Dynamics of large cooperative pulsed-coupled networks

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Abstract

We study the deterministic dynamics of networks $\mathcal{N}$ composed by m non identical, mutually pulse-coupled cells. We assume weighted, asymmetric and positive (cooperative) interactions among the cells, and arbitrarily large values of m. We consider two cases of the network's graph: the complete graph, and the existence of a large core (i.e. a large complete subgraph). First, we prove that the system periodically eventually synchronizes with a natural "spiking period" $p \geq 1$, and that if the cells are mutually structurally identical or similar, then the synchronization is complete ($p = 1$). Second, we prove that the amount of information H that $\mathcal{N}$ generates or processes, equals $\log p$. Therefore, if $\mathcal{N}$ completely synchronizes, the information is null. Finally, we prove that $\mathcal{N}$ protects the cells from their risk of death.

1 Introduction

The theory of deterministic dynamical systems composed by two or more coupled dynamical units assumes that each unit - which we call *cell* - has a proper own dynamics, and that the couplings among the units are *interactions* that depend on the instantaneous states of the cells [33, 7].

Among the systems of interacting units, we focus on those that are pulsed-coupled (i.e. the interactions are instantaneous). In particular, the global system can be understood as a multi-dimensional differential or difference equation with impulsive terms [18].

On the one hand, the theory of dynamically interacting units is a source of mathematical open questions [33]. In particular, those systems governed by impulsive differential equations [28] pose mathematical problems that are mostly open, except in particular cases or low dimensions.

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On the other hand, the network of interacting dynamical units is a particular model of a coalitional game, that evolves or changes on time. We will discuss this relation in Subsection 1.2.

The dynamical systems composed by mutually interacting units, and in particular those that are pulsed-coupled, have relevance in many applications. As examples in Physics, a two-dimensional impulsive differential equation models the joint dynamics of two or more coupled oscillators [27]. In particular, they are used in applications to Light-Controlled-Oscillators (LCO), [25]. The mathematical investigation of a genetic regulatory network of two antagonist genes, is also modeled as a network of pulse-coupled units [9]. In Neuroscience, among the theoretical methods of research, the mathematical analysis of the dynamics of pulse-coupled networks is applied [12, 16]. In Engineering, networks of coupled dynamical units are designed for control systems and communications [32]. Computational research on artificial intelligence, by means of artificial neuronal networks, is used to analyze, simulate, and investigate on data obtained from dynamical systems of interacting units with a large degree of complexity [8]. In Economics, networks of coupled units are used to investigate the equilibrium states in social systems of interacting agents [1]. Artificial neuronal networks are applied for the prediction of the exchange market [21], also to investigate on financial markets [17], and for the accounting of financial applications [15]. In Ecology, networks of interacting units model the dynamics of predator-prey communities of two or more species [30, 13]. The dynamics of an infectious disease, taking into account the interaction among populations of diverse infectious agents, is modelled as a neural network [26]. In Geosciences, the forecast of ozone peaks in weather prediction uses computational methods on artificial neuronal networks [10]. In Social Sciences, the dynamics of large WWW social networks is mathematically modelled by the interactions of their individuals [22, 31]; and the self- synchronization of many small clusters of cells in a low-dimensional network models the dynamics of a medieval social network [3].

1.1 The object and method of research

Along this paper we investigate, by exact mathematical analysis and deductive proofs, the global dynamics of certain pulse-coupled deterministic networks of m cooperative cells, for large values of m.

Each cell $i \in \{1, 2, \ldots, m\}$ is governed by a deterministic dynamical sub-system, which - if i were hypothetically isolated from the network - we call *the free dynamics* of i. Besides, each cell i *acts on* the other cells $j \neq i$ of the network at certain instants t_i, which we call the *spiking or milestone instants* of i. Conversely, each cell i *receives the actions from* the other cells $j \neq i$ at the spiking instants t_j of j.

The free dynamics of each cell evolves governed by a finite-dimensional ordinary *differential equation,* joint with an autonomous and instantaneous *reset or update rule.* The update rule applies when the state of i arrives to a pursued *goal or threshold level* θ_i. The update rule resets the state of i, or equivalently, it changes the velocity according to which the free dynamics evolves. The free dynamics of the many cells of the network may be mutually different. As a particular case, in Neuroscience the model of the free dynamics of each cell - by integration of a differential equation plus a reset or update rule - is called

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integrate and fire. Each reset or update event of the cell (the neuron) is called *a spike.* In brief, the spike of a cell i is produced when its state arrives to the goal θ_i.

The cells compose the network by *mutual interactions* between any ordered pair (i, j) such that $1 \leq i, j \leq m$ and $i \neq j$. These interactions exist in both directions (some interactions may be zero), but are neither necessarily symmetric nor simultaneous. Roughly speaking, the action from the cell i to the cell $j \neq i$ is a discontinuity jump $\Delta_{i,j}$ - applied on the state of j - that is produced at the instant t_i when i spikes. So, the instant t_i depends on the state of i. The rule is the same to define the action from the cell j to the cell i, but the matrix $(\Delta)_{i,j}$ is not necessarily symmetric. Besides, the instant t_i when the action $\Delta_{i,j}$ is applied, is in general different from the instant t_j when the action $\Delta_{j,i}$ is applied.

Our purpose of research is to find qualitative and quantitative relevant characteristics of the global dynamics of such an abstract network, while time t evolves to the future. As said above, the methodology is to find the exact abstract mathematical statements and their deductive proofs.

We take the main ideas from [4], in which a model of a network, composed by integrate and fire biological neurons, is studied by the exact mathematical method.

The main differences between the dynamical system that we study here and the one studied in [4], are the following: First, we assume that the cells are cooperative (which in Neuroscience are called excitatory). This means that $\Delta_{i,j} \geq 0$ for all ordered pairs (i, j) such that $i \neq j$, and the value zero is admitted in some of our results. In [4], any sign of $\Delta_{i,j}$ is admitted by hypothesis, but only nonzero values are assumed. Second, we do not assume that the free dynamics is the same for all the cells. In [4] all the cells are identical. Third, we neither assume the linearity of the differential equation that governs the free dynamics of each cell, nor the existence of a Lyapunov stable equilibrium state for the solution flow of this differential equation. In [4] these latter two conditions are assumed.

1.2   The network as a cooperative game that evolves on time

The mathematical model of the network $\mathcal{N}$ that we study in this paper is also an evolutive game represented by a graph whose m vertices are the players $i \in \{1, \dots, m\}$ (the cells), and whose edges $\Delta_{i,j}$ (the interactions) are directed and weighted. The hypothesis of cooperativity among the individuals, namely $\Delta_{i,j} \geq 0$ for all $i \neq j$ makes the game work in an *imitate the best* strategy, which produce players that adopt a myopic behaviour ([14]). In fact, by hypothesis, each cell or player i just knows its own actions to the other players $j \neq i$, the value of its own satisfaction variable, and the actions it receives from the other cells. But i ignores the global state and dynamical behaviour of the whole network.

The model of the network $\mathcal{N}$, as described in Subsection 1.1 from the dynamical viewpoint, is a cooperative or coalitional game, that changes on time. It disregards the individual strategies of its cells (or players) and instead, it focusses on the coalitions (which we call clusters of cells), defined as nonempty subsets of cells $i \in \{1, \dots, m\}$ whose satisfaction variables S_i arrive simultaneously to their respective goals or threshold levels $\theta_i > 0$.

The characteristic function $\nu : 2^m \mapsto \mathbb{R}^+$ of the coalitional game (i.e. the function

assigning the total gain or payment $\nu(A)$ to each coalition $A \subset \{1, \ldots, m\}$, with the agreement $\nu(\emptyset) = 0$) can be defined as the sum of the goal levels θ_i of the cells $i \in A$. In fact, at each instant t_n for which a nonempty coalition I_n is formed (i.e. a cluster of simultaneously spiking cells is exhibited), the satisfaction variable of each cells $i \in I_n$ equals its respective goal or threshold level θ_i. Since $\theta_i > 0$ for all $i \in \mathcal{N}$, the coalitional game is *convex*, i.e.

$$\nu(A \cup B) + \nu(A \cap B) \geq \nu(A) + \nu(B) \quad \forall\, A, B \subset \mathcal{N} \tag{1}$$

We note that, since in our case $\nu(A) = \sum_{i \in \mathcal{A}} \theta_i$, inequality (1) is indeed an equality. Thus:

$$\nu(A \cup \{i\}) - \nu(A) \leq \nu(B \cup \{i\}) - \nu(B) \quad \forall\, A \subset B \subset \mathcal{N} \setminus \{i\}, \quad \forall\, i \in \mathcal{N}.$$

In other words, the gain for a player i for joining a coalition B larger than A, is non negative.

As for any convex coalitional game, $\mathcal{N}$ has a nonempty core of solutions. In Game Theory, a solution in the core is a vector

$$(z_1, z_2, \ldots, z_i, \ldots, z_m) \in \mathbb{R}^m,$$

which is called "allocation" or "payoff vector", such that

$$\sum_{i \in I} z_i \geq \nu(I) \; \forall\, I \subset \mathcal{N}. \tag{2}$$

For our model, the payoff vector at the n-th. instant t_n when at least one cell arrives to its goal level and spikes, can be defined by the following formula:

$$z_i = S_i(t_n^-) + \sum_{j \neq i,\; j \in I_n} \Delta_{j,i},$$

where $S_i(t_n^-) \in [0, \theta_i]$ is the value of the satisfaction variable S_i of the cell i just before instant t_n (cf. Definition 2.1), I_n is the cluster at the spiking instant t_n (cf. Definition 2.7) and $\Delta_{j,i} \geq 0$ is the cooperative action from the cell $j \in I_n$ to $i \neq j$ at any instant for which j spikes. In Equation (10) we will precisely state the rule according to which the cooperative interactions $\Delta_{j,i}$ among the cells of the network increase their satisfaction variables. In brief, the cell i spikes at instant t_n, hence it belongs to the cluster I_n, if and only if one of the following conditions is satisfied: either the satisfaction variable S_i arrives to the goal value θ_i spontaneously (due to the free dynamics of i) at instant t_n^-, or the satisfaction variable at instant t_n^- is smaller than θ_i but suddenly increases to become larger or equal than θ_i, by the adding of some positive cooperative actions $\Delta_{j,i}$ from cells $j \neq i$ that spike at the same instant t_n. In any case, the component z_i of the payoff vector is $z_i \geq \theta_i$ if $z_i \in I_n$. Since $\nu(I_n) = \sum_{i \in I_n} \theta_i$ we deduce that Inequality (2) holds in the coalitional game $\mathcal{N}$.

Although the characteristic function ν is assumed to be invariant with time t, the payoff vector changes with t, because the satisfaction variables evolve on time, and the interactions among the cells are not constantly applied at any time. In fact $\Delta_{j,i}$ is effectively added to the satisfaction variable S_i just at the instants for which the cell j spikes. So $\mathcal{N}$ is an evolutive coalitional game: its solutions depend on time.

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In Definition 3.1 we define the global synchronization phenomenon of the network $\mathcal{N}$ as the recurrent exhibition - infinitely many times in the future - of the so called "grand coalition". Precisely, all the cells of the network compose a single cluster: all of them spike simultaneously, and this phenomenon occurs infinitely many times. In Theorem 3.5 we prove that, if the number m of players is large enough in relation with the minimum positive interaction (namely, m is large compared with the minimum payoff component at any instant), then the global synchronization occurs recurrently. Nevertheless, between two instants $t_2^* > t_1^*$ when the grand coalition is exhibited, there may appear many coalitions or clusters that are smaller than the grand coalition. In Theorem 3.20 we prove that if more different coalitions recurrently appear in the future, then the amount of information of the global dynamics of the network increases.

In spite that the network can be studied as a coalitional evolutive game, it has some important differences from the models that are usually investigated in Game Theory, even from those that focus on evolutive games (e.g. [14, 1, 19]): For instance, in our model, each cell or player i decides *the instants when it acts* in the network, by integrating the states of its own internal dynamics with the actions (or payoffs) that it had received from the other players. So, there does not exist a forcing external mechanism making the players interact sharing a resource. As a counterpart, neither the cell nor the coalition, has an alternative to decide how to choose among several strategies, its actions on the network. The cells and the coalitions are not optimizers. The network is just a deterministic dynamical system whose state suffers instantaneous changes according to the states of its cells and, as a result of the positive interactions among the cells, it shows patterns of coalitions that spike simultaneously infinitely many times in the future.

1.3   The hypothesis

We adopt three types of hypothesis. First, we assume that the number of cells - in a part or the whole network - is large enough with respect to certain other parameters of the network (Inequality (14) of Theorem 3.5 and a similar inequality in Corollary 3.24). Second, we study the cases for which the cells are cooperative (cf. Definition 2.2). Third, we assume that the network's graph is complete, or, if not, that there exists a large *cooperative core* (i.e., a complete subgraph of cells which are fully cooperative, cf. Definition 3.23).

1.4   The results to be proved

Under the hypothesis above described, we prove three results: the periodic eventual synchronization, the positiveness of the protection factor of the network against the risk of death of its cells, and the value of the total amount of information that the network is able to generate or process.

1.4.1   Results about synchronization

In Theorems 3.5, 3.6 and Corollary 3.24, we prove that the network necessarily self-synchronizes the spikes of clusters of cells periodically, after a transitory waiting time, and that this be-

haviour is robust (i.e. it persists under small perturbation of the numerical and functional parameters of the network). After the transients, the synchronized clusters are independent of the initial state. But the finite transitory time-interval itself, does depend on the initial state.

We give the word "synchronization" an exact mathematical meaning (cf. Definitions 3.2 and 3.1). By eventual *periodic spike-synchronization,* we mean that the interactions among *all* the cells within the same cluster are produced simultaneously, infinitely many times in the future and periodically - but *not necessarily at all the spiking instants of the network* (Definition 3.2). Nevertheless, this definition is rather non-standard in Physics. (For the concept of synchronization and phase locking in Physics see for instance [23].) In fact, the classical definition of synchronization requires all the cells be identical (e.g. [20]). But along this paper, we are not assuming the hypothesis of identical cells, neither in the whole network nor in a part of it.

Joining the results of Theorems 3.5 and Theorem 3.22, we prove that the events' synchronization of more than one cluster is produced only if the cells are mutually different. In other words, if all the cells are structurally identical - also if they are non identical but similar - we prove that all the events of the whole network synchronize, after a transitory time-interval and from any initial state (Theorem 3.22, Part (a)). In brief, networks of similar cooperative cells form a unique spike-synchronized cluster.

In their seminal article [20], Mirollo and Strogatz proved that networks of identical fully cooperative cells, with constant interaction $\Delta_{i,j} = \Delta > 0 \ \forall \ i \neq j$, synchronize all their spikes. In [2], Bottani proved a very general result under a certain stability hypothesis, stating the synchronization of complete networks composed by a large number of excitatory neurons (i.e. cooperative cells) that are modelled as integrate and fire oscillators: Either all the oscillators evolve synchronized in block, or subsets of synchronized oscillators appear always in stable avalanches. The synchronization of several clusters of cells, provide repetitive patterns that allow the network to organize the information. In Theorem 3.5 we generalize Bottani's results to mathematically abstract networks of pulse-coupled cooperative cells, that are neither necessarily identical nor similar, and without the stability hypothesis. In Corollary 3.24 we generalize the statements for some kind of not completely connected networks.

1.4.2   Results about the protection and the amount of information

In Theorem 3.15 we prove that cooperative networks with a large number of cells protect their cells from external negative interferences, diminishing their risk of death (cf. [11]). This protection is due to the synchronization of sufficiently large clusters. As a counterpart, in Part (c) of Theorem 3.22, we prove that if a complete synchronization is achieved, then the total amount of information that the network can generate, or process, is null. But if the synchronization is not complete - as said above this occurs if perceptible differences exist among the free dynamics of the cells - then the network is able to process a positive amount of information (see Theorem 3.20, in the case that the spiking period p is not 1). Nevertheless, this amount of information is necessarily much lower than the theoretical maximum that a general network could process, according to the number of cells that are

employed in the task. This is because, due to Theorem 3.5, in large cooperative networks there exist large clusters of cells whose spikes are produced all together. Therefore, they contribute to enlarge the number of different spiking-code patterns, as if each cluster had a unique cell.

The null amount of information of the cooperative networks that completely synchronize, opposes to many networks composed by all antagonist cells. These latter may exhibit a large amount of (rather unpredictable) information. See for instance the "virtual chaos" in [5, 6], or the "stable chaos" in [24]. As a counterpart, the networks composed by all antagonist cells unprotect their cells and so, they enlarge their risk of death (cf. Remark 3.13). As a consequence, if all the interactions were antagonist, the whole network would be in risk, since the death of many of its cells can drastically diminish the initial richness of its global dynamics.

Therefore, a large network that performs both features: protects their cells and processes a large amount of *not always structured and predictable* information, should theoretically be composed by some cooperative cells and by some antagonist ones, in a more or less balanced interplay among them. Nevertheless, networks of antagonist cells (which in Neuroscience are called inhibitory neurons) also may synchronize and exhibit null amount of information, if the hypothesis of instantaneous spikes is released to take into account the delays [29].

Large networks of all cooperative cells with instantaneous interactions- the ones that we study along this paper - also show a balanced interplay between the amount of information and the protection of the network against the risk of death of its cells. In fact, in Theorem 3.22, we prove that the network maximizes its protection to the cells, if they have mutually similar free dynamics. But if so, the total amount of information becomes zero (Part (c) of Theorem 3.22). In other words, if the network is able to process a positive amount of information, then the cells are mutually different. In particular, the time-constants of their respective free dynamics must be diverse. In such a case, the network may show many different clusters of mutually synchronized cells, and a long period p until the spike patterns of the whole network repeat. In Theorem 3.20, we prove that the amount of information H that a fully cooperative network is able to generate or process, equals $\log_2 p$. So, H is larger if the number of synchronized clusters and the period p are larger.

The paper is organized as follows:

In Section 2 we pose the previous definitions and hypothesis that are assumed for the mathematical model of a cooperative network of pulsed coupled dynamical units. In Section 3, we include the particular mathematical definitions and the statements of the theorems that we will prove along the paper. From Section 4 to Section 8 we write the proofs.

2   Previous definitions and assumptions

We call the scale of the cell the *micro-scale* relative to the whole network $\mathcal{N}$ under study. The state x_i of a cell i is - by hypothesis - governed by a continuous, deterministic and autonomous dynamical system Φ_i, on time $t \in \mathbb{R}$, $t \geq 0$, which we call *the free dynamics of the cell i*. It evolves on a finite-dimensional, differentiable, riemannian and compact

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manifold X_i. Therefore, x_i is a (maybe multidimensional) variable living on a compact metric space with a local euclidean structure, such that for each instant $t \geq 0$ and for each initial state $x_i(0)$, the continuous mapping Φ_i satisfies the following equalities:

$$x_i(t) = \Phi_i(x_i(0), t), \quad x_i(0) = \Phi_i(x_i(0), 0),$$
$$x_i(t + s) = \Phi_i(x_i(0), t + s) = \Phi_i(\Phi_i(x_i(0), t), s) = \Phi_i(x_i(t), s) \ \forall \ t, s \geq 0. \tag{3}$$

The flow defined on a finite dimensional manifold by an autonomous ordinary differential equation is an example of a continuous dynamical system, and may model each cell. Each cell i has its own free dynamics, governed by its own rules. The spaces X_i where the respective states x_i of the cells live, and their dimensions, may be mutually different.

We call the scale of the whole network the *macro-scale* or also, the *global scale.* It is defined by the interactions $\Delta_{i,j}$ among any ordered pair (i, j), $i \neq j$ of different cells. These interactions are produced, in our case, according to certain deterministic rules that we will state in Definition 2.1.

Each network $\mathcal{N}$ may be a cell of a larger hyper-macro network. We will not study this hyper-macro system, nor the role of $\mathcal{N}$ as a cell of it. Nevertheless, we assume that this hyper-macro system exists and may perturb the state x_i of each individual cell i in $\mathcal{N}$.

Two main concepts are the *spikes or milestones* and the eventual *death* of a cell.

Definition 2.1 (Spikes or milestones, satisfaction S_i and goal θ_i)
The *spikes* or *milestones* of a cell i are the instants t_i when i sends actions to the other cells j of the network $\mathcal{N}$. The spiking instants or milestones of the cell i are defined by i, according to the value of a real variable S_i, which depends on the (maybe multidimensional) state x_i of the cell. Thus, S_i depends on time t, i.e. $S_i = S_i(x_i(t))$. We call S_i *the satisfaction variable* of the cell i. This variable may be, for instance, the algebraic sum of several positive or negative components that depend on the state x_i.

By hypothesis, the satisfaction variable S_i, while not perturbed from the exterior of the cell i, satisfies the following two conditions, (4) and (6):

$$\frac{dS_i}{dt} = g_i(x_i), \quad \text{if } 0 \leq S_i \leq \theta_i, \tag{4}$$

where

$$g_i \in C^1(X_i, \mathbb{R}^+), \quad g_i(x_i) > 0 \ \forall \ x_i \leq X_i \quad \text{and} \quad \theta_i > 0. \tag{5}$$

We call θ_i the *goal level or threshold level*. It may be achieved at many different states x_i of the cell, i.e. the set $S_i^{-1}(\theta_i) \subset X_i$ is not necessarily reduced to a single point. Nevertheless, we assume, by hypothesis:

$$\exists \ \text{unique } x_{i,\text{reset}} \in X_i \quad \text{such that} \quad S_i(x_{i,\text{reset}}) = 0. \tag{6}$$

By hypothesis, at instant 0 the initial state $x_i(0)$ of each cell satisfies:

$$0 \leq S_i(x_i(0)) < \theta_i. \tag{7}$$

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On the one hand, at each instant t_i such that $\lim_{t \to t_i^-} S_i(x_i(t)) = \theta_i$, the cell i reacts in the following way:

First, the following *reset or update rule* holds:

$$\lim_{t \to t_i^-} S_i(x_i(t)) \geq \theta_i \quad \Rightarrow \quad x_i(t_i) = x_{i,\text{reset}} \quad \Rightarrow \quad S_i(x_i(t_i)) = 0, \tag{8}$$

i.e. the state x_i of the cell i jumps to $x_{i,\text{reset}}$ at intant t_i. Thus, the satisfaction variable resets to zero at each milestone-instant.

Second, the cell i "produces" a spike at each milestone-instant t_i, i.e. i plays in the game $\mathcal{N}$ when its satisfaction variable S_i arrives to the goal θ_i. At instant t_i the cell i sends instantaneous signals $\Delta_{i,j}$ to the other cells $j \neq i$.

On the other hand, when one and only one cell $j \neq i$ is spiking at an instant t_j (t_j may be different from t_i), it sends to i a signal $\Delta_{j,i} \geq 0$ (in general, $\Delta_{j,i} \neq \Delta_{i,j}$). In such a case, by hypothesis, the state x_i suffers a discontinuity jump such that:

$$\begin{aligned} S_i(x_i(t_j)) = \; & S_i(x_i(t_j^-)) + \Delta_{j,i} \quad \text{if} \quad S_i(x_i(t_j^-)) + \Delta_{j,i} < \theta_i, \\ & S_i(x_i(t_j)) = 0 \;\; \text{otherwise.} \end{aligned} \tag{9}$$

(We denote $S_i(x_i(t_j^-)) = \lim_{t \to t_j^-} S_i(t)$.)

Finally, by hypothesis of the model, if all the cells of a nonempty set $I_n = \{j_1, \ldots, j_k\}$ - of k different cells - spike simultaneously at some instant, say t_n, and if $\Delta_{j,i} \geq 0$ for all $j \in I_n$ and for all $i \notin I_n$, then the state x_i of any other cell $i \notin I_n$ suffers a discontinuity jump such that:

$$\begin{aligned} S_i(x_i(t_n)) = \; & S_i(x_i(t_n^-)) + \sum_{j \in I_n} \Delta_{j,i} \quad \text{if} \quad S_i(x_i(t_n^-)) + \sum_{j \in I_n} \Delta_{j,i} < \theta_i, \\ & S_i(x_i(t_n)) = 0 \;\; \text{otherwise.} \end{aligned} \tag{10}$$

Definition 2.2 (Cooperative and antagonist cells)
A cell j is:
cooperative if $\Delta_{j,i} \geq 0$ for all $i \neq j$ and $\max_i \Delta_{j,i} > 0$,
fully cooperative if $\Delta_{j,i} > 0$ for all $i \neq j$,
antagonist if $\Delta_{j,i} \leq 0$ for all $i \neq j$ and $\min_i \Delta_{j,i} < 0$,
fully antagonist if $\Delta_{j,i} < 0$ for all $i \neq j$,
mixed if there exist $i_1 \neq j$ and $i_2 \neq j$ such that $\Delta_{i_1,j} > 0$ and $\Delta_{i_2,j} < 0$.

We call a network *(fully) cooperative* if all its cells are (fully) cooperative.

Formulae (9) and (10) show that when each cooperative cell j spikes, then it contributes to enlarge the values of the satisfaction variables S_i of the other cells $i \neq j$. Thus, j helps the other cells i to approach to their respective goal levels, and so, it shortens the waiting times until the milestones of the others occur.

When an antagonist cell j spikes, it reduces the values of the satisfaction variables S_i of the other cells Thus, j enlarges the waiting times of the others to arrive to their respective goal levels.

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In this paper, we will focus on full cooperative networks or subnetworks.

By conditions (4) and (5), while the free dynamics of a cell is not negatively perturbed by external agents, and while the satisfaction variable S_i of the cell i does not arrive to its goal level, it strictly increases on time. In other words, each cell is "born optimist": if free, it approximates with positive velocity to its goal. Besides, the instantaneous velocity $g_i(x_i)$ is bounded away from zero, because $\min_{x_i \in X_i} g_i(x_i) > 0$ exists, due to the compactness of the space X_i and the continuity of g_i. So, if free, each cell arrives to its goal after a finite time. Nevertheless, if a negative term $-\delta_i < 0$ is added to the real function g_i, the velocity dS_i/dt will decrease, and may also become negative.

Definition 2.3 (Negative external interferences)
We call a negative number $-\delta < 0$ *negative differential interference* to the cell i from its external environment, if during an interval of time, the differential equation (4) is substituted by

$$\frac{dS_i}{dt} = g_i(x_i) - \delta \quad \text{if } 0 \leq S_i < \theta_i. \tag{11}$$

We call a negative number $-\Delta < 0$ *negative impulsive interference* to the cell i from its external environment, if at some instant t for which the satisfaction variable S_i has not arrived to its goal level, the state x_i of the cell suffers a discontinuity jump such that

$$S_i(x_i(t)) = S_i(x_i(t^-)) - \Delta.$$

Definition 2.4 (Death of a cell)
The *death* of a cell i (cf. [11]) occurs at an instant $T \geq 0$, if for any time $t > T$ the cell i does not spike. In other words, after a cell i dies, it does not arrive to its goal anymore, and so, it stops sending actions to the other cells of the network forever.

In Definition 3.7, we will state a numerical formula to measure the theoretical intrinsic risk R_i of death of any cell under eventual negative interferences, if it were not connected to the cooperative network. In Definitions 3.11 and 3.12, we pose formulae to measure the net risk of death $R_i' < R_i$, if the cell i is interacting in a cooperative network, and thus, the protection factor $P_i > 0$ that the network provides to i.

Definition 2.5 (Space of parameters)
Let $\mathcal{N}$ be a network with $m \geq 2$ fixed cells. The *parameters of the network* are:

$$\text{Param}(\mathcal{N}) := \left(\left\{ (\theta_i, g_i) \right\}_{1 \leq i \leq m}, \ \left\{ \Delta_{i,j} \right\}_{1 \leq i,j \leq m, \ i \neq j} \right), \tag{12}$$

where, according to Definition 2.1, g_i is the given C^1 real function in the second member of the differential equation (4) which governs the free dynamics of the cell i; $\theta_i \in \mathbb{R}^+$ is its goal level; and $\Delta_{i,j} \in \mathbb{R}$ is the interaction in the network from the cell i to the cell $j \neq i$.

In the space of all the parameters of a network with exactly m cells (for $m \geq 2$ fixed), we define the topology induced by the following metric (distance), where the parameters

of the networks $\mathcal{N}$ and $\mathcal{N}'$ are defined by Equality (12); those denoted without $'$ (with $'$) correspond to the network $\mathcal{N}$ (resp. $\mathcal{N}'$):

$$\mathrm{dist}\Big(\mathrm{Param}(\mathcal{N}),\ \mathrm{Param}(\mathcal{N}')\Big) :=$$

$$\max\Big\{ \max_{1 \leq i \leq m} \big\{ |\theta_i - \theta'_i|,\ \|g_i - g'_i\|_{C_1} \big\},\ \max_{i \neq j} |\Delta_{i,j} - \Delta'_{i,j}|\ \Big\}. \tag{13}$$

In the above equality, $\|\cdot\|_{C^1}$ is the C^1 norm in the space of all the C^1 real functions defined on the compact manifold X_i.

We say that a phenomenon of the global dynamics is *robust, or persistent, or structurally stable* if the set $\mathcal{G}$ of parameters of the networks for which the phenomenon occurs is open. In other words, if a phenomenon is robust and if $\mathrm{Param}(\mathcal{N}) \in \mathcal{G}$, then $\mathrm{Param}(\mathcal{N}')$ still belongs to $\mathcal{G}$ for any sufficiently small perturbation $\mathcal{N}'$ of the network $\mathcal{N}$ (openness condition).

Definition 2.6 (Spiking instants and interspike intervals of the network)

We denote by $\{t_n\}_{n \in \mathbb{N}^+}$ the strictly sequence of all the instants for which at least one neuron spikes. We call t_n the *n-th. spiking instant* of the network. We call $(t_n, t_{n+1}]$ the *n-th. interspike interval* of the network and denote it by $ISI^{(n)}$.

Definition 2.7 (Clusters or spiking codes)

We denote by I_n the set of neurons that spike at the instant t_n. We call I_n the *n-th. cluster* or also, the *n-th. spiking code.*

3 Mathematical statements

Definition 3.1 (Spike-synchronization)

We say that the network *eventually synchronizes spikes* if there exists a subsequence $\{t_{n_h}\}_{h \in \mathbb{N}}$ of spiking instants such that the respective clusters I_{n_h} are $\{1, \ldots, m\}$ (i.e. all the cells spike at instants t_{n_h}).

Definition 3.2 (Periodic spike-synchronization)

We say that all the cells of the network *eventually periodically synchronize spikes with period $p \geq 1$*, if there exists $n_0 \geq 0$ such that:

i) the subsequence $\{t_{n_h}\}_{h \in \mathbb{N}}$ of Definition 3.1 satisfies

$$t_{n_h} = t_{n_0 + hp} \ \ \forall\, h \in \mathbb{N},$$

ii) the sequence $\{I_n\}_{n \geq 0}$ of clusters satisfies

$$I_n = I_{n+p} \ \ \forall\, n \geq n_0,$$

iii) the sequence $\{t_{n+1} - t_n\}_{n \geq 0}$ of the interspike intervals' lengths satisfies:

$$t_{n+p+1} - t_{n+p} = t_{n+1} - t_n \quad \forall \, n \geq n_0.$$

We call p the *natural spiking period* of the network.

Note that the network eventually periodically synchronizes spikes, if and only if at least one cell spikes at instant t_n for all $n \geq 0$, no cell spikes at instants $t \in (t_n, t_{n+1})$ for all $n \geq 0$, and all the cells spike at instants t_{n_0+hp} for any natural number $h \geq 0$.

Definition 3.3 (Transitory time)
For each fixed initial state for which the network periodically synchronize spikes, we call t_{n_0} the *waiting time* or *transitory time* until the synchronization of the full network occurs.
Note that the occurrence of synchronization and the value of the waiting time depend on the initial state of the network.

Let $\mathcal{N}$ be a network composed by $m \geq 2$ fully cooperative cells (cf. Definition 2.2). Recall that θ_j denotes the goal level of the cell j, and $\Delta_{i,j}$ denotes the action from the cell i to the cell j (cf. Definition 2.1).

Definition 3.4 (Large cooperative network)
A fully cooperative network is called *large* if

$$\sqrt{m} > \max \left\{ \sqrt{3}, \quad \frac{\max\{\theta_j \colon \ j \in \mathcal{N}\}}{\min\{\Delta_{i,j} \colon \ i, j \in \mathcal{N}, \ i \neq j\}} + 1 \right\}, \tag{14}$$

Theorem 3.5 (Synchronization)
If $\mathcal{N}$ is a large fully cooperative network then:
(a) *From any initial state, all the cells of the network $\mathcal{N}$ eventually periodically synchronize spikes.*
(b) *The eventual periodic synchronization is a robust phenomenon.*

We prove Theorem 3.5 in Subsection 4.1.

Theorem 3.6 (Upper bounds for the transitory time and the spiking period)
Under the hypothesis of Theorem 3.5, the transitory time T and the natural spiking period p satisfy the following inequalities:

$$T \leq \max_{1 \leq i \leq m} \left\{ \frac{\theta_i}{\min \ \{g_i(x_i) \colon \ x_i \in X_i \ \}} \right\}, \tag{15}$$

$$p \leq 1 + \frac{\max\{\theta_j \colon \ j \in \mathcal{N}\}}{\min\{\Delta_{i,j} \colon \ i, j \in \mathcal{N}, \ i \neq j\}}. \tag{16}$$

We prove Theorem 3.6 in Subsection 4.2.

Now, we define the risk of death of each cell and the protection factor of the network.

In Definition 2.4 we say that a cell i dies if (due to external causes of the cell) there is an instant T such that for all $t > T$ the satisfaction variable S_i remains smaller than the goal level θ_i. In brief:

$$i \text{ dies} \quad \Leftrightarrow \quad S_i(x_i(t)) < \theta_i \ \ \forall\, t > T \text{ for some } T \geq 0.$$

Due to the rules of the interactions among the cells, if a cell i dies, it can not act on the network anymore (after time T).

We measure the risk of death of the cell i by the following definitions:

Definition 3.7 (Intrinsic risk of death R_i)

The *intrinsic risk of death R_i* of the cell i, relatively to the other cells of the network is defined by

$$R_i := \frac{\theta_i}{\max_{1 \leq j \leq m} \theta_j}, \tag{17}$$

where m is the number of cells of the network. Since $\theta_i > 0$ for all i, Equality (17) immediately implies the following:

$$0 < R_i \leq 1. \tag{18}$$

Recall Definition 3.16: the n-th. spiking-code I_n is the (nonempty) set of cells that spike at instant t_n. Let us fix a cell i.

Definition 3.8 (Spiking instants of each cell).

The sequence $\{t_i^{(h)}\}_{h \geq 1}$ of *spiking times* of the cell i is defined by the following equalities:

$$
\begin{aligned}
t_i^{(1)} &:= \min\{t_n > 0: \ \ i \in I_n\} && \text{if } i \in I_n \text{ for some } n \geq 0, \\
t_i^{(1)} &:= +\infty && \text{otherwise}, \\
t_i^{(h+1)} &:= \min\{t_n > t_i^{(h)}: \ \ i \in I_n\} && \text{if } i \in I_n \text{ for some } t_n > t_i^{(h)}, \\
t_i^{(h+1)} &:= +\infty && \text{otherwise}.
\end{aligned}
$$

We denote

$$t_i^{(0)} := 0.$$

For each cell i and for each natural number $h \geq 1$ such that $t_i^{(h)} < +\infty$, we call $t_i^{(h)}$ the *h-th. spiking instant* or the *h-th. milestone* of the cell i. Note that a cell i dies if and only if $t_i^{(h+1)} = +\infty$ for some $h \geq 0$.

Definition 3.9 (The inter-spike-intervals of each cell).

We call the time-interval

$$\mathrm{ISI}_i^{(h)} := \left(t_i^{(h)}, t_i^{(h+1)} \right] \ \ \forall\, h \geq 0$$

the *h-th. inter-spike-interval of the cell i*. Note that we include the instant $t_i^{(h+1)}$ in the h-th. inter-spike-interval of the cell i.

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When a cell i receives cooperative actions $\Delta_{j,i} > 0$ from the other cells $j \neq i$ of the network, then its satisfaction variable S_i is increased, to approach (or even reach) its goal θ_i. Equivalently, a positive action $\Delta_{j,i} > 0$ can be understood, as a reduction of the goal level θ_i substituting it by $\theta_i - \Delta_{i,j}$. Roughly speaking, when a cell i receives positive interactions from the other cells, its risk of death diminishes.

Definition 3.10 For each fixed cell i, and for each fixed natural number $h \geq 0$, we denote:

$$\Delta_{i,\mathcal{N}}^{(h)} := \sum_{j \neq i, \ j \in I_n, \ t_n \in \mathrm{ISI}_i^{(h)}} \Delta_{j,i}, \tag{19}$$

We call $\Delta_{i,\mathcal{N}}^{(h)}$ *the net action* that the cell i receives during its h-th. inter-spike-interval from the other cells of the network.

If the network is not cooperative, then some of the actions $\Delta_{j,i}$ in Formula (19) may be negative, so the sum of all of them may be negative or null. Note that, in general, the net action $\Delta_{i,\mathcal{N}}^{(h)}$ depends on the initial condition of the network.

Definition 3.11 (Net risk of death R_i')

The *net risk of death* $R'^{(h)}_{i,\mathcal{N}}$ of the cell i while connected to the network $\mathcal{N}$, during its h-th. inter-spike-interval, is

$$R'^{(h)}_{i,\mathcal{N}} := \max \left\{ 0, \ \min \left\{ 1, \ \frac{\theta_i - \Delta_{i,\mathcal{N}}^{(h)}}{\max\limits_{1 \leq j \leq m} \theta_j} \right\} \right\}. \tag{20}$$

In general the net risk of death $R'^{(h)}_{i,\mathcal{N}}$ depends on the initial condition. For simplicity in the notation, in the sequel we will write R_i' to denote the net risk of the cell i, when the network $\mathcal{N}$ is clear from the context, and when referring to some fixed $h \geq 0$.

Definition 3.12 (Protection factor P_i) The *protection factor* $P_{i,\mathcal{N}}^{(h)}$ of the network $\mathcal{N}$ to its cell i, during the h-th. inter-spike-interval of i, is

$$P_{i,\mathcal{N}}^{(h)} := \frac{\Delta_{i,\mathcal{N}}^{(h)}}{\theta_i}. \tag{21}$$

In the sequel, we will simply write P_i instead of $P_{i,\mathcal{N}}^{(h)}$, when the network $\mathcal{N}$ is clear from the context, and when referring to some fixed $h \geq 0$.

Remark 3.13 From Equality (21), it is immediate to deduce that the protection factor P_i is positive (negative) if and only if the minimum net sum of actions $\Delta_{i,\mathcal{N}}^{(h)}$ that the cell i receives from the other cells of the network is positive (resp. negative). Therefore, if the network is fully cooperative, its protection factor is strictly positive, and if the network is fully antagonist, its protection factor is strictly negative.

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In the following Proposition 3.14 we state that net risk R_i' of a cell that is connected to the network $\mathcal{N}$, is (essentially) the product of its intrinsic risk R_i, if i were not connected to the network, by $1 - P_i$.

In other words, if the protection factor P_i of the network were positive, then the net risk R_i' is smaller than the intrinsic risk of death, and if the protection factor P_i were 1, then the net risk of death R_i' would be zero. So, a cooperative network always reduces the intrinsic risk, and may also make it zero, if the cooperative interactions are large and frequent enough.

Proposition 3.14 (Formula of the protection factor of the network)
$$R_i' := \max \Big\{ 0, \ \min \big\{ 1, \ \big(1 - P_i \big) \, R_i \big\} \Big\}. \tag{22}$$

We prove Proposition 3.14 in Subsection 6.1.

Theorem 3.15 (Protection factor)
Under the hypothesis of Theorem 3.5, the protection factor P_i of the network to each cell i is positive. So, the net risk of death R_i' of each cell i under negative external interferences, when it is connected to the network, is strictly smaller than its intrinsic risk R_i when it is isolated from the network.

We prove Theorem 3.15 in Subsection 6.2.

Definition 3.16 (Code-patterns)
For any natural number $k \geq 1$, and for any fixed initial state

$$\mathbf{x}(0) = (x_1(0), \ldots, x_m(0))$$

of the network, we construct the following word of clusters with length k:

$$\Pi_{n,k} := (I_n, \ldots, I_{n+k-1}).$$

We call $\Pi_{n,k}$ the $n-th$. *code-pattern with length k from the initial state* $\mathbf{x}(0)$.

Definition 3.17 (Recurrent code-patterns)
We say that a code-pattern Π_k with length k is *recurrent,* if there exists $n_j \to +\infty$ such that

$$\Pi_{n_j,k} = \Pi_k \ \ \forall \, j \in \mathbb{N}.$$

We denote by $\mathcal{P}_k$ the set of all the recurrent code-patterns with length k, obtained from all the initial states of the network. We denote by

$$\#\mathcal{P}_k \geq 1$$

the cardinality of the set $\mathcal{P}_k$.

Definition 3.18 (Amount of information)
    We denote
$$H := \sup_{k \geq 1}\{\log_2(\#\mathcal{P}_k)\} \text{ bits} \in [0, +\infty], \tag{23}$$
where $\log_2$ is the logarithm in base 2.
    We call H *the total amount of information* that the network $\mathcal{N}$ is able to process.

    Note that, since $\#\mathcal{P}_k \geq 1$ for all k, then $H \geq 0$.

Definition 3.19 (Entropy)
    If $H = +\infty$, we define *the entropy h,* or the exponential rate of increasing information, by
$$h := \limsup_{k \to +\infty} \frac{H_k}{k}, \quad \text{where} \quad H_k := \log_2(\#\mathcal{P}_k).$$

Interpretation of the amount of information H.   The number $\#\mathcal{P}_k$ of different recurrent patterns Π_k with length k measures the dynamical richness of $\mathcal{N}$, with respect to the many possible finite words that the spiking-codes can show. Namely, H takes into account how many different clusters I_n are exhibited at the milestones t_n of the network. Thus, the total amount of information H is a quantitative index of the maximum global codified dynamical diversity that the network $\mathcal{N}$ is able to show along its recurrent evolution in the future, if the observer looks at *each cell with an individual role.* In fact, note that if $i \neq j$, then the event for which the cell i spikes and j does not, is *distinguished* from the converse event.

Theorem 3.20 (Amount of information)
    Under the hypothesis of Theorem 3.5*, the total amount of information H of the network is*
$$H = \log_2 p \leq \log_2 \left(1 + \frac{\max\{\theta_j: \ j \in \mathcal{N}\}}{\min\{\Delta_{i,j}: \ i, j \in \mathcal{N}, \ i \neq j\}}\right).$$

We prove Theorem 3.20 in Section 5.

Definition 3.21 (Mutually similar cells)
    We say that the cells of a fully cooperative network are *mutually similar* (with respect to the minimum interaction) if:

$$\frac{\left(\min\limits_{1 \leq i \leq m} \theta_i\right)\left(\min\limits_{1 \leq i \leq m} \min\{g_i(x_i): x_i \in X_i\}\right)}{\left(\max\limits_{1 \leq i \leq m} \theta_i\right)\left(\max\limits_{1 \leq i \leq m} \max\{g_i(x_i): x_i \in X_i\}\right)} > 1 - \frac{\min\{\Delta_{i,j}: \ i, j \in \mathcal{N}, \ i \neq j\}}{\max\limits_{1 \leq i \leq m} \theta_i}, \tag{24}$$

Theorem 3.22 (Cooperative networks of mutually similar cells) *Under the hypothesis of Theorem* 3.5*, if besides the cells are mutually similar, then*

(a) *From any initial state all the cells eventually periodically synchronize all their spikes with natural spiking period $p = 1$.*

(b) *After the transitory time T has elapsed, the protection factor P_i of the network to each cell i is 1. So, its net risk of death R_i' under negative external interferences is null.*

(c) *The total amount of information H of the network is zero*

We prove Theorem 3.22 in Section 7.

Remark: In particular Inequality (24) - and thus, also the assertions (a), (b) and (c) of Theorem 3.22 - holds if all the cells are mutually identical, i.e.

$$\theta_i = \theta_j = \theta, \quad g_i = g_j = g : X \mapsto \mathbb{R}^+ \quad \forall\, i \neq j,$$

and if the minimum positive interaction is large enough, i.e.:

$$\frac{\min\{\Delta_{i,j} :\ i, j \in \mathcal{N},\ i \neq j\}}{\theta} > 1 - \frac{\min\{g(x) \colon x \in X\}}{\max\{g(x) \colon x \in X\}}.$$

Now let us pose a result about large cooperative networks whose graphs are not necessarily complete.

Definition 3.23 (Full cooperative core)

Let $\mathcal{N}$ be a network. A *full cooperative core* in $\mathcal{N}$, if it exists, is a subnetwork $\mathcal{N}_1$ composed by fully cooperative cells, i.e.

$$i \in \mathcal{N}_1 \ \Rightarrow\ \Delta_{i,j} > 0 \ \forall\, j \in \mathcal{N},$$

and such that all the cells that do not belong to $\mathcal{N}_1$ have non negative actions on the cells of $\mathcal{N}$, i.e.

$$i \notin \mathcal{N}_1 \ \Rightarrow\ \Delta_{i,j} \geq 0 \ \forall\, j \in \mathcal{N}.$$

Corollary 3.24 (Networks with a large cooperative core.)

Let $\mathcal{N}$ be a network that has a full cooperative core $\mathcal{N}_1$. Assume that the number m of cells in $\mathcal{N}_1$ is large enough. Precisely:

$$\sqrt{m} > \max\left\{\sqrt{3}, \quad \frac{\max\{\theta_j \colon j \in \mathcal{N}\}}{\min\{\Delta_{i,j} \colon i \neq j,\ i \in \mathcal{N}_1,\ j \in \mathcal{N}\}} + 1 \right\}. \tag{25}$$

Then Assertions of Theorems 3.5, 3.6, 3.15, 3.20 and 3.22 hold, where $\min\{\Delta_{i,j} :\ i, j \in \mathcal{N},\ i \neq j\}$ must be substituted by $\min\{\Delta_{i,j} :\ i \in \mathcal{N}_1,\ j \in \mathcal{N},\ i \neq j\}$.

We prove Corollary 3.24 in Section 8.

4 The proof of synchronization

In this section we prove Theorems 3.5 and 3.6. To do so, we need the following previous result:

Lemma 4.1 (A) *If from some initial state of the network there are two different (minimal) instants $0 < t_0^* < t_p^*$ when all the cells spike simultaneously, then the network eventually periodically synchronizes spikes with some natural spiking period $p \geq 1$.*
(B) *If from all the initial states the network eventually periodically synchronizes spikes with the same period $p \geq 1$, then the number of all the possible recurrent code-patterns with length $k \geq 1$ is*

$$\#\mathcal{P}_k \leq p \quad \forall\, k \geq 1,$$

and

$$\max_{k \geq 1}(\#\mathcal{P}_k) = p.$$

(C) *If from some initial state the network eventually periodically synchronizes spikes with period $p = 1$, then, the sequence $\{I_n\}_{n \geq n_0}$ of clusters is constantly equal to the set of all the cells of the network, i.e.*

$$I_n = \{1, 2, \ldots, m\} \quad \forall\, n \geq n_0.$$

(D) *If from all the initial states the network eventually periodically synchronizes spikes with period $p = 1$, then the number of all the possible recurrent code-patterns with length $k \geq 1$ is*

$$\#\mathcal{P}_k = 1 \quad \forall\, k \geq 1.$$

Proof: Assertions (C) and (D) are immediate consequences of (A) and (B) respectively, in the particular case $p = 1$.

Let us prove Assertion (A). Recall that $m \geq 2$ denotes the number of cells in the network.

By hypothesis, all the cells of the network spike at instants $t_0^* > 0$ and $t_p^* > t_0^*$. Thus, due to the reset hypothesis in Definition 2.1, Formula (8), the state of the network at instant t_0^*, is

$$\mathbf{x}(t_0^*) = \mathbf{x}_{\text{reset}} := \big(x_{1,\text{reset}}, \ldots, x_{i,\text{reset}}, \ldots, x_{m,\text{reset}}\big),$$

where $x_{i,\text{reset}}$ is the unique point in the state-space X_i that satisfies Equality (6). Thus

$$\mathbf{S}(\mathbf{x}(t_0^*)) = \mathbf{0}, \quad \text{i.e. } S_i(x_i(t_0^*)) = 0 \ \forall\, 1 \leq i \leq m.$$

As the dynamics defined by (3) is deterministic, once fixed the unique state $\mathbf{x}_{\text{reset}}$ of the network, a unique orbit exists in the future. Thus, translating the origin 0 of time to t_0^*, and recalling Equalities (3), we obtain:

$$\mathbf{S}(\mathbf{x}(t)) = \mathbf{S}(\mathbf{x}(t_0^* + (t - t_0^*))) = \mathbf{S}(\mathbf{\Phi}(\mathbf{\Phi}(\mathbf{x}_0, t_0^*), t - t_0^*)) \ \forall\, t \geq t_0^*,$$

where $\mathbf{\Phi} = (\Phi_1, \ldots, \Phi_i, \ldots, \Phi_m)$. Since $S_i(x_i(t))$ satisfies the differential equation (4) for all $t > t_0^*$ such that $t \neq t_n$, and

$$\mathbf{S}(\mathbf{\Phi}(\mathbf{x}_0, t_0^*)) = \mathbf{0},$$

we have:
$$\mathbf{x}(t_0^*) = \mathbf{\Phi}(\mathbf{x}_0, t_0^*) = \mathbf{x}_{\text{reset}},$$
$$\mathbf{S}(\mathbf{x}(t)) = \mathbf{S}(\mathbf{\Phi}(\mathbf{\Phi}(\mathbf{x}_0, t_0^*), t - t_0^*)) = \mathbf{S}(\mathbf{\Phi}(\mathbf{x}_{\text{reset}}, t - t_0^*)) = \mathbf{S}(\mathbf{x}^*(t - t_0^*)) \ \ \forall\, t \geq t_0^*,$$

where $\mathbf{x}^* = (x_1^*, \ldots, x_i^*, \ldots, x_m^*)$ is the unique solution $\mathbf{x}^*(\cdot) = \mathbf{\Phi}(\mathbf{x}_{\text{reset}}, \cdot)$ of the deterministic dynamical system (3) with initial state $\mathbf{x}^*(0) = \mathbf{x}_{\text{reset}}$, plus the deterministic interactions' rule (10)during the time-interval (t_0^*, t_p^*).

By the hypothesis, all the cells spike simultaneously again at the instant $t_p^* > t_0^*$. Therefore, $\mathbf{S}(\mathbf{x}(t_p^*)) = \mathbf{S}(\mathbf{x}(t_0^*)) = \mathbf{0}$. Thus $\mathbf{x}(t_p^*) = \mathbf{x}(t_0^*) = \mathbf{x}_{\text{reset}}$. Then,

$$\mathbf{x}(t_p^*) = \mathbf{x}^*(t_p^* - t_0^*) = \mathbf{x}_{\text{reset}} = \mathbf{x}^*(0).$$

We deduce that the unique solution $\mathbf{x}^*$ which has initial condition $\mathbf{x}_{\text{reset}}$, is periodic with period $t_p^* - t_0^*$. Thus, the instants t_n and the spiking-codes I_n are determined recursively from the unique *periodic* orbit $\mathbf{x}^*$. We deduce

$$I_n = I_{n+p} \ \ \forall\, n \geq n_0. \tag{26}$$

Also,

$$I_{n_0+hp} = \{1, 2 \ldots, m\}, \ \ I_n \subset_{\neq} \{1, 2 \ldots, m\} \ \ \forall\, n_0 + hp < n < n_0 + (h+1)p, \ \ \forall\, h \geq 0, \tag{27}$$

and the sequence $\{t_n\}_{n \geq 0}$ of instants for which at least one cell spikes satisfies:

$$t_{n+p+1} - t_{n+p} = t_{n+1} - t_n \ \ \forall\, n \geq n_0. \tag{28}$$

Equalities (26), (27) and (28) end the proof of Assertion (A) of Lemma 4.1.

Let us prove Assertion (B):
By Assertion (A), the sequence $\{I_n\}_{n \geq n_0}$ is periodic with period p. Thus, from Definition 3.16, for any fixed natural number $k \geq 1$, all the recurrent code-patterns with length k are:

$$(I_{n_0}, I_{n_0+1}, \ldots, I_{n_0+k-1}), \ (I_{n_0+1}, I_{n_0+2}, \ldots, I_{n_0+k}), \ldots,$$
$$\ldots, (I_{n_0+r}, I_{n_0+r+1}, \ldots, I_{n_0+r+k-1}), \ldots, \tag{29}$$
$$\ldots, (I_{n_0+p-1}, I_{n_0+p}, \ldots, I_{n_0+p+k-2})$$

with $0 \leq r \leq p - 1$. In fact, for $n = n_0 + hp + r \geq p$ the code-pattern $(I_n, I_{n+1}, \ldots, I_{n+k-1})$ coincides with $(I_{n_0+r}, I_{n_0+r+1}, \ldots, I_{n_0+r+k-1})$, because $I_n = I_{n_0+hp+r} = I_{n_0+r}$. So, all the code-patterns in the list (29) are recurrent. Two or more code-patterns in the list (29) may coincide, so the number of different code-patterns with length k is at most equal to the number of items in the list (29). Thus:

$$\#\mathcal{P}_k \leq p \ \ \forall\, k \geq 1.$$

Now let us prove that, in the particular case that $k = p$, the code-patterns of the list (29) are pairwise different. To prove this assertion, with no loss of generality, we assume $n_0 = 0$ (if not, we translate the origin 0 of time to the instant t_{n_0}). Assume that

$$(I_r, I_{r+1}, \ldots, I_{r+p-1}) = (I_s, I_{s+1}, \ldots, I_{s+p-1}) \tag{30}$$

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for $0 \le r, s \le p - 1$. We must prove that $r = s$.

The code-pattern $I_0 = \{1, \ldots, m\}$ will appear once in both patterns (30), because they both have length p, which is the (minimum) period of the sequence $\{I_n\}_{n \ge 1}$. Say $I_0 = I_{r+h} = I_{s+k}$ with $0 \le h, k \le p - 1$.

Both patterns in Equality (30) coincide. Then the positions h and k are the same:

$$h = k.$$

Besides, since $0 \le r + h, s + k \le p - 1$ and $0 \le r, s \le p - 1$, we have

$$|(r + h) - (s + k)| = |r - s| \le p - 1 < p.$$

As $I_0 = I_{r+h} = I_{s+k}$, there are two indexes $r + h$ and $s + k$, whose difference is smaller than p, such that the respective patterns coincide with $I_0 = \{1, \ldots, m\}$. In other words, all the cells spike at two instants t_{r+h} and t_{s+k} such that $|(r + h) - (s + k)| < p$. But p is the minimum positive natural number such that all cells spike at instants t_n and t_{n+p} for some n. We deduce that $r + h = s + k$. Since we already know that $h = k$, we deduce that $r = s$, as wanted.

We conclude that all the code-patterns of the list (29) are pairwise different when the length k equals the period p. Thus, the number of different code-patterns with length p is p, ending the proof of Assertion (B) of Lemma 4.1. $\qquad\square$

4.1 Proof of Theorem 3.5

Proof: **Part a)**

From Lemma 4.1, to prove that all the cells of the network eventually periodically synchronize spikes, it is enough to prove that there exist two instants $0 < t_0 < t_p$ such that all the cells simultaneously spike at t_0 and at t_p.

If *for any initial condition* we find a single instant $t_0 > 0$ at which all the cells simultaneously spike, then, taking as new initial state $\mathbf{x}(t_0)) = \mathbf{x}_{\text{reset}}$, we deduce that there exists a second instant $t_p > t_0$ at which all the cells simultaneously spike. Thus, to prove Part (a) of Theorem 3.5, it is enough to show the following:

Assertion **(i)** to be proved: *For any initial condition, there exists an instant $t_0 > 0$ at which all the cells simultaneously spike.*

From any initial state $\mathbf{x}_0 = (x_1(0), \ldots, x_m(0))$ such that $0 \le S_i(x_i(0)) < \theta_i$ for all $1 \le i \le m$, consider the state

$$\mathbf{x}(t) = \Big(x_1(t), \ldots, x_i(t), \ldots, x_m(t) \Big),$$

and the m-dimensional vector whose components are the satisfaction variables $S_i(x_i(t))$ at instant $t > 0$.

Since each variable S_i is governed by the differential equation (4) with a strictly positive real function $g_i : X_i \mapsto \mathbb{R}^+$ (which is continuous on the compact space X_i), plus the eventual sum of interactions $\Delta_{j,i} \ge 0$ according to Equalities (10), we deduce:

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Property **(ii)** *While no interference from outside the network appears, for each cell i the real variable S_i is strictly increasing on t, for all $t \geq 0$ such that $S_i(x_i(t)) \in [0, \theta_i)$. Besides, except at those instants where S_i is discontinuous, its derivative respect to time t exists, is positive and bounded away from zero, and at the instants where S_i is discontinuous, the discontinuity jumps are positive.*

We are assuming that $S_i(x_i(0)) < \theta_i$. Thus, from Property (ii), we deduce:

Property **(iii)** *For each cell i, there exists a first finite time $t_i > 0$ such that $\lim_{t \to t_i^-} S_i(x_i(t)) = \theta_i$. Namely, for any $i \in \{1, \ldots, m\}$ there exists a first spiking instant $t_i > 0$.*

Consider the *minimum natural number $K \geq 1$* such that

$$K \geq \frac{\max\{\theta_j \colon \ j \in \mathcal{N}\}}{\min\{\Delta_{i,j} \colon i, j \in \mathcal{N} \text{ such that } i \neq j\}}. \tag{31}$$

From Inequalities (14) and (31), we deduce:

$$\sqrt{m} > K. \tag{32}$$

By hypothesis, $\theta_j > 0$ for all $1 \leq j \leq m$. Denote

$$0 < l_j := \frac{\theta_j}{K} \leq \theta_j \cdot \frac{\min\limits_{i \neq j} \Delta_{i,j}}{\max\limits_{1 \leq j \leq m} \theta_j} \leq \min\limits_{i \neq j} \Delta_{i,j}. \tag{33}$$

For later use, we note the following property:

$$\text{If} \ \ 1 \leq h \leq K-1, \qquad \text{then} \qquad h \, l_j = \frac{h}{K}\theta_j < \theta_j, \tag{34}$$

$$\text{and if} \ \ \ h = K, \qquad \text{then} \qquad K \, l_j = \frac{K}{K}\theta_j = \theta_j. \tag{35}$$

Assume that at the instant $t > 0$, at leat K cells are spiking, where $K \geq 1$ is the natural number defined by Inequality (31). Then, for any other cell $j \in \{1, \ldots, m\}$, applying Equalities (10), we have:

$$S_j(x_j(t)) \geq \ S_j(x_j(t^-)) \ + \ K \min\limits_{i \neq j} \Delta_{i,j} \geq$$

$$\geq \ \Big(\min\limits_{i \neq j} \Delta_{i,j}\Big) \cdot \frac{\max\limits_{1 \leq j \leq m} \theta_j}{\min\limits_{i \neq j} \Delta_{i,j}} = \max\limits_{1 \leq j \leq m} \theta_j \ \geq \theta_j.$$

Thus, any cell j will also spike at instant t, because its satisfaction variable $S_j(x_j(t))$ arrives to the goal level θ_j. Summarizing, we have proved:

Property **(iv)** *If at least K cells spike at an instant $t > 0$, then all the cells spike at the instant t.*

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Now, to end the proof of Assertion (i) it is enough to prove the following:

Assertion **(v)** to be proved: *There exists an instant $T > 0$ such that at least K cells spike simultaneously at T.*

Let us take $t_1 > 0$ equal to the first positive instant when at least one cell i_1 arrives to its goal level, i.e.

$$S_{i_1}(x_{i_1}(t_1)^-) = \theta_{i_1} \quad \text{for some} \quad i_1 \in \{1, \ldots, m\}. \tag{36}$$

Let us discuss according to two cases: either at least K cells spike at instant t_1 with the cell i_1, or at most $K - 1$ do.

FIRST CASE: At least K cells spike at instant $t_1 > 0$. Thus Assertion (v) holds.

SECOND CASE: At most $K - 1$ cells (and at least one cell i_1) spike at instant t_1. From Inequality (32), there exist at least $m - (K - 1) \geq K^2 - (K - 1) \geq 1$ cells that do not spike at instant t_1. Denote by A_1 this *set of non spiking cells* at instant t_1. We have:

$$\#A_1 \geq K^2 - (K - 1) \geq 1.$$

Using Inequality (7), for each cell $j \in A_1$ we know that $S_j(x_j(t_1^-)) \geq 0$. Since at instant t_1 at least the cell i_1 spikes, it sends a positive interaction $\Delta_{i_1, j}$ to any cell $j \in A_1$. Applying Inequalities (10) and (33), we deduce:

$$S_j(x_j(t_1)) \geq \; S_j(t_1^-) + \Delta_{i_1, j} \geq \min_{i \neq j} \Delta_{i,j} \geq l_j \qquad \forall \, j \in A_1.$$

In brief:

$$S_j(x_j(t_1)) \geq l_j \qquad \forall \, j \in A_1. \tag{37}$$

Denote by $t_2 > t_1$ the first instant after t_1 for which at least one cell i_2 arrives to its goal level.

Now, we discuss again two cases: either there exist at least K cells that spike at instant t_2 with the cell i_2, or there exists at most $K - 1$ cells that so do. In the first case, Assertion (v) holds. In the second case, denote by $A_2 \subset A_1$ the set of cells that did not spike at any instant in $[0, t_2]$. We have:

$$\#A_2 \geq \#A_1 - (K - 1) \geq K^2 - 2(K - 1).$$

Since $t_2 > t_1$, applying Property (ii) and Inequalities (10) and (37), we obtain:

$$S_j(x_j(t_2^-)) > S_j(x_j(t_1)) \geq l_j \;\; \forall \, j \in A_2.$$

Since at least one cell i_2 spikes at instant t_2, it adds a positive jump $\Delta_{i_2, j}$ to $S_j(x_j(t_2^-))$ for all $j \in A_2$. Thus, using Inequality (33), we deduce:

$$S_j(x_j(t_2)) \geq \; S_j(x_j(t_2^-)) + \Delta_{i_2, j} \geq l_j + \min_{i \neq j} \Delta_{i,j} \geq 2l_j \qquad \forall \, j \in A_2. \tag{38}$$

By induction on $h \in \mathbb{N}^+$, if $2 \leq h \leq K$, assume that t_h is the h-th. consecutive instant $t_h > t_{h-1}$ such that at least one cell i_h spikes, and no more than $K - 1$ cells have spiked

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simultaneously at each instant $t_1 < t_2 \ldots < t_{h-1}$. Denote by A_h the set of cells that have not spiked at those instants and also do not spike at instant t_h.

We have

$$\#A_h \geq m - h(K-1) \geq K^2 - h(K-1). \tag{39}$$

Arguing by induction as in Inequality (38), we obtain:

$$S_j(x_j(t_h)) \geq h\, l_j \qquad \forall\, j \in A_h. \tag{40}$$

For $h = K$, joining Inequality (40) and Equality (35), we deduce:

$$S_j(x_j(t_K)) \geq \theta_j \ \ \forall\, j \in A_K.$$

In other words, we have proved that, if at each instant $t_1 < t_2 < \ldots < t_{K-1}$, not more than $K-1$ cells spike simultaneously, then, at instant t_K the value of S_j arrives to the goal level θ_j, for any cell $j \in A_K$. This implies that all the cells of A_K spike simultaneously at instant $t_K > 0$. Besides, from Inequality (39) there exist at least $K^2 - K(K-1) = K$ cells in the set A_K. So, we have proved that at least K cells spike simultaneously at some instant $t_1 > 0$ or $t_2 > t_1$ or $\ldots t_{K-1}$ or t_K. This ends the proof of Assertion (v), as wanted, and thus the proof of Part (a) of Theorem 3.5 is complete. $\qquad \square$

Proof: **Part b)**

Since the hypothesis (14) is a strict inequality, it establishes an open condition in the space of parameters of the network, endowed with the topology of Definition 2.5. Therefore, the Inequality (14) joint with any dynamical property that is deduced from it, is a robust phenomenon. In particular, due to Part (a) of Theorem 3.5, Inequality (14) joint with the eventually periodic synchronization of the spikes, is a robust phenomenon. This proves Assertion (b) of Theorem 3.5. $\qquad \square$

4.2 Proof of Theorem 3.6

Proof: From the proof of Part (a) of Theorem 3.5, the waiting time $T > 0$ until the spike-synchronization of all the cells occurs, equals the time that takes the latest cell, say i, to arrive to its goal level θ_i from its initial state $x_i(0)$. The worst case occurs when the initial state $x_i(0)$ of this slowest cell i is such that $S_i(x_i(0))$ takes its lowest possible value 0 (cf. Inequality (7)). Thus, to consider the worst case, we assume

$$S_i(x_i(0)) = 0.$$

Therefore T is smaller or equal than the time constant t_i of the differential equation (4), for the solution $S_i(x_i(t))$ with initial state $S_i(x_i(0)) = 0$, such that $S_i(x_i(t_i^-)) = \theta_i$. This is because during the time-interval $[0, t_i)$, some other cells $j \neq i$ might have spiked. So they might have injected non negative jumps $\Delta_{j,i}$ to the instantaneous value of the variable S_i of the cell i.

Then, the worst case if when all those jumps are zero. Summarizing, we have

$$T \leq t_i$$

and the worst case occurs if S_i is only governed by the differential equation (4) for all $t \in [0, t_i)$. Due to the mean value theorem of the differential calculus, there exists a time $\tau_i \in [0, t_i)$ such that:

$$\left.\frac{dS_i}{dt}\right|_{t=\tau_i} = \frac{S_i(x_i(t_i^-)) - S_i(x_i(0))}{t_i} = \frac{\theta_i}{t_i}. \tag{41}$$

Using the differential equation (4), we have:

$$\left.\frac{dS_i}{dt}\right|_{t=\tau_i} = g_i(x_i(\tau)) \geq \min\{g_i(x_i) \colon x_i \in X_i\}. \tag{42}$$

Joining Equality (41) and Inequality (42), we obtain:

$$T \leq t_i \leq \frac{\theta_i}{\min\{g_i(x_i) \colon x_i \in X_i\}} \leq \max_{i \in \mathcal{N}} \left\{ \frac{\theta_i}{\min\{g_i(x_i) \colon x_i \in X_i\}} \right\},$$

proving the first assertion of Theorem 3.6.

Now, let us prove the second assertion of Theorem 3.6, finding an upper bound for the natural spiking period p. We revisit the proof of Part (a) of Theorem 3.5. We have defined the constant K as the minimum positive natural number that satisfies Inequality (31). Thus:

$$K \leq 1 + \frac{\max\{\theta_j \colon j \in \mathcal{N}\}}{\min\{\Delta_{i,j} \colon i, j \in \mathcal{N}, i \neq j\}}.$$

On the one hand, by Property (iv) in the proof of Part (a) of Theorem 3.5, if K cells spike simultaneously at instant t_i then all the cells spike simultaneously at instant t_i. On the other hand, at the end of the proof of Assertion (v), (second case), we found that from any initial state, after at most K spikes of some cells (i.e. at instant t_K as latest), there exist K cells that spike simultaneously. We conclude that, once all the cells had simultaneously spiked at instant t_0, the minimum next instant $t_p > t_0$ for which all the cells spike again simultaneously, is such that $p \leq K$. Therefore:

$$p \leq K \leq 1 + \frac{\max\{\theta_j \colon j \in \mathcal{N}\}}{\min\{\Delta_{i,j} \colon i, j \in \mathcal{N}, i \neq j\}},$$

ending the proof of Theorem 3.6. $\qquad\square$

5   The proof of the amount of information

In this Section, we prove Theorem 3.20. We compute the total amount of information of fully cooperative networks with a large number m of cells.

Proof: of Theorem 3.20

In Part (B) of Lemma 4.1 we have proved that

$$\max_{k \geq 0}\{\#\mathcal{P}_k\} = p,$$

where p is the natural spiking period, i.e the period of the sequence $\{I_n\}_{n \in \mathbb{N}}$ of spiking-codes. Therefore, from Formula (23), the total amount of information H that the network can generate or process is

$$H = \sup_{k \geq 0}\{\log_2(\#\mathcal{P}_k)\} = \log_2(\sup_{k \geq 0}\{\#\mathcal{P}_k\}) = \log_2(\max_{k \geq 0}\{\#\mathcal{P}_k\}) = \log_2 p.$$

(The first equality in the above chain holds because the real function $\log_2(x)$ is increasing on $x \in \mathbb{R}^+$.) We have proved that

$$H = \log_2 p,$$

which is the first assertion of Theorem 3.20. Now, let us prove the second assertion. We use the upper bound of the natural spiking period p that was proved in Theorem 3.20, Formula (16). We conclude that

$$H = \log_2 p \leq \log_2 \left(1 + \frac{\max\limits_{1 \leq j \leq m} \theta_j}{\min\limits_{i \neq j} \Delta_{i,j}}\right),$$

ending the proof of Theorem 3.20. □

6 The proofs of results on the protection factor

In this section we prove Proposition 3.14 and Theorem 3.15.

6.1 Proof of Proposition 3.14

Proof: From Equalities (17) and (21), we obtain

$$\left(1 - P_i\right) R_i = \frac{\left(\theta_i - \Delta_{i,\mathcal{N}}^{(h)}\right) \theta_i}{\theta_i \left(\max\limits_{1 \leq j \leq m} \theta_j\right)} = \frac{\theta_i - \Delta_{i,\mathcal{N}}^{(h)}}{\max\limits_{1 \leq j \leq m} \theta_j}$$

From Equality (20), $R_i' = \max\left\{0, \ \min\left\{1, \ \dfrac{\theta_i - \Delta_{i,\mathcal{N}}^{(h)}}{\max\limits_{1 \leq j \leq m} \theta_j}\right\}\right\}$,

and thus: $R_i' = \max\left\{0, \ \min\left\{1, \ \left(1 - P_i\right) R_i\right\}\right\}$. □

6.2 Proof of Theorem 3.15

Proof: By hypothesis the network is fully cooperative. Thus, $\Delta_{j,i} > 0$ for all $j \neq i$. By Equalities (19) and (21), the protection factor P_i of the network to each cell is positive.

Since $P_i > 0$, we obtain $1 - P_i < 1$. Applying Formula (22) of Lemma 3.14, and recalling that $0 < R_i \leq 1$ -cf. Inequality (18)- we obtain:

$$R_i' < \max\{0, \min\{1, R_i\}\} = R_i.$$

Therefore the net risk of death R_i' of the cell i when interacting in the network is strictly smaller than the intrinsic risk R_i that i would have if it were not connected to the network. This ends the proof of Theorem 3.15. □

7 Proof of the results on networks with similar cells

In this section we prove Theorem 3.22:

Proof: **Part a) of Theorem 3.22**

From Part (a) of Theorem 3.5, there exists a first instant $t_0 > 0$ such that all the cells spike simultaneously at t_0. So, to proof Part (a) of Theorem 3.22 it is enough to show the following assertion, under the additional hypothesis of Inequality (24):

Assertion (vi) to be proved. *If at some instant $t_0 > 0$ all the cells spike simultaneously, and if $t_1 > t_0$ is the first instant after t_0 when at least one cell spikes, then all the cells spike simultaneously at t_1.*

Fix i, one of the cells that spike at instant $t_1 > t_0$. By hypothesis, the cell i has also spiked at instant t_0, but not during the inter-spike-interval (t_0, t_1). Then, applying the reset rule (8), we have:

$$S_i(x_i(t_0)) = 0$$

Since i also spikes at instant $t_1 > t_0$, we have:

$$S_i(x_i(t_1^-)) = \theta_i.$$

Therefore:

$$S_i(x_i(t_1^-)) - S_i(x_i(t_0)) = \theta_i.$$

No cell spikes during the time-interval (t_0, t_1). Thus, S_i is governed by the differential equation (4) during such a time-interval. Applying the mean value theorem of the differential calculus, there exists $\tau_i \in (t_0, t_1)$ such that:

$$\left. \frac{dS_i}{dt} \right|_{t \,=\, \tau_i} = \frac{S_i(x_i(t_1^-)) - S_i(x_i(t_0))}{t_1 - t_0} = \frac{\theta_i}{t_1 - t_0}. \tag{43}$$

Using the differential equation (4), and recalling that $x_i \in X_i$, X_i is compact and g is continuous, we obtain:

$$\frac{dS_i}{dt} = g_i(x_i) \leq \max\{g_i(x_i) \colon x_i \in X_i\}. \tag{44}$$

Joining Equality (43) and Inequality (44), we deduce:

$$t_1 - t_0 \geq \frac{\theta_i}{\max\{g_i(x_i) \colon x_i \in X_i\}} \geq \frac{\min\limits_{1 \leq i \leq m} \theta_i}{\max\limits_{1 \leq i \leq m} \max\{g_i(x_i) \colon x_i \in X_i\}}. \tag{45}$$

Now denote by j any cell. It spikes at instant t_0. Then,

$$S_j(x_j(t_0)) = 0, \quad 0 \leq S_j(x_j(t_1^-)) \leq \theta_j.$$

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No cell spikes during the time-interval (t_0, t_i). Thus, S_j is governed by the differential equation (4) during such an interval of time. Applying the mean value theorem of the differential calculus, there exists $\tau_j \in (t_0, t_1)$ such that:

$$\left. \frac{dS_j}{dt} \right|_{t \, = \, \tau_j} = \frac{S_j(x_j(t_1)^-) - S_j(x_j(t_0))}{t_1 - t_0} = \frac{S_j(x_j(t_1)^-)}{t_1 - t_0}. \tag{46}$$

From the differential equation (4), we deduce:

$$\frac{dS_j}{dt} = g_j(x_j) \geq \min\{g_j(x_j) : \; 0 \leq x_j \leq \theta_j\} \geq \min_{1 \leq i \leq m} \min\{g_i(x_i) : \; x_i \in X_i\}. \tag{47}$$

Joining Inequalities (46) and (47), we deduce:

$$t_1 - t_0 \leq \frac{S_j(x_j(t_1^-))}{\min\limits_{1 \leq i \leq m} \min\{g_i(x_i) : \; x_i \in X_i\}\}}. \tag{48}$$

Joining Inequalities (45) and (48), we obtain:

$$S_j(x_j(t_1^-)) \geq \frac{\left(\min\limits_{1 \leq i \leq m} \min\{g_i(x_i) : \; x_i \in X_i\}\} \right) \left(\min\limits_{1 \leq i \leq m} \theta_i \right)}{\max\limits_{1 \leq i \leq m} \max\{g_i(x_i) : x_i \in X_i\}}$$

from where

$$S_j(x_j(t_1^-)) - \theta_j \geq$$

$$\geq \frac{\left(\min\limits_{1 \leq i \leq m} \min\{g_i(x_i) : \; x_i \in X_i\}\} \right) \left(\min\limits_{1 \leq i \leq m} \theta_i \right)}{\left(\max\limits_{1 \leq i \leq m} \max\{g_i(x_i) : x_i \in X_i\} \right) \left(\max\limits_{1 \leq i \leq m} \theta_i \right)} \cdot \left(\max\limits_{1 \leq i \leq m} \theta_i \right) - \theta_j \geq$$

$$\geq \max\limits_{1 \leq i \leq m} \theta_i \cdot \left(\frac{\left(\min\limits_{1 \leq i \leq m} \min\{g_i(x_i) : \; x_i \in X_i\}\} \right) \left(\min\limits_{1 \leq i \leq m} \theta_i \right)}{\left(\max\limits_{1 \leq i \leq m} \max\{g_i(x_i) : x_i \in X_i\} \right) \left(\max\limits_{1 \leq i \leq m} \theta_i \right)} - 1 \right) \tag{49}$$

By hypothesis, Inequality (24) holds. So, the factor at right (between large parenthesis) in Inequality (49) is bounded from below by $-(\min_{i \neq j} \Delta_{i,j}) / \max_{1 \leq i \leq m} \theta_i$. We deduce:

$$S_j(x_j(t_1^-)) - \theta_j \geq -\min_{i \neq j} \Delta_{i,j},$$

from where

$$S_j(x_j(t_1^-)) + \min_{i \neq j} \Delta_{i,j} \geq \theta_j. \tag{50}$$

Since at instant t_1 the cell i spikes, it sends an action $\Delta_{i,j}$ to each cell j. From Inequality (50), and from the interaction rule in Equalities (10), we get:

$$S_j(x_j(t_1)) = S_j(x_j(t_1^-)) + \Delta_{i,j} \geq S_j(x_j(t_1^-)) + \min_{i \neq j} \Delta_{i,j} \geq \theta_j.$$

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Therefore, the variable S_j of any cell j arrives to its goal level θ_j at the instant t_1. Thus, any cell j spikes at t_1, and Assertion (vi) is proved. This ends the proof of Part (a) of Theorem 3.22. $\qquad\square$

Proof: **Part b) of Theorem 3.22**

First, let us prove that the protection factor P_i of the network to each cell i, according to Definition 3.12, satisfies

$$\min\{1,\ P_i\} = 1 \qquad \text{(to be proved)}. \tag{51}$$

In fact, by Formula (19) the net sum of the actions $\Delta_{i,\mathcal{N}}^{(h)}$ that the cell i receives from the other cells of the network during its h-th. inter-spike-interval $(t_i^{(h)}, t_i^{(h+1)}]$ is

$$\Delta_{i,\mathcal{N}}^{(h)} = \sum_{j \neq i,\ \ j \in I_n,\ t_n \in (t_i^{(h)}, t_i^{(h+1)}]} \Delta_{j,i} \tag{52}$$

By Part (a) of Theorem 3.22, all the cells synchronize spikes after a transitory time T. So, for all $h \geq 1$ such that $t_i^{(h+1)} \geq T$, all the cells belong to I_n for the spiking time $t_n = t_i^{(h+1)}$. Thus, in Equality (52) the sum at right is realized in $j \in \{1, \ldots, m\}$ such that $j \neq i$. Then,

$$\Delta_{i,\mathcal{N}}^{(h)} = \sum_{j \neq i} \Delta_{j,i} \geq (m-1) \min_{j \neq i} \Delta_{j,i}, \tag{53}$$

where m is the number of cells of the network. By hypothesis, Inequality (14) holds. Therefore $m \geq 3$, which implies $m - 1 \geq \sqrt{m}$. We deduce that:

$$m - 1 \geq \frac{\max\limits_{1 \leq i \leq m} \theta_i}{\min\limits_{j \neq i} \Delta_{j,i}}.$$

Substituting in (53), we obtain: $\Delta_{i,\mathcal{N}}^{(h)} \geq \max\limits_{1 \leq i \leq m} \theta_i \geq \theta_i$. Thus, $\dfrac{\Delta_{i,\mathcal{N}}^{(h)}}{\theta_i} \geq 1$. Now, we apply Formula (21): $P_i = \dfrac{\Delta_{i,\mathcal{N}}^{(h)}}{\theta_i} \geq 1$, and thus Equality (51) is proved.

Second, we apply Lemma 3.14. From Equalities (22) and (51), and Inequality (18), we deduce:

$$R_i' = \max\big\{0,\ \min\{1,\ (100 - P_i)R_i\ \}\big\} = \max\{0,\ (1 - P_i)R_i\} = 0,$$

ending the proof of Part (b) of Theorem 3.22. $\qquad\square$

Proof: **Part c) of Theorem 3.22**

From Part (a) of Theorem 3.22, under the additional hypothesis stated by Inequality (24), the period p of the sequence $\{I_n\}_{n \in \mathbb{N}}$ of spiking-codes is $p = 1$. From Theorem 3.20, we know that the total amount of information H that the network can generate of process is $\log_2 p$. Therefore, $H = \log_2 p = \log_2 1 = 0$, as wanted. $\qquad\square$

8 Proof of Corollary 3.24

(Large full cooperative core)

Proof: Since by hypothesis, the number m of cells of the core $\mathcal{N}_1$ satisfies Inequality (25), we can repeat all the arguments in the proof of Part (a) of Theorem 3.5 in Subsection 4.1, by substituting the network $\mathcal{N}$ by the core $\mathcal{N}_1$. So, we deduce that there exists a strictly increasing sequence $\{t_n\}_{n \geq 0}$ of instants $t_n \to +\infty$, and a natural period $p \geq 1$, such that:

At least one cell of $\mathcal{N}_1$ spikes at each instant t_n, for all $n \geq 0$.

No cell of $\mathcal{N}_1$ spikes in the open time-intervals (t_n, t_{n+1}).

All the cells of $\mathcal{N}_1$ spike simultaneously at the instant t_{hp} for all $h \geq 0$.

Therefore, to prove the eventual periodic synchronization of the whole network $\mathcal{N}$, it is enough to prove that the cells of $\mathcal{N}$ spike altogether at each instant t such that all the cells of the core $\mathcal{N}_1$ spike.

In fact, we repeat, with slight changes, the proof of Property (iv) in Subsection 4.1: We define the *minimum natural number* $K \geq 1$ such that

$$K > \frac{\max\{\theta_j \colon \ j \in \mathcal{N}\}}{\min\{\Delta_{i,j} \colon i \in \mathcal{N}_1, \ j \in \mathcal{N} \ , \ i \neq j\}}.$$

Due to Inequality (25), the number m of the core $\mathcal{N}_1$ satisfies

$$\sqrt{m} > K. \tag{54}$$

Now, we repeat the same arguments of the proof in Subsection 4.1, starting at Equality (33) and ending just before Property (iv): Substituting $\min_{i \neq j} \Delta_{i,j}$ by

$$\min\{\Delta_{i,j} \colon \ i \in \mathcal{N}_1, \ j \in \mathcal{N} \ , i \neq j\}, \tag{55}$$

we deduce

Property **(iv)'**: *If at least K cells of the core $\mathcal{N}_1$ spike at instant $t > 0$, then all the cells of the network $\mathcal{N}$ spike at instant t.*

Since at instant t_{hp} all the m cells of the core $\mathcal{N}_1$ spike, and $m \geq 3$ satisfies Inequality (54), we obtain:

$$m > \sqrt{m} > K.$$

We conclude that the cells of the network spike altogether at instants t_{hp} for all $h \geq 0$. Thus, the whole network periodically synchronizes spikes from any initial state (after a finite transitory time).

Once we have proved the periodic synchronization of the network, the proof of the other assertions under the hypothesis of Corollary 3.24, follow the same arguments in the proofs of Sections 4, 5 and 6, after substituting $\min_{i \neq j} \Delta_{i,j}$ by the expression (55). $\qquad \square$

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