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RIGIDITY OF ISOMETRIC IMMERSIONS OF HIGHER CODIMENSION

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§1 Introduction

Let $f: M^n \to \mathbf{R}^N$ be an isometric immersion into the Euclidean space of an n-dimensional connected Riemannian manifold. Even assuming f to be rigid, a large set of non-congruent isometric immersions of M^n into $\mathbf{R}^{N+1}$ can be produced by composing f with elements of the infinite-dimensional family of (local) isometric immersions of $\mathbf{R}^N$ into $\mathbf{R}^{N+1}$. The main purpose of this paper is to provide sufficient conditions on f to ensure that any isometric immersion of M^n into $\mathbf{R}^{N+1}$ is a composition of the above isometric immersions.

A classical rigidity theorem due to Allendoerfer [All] states that f is rigid in $\mathbf{R}^N$ if the type number τ of his vector valued second fundamental form $\alpha: TM \times TM \to TM^\perp$ satisfies $\tau(x) \geq 3$ for all $x \in M$. Recall that the type number of a symmetric bilinear form $\beta: \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^p$ is the largest integer τ for which there are τ vectors $X_1, \ldots, X_\tau$ in $\mathbf{R}^n$ such that the τp vectors $B_{\xi_j}(X_i)$, $1 \leq i \leq 3$, $1 \leq j \leq p$, are linearly independent, where $\xi_1, \ldots, \xi_p$ is a basis of $\mathbf{R}^p$ and $B_{\xi_j}: \mathbf{R}^n \to \mathbf{R}^n$ is given by $\langle B_{\xi_j} X, Y \rangle = \langle \beta(X,Y), \xi_j \rangle$. We define the *rank* ρ of β as

$$\rho = \min\{rank\, B_\xi : \xi \in \mathbf{R}^p, \xi \neq 0\}.$$

Observe that $\rho \geq \tau$. The rank of an isometric immersion at a point is the rank of its second fundamental form at this point. Our main result is the following.

Theorem 1. *Let $f: M^n \to \mathbf{R}^N$ be an isometric embedding of a simply-connected Riemannian manifold with type number $\tau \geq 3$ and rank $\rho \geq 4$, everywhere. If $g: M^n \to \mathbf{R}^{N+1}$ is a 1-regular isometric immersion, then $g = h \circ f$, where $h: U \subset \mathbf{R}^N \to \mathbf{R}^{N+1}$ is isometric, U is open and $f(M) \subset U$.*

Recall that the assumption of 1-regularity means that the first normal space $N_1(x) = \text{span}\{\alpha(X, Y) : X, Y \in T_xM\}$ has constant dimension.

Theorem 1 was obtained by Erbacher [Er] for $M^n = S_c^n$ and $N = n+1$. Counterexamples exist if we drop either the 1-regularity or the rank hypothesis. Hencke [He] showed that local isometric immersions $S_c^n \supset U \to \mathbf{R}^{n+2}$ may not be a composition of immersions near umbilical points. As for the rank condition, there exists local isometric immersions of $S_c^3 \supset U \to \mathbf{R}^5$ which are not composition of immersions (see [D-T]). Moreover, the theorem is also not true if we do not ask f to be an embedding. It is not too difficult to verify that an n-dimensional tube with self intersections along a curve in $\mathbf{R}^{n+1}$ may admit a 1-regular isometric embedding in $\mathbf{R}^{n+2}$.

On the other hand, some of the assumptions of Theorem 1 may be dropped if we restrict ourselves to the class of minimal immersions.

Theorem 2. *Let $f: M^n \to \mathbf{R}^N$ be a minimal isometric immersion with type number $\tau(x_0) \geq 3$ at a point $x_0 \in M$. Then any minimal isometric immersion $g: M^n \to \mathbf{R}^{N+1}$ is congruent to f in $\mathbf{R}^{N+1}$.*

Finally, we apply Theorem 1 to the study of Riemannian manifolds which can be isometrically immersed in both, $\mathbf{R}^N$ and S_c^N. The case of codimension one was already considered by do Carmo-Dajczer [C-D], where it is shown that the manifold must be conformally flat. Conversely, a simply-connected conformally flat hypersurface L^N of S_c^{N+1}, $N \geq 4$, without umbilical points, can be isometrically immersed in $\mathbf{R}^{N+1}$ (see [C-Y]) as a 1-parameter envelope of spheres (see [A-D]). In particular, any submanifold M^n of L^N admits isometric immersions in S_c^{N+1} and $\mathbf{R}^{N+1}$. We say that an isometric immersion $f: M \to \tilde{M}$ has an *umbilical direction* $\xi \in T_xM^\perp$ at $x \in M$ if $A_\xi = \lambda I$, $0 \neq \lambda \in \mathbf{R}$.

Theorem 3. *Let $f: M^n \to \mathbf{R}^{n+p}$, $p \geq 2$, be an isometric embedding of a simply-connected Riemannian manifold with type number $\tau \geq 3$, rank $\rho \geq 4$, and free of umbilical direc-*

tions, everywhere. If M^n admits an isometric immersion $g: M^n \to S_c^{n+p}$, then there exists a conformally flat manifold N^{n+p-1} and isometric immersions $k: M^n \to N^{n+p-1}$, $h_1: N^{n+p-1} \to \mathbf{R}^{n+p}$, and $h_2: N^{n+p-1} \to S_c^{n+p}$ so that $f = h_1 \circ k$ and $g = h_2 \circ k$.

Although all our results reffer to submanifolds of the Euclidean space, they remain valid for submanifolds of the Euclidean sphere or the hyperbolic space by similar arguments.

§2 Some linear algebra results

We will make use of the theory of flat bilinear forms to prove some lemmas which describe the pointwise structure of the second fundamental forms of the immersions involved in our theorems.

Lemma 1. *Let $\alpha: \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^p$ be a symmetric bilinear form with type number $\tau \geq 3$ and rank $\rho \geq 4$. Assume that $\gamma: \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^{p+1}$ is a symmetric bilinear form which verifies*

$$\langle \alpha(X,Y), \alpha(Z,W) \rangle - \langle \alpha(X,W), \alpha(Z,Y) \rangle = \langle \gamma(X,Y), \gamma(Z,W) \rangle - \langle \gamma(X,W), \gamma(Z,Y) \rangle$$

for all $X, Y, Z, W \in \mathbf{R}^n$. Then there exists an orthogonal sum decomposition $\mathbf{R}^{p+1} = \mathbf{R}^p \oplus \mathbf{R}$, such that

(i) $\pi_{\mathbf{R}^p} \circ \gamma = \alpha$,

(ii) rank $\pi_{\mathbf{R}} \circ \gamma \leq 1$,

where $\pi_{\mathbf{R}^p}$ (respectively $\pi_{\mathbf{R}}$) denotes the orthogonal projection onto $\mathbf{R}^p$ (respectively $\mathbf{R}$).

Proof: Consider the symmetric bilinear form

$$\beta = \alpha \oplus \gamma: \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^p \oplus \mathbf{R}^{p+1} \approx \mathbf{R}^{2p+1},$$

where $\mathbf{R}^{2p+1}$ is endowed with the nondegenerate inner product $\langle\langle\ ,\ \rangle\rangle$ defined by

$$\ll (\xi, \xi'), (\eta, \eta') \gg = \langle \xi, \eta \rangle - \langle \xi', \eta' \rangle .$$

Then β is flat in the sense that it verifies

$$\ll \beta(X,Y),\beta(Z,W) \gg - \ll \beta(X,W),\beta(Z,Y) \gg = 0$$

for all $X, Y, Z, W \in \mathbf{R}^n$.

Given $X \in \mathbf{R}^n$, we define a linear transformation $B(X): \mathbf{R}^n \to \mathbf{R}^{2p+1}$ by $B(X)(Z) = \beta(X,Z)$. We say that $X \in \mathbf{R}^n$ belongs to the set $RE(\beta)$ of *regular elements* iff

$$\dim B(X)(\mathbf{R}^n) = q = \max\{\dim B(Y)(\mathbf{R}^n) : Y \in \mathbf{R}^n\}.$$

The basic property of an element $X \in RE(\beta)$ (see [Mo], p. 246) is that for any vector $n \in \operatorname{Ker} B(X)$,

$$\beta(\mathbf{R}^n, n) \subset \mathcal{U}(X) = B(X)(\mathbf{R}^n) \cap B(X)(\mathbf{R}^n)^{\perp}. \tag{1}$$

Let $k_0 = \min\{\dim \mathcal{U}(X) : X \in \mathbf{R}^n\}$, and define

$$RE^{*}(\beta) = \{X \in RE(\beta) : \dim \mathcal{U}(X) = k_0\}.$$

Since $RE^{*}(\beta)$ is open and dense in $\mathbf{R}^n$ (see [D-R], p. 214) and $r \geq 3$, there exist $X_1, X_2, X_3 \in RE^{*}(\beta)$ such that the vectors

$$\{B_{\xi_j} X_i : 1 \leq i \leq 3, 1 \leq j \leq p\}$$

are linearly independent for a basis $\xi_1, \dots, \xi_p$ of $\mathbf{R}^p$. Now, the subspace

$$S = \{Z \in \mathbf{R}^n : \alpha(X_i, Z) = 0, 1 \leq i \leq 3\}$$

satisfies

$$S = [\operatorname{span}\{B_{\xi_j} X_i : 1 \leq i \leq 3, 1 \leq j \leq p\}]^{\perp},$$

and therefore

$$\dim S = n - 3p. \tag{2}$$

Since $B(X_2)(\operatorname{Ker} B(X_1)) \subset \mathcal{U}(X_1)$ by (1), thus the linear transformation

$$D(X_2) = B(X_2)|_{\operatorname{Ker} B(X_1)} : \operatorname{Ker} B(X_1) \to \mathcal{U}(X_1)$$

satisfies

$$\operatorname{Ker} D(X_2) = \operatorname{Ker} B(X_1) \cap \operatorname{Ker} B(X_2).$$

and

$$\dim \operatorname{Ker} D(X_2) \geq \dim \operatorname{Ker} B(X_1) - k_0.$$

Similarly,

$$D(X_3) = B(X_3)|_{\operatorname{Ker} D(X_2)}: \operatorname{Ker} D(X_2) \to \mathcal{U}(X_1) \cap \mathcal{U}(X_2)$$

satisfies

$$\operatorname{Ker} D(X_3) = \bigcap_{j=1,2,3} \operatorname{Ker} B(X_j),$$

and

$$\begin{aligned} \dim \operatorname{Ker} D(X_3) &\geq \dim \operatorname{Ker} B(X_1) - \dim \mathcal{U}(X_1) \cap \mathcal{U}(X_2) \\ &\geq n - q - k_0 - \dim \mathcal{U}(X_1) \cap \mathcal{U}(X_2). \end{aligned} \tag{3}$$

Since $\bigcap_{j=1}^{3} \operatorname{Ker} B(X_j) \subset S$, we get from (2) and (3) that

$$n - 3p = n - q - k_0 - \dim \mathcal{U}(X_1) \cap \mathcal{U}(X_2).$$

Using

$$q \leq 2p + 1 - k_0, \quad \dim \mathcal{U}(X_1) \cap \mathcal{U}(X_2) \leq k_0, \tag{4}$$

we get $k_0 \geq p - 1$. Hence, either $k_0 = p - 1$ or $k_0 = p$.

Case $k_0 = p - 1$. Under this assumption, inequalities (4) must be equalities. We easily conclude that

(i) $q = p + 2$,

(ii) $\mathcal{U}(X_i) = \mathcal{U}(X_j)$, $1 \leq i, j \leq 3$.

Set $\mathcal{U}(X_j) = \mathcal{U}$. We claim that $\mathcal{U}^\perp = S(\beta)$, where

$$S(\beta) = \operatorname{span}\{\beta(Y, Z) : Y, Z \in \mathbf{R}^n\}.$$

From $\mathcal{U} \subset ImB(X_j)^\perp$, we have $\mathcal{U}^\perp \supset ImB(X_j)^{\perp\perp} = ImB(X_j)$. Since $\dim \mathcal{U}^\perp = 2p + 1 - \dim \mathcal{U} = p + 2$, we get.

$$ImB(X_i) = \mathcal{U}^\perp, \quad 1 \leq i \leq 3.$$

Clearly, given $Y \in \mathbf{R}^n$, there exists $\varepsilon > 0$ such that the set of vectors $\overline{X}_1 = X_1 + \varepsilon Y$, X_2, X_3 have the same properties than the set X_1, X_2, X_3. Hence, $ImB(\overline{X}_1) = \mathcal{U}^\perp$, and therefore $ImB(Y) \subset \mathcal{U}^\perp$. This proves the claim.

Consider orthonormal bases $\xi_1, \ldots, \xi_p$ of $\mathbf{R}^p$ and $\eta_1, \ldots, \eta_{p+1}$ of $\mathbf{R}^{p+1}$, such that

$$\mathcal{U} = \text{span}\{\delta_j = \xi_j + \eta_j, 1 \leq j \leq p-1\}.$$

It follows from the claim that

$$0 = \ll \beta(Y, Z), \delta_j \gg = \langle \alpha(Y, Z), \xi_j \rangle - \langle \gamma(Y, Z), \eta_j \rangle$$

for all $Y, Z \in \mathbf{R}^n$ and $1 \leq j \leq p-1$. Set

$$\phi = \sum_{j=1}^{p-1} \langle \alpha, \xi_j \rangle \xi_j, \quad \psi = \langle \alpha, \xi_p \rangle \xi_p, \quad \theta = \sum_{h=p}^{p+1} \langle \gamma, \eta_h \rangle \eta_h.$$

It follows from the claim and flatness of β that α and γ split orthogonally as

$$\alpha = \phi \oplus \psi, \quad \gamma = \phi \oplus \theta,$$

where the symmetric bilinear form

$$\psi \oplus \theta : \mathbf{R}^n \times \mathbf{R}^n \to \mathbb{L} = \text{span}\{\xi_p, \eta_p, \eta_{p+1}\}$$

is flat with respect to the induced Lorentizian metric on $\mathbb{L}$ as a subspace of $\mathbf{R}^{2p+1}$. Since rank $B_{\xi_p} \geq 4$, we have from Corollary 2 of [Mo] that $S(\psi \oplus \theta) \neq \mathbb{L}$. Hence Corollary 3 of [Mo] applies, and therefore η_p, η_{p+1} can be chosen so that

$$\langle \alpha, \xi_p \rangle = \langle \gamma, \eta_p \rangle, \quad rank\, \langle \gamma, \eta_{p+1} \rangle \leq 1.$$

This concludes the proof of this case.

Case $k_0 = p$. This case is now trivial. ∎

Lemma 2. *The same conclusions than in Lemma 1 remain valid if instead of $\rho \geq 4$, we assume trace α = trace $\gamma = 0$.*

Notice that, in this case, we have trace $\pi_R \circ \gamma = 0$, and consequently, $\pi_R \circ \gamma = 0$.

Proof: To prove Lemma 1, condition $\rho \geq 4$ was used only once, namely to deal with the bilinear form $\psi \oplus \theta$. In this case, instead, we can use the following result whose proof is part of the arguments in ([B-D-J], pp. 435–436).

Lemma 3. *Let $B\colon \mathbf{R}^n \to \mathbf{R}^n$ be a symmetric linear transformation with rank $B \geq 3$, and let $\beta\colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^p$ be a symmetric bilinear form. Assume*

$$\textit{trace } B = \textit{trace } \beta = 0,$$

and

$$\langle BX, Y\rangle \langle BZ, W\rangle - \langle BX, W\rangle \langle BZ, Y\rangle = \langle \beta(X,Y), \beta(Z,W)\rangle - \langle \beta(X,W), \beta(Z,Y)\rangle$$

for all $X, Y, Z, W \in \mathbf{R}^n$. Then $\dim S(\beta) = 1$ and

$$\pi_{S(\beta)}\beta(X,Y) = \langle BX, Y\rangle$$

for all $X, Y \in \mathbf{R}^n$. ∎

§3 Proofs of the theorems

For the proof of Theorem 1 we will need the following result.

Lemma 4. *Let $f\colon M^n \to \mathbf{R}^{n+p}$ be an isometric immersion whose second fundamental form α splits orthogonally at each point $x \in M$ as*

$$\alpha = \gamma \oplus \langle A_\eta, \ \rangle \eta$$

for same unit $\eta \in T_xM^\perp$. Assume rank $A_\eta \equiv 1$ and that $\|\gamma(X,Y)\|^2$ is a smooth function for any $X, Y \in TM$. Then

(i) If γ has rank $\rho \geq 3$, and η is chosen so that A_η has a positive eigenvalue, then η is smooth.

(ii) Suppose that γ has type number $\tau \geq 3$ and rank $\rho \geq 4$. Then η is constant along Δ in $\mathbf{R}^{n+p+1}$.

Proof: i) First we show that for any hyperplane $W \subset T_xM$, we have at $x \in M$,

$$S(\gamma) = \operatorname{span}\{\gamma(Z,Y) : Z, Y \in W\}.$$

Otherwise, there exists $\delta \in T_x M^\perp$, so that $\langle \gamma(Z, Y), \delta \rangle = 0$ for all $Z, Y \in W$. This implies that rank $A_\delta \leq 2$, which is a contradiction.

Fixed $x_0 \in M$, let $X_1, \ldots, X_n$ be an orthonormal basis of $T_{x_0}M$ so that $A_\eta X_1 = \mu X_1, \mu > 0$. Extend locally $X_1, \ldots, X_n$ to linearly independent vector fields and set $Y_j = X_1 + X_{j+1}$, $2 \leq j \leq n$. By the above

$$S(\gamma) = \text{span}\{\gamma(Y_{i_k}, Y_{j_k}) : 1 \leq k \leq p\}$$

at any point in a neighboorhod of x_0.

Clearly, the set of vector fields

$$\alpha(Y_{i_k}, Y_{j_k}) = \gamma(Y_{i_k}, Y_{j_k}) + \langle A_\eta Y_{i_k}, Y_{j_k} \rangle \eta, \quad 1 \leq k \leq p$$

are linearly independent, and the functions $\psi_k = \langle A_\eta Y_{i_k}, Y_{j_k} \rangle$ satisfy:

(1) $\psi_k(x_0) = \mu > 0$,

(2) $\psi_k^2 = \|\alpha(Y_{i_k}, Y_{j_k})\|^2 - \|\gamma(Y_{i_k}, Y_{j_k})\|^2 \in C^\infty(M)$.

Consequently, all functions ψ_k are positive and smooth in a neighboorhod U of x_0.

At each point $x \in U$, η is a solution of the system of linear equations

$$\frac{1}{\psi_k} \langle \alpha(Y_{i_k}, Y_{j_k}), \eta \rangle = 1, \quad 1 \leq k \leq p.$$

Hence, η belongs to the intersection of $p-1$ affine hyperplanes $H_k \perp \alpha(Y_{i_k}, Y_{j_k})$ and the unit sphere $S_1^{p-1} \subset \mathbf{R}^p$. The line $\mathbf{R} = \bigcap_{k=1}^{p-1} H_k$ is orthogonal to all $\alpha(Y_{i_k}, Y_{j_k})$ and meets S_1^{p-1} in at least one point, since solution exists. To conclude the proof it is sufficient to show that $\mathbf{R} \cap S_1^{p-1}$ is allways two different points. But if $\mathbf{R}$ is tangent to S_1^{p-1}, then η would be orthogonal to $\mathbf{R}$, and therefore $\eta \in \text{span}\{\alpha(Y_{i_k}, Y_{j_k}) : 1 \leq k \leq p\}$, which is a contradiction.

(ii) Choose η to be smooth and local vector fields $X_1, \ldots, X_n$ so that $AX_1 = \mu X_1$, $\mu > 0$. By Codazzi's equation

$$\mu \langle [Y, Z], X_1 \rangle X_1 = A_{\nabla_Z^\perp \eta} Y - A_{\nabla_Y^\perp \eta} Z \tag{5}$$

for all $Y, Z \in \Delta$. At $x \in M$, consider the linear map $\phi : \Delta \to L$ defined by

$$\phi(X) = \nabla^{\perp}_{X}\eta,$$

where L is the orthogonal complement to η in $T_xM^{\perp}$. Suppose that $r = \dim Im\phi$ is positive. For a basis $\delta_1, \ldots, \delta_r$ of $Im\phi$, we have from (5)

$$A_{\delta_j}Y \in \text{span}\{X_j\}, \quad 1 \leq j \leq r$$

for all $Y \in \text{Ker}\,\phi$. Hence $\dim \text{Ker}\, A_{\delta_j} \geq \dim \text{Ker}\,\phi - 1$. It follows that

$$\dim \bigcap_{j=1}^{r} \text{Ker}\, A_{\delta_j} \geq \dim \text{Ker}\,\phi - 1 - (r-1) \geq n - 2r - 1.$$

For $r = 1$ this is a contradiction to $\rho \geq 4$. The assumption $r \geq 3$ implies that $\dim \bigcap_{j=1}^{r} \text{Ker}\, A_{\delta_j} \leq n - 3r$. This provides a contradiction for $r \geq 2$. Hence $r = 0$, and this concludes the proof. ∎

Proof of Theorem 1: From Lemmas 1 and 4 we have that the second fundamental form α of g splits orthogonally and smoothly as

$$\alpha = \gamma \oplus \langle A\eta, \ \rangle \eta,$$

where γ is congruent to the second fundamental form of f and η is a smooth unit normal vector field so that rank $A_\eta \equiv 1$. Set $\Delta = \text{Ker}\, A_\eta$, and let $X \in TM$ be a proper unit tangent vector field so that $A_\eta X = \mu X$, $\mu > 0$.

Let Λ be the vector bundle whose p-dimensional fibre is the orthogonal complement to span $\{\eta, \tilde{\nabla}_X\eta = -\mu X + \nabla^{\perp}_{X}\eta\}$ in $T_gM^{\perp} \oplus \text{span}\{\eta\}$. Define a map $H : \Lambda \to \mathbb{R}^{n+p+1}$ by

$$H(x, \xi) = g(x) + \xi.$$

Since $\mu > 0$, it is clear that H parametrizes a regular hypersurface "near" g. By Lemma 4(ii), we have

$$\langle H_* Z, \eta \rangle = \left\langle Z + \tilde{\nabla}_Z \xi, \eta \right\rangle = -\left\langle \xi, \tilde{\nabla}_Z \eta \right\rangle = 0$$

for any $Z \in TM$. We easily conclude that the Gauss map $\mathcal{N}$ of H is $\mathcal{N}(x,\xi) = \eta(x)$. Again by Lemma 4(ii) the hypersurface is flat.

Let $r \in C^{\infty}(M)$ be a positive function so that H is regular at $\Lambda' = \{(x,\xi) \in \Lambda : \|\xi(x)\| < r(x)\}$. Since Λ' is flat and simply-connected, there exists an isometric immersion (developing) $K: \Lambda' \to \mathbf{R}^{n+p}$, which induces an isometric immersion $K: M^n \to \mathbf{R}^{n+p}$ by restricting K to the zero-section of Λ'. By assumption K is congruent to f. Since f is an embedding, we may choose the function $r(x)$ so that K is also an embedding. Let $K(\Lambda') = U \subset \mathbf{R}^{n+p}$ and define $h: U \to \mathbf{R}^{n+p+1}$ by $h(y) = H \circ K^{-1}(y)$. Clearly, $h \circ y = f$. This concludes the proof. ∎

Proof of Theorem 2: Since minimal immersions are real analitic, it is sufficient to argue for an open subset $V \subset M$ where g is 1-regular and has type number $\tau \geq 3$. Moreover, we may assume that $p = N - n \geq 2$, since case $p = 1$ was already proved in [B-D-J]. By Lemma 2, the first normal space N_1 of g has constant dimension p and his second fundamental form has type number $\tau \geq 3$ since on N_1 it is congruent to the second fundamental form of f. It is well known that N_1 must be parallel (see [Sp] V p. 362) and therefore g reduces codimension to p. Consequently, g is congruent to f. ∎

Proof of Theorem 3: Let $\tilde{g}: M^n \to \mathbf{R}^{n+p+1}$ be the isometric immersion obtained by composing g with the inclusion of S_c^{n+p} in $\mathbf{R}^{n+p+1}$. Because f is free of umbilical directions we argue that $\tilde{g}$ is 1-regular. In fact, if the dimension of the first normal space of $\tilde{g}$ at some point is less than $p+1$, then the second fundamental form of g has an umbilical direction, and so does the second fundamental form of $\tilde{g}$. But then the second fundamental forms of f and $\tilde{g}$ must be congruent and this is a contradiction. Since Theorem 1 applies, there exists an isometric immersion $h: U \subset \mathbf{R}^{n+p} \to \mathbf{R}^{n+p+1}$ such that $\tilde{g} = h \circ f$. Take N^{h+p-1} to be a regular neighboorhod of $g(M)$ in $h(U) \cap S_c^{n+p}$. ∎

The assumption that f is free of umbilical directions is not essencial in the sense that if there exists an umbilical vector field globally defined then the proof of the theorem still works.

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BILLARES PLANOS
RESUMEN DE RESULTADOS SOBRE SUS PROPIEDADES ERGÓDICAS

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Resúmen: Este trabajo presenta, para no especialistas en sistemas dinámicos, los resultados conocidos sobre propiedades ergódicas de las transformaciones definidas por el movimento libre de una masa puntual en una región acotada del plano, con choques elásticos en la frontera. Se incluyen los resultados más recientes del autor.

Abstract: This work presents, for not especialists in dynamical systems, the known results on ergodic properties of the maps defined by the free motion of a point mass inside a bounded region of the plane with elastic reflections at the boundary. Recent results of the author are included.

0. Introducción

Una masa puntual se mueve sin fricción en una región acotada del plano. La partícula es reflejada elásticamente en los bordes exteriores y en los obstáculos que puede encontrar por el camino. Pero tambiém puede suceder que el borde externo ser un cuadrado y la partícula al llegar a uno de sus lados desaparezea y reaparezca en el lado opuesto, con la misma velocidad (billar tórico).

El movimento de la partícula es claramente muy complicado y las trayectorias no varían con continuidad (cuando son tangentes a uno de los bordes) *o* no están definidas, si las trayectorias se clavan en un ángulo (Ver fig.2). Pueden suceder comportamientos patológicos aún más inesperados, por ejemplo, trayectorias que tienen un número infinito de rebotes en tiempo finito (Halpern, 1977), o trayectorias que se acercan todo lo que se quiera a la frontera de un billar convexo en tiempo infinito (Mather, 1982). E incluso se pueden construir billares con fronteras razonablemente regulares ($\mathbf{C}^1$, por ejemplo) que no verifican buenas propiedades globales (para casi todas sus trayectorias). Ver (Katok, Strelcyn, 1986), Part V.5.2 y, aunque lo estudiado no es propiamente un billar plano,(Galperin, 1981), §8.

Para sortear estos últimos casos se deben colocar condiciones de regularidad más fuertes sobre los arcos de la frontera; y para eliminar las trayectorias anormales se debe usar teoría de medida y caer, por tanto, en el campo de la teoría ergódica.

Las fuentes de interés para el estudio de billares son diversas y están en un proceso de expansión muy acelerado al ser un modelo simple y *físico* para estudiar fenómenos de caos, bifurcaciones, etc. que están fuertemente en la interfase de las relaciones entre la matemática y sus aplicaciones. Mencionaremos sólo las más clásicas, que influyeran fuertemente en los primeros resultados:

a) Estudio de movimientos periódicos en billares convexos, como aplicación del llamado teorema geométrico de Poincaré (Birkhoff, 1927a,b)

b) Estudio de propiedades ergódicas del modelo de los gases de Boltzmann-Gibbs, comenzado en (Sinai, 1963 y 1970), sobre la base de consideraciones heurísticas de Krylov. Ver también (Sinai, 1979).

El presente trabajo intenta presentar los resultados exactos conocidos de propiedades ergódicas de billares planos no poligonales en un lenguaje accesible para un no especialista en sistemas dinámicos. Los resultados sobre billares poligonales (todos los arcos de frontera son segmentos de recta) son notoriamente menos explicitos: está probado (Kerckhoff et al., 1986) que en el conjunto de los polígonos convexos de n lados (considerado como subconjunto de $\mathbf{R}^{2n}$) hay un subconjunto denso, intersección numerable de abiertos, de polígonos cuyas transformaciones de billar son ergódicas; pero no se conoce la descripción de un billar ergódico. Ver resumen de resultados en (Gutkin, 1986) y (Arnoux, 1988). Respecto de los billares en dimensiones más altas sólo se han estudiado bien los billares semidispersores generados por el modelo de los gases de las bolas duras, y la demonstración final de la ergodicidad de ese modelo, para cualquier número de bolas aún no se ha logrado, aunque hay resultados parciales muy importantes; por ejemplo en (Sinai, Chernov, 1987) y (Kramli et al., 1988).

El lector no interesado en las técnicas de demostración de algunos resultados presentados puede prescindir de la última parte de la Sección 2 (lo que sigue a la fórmula 3).

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1. Propiedades ergódicas

Se dan a continuación las definiciones precisas de las propiedades ergódicas a que haremos referencia, y algunos resultados relacionados con ellas.

Sea μ una medida de probabilidad absolutamente continua respecto de la medida de volumen en una variedad M, compacta, suave; N un subconjunto de medida nula, $H = M \setminus N$ y $f\colon H \to H$ las restricción a H de un $\mathbf{C}^r$ difeomorfismo, $r \geq 1$, definido sobre un abierto de M, que preserva la medida μ. Si f'_x es la derivada de f en x, supondremos, para poder aplicar más adelante el teorema de Oseledets, que $\int_H \log^+ \|f'_x\| \, d\mu < \infty$ donde $\log^+ s = \max\{\log s, 0\}$.

Decimos que x es un *punto regular* (Oseledets) de f si existen números $\lambda_1(x) > \lambda_2(x) > \cdots > \lambda_m(x)$ y una descomposición del espacio tangente en x, $T_x M = E_1(x) \oplus E_2(x) \oplus \cdots \oplus E_m(x)$ tales que $\lim_{n \to \pm\infty} \frac{1}{n} \log \|(f^n)'_x\| = \lambda_i(x)$ para todo $0 \neq w \in E_i(x)$ y todo $1 \leq i \leq m(x)$. $E_j(x)$ es el subespacio propio correspondiente al exponente de Liapunov $\lambda_j(x)$. El *teorema de Oseledets* establece que el conjunto de los puntos regulares tiene medida uno. Obsérvese que si $M = \mathbf{R}^d$ y f es lineal los exponentes de Liapunov en el origen corresponden a los logaritmos del módulo de los autovalores de f. En este sentido, el teorema de Oseledets es una generalización del teorema de la forma canónica de Jordan.

$\Lambda = \Lambda(f)$ indicará la *región de Pesin*, esto es, el conjunto de los puntos regulares que sólo tienen exponentes de Liapunov no nulos. La existencia de exponentes de Liapunov no nulos para f permite construir variedades invariantes locales.[1] Si todos los exponentes de Liapunov son diferentes de cero en x se obtiene una descomposición hiperbólica no uniforme del espacio T_xM y (si Λ tiene medida positiva) existe una familia numerable de conjuntos invariantes Λ_i tales que $f_{|\Lambda_i}$ es ergódica y $\sum \mu(\Lambda_i) = \mu(\Lambda)$. Para un resumen de estos resultados ver, por ejemplo (Pesin, Sinai, 1981).

Si $\mu(\Lambda(f)) = 1$ *o* sea, si la región de Pesin tiene medida uno, diremos que la transformación f (*o* el sistema dinámico definido por ella) es ***hiperbólico no uniforme***, *o* que tiene ***comportamiento caótico***.

La definición de ergodicidad es muy simple: f es ***ergódica*** (respecto de la medida μ) si los conjuntos invariantes por f tienen medida cero o uno; esto es, si $f^{-1}(A) = A$ implica $\mu(A) = 0$ ó 1.

Por tanto, si la región de Pesin tiene medida uno, el espacio H se descompone en un conjunto numerable de componentes ergódicas Λ_i que son maximales en el sentido que no existen dentro de ellos conjuntos de medida positiva invariantes.

Una propiedad íntimamente relacionada con las anteriores es la siguiente: f es *Mixing* (mezcladora) si dados cualesquiera dos conjuntos medibles $A, B \subset H$, entonces $\lim_{n\to+\infty} \mu(f^{-n}(A) \cap B) = \mu(A)\mu(B)$. O sea que en el límite, la proporción-en medida-de $f^{-n}(A)$ que está en B es la misma que la de A en H: $\frac{\mu(f^{-n}(A)\cap B)}{\mu(B)} \simeq \frac{\mu(A)}{\mu(H)} = \mu(A)$.

Es simple ver que toda transformación mixing es ergódica. Y que si μ es una medida positiva sobre los abiertos de H y f es mixing, entonces f es *topológicamente mixing*: para

[1] En verdad, se necesitan algunas condiciones adicionales sobre las derivadas de f, que permitan un control mayor cerca del conjunto N que contiéne las "singularidades". No creemos necesario explicitarlas aquí. Ellas son verificadas por las transformaciones de billar que estudiaremos en este trabajo. Ver (Katok, Strelcyn, 1986).

cualesquier abiertos U, V existe $N \in \mathbf{N}$ tal que $f^{-n}(U) \cap V \neq \phi$ para todo $n \geq N$. Sobre estos últimos resultados y la llamada *jerarquía ergódica*, que incluye otras propiedades relacionadas con los resultados expuestos, ver (Mañé, 1983), Chap.III, 8 y 11. Allí se prueba que Bernoulli$\subset$ Kolmogorov $\subset$ Lebesgue $\subset$ Mixing $\subset$ ergódico, donde cada palabra indica el conjunto de los automorfismos que verifica su denominación.

No daremos aquí las definiciónes de las tres primeras categorías.

En virtud de la *fórmula de Pesin*, si exigimos a f algunas condiciones adicionales; por ejemplo, si $N = \phi$, debe ser $\mathbf{C}^{1+\epsilon}$ (Hölder C^1); resulta que la *entropía* de f viene dada por $h(f) = \int_H \sum_{\lambda_j(x)>0} \lambda_j(x) \dim E_j(x) d\mu$. Esta expresión puede ser tomada como la definición de entropía, aunque ella no es ninguna de las cronológicamente iniciales, ni refleja directamente la producción de información que tal definición implica. Ver (Mañé, 1983), Cap. IV. Entonces si la región de Pesin tiene medida positiva, la entropía es positiva. Es de notar que la entropia positiva de f no implica ergodicidad ni comportamiento caótico desde que pueden existir *bolsones de estabilidad*, ó sea conjuntos invariantes donde no hay caos.

En este sentido (existencia de regiones de estabilidad), el resultado más importante en dimensión 2 es el *teorema de Kolmogorov-Arnold-Moser (KAM)*. Comencemos definiendo transformaciones *twist* en $\mathbf{R}^2$; en coordenadas polares ellas son de la forma $(r, \theta) \to (r, \theta + \alpha_0 + \alpha_1 r)$ con α_0, α_1 constantes reales, $\alpha_1 \neq 0$. O sea que un twist deja invariantes los circulos $r = cte$ y rota cada uno de ellos un ángulo que depende de r. Respecto de los resultados más importantes de los twists y el teorema KAM, dos referencias recientes son (Herman, 1983) y (Bost, 1986).

Sean ahora $r \geq 4$, x_0 un punto fijo elíptico (los valores propios de f'_{x_0} son complejos conjugados λ_1, λ_2 que verifican $\lambda_1 \lambda_2 = 1$), no degenerado (el argumento de $\lambda_1, \alpha_0 \neq 0$, $\pm\pi/2$, π, $\pm\frac{2\pi}{3}$ y f localmente conjugado a un twist). Esto último significa que existe un entorno U de x_0 y un difeomorfismo $\mathbf{C}^4$, h: $U \to \mathbf{R}^2$, $h(x_0) = 0$ tal que $(h \circ f \circ h^{-1})(r, \theta) = (r, \theta + \alpha_0 + \alpha_1 r) + F$ donde (r, θ) son las coordenadas polares de los puntos $h(U \cap f^{-1}(U))$ y todas las derivadas de F en el origen hasta el tercer orden se anulan. La expresión del segundo miembro es la llamada *forma normal de Birkhoff*. En estas condiciones el teorema KAM establece que dado $\varepsilon > 0$, existe un entorno V de x_0 y un conjunto $V_0 \subset V$ tales que $\mu(V \setminus V_0) \leq \varepsilon\mu(V)$, V_0 es la unión de curvas simples f-invariantes de clase $\mathbf{C}^{r-1}$ que contienen x_0 en su interior y la restricción de f a cada una de esas curvas es topológicamente equivalente a una rotación irracional.

En resumidas cuentas, el teorema KAM establece que si f es $\mathbf{C}^4$ y x_0 es un punto fijo elíptico no degenerado (recuérdese que f es μ-invariante), entonces existe una família de curvas invariantes, que cubren una región de medida positiva. En esa región V_0 hay exponentes de Liapunov nulos; f no es ergódica por ser V_0 invariante, de medida no nula ni total.

2. Billares planos

Un billar plano es el sistema dinámico que describe el movimiento libre de una masa puntual en una región Q acotada, conexa, del plano, con reflexiones elásticas en la frontera. Esta consiste de un número finito de curvas ∂Q_i, que tienen $r+1$ derivadas continuas ($\mathbf{C}^{r+1}$), $r \geq 2$, con curvatura ($|K|$) acotada. Las componentes regulares de la frontera $\partial\tilde{Q}_i = \partial Q_i \setminus \bigcup_{i \neq k} \partial Q_j$ pueden tener curvatura positiva (componentes focalizadoras), negativa (dispersoras) ó nula (neutras). El sentido de recorrido sobre $\partial\tilde{Q}_i$ y el signo de K quedan definidos por la normal: si $n(q)$ es la normal a $\partial\tilde{Q}_i$, en q, hacia adentro de Q, entonces $q(s)$ es la parametrización por la longitud de arco de $\partial\tilde{Q}_i$ tal que $q''(\bar{s}) = Kn(q) = Kiq'^{(\bar{s})}$ donde i indica el giro de ángulo $\pi/2$, $q(\bar{s}) = q$.

La referencia estandar para la formalización de la teoría de billares es (Cornfeld et al., 1982) ch 6. Definimos el conjunto $M_1 = \{(q,v) : q \in \partial\tilde{Q}_i, \ \ \|v\| = 1, \ \ \langle v, n(q)\rangle > 0\}$ de las parejas de puntos de cada componente regular y versores entrantes. Dado $x_1 = (q_1, v_1) \in M_1$, Tx_1 (si está definido) se obtiene moviéndose hacia adelante en la mesa de billar, una distancia (tiempo) t_1 hasta la intersección con $\partial\tilde{Q}_j$ en q_2. Formalmente $Tx_1 = (q_2, v_2)$ donde $v_2 = v_1 - 2\langle n(q_2), v_1\rangle n(q_2)$. Ver fig. 1. Sea $N \subset M_1$ el conjunto de los puntos donde T^k no está definido o no es continua par algún $k \in \mathbb{N}$ (Ver fig. 2).

La transformación de billar $T\colon H \to H$ es medible, biyectiva, C^r y μ-invariante, donde $d\mu = \cos\theta ds d\theta$ (normalizada) es la medida que resulta de medir las variaciones de v por su ángulo con $n(q)$, y las de q proyectando la curva frontera sobre la normal a v.

La manera natural de tomar vectores del espacio $T_x M_1$ es considerar *frentes de onda* dados por curvas $(q(s), v(s))$ em M_1, $(q(0), v(0)) = x$ *y* derivar: $(q', v') = (q'(0), v'(0))$.

Sean s_1, s_2 las longitudes de arco en $\partial\tilde{Q}_i$, $\partial\tilde{Q}_j$ en entornos de q_1, q_2, respectivamente (fig. 1); $q_2(s_2(s_1)) = q_1(s_1) + t_1(s_1)v(s_1)$, la evolución espacial del frente de onda; $K_j(s_j)$ la curvatura en $q_j(s_j)$; *y* $\theta_j(s_j)$ los ángulos de $v_j(s_j)$ con $n(q_j(s_j))$, $j = 1, 2$.

Si la variable $(\cdot)$ no está indicada, significa que estamos trabajando en el punto base; por ejemplo $K_2 = K_2(s_2(0))$.

Es simple probar las siguientes fórmulas, en las que $\frac{ds_2}{ds_1} = \sigma'$, $\frac{d\theta_2}{ds_2} = \theta_2'$, $\frac{d\theta_1}{ds_1} = \theta_1'$:

(1) $$K_1 + \theta_2' = (K_2 - \theta_2')\sigma'$$

(2) $$-\sigma' \cos\theta_2 = \cos\theta_1 - t_1(K_1 + \theta_1')$$

(3) $$\cos\theta_1 = [t_1(K_2 - \theta_2') - \cos\theta_2]\sigma'$$

Estas expresiones corresponden al Lema 2.3 de (Sinai, 1970), ajustadas en el Lema 3 de (Bunimovich, 1974), y son importantes para su construcción de las variedades locales contractoras y expansoras. Más precisamente, de (1) y (2) se deduce

(4) $$\frac{1}{\cos\theta_2}\,\theta_2' = \frac{K_2}{\cos\theta_2} + \frac{1}{-t_1 + \frac{1}{\frac{1}{\cos\theta_1}(K_1 + \theta_1')}}$$

que permite deducir immediatamente, si $K < 0$ (billares dispersores): $\theta_1' \leq 0$ (frente decreciente) implica $\theta_2' \leq 0$, y $(-K)_{\min} \leq -\theta_2' \leq (-K)_{\max} + \frac{1}{t_{\min}}$.

En la misma situación, (2) permite deducir

$$-\sigma' \frac{\cos\theta_2}{\cos\theta_1} \geq 1 + t_{\min}(-K)_{\min} \stackrel{\text{def}}{=} \lambda_0.$$

Entoces, si Γ, $T^{-1}\Gamma, \ldots T^{-k}\Gamma$ es una sucesión de curvas suaves tales que $T^{-k}\Gamma$ es decreciente (por tanto $T^{-i}\Gamma$ es decreciente para todo $0 \leq i \leq k$) y definimos las longitudes del arco $\Delta \subset M_1$ por

$$p(\Delta) = \int_\Delta \cos\theta ds, \quad S(\Delta) = \int_\Delta ds,$$

resulta

$$p(T^{-i}\Gamma \leq \lambda_o^{-i} p(\Gamma), \quad S(T^{-i}\Gamma) \leq \frac{\lambda_0^{(i-1)}}{\lambda_0^{-1}} \leq S(\Gamma)$$

Se deduce de la acotación uniforme de $-\theta_2'$ que la longitud euclideana de Δ en el plano (s, θ) es equivalent a la S-longitud (progección sobre el eje s). Y resulta natural buscar una fibra expansora como una curva Γ_ϵ tal que $T^{-1}\Gamma_\epsilon$, $T^{-2}\Gamma_\epsilon, \ldots$ es una sucesión de fibras decrecientes.

La construcción de la variedad in inestable local en x se hace por aproximaciones sucesivas (método de Hadamard-Perron). Si $\Gamma_\epsilon^0(y)$ es la solución de $\theta_1' = K_1(s)$ en un cierto intervalo (esta solución corresponde a un frente de onda que al llegar-antes de reflejarse-es plano) entonces $\Gamma_\epsilon^0(T^{-k}x))$ cumple la condición de tener todos sus iterados para atras decrecientes. Cuando $k \to \infty$, las curvas Γ_ϵ^k convergen en la topologia $\mathbf{C}^1$ y definen la variedad inestable local. Obsérvese que ésta en realidad aparece como el estado de un frente de onda que en el tiempo $-\infty$ salió plano. En (Gallavotti, 1975) hay una exposición en detalle de esta construcción y en (Markarian, 1990) hay una presentación en términos de la curvatura de los frentes de onda, que sigue más fielmente los trabajos de Sinai y Bunimovich.

La prueba de la ergodicidad de los billares dispersores o semidispersores ($K \leq 0$) luego de construidas las variedades estables e inestables locales y estudiada su suavidad (continuidad, etc.) pasa en primer lugar por analizar la continuidad absoluta de las foliaciones resultantes y la consecuente existencia de componentes ergódicas de médida positiva donde, incluso, la transformación de la billar cumple propiedades ergódicas más, fuertes (sistema de Kolmogorov).

Entonces, para deducir la ergodicidad hay que probar la existencia de una única componente ergódica a través del llamado *teorema fundamental de los billares dispersores* (Sinai, 1970), (Bunimovich, Sinai, 1973). La adaptación de los argumentos de Hopf utilizados en los casos de flujos geodésicos y extendidos a conjuntos hiperbólicos compactos presenta las siguientes dificultades:

a) al prolongar las variedades estables e inestables locales aparecen singularidades (cusps) donde ellas cambian de sentido como consecuencia de las trayectorias tangentes a la frontera.

b) la falta de control por bajo en la longitud de las componentes suaves de esas variedades.

La manera de resolver estos problemas pasa por observar que la probabilidad de que las componentes suaves sean muy pequeñas, es muy baja uniformemente.

3. Billares ergódicos

En base a las ideas que tan groseramente expresáramos antes, se prueba que los billares dispersores (todas las componentes regulares de la frontera con $K < 0$) ó aquellos que siendo semidispersores ($K \leq 0$) tienen casi todas sus trayectorias pasando eventualmente por puntos de la frontera con $K < 0$, son ergódicas (más aún, son K-sistemas: sistemas de Kolmogorov). En los puntos de intersección $\partial Q_i \cap \partial Q_j$ no debe haber normal común; esto es, la intersección debe ser transversal.

En esto trabajo el caracter *eventual* de una propiedad significa que para casi todo $x \in M_1$ existe $k > 0$ tal que $T^k x$ verifica la propiedad.

En (Gallavotti, Ornstein, 1974) se prueban propiedades ergódicas aún más fuertes (B-sistemas: sistemas de Bernoulli), para el caso $K < 0$ y en (Bunimovich, Sinai, 1980) se construye un conjunto numerable de particiones (de Markov), también para el caso $K < 0$, que permite estudiar otras propiedades estadísticas de los billares. Estos dos autores, junto con Chernov han anunciado (comunicación oral de Sinai,1989) mejorías a su construcción de tales particiones.

Estas construcciones facilitan también el estudio de órbitas periódicas, aunque existen estimaciones del número de ellas para billares semidispersores, calculadas de manera totalmente independiente, en (Stojanov, 1989). También cabe esperar que existan relaciones entre el valor limite del número de órbitas periódicas (cuando el período tiende a $+\infty$) y la entropía de los billares. Ver (Katok, 1980).

En (Bunimovich, 1974) se probó que la ergodicidad (en realidad el caracter de K-sistema) se mantiene si en los ángulos de un billar dispersor se colocan arcos de circunferencias Γ_i tales que $i \neq j$ implica que Γ_i, Γ_j son arcos de distintas circunferencias, y la parte de la circunferencia de Γ_i que no es de la frontera esta enteramente contenida en la superficie del billar.

En (Bunimovich, 1979) se muestra por último que se pueden armar billares ergódicas con arcos de circunferencia Γ_i y segmentos de recta (sin componentes dispersoras) siempre que se verifiquen las condiciones que se indican a continuación. Sean

$$\mathcal{U} = \{x = (q, v) \in M_1\colon\ q \in \Gamma_i \text{ para algún } i\} \text{ y}$$

$T^j x_1 = (q_{j+1}, v_{j+1})$. Si $x_1 \in \mathcal{U}$, sean $L(x_1)$ la longitud de la cuerda de Γ_i en que se apoya la trayectoria que sale (ó entra) de x_1 y $\tau(x_1) = \sum_{j=1}^{n-1} t_j$ siendo t_j la longitud del segmento de trajectoria (q_j, q_{j+1}), $q_n \in \Gamma_k$ para algún k, q_j en componentes neutras para $1 < j < n$. Por ejemplo, si $q_i, q_2 \in \Gamma_i$ entonces $L(x_1) = \tau(x_1) = t_1$. Sean $\tilde{\mathcal{U}} = \{x \in \mathcal{U}\colon\ L(x) < \tau(x)\}$ y para todo m natural, $\alpha > 0$,

$$A_n^\alpha = \{x\colon\ T^i x \in \tilde{\mathcal{U}} \text{ para } \quad o < i < n^\alpha\}$$

Las condiciones indicadas por Bunimovich para que el billar sea un K-sistema, son

a) $L(x) \leq \tau(x)$ para todo $x \in H$

b) $\mu(\tilde{\mathcal{U}}) > 0$

c) Existe $0 < \alpha < 1$ tal que la serie $\sum_{n=1}^{\infty} \mu(A_n^\alpha)$ es convergente.

La primera condición es verificada si toda la circunferencia que contiene un Γ_i está contenida en la superficie del billar (aunque hay otras situaciones), y la última, si los rebotes sucesivos sobre un mismo Γ_i ó entre componentes neutras, no son muchos.

Ejemplos de mesas de billar que verifican estas condiciones se muestran en la fig.3.

4. Billares sin comportamiento caótico

Los billares de Bunimovich presentan un caso extremo de rigidez en cuanto los bordes con $K \neq 0$ deben ser exactamente arcos de circunferencia y obviamente muchas de sus características se pierden si los arcos de frontera son levemente perturbados.

Diremos que un arco de curva en $\mathbf{R}^d$ es $\varepsilon - \mathrm{C}^k$*-perturbado* de otro C si sus puntos están a distancia menor que ε de C, y todas sus derivadas hasta las de orden k difieren de las de C en menos de ε. Diremos que una propiedad de un billar se mantiene por C^k*-perturbaciones* de alguna componente regular C de la frontera si existe $\varepsilon > 0$ tal que la sustitución de C por cualquier $\varepsilon - C^k$-perturbado no modifica esa propiedad.

Mostraremos que ni la ergodicidad ni el comportamiento caótico se mantienen por C^k-perturbaciones ($k \in \mathrm{N}$) de las componentes de la frontera de un billar de Bunimovich que contenga más de media circunferencia.

Antes resumiremos algunos resultados y definiciones que resultan muy útiles para entender estos problemas de no ergodicidad.

En primer lugar observemos que si $T^2 x_1 = x_1$, ó sea si x_1 es un punto periódico de período 2 (por ejemplo cualquier trayectoria que una los vértices de una elipse), esa trayectoria es linealmente estable (puntos elípticos de T^2) si y solo si (Wojtkowski, 1986), (Hayli, Dumont, 1986)

$$t_1 K_1 K_2 - K_1 - K_2 < 0 \ \ y \ \ (t_1 K_1 - 1)(t_1 K_2 - 1) > 0$$

ó, escrito en término de los radios de curvatura, para $K_i = \frac{1}{R_i} > 0$,

$$(5) \qquad t_1 < R_1 + R_2, \quad (t_1 - R_1)(t_1 - R_2) > 0$$

La sola existencia de un punto elíptico no alcanza para eliminar la posibilidad de tener ergodicidad, ver teorema KAM, pero es una situación a evitar en general. Por ello aparece como muy natural la condición de que, para tener algún tipo de propiedad ergódica (en particular, comportamento caótico) que se mantenga por perturbaciones, los círculos de semicurvatura de las componentes focalizadoras deben ser disjuntos: en este caso es $t_1 > R_1 + R_2$.

Llamamos *círculo de curvatura (osculatriz)* en q_1, $K_1 > 0$, al círculo por q_1 con centro en $q_1 + R_1 n(q)$. El *círculo de semicurvatura* pasa por q_1 y tene centro en $q_1 + \frac{R_1}{2}\, n(q_1)$. Llamaremos L_1 a la parte de la trayectoria contenida en el círculo de curvatura. Ver fig. 1. Si $K_1 > 0$, $L_1 = 2R_1 \cos\theta_1$.

Las condiciones (5) son verificadas por las trayectorias periódicas entre los vértices de los ejes menores de una elipse. La trayectorias diametrales de una circunferencia verifican las igualdades de las expresiones en (5). Por tanto, si bien cabe esperar no comportamiento caótico en el caso de la elipse, el asunto no es tan claro, por esta via, en la circunferencia. Para dar una respuesta a estos asuntos introduciremos el concepto de caustica.

Una curva suave Γ contenida em Q es una *cáustica* del billar si el hecho de ser un segmento de trayectoria q_k, q_{k+1} tangente a Γ significa que todo otro segmento de esa trayectoria tambien lo es.

Es obvio que todas las circunferencias con centro en el centro de un billar circular son cáusticas; y no es dificil probar usando el teorema de Poncelet que en un billar elíptico, todas las trayectorias tienen por cáusticas hipérbolas ó elipses confocales, según corten ó no el segmento entre los focos. Ver (Cornfeld et al., 1982), Ch.6. Th.1.

El resultado fundamental sobre la existencia de cáusticas es una consecuencia del teorema KAM y fue probado en (Lazutkin, 1973): si el borde de un billar focalizador ($K > 0$) es una curva C^6, existe un conjunto de Cantor de cáusticas que se acumulan sobre el borde. Esta versión corresponde a (Douady, 1982), donde se conjetura que alcanza con el borde ser C^4.

La existencia de tales cáusticas elimina la posibilidad de tener comportamiento caótico porque los exponentes de Liapunov se anulan en conjuntos de medida positiva.

Lo mismo sucede alrededor de la órbita 2-periódica que une los vértices del eje menor de una elipse, donde todas las órbitas son tangentes a cáusticas hiperbólicas. Es claro que por perturbaciones C^∞ tan pequeñas como se quiera de más de media circunferencia se puede obtener una parte de elipse que contenga los dos vértices del eje menor. Por tanto los billares de Bunimovich que en su frontera contengan más de media circunferencia Γ pierden su caracter ergódico (incluso su caoticidad) por C^k- perturbaciones ($k \in \mathbf{N}$) de Γ.

En (Hubacher, 1987) se estudian los billares focalizadores ($K > 0$) limitados por curvas C^1 cuyas derivadas segundas existen y son continuas excepto en un número finito de puntos dende los limites laterales existen, pero la curvatura es discontinua, sin limite cero, y acotada. Se prueba que no existen cáusticas cerca de la frontera y se observa que podría haberlas en el interior del billar.

En (Mather, 1982) se probó que un cero en la curvatura de un billar focalizador con borde C^2, implica la no existencia de cáusticas. Por tanto el comportamiento caótico de un tal billar no puede ser excluído a priori. En (Hayli et al, 1987) se describe un billar con borde analítico con curvatura positiva excepto en un punto que no tiene comportamiento caótico. Las ecuaciones paramétricas del borde son

$$x = \cos t + \lambda \cos 2t, \quad y = \operatorname{sen} t + \lambda \operatorname{sen} 2t, \quad \lambda = 1/4.$$

Este trabajo por otra parte muestra el extremo cuidado con que se deben interpretar los resultados numéricos. En (Robnik, 1983) se afirmaba que para $1/4 < \lambda < 1/2$ el billar (no focalizador) es ergódico. En (Hayli et al, 1987) se demuestra que para valores de λ levemente superiores a $1/4$ el sistema continúa no siendo ergódico.

Por ello se debe tomar con cuidado la aparente existencia de un billar estrictamente convexo, de borde C^1, formada por cuatro arcos de circunferencia, con comportamiento caótico, que fuera estudiado numericamente en (Hayli, Dumont, 1986)

Los estudios computacionales arriba indicados, y otros anteriores (recordamos sólo (Benettin, Strelcyn, 1978) y (Hénon, Wisdom, 1983) por ser los que marcaran ciertos rumbos posteriores) indican la coexistencia de fenómenos de caos y equilibrio, aún no muy bien estudiados. Ver, por ejemplo (Wojtkowski, 1981) y (Strelcyn, 1989).

5. Billares con comportamiento caótico

Los avances más reciente en el estudio de propiedades ergódicas de billares planos han estado relacionados con la prueba de teoremas generales sobre la no anulación de los exponentes de Liapunov. Estos teoremas se refieren a la construcción de famílias de conos invariantes ó de formas cuadráticas de Liapunov crecientes a lo largo de las trayectorias, métodos esencialmente equivalentes introducidos por Wojtkowski (1985, 1986) y el autor de este resumen (1988, 1990). La aplicabilidad de la teoria de Pesin al sistema de los billares que definimos en la Sección 2 fue demostrada por Strelcyn en Part *V* de (Katok, Strelcyn, 1986).[2]

Se ha probado la existencia de una variedad muy grande de billares planos con componentes regulares de la frontera focalizadoras que tienen comportamiento caótico. Estas componentes verifican ciertas condiciones abiertas lo cual permite hacer C^4 perturbaciones en ellas y mantener la hiperbolicidad no uniforme.

Los tipos de componentes focalizadoras C^4 que se ha demostrado pueden ser parte de un billar con comportamiento caótico son los siguientes:

a) Arcos de curva que verifican $\frac{d^2R}{ds^2} < 0$ (Wojtkowski, 1986). E n esta categoria están la epicicloide, la hipocicloide, la cicloide y en particular toda la cardioide (curva cerrada que en coordenadas polares tiene ecuación $r(t) = 1 + \cos t$, $-\pi \leq t \leq \pi$. También para la elipse $x = a\cos t$, $y = b\,\mathrm{sen}\,t$, $-\pi/2 \leq t < \frac{3\pi}{2}$ con $b^2 > a^2$, el arco $-\pi/4 < t < \pi/4$ (y su simétrico) verifican la condición de Wojtkowski.

[2] Hace menos de 10 años Pesin y Sinai (1981) escribieron que el estudio de propiedades ergódica de billares a traves de la teoría de los sistemas dinámicos discontinuos lleva al "no muy simple problema de determinar que los exponentes de Liapunov del sistema son distintos de cero. Hasta el momento el único método efectivo (pero no riguroso) de determinación es el de su cálculo por computadoras" (p. 84).

b) Arcos que verifican $L_2(t_1+t_2) < 2t_1t_2$ para los choques sucesivos sobre la misma componente focalizadora. Esta condición se verifica localmente (trayectorias con segmentos cortos) si $\frac{d^2(R^{1/3})}{ds^2} > 0$. Para trayectorias largas se deben agregar otras condiciones verificadas, por ejemplo, en la elipse anterior si $\text{sen}^2 t > \frac{b^2}{b^2+a^2}$, que corresponde a un arco disjunto con el anterior (fig. 4). Ver (Markarian, 1988), p. 93.

c) Cualquier pequeño arco focalizador, donde la pequeñez viene dada por la validez en el entorno de un punto de la negatividad de un desarrollo de Taylor cuyo primer sumando es de orden par e involucra hasta las derivadas cuartas de la curva. Ver (Markarian, 1989, 1990).

Cómo combinar los arcos regulares de frontera cumpliendo diferentes condiciones en su curvatura hasta cerrar una mesa de billar fue discutido en los trabajos citados en a), b). En (Markarian, 1990) se hace una descripción que engloba todas las anteriores. Las siguientes dos familias de billares tienen comportamiento caótico:

1. Las $\mathbf{C}^3$-componentes de la frontera pueden ser de cualquier tipo, excepto las focalizadoras que deben ser $\mathbf{C}^4$ y verificar a) ó c). Los círculos de semicurvatura en cada punto de cada componente regular focalizadora no contienen puntos de otra componente ni de otras componentes focalizadoras no adyacentes. Las componentes focalizadoras adyacentes forman ángulo interior mayor que π. Componentes focalizadoras y dispersoras adyacentes forman angulo no menor que π. Componentes focalizadoras, y neutras adyacentes tienen angulos mayor que $\pi/2$.

2. Las $\mathbf{C}^3$-componentes de la frontera pueden ser de cualquier tipo, excepto que las focalizadoras deben ser $\mathbf{C}^4$ y verificar b) ó c). Los círculos de curvatura de las componentes focalizadoras no deben cortar otras componentes de la frontera. Las condiciones sobre los arcos adyacentes son las mismas que en el caso 1.

Tal cual fuera mencionado al final de la Sección 2, la unicidad de la componente ergódica en los billares semidispersoras, luego de demostrada la no anulación de los exponentes de Liapunov (existencia de variedades locales transversales) surge de un estudio detallado de las singularidades de la transformación de billar. Esto, que fuera hecho en detalle para tales billares, por los billaristas de la escuela de Sinai, parece francamente más dificil y/o tedioso para los billares con arcos focalizadores, descritos en 1 ó 2, en los cuales la medida total de la región de Pesin asegura la existencia de un conjunto numerable de componentes ergódicas.

En (Wojtkowski, 1986), Appendix C, se describe un billar con comportamiento caótico y por lo menos dos componentes ergódicas.

Las curvas que verifican las igualdades de las condiciones en a) y b) pueden ser colocadas en los billares de los tipo 1. y 2., construyéndose billares con comportamiento caótico, pero si tales arcos son perturbados las propiedades ergódicas pueden dejar de cumplirse. En particular los arcos de circunferencia están en el borde las condiciones a) y b) y al perturbarlos los billares resultantes podrían dejar de tener comportamiento caótico. Es el caso de los billares de Bunimovich con más de media circunferencia o de los billares *cacahuete* estudiados numéricamente en (Hayli, Dumont, 1986) que verifican las descripciones de 1. ó 2. con $\frac{d^2R}{ds^2} = \frac{d^2(R^{1/3})}{ds^2} = 0$.

En (Markarian, 1989, 1990) se prueba que los arcos de menos de media circunferencia de los billares de Bunimovich pueden ser C^4 perturbados manteniendo el comportamiento caótico. En particular, el ejemplo más conocido de tales billares, el stadium (fig. 3a) puedo tener sus dos semicircunferencias C^4 perturbadas, manteniendo fijos los lados rectos y la tangencia de estos con las curvas perturbadas en los puntos de contacto. En (Bunimovich, 1985) hay algunas ideas heurísticas sobre este problema y Donnay (Princeton University) anunció en 1988 un resultado similar (sólo se admiten C^6 perturbaciones) usando coordenadas de Lazutkin.

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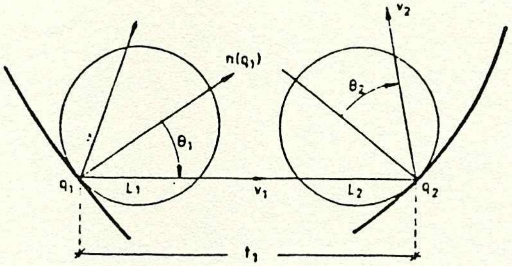


Fig. 1

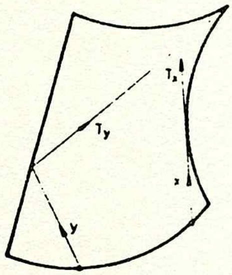
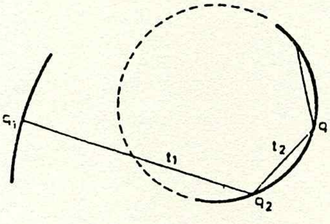


Fig. 2



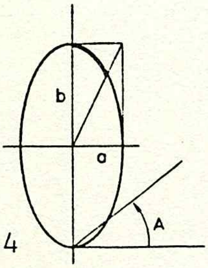


Fig. 4

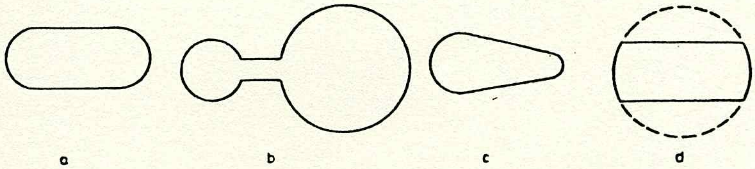
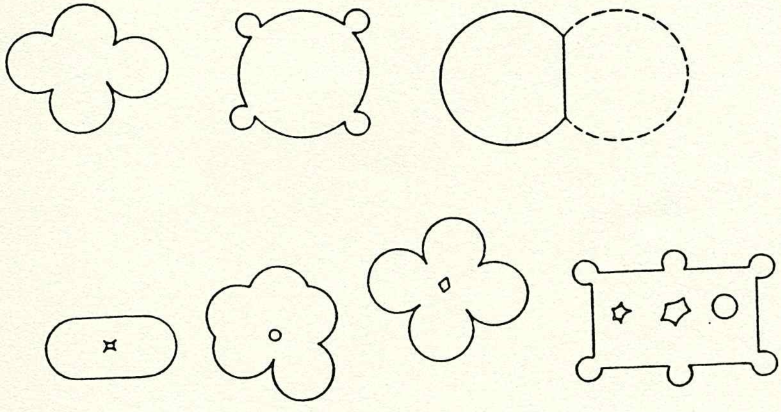


Fig. 3



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New Elliptic Potentials

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§ 1 - Introduction

1.1. In the framework of different Paris Seminars, I was warmly and patiently introduced by J.-L. Verdier to the whole subject of Soliton theory and in particular to the KP hierarchy and the *Krichever dictionary*. I began working out under his direction, Krichever's remarkable and seminal article on elliptic solutions of the KP equation ([K]). Having made some progress in its understanding we started what would turn out to be a full 4 years' collaboration abruptly ended by his death. Throughout this period we had a great time playing with and discussing about elliptic solitons and tangential polynomials ; guessing, testing and finding new ideas and results about hyperelliptic tangential covers ; seeking out rigorous proofs of every detail ; and in general just trying to understand what was going on. Fortunately we were also able to manage not to worry about getting something published before we got a clear picture of the subject.

1.2. Let Λ be a lattice of $\mathbb{C}$, q the origin of the elliptic curve $E = \mathbb{C}/\Lambda$ and $\wp(x)$ the corresponding Weierstrass function. We say that a finite pointed morphism $\pi:(\Gamma,p) \to (E,q)$ ($\pi(p)=q$) is a *tangential cover* if the solutions of the KP equation associated to (Γ,p), are Λ-periodic in the first variable x (c.f. : [T-2], [V]). Furthermore, if there exists a 2 to 1 projection $\Gamma \to \mathbb{P}^1$ ramified at p, we obtain Λ-periodic solutions of the KdV equation and π is called *hyperelliptic tangential cover*. We naturally associate to such a morphism π a finite-gap Λ-periodic complex potential u_π (so-called *source potential*), of the form

$$\sum_{1\le i\le l} m_i(m_i+1)\wp(x-\alpha_i), \text{ where } \sum_{1\le i\le l} m_i(m_i+1) = 2.\deg\pi \text{ and } \alpha_i \neq \alpha_j \ (\text{mod } \Lambda) \text{ if } i \neq j.$$

It is classically known that u_π uniquely determines the *spectral data* (Γ,p) and all Λ-periodic solutions of the KdV equation associated to (Γ,p) flow out from u_π by applying the KdV hierarchy. We say that u_π has genus (resp. : degree) equal to the arithmetic genus of Γ (resp. : deg π).

1.3. The first account of our joint work, written by J.-L. Verdier in 1987 ([V]), reformulated the *KP and KdV elliptic problem* in terms of tangential covers (unnecessarily supposed to be smooth ; see [T.2]) and answered the corresponding moduli problem, using a particular ruled surface over E. At the end he sketched an algorithmic construction of a particular type of tangential covers (see also § 6.4), which we conjectured to be hyperelliptic and announced (new) source potentials of degree 3, 4, 5 and 6. All the results reported or conjectured in [V] were completely proved and fully generalized by pure algebraic (surface) means in [T.2] and [T-V.1]. Our main purpose here is twofold : to give complete (down-to-earth) proofs of the latter results in the smooth case, and to explain their generalization, obtained together with J.-L. Verdier (c.f. : [T-V.3]).

1.4. The paper is organized as follows ; we start recalling the *Krichever dictionary* and defining the KP and KdV elliptic problem (§ 2). We also mention and give a simple, although partial, solution to the *KP elliptic initial-value* problem. After a general development of the (symmetric) tangential covers and their moduli by means of tangential polynomials (§ 3, § 4), we work out a criterion to detect the hyperelliptic ones, and study their basic properties (§ 5). The latter criterion is then applied to the setting of an algorithm liable to give rise to hyperelliptic tangential covers of any degree (§ 6). Having at hand the explicit equations of all tangential covers as (roughly speaking) zeroes of tangential polynomials (§ 6.1, 6.2) we are able to check in lower degrees (up to 6) that the tangential covers thus constructed are indeed hyperelliptic and find the corresponding (new) source potentials listed in [V] (§ 7). Last but not least we announce the construction of source potentials of any degree and their characterization, as suggested by the list of known ones (§ 7.4). For example let $\{\omega_0=q,\omega_1,\omega_2,\omega_3\}$ denote the half-periods of (E,q), then for any $(a_i)\in \mathbb{N}^4-\{(0)\}$ the function $\sum_{0\le i\le 3} a_i(a_i+1)\wp(x-\omega_i)$ is a source potential (of degree $\frac{1}{2}\sum_{0\le i\le 3} a_i(a_i+1)$ and genus a_0) if and only if $a_0=\sup\{a_i\}$ and $2\sup\{a_i\}\ge s-(1+(-1)^s)\inf\{a_i\}$ where $s=\sum_{0\le i\le 3} a_i$ (§ 7.6).

§ 2 - The elliptic KP (initial value) problem

2.1. Let Γ be an integral projective curve of arithmetic genus $g>0$, $\mathrm{Jac}\,\Gamma$ the jacobian of Γ and p in Γ^0, the open dense subset of smooth point of Γ. The Abel map $A_\Gamma:\Gamma^0\to\mathrm{Jac}\Gamma$ which associates to any $r\in\Gamma^0$ the isomorphism class of the line bundle $\mathcal{O}_\Gamma(r-p)$, is a closed immersion. We will denote $W(\Gamma)$ the compactified jacobian of Γ i.e. : the moduli space of rank 1, torsion free sheaves on Γ of degree $g-1$. The generic element of $W(\Gamma)$ has no holomorphic section and the closed subset $\Theta=\{\mathcal{L}\in W(\Gamma),\ h^0(\mathcal{L})\ge 1\}$ is the support of a divisor of $W(\Gamma)$.

Let $G(\Gamma,p)$ be the 1-parameter subgroup of Jac Γ generated by the tangent space to Γ at p. The group Jac Γ acts on $W(\Gamma)$ via tensor product and for any $\mathcal{L}\in W(\Gamma)$ the $G(\Gamma,p)$-orbit of $\mathcal{L}$ $(G(\Gamma,p).\mathcal{L})$ is not contained in the divisor Θ ([S-W], 8.6). Whenever Γ is a smooth curve, $W(\Gamma)$ is isomorphic to Jac Γ, Θ is the usual theta divisor and the KP solution $u_{\mathcal{D}}(x,y,t)$ is given in terms of the Riemann theta function. Yet is still a deep non-trivial result that Θ doesn't contain $G(\Gamma,p).\mathcal{L}$, the $G(\Gamma,p)$ orbit of $\mathcal{L}$ (cf. [F]).

2.2. Fix any local coordinate, λ, of Γ at p and consider the i-th derivative $(i\ge 1)$ of A_Γ at $\lambda=0$ as a tangent vector at the origin of $\mathrm{Jac}\Gamma$. Let us denote $U_i=-\frac{1}{i!}\frac{\partial^i}{\partial\lambda^i}A_\Gamma(\lambda)_{|\lambda=0}$ and $\partial/\partial x$ the invariant tangent vector field of $\mathrm{Jac}\Gamma$ generated by U_1. For any data $\mathcal{D}=(\Gamma,p,\lambda,\mathcal{L})$, $\mathcal{L}\in W(\Gamma)$ the corresponding *tau* function, denoted hereafter $\tau_{\mathcal{D}}$, is defined and holomorphic on the tangent space of $\mathrm{Jac}\Gamma.\mathcal{L}$ and satisfies the following properties :

(1) the zero divisor of $\tau_{\mathcal{D}}$ coincides with the divisor Θ ;

(2) the second logarithmic derivative of $\tau_{\mathcal{D}}$, $\frac{\partial^2}{\partial x^2}\log\tau_{\mathcal{D}}$, is meromorphic on $\mathrm{Jac}\Gamma.\mathcal{L}$

and restricts for any $\mathcal{F} \in \mathrm{Jac}\Gamma.\mathcal{L}$ to a rational function on $G(\Gamma,p).\mathcal{F}$ (since Θ doesn't contain any $G(\Gamma,p)$-orbit).

(3) the function

$$u_{\mathcal{D}}(x,y,t) = -2\frac{\partial^2}{\partial x^2}\log \tau_{\mathcal{D}}(\exp(xU_1+yU_2+tU_3).\mathcal{L}) \tag{2.2.1}$$

is a solution of the KP equation.

Remark 2.3.

1) If $G(\Gamma,p)$ is an elliptic curve then for all data $\mathcal{D} = (\Gamma,p,\lambda,\mathcal{L})$, the KP solution $u_{\mathcal{D}}(x,y,t)$, given by the (so called russian) formula (2.2.1), is doubly periodic in x.

2) If Γ is an hyperelliptic curve and p a Weierstrass point of Γ, there exists a local coordinate λ of Γ at p such that $U_2 = \frac{\partial^2}{\partial \lambda^2}(A_\Gamma)|_{\lambda=0}$ equals zero. It follows then, that for any data $\mathcal{D} = (\Gamma,p,\lambda,\mathcal{L})$, $u_{\mathcal{D}}$ is y-independent and solves the KdV equation :

$$\text{(KdV)} : \qquad 4u_t + 6uu_x - u_{xxx} = 0 .$$

2.4. At this point it is quite natural to propose the following so-called *KP* (resp. : *KdV) elliptic* problem. Study the general properties and construct explicity the moduli space of all $\mathcal{D} = (\Gamma,p,\lambda,\mathcal{L})$ such that the function $u_{\mathcal{D}}$ is doubly periodic in x (resp. : and a KdV solution). These problems have been developped and solved in [T.2] and [T-V.1]. More generally it has been proved in [T-V.2] that any KP meromorphic solution $u(x,y,t)$, doubly periodic in x, comes from an explicit data $\mathcal{D} = (\Gamma,p,\lambda,\mathcal{L})$ whose 1-parameter subgroup $G(\Gamma,p)$ is an elliptic curve. Our starting point was the following observation : for any pointed curve (Γ,p), the subgroup $G(\Gamma,p)$ of Jac Γ is an elliptic curve if and only if there exists a finite pointed morphism of (Γ,p) onto an elliptic curve (E,q) such that the pull-back image of E in Jac Γ is equal to $G(\Gamma,p)$ ([T.1]). These particular pointed morphisms, so called tangential covers, afford an intrinsic characterization by means of which we get the jacobians out of the picture. Finally we write down explicit equations of all tangential covers (solving the *KP elliptic* problem) as zeroes of tangential polynomials (see § 3.8).

2.5. Although the tangential polynomial approach doesn't solve the subtler KdV elliptic problem, it can be used as experimental field to develop the right tools to tackle it. Before we start dealing with tangential polynomials and their application to the construction of new finite-gap elliptic potentials, let me introduce a natural generalization of the KP elliptic problem. Our goal this time is to determine all data $\mathfrak{D} = (\Gamma, p, \lambda, \mathcal{L})$ such that the corresponding KP solution $u_{\mathfrak{D}}$ has as *initial-value* function $u_{\mathfrak{D}}(x,0,0)$ a doubly periodic function. This (*KP elliptic initial-value*) problem led us to interesting questions about orbits of subgroups of JacΓ inside the theta divisor $\Theta \subset W(\Gamma)$. We give hereafter a partial answer to this problem based upon the following 2 lemmas.

Lemma 2.6. Let A be a complex Lie group, D a 1-codimensional subvariety of A, C a 1-parameter subgroup and $\overline{C}$ the closure of C in A. Then $D \cap C$ is dense in $D \cap \overline{C}$.

Proof. We can suppose that $A = \overline{C}$. Let T be a non-zero left-invariant vector field on A parallel to C. There exists a dense open subset of D over which T is not tangent to D. Otherwise we could find an integral curve of T (a translate of C) contained in D and in particular C could not be dense in A.

Let V be an open dense and smooth subset of D, transverse everywhere to T and denote $f : V \times C \to A$ the map $(v,\alpha) \to v+\alpha$. To finish the proof of lemma 1 it is enough to check that $V \cap C$ is dense in $V \cap \overline{C}$. Let V' be any open subset of V contained in the complement of $V \cap C$. By transversality $f(V' \times C)$ contains a neighborhood of V' in A and should be in the complement of C, which is impossible since C is dense in A.

q.e.d.

Lemma 2.7. Let A be an abelian variety, D an ample divisor of A and B an abelian subvariety of A not contained in D. Then the subset
$H(D \cap B) = \{b \in B \,/\, b+(D \cap B) \subset D \cap B\}$ is finite.

Proof. The intersection $D \cap B$ is an ample divisor of B. It follows that $H(D \cap B)$ is finite (cf. : [M]).

Proposition 2.8. Let Γ be a compact smooth Riemann surface and $\mathcal{L} \in W(\Gamma)$. If $u_{\mathfrak{D}}(x,0,0) = -2\frac{\partial^2}{\partial x^2} \log \tau_{\mathfrak{D}}(\exp(xU_1)\mathcal{L})$ is doubly periodic of lattice Λ then the 1-parameter subgroup $G(\Gamma,p)$ is an elliptic curve isogeneous to $\mathbb{C}/\Lambda$.

Proof. Let us denote $\varphi : \mathbb{C} \to G(\Gamma,p)$ the surjective homomorphism $x \longrightarrow \exp(xU_1)$ $(U_1 = -\frac{\partial}{\partial\lambda}(A_\Gamma)_{|\lambda=0})$ and A the closure of $G(\Gamma,p).\mathcal{L}$ in $W(\Gamma)$. The jacobian Jac Γ is an abelian variety isomorphic to $W(\Gamma)$, Θ corresponds to the theta divisor of Jac Γ and A is an abelian subvariety of $W(\Gamma)$. The $G(\Gamma,p)$-orbit of $\mathcal{L}$ is not contained in Θ (cf. : [F]) and $D = \Theta \cap A$ is a divisor of A. Furthermore the intersection $D \cap (G(\Gamma,p).\mathcal{L})$ is clearly $\varphi(\Lambda)$-invariant and, by lemma 1, dense in D. It follows that D itself is $\varphi(\Lambda)$-invariant. On the other side, Γ being smooth, the couple $(W(\Gamma),\Theta)$ is a principally polarized abelian variety and in particular $D = \Theta \cap A$ is also an ample divisor of A. Hence, by lemma 2, $\varphi(\Lambda)$ is a finite subset of Jac Γ and therefore $G(\Gamma,p)$ is an elliptic curve isogeneous to $\mathbb{C}/\Lambda$. q.e.d.

§ 3 - Tangential covers and polynomials

We fix once for all an elliptic curve E and z, a local antisymmetric coordinate at q, the origin of E.

Definition 3.1. Let Γ be a projective integral curve (i.e. : a compact Riemann surface with, at most, a finite number of singularities), p a smooth point of Γ and $\pi : (\Gamma,p) \to (E,q)$ a finite pointed morphism. We say that π is :

1) *primitive* if and only if it doesn't factorize thru a non-trivial isogeny of elliptic curves or equivalently if the pull-back homomorphism $\pi^*: E \to \text{Jac}\,\Gamma$ is injective.

2) *tangential* or a *tangential cover* if and only if $\pi^*(E)$ is tangent to $A_\Gamma(\Gamma^0)$, the image of the Abel map (see § 2.1), at the origin of Jac Γ.

Remark 3.2. Let $j : (\hat{\Gamma},p) \to (\Gamma,p)$ be a partial desingularization (i.e. : j is a birational morphism). If $\pi : (\Gamma,p) \to (E,q)$ is tangential then $\pi \circ j$ is also tangential but the converse is not always true (c.f. : [T-2]).

Definition 3.3. A tangential cover π is called *minimal* if it can't be obtained as partial desingularization of another tangential cover.

Theorem 3.4 (*Tangency criterion*) A pointed morphism $\pi : (\Gamma,p) \to (E,q)$ is a tangential cover if and only if there exists a meromorphic function k on Γ, holomorphic on $\Gamma - \{\pi^*(q)\}$, having over a neighborhood of q the following property :

$k+\pi^*(z^{-1})$ is defined at every point different from p and has at p a simple pole (c.f. [T-2], 1.10).

3.5. The function k satisfying the tangency criterion is unique up to an additive constant and will be called *tangential function for* π.

Definition 3.6. Let $n\geq 1$ be a positive integer and $P(T) = T^n + \sum_{1\leq j\leq n} a_j T^{n-j}$ a unitary polynomial of degree n whose coefficients are meromorphic funtions on E. The polynomial $P(T)$ will be called *tangential* (with respect to z) if and only if the functions $\{a_j, j=1,...,n\}$ are holomorphic on $E-\{q\}$ and all the coefficients of $zP(T-z^{-1})$ are holomorphic at q.

The set of all tangential polynomials of degree n is an n-dimensional affine space denoted $\Theta(n,E,z)$ (§ 3.8).

3.7. Let us consider the variable T as a rational function on $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ having a simple pole at ∞. We choose $\{T^{-1},z\}$ as local coordinates of $\mathbb{P}^1\times E$ at $p_0 = (\infty,q)$ and denote p_1 the point *infinitely near* p_0 corresponding to the tangent direction -1. By blowing up p_0 and p_1 (and contracting the strict transform of $\mathbb{P}^1\times\{q\}$) we construct a rational morphism (over E) from $\mathbb{P}^1\times E \to E$ to a new ruled surface $S \to E$. The latter map is an isomorphism outside the fiber $\mathbb{P}^1\times\{q\}$.

The strict transform of $\{\infty\}\times E$, denoted C_0, is the unique section of the ruled surface $S\to E$, of zero self-intersection. This property characterizes the couple $(S\to E, C_0)$ up to isomorphism. Let S_q be the fiber of S over q and p the intersection $S_q \cap C_0$.

Theorem 3.8 ([T-2] § 3.8). For any $n \geq 1$ and any tangential polynomial $P \in \Theta(n,E,z)$ the strict transform in S of the divisor $\{P=0\} \subset \mathbb{P}^1\times E$, denoted $\Gamma(P)$, has the following properties :

1) $\Gamma(P)$ is a reduced and irreducible divisor of S linearly equivalent to nC_0+S_q ;

2) p belongs to $\Gamma(P)$ and the natural projection $\pi : (\Gamma(P),p) \to (E,q)$ is a minimal tangential cover of degree n.

Furthermore the map $P \to \Gamma(P)$ defines an affine isomorphism between $\Theta(n,E,z)$ and $I|nC_0+S_q|$, the open subset of integral divisors in the complete linear system $|nC_0+S_q|$.

Remark 3.9. Let k be a tangential function for a tangential cover π of degree n and denote P_k its characteristic polynomial. Then P_k belongs to $\Theta(n,E,z)$ and π is a partial desingularization of the minimal tangential cover $(\Gamma(P_k),p) \to (E,q)$.

§ 4 - Symmetric tangential covers

4.1. There exists an involution $\tau : S \to S$ fixing the divisors C_0 and S_q and inducing over $\mathbb{P}^1 \times E$ the natural map $(T,z) \to (-T,-z)$ ([T-V.1]). Let us denote $I|nC_0+S_q|^{\tau}$ the affine subset of τ-invariant integral divisors in $|nC_0+S_q|$. For any $\Gamma \in I|nC_0+S_q|^{\tau}$, τ restricts to an involution (of Γ) whose fixed points lie over the half-periods of (E,q).

4.2. Analogously we denote $\Theta(n,E,z)^{\tau}$ the affine subset of tangential polynomials $P(T,z) = T^n + \sum_{1 \le j \le n} \alpha_j(z) T^{n-j}$, so called *symmetric*, such that $P(-T,-z)=(-1)^n P(T,z)$ (i.e. : α_j is a $(-1)^j$-even function). The map $P \longrightarrow \Gamma(P)$ restricts to an affine isomorphism of $\Theta(n,E,z)^{\tau}$ onto $I|nC_0+S_q|^{\tau}$.

4.3. Let Δ represent the differentiation with respect to the variable T and Δ^{-1} the integration operator which associates to $R = \sum_{0 \le j \le m} \beta_j T^j$ the polynomial

$$\Delta^{-1}(R) = \sum_{0 \le j \le m} (j+1)^{-1} \beta_j T^{j+1}.$$

Proposition ([T-2, § 4]) 4.4.
1) For any integer $n \ge 1$ and any tangential polynomial of degree $n+1$, $P \in \Theta(n+1,E,z)$, $\Delta(P)$ is in $\Theta(n,E,z)$.
2) For any $R \in \Theta(n,E,z)$ there exists a meromorphic function $\alpha(R)$ such that for any $c \in \mathbb{C}$, $(n+1)\Delta^{-1}(R)+\alpha(R)+c$ is in $\Theta(n+1,E,z)$.

In other words we can differentiate and integrate tangential polynomials.

Remark 4.5. Let Q be in $\Theta(n+2,E,z)$, R in $\Theta(n,E,z)$ and $\{\Gamma(Q), \Gamma(R)\}$ the corresponding τ-invariant divisors of S. Then for any $c \in \mathbb{C}$, $Q+cR$ is in $\Theta(n+2,E,z)^{\tau}$ and $\Gamma(Q+cR)$ belongs to the pencil of τ-invariant divisors of S generated by $\Gamma(Q)$ and $\Gamma(R)+2C_0$. Furthermore it follows from the latter properties, using classical Bertini arguments and the adjunction and Hurwitz formula, that for any $n \ge 1$:

Proposition 4.6.

1) $I|nC_0+S_q|$ and $I|nC_0+S_q|^\tau$ are affine spaces of dimension n and $[\frac{n}{2}]$ respectively ;

2) the generic elements of $I|nC_0+S_q|$ and $I|nC_0+S_q|^\tau$ are smooth of genus n ;

3) the canonical involution of the generic element of $I|nC_0+S_q|^\tau$ has $2+(1-(-1)^n)$ fixed points and the quotient is a smooth curve of genus $[\frac{n}{2}]$.

Definition 4.7. Let $\{\omega_j, j=0,1,2,3\}$ $(\omega_0 = q)$ be the subset of half-periods of (E,q). Th involution $\tau : S \to S$ has 2 fixed points over ω_j : one in C_0, the other one denoted r $(j=0,1,2,3)$. We say that a τ-invariant curve $\Gamma \in I|nC_0+S_q|^\tau$ is of *type* $\mu = (\mu_j) \in \mathbb{N}^4$ i

μ_j is equal to $\mathrm{mult}_{r_j}(\Gamma)$, the multiplicity of Γ at r_j, $j=0,1,2,3$. In that case we mus have $\mu_0+1 = \mu_1=\mu_2=\mu_3=n$ (mod 2). Furthermore if r_j is an ordinary singular point (Γ then Γ has μ_j transverse τ-invariant local branches passing thru r_j $(j=0,1,2,3)$ In particular the desingularization at $\{r_j, j=0,\ldots,3\}$ of such a curve has an involution with exactly $1+\sum_{0\le j\le 3}\mu_j$ fixed points : μ_0+1 over $\omega_0 = q$ and μ_i over ω_i $(i=1,2,3)$.

§ 5. Hyperelliptic tangential covers - general properties

Definition 5.1. From now on we will say that a curve C is *hyperelliptic at* p if belongs to C^0, the open dense subset of smooth points of C and equivalently :

1) there exists a local coordinate λ of C at p such that λ^2 defines a global projectio $C \to \mathbb{P}^1$ of degree 2 (ramified at p) ;

2) the origin of Jac$C$ is an inflexion point of the curve $A_C(C^0)$, the image of th associated Abel map ;

3) there exists an involution $\sigma : C \to C$ fixing p such that the quotient by th equivalence relation σ, C/σ, is isomorphic to $\mathbb{P}^1$.

Remark 5.2. The property 2) amounts to say that the tangent vector $U_2 = -\frac{\partial^2}{\partial\lambda^2}(A_C)_{|\lambda=0}$ is zero (for λ as in 1)). In particular for any $\mathcal{L} \in W(C)$ the functior $u(x,t) = -2\frac{\partial^2}{\partial x^2}\log\tau(\exp(xU_1+tU_3)\mathcal{L})$ is a solution of the Korteweg-de Vries equatior

2.3 (KdV). Furthermore any wave (also callded Baker-Akhiezer) function ψ associated to $\mathcal{D} = (C,p,\lambda,\mathcal{L})$ (c.f. [S-W]) is an eigenfunction of the operator $\frac{\partial^2}{\partial x^2} + u(x,t)$, of eigenvalue λ^2 i.e.: $(\frac{\partial^2}{\partial x^2} + u)\psi = \lambda^2 \psi$.

Definition 5.3. A pointed morphism $\pi : (C,p) \to (E,q)$ is called an *hyperelliptic tangential* cover if and only if π is tangential and C is hyperelliptic at p. In the latter case for any $\mathcal{L} \in W(C)$ and λ as in 5.1,1) the function $u(x,t)$ above is a finite-gap potential doubly periodic in x.

The following result implies that any hyperelliptic tangential cover is associated to a unique symmetric tangential polynomial.

Proposition 5.4. Let $\pi : (C,p) \to (E,q)$ be an hyperelliptic tangential cover and $\sigma : C \to C$ the unique hyperelliptic involution ($\sigma(p) = p$). There exists a unique tangential function k for π such that $\sigma^*(k) = -k$. In particular the characteristic polynomial of k is a symmetric tangential polynomial, π factorizes thru a unique birational morphism $j : C \to \Gamma$, $\Gamma \in I|nC_0+S_q|^\tau$ and $\sigma : C \to C$ is the restriction to C of the natural involution $\tau : S \to S$.

Proof. We have indeed the equality $\pi\circ\sigma = [-1]\circ\pi$, [-1] being the inverse homomorphism of (E,q). Therefore any tangential function k (for π) has same polar parts as $-\sigma^*(k)$ and we can uniquely normalize it by imposing $\sigma^*(k) = -k$. q.e.d.

Definition 5.5. Let $\Gamma \in I|nC_0+S_q|^\tau$ be any integral τ-invariant divisor in $|nC_0+S_q|$ of type $\mu = (\mu_0,\mu_1,\mu_2,\mu_3)$, $j : \hat{\Gamma} \to \Gamma$ its desingularization and $\hat{\tau} : \hat{\Gamma} \to \hat{\Gamma}$ the involution induced by $\tau : S \to S$. We say that $\hat{\Gamma}$ has *smooth type* $m = (m_0,m_1,m_2,m_3) \in \mathbb{N}^4$ if and only if $\hat{\tau}$ has m_j fixed points over r_j ($j=0,1,2,3$).

Remark 5.6. The smooth point $p \in \Gamma$ is fixed by τ and $\hat{\tau}$. Any other point of $\hat{\Gamma}$ fixed by $\hat{\tau}$ corresponds to a unique local branch of Γ passing thru one of the points $\{r_j, j=0,1,2,3\}$. Hence $m_j = \mu_j \pmod 2$ and $m_j \leq \mu_j$ with equality if r_j is an ordinary singular point ($j=0,1,2,3$).

Proposition 5.7. The genus $\hat{g}$ of the desingularized curve $\hat{\Gamma}$ and the invariants μ, m and n above satisfy the following inequalities :

1) $2\hat{g} \le 2n - \sum_{0\le j\le 3} \mu_j(\mu_j-1) \le 2n - \sum_{0\le j\le 3} m_j(m_j-1)$;

2) $\sum_{0\le j\le 3} m_j \le 2\hat{g}+1$.

In particular 1) and 2) imply

3) $\sum_{0\le j\le 3} m_j^2 \le 2n+1$.

It follows that $\hat{\Gamma}$ is hyperelliptic at p (i.e.: the natural projection $\hat{\pi} : (\hat{\Gamma},p) \to (E,q)$ is an hyperelliptic tangential cover) if and only if $2\hat{g}+1 = \sum_{0\le j\le 3} m_j$.

Proof

1) The curve Γ is a divisor of S of arithmetic genus n (according to the adjunction formula) and multiplicity μ_j at $r_j \in S$ (j=0,1,2,3). By blowing up the set of point $\{r_j, j=0,1,2,3\}$ we get a (partial) desingularization of Γ of arithmetic genus not less than $\hat{g}$ and equal to $n - \frac{1}{2}\sum_{0\le j\le 3} \mu_j(\mu_j-1)$. q.e.d.

2) The quotient curve $X = \hat{\Gamma}/\hat{\tau}$ is smooth and the canonical projection $\hat{\Gamma} \to X$ is ramified at the $1+\sum_{0\le j\le 3} m_j$ points fixed by $\hat{\tau}$. Therefore, by the Hurewitz formula, the genus g_0 of X satisfies the following equation : $2\hat{g} = 4g_0 - 1+\sum_{0\le j\le 3} m_j$. Hence $2\hat{g} + 1 \ge \sum_{0\le j\le 3} m_j$ with equality if and only if $g_0 = 0$, i.e. : if and only if X is isomorphic to $\mathbb{P}^1$, in which case we get a 2 to 1 projection $\hat{\Gamma} \to \mathbb{P}^1$ ramified at p.

q.e.d.

Corollary 5.8. Let $\pi : (C,p) \to (E,q)$ be a smooth hyperelliptic tangential cover of degree n and genus g then : $\frac{1}{2}g(g+1) \le n$.

Proof. The inequality above is equivalent to $g(g-1) \le 2(n-g)$. On the other side follows from the proposition (5.7) that
$\sum_{0\le j\le 3} m_j(m_j-1) \le 2(n-g)$ and $4g(g-1) = (-1+\sum_{0\le j\le 3} m_j)(-3+\sum_{0\le j\le 3} m_j)$.
It is therefore enough to prove that $(-1+\sum_{0\le j\le 3} m_j)(-3+\sum_{0\le j\le 3} m_j) \le 4\sum_{0\le j\le 3} m_j(m_j-1)$ or equivalently that $3 \le \sum_{0\le j,k\le 3} (m_j-m_k)^2$. The latter is always true since m_0-m_i (i=1,2,3) is an odd integer (see remark 5.6). q.e.d.

We end up this paragraph showing how to construct for any $n \geq 3$, primitive hyperelliptic tangential covers of degree n. The complete program has been carried out in [T-V.1].

Definition 5.9. Let $S' \to S$ be the blow-up of S at $\{r_j, j=0,1,2,3\}$, n an integer ≥ 1, Γ any τ-invariant curve in $I|nC_0+S_q|^\tau$ and denote $\Gamma' \to \Gamma$ its strict transform in S'. We say that the type $\mu = (\mu_0,\mu_1,\mu_2,\mu_3) \in \mathbb{N}^4$ of Γ is *exceptional* if $\sum_{0\leq j\leq 3} \mu_j^2 = 2n+1$. In the latter case Γ'has negative self-intersection ($\Gamma'.\Gamma' = -1$) and is therefore uniquely determined by the *exceptional couple* (n,μ).

Corollary 5.10. (*hyperelliptic criterion*)

Let $\Gamma \in I|nC_0+S_q|^\tau$ be a τ-invariant curve of exceptional type μ (i.e.: $\sum_{0\leq j\leq 3} \mu_j^2 = 2n+1$) and Γ' its strict transform in S'. If Γ has ordinary singularities at $\{r_j, j=0,1,2,3\}$ (i.e. : μ_j local branches thru r_j, $j=0,1,2,3$) then the canonical projection $(\Gamma',p) \to (E,q)$ is an hyperelliptic tangential cover of degree n and Γ' is a smooth curve of genus $\frac{1}{2}(-1 + \sum_{0\leq j\leq 3} \mu_j)$. Furthermore $(\Gamma',p) \to (E,q)$ is primitive (see 3.1.1)) unless n is even and there exists $i_0 \in \{1,2,3\}$ such that $(\mu_1,\mu_2,\mu_3) = (\mu_{i_0},0,0)$. An exceptional type $\mu = (\mu_i)$ doesn't satisfy the latter condition, and is called *primitive*, if and only if at most one of the μ_i's vanishes.

Proof. The curve Γ has arithmetic genus n and multiplicity μ_j at r_j ($j=0,1,2,3$) while Γ'has arithmetic genus $g' = n - \frac{1}{2} \sum_{0\leq j\leq 3} \mu_j(\mu_j-1) = \frac{1}{2}[-1 + \sum_{0\leq j\leq 3} \mu_j]$ and μ_j smooth τ-fixed points over r_j corresponding to the μ_j different local branches of Γ passing thru r_j ($j=0,1,2,3$). In particular the desingularization of Γ' has genus $\hat{g}$, $\hat{g} \leq g' = \frac{1}{2}[-1 + \sum_{0\leq j\leq 3} \mu_j]$ as well as $1 + \sum_{0\leq j\leq 3} \mu_j$ fixed points (for the induced involution). It follows (see proposition 5.7) that $2\hat{g}+1 \geq \sum_{0\leq j\leq 3} \mu_j = 2g'+1$. Hence $\hat{g} = g'$ and Γ' is smooth and hyperelliptic at p (see proposition 5.7).

Let us suppose now that $\pi' : (\Gamma',p) \to (E,q)$ is not primitive and factorizes as a projection $\pi^* : (\Gamma',p) \to (E^*,q^*)$, followed by a non-trival isogeny $\varphi : (E^*,q^*) \to (E,q)$.

Let us denote d ($d > 1$) the degree of φ and ${}_2E^*$ (resp. : ${}_2E$) the set of half-periods of (E^*,q^*) (resp. : (E,q)). The pointed morphism π^* is an hyperelliptic tangential cover of degree n/d and the fixed points of Γ' for the induced involution (i.e. : all Weierstrass points of Γ') are sent into ${}_2E^*$. We have two different cases to be treated depending on whether $d = \deg \varphi$ is even or odd.

1) If d is odd, φ induces an isomorphism between ${}_2E^* = \{\omega_j^*\}$ and ${}_2E = \{\omega_j\}$. Therefore

π^* has μ_0+1 (resp. : μ_i ; $i = 1,2,3$) Weierstrass points over $\omega_0^* = q^*$ (resp. : over ω_i^* ; i=1,2,3) and (by 5.7) $2\frac{n}{d}+1 \geq \sum_{0\leq j\leq 3} \mu_j^2 = 2n+1$ which implies d=1 ; contradiction!

In particular whenever n is odd $\pi' : (\Gamma',p) \to (E,q)$ is primitive.

2) If d is even there exists $\omega_{i_0} \in {}_2E - \{q\}$ such that ${}_2E^*$ is contained in $\varphi^{-1}(\{q,\omega_{i_0}\})$.

Hence all Weierstrass points of Γ' lie over $\{q,\omega_{i_0}\}$ and μ_i=μ_j=0 for $i,j \in \{1,2,3\}-\{i_0\}$.

q.e.d.

§ 6 - Construction of hyperelliptic tangential covers

6.1. Any hyperelliptic tangential cover $\pi : (C,p) \to (E,q)$ is associated to a unique symmetric tangential polynomial P and uniquely factorizes thru the ruled surface $S \to E$, obtained by blowing up $(\infty,q) \in \mathbb{P}^1{\times}E$ and a tangent direction over (∞,q) (see § 3.8, 5.4). Let us denote Γ the τ-invariant image of C in S, $\mu \in \mathbb{N}^4$ its type and $D(P)$ the zero divisor of the tangential polynomial P viewed as a rational function on $\mathbb{P}^1{\times}E$. The ruled surfaces $S \to E$ and $\mathbb{P}^1{\times}E \to E$ are isomorphic over $E-\{q\}$, Γ is the strict transform of $D(P)$ in S and the natural morphism $\Gamma \to D(P)$ is a (partial) desingularization of $(\infty,q) \in D(P)$. Hence for i=1,2,3 the multiplicity μ_i of Γ at r_i is equal to (and easily calculated as) the multiplicity of $D(P)$ at $(0,\omega_i)$.

6.2. Over a suitable neighborhood U of $q \in E$, S is isomorphic to $\mathbb{P}^1{\times}U$ and Γ is given by the equation $zP(T-z^{-1}) = 0$, z being a local coordinate of E at q. Furthermore the points p and r_0 of S, when considered in (the trivialization) $\mathbb{P}^1{\times}U$ have local coordinates $\{z,T^{-1}\}$ and $\{z,T\}$ respectively. In particular the type of Γ, $\mu = (\mu_0, \mu_1, \mu_2, \mu_3)$, can be computed from $P(T)$ as follows :

a) $\mu_0 = \text{mult}_{r_0}(\Gamma)$, by developping $zP(T-z^{-1})$ as a formal series in T and z, and calculating the degree of its leading form ;

b) $\mu_j = \text{mult}_{r_j}(\Gamma)$ $(j=1,2,3)$, by developping $P(T)$ as a formal series in T and $z-e_j$ $(e_j = \wp(\omega_j))$ and calculating the degree of its leading form.

6.3. For any symmetric tangential polynomial $P(T)$ there exists a unique symmetric tangential polynomial, denoted $\int(P)$, satisfying the following conditions (see 4.2 and 4.6) :

a) $\frac{\partial}{\partial T}(\int(P)) = (n+1)P$, $n = \deg P$;

b) $\text{Res}_q\{(\int(P)_{|T=0})z^{-1}dz\} = 0$.

Whenever deg P is odd the condition a) determines $\int(P)$ only up to an additive constant which is fixed by b). It follows that the moduli space of symmetric tangential polynomials is an affine space of dimension $[\frac{n}{2}]$.

Let us denote $S_1 = T$ and for any $a \in \mathbb{C}$, $S_2(a) = \int(S_1)+a$ and $S_3(a) = \int(S_2(a))$ More generally, for any $h \geq 2$ and $a_1,...,a_h \in \mathbb{C}$ we define $S_{2h}(a_1,...,a_h) = \int(S_{2h-1}(a_1,...,a_{h-1})) + a_h$ and $S_{2h+1}(a_1,...,a_h) = \int(S_{2h}(a_1,...,a_h))$. In other words, fo any $n \geq 2$ $S_n(a_1,...,a_{[\frac{n}{2}]})$ is the general symmetric tangential polynomial of degree n.

We deduce from the properties listed above and the hyperelliptic criterion 5.10 the following algorithm to construct (primitive exceptional) hyperelliptic tangential covers of any degree.

P.E.H.T.C. Algorithm 6.4.

1) Fix an integer $n \geq 3$ and find the coefficients of $S_n(a_1,...,a_{[\frac{n}{2}]})$, the general symmetric tangential polynomial of degree n.

2) Choose any exceptional type $\mu \in \mathbb{N}^4$ for n (i.e. : $\mu_0 = n+1 \pmod 2$ and $\sum_{0\leq j\leq 3} \mu_j^2 = 2n+1$) such that at most one of the $\{\mu_i, i=1,2,3\}$ vanish (the latter types, so-called *primitive exceptional*, exist for each $n \geq 3$).

3) Develop $P = S_n(a_1,\ldots,a_{[\frac{n}{2}]})$ around $(0,\omega_i)$ and $zP(T-z^{-1})$ around r_0 (see 6.2) ($\{\omega_i, i = 1,2,3\}$ being the set of non-trivial half-periods of (E,q)).

4) Find the values of $(a_1,\ldots,a_{[\frac{n}{2}]})$ such that the divisor of zeroes $D(P)$ (and its strict transform in S, denoted Γ_μ) have multiplicites μ_i at $(0,\omega_i)$ $(i=1,2,3)$ (and μ_0 at r_0, respectively).

5) Finally check that $\{r_j, j=0,1,2,3\}$ are ordinary singular points of Γ_μ, i.e. : check that the corresponding leading forms of $P(T)$ around $(0,\omega_i)$, for $j = 1,2,3$, and of $zP(T-z^{-1})$ around r_0, don't have multiple roots (at least for a generic elliptic curve E).

6) If steps 4) and 5) above get a positive answer (i.e. : there exists a curve Γ_μ, satisfying those properties) then the tangential cover $(\Gamma'_\mu,p) \to (E,q)$, obtained by blowing up $\{r_i, i=0,1,2,3\}$, satisfies the hyperelliptic criterion 5.10. More precisely, the pointed projection $(\Gamma'_\mu,p) \to (E,q)$ thus constructed is a primitive hyperelliptic tangential cover of degree n, exceptional type μ and arithmetic genus $g_\mu = \frac{1}{2}(-1 + \sum_{0\le i\le 3} \mu_i)$ (and smooth for a generic elliptic curve E).

We have carried out the PEHTC program (or algorithm) for n up to 6 obtaining the following results :

6.5. Let Λ be a lattice of $\mathbb{C}$, E the elliptic curve $\mathbb{C}/\Lambda$, $q \in E$ its origin and z the canonical local coordinate of E at q. The Weierstrass function of Λ, $\wp(z)$, and its derivative $\wp'(z)$ satisfy the following relation :

$$\wp'^2 = 4\prod_1^3(\wp-e_i) = 4\wp^3 - g_2\wp - g_3$$

where $e_i = \wp(\omega_i)$, the value of $\wp$ at the half-period ω_i, $i=1,2,3$. The isomorphism class of the elliptic curve $E = \mathbb{C}/\Lambda$ is given by its J-invariant, $J(E) = g_2^3(g_2^3 - 27\,g_3^2)^{-1}$.

Let $S_n(a_1,\ldots,a_{[\frac{n}{2}]})$ denote the general symmetric tangential polynomial of degree n. Wet get for n up to 6 that :

$S_3(a_1)$	$= T^3 + 3(a_1-p)T + p'$
$S_4(a_1,a_2)$	$= T^4 + 6(a_1-p)T^2 + 4p'T^2 - (3p^2+6a_1p-a_2)$
$S_5(a_1,a_2)$	$= T^5 + 10(a_1-p)T^3 + 10p'T^2 - 5(3p^2+6a_1p-a_2)T + 2p'(p+5a_1)$
$S_6(a_1,a_2,a_3)$	$= T^6 + 15(a_1-p)T^4 + 20p'T^3 - 15(3p^2+6a_1p-a_2)T^2 + 12p'(p+5a_1)T - c$ where $c = 5p^3 + 45a_1p^2 + (15a_2-8g_2)p - a_3$.

6.6. For each couple (n,μ), n=3,4,5,6 and μ a primitive exceptional (p.e.) type for n, we write down hereafter a symmetric tangential polynomial $\mathrm{Exc}_\mu(T)$ giving rise to a primitive exceptional hyperelliptic tangential cover Γ'_μ, of degree n and type μ (6.1)

n	p.e. type μ for n	$\mathrm{Exc}_\mu(T)$
3	(2,1,1,1)	$S_3(0) = T^3-3pT+p'$
4	(1,2,2,0)	$S_4(\frac{e_3}{2},-3e_1e_2) = T^4+3(e_3-2p)T^2+4p'T-3(p-e_1)(p-e_2)$
4	(1,2,0,2)	$S_4(\frac{e_2}{2},-3e_1e_3)$
4	(1,0,2,2)	$S_4(\frac{e_1}{2},-3e_2e_3)$
5	(0,3,1,1)	$S_5(-\frac{e_1}{5},\frac{9e_1^2}{5})=T^5-2(5p+e_1)T^3+10p'T^2-3(p-e_1)(5p+3e_1)T+2p'(p-e_1)$
5	(0,1,3,1)	$S_5(-\frac{e_2}{5},\frac{9e_2^2}{5})$
5	(0,1,1,3)	$S_5(-\frac{e_3}{5},\frac{9e_3^2}{5})$
6	(1,2,2,2)	$T^6-15pT^4+20p'T^3-\frac{9}{4}(20p^2-3g_2)T^2+12pp'T-\frac{5}{4}p'^2$ i.e. : $\mathrm{Exc}_{(1,2,2,2)} = S_6(0,\frac{9}{20}g_2,\frac{5}{4}g_3)$.

We got the latter tangential polynomials $\mathrm{Exc}_{\mu}(T)$ just by following the P.E.H.T.C. algorithm 6.4 for n up to 6. In each case the first 4 steps are successfully completed. Last but not least we check the last step, i.e. : that for any (n,μ) the corresponding τ-invariant divisor Γ_{μ} of S has at most ordinary singularities at the points $\{r_j, j=0,1,2,3\}$, for a generic elliptic curve E. For example we got that $\Gamma_{(1,2,2,0)}$ (resp. : $\Gamma_{(1,2,2,2)}$) has at most ordinary singularities if and only if $(5e_1+7e_2)(7e_1+5e_2) \neq 0$ (resp. : $\prod_1^3 (5g_2-12e_i^2) \neq 0$).

6.7. Summing up we get for a generic elliptic curve E that Γ'_{μ}, the strict transform of Γ_{μ} by the blow-up of $\{r_j, j=0,1,2,3\} \subset S$, is a smooth curve of (arithmetic) genus $g_{\mu} = \frac{1}{2}(-1 + \sum_{0\leq j\leq 3} \mu_j)$ and the natural projection $(\Gamma'_{\mu},p) \to (E,q)$ is a primitive hyperelliptic tangential cover of degree n (n=3,4,5,6). Furthermore, when we let E vary over the moduli of elliptic curves we get, for each exceptional type μ above, a flat family of integral curves $\{\Gamma'_{\mu}\}$. We easily deduce then that, besides smoothness, all other properties of $(\Gamma'_{\mu},p) \to (E,q)$ are still true (for any elliptic curve E).

§ 7 - Source potentials

7.1. Let $\pi : (C,p) \to (E,q)$ be an hyperelliptic tangential cover, $\sigma : C \to C$ the hyperelliptic involution, z an anti-invariant local coordinate of E at q and k the unique tangential function for π such that $\sigma^*(k) = -k$ (c.f. (5.4)). Then $k+z^{-1}$ has a simple pole at p and its inverse, denoted $\lambda_C = (k+z^{-1})^{-1}$, defines a local σ-anti-invariant local coordinate of C at p. For any $\mathcal{L} \in W(C)$, the solution to the KdV equation corresponding to $\mathcal{D} = (C,p,\lambda_C,\mathcal{L})$ is a *KdV elliptic soliton* (c.f. : (5.2) and [T-V]-2, § 7.5), i.e. : it's equal to :

$$u_{\mathcal{D}}(x,t) = 2 \sum_{1\leq i\leq n} \wp(x-x_i(t)) \quad ,$$

where $n = \deg \pi$, $\wp$ is the Weierstrass function associated to (E,z) and $\{x_i\} : \mathbb{C} \to \mathrm{Sym}^n E$, $t \to \{x_i(t)\}$, is an analytic function. Let $g \geq 1$ be the arithmeti genus of C and $v(x)$ the initial-value function of the KdV elliptic soliton associated to

$\mathcal{D}_0 = (C,p,\lambda_C,\mathcal{O}_C((g-1)p))$ (i.e. : $v(x) = u_{\mathcal{D}_0}(x,0)$). The function $v(x)$, called hereaft *source potential* is a finite-gap elliptic (Λ-periodic) potential which uniquel determines its spectral pointed curve (C,p). Moreover, any KdV elliptic solito $u_{\mathcal{D}}(x,t)$ as above is deduced from $v(x)$ by applying the hierarchy of KdV flows.

If π is not primitive there exists a projection $\pi' : (C,p) \to (E',q')$ and a non-trivial isogeny of degree d ($d > 1$), $\varphi : (E',q') \to (E,q)$, such that $\pi = \varphi \circ \pi'$. Furthermore π' is also an hyperelliptic tangential cover (of degree $m = \frac{n}{d}$) and the function $u_{\mathfrak{D}}(x,t)$ above is still a KdV elliptic soliton with respect to π', i.e. : there exists a map $t \to \{a_j(t)\}$ from $\mathbb{C}$ into $\mathrm{Sym}^m E'$ such that

$$u_{\mathfrak{D}}(x,t) = 2 \sum_{1 \le j \le m} \wp_{E'}(x - a_j(t)) \quad ,$$

where $\wp_{E'}$ is the Weierstrass function associated to $(E', \varphi^*(z))$.

We gather hereafter enough properties satisfied by all *source potentials* to calculate them for each Γ'_μ ($n = 3,4,5,6$) of (6.6).

Lemma 7.2. Let $\pi : (C,p) \to (E,q)$ and λ_C as above, $\mathcal{L} \in W(C)$ any theta-characteristic (i.e. : $\mathcal{L} \otimes \mathcal{L} \sim \mathcal{O}_C((2g-2)p)$ and denote $\mathrm{Orb}_{\mathcal{L}} : E \to W(C)$ the orbital morphism $\alpha \longrightarrow \mathcal{L}(\pi^*(\alpha - q))$. Then the KdV elliptic soliton associated to $(C, p, \lambda_C, \mathcal{L})$ has an initial value of the form $2 \sum_{1 \le i \le l} m_{\alpha_i} \wp(x - \alpha_i)$ ($\alpha_i \neq \alpha_j$ if $i \neq j$) where :

a) the coefficients $\{m_{\alpha_i}, i = 1,\dots,l\}$ are triangular integers
   (i.e. : $2m_\alpha = p_\alpha(p_\alpha + 1)$, for some $p_\alpha \in \mathbb{N}$) and $\sum_{1 \le i \le l} m_{\alpha_i} = n$;

b) the E-divisor $\sum_{1 \le i \le l} m_{\alpha_i}(\alpha_i)$ is [-1]-invariant (i.e. : we have $\alpha \in \{\alpha_i, i=1,\dots,l\}$ if and only if $-\alpha \in \{\alpha_i, i=1,\dots,l\}$ and $m_\alpha = m_{-\alpha}$;

c) the origin q of E belongs to $\{\alpha_i\}$ if and only if $h^0(C, \mathcal{L}) \ge 1$, in which case $m_q \ge 3$.

d) More generally a half-period ω of E belongs to $\{\alpha_i\}$ if and only if $h^0(\mathcal{L}(\pi^*(\omega - q))) \ge 1$ in which case the difference $m_\omega - h^0(\mathcal{L}(\pi^*(\omega - q)))$ is an <u>even</u> non-negative integer.

Proof. a) Let us map the data $\mathfrak{D} = (C, p, \lambda_C, \mathcal{L})$ (up to the choice of a trivialization of $\mathcal{L}$ around p) into a point w of the infinite-dimensional grassmannian Gr and denote τ_w and ψ_w the tau and Baker functions associated to w (c.f. [S-W]). The KdV elliptic soliton $u(x,t)$ associated to $\mathfrak{D}$ is then equal to $u = -2 \frac{\partial^2}{\partial x^2} \log \tau_w$ while ψ_w is an eigenfunction of the differential operator $\frac{\partial^2}{\partial x^2} + u$. If we let $\mathcal{L}$ vary over the whole compactified jacobian $W(C)$ we get a map from $W(C)$ into Gr such that the pull-back image of τ_w vanishes to order 1 along the divisor Θ. Hence $u(x,t)$ is of the form $2 \sum_{1\le i\le n} \wp(x-x_i(t))$ and its initial value $u(x,0)$ is equal to $\sum_{1\le i\le l} m_{\alpha_i}\wp(x-\alpha_i)+cst$, $\sum_{1\le i\le l} m_i=n$ being the intersection number $\mathrm{Orb}_{\mathcal{L}}(E).\Theta$ and the E-divisor $\sum_{1\le i\le l} m_{\alpha_i}(\alpha_i)$

being the pull-back of Θ, $\mathrm{Orb}^*_{\mathcal{L}}(\Theta)$.We then check that m_{α_i} $(i=1,..,l)$ is triangular, by forcing a local development of ψ_w at $x=\alpha_i$ to be an eigenfunction of $\frac{\partial^2}{\partial x^2}+u(x,0)$.

q.e.d.

b) The canonical sheaf of C, ω_C, is isomorphic to $\mathcal{O}_C((2g-2)p)$ and the map $F\to \mathcal{H}om_C(\mathcal{F},\omega_C)$ (equal to $\omega_C\otimes\mathcal{F}^{-1}$ if $\mathcal{F}$ is a locally free sheaf) defines an involution of $W(C)$, $\sigma : W(C) \to W(C)$, which fixes all theta caracteristics, leaves Θ invariant and induces on Jac C the inverse morphism. The latter are classical results for a smooth hyperelliptic curve C. Their detailed proof in case C is singular (and integral) will be given elsewhere. It follows that $\mathrm{Orb}_{\mathcal{L}}(E)$ is σ-invariant and the pull-back of (the σ–invariant divisor) Θ is [-1]-invariant. q.e.d.

c) Let us consider E (and C) as naturally (rationnally) embedded in Jac C and let $\varphi_{\mathcal{L}}$: Jac $C \to W(C)$ denote the Jac C-orbital morphism $\mathcal{F} \to \mathcal{F}\otimes\mathcal{L}$. If $\mathcal{L}$ is in Θ then $\varphi_{\mathcal{L}}(C)$ is contained in Θ, and tangent at $\mathcal{L}$ to $\varphi_{\mathcal{L}}(E)$. In particular $\varphi_{\mathcal{L}}(E)$ intersects Θ at $\mathcal{L}$ with multiplicity $m_q > 1$. Hence $m_q \ge 3$ since it must be a triangular integer by a). q.e.d.

d) Let us denote $\mathcal{L}_\omega$ the sheaf $\mathcal{L}(\pi^*(\omega-q))$ and $\mathcal{L}_\omega$.Jac C the (open dense) Jac C-orbit of $\mathcal{L}_\omega$ in $W(C)$. The natural involution of $W(C)$ induces the inverse morphism $\mathcal{L}_\omega.\alpha \to \mathcal{L}_\omega.(-\alpha)$ on $\mathcal{L}_\omega$.Jac C (and on $\mathcal{L}_\omega.E$), fixes $\mathcal{L}_\omega$ and leaves the divisor Θ invariant. The function τ_w is defined and anti-invariant on $\mathcal{L}_\omega$.Jac C, and vanishes (with multiplicity 1) along the restriction of Θ. The development of τ_w around $\mathcal{L}_\omega$ in terms of local anti-invariant coordinates of $\mathcal{L}_\omega$.Jac C has the form $\sum_{0\le j} F_{d_\omega+2j}$ where $d_\omega = h^0(C,\mathcal{L}_\omega)$ and for every j, $F_{d_\omega+2j}$ is a homogeneous form of degree $d_\omega+2j$. It easily follows that m_ω, the multiplicity intersection at $\mathcal{L}_\omega$ between Θ and $\mathcal{L}_\omega.E$, is equal to $d_\omega+2r$ for some $r \ge 0$. q.e.d.

7.3. Let $\mu = (\mu_j) \in \mathbb{N}^4$ be a primitive exceptional type (p.e. type) for n, $n = 3,4,5,6$ and $\pi_\mu : (\Gamma'_\mu, p) \to (E,q)$ the (primitive exceptional) hyperelliptic tangential cover of type μ constructed above. The corresponding source potential $u_\mu(x)$ is associated to the theta characteristic $\mathcal{O}_{\Gamma'_\mu}((g_\mu - 1)p)$. Recall also that the image of the τ-fixed point of S, r_i (i=1,2,3), is a non-trivial half-period of (E,q) denoted ω_i.

By means of lemma 7.2 we construct the following new elliptic finite-gap potential of degrees 3, 4, 5 and 6 for <u>any</u> elliptic curve E.

Theorem 7.4. For any primitive exceptional type μ ($\sum_{0\le j\le 3} \mu_j^2 = 2n+1$, $\mu_0 \ne n \pmod 2$, $n = 3,4,5,6$) the source potential $u_\mu(x)$ is equal to :

n	p.e. type μ for n	source potential $u_\mu(x)$ for (Γ'_μ, p)	g_μ
3	(2,1,1,1)	$2.3\wp(x)$	2
4	(1,2,2,0)	$2(3\wp(x)+\wp(x-\omega_3))$	2
4	(1,2,0,2)	$2(3\wp(x)+\wp(x-\omega_2))$	2
4	(1,0,2,2)	$2(3\wp(x)+\wp(x-\omega_1))$	2
5	(0,3,1,1)	$2(3\wp(x)+\wp(x-\omega_2))+\wp(x-\omega_3))$	2
5	(0,1,3,1)	$2(3\wp(x)+\wp(x-\omega_1))+\wp(x-\omega_3))$	2
5	(0,1,1,3)	$2(3\wp(x)+\wp(x-\omega_2))+\wp(x-\omega_3))$	2
6	(1,2,2,2)	$2.6\wp(x)$	3

Proof. The arithmetic genus, g_μ, of Γ'_μ is equal to $\frac{1}{2}(\sum_{0\le j\le 3}\mu_j - 1)$ and its source potential $u_\mu(x) = 2\sum_{1\le i\le l} m_{\alpha_i}\wp(x-\alpha_i)(\sum_{1\le i\le l} m_{\alpha_i}=n)$ corresponds to $\mathcal{L}=\mathcal{O}_{\Gamma'_\mu}((g_\mu-1)p)\in \mathcal{J}(\Gamma'_\mu)$. Recall also that $\sum_{1\le i\le l} m_{\alpha_i}(\alpha_i)$ is obtained by intersecting the E-orbit of $\mathcal{L}$ with the divisor Θ, which is isomorphic to Γ'_μ whenever $g_\mu = 2$.

$n=3$; $\mu = (2,1,1,1)$; $g_\mu=2$. It follows from lemma 7.2 that $m_q \ge 3$ (and $\sum_i m_{\alpha_i}=3$). Hence $u_\mu(x) = 6\wp(x)$.

$n=4$; $\mu = (1,2,2,0)$; $g_\mu=2$. For any elliptic curve E, $p = C_0 \cap S_q$ and $r_0 \in S$ are smooth fixed points of the τ-invariant divisor $\Gamma_{(1,2,2,0)}$, and Weierstrass points of its strict transform $\Gamma'_{(1,2,2,0)}$. Moreover $\Gamma'_{(1,2,2,0)}$ has one couple (resp. : two couples) of smooth points over q (resp. : over ω_3), interchanged by the hyperelliptic involution induced by $\tau : S \to S$. It follows that $\omega_3 \in E$ and $r_0 \in \Gamma'_{(1,2,2,0)}$ have same canonical image in Jac $\Gamma'_{(1,2,2,0)}$. Hence $m_{\omega_3} \ge 1$ and (by lemma 7.2 ; a),c)) $u_\mu(x) = 2(3\wp(x) + \wp(x-\omega_3))$.

$n=4$; $\mu = (1,2,0,2)$ or $(1,0,2,2)$; $g_\mu=2$. Same proof as above.

$n=5$; $\mu = (0,3,1,1)$ or $(0,1,3,1)$ or $(0,1,1,3)$; $g_\mu=2$. Here again the same proof works in all three cases (and is analogous to the latter one). Let us choose for example $\mu = (0,3,1,1)$. Then $\{p,r_2,r_3\} \subset S$ are fixed points of the τ-invariant divisor Γ_μ and Weierstrass points of its strict transform Γ'_μ. Moreover Γ'_μ has two couples of smooth points over q, ω_2 and ω_3, exchanged by the hyperelliptic involution induced by $\tau : S \to S$. It follows that ω_2 and r_2, as well as ω_3 and r_3, have same canonical image in Jac Γ'_μ. Thus $m_{\omega_2} \ge 1$, $m_{\omega_3} \ge 1$ and
$u_{(0,3,1,1)}(x) = 2(3\wp(x)+\wp(x-\omega_2)+\wp(x-\omega_3))$. q.e.d.

$n=6$; $\mu = (1,2,2,2)$; $g_\mu=3$. For any elliptic curve E, $\Gamma'_{(1,2,2,2)}$ is hyperelliptic, p is a Weierstrass point and $\mathcal{L} = \mathcal{O}_{\Gamma'_{(1,2,2,2)}}(2p)$ is a theta-characteristic such that

$h^0(\mathcal{L}) = 2$ (6.7). It easily follows from lemma 7.2, a),d) that the coefficient m_q is an even triangular integer ≤ 6. Hence $m_q = 6$ and $u_{(1,2,2,2)}(x) = 12\mathrm{p}(x)$. q.e.d.

7.5. Let $n \geq 1$ be an integer and denote I_n the set of all exceptional types for n, i.e. : $\mu = (\mu_i) \in \mathbb{N}^4$ is in I_n if and only if $n-\mu_0 = 1 \pmod 2$ and $\sum_{0\leq i\leq 3} \mu_i^2 = 2n+1$. On the other side let us denote $II_n = \{ (a_i) \in \mathbb{N}^4 / \sum_{0\leq i\leq 3} a_i(a_i+1) = 2n \}$ and $II_n^{\geq}$ (resp. : $II_n^{\leq}$) the subset of those (a_i) in II_n sucht that $2\sup\{a_i\} \geq s - (1+(-1)^s)\inf\{a_i\}$ (resp. : $2\sup\{a_i\} \leq s - (1+(-1)^s)\inf\{a_i\}$) where $s = \sum_{0\leq i\leq 3} a_i$. We finish stating a complete generalization of the latter results, obtained in collaboration with J.-L. Verdier (c.f. : [T-V.3] ; detailed proofs will appear elsewhere).

Theorem 7.6 (*exceptional* **source potential characterization)**

For any $\mu \in I_n$ let $\pi_\mu : (\Gamma'_\mu, p) \to (E,q)$ be the unique hyperelliptic tangential cover of degree n, exceptional type μ and arithmetic genus $g_\mu = \frac{1}{2}(-1 + \sum_{0\leq i\leq 3} \mu_i)$ (cf. : [T-V-1]). If we denote λ_μ the canonical local coordinate of Γ'_μ at p (c.f. : 7.1) and $\mathcal{D}_\mu$ the data $(\Gamma'_\mu, p, \lambda_\mu, \mathcal{O}_{\Gamma'_\mu}((g_\mu-1)p))$, then :

A) The source potential $u_\mu(x)$ associated to $\mathcal{D}_\mu$ is equal to :

$$u_\mu(x) = g_\mu(g_\mu+1)\mathrm{p}(x) + \sum_{1\leq j\leq 3} (g_\mu-\mu_0-\mu_j)(g_\mu-\mu_0-\mu_j+1)\mathrm{p}(x-\omega_j) .$$

In other words we have :

$$u_\mu(x) = \sum_{0\leq i\leq 3} a_i(a_i+1)\mathrm{p}(x-\omega_i) \quad ,$$

where $a_0 = g_\mu$ and $a_j = \left| g_\mu-\mu_0-\mu_j + \frac{1}{2} \right| - \frac{1}{2}$ for $j = 1,2,3$.

Furthermore $a_0 = \sup\{a_i\}$ and (a_i) belongs to $II_n^{\geq}$.

B) Conversely, for any (a_i) in $II_n^{\geq}$ such that $a_0 = \sup\{a_i\}$, $\sum_{0\leq i\leq 3} a_i(a_i+1)\mathrm{p}(x-\omega_i)$ is the source potential associated to the pointed curve (Γ'_μ, p) (of genus a_0), for a uniquely determined $\mu \in I_n$.

It's quite natural at this point to ask, for any $(a_i) \in \mathbb{N}^4 - (0)$, whether $\sum_{0 \le i \le 3} a_i(a_i+1)\wp(x-\omega_i)$ is at least a finite-gap potential or the initial-value function of a KdV elliptic soliton (§ 2). The answer is positive and explicitly stated hereafter (c.f. : [T-V-3]).

Theorem 7.7. For any $n \ge 1$ and $(a_i) \in \mathbb{N}^4$ such that $\sum_{0 \le i \le 3} a_i(a_i+1) = 2n$ there exists a uniquely determined hyperelliptic tangential cover of exceptional type

$\mu \in I_n$, $\pi_\mu : (\Gamma'_\mu, p) \to (E,q)$, and a half-period $\omega \in \{\omega_0 = q, \omega_1, \omega_2, \omega_3\}$ such that the function $\sum_{0 \le i \le 3} a_i(a_i+1)\wp(x-\omega_i)$ is the initial-value function of the KdV elliptic soliton

associated to the data $(\Gamma'_\mu, p, \lambda_\mu, \mathcal{L}(\pi^*_\mu(\omega - q)))$ where

1) $\mathcal{L} = \mathcal{O}_{\Gamma'_\mu}((g_\mu - 1)p)$ if $(a_i) \in \Pi^{\ge}_n$

2) $\mathcal{L} = \mathcal{O}_{\Gamma'_\mu}(\pi^*_\mu(q) - (n+1-g_\mu)p))$ if $(a_i) \in \Pi^{\le}_n$.

Remark 7.8.

1) Every data $(\Gamma'_\mu, p, \lambda_\mu, \mathcal{L}(\pi^*_\mu(\omega - q)))$ as in 7.7, 1) or 2) $(\mu \in I_n)$ corresponds uniquely to some (initial-value) function $\sum_{0 \le i \le 3} a_i(a_i+1)\wp(x-\omega_i)$ $(\sum_{0 \le i \le 3} a_i(a_i+1) = 2n)$.
Conservely for any $(a_i) \in \mathbb{N}^4$ such that $\sum_{0 \le i \le 3} a_i(a_i+1) = 2n$, the function $\sum_{0 \le i \le 3} a_i(a_i+1)\wp(x-\omega_i)$ corresponds to one of those data.

2) For any exceptional type $\mu = (\mu_i) \in I_n$ the projection π_μ is not primitive if and only if n is even and at least two of the μ_i's vanish. In the latter case (and only in that case) π_μ factorizes thru a degree-2 isogeny $\varphi : (E^\sim, q^\sim) \to (E,q)$ and the correspondin initial-value functions are equal to

$$\alpha(\alpha+1)[\wp(x)+\wp(x-\omega)] + \beta(\beta+1)[\wp(x-\omega')+\wp(x-\omega'')],$$

as well as to

$$\alpha(\alpha+1)\wp^\sim(x)+\beta(\beta+1)\wp^\sim(x-\omega^\sim),$$

for some $\alpha, \beta \in \mathbb{N}$, where $\{\omega, \omega', \omega''\} = \{\omega_1, \omega_2, \omega_3\}$, $\omega^\sim$ is a half-period of $(E^\sim, q^\sim)$, an $\wp^\sim$ is the Weierstrass function associated to $(E^\sim, \varphi^*(z))$.

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ON EXPANSIVE COVERING MAPS

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Abstract: Liapunov functions are obtained for the universal cover of an expansive covering map. It is proved that any expansive covering map does not have stable points. It is also shown that the stable set of a point x of an expansive covering map with no reppelling points consist of two arcs that meet only at x.

Introduction.

In this article we study dynamical properties of expansive covering maps of manifolds. Observe that as a consequence of the Hurwitz's formula, if $f: M \to M$ is a nontrivial covering map then the Euler-Poincaré characteristic of M is zero, then, for instance, the only surface that supports such a map is the torus.

It is simple to show that expansive covering maps of manifolds have positive entropy. Examples of this maps are expanding endomorphisms, weakly Anosov endomorphisms [5] and, of course, expansive homeomorphisms.

In [2] Lewowicz[1] proved that an expansive homeomorphism of a surface is topologically conjugate to an Anosov diffeomorphism or to a pseudo-Anosov "diffeomorphism" and in [1] Hiraide proved that a positively expansive map of a manifold is topologically conjugate to an expanding infra-nil-endomorphism. Here, we do not give a topological classification of expansive covering maps of the 2-torus; however, by making use of Lewowicz's methods ([2]) we obtain a description of the stable sets of such a map (see Proposition 3.4).

In order to use Lewowicz's methods we need Lianupov function of two variables. In section 1 we construct a Liapunov function for a lift of f to the universal cover (see [2] section 1 and [3] section 4). This allows us

[1] K. Hiraide has obtained, independently, the same result.

to prove, in section 2, proposition 2.2, that expansive covering maps do not have stable points (see [4] lemma 2.7)

Thank to those propositions we can show, in section 3, that, if an expansive covering map of a surface (the 2-torus) has no reppelling points, a lift to the universal cover has local product structure for all points.

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1

Definition 1.1: A map $f: M \to M$ of a metric space (M,d) is called expansive if there exists a constants $\alpha > 0$ such that if orbits (x_n), (y_n) (i.e. $f(x_n) = x_{n-1}$, $f(y_n) = y_{n-1}$ $\forall n \in \mathbb{Z}$) verify $d(x_n, y_n) \leq \alpha$ $\forall n \in \mathbb{Z}$ then $x_n = y_n$ $\forall n \in \mathbb{Z}$.

In the sequel we will suppose that f is an expansive covering map with constant of expansivity α and M is a compact connected boundaryless riemannian manifold. Let $(\tilde{M},\pi)$ be the universal cover of M and $\tilde{d}$ distance induced on $\tilde{M}$. Let $\tilde{f}: \tilde{M} \to \tilde{M}$ be a lift of f. Like in [5], pag. 176, $\tilde{f}$ is a homeomorphism.

Hereafter, let α be so small that if $d(x,y) < \alpha$ then x,y are contained in a admissible neighbourhood of $(\tilde{M},\pi)$ and if $x', x'' \in f^{-1}(x)$ then $d(x',x'') > \alpha$.

Consider $\delta > 0$, $\delta > \alpha/2$. We call stable set to

$C^+ = \{(x,y) \in M \times M;\ d(f^n(x), f^n(y)) \leq \delta\ \forall\ n \geq 0\}$

and the unstable set to

$C^- = \{(x,y) \in M \times M;\ \exists\ (x_n), (y_n)$ orbits of x,y such that $d(x_n,y_n) \leq \delta\ \forall\ n \leq 0\}$

Lemma 1.2: C^+ and C^- are compact sets.

Proof: Suppose that (x^n,y^n) is a sequence in C^- which converges to (x,y). For all $n \in \mathbb{N}$, there exist (x_i^n), (y_i^n) orbits of x^n,y^n such that $d(x_i^n, y_i^n) \leq \delta \ \forall\ i \leq 0$. There exists a sequence $\{n_k\}$ such that $x_{-1}^{n_k} \to x_{-1}$ and $y_{-1}^{n_k} \to y_{-1}$ where x_{-1} and y_{-1} are preimages of x and y and $d(x_{-1}, y_{-1}) \leq \delta$. Reasoning by induction we deduce $(x,y) \in C^-$. The proof for C^+ is simpler. •

Consider a function $h: M \times M \to [0,1]$ such that

$$h(x,y) = 0 \text{ if } (x,y) \in C^+$$
$$h(x,y) = 0 \text{ if } d(x,y) \geq \delta$$
$$h(x,y) > 0 \text{ elsewhere}$$

and another function $H: M \times M \to [0,1]$ such that

$$H(x,y) = 1 \text{ if } d(x,y) \leq \sigma < \delta$$
$$H(x,y) = 0 \text{ if } d(x,y) \geq \delta$$
$$H(x,y) > 0 \text{ elsewhere.}$$

Then we can define the following functions:

$\tilde{h}: \tilde{M} \times \tilde{M} \to [0,1]$ such that

$$\tilde{h}(\tilde{x},\tilde{y}) = h(\pi\tilde{x},\pi\tilde{y}) \text{ if } \tilde{d}(\tilde{x},\tilde{y}) \leq \delta$$
$$\tilde{h}(\tilde{x},\tilde{y}) = 0 \text{ if } \tilde{d}(\tilde{x},\tilde{y}) > \delta$$

and $\tilde{H}: \tilde{M} \times \tilde{M} \to [0,1]$ such that

$$\tilde{H}(\tilde{x},\tilde{y}) = H(\pi\tilde{x},\pi\tilde{y}) \text{ if } \tilde{d}(\tilde{x},\tilde{y}) \leq \delta$$
$$\tilde{H}(\tilde{x},\tilde{y}) = 0 \text{ if } \tilde{d}(\tilde{x},\tilde{y}) > \delta$$

Clearly $\tilde{h}$ and $\tilde{H}$ are uniformly continuous functions.

Define

$$a_{-n}(\tilde{x},\tilde{y}) = \tilde{h}(\tilde{f}^{-n}(\tilde{x}), \tilde{f}^{-n}(\tilde{y})) \prod_{i=0}^{n} \tilde{H}(\tilde{f}^{-i}(\tilde{x}), \tilde{f}^{-i}(\tilde{y}))$$

Lemma 1.3: For any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n \geq N$, $a_{-n}(\tilde{x},\tilde{y}) < \varepsilon \ \forall\ (\tilde{x},\tilde{y}) \in \tilde{M} \times \tilde{M}$.

Proof: Assume this is not the case. Then there exists $\varepsilon_0 > 0$ such that exist $\{n_k\}$ and $(\tilde{x}^k,\tilde{y}^k)$ with the property that $a_{-n_k}(\tilde{x}^k,\tilde{y}^k) > \varepsilon_0\ \forall\ k \in \mathbb{N}$ (1).

This implies

$$\tilde{h}(\tilde{f}^{-n_k}(\tilde{x}^k),\ \tilde{f}^{-n_k}(\tilde{y}^k)) > \varepsilon_0$$

thus

$$\tilde{h}(\pi\tilde{f}^{-n_k}(\tilde{x}^k), \pi\tilde{f}^{-n_k}(\tilde{y}^k)) > \varepsilon_0$$

There exists an orbit (x^k_n) of $x^k = \pi\,\tilde{x}^k$ such that $\pi\tilde{f}^{-j}(\tilde{x}^k) = x^k_{-j}$ for $0 \le j \le n_k$ and analogously for $\tilde{y}^k$, which verify

$$h(x^k_{-n_k}, y^k_{-n_k}) > \varepsilon_0 \quad (2)$$

Inequality (1) implies $d(x^k_{-i},\ y^k_{-i}) \le \delta$ for $0 \le i \le n_k$ and all limit points of $(x^k_{-n_k}, y^k_{-n_k})$ are in C^+ which contradicts (2). •

As a consequence of this lemma we have that there exists a C^∞ function $e: \mathbb{R}^+ \to \mathbb{R}^+$, $e(0) = 0$, $e'(s) > 0$ if $s \neq 0$, with the property that

$$v^-(\tilde{x},\tilde{y}) = \sum_{n=0}^{\infty} e(a_{-n}(\tilde{x},\tilde{y})) \text{ uniformly converges.}$$

We call $V^- = v^-|_A$ where $A = \{ (\tilde{x},\tilde{y}) \in \tilde{M} \times \tilde{M}\ ;\ \tilde{d}(\tilde{x},\tilde{y}) < \sigma \}$.

We remark that $v^-(\tilde{x},\tilde{y}) = 0$ if and only if $\tilde{d}(\tilde{x},\tilde{y}) \ge \delta$ or $(\pi\tilde{x},\pi\tilde{y}) \in C^+$;

and

$$V^-(\tilde{f}(\tilde{x}),\tilde{f}(\tilde{y})) - V^-(\tilde{x},\tilde{y}) = e\,(\tilde{h}(\tilde{f}(\tilde{x}),\tilde{f}(\tilde{y})))$$

Furthermore, it is clear that v^- is uniformly continuos.

Define

$$b_{-n}(\tilde{x},\tilde{y}) = v^-(\tilde{f}^{-n}(\tilde{x})\ ,\ \tilde{f}^{-n}(\tilde{y}))\cdot\prod_{i=0}^{n} \tilde{H}\,(\tilde{f}^{-i}(\tilde{x})\ ,\ \tilde{f}^{-i}(\tilde{y}))$$

Lemma 1.4: For any $\varepsilon > 0$ there exists N such that,

$$\forall\ n \ge N \text{ and } \forall(\tilde{x},\tilde{y}) \in \tilde{M} \times \tilde{M},\ b_{-n}(\tilde{x},\tilde{y}) < \varepsilon.$$

Proof: Suppose there exist $\varepsilon_0 > 0$, $\{n_k\}$ and $(\tilde{x}^k,\tilde{y}^k)$ such that $b_{-n_k}(\tilde{x}^k,\tilde{y}^k) \geq \varepsilon_0 \; \forall \, k \in \mathbb{N}$. (3)

Then

$$v^-(\tilde{f}^{-n_k}(\tilde{x}^k), (\tilde{f}^{-n_k}(\tilde{y}^k)) > \varepsilon_0$$

Analogously to lemma 1.3, (3) implies $d(x^k_{-t}, y^k_{-t}) \leq \delta$ for $0 \leq t \leq n_k$ and $\forall \, k \in \mathbb{N}$.

Then a limit point (x,y) of $(x^k_{-n_k}, y^k_{-n_k})$ must be in C^+, because $d(f^m(x), f^m(y)) = \lim_{k\to\infty} d(f^m(x^k_{-m_k}), f^m(y^k_{-m_k})) = \lim d(x^k_{-n_k+m}, y^k_{-n_k}) \leq \rho$.

Thus for k large enough and arbitrary $\rho > 0$, there exists $(\tilde{x},\tilde{y})$ such that $\pi\tilde{x}=x$ and $\pi\tilde{y}=y$ and $\tilde{d}(\tilde{x},\tilde{f}^{-n_k}(\tilde{y}^k)) < \rho$, but $v^-(\tilde{x},\tilde{y}) = 0$, a contradiction.

Define $U^-: A \to \mathbb{R}$ such that $U^-(\tilde{x},\tilde{y}) = \sum_{n=0}^{\infty} g(b_{-n}(\tilde{x},\tilde{y}))$ where g is a analogous to e.

We have

$$U^-(\tilde{f}(\tilde{x}),\tilde{f}(\tilde{y})) - U^-(\tilde{x},\tilde{y}) = g(V^-(\tilde{f}(\tilde{x}),\tilde{f}(\tilde{y})))$$

and

$$U^-(\tilde{f}^2(\tilde{x}),\tilde{f}^2(\tilde{y})) - 2U^-(\tilde{f}(\tilde{x}),\tilde{f}(\tilde{y})) + U^-(\tilde{x},\tilde{y}) =$$
$$g(V^-(\tilde{f}^2(\tilde{x}),\tilde{f}^2(\tilde{y})) - g(V^-(\tilde{f}(\tilde{x}),\tilde{f}(\tilde{y}))) \geq 0$$

and it takes the value 0 if and only if $(\pi\tilde{x},\pi\tilde{y}) \in C^+$.

Furthermore, analogously to V^-, U^- is uniformly continous.

A similar argument let us obtain $U: A^+ \to \mathbb{R}$ such that

$$U^+(\tilde{f}^2(\tilde{x}),\tilde{f}^2(\tilde{y})) - 2U^+(\tilde{f}(\tilde{x}),\tilde{f}(\tilde{y})) + U^+(\tilde{x},\tilde{y}) \geq 0$$

and the equality holds if and only if $(\pi\tilde{x},\pi\tilde{y}) \in C^-$.

Hence, if we define $U = U^+ + U^-$,

$$U(\tilde{f}^2(\tilde{x}),\tilde{f}^2(\tilde{y})) - 2U(\tilde{f}(\tilde{x}),\tilde{f}(\tilde{y})) + U(\tilde{x},\tilde{y}) \geq 0 \quad (4)$$

and it is equal to 0 if and anly if $\tilde{x} = \tilde{y}$.

Remarks: i) $U^+(\tilde f(\tilde x),\tilde f(\tilde y)) - U^+(\tilde x,\tilde y) \le 0$.

ii) If f was a homemorphism then the same method would give us a function $U: A\to\mathbb{R}^+$ with property (4) and $U(x,y)=0$ if and only if $(x,y)\in C^+\cap C^-$ (A is a suitable neighbourhood of the diagonal of $M\times M$).

2

Definition 2.1: A point $x\in M$ is called stable if $\forall\ \varepsilon>0\ \exists\ \rho>0$ such that $d(x,y)\le\rho$ implies $d(f^n(x), f^n(y))\le\varepsilon\ \forall\ n\ge 0$.

Proposition 2.2: There are no stable points for f.

Proof: Assume that x is a stable point for f, then $\tilde x\in\pi^{-1}(x)$ is a stable point for $\tilde f$.

Choose an ε, $0<\varepsilon<\alpha/2$, then $\exists\,\rho>0$ such that if $\tilde d(\tilde x,\tilde y)<\rho$, $\tilde d(\tilde f^{-n}(\tilde x),\tilde f^{-n}(\tilde y))<\varepsilon\ \forall\ n\ge 0$.

Let $(x_n)=(\pi\tilde f^{-n}(\tilde x))\ \forall\ n\in\mathbb{Z}$.

Given $\rho>0$, $\exists\gamma>0$ such that $U(\tilde x,\tilde y)<\gamma$ implies $\tilde d(\tilde x,\tilde y)<\rho$. Call $K_\gamma(\tilde x)$ the connected component of $\{\tilde y\in\tilde M : U(\tilde x,\tilde y)<\gamma\}$ that contains $\tilde x$.

There exists $n_0<0$ such that $\forall\ \tilde y\in K_\gamma(\tilde f^N(\tilde x))$, $N<n_0$, $\tilde f^m(\tilde y)\in K_\gamma(\tilde f^{N+m}(\tilde x))$ for $0\le m\le -N$. Assume it is not the case, then there exists a sequence $n_k\to-\infty$ and $\tilde y_k\in K_\gamma(\tilde f^{n_k}(\tilde x))$ such that $\tilde f^m(\tilde y)^k\notin K_\gamma(\tilde f^{n_k+m}(\tilde x))$ for $0\le m\le n_k$. Choose an arc $\gamma_k:([0,1])$ such that $\tilde f^{-n}(\xi_k)\in\partial K_\gamma(\tilde x)$ and $\tilde f^m(\xi_k)\in K_\gamma(\tilde f^{n_k+m}(\tilde x))$, $0\le m\le -n$. Let ξ be a limit point of $\tilde f^{-n}(\xi_k)$. Obviously, $\tilde d(\tilde f^n(\xi),\tilde f^n(\tilde x))\le\varepsilon\ \forall\ n\ge 0$ and $\tilde d(\tilde f^n(\xi),\tilde f^n(\tilde x))\le\rho\ \forall\ n\le 0$ which is absurd.

Let $(x_n) = (\pi \tilde{f}^n(\tilde{x}))\ \forall\ n \in \mathbb{Z}$ and z an α-limit point of (x_n) (i.e. $\exists\ \{n_k\}$ such that $n_k \to -\infty$ and $x_{n_k} \to z$). In acount of the uniform continuity of U, there exists $n_k < n_0$ and $\delta' > 0$ such that if $d(z,y) < \delta'$, for each $k \geq k_0$ there exists $\tilde{z}_k \in \pi^{-1}(z)$ and $\tilde{y}_k \in \pi^{-1}(y)$ such that $U(\tilde{f}^{n_k}(\tilde{x}),\tilde{z}_k) < \gamma$ and $h(\tilde{f}^{n_k}(\tilde{x}),\tilde{y}_k) < \gamma$; this implies that $\tilde{d}(\tilde{f}^m(\tilde{z}_k),\tilde{f}^m(\tilde{y}_k)) < \varepsilon\ \forall\ m \geq 0$, then $d(f^m(z), f^m(y)) < \varepsilon\ \forall\ m \geq 0$ and z is stable.

Moreover, for any $K > 0$ $\exists j \geq K$ and $\tilde{z}_j \in \pi^{-1}(z)$ such that $\tilde{f}^j(\tilde{z}_j) \in K_\gamma(\tilde{x})$, then $x \in \omega(z)$.

If $d(x,x') < \rho$ then $\lim_{n\to\infty} d(f^n(x),f^n(x')) = 0$, because if there exists $n_k \to +\infty$ such that $d(f^{n_k}(x),f^{n_k}(x')) > n > 0$, we have x_∞ and x'_∞ limit points such that $d(x_\infty,x'_\infty) \geq n$ and orbits $(x_{\infty n})$, $(x'_{\infty n})$ with $d(x_\infty,x'_\infty) \leq \varepsilon \forall n \in \mathbb{Z}$, a contradiction. Then we deduce that $x \in \omega(z) = \omega(x)$.

Moreover, if y is in a small enough open neighbourdhood of x then $\omega(y) = \omega(x)$ and, by the same reasoning, $y \in \omega(y) = \omega(x)$.

Finally, if $y' \in \omega(x)$, $y' \in \omega(z)$ then y' is in the α-limit set of (x_n) which implies that y' is stable. Then there exists an open neighbourhood of y' contained in $\omega(y') \subset \omega(x)$; $\omega(x)$ is open and closed, so $\omega(x) = M$ and all points are stable. This implies that there exists a sequence of iterates that uniformly converges to a constant function, a contradiction.

Remark: This proposition is a consequence of the relation that exists between topology and dynamics. The local connection is here esential; in order to show this, consider $\sigma: 2^{\mathbb{Z}} \to 2^{\mathbb{Z}}$ the shift. Call K to $\{(x_n) \in 2^{\mathbb{Z}};$ if $x_i = x_{i+1} = 0$ then $x_j = 0$ for $j < i\}$. K is an invariant compact perfect set. It is easy to see that the sequence $(0) \in 2^{\mathbb{Z}}$ is a stable point for $\sigma|_{K}$.

3

Definition 3.1: A continuos map $f: M \to M$ is *positively expansive* if there is a constant $\beta > 0$ such that if $x \neq y$ then $d(f^n(x),f^n(y)) \geq \beta$ for some $n \geq 0$; x is a *reppelling point* if there exists $\beta > 0$ such that $x \neq y$ implies $\exists n \geq 0$ such that $d(f^n(x), f^n(y)) \geq \beta$.

From now on, assume that f do not have any reppelling point.
Let

$$S_\delta(\tilde{x}) = \{\tilde{y} \in \tilde{M};\ \tilde{d}(\tilde{f}^n(\tilde{x}), \tilde{f}^n(\tilde{y})) \le \delta\ \forall n \ge 0\}$$

and

$$U_\delta = \{\tilde{y} \in \tilde{M};\ \tilde{d}(\tilde{f}^n(\tilde{x}), \tilde{f}^n(\tilde{y})) \le \delta\ \forall n \le 0\}$$

Let $\delta_1, \delta_2,\ k$ be such that $0 < \delta_1 < \delta_2 < \alpha,\quad k > 0$ and $B_{\delta_1}(\tilde{x}) = \{\tilde{y} \in \tilde{M};\ \tilde{d}(\tilde{x}, \tilde{y}) \le \delta_1\} \subset \{\tilde{y} \in \tilde{M};\ U(\tilde{x}, \tilde{y}) < k\} \subset B_{\delta_2}(\tilde{x})$.

Lemma 3.1: Let $A \subset \tilde{M}$ be an open set, $\tilde{x} \in A \subset B_{\delta_1}(\tilde{x})$.

Then there exists a compact connected set C, $\tilde{x} \in C \subset \bar{A}$, $C \cap \partial A \neq \emptyset$ such that $C \subset S_{\delta_2}(\tilde{x})$ $(U_{\delta_2}(\tilde{x}))$.

Proof: Assume this is not the case. Then there exists $N > 0$ such that for every compact connected $D \subset \bar{A}$ joining $\tilde{x}$ to ∂A there exist $z \in D$ and n, $0 \le n \le N$, such that $\tilde{d}(\tilde{f}^n(\tilde{x}), \tilde{f}^n(\tilde{y})) > \delta$. This is proved in the first paragraph of [3] lemma 2.1.

For arbitrarely large n, there exists $\tilde{y} \in K_k(\tilde{f}^n(\tilde{x}))$ such that $\tilde{f}^{-m}(\tilde{y}) \notin K_k(\tilde{f}^{n-m}(\tilde{x}))$ for some m, $0 \le m \le n$, because if this were not the case, a point of $\omega(\pi\tilde{x})$ would be a reppelling point.

Let $n_0 > N$ and such that $\exists m$ and $\tilde{y}$ which verify $\tilde{f}^{-m}(\tilde{y}) \notin K_k(\tilde{f}^{n-m}(\tilde{x}))$. Consider an arc $\gamma: [0,1] \to K_k(\tilde{f}^{n_0}(\tilde{x}))$ such that $\gamma(0) = \tilde{f}^{n_0}(\tilde{x})$ and $\gamma(1) = \tilde{y}$, then, because a connection argument, there exists $s_0 \in (0,1)$ such that $\tilde{f}^{-m}(\gamma(s_0)) \in \partial A$, $\tilde{f}^{-m}(\gamma(s)) \in A$ for $0 \le s \le s_0$, and $\tilde{f}^{-n}(\gamma(s)) \in K_k(\tilde{f}^{n_0-m}(\tilde{x}))$ for $0 \le s \le s_0$ and $0 \le n \le n_0$, a contradiction. •

In the sequel we will suppose that M is the 2-torus and f an expansive covering map such that it is not a homeomorphism.

Reasoning like in [3] section 2 and 3, since the connected stable (unstable) set of $\tilde{f}$ at most at one point, we can prove, using the topological properties of the plane, that they are locally connected and, then, are path connected. As a consequence of this we have.

Lemma 3.3: Let $\tilde{x} \in \tilde{M}$. Then there exists a neighbourhood N of $\tilde{x}$ such that each $\tilde{y}$ in N, $\tilde{y} \neq \tilde{x}$, has local product structure. If $\tilde{x}$ is a point without local product structure (singular point) it is local stable (unstable) set consist of the union of r arcs that meet only at $\tilde{x}$, $r \geq 3$. The stable (unstable) arcs separate unstable (stable) arcs.

If $x = \pi\tilde{x}$ then the stable arcs of $\tilde{x}$ project homeomorphically onto the stable set of x (observe that this is not true for the unstable arcs).

Moreover, M cannot have singular points because if the singularities are not finite, they must accumulate which contradicts lemma 3.3, but it cannot be finite because all points have at least one nonperiodic orbit.

Proposition 3.4: If $f: M \to M$ is an expansive covering map which is not a homeomorphism and such that it do not have any reppelling point then the stable set of any point $x \in M$ consists of the union of two arcs that meet only at x.

For the unstable set we can say the same, but only on the universal cover, because his structure on M can be complicated (see [6]).

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Crossings and local times of one dimensional diffusions.

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§1.- Introduction.

Let

$$(1) \qquad \begin{aligned} dX &= b(X)\,dt + \sigma(X)\,dW \\ X(0) &= x_0 \end{aligned}$$

be a one-dimensional stochastic differential equation, where $b, \sigma : \mathbb{R} \mapsto \mathbb{R}$ are C^1-functions with bounded derivatives, $\sigma(x) > 0 \ \forall x \in \mathbb{R}$ and W is Brownian motion. $\{X(t): t \geq 0\}$ will denote a strong solution of (1).

Suppose that $\Psi: \mathbb{R} \mapsto \mathbb{R}$ is a non-negative C^{∞}- function with support in $[-1,1]$, $\int_{-\infty}^{\infty} \Psi(x)\,dx = 1$, $\Psi_{\varepsilon}(t) = (1/\varepsilon)\Psi(t/\varepsilon)$ $(\varepsilon > 0)$ and

$$X_{\varepsilon}(t) = (\Psi_{\varepsilon} * X)(t) = \int_{0}^{\infty} \Psi_{\varepsilon}(t-s)X(s)ds$$

the regularization of X(.) by means of the convolution with Ψ_{ε}.

These notes concern Theorem 1 below recently published by Jean-Marc Azais [1]; they contain some small improvements in the statement and changes in the proof even though the general framework is the one in Azais' work.

We shall assume throughout that Ψ' is positive in some interval (s_0,s_1) where $s_0 = \inf\{s: \Psi(s) > 0\}$ and $s_0 < s_1 < 1$.

Theorem 1. *Let* $T = (T_1,T_2)$ $0 < T_1 < T_2 < \infty$. *Then for each* $u \in \mathbb{R}$, *as*

$\varepsilon \mapsto 0$:

$$(\pi\varepsilon/2)^{1/2}\,[\sigma(u)\,\|\Psi\|_2]^{-1}\,N_u^{X_\varepsilon}(T) \quad \textit{converges in probability to } L^X(u,T),$$

where

$$N_u^{X_\varepsilon}(T) = \# \{t: t \in T, X_\varepsilon(t) = u\}$$

is the number of crossings of X_ε *with the level* u *on the set* T, $L^X(u,T)$ *is the local time of the process* X(.) *at the level* u *corresponding to the parameter set* T *and* $\|\Psi\|_2$ *denotes the* L^2*-norm of* $\Psi(.)$ *with respect to the Lebesgue measure.*

Recall that the "occupation measure" μ_T^X of the function X(.) on the Borel set T is defined as:

$$\mu_T^X(A) = \lambda(\{t: t \in T, X(t) \in A\})$$

(λ is Lebesgue measure and A a Borel set) and the local time, when it exists, is the Radon-Nikodym derivative

$$L^X(u,T) = (d\mu_T^X/d\lambda)(u).$$

The fact that almost surely the solution X(.) of (1) has a local time and the function $L^X(u,[0,t])$ is a continuous function of the pair (u,t) is well known and has various proofs. When X(.) is the Wiener process it is the classical Lévy-Trotter theorem ([5],[6]) and for general X(.) it can be obtained - for example - from it toghether with the Girsanov density theorem (see also [2],[4]).

The content of this paper was the object of a series of talks at the Central and Autonomous Universities of Barcelone on june 1989 on the invitation of the Centre for Mathematics Research of the Institute for Catalan Studies. I want to thank here the Institute, especially the Director Professor Castellet, as well as my colleagues Professors Marta Sanz and David Nualart for the nice time I was able to spend in Barcelone.

§2.- Reductions.

For the proof of theorem 1 the following two reductions will be convenient:

A) A standard localization shows that it is enough to prove the theorem when the functions $b(.)$ and $\sigma(.)$ are bounded and have bounded derivatives and $\sigma(.)$ is bounded below by a positive constant.

B) Instead of equation (1) consider:

(2) $$dY = (1/2)(\sigma\sigma')(Y)\,dt + \sigma(Y)\,dW$$
$$Y(0) = x_0$$

The measures μ_X and μ_Y induced on $C(T)$ respectively by the solutions $X(.)$ of (1) and $Y(.)$ of (2) are equivalent because of Girsanov's theorem; moreover $d\mu_X/d\mu_Y$ is bounded and bounded below by a positive constant under reduction A) above. Since $N_u^X\varepsilon(T)$ and $L^X(u,T)$ are both defined pathwise it is enough to prove the theorem for the process $Y(.)$.

Or the process $\{s(Y(t)): t \geq 0\}$ with:

$$s(\alpha) = \int_0^\alpha [\sigma(y)]^{-1}dy$$

is a Wiener process. In fact using Ito's formula:

$$d(s°Y) = s'(Y)dY + (1/2)s''(Y)(dY)^2 = [\sigma(Y)]^{-1}\{(1/2)\sigma(Y)\sigma'(Y)dt + \sigma(Y)dW\} -$$
$$-(1/2)\sigma'(Y)dt = dW.$$

It follows that it suffices to prove the theorem for a process of the form:

$$X(t) = S(W(t)) \quad (t \geq 0)$$

where S is a C^2-function, S' and S" are bounded and $\inf\{S'(x): x\in \mathbb{R}\} > 0$ (Take $S(x) = s^{-1}(x)$.

§3.- Moments of number of crossings.

3.1.- Rice' formula.

After the reductions in §2. the proof of theorem 1 will consist in showing that the convergence containedin the statement takes place in L^2. Our first task will be to compute the first two moments of the random variable $N_u^{X_\varepsilon}(T)$ by means of the so called Rice' formula for which a possible statement is as follows:

Theorem 2. *Suppose* $\{Z(t): t\in \mathbb{R}\}$ *is a real random process,* k *a positive integer,* $V[k,m] = m(m-1)....(m-k+1)$ *and*

$$N_u^Z(T) = \# \{t: t \in T, Z(t) = u\}, \quad T \text{ an open set in the line.}$$

Then, under an appropriate set of hypothesis on the process (that we will not list here, see for example [7], Ch.3) *one has the formula*

$$(3)\ E(V[k,N_u^Z(T)]) = \int_{T^k} dt_1....dt_k \int_{R^k} \left(\prod_{1\le j\le k} |x_j'|\right) q_{t_1...t_k}(u,....,u;x_1',....,x_k')dx_1',....,dx_k' =$$

$$= \int_{T^k} E\left(\prod_{1\le j\le k} |Z'(t_j)| / Z(t_1)=....=Z(t_k)=u\right) q_{t_1...t_k}(u,....,u)dt_1....dt_k$$

where $q_{t_1...t_k}(x_1,....,x_k;x_1',....,x_k')$ *denotes the joint density of the random variables*

$$Z(t_1),.....,Z(t_k);\ Z'(t_1),......,Z'(t_k)$$

and similarly

$$q_{t_1 \dots t_k}(x_1, \dots, x_k)$$

denotes the joint density of $Z(t_1), \dots, Z(t_k)$.

We shall substitute formula (3) above by a weaker one since to apply theorem 2 or some similar statement we need the existence and also some regularity of the densities involved when we put $X_\varepsilon(.)$ instead of $Z(.)$. Unhappily we are only able to study these densities (for $k=2$) when the pair (t_1, t_2) is not too near the diagonal $\{t_1 = t_2\}$ and this inhibites a straightforward application of (3).

The new theorem - that we prove below and for which we give a precise statement - is:

Theorem 2'. *Suppose that the random process* $\{Z(t): t \in \mathbb{R}\}$ *satisfies the following hypothesis*

(a) $Z(.) \in C^1$.

(b) *Let* k *be a positive integer and* D *the "diagonal") be the set*

$$D = \{(t_1, \dots, t_k): t_i = t_j \text{ for some pair } i,j,\ i \neq j\}.$$

Let T *be a bounded open set in the real line and* B *an open set in* $\mathbb{R}^k$ *such that*

$$B \subset T^k \setminus D.$$

We assume that $Z(t_1), \dots, Z(t_k)$ *have a joint density*

$$q_{t_1 \dots t_k}(x_1, \dots, x_k)$$

for $(t_1, \dots, t_k) \in B$, *and that the function*

$$A_{t_1 \dots t_k}(x_1, \dots, x_k) = E\{\prod_{1 \le j \le k} |Z'(t_j)| / Z(t_1) = x_1 \dots = Z(t_k) = x_k\} q_{t_1 \dots t_k}(x_1, \dots, x_k)$$

can be defined as a bounded continuous function of its arguments for $(t_1, \dots, t_k) \in B$ *and* x_i $(i = 1, \dots, k)$ *in a neighbourhood of* u.

(c) $E([N_0^{Z'}(T)]^k) < \infty$.

Then

$$E(\nu^Z(u,k,B)) = \int_B A_{t_1 \dots t_k}(u,\dots,u)\, dt_1 \dots dt_k$$

where

$$\nu^Z(u,k,B) = \# \{(t_1,\dots,t_k) \in B : Z(t_1)=u, \dots, Z(t_k)=u\}.$$

Proof. We have:

$$\nu^Z(u,k,B) \le [N_u^Z(T)]^k \le [N_0^{Z'}(T)+1]^k$$

so that $\nu^Z(k,B)$ has a finite expectation according to (c). So,

(4) $\nu^Z(u,k,B) = \lim_{\delta \to 0} (1/2\delta)^k \int_B \prod_{1 \le j \le k} [\mathbb{1}_{\{|Z(t_j)-u|<\delta\}} |Z'(t_j)|]\, dt_1 \dots dt_k$

holds true. (4) is easily verified by observing that if $(t_1,\dots,t_k) \in B$ is an isolated k-tuple in which $Z(t_1)=u, \dots, Z(t_k)=u$ and $J = J_1 \times \dots \times J_k$ is a small enough rectangle containing $(t_1,\dots,t_k)$ and no other point of this sort, then

$$(1/2\delta)^k \int_J \prod_{1 \le j \le k} [\mathbb{1}_{\{|Z(t_j)-u|<\delta\}} |Z'(t_j)|]\, dt_1 \dots dt_k =$$

$$= \prod_{1 \le j \le k} [\, (1/2\delta) \int_{J_j} \mathbb{1}_{\{|Z(t_j)-u|<\delta\}} |Z'(t_j)|\, dt_j\,] = 1$$

if δ is small enough.

Also a direct computation shows that

$$(1/2\delta) \int_T \mathbb{1}_{\{|Z(t)-u|<\delta\}} |Z'(t)|\, dt \le N_0^{Z'}(T)+1$$

and repeated application of dominated convergence gives:

$$E(\nu^Z(u,k,B)) = \lim_{\delta\mapsto 0}(1/2\delta)^k \int_B dt_1....dt_k \int_{u-\delta<x_j<u+\delta} A_{t_1...t_k}(x_1,....,x_k)dx_1....dx_k =$$

$$= \int_B A_{t_1...t_k}(u,....,u)\, dt_1....dt_k.$$

3.2.- Bounds for moments.

The following theorem is a modification of known bounds for the moments of the number of crossings of regular processes although in its present form it is new (see for example [3] and [7] for close results).

Theorem 3. *Let* $\{Z(t): t\in I\}$ *be a real random process defined on a fixed interval* I *of length equal to one in the real line. We assume*

(a) *For each* $t\in I$, $Z(t)$ *has a density* $q_t(.)$ *and*

$$C = \lim\sup_{x\mapsto u}\sup\{q_t(x): t\in I\} < \infty$$

(b) $Z(.)$ *is of class* C^{p+1}, p *a positive integer. We denote*

$$Z_h = \sup\{|Z^{(h)}(t)|: t\in I\} \quad (h=1,....p).$$

Then, if $m < p/2$:

(5) $$E([N_u^Z(I)]^m) < C_{p,m} + C_{p,m}[E(Z_{p+1}) + C]$$

where $C_{p,m}$ *is a constant depending only on* p *and on* m.

Proof. We have:

(6) $$E([N_u^Z(I)]^m) = \sum_{n=1}^{\infty} P([N_u^Z(I)]^m \geq n) < (p^m+1)(p+1)^m +$$

$$+ (p+1)^m \sum_{k>p^m} P(N_u^Z(I) \geq k^{1/m}(p+1))$$

where we have replaced in the second member of (6) all the terms with

$n < (p^m+1)(p+1)^m$ by the rough bound 1, and on each interval of the integers of the form $k(p+1)^m \leq n \leq (k+1)(p+1)^m - 1$ $(k \geq p^m+1)$, we have replaced $P([N_u^Z(I)]^m \geq n)$ by its maximum value located at $n = k(p+1)^m$.

Denote $\omega_f(\delta) = \sup\{|f(t') - f(t'')|: t',t'' \in I, |t'-t''| < \delta\}$ the continuity modulus of the real valued function $f(.)$ on the interval I. For each term in the sum at the last member of (6) we have:

$$(7)\ P(N_u^Z(I) \geq k^{1/m}(p+1)) < P(\omega_{Z^{(p)}}(k^{-1/m}) \geq \varepsilon_k) + P(N_u^Z(I) \geq k^{1/m}(p+1), \omega_{Z^{(p)}}(k^{-1/m}) < \varepsilon_k).$$

For the first term in (7) we have the simple bound:

$$(8)\quad P(\omega_{Z^{(p)}}(k^{-1/m}) \geq \varepsilon_k) \leq P(Z_{p+1} > \varepsilon_k k^{1/m}) \leq (1/\varepsilon_k k^{1/m})E(Z_{p+1}).$$

For the second term in (7) introduce the notations:

$$V = \{t: |Z'(t)| < \varepsilon_k k^{-(p-1)/m}\}$$

$$F = \{N_u^Z(I) \geq k^{1/m}(p+1), \omega_{Z^{(p)}}(k^{-1/m}) < \varepsilon_k\}$$

$$G = \{N_u^Z(I \cap V) \geq p+1\}.$$

We prove that the events F,G verify the inclusion:

$$(9)\quad F \subset G.$$

In fact, if F holds true there exists an interval $J \subset I$, $|J| \leq k^{-1/m}$ ($|J|$ denotes the length of J) such that:

$$N_u^Z(J) \geq (p+1),$$

because if this were not true one would be able to cover I with a set of

$M = [k^{1/m}] + 1$ ([.] denotes integer part) intervals $J_1,\dots,J_M$ of length $k^{-1/m}$ each and $N_u^Z(J_h) \le p$ for $h=1,\dots,M$, which would imply

$$k^{1/m}(p+1) \le N_u^Z(I) \le p.M \le p(k^{1/m}+1)$$

which is not possible for $k \ge p^m+1$.

Rolle's theorem implies that on J the derivative $Z^{(h)}$ $(h=1,\dots,p)$ must have at least $p+1-h$ vanishing points. An elementary inductive argument shows now that

$$|Z^{(h)}(t)| < \varepsilon_k k^{-(p-h)/m} \text{ for } h=p,p-1,\dots,1 \text{ and every } t \in J.$$

Applying this inequality for $h=1$ (9) follows. Hence,

$$P(F) \le P(G) \le (1/(p+1))\, E\{N_u^Z(I \cap V)\}.$$

On the other hand it is well known and easily proved (Bulinskaya's lemma [ ]) that almost surely $Z'(.)$ does not vanish when $Z(t) = u$ and so, one can prove readily that

$$N_u^Z(U) \le \lim_{\delta \mapsto 0}(1/2\delta) \int_U \mathbf{1}_{\{|Z(t)-u|<\delta\}} |Z'(t)|\, dt$$

for each open set U.

Summing up and applying Fatou's lemma:

$$(10)\ P(F) \le (1/(p+1)) \liminf_{\delta \mapsto 0}(1/2\delta) E\Big(\int_{I \cap V} \mathbf{1}_{\{|Z(t)-u|<\delta\}} |Z'(t)|\, dt \Big) \le$$

$$\le (1/(p+1))\, \varepsilon_k k^{-(p-1)/m} \liminf_{\delta \mapsto 0}(1/2\delta) \int_T dt \int_{[u-\delta,u+\delta]} q_t(x)\, dx \le$$

$$\le C/(p+1)\, \varepsilon_k\, k^{-(p-1)/m},$$

according to the hypothesis.

Replacing the various bounds into (6) we get:

$$E([N_u^{Z}(I)]^m) < C_{p,m} + C_{p,m}\left[E(Z_{p+1})\Sigma_{k>0}\varepsilon_k^{-1}k^{-1/m} + C/(p+1)\,\Sigma_{k>0}\varepsilon_k k^{-(p-1)/m}\right]$$

Choose $\varepsilon_k = k^{(p-2)/2m}$ and both series will converge. Changing the name of the constants we obtain (5).

§4.- General scheme of the proof.

As we have already mentioned with the additional hypothesis contained in §2 we shall prove that convergence in theorem 1 takes place in L^2. Put:

$$c_\varepsilon = (\pi\varepsilon/2)^{1/2}\,[\sigma(u)\,\|\Psi\|_2]^{-1}$$

and

(11) $E([c_\varepsilon N_u^{X_\varepsilon}(T) - L^X(u,T)]^2) \le$

$$\le \liminf_{\delta\mapsto 0} E\{[c_\varepsilon N_u^{X_\varepsilon}(T) - (1/2\delta)\int_T \mathbf{1}_{(|X(t)-u|<\delta)}\,dt]^2\} \le$$

$$= \liminf_{\delta\mapsto 0}\,[E([c_\varepsilon N_u^{X_\varepsilon}(T)]^2) -$$

$$-2\,E(c_\varepsilon N_u^{X_\varepsilon}(T)\,(1/2\delta)\int_T \mathbf{1}_{(|X(t)-u|<\delta)}\,dt) +$$

$$+(1/2\delta)^2\int_{T\times T} E(\mathbf{1}_{(|X(t)-u|<\delta)}\,\mathbf{1}_{(|X(s)-u|<\delta)})ds\,dt]$$

We introduce the following notations:

- $p_{t_1\ldots t_k}(x_1,\ldots,x_k)$ denotes the joint density of $X(t_1),\ldots,X(t_k)$

- $p_{t_1\ldots t_k;\varepsilon}(x_1,\ldots,x_k;x_1',\ldots,x_k')$ denotes the joint density of

$$X_\varepsilon(t_1),\ldots,X_\varepsilon(t_k);\ \varepsilon^{1/2}X'_\varepsilon(t_1),\ldots,\ \varepsilon^{1/2}X'_\varepsilon(t_k)$$

when it exists.

We shall be mainly concerned with the first term on the right-hand side of (11). The remainder will be simpler. We split the proof into several parts:

(a) Prove that

$$E(\varepsilon\ \nu^{X_\varepsilon}(u,1,\Delta(\varepsilon))) = o\,(\varepsilon^{1/2-\eta}) \quad \text{as } \varepsilon\mapsto 0$$

for every $\eta > 0$ where:

$$\Delta(\varepsilon) = \{(t_1,t_2): |t_1-t_2| < 2\varepsilon\} \cap (T\times T)$$

and $\nu^{X_\varepsilon}(u,2,\Delta(\varepsilon))$ is defined in §3.

(b) For each $s,t > 0$ $(X_\varepsilon(s), \varepsilon^{1/2}\,\|\Psi\|_2^{-1}X'_\varepsilon(t))$ converges in distribution to the law of the pair $(X(s), \sigma(X(t))\,N)$ as $\varepsilon\mapsto 0$, where N is standard normal and independent of the pair $(X(s),X(t))$.

For each pair t_1,t_2, $t_1 \neq t_2$ the random vector

$$(X_\varepsilon(t_1),\ X_\varepsilon(t_2),\ \varepsilon^{1/2}\|\Psi\|_2^{-1}\ X'_\varepsilon(t_1),\ \varepsilon^{1/2}\|\Psi\|_2^{-1}\ X'_\varepsilon(t_2))$$

converges in distribution to the law of

$$(X(t_1),\ X(t_2),\ \sigma(X(t_1))\ N_1,\ \sigma(X(t_2))\ N_2)$$

where N_1,N_2 are standard normal and $(X(t_1), X(t_2))$, N_1,N_2 are independent.

(c) The pair $(X_\varepsilon(t), \varepsilon^{1/2}\ X'_\varepsilon(t))$ $(t > \varepsilon)$ has a density and

$$p_{t;\varepsilon}(x;x') < (\text{const})\ (t-\varepsilon)^{-1/2}\varphi_1(x,x')$$

where φ_1 is a bounded function such that

$$\sup\{ \int_{-\infty}^{\infty} \varphi_1(x,x')\, dx : x' \in \mathbb{R}\} < \infty \quad \text{and}$$

$$\sup\{ \int_{-\infty}^{\infty} |x'|\, \varphi_1(x,x')\, dx' : x \in \mathbb{R}\} < \infty.$$

Moreover for $0 < \alpha < \beta < \infty$ and K_0 a compact subset of $\mathbb{R}^2$,

$$\{p_{t;\varepsilon}(x;x') : \alpha \le t - \varepsilon \le \beta,\ (x;x') \in K_0\}$$

is an equicontinuous and equibounded set of functions.

(d) The random vector

$$(X_\varepsilon(t_1), X_\varepsilon(t_2), \varepsilon^{1/2} X'_\varepsilon(t_1), \varepsilon^{1/2} X'_\varepsilon(t_2)) \quad (t_1 > \varepsilon,\ t_2 - t_1 > 2\varepsilon)$$

has a density and

$$p_{t_1,t_2;\varepsilon}(x_1,x_2;x_1',x_2') \le (\text{const})\, (t_1 - \varepsilon)^{-1/2}(t_2 - t_1 - 2\varepsilon)^{-1/2}\varphi_2(x_1,x_2;x_1',x_2')$$

where

$$\sup\{\int_{R^2} \varphi_2(x_1,x_2;x_1',x_2')\, dx_1 dx_2 :\ x_1',x_2' \in \mathbb{R}\} < \infty$$

$$\sup\{\int_{R^2} |x_1'|.|x_2'|\varphi_2(x_1,x_2;x_1',x_2')\, dx_1' dx_2' :\ x_1,x_2 \in \mathbb{R}\} < \infty.$$

Moreover, for $0 < \alpha < \beta < \infty$ and K_1 a compact subset of $\mathbb{R}^4$

$$\{p_{t_1,t_2;\varepsilon}(x_1,x_2;x_1',x_2') : \alpha \le t_1 - \varepsilon,\ t_2 - t_1 - 2\varepsilon \le \beta,\ (x_1,x_2;x_1',x_2') \in K_1\}$$

is equicontinuous and equibounded.

(e) $E([c_\varepsilon N_u^{X_\varepsilon}(T)]^2)$ converges to $E_L = \int_{T\times T} p_{t_1,t_2}(u,u)\, dt_1 dt_2$

as $\varepsilon \mapsto 0$. (This settles the behaviour of the first term in (11)).

(f) Second and third terms in (11) and conclusion that the left-hand member in (11) tends to zero as $\varepsilon \mapsto 0$.

§5.- Proofs.

Proof of (a). We shall assume that (c) has been already proved (The proof of (c) will be independent of (a)).

It is obvious that $\Delta(\varepsilon)$ can be covered by a union of squares of the form

$$\bigcup_1^N J_h \times J_h,$$

where the length of each interval J_h is 4ε and the number N of squares is $(1/\varepsilon)$ times a fixed constant. So:

$$\nu^{X_\varepsilon}(u,2,\Delta(\varepsilon)) \leq \sum_1^N [N_u^{X_\varepsilon}(J_h)]^2.$$

(a) will be proved if we prove that:

$$\text{(12)} \qquad \sup\{ E([N_u^{X_\varepsilon}((a,a+\varepsilon))]^2) : 0 < \delta < a,\ (a,a+\varepsilon) \subset T\} < C_\eta\, \varepsilon^{1/2-\eta}$$

for every $\eta > 0$, C_η being a constant that may depend on η.

Apply Holder's inequality $((1/m)+(1/m') = 1)$ and put for simplicity $J = (a,a+\varepsilon)$:

$$\text{(13)}\ E([N_u^{X_\varepsilon}(J)]^2) \leq [E([N_u^{X_\varepsilon}(J)]^{2m})]^{1/m}[E(1_{\{N_u^{X_\varepsilon}(J) \geq 1\}})]^{1/m'} \leq$$

$$\leq [E([N_u^{X_\varepsilon}(J)]^{2m})]^{1/m}\, [E(N_u^{X_\varepsilon}(J))]^{1/m'}.$$

We use again the inequality

$$N_u^{X_\varepsilon}(J) \le \lim_{\delta \mapsto 0}(1/2\delta) \int_J \mathbf{1}_{(|X_\varepsilon(t)-u|<\delta)} |X_\varepsilon'(t)|\, dt \qquad \text{a.s.}$$

as in theorem 3 and it follows that:

$$E(\varepsilon^{1/2} N_u^{X_\varepsilon}(J)) \le \liminf_{\delta \mapsto 0}(1/2\delta) \int_J E(\varepsilon^{1/2} \mathbf{1}_{(|X_\varepsilon(t)-u|<\delta)} |X_\varepsilon'(t)|)\, dt =$$

$$\le \liminf_{\delta \mapsto 0}(1/2\delta) \int_J dt \int_{[u-\delta,u+\delta]} dx \int_{-\infty}^{\infty} |x'|\, p_{t;\varepsilon}(x;x')dx' \le$$

$$\le (\text{const})\, \varepsilon$$

if ε is small enough, using the result in (c).

(Here, and in the remainder of this paper "const" denotes a positive constant that may vary from line to line)

As for the first factor on the right hand side of (13) put:

$$Y_\varepsilon(\tau) = X_\varepsilon(a+\varepsilon\tau) \qquad (0 \le \tau \le 1).$$

Clearly:

$$Y_\varepsilon^{(p)}(\tau) = \varepsilon^p X_\varepsilon^{(p)}(a+\varepsilon\tau) = \int_{-1}^{1} \Psi^{(p)}(w)\, X(a+\varepsilon(\tau-w))\, dw.$$

Assume $0 < \varepsilon < \delta$. Then:

$$E\{\sup(Y_\varepsilon^{(p)}(\tau):0 \le \tau \le 1)\} \le \|\Psi^{(p)}\|_1\, E\{\sup(|X(t)|:0 \le t \le T+\delta)\} \le (\text{const})$$

since $X(t) = S(W(t))$ and $|X(t)| < \|S'\|_\infty.|W(t)| + |S(0)|$.

On the other hand (c) implies that the density of $X_\varepsilon(t)$ is bounded by a constant and we are able to apply theorem 3 to show that

$$E([N_u^{X_\varepsilon}(J)]^{2m})$$

is bounded by a constant depending only on m. Substituting into (13) we obtain:

$$E([N_u^{X_\varepsilon}(J)]^2) \leq (\text{const})\, \varepsilon^{1/(2m')},$$

and since $m'>1$ can be chosen arbitrarily close to 1, (a) follows.

Proof of (b). Note that:

$$X'_\varepsilon(t) = \int_0^\infty \Psi_\varepsilon(t-y)\, dX(y) = \int_0^\infty \Psi_\varepsilon(t-y)\, [\sigma(X(y))\, dW(y) + (1/2)(\sigma\sigma')(X(y))]\, dy$$

so that we can write:

$$\varepsilon^{1/2} \|\Psi\|_2^{-1} X'_\varepsilon(t) = A_\varepsilon + B_\varepsilon + C_\varepsilon \quad \text{where:}$$

$$A_\varepsilon = \varepsilon^{1/2} \|\Psi\|_2^{-1} \sigma(X(t-\varepsilon)) \int_{(t-\varepsilon,t+\varepsilon)} \Psi_\varepsilon(t-y)\, dW(y)$$

$$B_\varepsilon = \varepsilon^{1/2} \|\Psi\|_2^{-1} \int_{(t-\varepsilon,t+\varepsilon)} \Psi_\varepsilon(t-y)[\sigma(X(y)) - \sigma(X(t-\varepsilon))]\, dW(y)$$

$$C_\varepsilon = \varepsilon^{1/2} \|\Psi\|_2^{-1} (1/2) \int_{(t-\varepsilon,t+\varepsilon)} \Psi_\varepsilon(t-y)\, (\sigma\sigma')(X(y))\, dy.$$

So, $C_\varepsilon \mapsto 0$ as $\varepsilon \mapsto 0$ and a.s.

$|B_\varepsilon| \leq (\text{const})\, \varepsilon^{1/2} \sup\{|X(t'')-X(t')|: t',t'' \in [0,T_2+\varepsilon], |t''-t'| < 2\varepsilon\} \mapsto 0$ as $\varepsilon \mapsto 0$

On the other hand if we denote

$$W_\varepsilon(t) = \varepsilon^{1/2} \|\Psi\|_2^{-1} \int_{(t-\varepsilon,t+\varepsilon)} \Psi_\varepsilon(t-y)\, dW(y)$$

then it is clear that $W_\varepsilon(t)$ is standard normal. Also,

$$\mathbb{E}(W_\varepsilon(t)W(s)| = \varepsilon^{1/2}\,\|\Psi\|_2^{-1}\Big|\int_{(t-\varepsilon,t+\varepsilon)}\Psi_\varepsilon(t-y)\,\mathbf{1}_{[0,s]}(y)\,dy\Big| \le \varepsilon^{1/2}\,\|\Psi\|_2^{-1} \mapsto 0$$

as $\varepsilon\mapsto 0$,

which means that $W_\varepsilon(t)$ is asymptotically independent from $(W(s): s\geq 0)$ and as a consequence from $(X(t): t\geq 0)$. (b) follows now from the fact that a.s. $X_\varepsilon(t)$ and $X(t-\varepsilon)$ tend to $X(t)$ as $\varepsilon\mapsto 0$.

The second statement in (b) can be proved by writing similar decompositions for $\varepsilon^{1/2}\|\Psi\|_2]^{-1}\, X'_\varepsilon(t_1)$ and $\varepsilon^{1/2}\|\Psi\|_2]^{-1}\, X'_\varepsilon(t_2)$ than the one employed above.

Proof of (c). We introduce the following notations:

- $W_1(\tau) = \varepsilon^{-1/2}[W(t+\varepsilon\tau)-W(t-\varepsilon)]$, $-1 \le \tau \le 1$. $W_1(.)$ is a Wiener process starting at $W_1(-1) = 0$.

- $U = W(t-\varepsilon)$.

- a,b are fixed real numbers $-1 < a, b < 1$ to be chosen later on in a convenient manner; $V = W_1(b)-W_1(a)$.

- $\mathcal{K}(\tau) = W_1(\tau) - V.h(\tau)$ where h is the function defined on $[-1,1]$ by

$$h(x) = [(x-a)/(b-a)] \vee 0 \wedge 1.$$

With these definitions:

1) $W(y) = U + \varepsilon^{1/2}(\mathcal{K}+Vh)((y-t)/\varepsilon)$ $\quad (t-\varepsilon \le y \le t+\varepsilon)$

2) U, V, $\mathcal{K}(\tau)$ are independent centered gaussian for every $\tau \in [-1,1]$.

3) Putting (c.f. § 2 for the notation)

$$X(u,v) = \int_0^\infty \Psi_\varepsilon(t-y)\, S[u + \varepsilon^{1/2}(\mathcal{K}+vh)((y-t)/\varepsilon)]\, dy$$

$$X'(u,v) = \varepsilon^{1/2}\int_0^{\infty} \Psi'_{\varepsilon}(t-y)\, S[u + \varepsilon^{1/2}(\mathcal{K}+vh)((y-t)/\varepsilon)]\, dy$$

it turns out that:

$$X_{\varepsilon}(t) = X(U,V)$$

$$\varepsilon^{1/2} X'_{\varepsilon}(t) = X'(U,V).$$

The proof of (c) will be deduced from the properties of the application:

$$(u,v)^T \mapsto L(u,v) = (X(u,v), X'(u,v))^T$$

and the joint density of the pair of random variables U,V.

Fix ε and $\mathcal{K}$. Denote $\mathcal{J} = \mathcal{J}(u,v)$ the Jacobian matrix of L and

$$g(x) = S'[u + \varepsilon^{1/2}(\mathcal{K}+vh)(x)].$$

We have:

$$X_u = \int_{-1}^{1} \Psi(-x)\, g(x)\, dx$$

$$X_v = \varepsilon^{1/2}\int_{-1}^{1} \Psi(-x)\, g(x)\, h(x)\, dx$$

$$X'_u = \varepsilon^{-1/2}\int_{-1}^{1} \Psi'(-x)\, g(x)\, dx$$

$$X'_v = \int_{-1}^{1} \Psi'(-x)\, g(x)\, h(x)\, dx.$$

We shall prove that given any positive number A_0 one can choose a and b in the definition of the function h so that:

(14) $X'_v > A_0\varepsilon^{-1/2}X_v$ and $\int_{-1}^{1} \Psi(-x)\, h(x)\, dx > 0.$

This will imply that

$$(15)\quad \det(\mathcal{J}) = X_u X'_v - X_v X'_u > [\inf S']\left\{\int_{-1}^{1}\Psi(-x)h(x)dx\right\}\left\{A_0[\inf S'] - \|\Psi\|_1[\inf S']\right\}.$$

If A_0 is large enough the right-hand side of (15) is a fixed positive constant depending on Ψ, a and b and the upper and lower bounds of S', but not on u, v, t, ε and $\mathcal{K}$.

To verify (14) denote (as in the beginning):

$$s_0 = \inf\{s: \Psi(s) > 0\}$$

and suppose $\Psi'(s) > 0$ on the interval (s_0, s_1). We choose a and b so that $-s_1 < a < b < -s_0$; the second condition in (14) is then obviously satisfied. As for the first:

$$X'_v = (1/(b-a))\int_{[a,b]}\Psi'(-x)\,g(x)\,(x-a)dx + \int_{[b,1]}\Psi'(-x)\,g(x)dx > \int_{[b,1]}\Psi'(-x)g(x)dx$$

$$\varepsilon^{-1/2}X_v = (1/(b-a))\int_{[a,b]}\Psi(-x)\,g(x)\,(x-a)dx + \int_{[b,1]}\Psi(-x)\,g(x)dx <$$

$$< 2\int_{[b,1]}\Psi(-x)g(x)dx$$

where the last inequality is obtained simply by choosing a after having chosen b and sufficiently close to b.

So,

$$X'_v/(\varepsilon^{-1/2}X_v) \geq [\inf S'].\Psi(-b)/\left\{2[\sup S']\int_{[b,1]}\Psi(-x)dx\right\} \geq$$

$$\geq [\inf S'].\Psi(-\xi)/\left\{2[\sup S'](-s_0-b)\Psi(-\xi)\right\} > (\text{const})/(-s_0-b)\quad (b<\xi<-s_0).$$

Letting b approach $-s_0$ we can make $X'_v/(\varepsilon^{-1/2}X_v)$ to be larger than A_0 as stated.

Note that the application $L: \mathbb{R}^2 \mapsto \mathbb{R}^2$ is not only a local diffeomorfism for $t, \varepsilon, \mathcal{K}$ fixed but also globally injective. In fact, if

$$X(u,v) = X(u',v')$$

$$X'(u,v) = X'(u',v')$$

there exist $(u_1,v_1), (u_2,v_2)$ such that:

$$X_u(u_1,v_1)(u'-u) + X_v(u_1,v_1)(v'-v) = 0$$

$$X'_u(u_2,v_2)(u'-u) + X'_v(u_2,v_2)(v'-v) = 0.$$

Since the lower bound for $\det(\mathcal{J})$ also works for the determinant of the coefficients of this system of equations we get:

$$u' = u \text{ and } v' = v.$$

After the above preparation let us turn to the proof of (c).

Conditionally on $\mathcal{K}$, we have in fact proved that on the (open) range of the application L the pair

$$(X_\varepsilon(t), \varepsilon^{1/2} X'_\varepsilon(t))$$

has a density $p_{t;\varepsilon}(x;x'/\mathcal{K})$ given by:

$$(16) \qquad p_{t;\varepsilon}(x;x'/\mathcal{K}) = \Phi_{t;\varepsilon}(L^{-1}(x,x'))/\det[\mathcal{J}(L^{-1}(x,x'))]$$

where $\Phi_{t;\varepsilon}(u,v)$ stands for the joint density of U and V, that is:

$$\Phi_{t;\varepsilon}(u,v) = (2\pi)^{-1}(t_1-\varepsilon)^{-1/2}(b-a)^{-1/2}\exp\{(-1/2)[(u-w_0)^2/(t-\varepsilon) + v^2/(b-a)]\}.$$

$w_0 = S^{-1}(x_0)$ is the initial value of $W(.)$.

We shall use the notation:

$$L^{-1}(x,x') = (\xi(x,x'),\eta(x,x')).$$

It follows from (16) and the lower bound for det($\mathfrak{J}$) that:

(17) $p_{t;\varepsilon}(x;x'/\mathcal{K}) \leq (\text{const})\,(t-\varepsilon)^{-1/2}.$

$$.\exp\{(-1/2)[(\xi(x,x')-w_0)^2/(t-\varepsilon) + \eta^2(x,x')/(b-a)]\} \leq$$

$$\leq (\text{const})\,(t-\varepsilon)^{-1/2}\exp\{-(\text{const})E\}$$

where the last inequality holds if t is restricted to be bounded above and

$$E = (\xi(x,x')-w_0)^2 + \eta^2(x,x').$$

Our next task will be to give two lower bounds for E.

The first one is obtained as follows.

$$X'_u = \varepsilon^{-1/2}\int_{-1}^{1}\Psi'(-x)\,(S'[u + \varepsilon^{1/2}(\mathcal{K}+vh)(x)] - S'(u))dx,$$

so that:

$$|X'_u| \leq \varepsilon^{-1/2}\int_{-1}^{1}|\Psi'(-x)|\,\|S''\|_\infty\varepsilon^{1/2}|\mathcal{K}+vh|(x)\,dx \leq (\text{const})\,\|\mathcal{K}+vh\|_\infty.$$

With the notation $u = \xi(x,x')$, $v = \eta(x,x')$ we get:

(18) $|x'| \leq |X'(u,v) - X'(w_0,0)| + |X'(w_0,0)|.$

The first term on the right-hand side is bounded by

$$(\text{const})\{[\|\mathcal{K}\|_\infty + |v|]|u-w_0| + |v|\}$$

and the second,

$$|X'(u,0)| \leq \varepsilon^{1/2}\int_0^\infty|\Psi'_\varepsilon(t-y)|\,\{S[u + \varepsilon^{1/2}\mathcal{K}((y-t)/\varepsilon)] - S(u)\}\,dy \leq (\text{const})\|\mathcal{K}\|_\infty.$$

Substituting into (18):

$|x'| \leq (\text{const})\,(\|\mathbb{K}\|_{\infty}+1)(E+1).$

Hence,

(19) $$E \geq (\text{const})|x'|/(\|\mathbb{K}\|_{\infty}+1) - 1$$

Our second lower bound for E is obtained in a similar way.

$$X(u,v) - x_0 = \int_0^{\infty} \Psi_{\varepsilon}(t-y)\,\{S[u + \varepsilon^{1/2}(\mathbb{K}+vh)((y-t)/\varepsilon)] - S(w_0)\}\,dy$$

so that:

$$|x-x_0| \leq \|S'\|_{\infty}\{|u-w_0| + \|\mathbb{K}\|_{\infty} + |v|\} \leq (\text{const})\,(\|\mathbb{K}\|_{\infty}+1)(E+1)$$

and

(20) $$E \geq (\text{const})|x-x_0|/(\|\mathbb{K}\|_{\infty}+1) - 1.$$

Using (19) and (20) we get from (17):

$$p_{t;\varepsilon}(x;x'/\mathbb{K}) \leq (\text{const})\,(t-\varepsilon)^{-1/2}.\exp\{-(\text{const})[|x'|+|x-x_0|]/(\|\mathbb{K}\|_{\infty}+1)\}.$$

Put:

$$\varphi_1(x,x') = E\left[\exp\{-(\text{const})[|x'|+|x-x_0|]/(\|\mathbb{K}\|_{\infty}+1)\}\right].$$

The inequality

$$\|\mathbb{K}\|_{\infty} \leq 3\,\sup(|W_1(t)|: -1 \leq t \leq 1)$$

implies that $\|\mathbb{K}\|_{\infty}$ has finite moments of all orders. It follows that:

$$\int_{-\infty}^{\infty}\varphi_1(x,x')dx \leq E\left[\int_{-\infty}^{\infty}\exp\{-(\text{const})|x-x_0|/(\|\mathcal{K}\|_\infty+1)\}dx\right] \leq (\text{const})E(\|\mathcal{K}\|_\infty+1)<\infty$$

$$\int_{-\infty}^{\infty}|x'|\varphi_1(x,x')dx\leq E\left[\int_{-\infty}^{\infty}|x'|\exp\{-(\text{const})|x'|/(\|\mathcal{K}\|_\infty+1)\}dx\right]\leq(\text{cns})E(\|\mathcal{K}\|_\infty+1]^2)<\infty$$

This proves the first part of (c).

Let us now move to the second part of (c). The equiboundedness of the family of densities $\{p_{t;\varepsilon}(x;x')\}$ is already proved. For the equicontinuity we shall prove that $p_{t;\varepsilon}(x;x'/\mathcal{K})$ has bounded derivatives as a function of (x,x') for $|x|+|x'|\leq r$, $\alpha \leq t-\varepsilon \leq \beta$ and we shall give estimates for these derivatives in terms of $\mathcal{K}(.)$.

For this purpose consider expression (16) for $p_{t;\varepsilon}(x;x'/\mathcal{K})$ and observe that the function

$$\Phi_{t;\varepsilon}(u,v)/\det[\mathcal{J}(u,v)]$$

has uniformly bounded derivatives because $\Phi_{t;\varepsilon}$ is C^∞ with bounded derivatives in the considered region, det(J) is bounded below by a positive constant and the partial derivatives

$$\partial[\det(\mathcal{J})]/\partial u = X_{uu}X'_v + X_uX'_{vu} - X_{vu}X'_u - X_vX'_{uu}$$

$$\partial[\det(\mathcal{J})]/\partial v = X_{uv}X'_v + X_uX'_{vv} - X_{vv}X'_u - X_vX'_{uv}$$

are bounded (this follows easily by inspection of the derivatives of the functions $X(u,v)$, $X'(u,v)$).

Further, let us analyze the derivatives of the functions

$$u = \xi(x,x')$$

$$v = \eta(x,x')$$

We have:

$$dL^{-1} = ((d_{ij}))_{i,j=1,2}$$

with:

$$d_{11} = \xi_x = X'_v/\det(\mathfrak{J})$$

$$d_{12} = \xi_{x'} = - X_v/\det(\mathfrak{J})$$

$$d_{21} = \eta_x = - X'_u/\det(\mathfrak{J})$$

$$d_{22} = \eta_{x'} = X_u/\det(\mathfrak{J}).$$

X_u, X_v, X'_v are uniformly bounded so that our only problem is d_{21}.

We have already the bound

$$|X'_u| \leq (\text{const}) [\|\mathbb{K}\|_\infty + |v|],$$

so that

$$|\eta_x| \leq (\text{const}) [\|\mathbb{K}\|_\infty + |\eta(x,x')|]$$

Put:

$$x = X(0,0), \quad x' = X'(0,0)$$

which imply

$$\xi(x, x') = 0, \quad \eta(x, x') = 0$$

It follows that for $x < x$:

$$\theta(x) = |\eta(x,x')-\eta(x,x')| = \left|\int_{[x,x]} \eta_x(y,x')dy\right| \leq (\text{const})\int_{[x,x]} [\|\mathbb{K}\|_\infty + |\eta(y,x')|]dy.$$

On the other hand:

$$|\eta(x,x')| = |\eta(x,x')-\eta(x,x')| \leq (\text{const})|x'-x'| \leq (\text{const}) [r+|x'|]$$

and

$$|x'| - |X'(0,0)| - \left|\varepsilon^{1/2}\int_0^\infty \Psi'_\varepsilon(t-y)\,[S(\varepsilon^{1/2}\mathbb{K}((y-t)/\varepsilon) - S(0)]\,dy\right| \le (\text{const})\|\mathbb{K}\|_\infty.$$

So

$$\theta(\mathbf{x}) \le (\text{const})\int_{[x,\mathbf{x}]}[r + \|\mathbb{K}\|_\infty + \theta(y)]dy$$

which implies (Gronwall's lemma):

(21) $$\theta(\mathbf{x}) \le (r + \|\mathbb{K}\|_\infty)\exp\{(\text{const})|\mathbf{x}-x|$$

for $x < \mathbf{x}$. (21) also holds and is similarly porved if $x \ge \mathbf{x}$. From (21):

$$\theta(\mathbf{x}) \le (\text{const})\exp\{(\text{const})(r + \|\mathbb{K}\|_\infty)\},$$

using that

$$|x| - |X(0,0)| \le (\text{const})\|\mathbb{K}\|_\infty + (\text{const})$$

and $|\mathbf{x}| \le r$.

Finally,

$$|\eta_{\mathbf{x}}| < (\text{const})\,[\|\mathbb{K}\|_\infty + \theta(\mathbf{x}) + |\eta(x,\mathbf{x}')|] < (\text{const})\exp\{(\text{const})(r + \|\mathbb{K}\|_\infty)\}$$

and we get:

$$|p_{t;\varepsilon}(\mathbf{x}_1;\mathbf{x}'_1/\mathbb{K}) - p_{t;\varepsilon}(\mathbf{x}_2;\mathbf{x}'_2/\mathbb{K})| \le [|\mathbf{x}_1-\mathbf{x}_2| + |\mathbf{x}'_1-\mathbf{x}'_2|]\exp\{(\text{const})(r + \|\mathbb{K}\|_\infty)\}$$

for $|\mathbf{x}_i|+|\mathbf{x}'_i| \le r$, $\alpha \le t-\varepsilon \le \beta$ $(i-1,2)$.

Given that

$$E\{\exp((\text{const})\,\|\mathcal{K}\|_\infty)\} < \infty$$

we conclude the equicontinuity in the statement of (c).

Proof of (d). Take t_1, t_2 as in the statement and denote $t_3=(t_1+t_2)/2$. Since the process $X(.)$ is Markovian and the support of Ψ is contained in $[-1,1]$ the joint law of

$$(X_\varepsilon(t_2),\ \varepsilon^{1/2}\, X'_\varepsilon(t_2))$$

given the past of the Wiener process until time t_3 has the density

$$p_{(t_2-t_1)/2;\varepsilon}(x;x'|x_3)$$

where we have modified our notation introducing the initial value x_3 (instead of x_0 in the proof of (c) that did not appear explicitly) and

$$p_{(t_2-t_1)/2;\varepsilon}(x_2;x'_2|x_3/\mathcal{K})$$

is the density of the same pair conditionally on the past of the Wiener process until time t_3 and the function $\mathcal{K}(.)$.

We slightly modify the decomposition and the conditioning of the process with respect to the decomposition we used in the proof of (c), in the following way:

- Repeat the definitions of $W_1(\tau)$, U, V, $\mathcal{K}(\tau)$, $\xi(x,x')$, $\eta(x,x')$ and the choice of the numbers a and b, *replacing* t *by* t_1.

- Put $W = W(t_3) - W(t_1+\varepsilon)$ and

$$L^*(u,v,w) = \left(X(u,v),\ X'(u,v),\ S[u+\varepsilon^{1/2}(\mathcal{K}(1)+v)+w]\right)$$

where $X(u,v)$, $X'(u,v)$ are as in the proof of (c). Then:

$$L^*(U,V,W,) = \left(X_\varepsilon(t_1),\ \varepsilon^{1/2}\, X'_\varepsilon(t_1),\ X(t_3)\right).$$

The existence of the conditional density (for given $\mathcal{K}(.)$) of the random vector $L^*(U,V,W,)$ can be obtained along the same lines as in (c). In fact, we have:

$$\mathcal{J}^* = dL^* = ((e_{ij}))_{i,j=1,2,3}$$

where

$$((e_{ij}))_{i,j=1,2} = dL$$

and

$$e_{13} = e_{23} = 0,$$

$$e_{31} = S'(z),\ e_{32} = \varepsilon^{1/2}S'(z),\ e_{33} = S'(z)$$

with $z = u + \varepsilon^{1/2}(\mathcal{K}(1)+v) + w$. This implies that:

$$\det(\mathcal{J}^*) = \det(\mathcal{J}).\ S'(z) \geq \inf(S').\ \inf[\det(\mathcal{J})] > 0$$

and the joint density of $(X_\varepsilon(t_1),\ \varepsilon^{1/2}\ X'_\varepsilon(t_1),\ X(t_3))$ conditionally on $\mathcal{K}(.)$ exists and is equal to:

$$\Phi_{t_1,t_3;\varepsilon}(L^{*-1}(x_1,x'_1,x_3))/\det[\mathcal{J}^*(L^{*-1}(x_1,x'_1,x_3))]$$

where we have put

$$\Phi_{t_1,t_3;\varepsilon}(u,v,w) = \Phi_{t_1;\varepsilon}(u,v)\ (t_3-t_1-\varepsilon)^{-1/2}\varphi(w.(t_3-t_1-\varepsilon)^{-1/2})$$

(φ stands for the standard normal density function) and

$$L^{*-1}(x_1,x'_1,x_3) = (\xi(x_1,x'_1),\ \eta(x_1,x'_1),\ S^{-1}(x_3)-\xi(x_1,x'_1)-\varepsilon^{1/2}[\mathcal{K}(1)+\eta(x_1,x'_1)]).$$

It follows that the random vector

$$(X_\varepsilon(t_1),\ X_\varepsilon(t_2),\ \varepsilon^{1/2}\ X'_\varepsilon(t_1),\ \varepsilon^{1/2}\ X'_\varepsilon(t_2),\ X(t_3))$$

has (conditionally on $\mathcal{K}(.)$) the joint density:

(22) $p_{(t_2-t_1)/2;\varepsilon}(x_2;x'_2|x_3/\mathcal{K})\Phi_{t_1,t_3;\varepsilon}(L^{*-1}(x_1,x'_1,x_3))/\det[\mathcal{J}^*(L^{*-1}(x_1,x'_1,x_3))]$

and using our bound in (c) it turns out that the expression in (22) is bounded by:

(23) $(\text{const})(t_1-\varepsilon)^{-1/2}(t_2-t_1-2\varepsilon)^{-1}.$

$$.\exp\{(-1/2)[(u-w_0)^2/(t-\varepsilon)+v^2/(b-a)+w^2/(t_2-t_1-2\varepsilon)]\}.$$

$$.\exp\{-(\text{const})[|x'_2|+|x_2-x_3|]/(\|\mathcal{K}\|_\infty+1)\}$$

where $u = \xi(x_1,x'_1)$, $v = \eta(x_1,x'_1)$, $w = S^{-1}(x_3)- w_1$,

$w_1 = \xi(x_1,x'_1)+\varepsilon^{1/2}[\mathcal{K}(1)+\eta(x_1,x'_1)])$.

Introduce now the trivial inequality

$$|x_3| = |S(w+w_1)| \leq |S(0)|+(|w|+|w_1|)\sup(S')$$

and bound the integral of the expression (23) with respect to x_3 over the real line by

(24) $(\text{const})(t_1-\varepsilon)^{-1/2}(t_2-t_1-2\varepsilon)^{-1/2}.$

$$.\exp\{(-1/2)E-(\text{const})[|x'_2|+|x_2|]/(\|\mathcal{K}\|_\infty+1)+(\text{const})E^{1/2}\}$$

with the notation we introduced in the proof of (c) i.e. $E=(u-w_0)^2+v^2$.

To finish with this part we get a bound for the joint density of

$$(X_\varepsilon(t_1), X_\varepsilon(t_2), \varepsilon^{1/2} X'_\varepsilon(t_1), \varepsilon^{1/2} X'_\varepsilon(t_2))$$

using the inequalities (19) and (20) for E and then taking expectation (with respect to $\mathcal{K}(.)$. So,

$$p_{t_1,t_2;\varepsilon}(x_1,x_2;x_1',x_2') \leq (\text{const})\,(t_1-\varepsilon)^{-1/2}(t_2-t_1-2\varepsilon)^{-1/2}\varphi_2(x_1,x_2;x_1',x_2')$$

where

$$\varphi_2(x_1,x_2;x_1',x_2') = E\{\exp[-(\text{const})[|x_1'|+|x_1-x_0|+|x'_2|+|x_2|]/(\|K\|_\infty+1)]\}$$

The fact that the function φ_2 verifies the requirements in (d) is straightforward. It is also clear that the equiboundedness stated in (d) is already contained in the computation above. In what concerns the proof of equicontinuity one can use expression (22) and similar arguments to those in (c).

Proof of (e). Let $T = (T_1,T_2)$, $0 < T_1 < T_2 < \infty$ as in the statement of theorem 1 and $\varepsilon > 0$ such that $T_1 - \varepsilon > 0$.

Note that (see §3 and §4 for the notation):

(25) $$[N_u^{X_\varepsilon}(T)]^2 = \nu^{X_\varepsilon}(u,2,\Delta^-(\varepsilon)) + \nu^{X_\varepsilon}(u,2,(T\times T)\setminus\Delta^-(\varepsilon))$$

($\Delta^-(\varepsilon)$ denotes the closure of $\Delta(\varepsilon)$).

Statement (a) implies that for any $\eta > 0$:

(26) $$E(c_\varepsilon^2\nu^{X_\varepsilon}(u,2,\Delta^-(\varepsilon))) = o(\varepsilon^{1/2-\eta}) \quad \text{as } \varepsilon\mapsto 0.$$

For the expectation of the second term on the right-hand side of (25) let us verify that theorem 2' in §3 applies for $k=2$ and $B = (T\times T)\setminus\Delta^-(\varepsilon)$

In fact, $X_\varepsilon \in C^\infty$ and if $t \in T$:

$$|X_\varepsilon^{(p)}(t)| = \left|\int_0^\infty \Psi_\varepsilon^{(p)}(t-s)X(s)ds\right| < \varepsilon^{-p}\|\Psi^{(p)}\|_1 \sup\{|X(t)|: T_1-\varepsilon \le t \le T_2+\varepsilon\}$$

which implies

$$E\{\sup\{|X_\varepsilon^{(p)}(t)|: t \in T\} < \infty \quad \text{for every } \varepsilon > 0 \text{ and } p=1,2,...$$

Moreover part (c) implies that for each $\varepsilon > 0$ $X'_\varepsilon(t)$ has a density for $t \in T$ and that it is a bounded function of t. We are then able to apply theorem 3 and conclude that

$$N_0^{X'}\varepsilon(T)$$

has finite moments of all orders. This, plus the existence, continuity and boundedness proved in (d) implies (theorem 2') that one can write for each $\varepsilon > 0$ the following Rice-type formula:

$$(27) \quad E(c_\varepsilon{}^2 \nu^{X_\varepsilon}(u,2,(T\times T)\setminus\Delta^-(\varepsilon))) = (\pi/2)[\sigma(u)\,\|\Psi\|_2]^{-2}\int_{T\times T} dt_1 dt_2 \cdot$$

$$\cdot\int_{R^2} |x_1' . x_2'| p_{t_1,t_2;\varepsilon}(u,u;x_1',x_2')\, dx_1' dx_2'.$$

For each fixed pair (t_1,t_2) $t_1 \neq t_2$, use now a standard Arzela-Ascoli argument based on (d) and the convergence in distribution in (b) to show that as $\varepsilon \mapsto 0$

$$p_{t_1,t_2;\varepsilon}(x_1,x_2;x_1',x_2')$$

converges uniformly on each compact set for $(x_1,x_2;x_1',x_2')$ to the joint density of

$$X(t_1),\ X(t_2),\ \sigma(u)\,\|\Psi\|_2 N_1,\ \sigma(u)\,\|\Psi\|_2 N_2$$

with N_1, N_2 standard normal and $(X(t_1), X(t_2))$, N_1, N_2 independent. The bound in (d) permits to pass to the limit under the integral sign in (27) thus obtaining:

$$(28) \quad \lim_{\varepsilon\mapsto 0} E(c_\varepsilon{}^2 \nu^{X_\varepsilon}(u,2,(T\times T)\setminus\Delta^-(\varepsilon))) = E_L = \int_{T\times T} p_{t_1,t_2}(u,u)\, dt_1 dt_2.$$

(26) toghether with (28) imply:

$$\lim_{\varepsilon\mapsto 0} E(c_\varepsilon{}^2 [N_u^{X_\varepsilon}(T)]^2) = E_L.$$

Proof of (f). It is easy to see that one can pass to the limit under the integral sign in the third term of (11) and get:

$$\lim_{\delta\mapsto 0}(1/2\delta)^2 \int_{T\times T} E(\mathbf{1}_{(|X(t)-u|<\delta)}\, \mathbf{1}_{(|X(s)-u|<\delta)})ds\, dt = E_L.$$

Let us look at the modulus of the middle term in (11) and write it under the form:

$$(\pi/2)^{1/2}\,[\sigma(u)\,\|\Psi\|_2]^{-1}(1/\delta)\int_T E(\varepsilon^{1/2} N_u{}^{X_\varepsilon}(T)\, \mathbf{1}_{(|X(t)-u|<\delta)})\, dt.$$

The expression inside the expectation is bounded below by the sum

$$Z_1 + Z_2$$

where

$$Z_1 = \varepsilon^{1/2} N_u{}^{X_\varepsilon}[T\cap(T_1,t-\varepsilon)]\, \mathbf{1}_{(|X(t)-u|<\delta)}$$

$$Z_2 = \varepsilon^{1/2} N_u{}^{X_\varepsilon}[T\cap(t+\varepsilon,T_2)]\, \mathbf{1}_{(|X(t)-u|<\delta)}.$$

Note that a Rice-type formula holds true to compute $E(Z_1)$ and $E(Z_2)$ that is for $i = 1,2$ and each $\varepsilon > 0$:

$$(29)\quad E(Z_i) = \int \mathbf{1}_{U_i}(s)ds \int_{[u-\delta,u+\delta]} dy \int_{-\infty}^{\infty} |x'|\, p^*_{s,t;\varepsilon}(x;x',y)dx'$$

where

$$U_1 = T\cap(T_1,t-\varepsilon)$$

$$U_2 = T\cap(t+\varepsilon,T_2)$$

and

$$p^*_{s,t;\varepsilon}(x,x',y)$$

is the joint density of

$$(X_\varepsilon(s),\ \varepsilon^{1/2} X'_\varepsilon(s),\ X(t),).$$

In fact, first for <u>i=2,</u> since $t < s-\varepsilon$ we have:

$$p^*_{s,t;\varepsilon}(x,x',y) = p_t(y)\, p_{s-t;\varepsilon}(x;x'|y)$$

(the notation for the second factor is the one introduced in the proof of (d) for the joint density of $(X_\varepsilon(s), \varepsilon^{1/2} X'_\varepsilon(s))$ conditionally on the past until time t, $X(t)=y$).

Formula (29) follows in the same way as theorem 2' in §3 for $k=1$, on account of the finiteness of the moments of $N_0^{X'}\varepsilon(T)$ and the regularity and boundedness conditions for $p_t(y)$ and $p_{s-t;\varepsilon}(x;x'|y)$.

As for i=1, we may study the properties of $p^*_{s,t;\varepsilon}(x,x',y)$ as we did for the joint density of

$$(X_\varepsilon(t_1), \varepsilon^{1/2} X'_\varepsilon(t_1), X(t_3))$$

in the proof of part (d). (29) follows again from an entirely similar argument to that in theorem 2'.

Introducing these results into (11) we have:

$$(30)\; E([c_\varepsilon N_u^{X}\varepsilon(T) - L^X(u,T)]^2) \le E([c_\varepsilon N_u^{X}\varepsilon(T)]^2) -$$

$$-2(\pi/2)^{1/2} [\sigma(u) \|\Psi\|_2]^{-1} \int_T dt \int_T \mathbf{1}_{\{|s-t|\ge\varepsilon\}} ds \int_{-\infty}^{\infty} |x'| p^*_{s,t;\varepsilon}(u,x',u)dx' + E_L$$

where the limit as $\delta \mapsto 0$ in the middle term is justified by the regularity and the boudedness of $p^*_{s,t;\varepsilon}(x,x',y)$.

To finish the proof we pass to the limit in (30) as $\varepsilon \mapsto 0$. The limit of the first term is E_L and has been computed in part (e). The middle term tends to $-2E_L$; to prove this it suffices to use the bound on $p^*_{s,t;\varepsilon}(u,x',u)$ to have dominated convergence as well as the pointwise convergence of $p^*_{s,t;\varepsilon}(x,x',y)$ for $s \neq t$ - which follows in the same way as (b) and (d) - to the joint density of

$$(X(s), \sigma(X(s)) \|\Psi\|_2 N, X(t))$$

where N stands for a standard normal random variable independent of $(X(s), X(t))$.

Hence, we obtain

$$\lim_{\varepsilon \mapsto 0} E(|c_\varepsilon N_u^{X_\varepsilon}(T) - L^X(u,T)|^2) = 0$$

as it was announced.

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