasing with r) family $\{K_r\}_{r>0}$ is the non empty compact set

$$\mathcal{K}_0 = \bigcap_{n=1}^{\infty} \mathcal{K}_{\frac{1}{n}}$$

² The needed hypothesis are restated according to “Corrections to Equilibrium states in ergodic theory” in <http://www.mi.uni-erlangen.de/~keller/publications/equibook-corrections.pdf> published by the author of the book.

It is enough to prove that its basin of attraction

$$B(\mathcal{K}_0) = \{x \in S_1 : pw(x) \subset \mathcal{K}_0\}$$

has total Lebesgue measure. In fact, if we prove that $m(B(\mathcal{K}_0)) = 1$, then applying Theorem 1.5, $\mathcal{O}_f \subset \mathcal{K}_0 = ES_f$ as wanted.

Now, we reproduce the proof of the part a. of Theorem 6.1.8 of [K98]. Let $r > 0$. We fix $0 < \varepsilon < r$. After Lemma 5.11, for any weak* neighborhood $\mathcal{N}$ of $\mathcal{K}_r \subset \mathcal{P}$, there exists n_0 such that for any $n > n_0$: $m\{x : \sigma_{n,x} \in \mathcal{P} \setminus \mathcal{N}\} \leq e^{-n(r-\varepsilon)}$. This implies that $\sum_{n=1}^{\infty} m\{x : \sigma_{n,x} \in \mathcal{P} \setminus \mathcal{N}\} < +\infty$. After the Borel-Cantelli Lemma it follows that

$$m\left(\bigcap_{n_0=1}^{\infty} \bigcup_{n=n_0}^{\infty} \{x : \sigma_{n,x} \in \mathcal{P} \setminus \mathcal{N}\}\right) = 0.$$

In other words, for each open neighborhood $\mathcal{N}$ of $\mathcal{K}_r$, for m -a.e. $x \in S^1$ there exists $n_0 \geq 1$ such that $\varepsilon_{n,x} \in \mathcal{N}$ for all $n \geq n_0$. Hence, $pw(x) \subset \mathcal{K}_r$ for m -a.e. $x \in S^1$. It follows that $pw(x) \subset \bigcap_{1 \leq n \in \mathbb{N}} \mathcal{K}_{1/n} = \mathcal{K}_0 = \{\mu \in \mathcal{P}_f : V_f(\mu) = 0\}$ for m -a.e. $x \in S^1$ as wanted. $\square$

6 Cardinality and decomposition of $\mathcal{O}$.

The first aim of this section is to state some results that characterize the maps having physical measures whose basins attract Lebesgue a.e., in terms of the cardinality of the set of its observable measures. In particular we prove the part (b) of Theorem 1.3 and Theorem 2.5, that were delayed up to this section.

The second aim of this section, for a seek of completeness, is to cover in the theory, all the possible cases in $C^0(M)$, including those maps that are singular respect to Lebesgue. To do that we analyze the chains and co-chains of reductions of the space of observable measures, even if no physical measures exists, or if one or more than one physical measure exists, but the union of their basins of attraction do not cover Lebesgue a.e.

Theorem 6.1 Cardinality of $\mathcal{O}$ and physical measures.

Let $f: M \mapsto M$ be any continuous map in the compact manifold M .

If the set $\mathcal{O}$ of the observable measures for f , or some proper reduction $\mathcal{O}_1$ of $\mathcal{O}$, is finite or countable infinite then there exists physical measures μ for f , precisely of $f|_{B(\mathcal{O}_1)}$, where $B(\mathcal{O}_1)$ denotes the basin of attraction in M of $\mathcal{O}_1$.

Conversely, if there exists a physical measure μ for f then, either the space $\mathcal{O}$ has a single probability measure μ , or it is reducible and there exists a proper reduction $\mathcal{O}_2$ of $\mathcal{O}$ with a single element.

Proof: The converse assertion is immediate. In fact, if μ is physical then the basin of attraction of μ has positive Lebesgue measure, and thus $\{\mu\}$ is a trivial reduction of $\mathcal{O}$. It is either a proper reduction or not. If not then $\mathcal{O} = \{\mu\}$.

Let us prove the direct assertion. Denote extensively as $\{\mu_n : n \in N\}$ the finite or countable infinite reduction $\mathcal{O}_1$ (proper or not) that is given in the hypothesis. (If it has only a finite cardinality, then repeat one or more of its elements in the extensive notation, but include all of them at least once).

By Proposition 3.11 the space $\mathcal{O}_1$ is the set of all observable measures for the restriction $f|_C$ of f to some forward invariant set with positive Lebesgue measure, say $m(C) > 0$. Thus $C \subset B(\mathcal{O}_1)$ and it is not restrictive to assume that $C = B(\mathcal{O}_1)$.

Rename if necessary $f|_C$ as f , $\mathcal{O}_1$ as $\mathcal{O}$, and rename as m the Lebesgue measure in C , (i.e: the restriction to C of the Lebesgue measure in M , which is then renormalized to be a probability measure in C). Resuming:

$$\text{For every } x \in C : p\omega(x) \subset \mathcal{O} = \{\mu_n : n \in \mathbb{N}\} \quad (5)$$

Let us define $C_n \in C$ to be candidates of the basins of attraction for the measures μ_n , and relate their respective Lebesgue measures $m(C_n)$ as follows:

$$C_n = \{x \in C : \mu_n \in p\omega(x)\}; \quad C = \bigcup_{n=1}^{\infty} C_n; \quad \sum_{n=1}^{\infty} m(C_n) \geq m(C) = 1 \quad (6)$$

So $m(C_n) > 0$ for some $n \in \mathbb{N}$.

To end the proof we shall show that for all $x \in C_n : \{\mu_n\} = p\omega(x)$. Due to (5) and (6), it is enough to prove that $C_n \cap C_k = \emptyset$ if $\mu_n \neq \mu_k$.

By contradiction, suppose that for some $\mu_n \neq \mu_k$ there exists a point $x \in C_n \cap C_k \subset C$. Then, from (6) we have $\mu_n, \mu_k \in p\omega(x)$. Now we apply Theorem 3.3 and (5) to conclude that the space $\mathcal{O}$ is non-countably infinite. $\square$

6.2 Proof of Theorem 1.3 (b) It is straightforward consequence of Theorems 1.5 and 6.1. $\square$

Now, let us analyze in an abstract theory, the complementary case: the set of all observable measures is a non countable infinite compact subset of the space of invariant probabilities.

Definition 6.3 (Chains of reductions.) A *chain of reductions* of the space $\mathcal{O}$ of the observable measures for f is a (finite or countable infinite) sequence $\{\mathcal{O}_n\}_{n \in I \subset \mathbb{N}}$ of reductions or *generalized attractors* (see Definition 2.1) such that $\mathcal{O}_i \subsetneq \mathcal{O}_j$ if $i > j$ in the set I of natural indexes.

We call *length of the chain* to its finite or countable infinite cardinality $\#I$.

Recall that, by definition of reductions (i.e. generalized attractors) each $\mathcal{O}_n$ is a compact part of the set of the observable measures.

For any chain $\{\mathcal{O}_n\}_{n \in I \subset \mathbb{N}}$ of reductions of the space of observable measures for f , let

$$d_n = \text{diam}(\mathcal{O}_n); \quad s_n = \text{attrSize}(\mathcal{O}_n)$$

where diam and attrSize denote respectively the diameter and the attracting size, defined in 3.9.

Observe that d_n and s_n are non negative decreasing sequences. We denote $\underline{d} = \lim d_n$ and $\underline{s} = \lim s_n$.

Theorem 6.4 (Physical measures and chains.) *There exists a physical measure for f if and only if there exists a chain $\mathcal{O}_n$ of reductions of the space $\mathcal{O}$ of the observable measures of f , such that the sequence of its diameters converges to zero and the sequence of its attracting sizes converges to some $\alpha > 0$.*

Proof: The converse statement is immediate defining the length 1-chain $\{\{\mu\}\}$, where μ is the given physical measure. The direct result is also immediate if the length of the given chain is finite. Let us see now the case when the chain $\mathcal{O}_n$; $n \in \mathbb{N}$ is infinite. As the sequence of its diameters converges to zero, then $\cap_{n \in \mathbb{N}} \mathcal{O}_n = \{\mu\}$ for some μ . It is enough to show that the attracting size of μ is positive. Note that from the construction of such μ we have that $C(\{\mu\}) = \cap_{n \in \mathbb{N}} C_n$ where C_n denotes the basins of attractions $C(\mathcal{O}_n)$. These basins are a countably infinite decreasing family of sets in M with positive Lebesgue measures s_n . Therefore, $\text{attrSize}(\{\mu\}) = m(C(\{\mu\})) = \lim s_n = \alpha > 0$, as wanted. $\square$

Definition 6.5 (Independence of generalized attractors and chains.) We say that two generalized attractors or reductions of the space of observable measures *are independent* if the basin of attraction of their intersection has zero Lebesgue measure.

We note from Definition 1.4 that the basin of attractions of two reductions $\mathcal{O}_1$ and $\mathcal{O}_2$ intersect exactly in the basin of attraction of $\mathcal{O}_1 \cap \mathcal{O}_2$. Therefore:

Two ergodic attractors are independent if and only if the intersection of their basins has zero Lebesgue measure.

We say that *two chains of reductions are independent* if each one of the chains has a reduction that is independent with *some* reduction of the other chain.

Definition 6.6 (Co-chains of reductions.) A *co-chain* of reductions of the space $\mathcal{O}$ of the observable measures for f is a (finite or countable infinite) family $\{\mathcal{O}_n$; $n \in I \subset \mathbb{N}\}$ of reductions or *generalized attractors* (see Definition 2.1) *that are pairwise independent*.

We call *length of the co-chain* to its finite or countable infinite cardinality $\#I$.

Remark: If the space $\mathcal{O}$ of all the observable measures for f is irreducible, then $\{\mathcal{O}\}$ is the unique chain of reductions and also the unique co-chain.

Now we state a slightly generalized version of a known result in the theory of Discrete Mathematics, the Theorem of Dilworth ([L85]), applied to the chains and co-chains of generalized attractors:

Theorem 6.7 (Reformulation of Dilworth Theorem.) *For any continuous map f the supreme k of the lengths of the co-chains of reductions in the space of observable measures for f , is equal to the supreme h of the number of pairwise independent chains.*

Moreover: for any co-chain of length l there is a family of l pairwise independent chains, and conversely.

Proof: Any co-chain $\{\mathcal{O}_j, j \in J\}$ with length $l = \#J$ can be seen as a collection $\{\{\mathcal{O}_j\}, j \in J\}$ of l pairwise independent chains $P_j = \{\mathcal{O}_j\}$, each chain P_j with length one. So $k \leq h$.

Conversely, given any collection $\{P_j, j \in J\}$ of pairwise independent chains, take a reduction $\mathcal{O}_1 \in P_1$ independent to some $\hat{\mathcal{O}}_2 \in P_2$, and take $\check{\mathcal{O}}_2 \in P_2$ independent with $\hat{\mathcal{O}}_3 \in P_3$. As both reductions $\hat{\mathcal{O}}_2$ and $\check{\mathcal{O}}_2$ belong to the chain P_2 , one of them must be contained in the other; thus their intersection, say $\mathcal{O}_2$, is also a reduction of the chain P_2 . Besides $\mathcal{O}_2$ is independent with $\mathcal{O}_1 \in P_1$ and with $\hat{\mathcal{O}}_3 \in P_3$. Analogously construct by induction a (finite or infinite) sequence $\{\mathcal{O}_j : j \in J\}$ of pairwise independent reductions, such that $\mathcal{O}_j \in P_j$. This sequence of reductions is by definition a co-chain. Therefore, $h \leq k$. $\square$

When no physical measure exist, or when some of them exist but their basins do not cover Lebesgue almost every point of the phase space M , we will still state a equivalent condition for the space being partitioned in (up to countably many) irreducible generalized attractors, whose basins cover Lebesgue all orbit.

Theorem 6.8 (Co-Chains and irreducible attractors.)

1. A map $f: M \mapsto M$ has (up to countable infinitely many) irreducible attractors whose basins of attraction cover Lebesgue almost all point in M if and only if there exist a co-chain of reductions of the space $\mathcal{O}$ of the observable measures for f such that the (finite or countable infinite) sum of its attracting sizes is 1.

In this case:

2. The irreducible attractors are all physical measures if and only if the diameter of the reductions are all zero.
3. The (finite or countable infinite) number of such irreducible generalized attractors is equal to the supreme k of the lengths of the co-chains of reductions for f and to the supreme h of the number of independent chains for f .

Proof: From Definition 3.9 we obtain that f has generalized attractors whose basins cover Lebesgue almost all points, if and only if the following statement holds:

- i. There exist (up to countable many) $\mathcal{O}_n \in \mathcal{O}$ such that $s_n = \text{attrSize}(\mathcal{O}_n) > 0$ and $\sum s_n = 1$.

Note that two different trivial reductions of $\mathcal{O}$ are always mutually independent. Therefore, i is equivalent to the following:

- ii. The family $\mathcal{O}_n$ is a co-chain of trivial reductions such that $\sum s_n = 1$.

So the first assertion of Theorem 6.8 is proved.

The second assertion is trivial from the characterization of physical measures as those reductions of the space of observable measures that have zero diameter.

To prove the third assertion observe that each reduction of any co-chain must contain at least one of the generalized attractors $\mathcal{O}_n$ because $\sum s_n = 1$, and that two different reductions of the same co-chain can not contain any common reduction, because they must be independent. Then k is smaller than or equal to the number of independent attractors. On the other hand, the set of all the independent reductions form itself a co-chain, so k is greater than or equal to the number of independent attractors. Finally apply Theorem 6.7 to show that the number of irreducible independent attractors is also equal to h . $\square$

Corollary 6.9 *A map $f: M \mapsto M$ has (up to countable infinitely many) physical measures whose basins of attraction cover Lebesgue almost all point in M , if and only if there exist a (finite or infinite) family*

$$\{\mathcal{O}_i^j, i \in I \subset \mathbb{N}, j \in J \subset \mathbb{N}\}$$

of generalized attractors $\mathcal{O}_i^j$ for f such that for all $i \in I$ and $j, k \in J, j \neq k$:

$$\mathcal{O}_{i+1}^j \subset \mathcal{O}_i^j, \quad \lim_i d_i^j = 0 \quad \lim_i s_i^{j,k} = 0 \quad \text{and} \quad \lim_i \sum_{j \in J} s_i^j \geq 1$$

where d_i^j and s_i^j denote respectively the diameter and attracting size of $\mathcal{O}_i^j$ and $s_i^{j,k}$ is the Lebesgue measure of the basin of attraction of $\mathcal{O}_i^j \cap \mathcal{O}_i^k$.

Proof: If there exist such physical measures μ_j for $j \in J \subset N$, simply define the family $\mathcal{O}_i^j = \{\mu_j\}$ for $i = 1$ and $j \in J$. This family of ergodic attractors verify all stated conditions.

To prove the converse statement let us first apply Theorem 6.6 to the chains $\{\mathcal{O}_i, i \in \hat{I} \subset I\}$, for each fixed $j \in J$ such that $\lim_i s_i^j > 0$, (while $\lim_i d_i^j = 0$). Then each of such chains has a intersection $\{\mu_j\}$ where μ_j is a physical measure.

For $j \neq k$ the basins of attraction of μ_j and μ_k are Lebesgue almost disjoint, because its Lebesgue measure is $\lim s_i^{j,k} = 0$. Thus $\mu_j \neq \mu_k$. Finally consider the co-chain $\{\{\mu_j\}, j \in J\}$ and apply Theorem 6.8. $\square$

6.10 Proof of Theorem 2.5: Decomposition in independent generalized attractors. We will prove Theorem 2.5 in the following version 6.11, that gives an upper bound to the number of generalized independent ergodic attractors in which the space $\mathcal{P}$ can be decomposed, and besides states a sufficient condition for the decomposition be unique. From the following Theorem 6.11, it follows also the last assertion of Theorem 2.5, using that the space $\mathcal{O}$ of all observable measures is compact, and thus, for all $\varepsilon > 0$ it can be covered by a finite number of weak* balls of size $\varepsilon > 0$.

Theorem 6.11 (Reformulation of Theorem 2.5.)

Any continuous map $f: M \mapsto M$ has a collection S formed by (up to countable many) pairwise independent generalized attractors (that are not necessarily irreducible) whose basins of attraction cover Lebesgue almost all points in M .

The supreme a of the number of such generalized attractors verifies $a \leq k$ (where k , may be infinite, is the supreme of the lengths of the co-chains of reductions in the space of the observable measures for f).

If there exists such a collection S whose generalized attractors are besides all irreducible, then such S is unique and besides $a = k = l$, where l is the cardinality of S .

Proof: To prove the first and second statements note that, by definition of the independence of the reductions, any co-chain P of reductions of the space $\mathcal{O}$ of the observable measures, verifies $\sum s_j \leq 1$, where s_j denotes the attracting size of the reduction $\mathcal{O}_j \in P$. Now take the family $\mathcal{F}$ of all the co-chains S such that $\sum s_j = 1$. (There exists always at least one such co-chain: in fact the length-1 co-chain $\{\mathcal{O}\}$ verifies $\sum s_j = s(\mathcal{O}) = 1$, due to the results of Theorem 3.8). By construction each $S \in \mathcal{F}$ verifies the wanted conditions. As the family $\mathcal{F}$ is a subfamily of all the co-chains of reductions, we obtain $a \leq k$.

Let us prove now the last assertions of the theorem. If there exists in $\mathcal{F}$ a co-chain $S = \{\hat{\mathcal{O}}_h : h \in H \subset \mathbb{N}\}$ with cardinality $l = \#H$ and whose reductions $\hat{\mathcal{O}}_h$ are all irreducible, to prove that $a = k = l$ it is enough to show that $l \geq k$.

Take any co-chain $P = \{\mathcal{O}_j, j \in J \subset \mathbb{N}\}$ (P is not necessarily in $\mathcal{F}$).

It is enough to exhibit an injective application from each $j \in J$ to some $h \in H$.

In fact, let us fix any $j \in J$ and consider the basin of attraction $C(\mathcal{O}_j)$. By definition of reduction, this basin has positive Lebesgue measure $m(C(\mathcal{O}_j))$. But $S \in \mathcal{F}$, so $\sum_{h \in H} m(C(\hat{\mathcal{O}}_h)) = 1$. Then we deduce that

$$0 < m(C(\mathcal{O}_j)) = \sum_{h \in H} m\left(C(\mathcal{O}_j) \cap C(\hat{\mathcal{O}}_h)\right) = \sum_{h \in H} m\left(C(\mathcal{O}_j \cap \hat{\mathcal{O}}_h)\right)$$

Therefore some of the intersections in the sum above at right has positive Lebesgue measure. We obtain that for all $j \in J$ there exist some $h = h(j) \in H$ such that $\mathcal{O}_j \cap \hat{\mathcal{O}}_h$ is a reduction. But as $\hat{\mathcal{O}}_h$ is irreducible then $\mathcal{O}_j \supset \hat{\mathcal{O}}_h$.

To end the proof it remains to show that for $j \neq i \in J$ the sets $\hat{\mathcal{O}}_{h(i)}$ and $\hat{\mathcal{O}}_{h(j)}$ in S are different reductions. By contradiction, if they were the same reduction in S , they both would be contained in two different reductions $\mathcal{O}_i$ and $\mathcal{O}_j$ in the chain P and therefore these two last reductions would not be independent and P would not be a co-chain.

Let us prove now the unicity of S , if there exists one, such that $S = \{\hat{\mathcal{O}}_h : h \in H \subset \mathbb{N}\} \in \mathcal{F}$ and $\mathcal{O}_h$ are all irreducible. If there were two such collections S_1 and S_2 , then repeating the construction above in this proof with $S = S_1$ and $P = S_2$ we conclude any $\mathcal{O}_j \in S_2$ contains one and only one $\hat{\mathcal{O}}_{h(j)} \in S_1$. But as all $\mathcal{O}_j \in S_2$ are irreducible we must have $\hat{\mathcal{O}}_{h(j)} = \mathcal{O}_j \in S_1$. So $S_2 \subset S_1$. The symmetric relation is obtained taking $S = S_2$ and $P = S_1$. $\square$

7 Appendix.

7.1 Topological attraction in mean of the generalized attractors.

Let $f \in C^0(M)$. Recall Definition 2.1 of Generalized Attractor. Let $(A, \mathcal{A}) \subset M \times \mathcal{O}$ be a generalized Attractor, and let $B \subset A$ be its basin of attraction, which by definition

has positive Lebesgue measure $m(B) > 0$. Recall that $A \subset M$ is the minimum compact subset in M that contains the support of all the observable measures $\mu \in \mathcal{A}$. In the next statement we will call $A \subset M$ as the attractor.

Proposition 7.2 For all $\varepsilon > 0$ there exists $N \geq 1$ and a subset $C \subset B$ of the basin of attraction B of the generalized attractor A such that $m(B \setminus C) < \varepsilon m(B)$, and for all $x \in C$ and for all $n \geq N$, more than $(1 - \varepsilon)100\%$ of the iterates of the finite piece $\{f^j(x)\}_{0 \leq j \leq n-1}$ of the future orbit of x , lay in the ε -neighborhood of the attractor A .

Proof: The attractor A is compact. Call $V \supset A$ to the ε -neighborhood of A . Construct a continuous function $\phi: M \mapsto [0, 1]$ such that $\phi(x) = 1$ for all $x \in A$ and $\phi(x) = 0$ for all $x \notin V$. By definition of the generalized attractor, for all $x \in B$ the convergent subsequences of (1) converge to a probability supported in A . So $\frac{1}{n} \sum_{j=0}^{n-1} \phi(f^j(x))$ has all its convergent subsequences converging to 1. Then:

$$B \subset \bigcup_{N \geq 1} \bigcap_{n \geq N} \{x \in M : \frac{1}{n} \sum_{j=0}^{n-1} \phi(f^j(x)) > 1 - \varepsilon\}$$

$$m(B) \leq \lim_{N \rightarrow +\infty} m\left(\bigcap_{n \geq N} \{x \in M : \frac{1}{n} \sum_{j=0}^{n-1} \phi(f^j(x)) > 1 - \varepsilon\}\right)$$

Therefore, there exists $N \geq 1$ such that $m(B \setminus C) > 1 - \varepsilon$ where:

$$C = \bigcap_{n \geq N} \{x \in B : \frac{1}{n} \sum_{j=0}^{n-1} \phi(f^j(x)) > 1 - \varepsilon\}$$

Call χ_V to the characteristic function of the open set V . By construction $0 \leq \chi_V \leq \phi$. Then, for all $n \geq N$, and for all $x \in C$:

$$\frac{\#\{0 \leq j \leq n-1 : f^j(x) \in V\}}{n} = \frac{1}{n} \sum_{j=0}^{n-1} \chi(f^j(x)) \geq \frac{1}{n} \sum_{j=0}^{n-1} \phi(f^j(x)) > 1 - \varepsilon \quad \square$$

7.3 Proof of Theorem 3.3. We shall prove the following:

- i. If $\mu, \nu \in p\omega(x)$ then for each real number $0 \leq \lambda \leq 1$ there exists a measure $\mu_\lambda \in p\omega(x)$ such that

$$\text{dist}(\mu_\lambda, \mu) = \lambda \text{dist}(\mu, \nu)$$

- ii. The set $p\omega(x)$ either has a single element or non-countable infinitely many.

Proof: First let us deduce ii from i: Suppose that $p\omega(x)$ has at least two different values μ and ν . It is enough to note that the application $\lambda \in [0, 1] \mapsto \mu_\lambda \in p\omega(x)$ that verifies thesis i, is injective. Therefore $p\omega(x)$ has non-countable infinitely many elements.

To prove i consider the sequence (1) $\mu_n = \left\{ \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)} \right\}_{n \in \mathbb{N}}$ of time averages. Either it is convergent, or has at least two convergent subsequences, say $\mu_{m_j} \rightarrow \mu$ and $\mu_{n_j} \rightarrow \nu$, with $\mu \neq \nu$. It is enough to exhibit in the case $\mu \neq \nu$ a convergent subsequence of (1) whose limit μ_λ verifies the thesis i.

Assertion A: *For any given $\varepsilon > 0$ and $K > 0$ there exists a natural number $h = h(\varepsilon, K) > K$ such that*

$$|\text{dist}(\mu_h, \mu) - \lambda \text{dist}(\nu, \mu)| \leq \varepsilon$$

Let us first prove that Assertion A implies thesis i: Take in assertion A: $h_0 = 1$ and by induction, for $j \geq 1$ take h_j given $\varepsilon_j = 1/j$ and $K_j = h_{j-1}$. Then we obtain a sequence μ_{h_j} , subsequence of (1), that verifies $\text{dist}(\mu_{h_j}, \mu) \rightarrow \lambda(\text{dist}(\nu, \mu))$. Any convergent subsequence of μ_{h_j} (that do exist $\mathcal{P}$ is compact in the weak* topology) verify i.

Now, let us prove Assertion A:

As $\mu_{m_j} \rightarrow \mu$ and $\mu_{n_j} \rightarrow \nu$ let us choose first m_j and then n_j such that

$$m_j > K; \quad \frac{1}{m_j} < \varepsilon/4; \quad \text{dist}(\mu, \mu_{m_j}) < \varepsilon/4; \quad n_j > m_j; \quad \text{dist}(\nu, \mu_{n_j}) < \varepsilon/4$$

To exhibit the computations let us explicit some metric structure giving the weak* topology of $\mathcal{P}$. We will use for instance the following distance:

$$\text{dist}(\rho, \delta) = \sum_{i=1}^{\infty} \frac{1}{2^i} \left| \int g_i d\rho - \int g_i d\delta \right|$$

for any $\rho, \delta \in \mathcal{P}$, where $\{g_i\}_{i \in \mathbb{N}}$ is a countable set of functions $g_i \in C(M)$ such that $|g_i| \leq 1$, dense in the unitary ball of $C(M)$.

Note from the sequence (1) that $|\int g d\mu_n - \int g d\mu_{n+1}| \leq (1/n)||g||$ for all $g \in C(M)$ and all $n \geq 1$. Then in particular for $n = m_j + k$, we obtain

$$\text{dist}(\mu_{m_j+k}, \mu_{m_j+k+1}) \leq \frac{1}{m_j} < \varepsilon/4 \quad \text{for all } k \geq 0 \quad (7)$$

Now let us choose a natural number $0 \leq k \leq n_j - m_j$ such that

$$|\text{dist}(\mu_{m_j}, \mu_{m_j+k}) - \lambda \text{dist}(\mu_{m_j}, \mu_{n_j})| < \varepsilon/4 \quad \text{for the given } \lambda \in [0, 1]$$

Such k does exist because inequality (7) is verified for all $k \geq 0$ and besides:

- If $k = 0$ then $\text{dist}(\mu_{m_j}, \mu_{m_j+k}) = 0$ and
- if $k = n_j - m_j$ then $\text{dist}(\mu_{m_j}, \mu_{m_j+k}) = \text{dist}(\mu_{m_j}, \mu_{n_j})$

Now renaming $h = m_j + k$, joining all the inequalities above, and applying the triangular property, we deduce:

$$\begin{aligned} |\text{dist}(\mu_h, \mu) - \lambda \text{dist}(\nu, \mu)| &\leq |\text{dist}(\mu_h, \mu_{m_j}) - \lambda \text{dist}(\mu_{m_j}, \mu_{n_j})| + \\ &\quad + |\text{dist}(\mu_h, \mu) - \text{dist}(\mu_h, \mu_{m_j})| + \\ &\quad + \lambda |\text{dist}(\mu_{m_j}, \mu_{n_j}) - \text{dist}(\mu_{m_j}, \nu)| + \lambda |\text{dist}(\mu_{m_j}, \nu) - \text{dist}(\mu, \nu)| < \varepsilon \quad \square \end{aligned}$$

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