PUBLICACIONES MATEMATICAS DEL URUGUAY

VOLUMEN 4

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EDITADAS POR EL CENTRO DE MATEMATICA DE LA FACULTAD DE CIENCIAS, UNIVERSIDAD DE LA REPUBLICA, CON EL APOYO DEL PROGRAMA PARA EL DESARROLLO DE LAS CIENCIAS BASICAS (PEDECIBA).

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ESTE NUMERO

En noviembre de 1991, la Universidad de la República Oriental del Uruguay otorgó el título de Doctor Honoris Causa a José Luis Massera. La ceremonia tuvo lugar en un Paraninfo desbordante, de público y emoción. Este número de las "PMU" recoge dos de los discursos pronunciados en esa oportunidad, el del Profesor Jacob Palis y el del propio Massera. Más adelante publicaremos trabajos que fueron presentados en el coloquio científico que por tal motivo tuvo lugar. Nos asociamos así al homenaje a un colega y maestro del cual, como matemáticos y como uruguayos, nos sentimos orgullosos.

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DISCURSO DE JOSE LUIS MASSERA

Señor Ministro de Educación y Cultura, Dr. Guillermo García Costa;

Señor Rector de la Universidad, Ing. Quím. Jorge Brovetto;

Señores miembros del Consejo Directivo Central y de los Consejos de las diversas Facultades, particularmente de Ingeniería, de Ciencias, y de Humanidades y Ciencias de la Educación;

Colegas matemáticos, uruguayos y extranjeros, que han aceptado la invitación de concurrir a este evento, algunos de ellos teniendo que viajar desde Europa y los Estados Unidos, entre los cuales se encuentran científicos de muy alto nivel internacional;

Querido amigo Misha Cotlar, que naturalmente integra el elenco precedente, pero que se distingue de los demás por el hecho de que formó parte, hace más de 50 años, junto con Rafael Laguardia, conmigo y unos pocos más, de ese pequeño grupo de jóvenes entusiastas, básicamente autodidactas, que desafiando dificultades de todo orden, incluso las que derivaban de la falta de una formación académica suficiente, nos lanzamos con decisión y audacia a la empresa casi quijotesca de iniciar el camino de la investigación matemática en el Uruguay. Sin que lo dicho signifique menospreciar el papel que jugaron en ese proceso personalidades de la talla del Ingeniero Don Eduardo García de Zúñiga y del matemático español Don Julio Rey Pastor;

Queridos matemáticos uruguayos que, partiendo de aquellas raíces, se incorporaron sucesivamente y se siguen incorporando en escala creciente a las sucesivas generaciones que forman el tronco, las ramas y las hojas y flores de ese árbol que se ha dado en llamar la escuela matemática uruguaya;

Queridos estudiantes que ya anuncian nuevos brotes fecundos de aquel árbol;

Queridos funcionarios administrativos, que han tomado parte imprescindible en este proceso, sintiéndolo también como cosa suya, y que hoy han hecho posible el éxito de esta reunión;

Queridos familiares, amigas y amigos, compañeras y compañeros:

Antes que nada, quiero agradecer a todos los que, con su iniciativa y esfuerzo, jugaron algún papel para que los órganos de gobierno universitarios adoptaran la decisión de concederme el título de Doctor Honoris Causa. He recibido otros, de parte de diversas Universidades, algunas de gran tradición y renombre, que fueron concedidos, sin duda por la valoración de méritos científicos, pero cuya motivación concreta fue la campaña por mi libertad -y esa motivación honra a dichas casas de estudio por la sensibilidad demostrada ante el problema del respeto a los derechos humanos-, en momentos en que estaba preso por la dictadura que entonces sufríamos. Pero, sin mengua de aquellos otros, valoro este título, entre otras cosas, porque procede de mi muy querida Universidad de la República en la que he vivido, trabajado y estudiado durante tantos años de mi vida.

Agradezco también los elogios que aquí se han pronunciado. Fuera de lo estrictamente científico, pienso que en ellos pesa la relación de fraternidad, amistad y estimación que me unen personalmente a todos estos colegas y amigos. Entendiéndolo así, hasta me complace lo que puedan tener de excesivo, en tanto ello esté motivado por otros valores de la relación humana que para mí son muy importantes, como también lo eran para Laguardia. Esa relación mutua de exigencia y aprecio, camaradería y respeto, es un rasgo típico de nuestra escuela, es el cemento fuerte que le da unidad y fortaleza y que esperemos se conserve sin fisuras en su sucesivo desarrollo.

Más allá de que, cuando plantábamos las primeras semillas, no podíamos prever la frondosidad que adquiriría y que sigue desarrollándose, ese árbol de que hablaba fue el producto de una decisión muy consciente del papel que la investigación científica debía jugar en una Universidad digna de merecer su nombre. En aquella época, la Facultad de Medicina y alguna otra daban ejemplo de institutos que, sin mengua del desarrollo de cursos curriculares, dedicaban no pocos esfuerzos a la investigación. Por otro lado, Rectores como Cassinoni y Maggiolo desarrollaron ampliamente el tema; este último elaboró el Plan que lleva su nombre, que propone una reestructura total de la Universidad, una de cuyas aristas destacadas es precisamente el papel central de la investigación, dando él mismo el ejemplo en el Instituto de Máquinas que dirigía. Ejemplo digno de destacarse, además, porque tomaba como objeto de la investigación una obra concreta, de enorme importancia económica, tanto para el Uruguay como para la Argentina, como lo es el estudio del comportamiento del río Uruguay y de la proyectada represa y usina hidroeléctrica de Salto Grande.

El ejemplo es también ilustrativo en relación a otro aspecto que vale la pena destacar. Aún hoy, hay docentes que piensan que la investigación debe autolimitarse dentro de niveles muy modestos; si se me permite la expresión, con espíritu de "pago chico". Creemos que esto es profundamente erróneo, y la escuela matemática uruguaya nunca ha aceptado semejantes limitaciones. La meta ideal que buscamos es colocarse a nivel de la matemática mundial, si es posible sobrepasando ese nivel. Y nos enorgullecemos de que hoy mismo haya jóvenes de menos de treinta años de edad -que, por añadidura, se formaron en parte en el clima adverso de la dictadura- alcancen estas metas. Naturalmente, no en todas las ramas de la ciencia -que sería una pretensión inalcanzable en un país tan pequeño como el nuestro-, pero sí en los campos en que nos proponemos concentrar el esfuerzo. Sólo apuntando alto y lejos, creando un clima de alta exigencia, podemos esperar tales resultados. Y es obvio que ello no significa descalificar investigaciones originales de menor nivel, cosa que puede ocurrir cuando se persiguen aplicaciones que, sin embargo, son tecnológicamente importantes.

Ahora que el Uruguay encara el Mercosur -desafío de grandes proporciones-, es con esa mentalidad que debemos afrontarlo. Hay que comprender que el desafío no es sólo en el campo estrictamente económico, sino también científico y tecnológico. Y sobre la Universidad y los universitarios recae, por ende, una parte importante de la responsabilidad nacional.

Termino. Ha sido un rasgo permanente de mi vida el que no pudiera nunca separar mi actividad científica de aspectos sociales y políticos que me son muy caros. Durante bastantes años, esas dos esferas pudieron coexistir sin demasiados conflictos. Pero a ello se debe que nunca he trabajado en régimen de dedicación total pese a que haya sido y siga siendo ferviente partidario de que ese régimen se extienda a la gran mayoría de los universitarios. Como era inevitable, en determinado momento la contradicción hizo crisis: fue cuando salí electo diputado, cargo que mantuve durante nueve años; aún así, nunca dejé de dar clases. La crisis siguiente fue más tajante, cuando la dictadura me expulsó de la Universidad y posteriormente me privó de la libertad. Al reconquistarla, luego de casi diez años, sin vacilar asumí, junto con otros colegas, lo que eran tareas absolutamente prioritarias, de reconstruir lo que había sido devastado. Sucesivamente concentré mi esfuerzo en llevar a buen fin el proyecto esencial del PEDECIBA, en restaurar condiciones de buen funcionamiento del IMERL y de la Facultad de Ingeniería, cuyo Consejo integré, y finalmente en poner en marcha el proyecto de creación de la Facultad de Ciencias.

Insisto en que en esas tareas nunca ocupé posiciones de primer plano sino que trabajé como participante de esfuerzos colectivos. Sea como fuere, sumando todo, fueron quizás veinticinco años en los que debí estar apartado de la investigación matemática. Es demasiado tiempo, sobre todo si se tiene en cuenta los ritmos con que, como todas las ciencias, se desarrolla en esta época en que vivimos. Estoy convencido de que me era imposible retornar a ella. Me pareció preferible emprender un nuevo camino: la filosofía, con la esperanza de que pudiera hacer algún aporte a la historia y la filosofía de la matemática, y resolver en ese terreno algunas cuestiones que me inquietaban y probablemente inquieten a otros. Sin saber, como es lógico, qué me deparará el destino en esta nueva ruta, estoy satisfecho por haber tomado esa decisión. Al fin de cuentas, se hace camino al andar...

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DISCURSO DE JACOBO PALIS

Señor Ministro de Educación y Cultura, Señor Rector de la Universidad, Señores Decanos, Señoras y Señores, Caros Colegas, José Luis Massera.

O matemático e sua dignidade

José Luis Massera é um exemplo, quase uma lenda, para várias gerações de matemáticos latino-americanos. Quando iniciava meus primeiros passos como matemático, dele ouvi falar, com respeito e fascínio, como o cientista que pioneiramente formava, quase que por milagre, uma escola matemática uruguaia, cujos trabalhos eram admirados nos centros mais avançados da America do Norte e Europa. Também ouvi falar de seus ideais sociais e de sua dignidade...

Em sua matemática, Massera exibiu um talento nato, autodidata, vigoroso, original. Desbravou novas trilhas nessa imensa e bela floresta dos Sistemas Dinâmicos -Equaçoês Diferenciais-, área maior da Matemática Contemporânea. Assim, desenvolveu uma obra definitiva de grande interesse atual e futuro sobre a estabilidade assintótica dos sistemas dinâmicos em termos da existência de funçoês de Lyapunov, obra em parte publicada em Annals of Mathematics em 1949 e 1956. Repetiríamos, vinte, trinta anos depois seus métodos... Assim é que nos anos sessenta surge a idéia de filtraçoês e no início dos anos setenta aparece no mesmo Annals of Mathematics um artigo de Smale e Shub sobre o tema e, finalmente, chegamos à compreensão, talvez mais bem sintetizada por Conley, de que em geral um sistema dinâmico consiste de peças recorrentes e ciclos entre elas, as quais são entaõ "ordenadas" através da existência de funções de Lyapunov, precisamente à la façon de Massera.

Como explicar o fenômeno de que parte dos fundamentos desta área central da Matemática, tenha sido feita em nosso continente com relativamente pequena tradiçao científica, em seu extremo sul, aqui, em um canto (por sinal dos mais belos) do mundo e, sobretudo, com tanta originalidade e finesse que tornar-se-ia definitiva?. Só um talento exuberante, transbordante como o de Massera!

A rica, notável, pioneira contribução matemática de Massera permeia outros tópicos de grande interesse matemático, como por exemplo:

1)sua demonstração do teorema da variedade estável, com seu enunciado geral, como o ensinamos hoje, feita no início dos anos cinquenta e publicada no Boletín de la Facultad de Ingeniería, Montevideo,

2)seus resultados sobre a existência de soluções subharmônicas de equações de segunda ordem e de soluções periódicas de equações diferenciais, publicados em 1949-1950 em Annals of Mathematics e Duke Mathematical Journal,

3)a teoria, construida com Schäffer,para equações lineares ou quasi-lineares onde introduzem-se conceitos como o de dicotomia exponencial, precurssor do conceito de hiperbolicidade e por isto mencionado no trabalho clássico de Anosov sobre hiperbolicidade global publicado vários anos depois. Seguiu-se, então, a construção da teoria hiperbólica, de importância central nesta area, por Smale e outros matemáticos. Também nos trabalhos de Massera e Schäffer aparece uma forma "linear" de estabilidade estrutural ligada à dicotomia exponencial, assim como a estabilidade estrutural è ligada à hiperbolicidade em geral, como proposto por Smale e eu próprio uma década depois, ao final dos anos sessenta, e comprovado por Mañé há apenas alguns anos. Mais ainda, o contexto de Massera e Schäffer é infinito-dimensional e seus métodos, em particular aqueles relativos à geometria do espaço, inspiraram inúmeros trabalhos de pesquisa. A obra foi publicada em Annals of Mathematics em 1958 e 1959 e em Mathematischen Annalen em 1960, bem como no livro "Linear Differential Equations and Function Spaces", Accademic Press, 1966.

Os trabalhos de Massera tiveram especial destaque em vários livros clássicos de equações diferenciais como os de Lefschetz, Hartman e o de Reissig-Sansone-Conti e, posteriormente, no trabalho de Anosov acima mencionado. E o seu fino espírito indagativo e de visão ao mesmo tempo ampla e profunda o levaram, nos dias de hoje, à História e Filosofia da Ciência e, em particular, da Matemática.

De tanta riqueza científica e extraordinária personalidade usufruiram, naturalmente, jovens matemáticos uruguaios de várias gerações como Lumer, Schäffer, Gandulfo, Lewowicz e ainda o brasileiro Onuchic, que veio especialmente para trabalhar com Massera. O ambiente matemático que conseguiu criar, com Laguardia, no Instituto de Matemática y Estadística de la Facultad de Ingeniería nos anos cinquenta ainda causam admiração. Hoje, reconstruido e ampliado este ambiente, volta a Matemática uruguaia a se destacar no cenário mundial. Constitúise em exemplo maior para todos nós que lutamos por uma matemática e, em geral, uma Ciência, de alta qualidade em todo o mundo e não apenas nos paises ditos do Primeiro Mundo, concientes que somos de sua importância para o desenvolvimento econômico e social e a integridade científico-cultural de uma nação. A atividade de pesquisa básica e aplicada,sem compromissos de qualidade como sempre proclamou Massera, influenciam diretamente o nível de competência dos quadros técnicos de um país e não pode ser considerado, por simplismos de eventuais dirigentes, como atividade de luxo de uma nação rica!. Tamanho absurdo parece ganhar força em alguns de nossos paises como se fora "conventional wisdom", arriscando uma frágil mas já rica estrutura científica construida com tanto esforço, humano e econômico, através décadas de trabalho, tantas vezes heróico, de teimosos cientistas nativos como José Luis Massera.

Figura maior da Matemática e da Ciência Latino-Americana, Massera tem sua obra e sua humanidade reconhecidas em todo o mundo, tendo sído homenageado pelas Universidades de Roma (La Sapienza), Humboldt de Berlín, Quito, Budapest, Puebla, San Andrés (Bolivia), Habana e, para meu orgulho, Federal do Rio de Janeiro. Reconhecimento que com toda justiça é hoje ampliado por sua Universidade, la Universidad de la República, e em cuja homenagem nós, seus amigos, colegas e admiradores temos a honra, a alegria, uma imensa alegria, de participar. Pequeno tributo a uma grande pessoa que brutalizada por seus ideais sociais, respondeu com o destemor e a dignidade. Exemplo maior de humanismo, de pessoa-integridade, de pessoa-Ciência, nós o admiramos e queremos muito, José Luis Massera, e sua vida, esteja certo, não foi e não será em vão. Ela marcará uma etapa de luta e sofrimento quase inacreditável, neste mundo por vezes absurdo.

Mas também da grande, da imensa alegria pela feitura por suas e por outras poucas mãos, da melhor Matemática e pela construção de ambiente científico em nossos paises, em nosso continente, como ainda de um exemplo maior de dignidade humana e finesse de espírito como o seu.

José Luis Massera, obrigado por seu legado. Obrigado por seres.

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Integrals and Invariant Theory

Walter R. Ferrer Santos
Mathematical Sciences Research Institute
1000 Centennial Drive. Berkeley. CA 94720. USA*

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Abstract

In this paper we describe how -since Hilbert's work in Invariant Theory in 1890- the concept of integral has been a basic tool in Representation and Invariant Theory. We describe also the limitations and the overcoming of some of the limitations of this tool. We end by presenting an extension of the concept and of some of the results.

1 The case of a finite group

Let k be a fixed field of characteristic p. If G is a finite group we call $\mathcal{G}$ the category of finite dimensional k-representations of G. In other words $\mathcal{G}$ is the category whose objects are finite dimensional k vector spaces V equipped with right linear actions of G on V and whose morphisms are the G-equivariant k-linear maps. We consider k as an object of $\mathcal{G}$ by equipping it with the trivial G-action. An object S in $\mathcal{G}$ is called simple if the only subobjects of S are S and $\{0\}$. An object of $\mathcal{G}$ is called semisimple if it is the direct sum of simple objects or equivalently if any of its subobjects has a $\mathcal{G}$-complement. If $M \in \mathcal{G}$ we denote as $M^G = \{m \in M : m.g = m \; \forall g \in G\}$.

We call $F(G)$ the k-algebra of all functions of G into k with the operations defined at every point. Clearly $F(G)$ is an object of $\mathcal{G}$ if we define an action as follows. For $x \in G$ and $f \in F(G)$ $(f.x)(y) = f(xy)$.

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Definiton 1.1 *An integral for G is a map $\mathbf{I} : F(G) \to k \in \mathcal{G}$. We denote as $\int$ the subspace of the linear dual of $F(G)$ consisting of all the integrals.*

The linear map $\mathbf{I_0}$ given as $\mathbf{I_0}(f) = \sum_{g \in G} f(g)$ is a non zero element of $\int$. One can easily show that $\int$ has dimension 1 over k.

In fact, if we call δ_g the element of $F(G)$ that takes the value 1 at $g \in G$ and 0 at all the other points of G, we have that $\delta_g.h = \delta_{h^{-1}g}$. Then if $\mathbf{I}$ is an integral $\mathbf{I}(\delta_x) = \mathbf{I}(\delta_y)$ for all $x, y \in G$. We call $\mathbf{i}$ this common value. If $f \in F(G)$ we have that $f = \sum_{g \in G} f(g)\delta_g$. Applying $\mathbf{I}$ to the above equality we obtain that

$$\mathbf{I}(f) = \mathbf{i} \sum_{g \in G} f(g) = \mathbf{i}\mathbf{I_0}(f) \tag{1}$$

Then every integral is a constant multiple of $\mathbf{I_0}$.

We say that the group G admits a normalized integral if there is an element $\mathbf{J} \in \int$ such that $\mathbf{J}(\mathbf{1}) = 1$. Here $\mathbf{1}$ is the unit elment of $F(G)$, i.e., the function on G that takes the constant value $1 \in k$.

It is easy to see that G admits a normalized integral if and only if p does not divide the order of G (that will be denoted as $|G|$).

In fact: equation 1 shows that for any $\mathbf{J} \in \int$, $\mathbf{J}(\mathbf{1})$ is a non-zero constant multiple of $|G| = \mathbf{I_0}(\mathbf{1})$. If moreover $\mathbf{J}$ is normalized $\mathbf{J}(\mathbf{1}) \neq 0$ and our conclusion follows.

In what follows we show how the existence of a normalized integral for a group implies that:

EX If $f : M \to N$ is a surjective map in $\mathcal{G}$, then $f(M^G) = N^G$.

SP If $\lambda : M \to k$ is a surjective morphism in $\mathcal{G}$ there exists an element $m \in M^G$ such that $\lambda(m) = 1$.

SS All objects of $\mathcal{G}$ are semisimple.

RO There exists a family of linear maps $R_M \in \mathcal{G}$ for $M \in \mathcal{G}$ such that:

- $R_M : M \to M^G$.
- If $f : M \to N \in \mathcal{G}$ then $fR_N = R_M f$.
- If $m \in M^G$ then $R_M(m) = m$.

FG Let A be an **N**-graded commutative k-algebra in which G acts by algebra automorphisms that preserve the grading. Suppose also that the part of degree zero is the base field k. Then if A is finitely generated over k so is A^G.

The validity of property **SS** in the case in which G is invertible in k is known as Maschke's Theorem (see [24]). A family of maps as in **RO** is called a family of Reynolds Operators. This name appeared for the first time in the mathematical literature in a paper by Garret Birkhoff (see [1])[1] and refers to the engineer Osborne Reynolds who used "averaging operators" to study certain problems in fluid dynamics. The fact that the existence of a normailzed integral implies condition **FG** is nothing but E. Noether's theorem (see [29] and [30]) on finite generation of invariants for a finite group in a particular case in which the proof besides becoming extremely elementary can be easily generalized to other contexts.

We indicate briefly the main steps in the proofs of the mentioned results.

It is clear that **RO** implies **EX** and that **EX** implies **SP**. Moreover, **SS** implies **RO** (just use the semismplicity of M to construct the projection of M onto M^G) and **RO** implies the existence of a normalized integral (a normalized integral is nothing but the Reynolds Operator corresponding to $M = F(G)$). To prove the equivalence of the existence of a normalized integral with conditions **EX**, **SP**, **SS** and **RO**, we need to verify that:

- **SP** implies **SS**.
- The existence of a normalized integral implies condition **SP**.

The first implication follows by considering the restriction map from $Hom_k(M,N)$ to $Hom_k(N,N)$ where N is a subobject of M. Consider the G-submodule of $Hom_k(N,N)$ given by the multiples of the identity map and call X its inverse image by the restriction. An element of $X^G \subset Hom_k(M,N)^G$ that is sent by the restriction to the identity map on N will split the inclusion of N in M.

It is convenient to introduce the viewpoint of comodules (that will be developped in more deatils in Section 4) to prove the second implication. To

[1]The author would like to thank Prof. I. Kaplansky for providing the above reference and for helping him find a path through the "maze" of the classical literature on the subject.

an arbitrary object $M \in \mathcal{G}$ we can associate a map $\chi_M : M \to M \otimes F(G)$ in the following way $\chi_M(m) = \sum m.g \otimes \delta_g$. In particular if we apply this construction to $F(G)$ itself we obtain a map $\Delta : F(G) \to F(G) \otimes F(G)$ that together with the multiplication of functions in $F(G)$, the unit and evaluation at the identity, gives to $F(G)$ a structure of bialgebra. The map χ_M defined above is a comodule structure on M. The construction of χ_M from the action of G on M can easily be reversed to obtain the action from the comodule structure. In that way we obtain a bijective correspondence between the $F(G)$-comodule structures in a vector space M and the G-actions on M. If $\mathbf{J}$ is a normalized integral and M is an arbitrary object of $\mathcal{G}$, we can define the map $\mathbf{J}_M = (id \otimes \mathbf{J})\chi_M$. In explicit terms $\mathbf{J}_M(m) = |G|^{-1} \sum m.g$.

Now, as to the proof of condition **SP**, if $\lambda : M \to k \in \mathcal{G}$ is surjective and m is an element of M such that $\lambda(m) = 1$, the element $\mathbf{J}_M(m)$ is in M^G and sent to 1 by λ.

It is worth noticing that the map $\mathbf{J}_M$ that is defined in terms of the normalized integral $\mathbf{J}$ can be though of as an "averaging process" in M. The existence of this "averaging process" (that in fact is an operator as in **RO**) is the crucial ingredient of the above proof.

As to the proof of **FG** we observe first that as A is finitely generated over k all the homogeneous components A_n are finite dimensional k-spaces. There is a Reynolds Operator for each A_n and all of them can be put together to define a Reynolds Operator for A. It is not hard to prove that this operator can be chosen so that it verifies the following multiplicativity condition:

$$R_A(fg) = fR_A(g) \, \forall \, f \in A^G, \, g \in A \tag{2}$$

Call $A_+ = \bigoplus_{n>0} A_n$ the null ideal of A. Consider A^G_+ and call $\mathcal{I}$ the ideal that it generates in A. By Hilbert Basis Theorem (that Hilbert proved in order to be able to conclude that certain rings of invariants were finitely generated, see [15]) there exists a finite set of elements $\mathcal{F}$ that generate $\mathcal{I}$. We prove that the k-algebra generated by $\mathcal{F}$ is all of A^G. If $a \in A^G$ is an homogeneous invariant that is not in k it will be in A^G_+. Then $a = \sum_{f \in \mathcal{F}} a_f f$ with $a_f \in A$. Apply R_A to this equation and call $b_f = R_A(a_f)$. Using equation 2 one has that $a = \sum_{f \in \mathcal{F}} b_f f$. The elements b_f have smallest degree than a. Then, by induction, we can conclude that they belong to the k-algebra generated by $\mathcal{F}$. Then, the same happens to a.

2 Integrals in the work of Hilbert and Weyl on Invariant Theory

The proof that the existence of a normalized integral for a finite group implies condition **FG** that we just presented is an adaptation of Hilbert's proof (see [15]) of Gordan's theorem (see [12]) on the finite generation of the invariants of binary forms (in the language of XIX century invariant theory "on the finiteness of the independent invariants of quantics"). The same idea was applied later by Hilbert (see [16]) to the case of n-ary forms (see [14] for a historical analysis of these –and other– mathematical concepts related to classical invariant theory and the excellent survey by A. Borel ([2]) –from which we borrowed heavily– on the work on these subjects of Weyl and also of Hurwitz and Schur). These n-forms were intractable by the classical methods: either the "symbolic methods" developped by the German school (Aronhold, Clebsch and Gordon) or the "algorithmic methods" developped by the British (and North American) School (Boole, Cayley, Sylvester and Salmon).

The strengths and limitations of both XIX century schools on Invariant Theory have been extensively studied in articles dealing with the history and philosophy of mathematics (see for example the comments in H. Weyl's book "The Classical Groups: Invariants and Representations" [37, pg 27-29] or [31] for a comparative analysis of both schools). It is interesting to note that recently some of the classical methods have been revitalized with success (see for example [20] and [18]).

In what follows we sketchly describe some of the aspects of the work done in Representation and Invariant Theory around the period 1890-1930 that are relevant to our presentation.

Consider the group $SL_2(C)$ acting by algebra automorphisms on the algebra $C[X,Y]$ as follows: (note that it is only necessary to define the action on the generators X and Y)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .X = dX - bY$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .Y = -cX + aY$$

If we fix a natural number d the k-subspace S_d of $k[X,Y]$ generated by $X^d, X^{d-1}Y, \ldots, Y^d$ is $SL_2(C)$-invariant.

Gordon's Theorem "on the finiteness of the independent invariants of quantics" states that the algebra $S(S_d)^{SL_2(C)}$ is finitely generated.

In a remarkable paper published in 1854 (see [6]) Cayley abandoned his old methods to produce invariants (based on the so called "hyperdeterminants") in favor of the method of differential operators (see [31] for a description of the work of the British School on Invariant Theory).

Consider the derivations $\xi_i : S(S_d) \to S(S_d)$, $i = 0, .., d$ given (on a sistem of generators of S_d) as $\xi_i(X^{d-j}Y^j) = \delta_{i,j}$.

It is well known that if A is an arbitrary commutative algebra and $\mathcal{D}(A)$ denotes the (k vector space) of all k-derivations of A, then $\mathcal{D}(A)$ has a natural structure of A-module with respect to the usual multiplication.

Taking special $S(S_d)$-linear combinations of the derivations ξ_i, Cayley defined two elements $\mathcal{X}, \mathcal{Y} \in \mathcal{D}(S(S_d))$ and proved (see [14] for a precise description of the above differential operators) that $S(S_d)^{SL_2(C)} = \{\alpha \in S(S_d) : \mathcal{X}(\alpha) = \mathcal{Y}(\alpha) = 0\}$.

Hilbert's proof of the finite generation of the ring $S(S_d)^{SL_2(C)}$ used $\mathcal{X}$ and $\mathcal{Y}$ to construct a map with the properties of (2) and with that, together with his Basis Theorem proceeded in the same fashion than in Section 1. For n-ary forms in a later paper [16] he used what become known as Cayley Ω-process to perform the same steps.

Hilbert himself was aware that his Basis Theorem and the existence of a map with the properties of (2) were all that was needed to prove the finite generation of invariants. In particular other maps R have to be constructed if one is dealing with other group actions.

In accordance with [14], Hilbert was able to apply his method to other groups than SL_n, in particular he suceeded constructing an analog to the Ω-process for the rotation group in the real 3-dimensional space, i.e., the group of real orthogonal transformations.

Hurwitz, which was a student of Klein and a former teacher of Hilbert, solved in 1897 the problem of finite generation of invariants for the real orthogonal group in n-space in 1897, by constructing the required Reynolds operators by integration (see [19]). The extension to compact Lie Groups was immediately observed.

Moreover it had already been observed (by E.H.Moore and Maschke among others) that in the case of a finite group the "averaging process" besides the finite generation of invariants also yielded the semisimplicity of the representations.

I. Schur, in a paper in 1924, extended this result on the semisimplicity of the representations to the real orthogonal group (see [33]) and observed that the theory could be extended to other groups as long as an "averaging process" could be constructed. He didn't develop the general theory because in his own words (see [14]) " [the rotation and orthogonal groups] stand out, not only by virtue of the important role they play in applications but also by virtue of the fact that here the integral calculus provides a solution of the counting problem that is practically useful". The "counting problem" was a problem proposed (and solved) by Cayley "on the number of independent covariants" of fixed degrees.

In 1924-26 Hermann Weyl, with the aid of E. Cartan's results on Lie Algebras, extended Schur's theory to all complex semisimple Lie groups (see [35] and [36]). His methods consisted in using again an "averaging process" of integration via what he called first the "unitarian restriction" ("unitäre Beschränkung") and later the "unitarian trick". If G is an arbitrary complex semisimple Lie Group and K is a maximal compact subgroup it can be proved that if V is a G-module, the G-submodules of V coincide with the K-submodules. Being K compact the integration can be carried along K and the results about the representations and invariants for G can be obtained from the corresponding results for K. In the case in which G is the special linear group $SL_n(C)$, K can be taken to be the special unitary group $SU_n(C)$ and that is the reason for the name of the trick. It is worth noticing that particular cases of this "trick" had already been used by Hurwitz and Schur in the papers just mentioned. Weyl proved (among many other things) what is now called Weyl's Theorem: All the representations of a semisimple Lie Algebra are completely reducible. The method of his proof was to pass from the Lie Algebra to a connected and simply connected complex Lie Group and then to apply to this group his "unitarian trick" to reduce the problem to the situation of a compact group.

It was observed later by Schiffer (1933 unpublished) that the existence of Reynolds operators (as was mentioned in the particular case of a finite group in Section 1) could be deduced by purely algebraic means from the semisimplicity. This appears as an Appendix to the Second Edition of [37](see Appendix C).

Completely algebraic proofs of Weyl's Theorem were obtained later (for arbitrary fields of characteristic zero) and we shall see that some type of integral or "averaging process" plays an important role in this algebraic approach.

3 Integrals and Lie Algebras

The historical comments that follow are based on [3, Note Historique, Chapitres I à III] and [2]. Results on the complete reducibility of Lie Algebras seem to have appeared for the first time around 1890 in some unpublished work of Study. His work was cited by Lie and Engel in their joint book [23]. Study proved that the representations of $\mathbf{sl_2}(C)$ are completely reducible and Lie and Engel conjectured that the same was true for $\mathbf{sl_n}(C)$ (in accordance to Borel –see [2]– who refers to Hawkins for this fact, Study, in a letter to Sophus Lie in 1890, conjectured the full reducibility of the representations of an arbitrary semismple Lie Algebra). In 1926 Weyl (see [36]) proved (and remarked that E. Cartan had used the result implicitly) the complete reducibility of the representations of a semisimple Lie Algebra using his "unitarian trick" (see [2] for an interesting discussion whether E. Cartan was aware or not at that time of the complete reducibility).

Casimir (a physicist) realized around 1932 that an operator that plays a role on quantum theory (the "square of the magnitude of the moment of momentum" – see [2, pg 63] –) and that he had generalized from $\mathbf{sl_2}$ to a general semisimple Lie Algebra (and is now called the Casimir operator), could be used to produce an algebraic proof of the full reducibility of the representations of $\mathbf{sl_2}$. This was generalized in 1935 by Casimir and van der Waerden (see [5]) to produce the first purely algebraic proof of the complete reducibility.

The proof that is frequently presented in modern literature (see for example [3] or [11]) is Brauer's proof that appeared in 1936 (see [4]).

A cohomological proof was presented in 1937 in [38]. In the mentioned papers J. H. C. Whitehead defined the first two cohomology groups of a Lie Algebra and used them to prove the mentioned semisimplicity result.

The authors mentioned above worked with real or complex base fields. The development of the theory of Lie Algebras (in particular the generalization of some of the above results to the case of other base fields) is due mainly to N. Jacobson (see for example [21]).

In order to illustrate the use of integrals in the algebraic proof of Weyl's Theorem we proceed as follows.

It is not hard to see (one has to proceed in a similar way as in Section 1) that the complete reducibility of the representations of a Lie algebra $\mathcal{L}$ is equivalent to the following condition (that in analogy with the considerations of Section 1 we call analogous **SP**): **(SP)** For every surjective morphism of $\mathcal{L}$-modules $\lambda : V \to k$ there is an element $v \in V^{\mathcal{L}}$ such that $\lambda(v) = 1$.

We prove that if $\mathcal{L}$ is a semisimple lie algebra over an algebraically closed field of characteristic zero condition **(SP)** is verified.

Consider λ and V as above, call W the kernel of λ and form the exact sequence of $\mathcal{L}$-modules:

$$0 \to W \to V \xrightarrow{\lambda} k \to 0 \tag{3}$$

where $\mathcal{L}$ acts trivially on k (being semisimple this is the only way it can act on k).

Without much labour one can reduce the problem to the case where W is a simple faithful $\mathcal{L}$-module. We call $\rho : \mathcal{L} \to End(W)$ the corresponding representation. The reduction to the case in which W is simple is obtained by reasoning by induction in the dimension of W. To be able to assume that W is faithfull we proceed as follows: take $\mathcal{L}'$ the kernel of the representation of $\mathcal{L}$ in W, it follows that $\mathcal{L}'$ also acts trivially on V (here we have to use that $\mathcal{L}'$ coincides with its derived subalgebra). In this way (3) becomes an exact sequence of $\mathcal{L}/\mathcal{L}'$-modules and W is now faithfull as an $\mathcal{L}/\mathcal{L}'$-module. The validity of the result in this case implies the validity of the general result.

In Section 1 we used an averaging process to produce the element we needed in V^G. Here we will show how a basic ingredient of the classical proofs, the "Casimir element", can be used to perform the same "averaging process" in our category.

We start by defining a bilinear form B_W in $\mathcal{L}$ by the following formula:

$$B_W(x, y) = tr_W(\rho(x)\rho(y))$$

A direct verification shows that for any $z, x, y \in \mathcal{L}$ the bilinear form B_W verifies $B_W(ad(z)x, y) = B_W(x, ad(y)z)$.

Consider the ideal of $\mathcal{L}$ defined as $\{x \in \mathcal{L} : B_W(x, y) = 0 \; \forall \, y \in \mathcal{L}\}$. The above ideal is solvable (by Cartan's criterion) and because of the semisimplicity of $\mathcal{L}$ it has to be zero. In other words, the form B_W is non degenerate.

We use B_W to establish an identification of $\mathcal{L}$ and $\mathcal{L}^*$. Using this identification we can construct an isomorphism of $\mathcal{L}$-modules between $Hom_k(\mathcal{L},\mathcal{L}) \cong \mathcal{L}\otimes\mathcal{L}^* \cong \mathcal{L}\otimes\mathcal{L}$.

Call b_W the element of $\mathcal{L}\otimes\mathcal{L}$ that is the image of the identity map on $\mathcal{L}$ via this isomorphism. As the identity on $\mathcal{L}$ is annihilated by the action of $\mathcal{L}$ so is b_W. Call $U(\mathcal{L})$ the universal envelopping algebra of $\mathcal{L}$ and consider the element c_W of $U(\mathcal{L})$ given as the image of b_W by the multiplication map from $\mathcal{L}\otimes\mathcal{L}$ into $U(\mathcal{L})$.

The element c_W is called the Casimir element of W and belongs to the center of $U(\mathcal{L})$ (this because b_W is annihilated by $\mathcal{L}$).

If T is an arbitrary $\mathcal{L}$-module the element c_W defines by multiplication an $\mathcal{L}$-endomorphism of T. We denote this endomorphism as $c_{W,T}$.

More explicitly: if $\{x_i\},\{y_i\}; i = 1,..,dim\,\mathcal{L}$ are dual basis of $\mathcal{L}$ with respect to B_W then $c_W = \sum x_i y_i$. Considered as an operator in W, the element c_W is equal to $\sum_i \rho(x_i)\rho(y_i)$, in other words $c_{W,W} = \sum_i \rho(x_i)\rho(y_i)$.

Then, $tr_W(c_{W,W}) = \sum tr_W(\rho(x_i)\rho(y_i)) = \sum B_W(x_i,y_i) = dim\,\mathcal{L}$. Thus, the operator $c_{W,W}$ is not zero. By Schur's Lemma we conclude that $c_{W,W} = r\,id_W$ for some $r \in k^*$.

We consider now the maps $c_{W,V}$ and $\tau_W = id - r^{-1}c_{W,V} : V \to V$.

The last map, that is closely related to the constructions of [5] and [4], sends V into $V^{\mathcal{L}}$ and plays the role of the "averaging process" of Section 1.

It is clear that $U(\mathcal{L})V \subset W$ (it is enough to check that if $v \in V$ and $x \in \mathcal{L}$ then $\lambda(xv) = 0$ and this is evident because $\mathcal{L}$ acts trivially on k).

Suppose now that v is an arbitrary element of V, we prove that $\tau_W(v) = v - r^{-1}c_W v \in V^{\mathcal{L}}$. In fact, $x(v - r^{-1}c_W v) = xv - r^{-1}xc_W v = xv - r^{-1}c_W xv = xv - r^{-1}rxv = 0$. The equality before the last is true because $xv \in W$ and in W multiplication by c_W amounts to multiplication by the scalar r.

If v is chosen in such a way that $\lambda(v) = 1$ then $\lambda(\tau_W(v)) = \lambda(v - r^{-1}c_W v) = 1 - r^{-1}\lambda(c_W v) = 1$. The last equality is a consequence of the fact that $c_W v \in W$. Thus, the element $\tau_W(v)$ verifies the required conditions.

It is important to note the following: in the case of a finite group the "averaging processes" for the G-modules were constructed (see the construction of $\mathbf{J}_M$ from $\mathbf{J}$ in Section 1) from a normalized integral for $F(G)$. An analogous construction that would give all maps τ_W in terms of a "normalized integral" could be developped here with the continuous dual of the universal envelopping algebra of $\mathcal{L}$ playing the role of $F(G)$. Being this envelopping algebra an infinite dimensional k-space, the constructions are more elaborate and will be omitted. See [17] for details.

In the next Section we consider the situation of an arbitrary Hopf Algebra. The case of a finite group and of a Lie Algebra appear as specializations of this situation.

4 Integrals for Hopf Algebras

The fact that some of the above considerations about integrals, representation theory and finite generation of invariants can be generalized to the context of Hopf Algebras seems to have been observed for the first time by Sweedler and Larson around 1968 (see [22] and [34]). Their considerations were motivated (see [34, Introduction]) by some results of Hochschild (see [17, page 63-64]). In [17] it is proved that if $\mathcal{L}$ is a finite dimensional Lie Algebra over a field of characteristic zero and we call $\mathcal{K}$ the continuous dual of the universal envelopping algebra $U(\mathcal{L})$ of $\mathcal{L}$, then $\mathcal{L}$ is semisimple if and only if there exists an $\mathcal{L}$-morphism $\mathbf{J} : \mathcal{K} \to k$ that sends the unit of $\mathcal{K}$ into the unit of the base field. The map $\mathbf{J}$ was called a *gauge* for the Hopf algebra $\mathcal{K}$, and the concept of *gauge* was defined for an arbitrary Hopf Algebra. A *gauge* is what later was called an integral except that it verifies the additional condition of sending 1 into 1. In [17] it was also observed (without proof) that ".. an affine algebraic group is fully reducible if and only if its Hopf Algebra of polynomial functions has a gauge".

In this section we will present Sweedler's arguments relating the existence of a "normalized integral" for a Hopf Algebra H with the analogue of what we call in Section 1 condition **SP**.

Let H be a Hopf algebra defined over an arbitrary field k and call $\Delta, \varepsilon, \mu, u$ and σ its comultiplication, counit, multiplication, unit and antipode respectively. The element $u(1_k)$ will be written as 1.

Definiton 4.1 *A linear map* $\mathbf{J} : H \to k$ *such that* $u \circ \mathbf{J} = (id_H \otimes \mathbf{J}) \circ \Delta$ *is called an integral for* H. *If* $\mathbf{J}$ *also verifies* $\mathbf{J} \circ u = id_k$ *it is called a normalized integral.*

In the definition above $\circ$ denotes the composition of functions. In the future we will omit this symbol and represent the composition by juxtaposition.

If M is an H-comodule its structure map will be denoted as χ_M. If M is an H-comodule we call $M^H = \{m \in M : \chi_M(m) = m \otimes 1\}$. We consider k as an H-comodule with the trivial structure (i.e. $\chi_k(a) = a \otimes u(1_k)$).

We consider the analogue of condition **SP** of Section 1.

Definiton 4.2 *We say that an H-comodule M verifies condition* **SP** *if for every non zero morphism $\lambda : M \to k$ there exists an element $m \in M^H$ such that $\lambda(m) = 1$.*

The following result generalizes the considerations of Section 1 and is due (in another formulation) to Sweedler ([34]).

Theorem 4.1 *The Hopf Algebra H admits a normalized integral if and only if all H comodules verify condition* **SP**.

Proof : Note first that by the application of condition **SP** to an appropriate comodule of homomorphisms one can easily prove that the validity of condition **SP** for all comodules is equivalent to the condition that all the H-comodules are semisimple (see Section 1 for the case of a finite group).

Suppose that condition **SP** is valid for all H-comodules. If we consider the unit map $u : k \to H$ as an injective H-comodule map the complete reducibility implies that there exists an H-map $\mathbf{J} : H \to k$ that splits u. The map $\mathbf{J}$ verifies the definition of a normalized integral for H.

Conversely, suppose there is a normalized integral $\mathbf{J}$. If $\lambda : M \to k$ is a surjective H-map and $m \in M$ is such that $\lambda(m) = 1$ the element $n = (id \otimes \mathbf{J})\chi_M(m)$ belongs to M^H.

Moreover $\lambda(n) = \lambda(id \otimes \mathbf{J})\chi_M(m) = (id \otimes \mathbf{J})(\lambda \otimes id)\chi_M(m) = (id \otimes \mathbf{J})(\lambda(m) \otimes 1) = 1$.

Q.E.D.

Once this point is settled we can proceed in the same way as in Section 1 and thus prove the first fundamental theorem on invariants for the action of a "co-semisimple" (i.e. with all the comodules semisimple) Hopf Algebra on a graded finitely generated algebra. This, in a certain sense finishes completely the subject and in that sense can be considered as a culmination of the line initiated by Hilbert in 1890 in order to prove finiteness of invariants using conveniently constructed "normalized integrals".

Hilbert's technique works extremely well in characteristic zero where it can be applied to all semisimple groups but has a very serious drawback in arbitrary characteristic.

It was proved by Nagata in 1964 (see [27]) that the only connected algebraic groups in positive characteristic whose algebras of polynomial functions have a normalized integral are the tori (Nagata didn't formulate the results in terms of integrals but in an equivalent form).

In 1964-65 Mumford introduced some basic ideas that were the key to the overcoming of the mentioned drawbacks. In dealing with Geometric Invariant Theory, Mumford introduced a concept weaker than the concept of complete reducibility –that of "geometric reductivity"– and conjectured that in characteristic p every reductive group is geometrically reductive (see [26]).

It was immediately proved by Nagata that for any geometrically reductive group the first theorem on invariants is true (see [28]) and by Nagata and Miyata that any geometrically reductive group is reductive (see [25]).

It took longer to prove Mumford's Conjecture. In 1975 Haboush, using some of Steinberg's ideas about representations of semisimple algebraic groups, proved that every reductive affine algebraic group is geometrically reductive and in that way (because of the results of Nagata just mentioned) settled the problem of finiteness of invariants for all reductive groups (see [13]).

It is interesting to note that the proof of Nagata uses the condition of "geometric reductivity" in an extremely ingenious (but rather obscure) way to make up for the lack of an integral. Even though Mumford's Conjecture has been settled for more that 10 years, there aren't in the literature available proofs that "jump" directly form the reductivity of G to the finiteness of the invariants without using the intermediate step of the geometric reductivity in the same way as did Nagata.

Any attempt to search for a family of algebraic groups for which the first theorem on invariants is true and that is larger than the family of reductive groups was shown to be fruitless by Popov in 1979 (see [32]) by showing thatif G is an affine algebraic group such that for any finitely generated k-algebra A the subalgebra A^G is finitely generated, then G is reductive.

In the next Section we show that a "relative approach" to the problem of the finite generation of invariants can be of a certain interest. We prove the finite generation of invariants for a family of finitely generated algebras under conditions that guarantee the existence of a certain type of "generalized integral" in situations in which the given group is not necessarily reductive.

5 A Relative Approach to Invariants and Integrals

In some special cases we may want to study invariants for non reductive groups. Consider for example the following situation.

Let G be an affine algebraic group (defined over an algebraically closed field of arbitrary characteristic) and K a closed subgroup of G. The problem of giving a representation theoretical condition equivalent to the geometric condition that G/K is affine has been completely solved by Cline, Parshall and Scott (see [7] for the original proof or [8] for a more elementary proof). In the mentioned paper the authors generalize the concept of induced representation to the category of algebraic groups and prove that G/K is an affine variety if and only if the induced representation functor from K-modules to G-modules is exact. If the induction functor is exact we say that K is exact in G.

A first step in the proof that G/K is an affine variety is the proof that the algebra $P(G)^K$ of K-invariant polynomial functions on G is finitely generated.

A possible approach to the finite generation of $P(G)^K$ is the following: the exactness hypothesis is easily seen to be equivalent to $P(G)$ being injective as a K-module. It follows from the very definition of injectivity that if $P(G)$ is injective as a K-module there exists a map $\mathbf{J} : P(K) \to P(G)$ that sends 1 into 1 and that is a morphism of K-modules. This morphism should play the role of a normalized integral and allow us to prove that $P(G)^K$ is a finitely generated algebra.

This approach is relative in the sense that the "normalized integral" takes values in $P(G)$ and it will only help us to prove that *certain* k-algebras (that are related to $P(G)$ in a sense we formalize later) have finitely generated invariants.

The study of this situation in the case of an affine group K acting on an affine variety X and the corresponding proof of the finite generation of the invariants as well as other considerations about the existence of quotient varieties will appear in [9].

In what follows we present a generalization of some of these results to the case of a Hopf Algebra. At the same time we introduce some simplifications of the arguments in [9].

Definiton 5.1 *Let H be a commutative Hopf Algebra defined over a field k and A an H-comodule algebra. An A-integral for H is a morphism of H-comodules $\mathbf{J} : H \to A$. A normalized A-integral is an A-integral such that $\mathbf{J}(1) = 1$.*

Definiton 5.2 *Let A be an H-comodule algebra as above and M a right A-module that is at the same time a right H-comodule. We say that M is an (A, H)-odule if for all $m \in M, a \in A$ we have that $\chi_M(ma) = \chi_M(m)\chi_A(a)$ where χ_A and χ_M denote the corresponding comodule structure maps (if $\xi = \sum a_i \otimes h_i \in A \otimes H$ and $\eta = \sum m_j \otimes k_j \in M \otimes H$, $\eta\xi$ we denotes the following element of $M \otimes H$: $\eta\xi = \sum m_j.a_i \otimes k_j h_i$). In the case in which the (A, H)-odule is also an A-algebra R in such a way that the multiplication map of R and the action of A on R are H-comodule maps, we say that it is an (A, H)-odule algebra.*

For the rest of this section H will be a commutative Hopf Algebra defined over a field k and A a commutative H-comodule algebra. The structure maps for H will be denoted as in Section 4 (and σ will be the antipode). The comodule structure map for A will be denoted as χ_A and the multiplication as μ_A. If M is an arbitrary H-comodule its structure map will be denoted as χ_M and the action of A on M as μ_M.

The role of a Reynolds operator for an (A, H)-odule M is played by the map $\mathcal{R}_M$ constructed below.

Lemma 5.1 *Suppose that H admits a normalized A-integral that we call $\mathbf{J}$ and let M be an arbitrary (A, H)-odule. The k-linear map $\mathcal{R}_M : M \to M$ defined as $\mathcal{R}_M = \mu_M(id \otimes \mathbf{J})(id \otimes \sigma)\chi_M$ verifies:*

1. *$\mathcal{R}_M^2 = \mathcal{R}_M$ and $\mathcal{R}_M(M) = M^H$.*
2. *If $f : M \to N$ is a morphism of (A, H)-odules and $\mathcal{R}_M, \mathcal{R}_N$ are the corresponding maps then : $\mathcal{R}_N f = f\mathcal{R}_M$.*

3. *If $m \in M$ and $a \in A^H$ then $\mathcal{R}_M(ma) = \mathcal{R}_M(m)a$.*

4. *If R is an (A, H)-odule algebra $r \in R$ and $s \in R^H$, we have $\mathcal{R}_R(rs) = \mathcal{R}(r)s$.*

Proof :

1. We want to prove that $\chi_M(\mathcal{R}_M(m)) = \mathcal{R}_M(m)\otimes 1$ for all $m \in M$. Using the fact that μ_M is a morphism of H-comodules and then the definition of normalized integral we deduce that: $\chi_M\mathcal{R}_M = (id \otimes \mu)(\mu_M \otimes id \otimes id)(id \otimes s \otimes id)(id \otimes id \otimes \mathbf{J} \otimes id)(id \otimes id \otimes \Delta)(\chi_M \otimes id)(id \otimes \sigma)\chi_M$.

   Using now the coassociativity and the properties of the antipode we deduce that: $\chi_M\mathcal{R}_M = (id \otimes \mu)(\mu_M \otimes id \otimes id)(id \otimes s \otimes id)(id \otimes id \otimes \mathbf{J} \otimes id)(id \otimes id \otimes \sigma \otimes \sigma)(id \otimes id \otimes s)(id \otimes \Delta \otimes id)(id \otimes \Delta)\chi_M$.

   By direct verification we see that the last expression can be written as follows: $\chi_M\mathcal{R}_M = (id \otimes \mu)(\mu_M \otimes id \otimes id)(id \otimes \mathbf{J} \otimes id \otimes id)(id \otimes \sigma \otimes id \otimes \sigma)(id \otimes id \otimes \Delta)(id \otimes s)(id \otimes \Delta)\chi_M = (\mu_M(id \otimes \mathbf{J}) \otimes id)(id \otimes \sigma \otimes \mu(id \otimes \sigma)\Delta)(id \otimes s)(id \otimes \Delta)\chi_M$.

   Now, $\mu(id \otimes \sigma)\Delta = u\varepsilon$ by the very definition of σ. If we substitute this formula in the last equality for $\chi_M\mathcal{R}_M$ we obtain that: $\chi_M\mathcal{R}_M = (\mu_M(id \otimes \mathbf{J})(id \otimes \sigma) \otimes id)(id \otimes s)(id \otimes u\varepsilon \otimes id)(id \otimes \Delta)\chi_M$.

   Using the fact that $(\varepsilon \otimes id)\Delta = id$ and computing the above expression at an arbitrary $m \in M$ we conclude that $\chi_M\mathcal{R}_M(m) = \mathcal{R}_M(m) \otimes 1$.

   If we start with an element m of M^H we see that $\mathcal{R}_M(m) = \mu_M(id \otimes \mathbf{J})\chi_M(m) = \mu_M(m \otimes \mathbf{J}(1)) = \mu_M(m \otimes 1) = m$.

   In this way the proof of the first assertion is finished.

2. The following chain of equalities follow immediately from the hypothesis about f and will prove our assertion: $\mathcal{R}_N f = \mu_N(id \otimes \mathbf{J}\sigma)\chi_N f = \mu_N(id \otimes \mathbf{J}\sigma)(f \otimes id)\chi_M = \mu_N(f \otimes id)(id \otimes \mathbf{J}\sigma)\chi_M = f\mu_M(id \otimes \mathbf{J}\sigma)\chi_M = f\mathcal{R}_M$.

3. Using the fact that the action of A on M is compatible with the comodule structures on M and A respectively, one gets that $\mathcal{R}_M(ma) = \mu_M(id \otimes \mathbf{J}\sigma)(\chi_M(m)\chi_A(a))$ Using the commutativity of A we conclude that $\mu_M(id \otimes \mathbf{J}\sigma)(\chi_M(m)(a \otimes 1) = (\mu_M(id \otimes \mathbf{J}\sigma)\chi_M(m))a = \mathcal{R}_M(m)a$.

4. The proof of this assertion is similar to the one we just wrote.

Q.E.D.

We have the tools to prove (in a way that is similar to the original proof by Hilbert as sketched in Section 1) a first approximation to the finite generation of the rings of invariants for the (A, H)-odule algebras provided that H admits a normalized A-integral.

Lemma 5.2 *Suppose that A and H are as above and also that A is a Noetherian ring. Let R be a commutative (A, H)-odule algebra that as an A-algebra is finitely generated. Suppose moreover that R is graded by the natural numbers in such a way that $R_0 = A$ and that the structure of H-comodule of R is compatible with the grading. If H has an A-normalized integral then R^H is a finitely generated A^H-algebra.*

Proof : Call $R_+ = \bigoplus_{n>0} R_n$ the null ideal of R and call $\mathcal{I}$ the ideal generated by R_+^H in R. As R is finitely generated over A it is Noetherian, therefore $\mathcal{I}$ can be generated by a finite set of H-fixed elements $\mathcal{F}$ that we can assume are homogeneous. Any $f \in \mathcal{F}$ will be in $f \in R_{d(f)}^H$ with $d(f) > 0$.

We prove by induction on the grading that $R^H = A^H[\mathcal{F}]$. For the elements of degree zero (i.e. the elements of A^H) there is nothing to prove. Suppose we know that for all $n \leq l$ we have that $R_n^H \subset A^H[\mathcal{F}]$. Take $r \in R_{l+1}^H$. As $r \in \mathcal{I}$ we have $r = \sum_{f\in\mathcal{F}} r_f f$ with $r_f \in R$. Now we apply the map $\mathcal{R}_R$ to the above equality and let $\rho_f = \mathcal{R}_R(r_f) \in R^H$. After decomposing each ρ_f into its homogeneous components we can assume that we have an equality of the form $r = \sum_{f\in\mathcal{F}} s_f f$ with each s_f homogenous of positive degree and H-fixed. Comparing degrees in the above equality we deduce that $s_f \in R_{l+1-d(f)}^H$. Using the inductive hypothesis we see that $R_{l+1-d(f)}^H \subset A^H[\mathcal{F}]$ and so that $r \in A^H[\mathcal{F}]$. **Q.E.D.**

The usual method of going from the graded to the non graded situation can also be adapted to our context.

Theorem 5.1 *Let H be a commutative Hopf Algebra and A a Noetherian H-comodule algebra. Suppose also that H has a normalized A-integral. Let R be a commutative (A, H)-comodule algebra that as an A-algebra is finitely generated. Then R^H is a finitely generated A^H-algebra.*

Proof : Let V denote the finite-dimensional H-comodule spanned (as a k-vector space) by a finite set of A-generators of R. Call S the k-symmetric algebra built on V and consider the (A, H)-odule algebra $S \otimes_k A$. Applying the universal property of the symmetric algebra to the map given by the inclusion of V into R, we construct a map of H-comodules from $S \to R$ that after tensoring with the cannonical map from A into R will become a surjective morphism of (A, H)-odule algebras from $S \otimes A$ into R. We call this map Φ. By Lemma 5.1–2. we have that $\mathcal{R}_R\Phi = \Phi\mathcal{R}_{S\otimes A}$. Using the above equality and Lemma 5.1–1 we deduce that the map Φ when restricted to $(S \otimes H)^H$ is an algebra homomorphism *onto* R^H. Applying Lemma 5.2 to $S \otimes A$ we conclude that $(S \otimes A)^H$ is a finitely generated A^H-algebra. Then, the same happens to R^H.

Q.E.D.

In the case in which A^H itself is finitely generated as a k-algebra and all the hypothesis of Theorem 5.1 remain valid we conclude that R^H is finitely generated over k. Clearly if A is finitely generated over k the Noetherian hypothesis of Theorem 5.1 is verified.

So that the (A, H)-odule algebra A is a "testing object" for the validity of the first theorem of invariants in our particular context.

In what follows we describe two particular cases in which the finite generation of A^H as a k-algebra can be guaranteed.

We first recall a result from [10]: If H and K are commutative Hopf Algebras defined over a field k and $\pi : H \to K$ is a surjective normal bialgebra map (see [10] for the definitions) then H is injective when considered as a K module via the usual restricition of scalars functor.

We observed at the beginning of Section 5 (in the particular case of the Hopf Algebra of an affine group) that if H is a Hopf Algebra and A an H-comodule algebra, the injectivity of A as an H-comodule implies the existence of an A-normalized integral for H.

Applying this to the case of K and H as above, we conclude that there exists an H-normalized integral for K.

Now, if H itself has an A-normalized integral by composition we obtain an A-normalized integral for K.

We apply the above considerations to the case in which $H = P(G)$ the algebra of polynomial functions of an affine algebraic group defined over an algebraically closed field k and $K = P(G_u)$ is the algebra of polynomial functions of its unipotent radical.

The result that follows appeared in [9]) in an equivalent context. We write down here the (adapted) proof for the sake of completness.

Theorem 5.2 *Let G be an affine algebraic group defined over an algebraically closed field k and let A be a commutative finitely generated G-module algebra. If $P(G)$ has a normalized A-integral then A^G is a finitely generated k-algebra.*

Proof : Let G_u be the unipotent radical of G. As we observed before $P(G_u)$ has a normalized A-integral that we call $\mathbf{J}$. Following a trick we learnt from [7] we transform the map $\mathbf{J} : P(G_u) \to A$ that is a map of G_u-modules and sends one into one into another map $\mathbf{I} : P(G_u) \to A$ that is a morphism of G_u-modules and of *k-algebras*, in other words a multiplicative A-integral. In this case the map $\mathcal{R}_A : A \to A$ constructed in Lemma 5.1. is an algebra homomorphism that sends A *onto* A^{G_u}.

We conclude then that A^{G_u} is a finitely generated k-algebra. As the quotient G/G_u is reductive, the Nagata-Mumford theory (see [28]) guarantees that $A^G = (A^{G_u})^{G/G_u}$ is finitely generated as a k-algebra.

Q.E.D.

The author doesn't know if the existence of a normalized A-integral for the Hopf Algebra H implies in general that A^H is finitely generated. The above Theorem gives an affirmative answer for the case in which $H = P(G)$ for some affine group G. The obstructions to the generalization of the above proof to the case of an arbitrary commutative Hopf Algebra are multiple. The more serious one seems to be the lack of a theory of "reductive Hopf Algebras".

Let us finally say that another case in which we can prove that the existence of the A-integral and the finite generation of A implies the finite generation of A^H is the case in which A is graded with $A_0 = k$ and the H-comodule structure preserves the grading. The proof will be omitted because it is a copy of the ones already written. We use the integral to construct a map with the required properties going from A to A^H and then we proceed by induction on the degree.

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Walter Ferrer
Current Adress: Centro de Matemática
Facultad de Ciencias
Eduardo Acevedo 1139
C. P. 11200
Montevideo - Uruguay

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NONPARAMETRIC CONSERVATIVE BANDS FOR THE TREND OF GAUSSIAN AR(p) MODELS.

Ricardo Fraiman
and
Gonzalo Pérez Iribarren

Centro de Matemática. Universidad de la República. Montevideo, Uruguay.

RESUMEN

En lo que sigue se proponen bandas de confianza conservativas para la tendencia de un modelo gaussiano, autoregresivo de orden p . Los resultados son válidos para muestras finitas. Las bandas son conservativas en el sentido de que la probabilidad de que las bandas cubran a la funcion considerada es al menos el nivel prefijado.

1. Introducción. In this preprint we look for nonparametric conservative bands for the trend function $g(t)$ of a gaussian stationary autoregressive model of order p based on observations $Y_1, \dots, Y_N$ verifying

$$Y_j = Y_{t_j} = g(t_j) + X_j, \quad a \le t_j \le b, \quad (j=1, \dots, N) \tag{1.1}$$

where $\{ X_j : j \ge 1\}$ is an autoregressive stationary process of order p, i.e.,

$$X_j = \phi_1 X_{j-1} + \phi_2 X_{j-2} + \dots + \phi_p X_{j-p} + U_j \tag{1.2}$$

and $\{ U_j : j \ge 1 \}$ is a gaussian white noise.

Conditions under which there is a stationary solution to equation (1.2) are well known, and will be assummed in what follows, i.e. we will suppose that the coeficients $\phi_1, \ldots, \phi_p$, satisfy

$$h(z)= 1- \phi_1 z - \phi_2 z^2 - \ldots - \phi_p z^p \neq 0 \quad \text{for} \quad |z| \leq 1. \qquad (1.3)$$

The bands will be conservative in the sense that the probability that the true function g lies on the band is at least the prefixed level $1-\alpha$, and the results are valid for finite sample sizes.

The case where $\{ X_j : j \geq 1 \}$ is a gaussian white noise has been considered by Hall and Titterington (1988) where conservative confidence bands for the regression function, related to those of Knafl, Sacks and Ylvisaker (1988) in that they are based on linear (in the data) estimates of the regression function at any given point, are proposed. Their proposal gives confidence bands for which calculation of widths are very easy. Some related work have also be done by Wahba (1983) and by Silverman (1985) from a Bayesian point of view, that leads to spline methods to construct confidence bands.

Basically the idea is to divide the interval where we are working, let say the interval [0,1] into m subintervals I_i, $1 \leq i \leq m$ for m an integer that satisfy $N=2mr$, where N is the sample size. At each of these subintervals using an average of the response variables in the subinterval, construct a confidence interval for the average $\bar{g}_i$ of the regression function g on the cell, and obtain from it a confidence band for the function $g(t)$, $t \in I_i$ using some constrain in the local behaviour of the function g. Finally some bound to the joint coverage probability will provide the conservative bands. Then the width of the confidence band will have two components: a deterministic one from the smoothness constrain or "interpolation error" and another one from the confidence interval for the averages of the response variables or "stochastic component".

With the same idea, Fraiman and Pérez-Iribarren (1991) consider two extensions for the i.i.d. case, one of them that allows overlapping between the observations at each local average and the other one by using local medians instead of local means. In both cases some optimal election of the number of observations at each local average (or local median), and the number of subintervals was possible.

Following the same approach we will provide conservative confidence bands for the trend function g verifying the model defined through (1.1) and (1.2), based on local means in the gaussian case.

<u>2. Main Results.</u> For the sake of simplicity we will begin considering the AR(1) case. Let $X_1, \ldots, X_{2r}$, $r > 1$ be observations verfying

$$X_j = \rho X_{j-1} + U_j \quad (j=1,\ldots,2r) \tag{2.1}$$

$|\rho|<1$, where U_j is a gaussian white noise $E(U_1)=0$, $var(U_1)=\sigma^2 = (1-\rho^2)var(X_j)$.

Define $W_j = X_{r+j+1} - X_{r-j}$ $(j=0,\ldots,r-1)$ and $V_j = W_j - \rho W_{j-1}$ $(j=1,\ldots,r-1)$.

The following lemmas will be proved in the Appendix. Lemma 2.1 provides an estimation of the variance of X_j which is independent of the average of the response variables at each subinterval. Therefore without loss of generality we will assume in what follows that $var(X_j)=1$. Lemma 2.2 will deal with the "stochastic component " while Lemma 2.3 does it with the "deterministic component".

<u>Lemma 2.1</u> If $\{X_j : j=1, \ldots, 2r\}$ verify (2.1) we have that:

a) $E(W_j)=0$, $var(W_j) = 2(1-\rho^{2j+1})$ $(j=0, \ldots, r-1)$, and

$$E(V_j)=0 \qquad \mathrm{var}(V_j) = 2(1-\rho^2) \qquad (j=1, \ldots, r-1).$$

b) $\sum_{i=1}^{2r} x_i$ is independent of W_k $(k=0, \ldots, r-1)$, and therefore independent of the vector $(W_0, W_1, \ldots, W_{r-1})$.

c) $\{ \sum_{i=1}^{2r} x_i, V_1, \ldots, V_{r-1}\}$ is a set of independent random variables.

d) $$\sigma^2 v(\rho,r) = \mathrm{var}\left(\sum_{i=1}^{2r} x_i / (2r)^{1/2} \right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\mathrm{sen}^2 \lambda r}{2r\, \mathrm{sen}^2 \lambda/2} \frac{\sigma^2}{|1-\rho e^{i\lambda}|^2} d\lambda$$
$$\leq (1-|\rho|)^{-2}\sigma^2 = (1+|\rho|)/(1-|\rho|).$$

e) $(\sum_{i=1}^{r-1} V_j^2) / 2(1-\rho^2)$ has a Chi-square distribution with r-1 degrees of freedom, and $\left\{(1-\rho^2)^{1/2}(1-1/r)^{1/2} \sum_{i=1}^{2r} x_i\right\} / \left\{v(\rho,r)\sum_{j=1}^{r-1} V_j^2\right\}^{1/2}$ has a Student distribution with r-1 degrees of freedom.

Let $\rho^* \in (-1,1)$, and define $V_j^* = W_j - \rho^* W_{j-1}$, $\Delta j = g(t_{r+j+1}) - g(t_{r-j})$,

$\tilde{W}_j = Y_{r+j+1} - Y_{r-j} = W_j + \Delta j$ and $\tilde{V}_j = \tilde{W}_j - \rho^* \tilde{W}_{j-1}$.

<u>Lemma 2.2</u> Under the assumptions of Lemma 2.1 we have

a) $P\left(\sum_{j=1}^{r-1} V_j^{*2} \geq a \right) \geq P\left(\sum_{j=1}^{r-1} V_j^2 \geq a \right)$ for all $a \geq 0$.

b) Moreover, we have that $P\left(\sum_{j=1}^{r-1} \tilde{V}_j^2 \geq a \right) \geq P\left(\sum_{j=1}^{r-1} V_j^2 \geq a\right)$ for all $a \geq 0$.

c) If a is such that

$$1 - \alpha = P\left(\left| \left\{(1-\rho^2)^{1/2}(1-1/r)^{1/2} \sum_{i=1}^{2r} x_i\right\} / \sigma\left\{\left(\sum_{j=1}^{r-1} V_j^2 \right) v(\rho,r) \right\}^{1/2} \right| \leq a \right)$$

, and ρ^* is such that $|\rho| < |\rho^*| < 1$ then

$$P\left(\left|(1-|\rho^*|)\,(1-1/r)^{1/2}\sum_{i=1}^{2r} x_i \Big/ \left(\sum_{j=1}^{r-1}\tilde{V}_j^2\right)^{1/2}\right| \le a\right) \ge 1-\alpha .$$

A proof of Lemma 2.3, that will deal with the interpolation error can be found in Hall and Titterington (1988), or in Fraiman and Pérez-Iribarren (1991).

<u>Lemma 2.3</u> If $t_{j+1}-t_j = \delta > 0$ for all $j= 1, \ldots, 2r$ and g verifies a Lipschitz condition of order one on the interval $[t_1, t_{2r}]$ with constant $C > 0$ we have that $| g(t) - \bar{g} | \le C \{ \delta(2r+1)/2\}$ for all $t \in [t_1, t_{2r}]$ where $\bar{g} = (1/2r) \sum_{j=1}^{2r} g(t_j)$.

We are now ready to construct a conservative confidence band for the trend of a gaussian AR(1) process. Let subdivide the interval where we are working, that we will assume without loss of generality the interval [0,1] , into m subintervals I_i $i = 1, \ldots, m$ each of them containing $2r_i$ observations corresponding to values $t_i \in I_i$. Let

$$\bar{Y}_i = (1/2r_i) \sum_{t_j \in I_i} Y(t_j) \qquad (i = 1, \ldots, m).$$

Thus, $\bar{Y}_i = (1/2r_i) \sum_{t_j \in I_i} g(t_j) + (1/2r_i) \sum_{t_j \in I_i} X(t_j) =$

$\bar{g}_i + (1/2r_i) \sum_{t_j \in I_i} X(t_j)$

In order to obtain conservative confidence bands it is reasonable to assume that $\mathrm{var}(X_j)$ and ρ are unknown. However we will need an upper bound of $|\rho|$ as well as an upper bound of the Lipschitz condition constant for the function g. The value $|\rho^*|$ will play the role of the upper bound of $|\rho|$. More precisely we will assume the following hypothesis.

<u>H1</u>. ρ^* is such that $|\rho| < |\rho^*| < 1$ holds.

<u>H2</u>. The function g verifies a Lipschitz condition of order one, at each subinterval I_i, i.e. there exists $C_i > 0$ such that $|g(x)-g(x')| \le C_i |x-x'|$ for all $x,x' \in I_i$, $i=1, \ldots, m$.

For each i=1, ... ,m, let a_i be such that $P(|t_{r_i-1}| \le a_i) = 1-\alpha / m$ where t_{r_i-1} denotes a random variable with Student distribution with r_i-1 degrees of freedom. The following theorem provides the confidence band construction, and is a consequence of Lemma 2.2 (c), Lemma 2.3, and Bonferroni inequality.

Theorem 2.1 Under H1 and H2, if model (2.1) is verified we have that

$$P (g(t) \in J_i \quad t \in I_i, \ i=1, ... , m) \ge 1-\alpha$$

where $J_i = \left[L_i, R_i\right]$, $L_i = \overline{Y}_i - \{a_i\tilde{\sigma}_i /2r_i(1-|\rho^*|)\} \ \{r_i/(r_i-1)\}^{1/2}$ - $\{C_i\delta \ (2r_i+1)/2\}$, $R_i = \overline{Y}_i + \{a_i\tilde{\sigma}_i / 2r_i(1-|\rho^*|)\} \ \{r_i /(r_i-1)\}^{1/2}$ + $\{C_i \ \delta \ (2r_i+1) /2\}$, (i = 1, . . . ,m),
$\tilde{\sigma}_i = (\sum_{j=1}^{r_i-1} \tilde{V}_{j,i}^2)^{1/2}$ and $\tilde{V}_{j,i}$ are the corresponding variables $\tilde{V}_j$ for the I_i interval .

We now turn to the AR(p) case.
Let (Y_j, X_j) j=1, . . . , N be observations verifying (1.1) and (1.2), r>p and consider $W_j = X_{r+j+1} - X_{r-j}$ (j=0, . . . ,r-1) as before.
Define $V_j = W_{j-1+p} - \phi_1 W_{j-2+p} - \cdots - \phi_p W_{j-1}$ (j=1, . . . ,r-p) ; $\rho_l = E(X_{j+l} \ X_j)$
. We may assume without loss of generality that $var(X_j)=1$.
Let $\underset{\sim}{\phi}^* = (\phi_1^*, ..., \phi_p^*)^t$ and $\underset{\sim}{\phi} = (\phi_1, ..., \phi_p)^t$. For the AR(p) case p≥1 H1 will be the following assumption.
H1: $\underset{\sim}{\phi}^*$ is such that $0 \le 1 - \sum_{l=1}^{p} |\phi_l^*| \le 1 - \sum_{l=1}^{p} |\phi_l|$, $\sum_{i=1}^{p} |\phi_i| < 1$.

Remark 2.1. We use the condition $\sum_{l=1}^{p}|\phi_l|<1$ in order to prove (d) of the following Lemma. This condition follows from (1.3) if p=1. Moreover, if $\phi_i \geq 0$ for all i=1,...,p is also a consequence of (1.3). The following lemmas will be proved in the Appendix.

Lemma 2.4 If $\{ X_j : j= 1, \ldots ,2r \}$ verifies (1.2) we have

a) $E(V_j)=0$, $\operatorname{var}(V_j)=2(1-\sum_{l=1}^{p}\phi_l\rho_l)=2\sigma^2$ for $j=1, \ldots ,r-p$ where $\sigma^2= \operatorname{var}(U_1)$.

b) $\sum_{i=1}^{2r} X_i$ is independent of W_k, $(k=0, \ldots ,r-1)$

c) $\{ \sum_{i=1}^{2r} X_i, V_1, \ldots, V_{r-p} \}$ is a set of independent random variables.

d) $\sigma^2 v(\underset{\sim}{\phi},r) = \operatorname{var}\{ \sum_{i=1}^{2r} X_i / (2r)^{1/2} \} =$

$$= (1/2\pi)\int_{-\pi}^{\pi}\{\operatorname{sen}^2\lambda r / (2r \operatorname{sen}^2\lambda/2)\}\left(\sigma^2/ |1 - \sum_{l=1}^{p}\phi_l e^{il\lambda}|^2\right) d\lambda \leq$$

$$\leq \sigma^2/ (1 - \sum_{l=1}^{p}|\phi_l|)^2 \quad \text{if} \quad \sum_{l=1}^{p}|\phi_l| < 1.$$

e) $\sum_{j=1}^{r-p} V_j^2/(2\sigma^2)$ has a chi-square distribution with r-p degrees of freedom, and $(1-p/r)^{1/2} \sum_{i=1}^{2r} X_i / \{(\sum_{j=1}^{r-p} V_j^2)^{1/2} (v(\underset{\sim}{\phi},r))^{1/2}\}$ has a Student distribution with r-p degrees of freedom.

Let $\underset{\sim}{W}_{j-2+p} = (W_{j-2+p}, \ldots , W_{j-1})^t$, $V_j^* = W_{j-1+p} - \underset{\sim}{\phi}^{*t}\underset{\sim}{W}_{j-2+p}$,

$\tilde{W}_j = Y_{r+j+1} - Y_{r-j} = W_j + \Delta_j$ and $\tilde{V}_j = \tilde{W}_{j-1+p} - \phi_1^*\tilde{W}_{j-2+p} - \ldots - \phi_p^*\tilde{W}_{j-1}$.

Lemma 2.5 Under the assumptions of Lemma 2.4 we have

a) $P\left(\sum_{j=1}^{r-1}(V_j^*)^2 \geq a\right) \geq P\left(\sum_{j=1}^{r-1} V_j^2 \geq a\right)$ for all $a \geq 0$.

b) Moreover we have that

$$P\left(\sum_{j=1}^{r-1} \tilde{V}_j^2 \geq a\right) \geq P\left(\sum_{j=1}^{r-1} V_j^2 \geq a\right) \quad \text{for all } a \geq 0.$$

c) Let $\underset{\sim}{\phi}^*$ verifying H1 and a such that

$$1-\alpha = P\left(\{(r-p)/r\}^{1/2} \sum_{i=1}^{2r} X_i / \{(\sum_{j=1}^{r-p} V_j^2)\, v(\phi,r)\}^{1/2}\right).$$

Then $P\left(\left| \{(r-p)/r\}^{1/2} (1-\sum_{l=1}^{p} |\phi_l|)\sum_{i=1}^{2r} X_i \;/(\sum_{j=1}^{r-p} \tilde{V}_j^2)^{1/2}\right| \leq a\right) \geq$

$P\left(\left|\{(r-p)/r\}^{1/2} \sum_{i=1}^{2r} X_i / \{ v(\phi,r) \;(\sum_{j=1}^{r-p} \tilde{V}_j^2)\}^{1/2}\right| \leq a\right) \geq 1-\alpha$.

Finally in the same way as for the AR(1) case we get the following result.

Theorem 2.2 Under H1 and H2, if the model (1.1) and (1.2) is verified we have that if

$J_l = [\, \overline{Y}_l - \{a_l\tilde{\sigma}_l (1-p/r_l)^{1/2}\}/ \{2r_l(1-\sum_{l=1}^{p} |\phi_l^*|)\} - C_l\,\delta\,(2r_l+1)/2 \;;\; \overline{Y}_l +$

$+ \{a_l\tilde{\sigma}_l (1-p/r_l)^{1/2}\}/ \{2r_l(1-\sum_{l=1}^{p} |\phi_l^*|)\} + C_l\,\delta\,(2r_l+1)/2 \,]$

$P(g(t) \in J_l,\ t \in I_l,\ i=1, \ldots, m) \geq 1-\alpha$

where $\tilde{\sigma}_l = (\sum_{j=1}^{r-p} \tilde{V}_{j,i}^2)^{1/2}$ and $\tilde{V}_{j,i}$ are the corresponding $\tilde{V}_j$ variables for the interval I_l; a_l are such that $P(|t_{r_i-p}| \leq a_l) \geq 1-\alpha/m$ and t_{r_i-p} is a random variable with Student distribution with r_i-p degrees of freedom.

3. Some examples. In this section we give some artificial examples from simulated data. Figures 3.1 and 3.2 shows respectively, 2000 observations from a gaussian AR(1) process with $\rho = 0.1$, $\sigma = 0.2$ and trend function $Y = 0.5 + |x-0.5|$ and a 95% conservative

confidence band constructed using 20 subintervals, a Lipschitz constant c=1 and a overestimation for ρ , $\rho^* = 0.11$. The graph in Figure a 3.2 corresponds to a step functions taking values B_i^+ and B_i^- , i=1, . . . ,20 , where B_i^+ and B_i^- denotes the upper and lower bounds of a 95% confidence band, which are plotted by joining succesive values by straight lines. The graph of the true function is also given. We can see, for instance, that there is clear evidence against the hypothesis that the true trend function is linear.

Figure 3.1 Figure 3.2

Figures 3.3 corresponds to 400 data of an AR(2) process with $\phi_1=0.1$, $\phi_2=-0.05$, $\sigma=0.3$ and trend function $y= | x-0.5 |$. A 95% uniform conservative confidence band constructed using 8 subintervals, a Lipschitz constant c=1 and a overestimation for $|\phi_1| + |\phi_2|$, $|\phi_1^*| + |\phi_2^*| = 0.165$ is given in Figure 3.4.

Figure 3.3 Figure 3.4

Finally Figure 3.5 and 3.6 corresponds to 2400 data from an AR(2) process with $\phi_1=0.25$, $\phi_2=0.1$, $\sigma=0.2$ and trend function $y= 0.5 + | x-0.5 |$. The overestimation of $|\phi_1| + |\phi_2|$ used on figure 3.6 was $|\phi_1^*| + |\phi_2^*| = 0.37$, and we use 10 subintervals and c=1.

Figure 3.5 Figure 3.6

Appendix. We will give now the proofs of Lemmas 2.2, 2.4 and 2.5.

Proof of Lemma 2.4

a) $E(V_j) = 0$ since $E(W_j) = 0$. $E(W_j W_{j+i}) = 2\rho_i - 2\rho_{2j+i+1}$ $i \geq 0$. if we define $\rho_0 = 1$.

Therefore

$$\rho_i - \sum_{l=1}^{p} \phi_l \rho_{|i-l|} = 0, \quad \text{implies that}$$

$E(W_{k-i}(W_k - \phi_1 W_{k-1} - \dots - \phi_p W_{k-p})) =$

$$2(\rho_i - \rho_{2k-i+1}) - 2\sum_{l=1}^{p} \phi_l(\rho_{|i-l|} - \rho_{2k-i-l+1}) = 0. \quad (4.1)$$

Thus,

$$\mathrm{Var}(V_j) = E\left(\left(W_{j-1+p} - \sum_{l'=1}^{p} \phi_{l'} W_{j-l'-1+p} \right) \left(W_{j-1+p} - \sum_{l=1}^{p} \phi_l W_{j-l-1+p} \right) \right)$$

$$= E\left(W_{j-1+p} \left(W_{j-1+p} - \sum_{l=1}^{p} \phi_l W_{j-l-1+p} \right) \right) =$$

$$2(1 - \rho_{2j+2p-1}) - 2\sum_{l=1}^{p} \phi_l(\rho_l - \rho_{2j+2p-l-1}) = 2\left(1 - \sum_{l=1}^{p} \phi_l \rho_l\right) = 2\sigma^2 \quad (j = 1,, \; r-p).$$

b) $$E\left(\left(\sum_{j=1}^{2r} X_j \right) (X_{r+k+1} - X_{r-k}) \right) = \sum_{j=1}^{2r} \rho_{|r+k+1-j|} - \sum_{j=1}^{2r} \rho_{|r-k-j|} = 0$$

since $\sum_{j=1}^{2r} \rho_{|r+k+1-j|} = \sum_{j'=1}^{2r} \rho_{|k+j'-r|}$ taking $j' = 2r-j+1$; and independence follows since the variables are gaussian.

c) Since $(V_1, \dots, V_{r-p})$ depends on the vector $(W_0, \dots, W_{r-p})$ which is independent of $\sum_{i=1}^{2r} X_i$, (c) will holds if we show that $E(V_j V_{j+i}) = 0$ for $i \geq 1$, which follows easily from (4.1).

d) If $f(\lambda)$ denotes the spectral density of a stationary process $\{X_t : t \geq 1\}$ we have that

$$\text{var}\left(\sum_{t=1}^{T} X_t / T^{1/2} \right) = \int_{-\pi}^{\pi} \frac{\text{sen}^2\lambda T/2}{T\,\text{sen}^2\lambda/2} f(\lambda)\, d\lambda$$

(see for instance, Anderson ((1971) pag 459). Therefore

$$\text{var}\left(\sum_{i=1}^{2r} X_i / (2r)^{1/2} \right) = 1/2\pi \int_{-\pi}^{\pi} \frac{\text{sen}^2\lambda r}{2r\,\text{sen}^2\lambda/2} \sigma^2 \left|1-\sum_{l=1}^{p} \phi_l e^{i\lambda l}\right|^{-2} d\lambda$$

$$\leq \sigma^2 \left(1-\sum_{l=1}^{p} |\phi_l|\right)^{-2} 1/2\pi \int_{-\pi}^{\pi} \frac{\text{sen}^2\lambda r}{2r\,\text{sen}^2\lambda/2} d\lambda = \sigma^2 \left(1-\sum_{l=1}^{p} |\phi_l|\right)^{-2}$$

since $\left| 1-\sum_{l=1}^{p} \phi_l e^{i\lambda l}\right|^2 \geq \{1- R_e(\sum_{l=1}^{p} \phi_l e^{i\lambda l})\}^2 \geq (1-\sum_{l=1}^{p} |\phi_l|)^2$

and $\sum_{l=1}^{p} |\phi_l| < 1$.

e) Follows from a) and c). ⋄

<u>Proof of Lemma 2.2.</u> a) We have that

$$\sum_{j=1}^{r-1} V_j^{*2} = \sum_{j=1}^{r-1} \{V_j + (\varrho-\varrho^*)\, W_{j-1}\}^2.$$

Let $Z_1 = (\varrho-\varrho^*)W_0$

$Z_2 = (\varrho-\varrho^*)W_1 = (\varrho-\varrho^*)V_1 + \varrho Z_1$

. . . .

$Z_j = (\varrho-\varrho^*)W_{j-1} = (\varrho-\varrho^*) \sum_{i=1}^{j-1} \varrho^{i-1} V_{j-i} + \varrho^{j-1} Z_1 \quad (j=2,\ldots,r-1).$

$$\sum_{j=1}^{r-1} V_j^{*2} = \sum_{j=1}^{r-1} [V_j + (\varrho-\varrho^*) \sum_{i=1}^{j-1} \varrho^{i-1} V_{j-i} + \varrho^{j-1} Z_1]^2 = f_0 (V_1, \ldots, V_{r-1}, Z_1) \tag{4.2}$$

Let $B = \{ \sum_{j=1}^{r-1} V_j^{*2} \geq a \}$, $1\!\!1_B$ the indicator function of the set B and

$f_K (V_1, \ldots, V_{r-k-1}, Z_1) = \sum_{j=1}^{r-k-1} V_j^{*2}$ defined as in (4.2). for, k= 0,...,r-2.

$P(B) = E(E (1\!\!1_B / V_1, \ldots, V_{r-2}, Z_1)) =$

$= E(E(1\!\!1_{[a,+\infty)} \{f_0 (V_1, \ldots, V_{r-1}, Z_1)\} / V_1, \ldots, V_{r-2}, Z_1)) \geq$

$E(E(1\!\!1_{[a,+\infty)} \{f_1 (V_1, \ldots, V_{r-2}, Z_1) + V_{r-1}^2\} / V_1, \ldots, V_{r-2}, Z_1))$

since conditional to $V_1, \ldots, V_{r-2}, Z_1$ we have that $f_0 (V_1, \ldots, V_{r-1}, Z_1) = (V_{r-1} + c)^2 + b$ with $b = f_1 (V_1, \ldots, V_{r-2}, Z_1)$ and

$$c = (\rho - \rho^*) \sum_{i=1}^{r-2} \rho^{i-1} . V_{r-1-i} + \rho^{r-2} Z_1 \qquad \text{and therefore}$$

$$P(V_{r-1}^2 + b \geq a) \leq P((V_{r-1} + c)^2 + b \geq a) \tag{4.3}$$

which follows from the fact that V_{r-1} is a zero mean normally distributed random variable. Therefore

$$E(1\!\!1_{[a,+\infty)} \{f_0 (V_1, \ldots, V_{r-1}, Z_1)\}) \geq E(1\!\!1_{[a,+\infty)} \{f_1 (V_1, \ldots, V_{r-2}, Z_1) + V_{r-1}^2\})$$

$$= E(E(1\!\!1_{[a,+\infty)} \{f_1 (V_1, \ldots, V_{r-2}, Z_1) + V_{r-1}^2\} / V_1, \ldots, V_{r-3}, Z_1, V_{r-1}))$$

$$\geq E(E(1\!\!1_{[a,+\infty)} \{f_2 (V_1, \ldots, V_{r-3}, Z_1) + V_{r-2}^2 + V_{r-1}^2\} / V_1, \ldots, V_{r-3}, Z_1, V_{r-1}))$$

by using the same argument, so we get that

$$E(1\!\!1_{[a,+\infty)} \{f_0 (V_1, \ldots, V_{r-1}, Z_1\}) \geq$$
$$E(1\!\!1_{[a,+\infty)} \{f_2 (V_1, \ldots, V_{r-3}, Z_1) + V_{r-2}^2 + V_{r-1}^2\}).$$

Finally ve get a) by iterating the argument already used.
With analogous argument we get b)
c) will be shown directly in Lemma 2.5. ◆

<u>Proof of Lemma 2.5.</u> a) Let $\underset{\sim}{W}_{p-1} = (W_{p-1}, W_{p-2}, \ldots, W_o)$, B be the backward operator and $P(B, \phi) = \sum_{i=1}^{p} \phi_i B^i$. We will first prove that for all j

$$V_j^* = V_j + a_j (V_1, \ldots, V_{j-1}, \underset{\sim}{W}_{p-1})$$

where $a_j : R^{j+p-1} \to R$ is a real function. Moreover, is a linear function in all variables.

Effectively, we have that

$$V_1^* = \{I-P(B,\phi^*)\}W_p = \{I-P(B,\phi)\}W_p + P(B,\phi^*-\phi)W_p = V_1 + a_1(\underset{\sim}{W}_{p-1}) .$$

$$V_2^* = \{I-P(B,\phi^*)\}W_{p+1} = V_2 + P(B,\phi^*-\phi)W_{p+1} =$$

$$= V_2 + (\phi_1^*-\phi_1)W_p + (\phi_2^*-\phi_2)W_{p-1} + \ldots + (\phi_p^*-\phi_p)W_1 =$$

$$V_2 + (\phi_1^*-\phi_1)V_1 + (\phi_1^*-\phi_1)\, P(B,\phi)W_p + (\phi_2^*-\phi_2)W_{p-1} + \ldots + (\phi_p^*-\phi_p)W_1 =$$

$$V_2 + a_2(V_1,\underset{\sim}{W}_{p-1})$$

since $V_1 = W_p - P(B,\phi)W_p$.

Finally, since

$$V_{j+1}^* = \{I-P(B,\phi^*)\}W_{p+j} = V_{j+1} + P(B,\phi^*-\phi)W_{p+j} =$$

$$V_{j+1} + (\phi_1^*-\phi_1)W_{p+j-1} + \ldots + (\phi_p^*-\phi_p)W_j$$

the desired result follows by using recursively

that $W_{p+j-s} = V_{j-s+1} - \sum_{i=1}^{p} \phi_i W_{p+j-s-i}$ as in V_2^*.

Now we get a) using a similar argument as those used in the AR(1) case (Lemma 2.2), where the function $a_j(V_1, \ldots, V_{j-1},\underset{\sim}{W}_{p-1})$ plays the role of $(\rho-\rho^*)\sum_{i=1}^{j-1} \rho^{i-1} V_{j-i} + \rho^{j-1} Z_1$.

b) follows as in Lemma 2.2.

c) Let $t_{r-p} = \sigma \{ (r-p)/r \}^{1/2} |\sum_{i=1}^{2r} X_i| \left\{(\mathrm{Var}(\sum_{i=1}^{2r} X_i)/2r)\right\}^{-1/2} \left(\sum_{j=1}^{r-p} V_j^2\right)^{-1/2}$

Then, Lemma 2.4 d) implies that

$$t_{r-p} \geq \{ (r-p)/r \}^{1/2} \left| \sum_{i=1}^{2r} X_i \right| \left(\sum_{j=1}^{r-p} V_j^2 \right)^{-1/2} \left(1 - \sum_{r=1}^{p} |\phi_r|\right)$$

and the conclusion follows easily from b) and Lemma 2.1 b) and c) ♦

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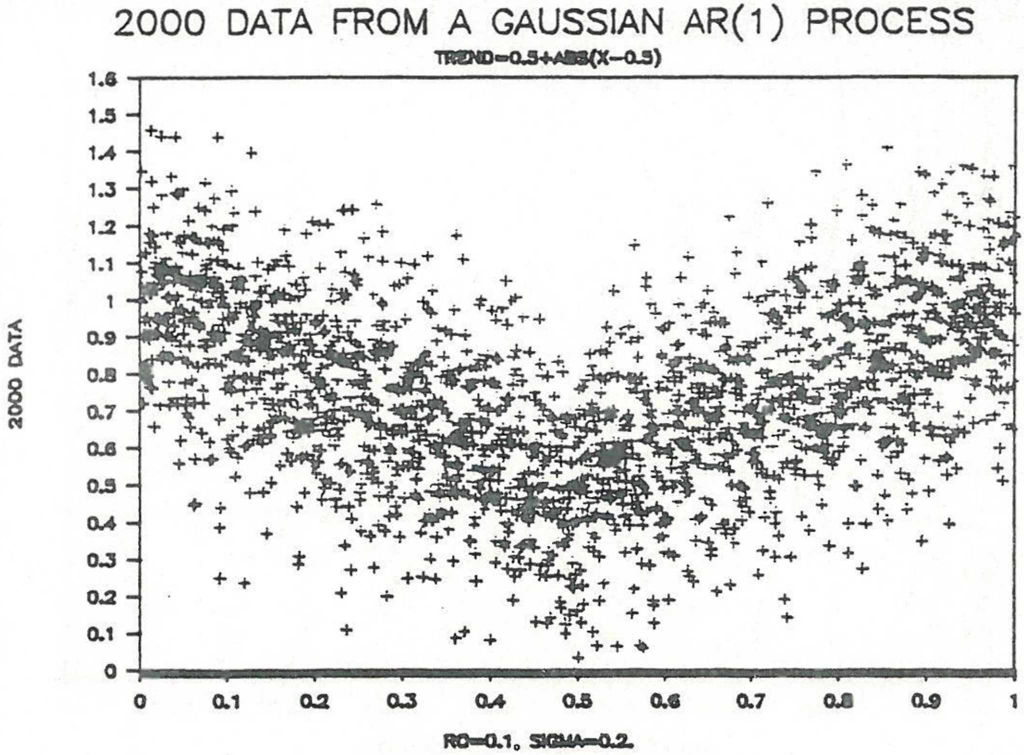

Figure 3.1

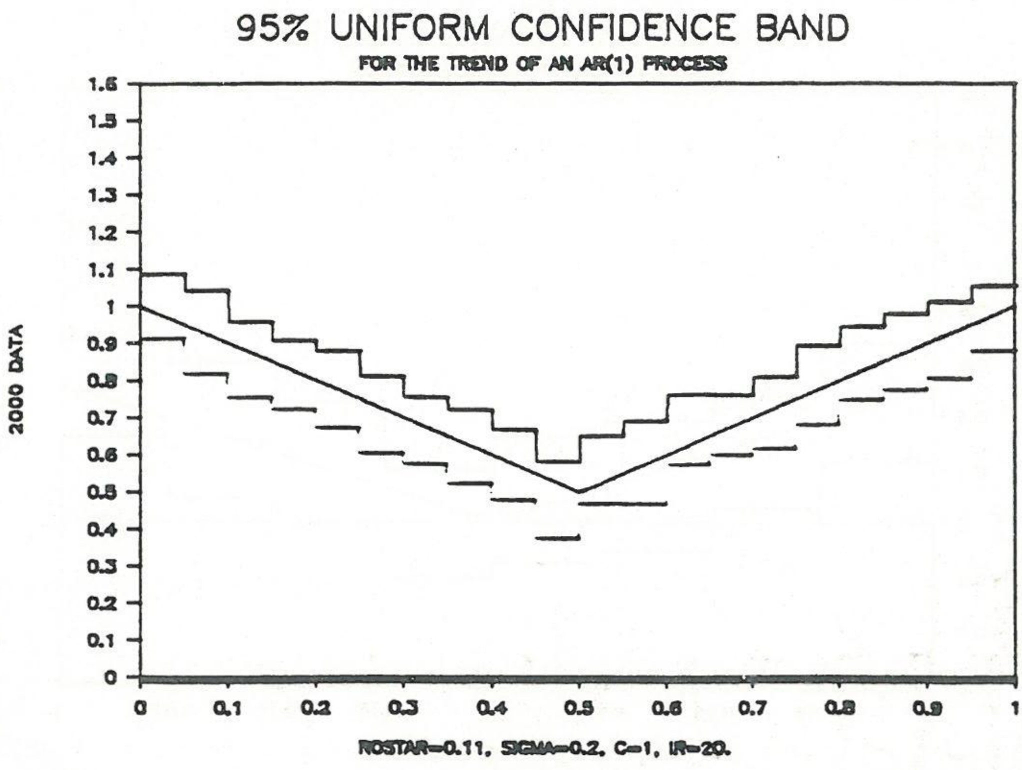

Figure 3.2

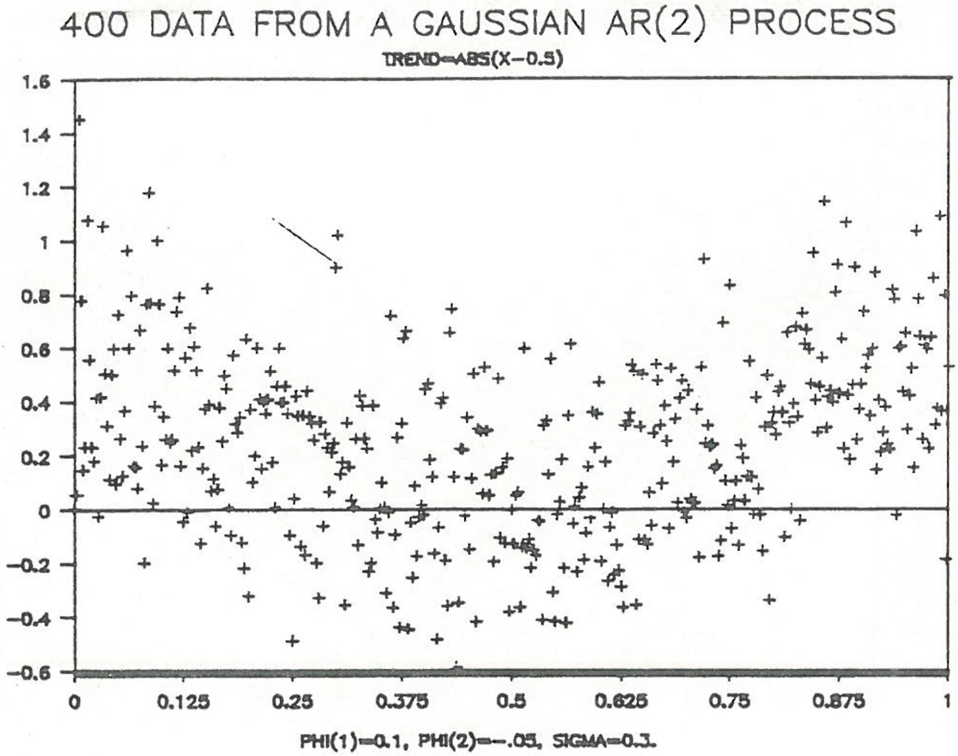

Figure 3.3

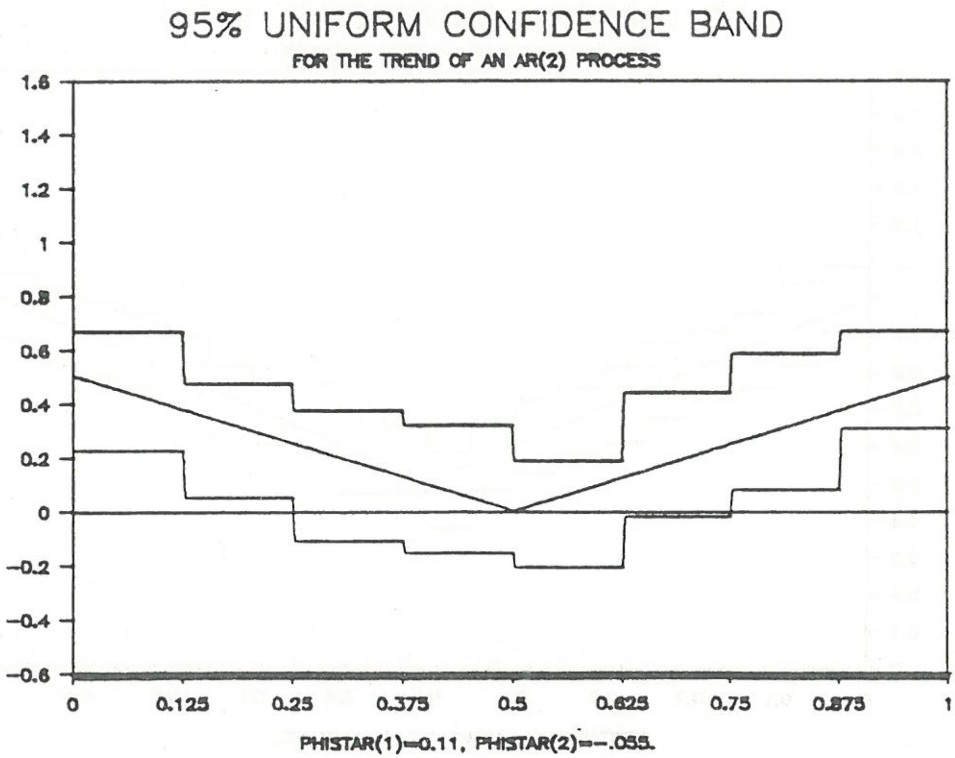

Figure 3.4

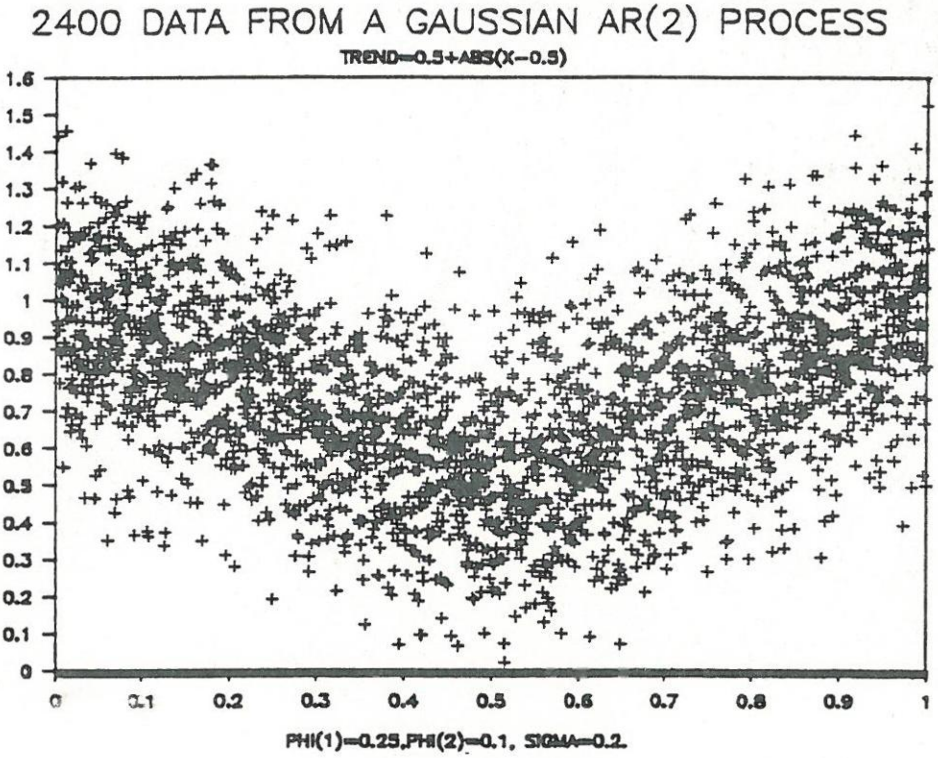

Figure 3.5

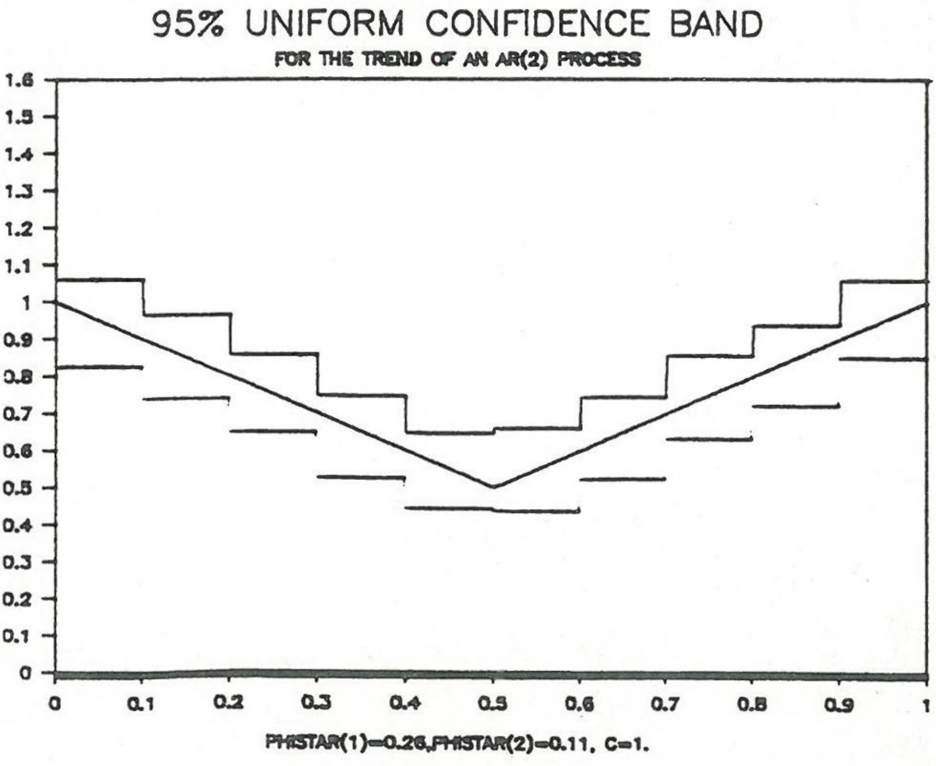

Figure 3.6

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ON STABLE AND UNSTABLE SETS

by J. Lewowicz and J. Tolosa

0. Introduction

Let $f:M\longrightarrow M$ be an homeomorphism M of the smooth compact connected riemannian manifold M. For $\varepsilon > 0$, $x \in M$ call

$$S_\varepsilon(x) = \left\{y \in M: \text{dist}(f^n(x),f^n(y)) \le \varepsilon,\ n \ge 0\right\}$$

and

$$V_\varepsilon(x) = \left\{y \in M: \text{dist}(f^n(x),f^n(y)) \le \varepsilon,\ n \le 0\right\}$$

the ε-stable and ε-unstable sets of x.

Stable and unstable sets are basic elements of the dynamical structure of f and a fundamental tool to face problems of classification of dynamical systems under conjugacy. See, for intance, [F], for the case of Anosov diffeomorphisms, and [H], [L] in connection with the topological equivalence of expansive homeomorphisms of surfaces. In case f is expansive, for any $x \in M$, these stable and unstable sets contain non-trivial (infinite) connected pieces.

In this paper we obtain some general results on the existence of such connected pieces at each $x \in M$ (Proposition 1.1). When x is a periodic point which is not a repellor (attractor) it is easy to show that $S_\varepsilon(x)$ $(U_\varepsilon(x))$ contains such a piece, for any $\varepsilon > 0$. However, points x in a minimal set may have trivial $S_\varepsilon(x)$ and $U_\varepsilon(x)$ for small $\varepsilon > 0$. Consider the Denjois map of S^1, i.e., take a rotation of S^1 by an angle $2\pi\alpha$, where α is irrational, and replace the points of a dense orbit $\{x_n,\ n \in \mathbb{Z}\}$ by intervals of size decreasing with $|n|$, in order to get a new space also homeomorphic to S^1. The Denjois transformation may be defined by asigning to each point that was not on the added intervals, the previous image under the rotation, and mapping linearly the interval we put instead of x_n, into the one replacing x_{n+1}, $n \in \mathbb{Z}$. Any two points which do not lie in the same added interval will be, under some positive and some negative iteration at a distance larger than the length of the interval replacing x_0. In fact, between these two points we find, in the original rotation, positive and negative iterates of x_0. Thus, for any x in the

Denjois -minimal set which is not an end-point of some added interval, $S_\varepsilon(x) = U_\varepsilon(x) = \{x\}$ if $\varepsilon > 0$, is small enough.

For each end point of the interval replacing x_0, that also belongs to the Denjois minmal set, there is a non-trivial connected set which is at the same time the ε-stable and ε-unstable set (ε small) of it. Moreover, this connected set has the property that its diameter decreases under positive and negative iteration. Proposition 1.1 shows that if for arbitrarily small ε no limit point of f has an ε-stable (and ε-unstable) set with this property, then for each $x \in M$ which is not a periodic repellor (attractor) there is a non-trivial connected set included in $S_\varepsilon(x)$ (resp. $U_\varepsilon(x)$). Theorem 2.2 shows that this property on the limit set of f is C^0-gene- ric; thus, for f in a C^0-residual subset of Hom (M), and each $x \in M$, $S_\varepsilon(x)$, $U_\varepsilon(x)$ contain non-trivial connected pieces.

Let now dim M=2; the description of local stable and unstable sets and the classification results of [H], [L] are based chiefly on the existence of those connected pieces, and on the fact that two such pieces meet at most at one point. As a matter of fact the same description of local stable and unstable sets may be obtained, even for non-expansive f, at points where the above mentioned properties of these stable and unstable pieces hold for them and for neighboring points. This is the case for instance if we take the homeomorphism of S^2 defined, after indentifying, x to -x on T^2,applying the usual linear Anosov map $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ of T^2. We get a non-expansive homeomorphism of S^2 which, except at the image of branch points under the canonical projection, the stable and unstable sets are topological manifolds.

We show that for f in a C^0 residual set of Hom (M), if f has a non-trivial attractor $C \subset \Omega(f)$, these connected pieces $C_\varepsilon(x) \subset S_\varepsilon(x)$, $D_\varepsilon(x) \subset U_\varepsilon(x)$ meet only at x, provided $x \in C$. Then the arguments in [L] permit to show to $D_\varepsilon(x)$ is an arc.

Thus, for almost all attractos, the connected unstable sets of each of its points are arcs.

Finally, we want to thank the participants of the Seminar on Dynamical Systems of the IMERL, for useful conversations on these topics.

1. General results

Let M be a compact conneced smooth riemannian manifold and f a homeomorphism of M onto M. Let $\omega(f)(\alpha(f))$ denote the set of all ω-limit points (α-limit points) of f, i.e.

$$\omega(f) = \bigcup_{x \in M} \omega(x) \quad , \quad \alpha(f) = \bigcup_{x \in M} \alpha(x)$$

where $\omega(x)(\alpha(x))$ is the set of ω-limit points (resp. α-limit points) of x. Obviously $\omega(f) \cup \alpha(f) \subset \Omega(f)$; as usual $\Omega(f)$ stands for the set of non-wandering points of f.

Proposition 1.1

Assume there is a sequence of positive numbers ρ_m, $\rho_m \to 0$, such that for any $\rho = \rho_m$, there is no point $x \in \omega(f)$ and connected set C containing x such that dist $(f^n(x), f^n(y)) \le \rho$ for every $y \in C$ and all $n \in Z$ and that dist $(x,z) = \rho$, for some $z \in C$. Then, for any $\varepsilon > 0$ and $x \in M$, there is a compact connected set $C_\varepsilon(x)$, $x \in C_\varepsilon(x) \neq \{x\}$ such that $\mathrm{dist}(f^n(x), f^n(y)) \le \varepsilon$ for every $y \in C_\varepsilon(x)$ and all $n \ge 0$, unless x is a periodic repellor.

Proof Let ε be a positive number and $x \in M$; we shall assume first that for some m with $\rho_m = \rho < \varepsilon$, f has the following property: for each k=1,2,..., there is $n_k > 0$ such that, for some ν, $0 \le \nu \le n_k$, $f^{-\nu}(B_{1/k}(f^{n_k}(x)))$ is not included in $B_\rho(f^{n_k-\nu}(x))$. (For $\sigma > 0$, $x \in M$, $B_\sigma(x)$ denotes the ball $\left\{y \in M: \mathrm{dist}(x,y) \le \sigma\right\}$).

Let $y \in B_{1/k}(f^{n_k}(x))$ be such that $f^{-\nu}(y) \notin B_\rho(f^{n_k-\nu}(x))$. Take an arc joining $f^{n_k}(x)$ to y within $B_{1/k}(f^{n_k}(x))$, say $a_k(t)$, $0 \le t \le 1$, $a_k(0) = f^{n_k}(x)$, $a_k(1) = y$, and let t^*_k be the supremum of those $t \in [0,1]$ such that $f^{-\nu}(a_k([0,t]))$ is contained in the interior of $B_\rho(f^{n_k-\nu}(x))$ for every ν, $0 \le \nu \le n_k$. Thus, for these ν,

$$f^{-\nu}(a_k([0,t^*_k])) \subset B_\rho(f^{n_k-\nu}(x)),$$

and for some ν_k, $0 \le \nu_k \le n_k$,

$$f^{-\nu_k}(a_k([0,t^\circ_k])) \cap \partial B_\rho(f^{n_k-\nu_k}(x)) \neq \phi$$

For each k, we choose then such a ν_k and show that $\lim_{k\to\infty} \nu_k = +\infty$, In fact, if some sub-sequence of ν_k were bounded, we would have that for infinitely many k, $\nu_k = N$, for some fixed N, and therefore f^N would map sets of diameter at least ρ onto sets of arbitrarily small diameter, which is absurd.

On the other hand, if $n_k - \nu_k$ were unbounded we could find $y \in w_f(x)$ and a compact connected set C, $y \in C$, dist $(y,z) = \rho$, for some $z \in C$, such that dist $(f^n(y), f^n(u)) \le \rho$ for every $u \in C$ and all $n \in Z$, in contradiction with the hypothesis of the proposition. Indeed, such a connected set C may be obtained as follows: assume that $f^{n_k-\nu_k}(x))$ converges to say, y (the construction is the same in case we have to replace $n_k - \nu_k$ by a convergent sub-sequence) and take

$$\mathbb{C} = \bigcap_{k=1}^{\infty} \text{clos}\left(\bigcup_{j\ge k} f^{-\nu_j}(a_j([0,t^\bullet_j]))\right)$$

Thus, $n_k - \nu_k$ is bounded, and therefore the arcs $f^{-n_k}(a_k([0,t^\bullet_k])$ have diameters bounded away from zero. The set

$$\mathbb{C}_\varepsilon(x) = \bigcap_{k=1}^{\infty} \text{clos}\left(\bigcup_{j=k}^{\infty} f^{-n_j}(a_j([0,t^\bullet_j]))\right)$$

satisfies clearly the requirements of the thesis of the proposition.

Let us suppose now that the assumption made in the first paragraph of this proof does not hold. Then, for every m with $\rho_m = \rho < \varepsilon$, there is $k = k(m) > 0$ such that for every $n \ge 0$.

$$f^{-\nu}\left(B_{1/k}(f^n(x))\right) \subset B_\rho(f^{n-\nu}(x))$$

if $o \le \nu \le n$. Consequently, for any $y \in w_f(x)$, we have that

$$f^n(B_{1/k}(y)) \subset B_\rho(f^n(y)),\ n \le 0$$

As $\rho_m \to 0$, we get that $w_f(x)$ is uniformly Lyapunov stable in the past. Take $y \in w_f(x)$; since $\text{dist}\left(x, \text{clos}(\{f^n(y) : n \in \mathbb{Z}\})\right) > 0$ contradicts the stability in the past of $w_f(x)$, it follows that $x \in w_f(x)$ and that $w_f(x)$ is a minimal set which, because of its uniform stability properties, consists of almost-periodic motions ([N.S] p. 390).

If x is not a periodic point, we choose $\rho_m < \varepsilon$, the corresponding $k_m = k$ and a point $y \in B_{1/k}(x)$, $y \neq x$, $y \in w_f(x)$. Join x to y through an arc contained in $B_{1/k}(x)$, and let $n_i \longrightarrow -\infty$ be a sequence of negative integers such that $f^{n_i}(x) \longrightarrow x$. Since on account of the uniform stability of $w_f(x)$ in both senses, the diameters of the n_i-iterates of this arcs are bounded away from zero, we take $\mathbb{C}_\varepsilon(x)$ as the usual intersection, for $i \geq 0$, of the closures of the unions of the n_j-iterates of the arc, $j \geq i$.

Clearly, $\mathbb{C}_\varepsilon(x)$ satisfies the required properties.

If x is periodic, and the diameters of $f^{-n}(B_{1/k}(x))$, $n \leq 0$ do not tend to zero, the previous arguments apply and permit to construct a set $\mathbb{C}_\varepsilon(x)$ as required. This complete the proof of the proposition.

Remark The same arguments prove that, unless x is a periodic attractor, there exits a compact connected set $D_\varepsilon(x)$, $x \in D_\varepsilon(x) \neq \{x\}$, such that $\mathrm{dist}(f^n(x), f^n(y)) \leq \varepsilon$ for every $y \in D_\varepsilon(x)$ and $n \leq 0$.

2. Generic Properties

Let M and f be as before. Call

$$S^f_\varepsilon(x) = \left\{y \in M: \text{dist}(f^n(x), f^n(y)) \le \varepsilon,\ n \ge 0\right\}$$

and $$U^f_\varepsilon(x) = \left\{y \in M: \text{dist}(f^n(x), f^n(y)) \le \varepsilon,\ n \le 0\right\}$$

Let f satisfy axiom A. Since every basic set of f is isolated and $f/\Omega(f)$ expansive, we may choose $\varepsilon = \varepsilon_f > 0$, such that $S_\varepsilon(x) \cap U_\varepsilon(x) = \{x\}$ for evry $x \in \Omega(f)$. For every f satisfying axiom A we choose once for all such an ε_f.

<u>Lemma 2.1</u>

Let f satisfy the axiom A and strong-transversality conditions. Then for every m=1,2,..., there is a C^0-neighbourhood $\mathcal{U}(f,m)$ of f such that if $g \in \mathcal{U}(f,m)$, for any $x \in \Omega(g)$ we have that

$$\left(S^g_\varepsilon(x) - S^g_{\varepsilon/m}(x)\right) \cap U^g_\varepsilon(x) = \phi,$$

where $\varepsilon = \varepsilon_f$.

<u>Proof</u> Arguing by contradiction, let us assume that for some $m > 0$ there is a sequence $\{g_\nu\}$ of homeomorphisms of M that converges to f in the C^0 topology and such that for each $\nu = 1,2,\ldots$, there exists $x_\nu \in \Omega(g_\nu)$, and

$$y_\nu \in \left(S^{g_\nu}_\varepsilon(x_\nu) - S^{g_\nu}_{\varepsilon/m}(x_\nu)\right) \cap U^{g_\nu}_\varepsilon(x) .$$

For these ν, let h_ν denote a semi-conjugacy between f and g_ν, i.e.. a continuous map of M onto M, such that $f\circ h_\nu = h_\nu \circ g_\nu$. Furthermore let the h_ν converge in the C^0 topology to the identity map of M [Hu].

We have that $\text{dist}\left(g^n_\nu(x_\nu),\ g^n_\nu(y_\nu)\right) \le \varepsilon$ for every $n \in \mathbb{Z}$ and that

$$\text{dist}\left(g_\nu^{n_\nu}(x_\nu), g_\nu^{n_\nu}(y_\nu)\right) > \varepsilon/m,$$

for some $n_\nu > 0$. Let $z_\nu = g_\nu^{n_\nu}(x_\nu)$, $u_\nu = g_\nu^{n_\nu}(y_\nu)$ and call (z_∞, u_∞) a limite pair of (z_ν, u_ν). Clearly $z_\infty \ne u_\infty$ and dist

$(f^n(z_\infty), f^n(u_\infty)) \le \varepsilon$, $n \in \mathbb{Z}$. But since $z_\nu \in \Omega(g_\nu)$, $h_\nu(\Omega(g_\nu)) \subset \Omega(f)$ and $\mathrm{dist}(z_\nu, h_\nu(z_\nu)) \to 0$ we get that $h_\nu(z_\nu) \to z_\infty$, and that $z_\infty \in \Omega(f)$; a contradiction.

Theorem 2.2

There is a C^0-residual set Σ such that if $g \in \Sigma$, $\varepsilon > 0$, and $x \in M$, then $S^g_\varepsilon(x)$ ($U^g_\varepsilon(x)$) contains a compact connected set $C_\varepsilon(x)$ ($D_\varepsilon(x)$), $x \in C_\varepsilon(x) \neq \{x\}$ ($x \in D_\varepsilon(x) \neq \{x\}$), unless x is a periodic repellor (resp. attractor).

Proof For f satisfying the axiom A and strong transversality conditions, take the chosen $\varepsilon_f > 0$ such that for $x \in \Omega(f)$, $S^f_\varepsilon(x) \cap U^f_\varepsilon(x) = \{x\}$ where, as before, $\varepsilon = \varepsilon_f$. Let m be a positive integer and let $\mathcal{U}(f,m)$ be the C^0-neighbourhood of f given by Lemma 2.1. call $\mathcal{N}_m$ the union of the $\mathcal{U}(f,m)$ for all f satisfying the above mentioned conditions; then $\Sigma = \bigcap_{m=1}^{\infty} \mathcal{N}_m$ is a C^0-residual set [S].

If $g \in \Sigma$, for each m, g belongs to some $\mathcal{U}(f,m)$. Choose $\rho_m > 0$, $\frac{\varepsilon}{m} < \rho_m < \varepsilon$ in such a way that, when $m \to \infty$, $\lim \rho_m = 0$.

Then, if $x \in \Omega(g)$ and C is a compact connected set containing x such that $\mathrm{dist}(g^n(x), g^n(y) \le \rho_m$ for every $y \in C$ and all $n \in \mathbb{Z}$, we have that $\mathrm{dist}(x,z) < \rho_m$ for any $z \in C$, for otherwise, $z \in \left(S^g_\varepsilon(x) - S^g_{\varepsilon/m}(x)\right) \cap U^g_\varepsilon(x)$ in contradiction with Lemma 2.1. Therefore the thesis of the theorem follows from Proposition 1.1 and the fact that there is a C^0-residual subset of Hom M, such that each homeomorphism in this set has no periodic attractors or repellors [PPSS].

3. The size of stable and unstable sets

In [L_1], section 1, it is shown that if f is an Anosov diffeomorphism of a compact connected riemannian manifold there exists a positive integer m such that, either $\|(f^m)'_x u\| \geq 2\|u\|$ or $\|(f^{-m})'_x u\| \geq 2\|u\|$, for each $u \in T_xM$ and every $x \in M$. But the arguments there, also show the existence of such an m, with the same property on the restriction of a diffeomorphism f of M, to a compact f-invariant hyperbolic subset C of M. Let $A: \bigcup_{x \in C} T_xM \to \mathbb{R}$ be the positive quadratic form defined, for $u \in T_xM$, $x \in C$, by $A(u) = \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \|(f^{-i+j})'_x u\|^2$. Then it is easy to check that

$$A(f'_x u) - 2A(u) + A((f^{-1})'_x u) =$$

$$= \|(f^m)'_x u\|^2 - 2\|u\|^2 + \|(f^{-m})'u\|^2$$

that is positive for every $u \in T_xM$, $\|u\| \neq 0$, $x \in C$. Because of the continuity of f', $(f^{-1})'$, this quadratic form satisfies the same properties on a neighbourhood of C, and moreover, we can define for x,y in some neighbourhood of C a function $V(x,y) = A(u)$, where $\exp_x u = y$, provided y is close enough to x. For some $\alpha > 0$ we will have again on account of the continuity of f', $(f^{-1})'$ that if $0 < \mathrm{dist}(x,y) < \alpha$,

$$V(f(x),f(y)) - 2V(x,y) + V(f^{-1}(x),f^{-1}(y)) > 0.$$

Let f now satisfy axiom A, B_i, $i=1,2,\dots r$ being its basic sets, $\Omega(f) = \bigcup_{i=1}^{r} B_i$. Let $\rho > 0$, $\rho < \min_{i \neq j} (\min_{x_i \in B_i, x_j \in B_j} \mathrm{dist}(x_i,x_j))$, and $\alpha > 0$ be so that on $\{x \in M: \mathrm{dist}(x,\Omega(f)) < \rho\}$ we may define a quadratic function V with the above mentioned property for $0 < \mathrm{dist}(x,y) < \alpha$. Let $\varepsilon > 0$, and let $k > 0$ and $\delta > 0$ be chosen so that $V(x,y) \leq k$ implies $\mathrm{dist}(x,y) \leq \varepsilon$; and $\mathrm{dist}(x,y) > \delta$ if $V(x,y) \geq k$. Let $\mathcal{U} = \mathcal{U}(f,\rho,k)$ be a C^0-neighbourhood of f such that for $g \in \mathcal{U}$

$$\left\{x \in M: \mathrm{dist}(x,\Omega(g)) \leq \frac{\rho}{2}\right\} \subset \left\{x \in M: \mathrm{dist}(x,\Omega(f)) < \rho\right\}$$

and that if $\mathrm{dist}(x,B) < \rho$, $\mathrm{dist}(y,B) < \rho$, for B a basic set of f that is not a periodic repellor, and if $0 < \mathrm{dist}(x,y) < \alpha$, we have

i) $V(g^{-1}(x), g^{-1}(y)) > k$, for some y with $V(x,y)=k$.

ii) $V(g(x),g(y)) - 2V(x,y) + V(g^{-1}(x),g^{-1}(y)) > 0$ for every y with $V(x,y) \geq k$.

Let g be a homeomorphism of M and let $C_\varepsilon(x)$, $D_\varepsilon(x)$ denote the connected components containing x of the g-stable set $S_\varepsilon(x)$ and the g-unstable set $U_\varepsilon(x)$ of x.

Lemma 3.1

Let $g \in \mathcal{U}(f,\rho,k)$. Assume that for some $x \in M$ and some f-basic set B that is not a periodic repellor (atracttor) we have that for $n \geq 0$ ($n \leq 0$), $\mathrm{dist}(g^n(x),B) \leq \rho/2$.

Then $C_\varepsilon(x)$ (resp. $D_\varepsilon(x)$) contains a point y such that $\mathrm{dist}(x,y) = \delta$.

Proof We prove that $C_\varepsilon(x)$ contains such an y, arguing by contradiction. Assume then that each connected set joining x to $\partial V_k(x)$, where

$$V_k(x) = \left\{ y \in M: V(x,y) \leq k \right\},$$

contains a point y such that for some $n > 0$, $g^n(y) \notin V_k(g^n(x))$. Because of the compactness of $V_k(x)$ we may assume that all those n are less than some $N > 0$. Choose $\nu > N$; then for some $z \in \partial V_k(g^\nu(x))$ $g^{-1}(z) \notin V_k(g^{\nu-1}(x))$ because of i). Join $g^\nu(x)$ to z through an arc $a:[0,1] \longrightarrow V_k(g^\nu(x))$. Let t^* be the supremum of those t for which

$$g^{-n}\left(a[0,t]\right) \subset V_k\left(g^{\nu-n}(x)\right),$$

$0 \leq n \leq \nu$. Then because of the contradiction assumption, for some p, $0 < p < \nu$; $g^{-p}(a(t^*)) \in \partial V_k(g^{\nu-p}(x))$, and, at the same time, $g^{-n}(a(t^*)) \in V_k(g^{\nu-n}(x))$, for $0 \leq n \leq \nu$, which is absurd on account of ii).

For f satisfying axiom A and the strong transversality property, and for $n > 0$ choose ρ_n, $0 < \rho_n < \frac{1}{n}$ and $k_n > 0$, $\mathrm{dist}(x,y) \leq \frac{1}{n}$ if $V(x,y) \leq k_n$, in order that each $g \in \mathcal{U}(f,\rho_n,k_n)$ fullfils conditions i) and ii).

Denote $\mathcal{V}(f,n)$ a C^0-neighbourhood of included in $\mathcal{U}(f,\rho_n,k_n)$ such that for $g \in \mathcal{V}(f,n)$ there is a semiconjugacy h, $f \circ h = h \circ g$ such that $\mathrm{dist}(x,h(x)) < \frac{1}{n}$ for $x \in M$. Let $\mathcal{N}_n$ be the union, for f satisfying axiom A and the strong transversality conditions, of the $\mathcal{V}(f,n)$. Then $\Sigma = \bigcap_{n>0} \mathcal{N}_n$ is C^0-residual.

<u>Proposition 3.2</u>

Let $g \in \Sigma$ and let C be a compact g-invariant susbset of M, $C \subset \Omega(g)$, that is either connected or transitive. Assume moreover that C is an attractor, i.e., there exists a neighbourhood U of C, $g(\mathrm{clos}\ U) \subset \mathrm{int}\ U$, such that $\bigcap_{n\geq 0} g^n(U) = C$. Then for each $\varepsilon > 0$, the diameters of $C_\varepsilon(x)$ are bounded away from zero, on U. The diameters of $D_\varepsilon(x)$ are bounded away from zero on C.

<u>Proof</u> Let $g \in \Sigma$. Given $\varepsilon > 0$, choose n such that $\frac{1}{n} < \varepsilon$, $\frac{2}{n} < \mathrm{dist}(C,M-U)$. $\frac{2}{n} < \mathrm{diam}\ C$. Then $g \in \mathcal{V}(f,n)$ for some f with the axiom A and strong transversality properties. Since $C \subset \Omega(g)$, $h(C) \subset \Omega(f)$ and as h(C) is connected or f-transitive, it is included in some basic set B of f. Since $\frac{2}{n} < \mathrm{dist}(C,M-U)$ there is a neighbourhood W of h(C) such that $\mathrm{clos}(h^{-1}(W)) \subset U$. From this remark it follows that B = h(C) is also an attractor. Then Lemma 3.1 applies and permits to obtain easily the thesis of the proposition since B, being connected and infinite ($\frac{2}{n} < \mathrm{diam}\ C$) can not be the orbit of a periodic point.

4. Dim M = 2

Let $g:M\longrightarrow M$, $g \in \Sigma$, have a compact attractor $C \subset \Omega(g)$, diam $C = d > 0$, and let U be an open neighbourhood of C such that $\bigcap_{n\geq 0} g^n(U) = C$, and ρ a positive number so that the ball of radius 10ρ centered at each $x \in M$, is homeomorphic to a disk in $\mathbb{R}^2$ and that $\left\{x \in M\colon \operatorname{dist}(x,C) \leq 10\,\rho\right\} \subset U$.

Let ε, $0 < \varepsilon < \frac{\rho}{10}$ and let $\sigma > 0$, $\sigma < \varepsilon$ be so that if $\operatorname{dist}(x,C) < \frac{\rho}{10}$, $C_\varepsilon(x)$ contains a point y_0, $\operatorname{dist}(x,y_0) = \sigma$, and that for $x \in C$, $D_\varepsilon(x)$ contains z_0, $\operatorname{dist}(x,z_0) = \sigma$.

Since C is an attractor $D_\varepsilon(x) \subset C$ for each $x \in C$, since $\operatorname{dist}(g^{-n}(x),g^{-n}(z)) \leq \varepsilon$, $n \geq 0$, for $z \in D_\varepsilon(x)$, implies $z \in \bigcap_{n\geq 0} g^n(U) = C$.

Let $f:M\longrightarrow M$ satisfying axiom A and strong transversality be such that g is semi-conjugate to f through h, $f\circ h = h\circ g$, $\operatorname{dist}(x,h(x)) < \frac{1}{n}$, for every $x \in M$; $\frac{2}{n} < \sigma$. $h(C) = B$ is an attracting basic set of f as we have shown before; moreover $h^{-1}(h(C)) = C$, for otherwise there would exist $y \notin C$ such that $h(g^{-n}(y)) = f^{-n}(h(y)) \in C$ for $n \geq 0$; thus $g^{-n}(y) \in U$ for $n \geq 0$ which is absurd.

For f satisfying axiom A and $x \in M$, we will denotes as usual,

$W^s(x) = \left\{y \in M\colon \operatorname{dist}(f^n(x),f^n(y))\longrightarrow 0 \text{ as } n\longrightarrow \infty\right\}$, and $W^u(x)$ the unstable manifold. For $\varepsilon > 0$,

$W^s_\varepsilon(x) = \left\{y \in W^s(x)\colon \operatorname{dist}(f^n(x),f^n(y)) \leq \varepsilon,\ n \geq 0\right\}$. $W^u_\varepsilon(x)$ is defined similarly.

<u>Lemma 4.1</u>

$h(D_\varepsilon(x)) \subset W^u(h(x))$ if $x \in C$.

<u>Proof</u> Consider, for $x \in C$, $h(D_\varepsilon(x)) \subset B$, and let β,γ be the end points of the maximum arc containing $h(x) = \xi$ and included in $h(D_\varepsilon(x)) \cap W^u(\xi)$. Construct a neighbourhood of this maximum arc by

taking a very close but strictly larger arc of $W^u(\xi)$ with end points $\beta' \notin h(D_\varepsilon(x))$ to the left of β and $\gamma' \notin h(D_\varepsilon(x))$ to the right of γ and by tracing through each point η of this new arc the local stable manifolds $W^s_{\theta'}(\eta)$ where θ' is chosen so small that the neighbourbood constrcuted in this way is homeomorphic to a rectangle $r = b \times c$ where $b, c \subset \mathbb{R}$ are intervals, b homeomorphic to the arc with end points β', γ' and c homeomorphic to $\omega^s_{\theta'}(\xi)$. We may assume that through each point ζ included in B and in this rectangle we may trace for some θ, $0 < \theta < \varepsilon$, $W^u_\theta(\zeta)$ and that this arc meets $W^s_\theta(\xi)$: if this were not the case we may take negative iterates of f in order to get that the maximum arc with end points β, γ, becomes small enough to apply the local product structure on B.

If $h(D_\varepsilon(x))$ does not coincide with this maximum arc, we may assume that the rectangle is so small that $h(D_\varepsilon(x))$ contains some points in the exterior of the rectangle. Thus, the connected component of $h(D_\varepsilon(x)) \cap r$ containing ξ must reach the boundary of the rectangle. Through each ζ that belongs to this connected component we trace $W^u_\theta(\zeta)$ and find the intersection $W^S_\theta(\xi) \cap W^u_\theta(\zeta)$.

We claim that the range of the mapping $\zeta \longrightarrow W^s_\theta(\xi) \cap W^u_\theta(\zeta)$ is $\{\xi\}$ which is absurd.

If not we would get a non-trivial subarc δ of $W^s_\theta(\xi)$ contained in B. Since the unstable arc through ξ is also included in B we obtain that B contains open sets. Since stable manifolds of points in int $B = B^0$ are also included in B^0 as it is easy to show inasmuch as this happens for stable manifolds of the interior periodic points, we get on account of the local product structure on B that $\partial B^0 = \phi$, i.e., $B = M$. But this implies that f is Anosov; on the other hand this arc δ on $W^s_\theta(\xi)$ has the property that for $n \le 0$, $f^{-n}(\delta)$ is contained in a disk of radius ρ, which is impossible.

Thus, $h(D_\varepsilon(x))$ coincides with an arc of $W^u(h(x))$.

<u>Lemma 4.2</u>

$h(C_\varepsilon(x)) \subset W^s(h(x))$.

$h(D_\varepsilon(x)) \cap h(C_\varepsilon(x)) = \{h(x)\}$.

Proof Let again β,γ denote the end-points of the maximum arc $\beta\gamma$ containing $h(x)=\xi$ of $h(C_\varepsilon(x) \cap W^s_\theta(\xi)$. We iterate f forward in order to get $f^n(\beta\gamma) \subset W^s_\theta(f^n(\xi))$, for some $n > 0$, where θ, $0< \theta <\varepsilon$, is such that $W^s_\theta(\zeta) \cap \omega^u_\theta(f^n(\xi)) = \{\zeta\}$ for $\zeta \in W^u_\theta(f^n(\xi))$. If $h(C_\varepsilon(x))$ had other points than those of $\beta\gamma$, then, as in the previous lemma, we would get projecting through $W^s_\theta(\zeta)$ on $W^u_\theta(f^n(\xi))$ a non-trivial arc or $W^u_\theta(f^n(\xi))$ whose forward iterates have diameter less than $4\varepsilon < \rho$. Let η be an f-periodic point $\eta \in B$, so close to $f^n(\xi)$ that by projection through $W^s_\theta(\zeta)$, for ζ in that arc, we get another non-trivial arc δ, $\delta \subset W^u(\eta)$; the diameter of $f^n(\delta)$, $n \geq 0$ is this time, less than $6\varepsilon < \rho$. The unstable manifold through η can be obtained as $\bigcup_{k\geq 0} f^{k\mu}(\delta)$, μ being the period of η. Let τ an accumulation point of $W^u(\eta)$. This implies that $W^s_\theta(\tau)$ meets twice $W^u(\eta)$, and we get, therefore, a disk of radius 2ρ centered at τ containing another disk D bordered by an unstable arc and a stable one. Now we finish the proof of both assertions of the lemma by showing that this is impossible. Since at the border of D the diameter is less than 4ρ, for some $n > 0$, $f^n(D)$ is so close to B that we may define on $f^n(D)$ and consequently on D, a stable vector field which never vanishes. Take half stable manifolds entering D and starting on the unstable border of D. Since no half stable manifold can neither stay in the interior of D nor meet the stable border of D, we get that the continuous map that sends a point on the unstable border of D to the first point where the half stable manifold through it meets again this unstable border, has a fixed point, which is absurd.

Lemma 4.3

$$C_\varepsilon(x) \cap D_\varepsilon(x) = \{x\}, \text{ for } x \in C.$$

Proof Let $y \in C_\varepsilon(x) \cap D_\varepsilon(x)$ and suppose $\mathrm{dist}(x,y) > 0$. Choose f satisfying axiom A and strong-transversality, such that $f\circ h = h\circ g$ and $2\mathrm{dist}(x,h(x)) < \mathrm{dist}(x,y)$.

Then $h(x) \neq h(y) \in h(D_\varepsilon(x)) \cap h(C_\varepsilon(x))$.

Corollary 4.4

Let $x \in C$ and $y \in C_\varepsilon(x)(D_\varepsilon(x))$. Then $\text{diam}(g^n(C_\varepsilon(x))\longrightarrow 0$ ($\text{diam}\ g^n(D_\varepsilon(x))\longrightarrow 0$) when $n\longrightarrow +\infty$ (resp., $n\longrightarrow -\infty$).

Proof Otherwise we would get a point $z \in \omega(x)$ and a non-trivial connected set containing z and included in $C_\varepsilon(x) \cap D_\varepsilon(z)$.

Proposition 4.5

For $x \in C$, $D_\varepsilon(x)$ is compact connected and locally conected.

Proof It follows from the previous corollary that given ε', $0 < \varepsilon' < \varepsilon$, there exists $\delta > 0$ such that, if $y \in C_\varepsilon(x)(D_\varepsilon(x))$ and dist $(x,y) < \delta$, then $y \in C_{\varepsilon'}(x)$ (resp. $y \in D_{\varepsilon'}(x)$). The proof of the proposition is now tha same as that of Corollary 2.4, (p.121) of [L].

Theorem 4.6

There is a C^0-residual set of Hom(M), such that if $g \in \Sigma$ has a connected attractor, $C \subset \Omega(g)$, diam $C > 0$, then there exists $\varepsilon_0 > 0$, such that if $\varepsilon < \varepsilon_0$, $x \in C$, the connected component $D_\varepsilon(x)$ containing x of $U_\varepsilon(x)$ is a homeomorphic image of an interval. Furthermore, $\lim_{n\to-\infty} \text{dist}(x,y) = 0$ for $y \in D_\varepsilon(x)$.

Proof Choose ε_0 as in the second paragraph of this section, and let $\varepsilon < \varepsilon_0$. Since, by the previous proposition, $D_\varepsilon(x)$ is locally connected, any two points may be joined by an arc within $D_\varepsilon(x)$. Assume that for some $\sigma > 0$, $\sigma < \varepsilon$ there are three arcs a,b,c in $D_\varepsilon(x)$ with origin x, joining x to $\partial B_\sigma(x)$ and such that $a \cap b = b \cap c = a \cap c = \{x\}$.

Take f satisfying axiom A and strong tranversality, and semi-conjugate to g through h, where $\text{dist}(x,h(x)) < \delta$ for $x \in M$; here $\delta > 0$ is chosen so small that the end points of each one of these arcs has a distance not less than 10δ to the other two arcs. Then we can not have that the h-image of an end point of some arc lies on the h-image of the other arcs; but this is impossible.

This argument proves that an interior point of an arc like, say, a, can not be joined to another point of $D_\varepsilon(x)$ through an arc that meets a only at that point. This proves the first assertion; on account of Corollary 4.4, this completes the proof.

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Collective complete integrability and loop space homology

Gabriel P. Paternain
Department of Mathematics
SUNY at Stony Brook
Stony Brook, NY 11794

Abstract

We show that if M is a simply connected compact riemannian manifold whose geodesic flow is completely integrable with collective integrals, then the loop space homology of M with coefficients on any field grows sub-exponentially.

1 *Introduction and results*

The study of completely integrable geodesic flows (and Hamiltonian systems in general) has regained momentum in recent years, as new techniques have been discovered to construct examples. Let us recall that a geodesic flow is said to be completely integrable if it admits a maximal number of independent conservation laws (i.e. first integrals) that Poisson-commute. Classical examples are given by n-dimensional ellipsoids with different principal axes (Jacobi, 1838), left invariant metrics on $SO(3)$ (Euler, 1765), surfaces of revolution ("Clairaut's first integral"), and flat tori.

In part due to Poincaré's realization that complete integrability was a rare phenomenon, the subject went through a period in which very little development occured. In the past decades the study of Hamiltonian actions and the geometry of the moment map provided the necessary framework for a solid theory of symmetries. As a consequence, new examples appeared. In 1978, Mishchenko and Fomenko [10] constructed left invariant metrics on semi-simple Lie groups with completely integrable geodesic flows. Then Thimm [13] devised a new method for constructing first integrals in involution on homogeneous spaces. In particular he was able to show that the geodesic flow on real or complex Grassmannians is completely integrable. Guillemin and Sternberg [8] strengthened this method and obtained further examples. Very recently Spatzier and the author [11] constructed the first non-homogeneous examples using riemannian submersions. We were able to show that spaces like Eschenburg's strongly inhomogeneous 7-manifold [2], $\mathbf{CP^n}\#\mathbf{CP^n}$ for n odd and the exotic 7-sphere constructed by Gromoll and Meyer [3], have metrics with completely integrable geodesic flows.

A natural question arises: What are the geometric and topological properties of a compact riemannian manifold whose geodesic flow is completely integrable? Some topological features are shared by all the previous examples and we would like to draw attention to them. Following Grove and Halperin [6] we will say that a simply connected compact manifold M^n is *rationally elliptic* if the sum of the Betti numbers of the loop space of M with rational coefficients grows sub-exponentially or equivalentely, if the rational homotopy of M, $\pi_*(M)\otimes\mathbf{Q}$ is finite dimensional. Homogeneous spaces are known to have this property, although is rather restrictive [6]. Rational ellipticity is shared by all the known examples of manifolds with completely integrable geodesic flows, but in fact they verify the stronger property that their loop space homology grows sub-exponentially even when the coefficient field has positive characteristic.

Before we state our results let us set some terminology.

Let G be a compact connected Lie group acting by Hamiltonian transformations on a symplectic manifold X with moment map $\phi : X \to g^*$ (cf. [7] for defintion and properties of the moment map). We will say that the action has *multiplicity* k if for generic $x \in X$, the symplectic reduction of $Kerd\ \phi_x$ (i.e. the quotient of $Ker\ d\phi_x$ by its null subspace) has dimension k. Since the symplectic reduction of a subspace is naturally symplectic, k can only take even values. If $k = 0$, then $Ker\ d\phi_x$ is isotropic for generic $x \in X$ and we obtain the notion of *multiplicity free action* introduced and studied by Guillemin and Sternberg in [8, 9].

Let H be a G-invariant Hamiltonian, ξ_H its Hamiltonian vector field and $H^{-1}(a) = N$ a compact regular level surface. Let $h_{top}(H)$ denote the topological entropy of the flow of ξ_H restricted to N. In this note we want to announce:

Theorem 1.1 *If the action of G has multiplicity zero or two, then* $h_{top}(H) = 0$.

Examples of homogeneous spaces G/H such that action of G on $T^*(G/H)$ has multiplicity two are the Stiefel manifold $SO(n+1)/SO(n-1)$ and the Wallach manifold $SU(3)/T^2$.

Let us now describe some of the interesting consequences that Theorem 1.1 has in the case of geodesic flows. Let M be a simply connected compact riemannian manifold. If the topological entropy of the geodesic flow is zero then the Morse Theory of the loop space implies that the loop space homology of M with coefficients on any field grows sub-exponentially, via results of Yomdin and Gromov [4, 5, 14, 12]. Thus from Theorem 1.1 we obtain:

Theorem 1.2 *Let M be a simply connected compact manifold whose cotangent bundle admits a compact Hamiltonian G-action with multiplicity $k \leq 2$. Assume the set of G-invariant functions on T^*M contains the Hamiltonian associated with some riemannian metric. Then the loop space homology of M with coefficients on any field grows sub-exponentially.*

Observe that Theorem 1.2 and thus Theorem 1.1 are false for $k \geq 4$. For example $M = S^2 \times S^2 \# S^2 \times S^2$ is a non-elliptic manifold, which admits a 2-torus action . The lift of this action to the cotangent bundle of M has mutiplicity $k = 4$. Any riemannian metric invariant under the torus action, gives rise to a geodesic flow with positive topological entropy.

The idea behind Theorem 1.2 is very simple. If the geodesic flow admits a sufficiently large group of symmetries ($k = 0, 2$), then M has severe topological restrictions (ellipticity).

Let us now describe briefly why actions with multiplicity ≤ 2 are relevant to complete integrability. A function of the form $f \circ \phi$, for $f : g^* \to \mathbf{R}$ is called *collective* (cf. [7]). We can prove the following lemma.

Lemma 1.3 *If there exist $f_1, ..., f_s$ in $C^\infty(g^*)$ such that $f_1 \circ \phi, ..., f_s \circ \phi$ are s-independent functions that Poisson-commute on X^{2n}, then the multiplicity of the action is $\leq 2(n-s)$.*

Observe that if $s = n$, that is, if we can find a full set of commutative collective Hamiltonians, then the action is multiplicity free. This was proved in [8]. Note also that a G-invariant Hamiltonian H is also completely integrable if it admits $n-1$ independent commuting collective integrals besides H. In this case the action has multiplicity ≤ 2.

Most of the known examples of completely integrable geodesic flows arise by considering collective integrals as above. The Thimm method (cf. [8, 13]) fits into this framework.

Let (M^n, g) be a compact riemannian manifold whose geodesic flow is completely integrable with first integrals $F_1 = \| \cdot \|_g, F_2, \ldots, F_n$. We will say that the geodesic flow is completely integrable with *collective integrals* if the functions F_i, $2 \leq i \leq n$ are collective with respect to the action of some compact Lie group G that leaves the Hamiltonian associated with the riemannian metric invariant. Combining Theorem 1.2 with Lemma 1.3 we obtain:

Theorem 1.4 *Let M^n be a simply connected compact riemannian manifold whose geodesic flow is completely integrable with collective integrals. Then the loop space homology of M with coefficients on any field grows subexponentially.* .

It is a pleasure to thank my advisor Detlef Gromoll for his permanent encouragement. His suggestions have been a constant source of ideas. I also would like to thank Ralf Spatzier and Steve Halperin for several stimulating discussions.

2 *Sketch of the proof of Theorem 1.1*

Let H be a G-invariant Hamiltonian, ξ_H its Hamiltonian vector field and $H^{-1}(a) = N$ a compact regular level surface. If g_t denotes the flow of ξ_H, then G and g_t leave N invariant. Set $\varphi = \phi/N$, where ϕ is the moment map associated with the action of G.

We say that $x \in X$ defines a *stationary motion* if there exists a 1-parameter subgroup ψ_t of G such that $\psi_t x = g_t x$. We denote by $St(G)$ the set of all $x \in X$ that define stationary motions.

Lemma 2.1 *If g_x° denotes the annihilator of g_x in $g*$, then*

$$Im\ d\varphi_x = g_x^\circ$$

if x is not in $St(G)$.

Now let $H \subset G$ be a closed subgroup and let $X_H = \{x \in X : G_x = H\}$. It is known that X_H is a symplectic submanifold of X. Moreover ϕ maps each connected component of X_H into an affine subspace of g^* of the form $p + h^o$, where h^o denotes the annihilator of h in g^* [7]. Let N_H denote the normalizer of H in G. The following is a crucial lemma.

Lemma 2.2 *Suppose the action of G on X has multiplicity k. Then the action of N_H on X_H has multiplicity $\leq k$.*

Let us now start with the proof of the theorem. Let $Y = N/G$, call π the canonical projection and let $\hat{g}_t$ be the induced family of homeomorphisms on Y. According to [1, Theorem 19] we only need to show that $h_{top}(\hat{g}) = 0$. Suppose first that the action is multiplicity free. Then it is easy to see that $St(G) = X$ and the result follows.

Next, let us prove the theorem in the multiplicity two case. We will actually prove more: $\hat{g}_t$ has only trivial recurrence.

Let $\hat{\gamma}$ denote an orbit of $\hat{g}_t$ i.e. $\hat{\gamma}(t) = \hat{g}_t\hat{x}$ for some $\hat{x} \in Y$. Take $x \in \pi^{-1}(\hat{x})$ and consider the orbit of g_t through x. Thus $\pi \circ \gamma(t) = \hat{\gamma}(t)$. Let $H = G_x$. Then since g_t commutes with the G-action, we deduce that $\gamma \subset X_H$. Let $\phi_{N_H} : X_H \to n_H^*$ denote the moment map corresponding to the action of N_H on X_H. In fact ϕ_{N_H} takes values on a subspace of n_H^* of the form $p + h^o$ where h^o is the annihilator of h in n_H^*. Set $c = \phi_{N_H}(\gamma)$ and $\varphi = \phi_{N_H}/X_H \cap N$.

Observe now that Lemma 2.1 says that c is a regular value of φ if $\varphi^{-1}(c) \cap St(N_H)$ is empty. Set $Q_c = \varphi^{-1}(c) - (\varphi^{-1}(c) \cap St(N_H))$. We have now two possible cases:

(a) $x \in St(N_H)$. If this happens, then clearly $\hat{\gamma}$ is a fixed point and hence trivially recurrent.

(b) $x \notin St(N_H)$. In this case Q_c is a non-empty submanifold of $X_H \cap N$ and $\gamma \subset Q_c$. From now on we will work with the connected component of Q_c containing γ. Let K_c denote the identity component of the stabilizer at c of the coadjoint action of N_H on n_H^*. Since the action of G on X has multiplicity two by Lemma 2.2, the action of N_H on X_H has multiplicity at most two. But it cannot be zero if $x \notin St(N_H)$. Thus $dim\, Q_c/K_c = 1$. Now we also have two possible cases:

(b1) Q_c/K_c is a circle. In this case it follows inmediately that $\hat{\gamma}$ is a closed orbit and hence trivially recurrent.

(b2) Q_c/K_c is an open interval I. Then Q_c is diffeomorphic to $\mathcal{O} \times I$, where $\mathcal{O}$ denotes a principal orbit for the action of K_c on Q_c. Also γ intersects every orbit of K_c once and only once. Thus if we assume that $\hat{\gamma}$ is not a closed orbit it follows that every G-orbit in X that intersects Q_c, does it in a single K_c-orbit. Hence we can find a G-invariant neighborhood W of x in X so that there exists $T > 0$ with the property that $\gamma(t) \notin W$ for $t \geq T$. But this implies that $\hat{\gamma}(t) \notin \pi(W)$ for $t \geq T$ and thus $\hat{x} \notin \omega(\hat{\gamma})$, proving that $\hat{\gamma}$ is not recurrent.

◇

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Gabriel Paternain
Current Adress: Centro de Matemática
Facultad de Ciencias
Eduardo Acevedo 1139
C. P. 11200
Montevideo - Uruguay

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Expansivity and Length Expansivity for Geodesic Flows on Surfaces

Miguel Paternain

I. INTRODUCTION

A continuous flow ϕ on a metric space K is said to be expansive if for every $\varepsilon > 0$ there is $\delta > 0$ with the property that if $\text{dist}(\phi_t(x), \phi_{\sigma(t)}(y)) < \delta$ for every $t \in \mathbb{R}, x, y \in K$ and a continuous map $\sigma: \mathbb{R} \to \mathbb{R}$ with $\sigma(0) = 0$, then $y = \phi_t(x)$, where $|t| < \varepsilon$.

Related to expansivity we have the concept of length expansive flow. A flow $\phi_t: M \to M$ of diffeomorphisms of a riemannian manifold M is said to be length expansive if every rectifiable curve ξ not contained in an orbit of the flow satisfies

$$\sup_{t \in \mathbb{R}} \text{ length } \phi_t(\xi) = \infty \quad .$$

Expansivity implies length expansivity. We show this in section IV of this paper. The converse property is false by the example of Remark 1.4 of [8]. However it holds for certain geodesic flows. This is one of the properties contained in the following theorem

THEOREM 1

Given a closed riemannian surface M:and denoting $\phi_t: UM \to UM$ its geodesic flow, where UM is the unit tangent bundle of M, the following properties are equivalent

a) ϕ is expansive

b) ϕ is length expansive and M has no conjugate points

c) M has no conjugate points and for any two disjoint geodesics γ_1 and γ_2 of the universal covering of M

$$\sup_{t \in \mathbb{R}} \ \mathrm{dist}(\gamma_1(t), \gamma_2(t)) = \infty .$$

Problem. Does any one of the implications above hold if dim M >2 ?

Our second result requires to recall certain basic concepts of the geometry of a riemannian manifold M without conjugate points.

Given $p \in M$ and $v \in U_p M$, consider a point $\tilde{p} \in \tilde{M}$, where $\Pi: \tilde{M} \to M$ is the universal covering, and take $\tilde{v} \in U_{\tilde{p}} \tilde{M}$ satisfying $\Pi'(\tilde{p})\tilde{v} = v$. Denote $B_r(x)$ the ball of radius r and centre $x \in \tilde{M}$. Its boundary $\partial B_r(x)$ is, by the absence of conjugate points, a submanifold diffeomorphic to a sphere. Let γ be the geodesic with $\gamma(0) = \tilde{p}$ and $\dot{\gamma}(0) = \tilde{v}$.

Denote $u^+(p,v,r)$ and $u^-(p,v,r)$ the scalar curvatures at $\tilde{p}$ in the direction of $\tilde{v}$ of the submanifolds $\partial B_r(\gamma(-r))$ $\partial B_r(\gamma(r))$. Clearly $u^+(p,v,r) \geq u^-(p,v,r)$ and $u^+(p,v,r)$ decreases (and $u^-(p,v,r)$ increases) when r increases. Define $u^+(p,v) = \lim_{r\to+\infty} u^+(p,v,r)$ and $u^-(p,v) = \lim_{r\to+\infty} u^-(p,v,r)$. Then

$u^+(p,v) \geq u^-(p,v)$.

These curvatures play a central rôle in the study of manifolds without conjugate points. For instance $u^+(p,v) > u^-(p,v)$ for all $(p,v) \in UM$ if and only if the geodesic flow is Anosov (Eberlein [2]). The continuity of u^+ and u^- was an open problem until Ballmann, Brin and Burns([1]) gave the first example of a closed surface for which u^+ and u^- are discontinuous. When the manifold has no focal points, then u^+ and u^- are continuous (see [3] and [10]).

In the example of [1], u^+ and u^- coincide only at one orbit of the geodesic flow. Compare this with the following result

THEOREM 2

If M is a closed surface without conjugate points and its geodesic flow is not expansive, then there exists a curve $\xi:[a,b]\to UM$,not contained in an orbit for which

$$u^{+}(\xi(t)) = u^{-}(\xi(t)) \quad \text{for almost every } t \in [a,b] .$$

Hence, the example of [1] is expansive, and then through the techniques of Ghys [4] we get

COROLLARY

The example of Ballmann, Brin and Burns is topologically equivalent to the geodesic flow of the constant negative curvature riemannian structure.

I am grateful to Jorge Lewowicz for helpful comments on these problems.

II. PRELIMINARIES

For the sequel, assume that M is a compact oriented riemannian surface with no conjugate points. Let $\tilde{M}$ stand for its universal riemannian covering and UM for the unit bundle of M endowed with its standard metric and the canonical projection $\pi : UM \to M$.

The geodesic flow $\phi_t : UM \to UM$ is defined as $\phi(\zeta,t) = \phi_t(\zeta) = (\gamma_\zeta(t), \dot{\gamma}_\zeta(t))$ where $\zeta = (p,v)$ and γ_ζ is the geodesic with initial conditions $\gamma_\zeta(0) = p$, $\dot{\gamma}_\zeta(0) = v$.
Denote by $\tilde{\phi}_t : U\tilde{M} \to U\tilde{M}$ the geodesic flow of the universal covering. Since M is oriented it is possible to define $e^{i\theta}\zeta = (p, e^{i\theta}v)$ ($e^{i\theta}v$ is the rotation of v by an angle θ) for real θ.

For $\zeta = (p,v)$ define

$$S(\zeta) = \left\{ W \in T_\zeta(UM) \ / \ \langle \pi'_\zeta(W), v\rangle = 0 \right\}$$

It is well known (see [2]) that $S(\zeta)$ is invariant in the following sense:

$$\phi'_t(S(\zeta)) = S(\phi_t(\zeta))$$

The vertical subspace is defined as $V(\zeta) = \operatorname{Ker}\left(\pi'_\zeta / S(\zeta)\right)$ and the horizontal subspace is the ortogonal complement of $V(\zeta)$ in $S(\zeta)$.

Put $Y(\zeta) = \left\{ w \in U_\zeta M \ / \ \langle w, v\rangle = 0 \right\}$

It is also well known (see [2]) that $S(\zeta)$ can be identified with $Y(\zeta) \times Y(\zeta)$ and that, with such an identification, we can write

$\phi'_t(V,W) = (J(t), \dot{J}(t))$, where J is the solution of the Jacobi equation

$$\ddot{J} + K(\gamma_\zeta(t))J = 0$$

with initial conditions $J(0) = V$, $\dot{J}(0) = W$

(here, K is the gaussian curvature of M).

Recall that $u = \dot{J}/J$ satisfies the Riccati equation

$$\dot{u} + u^2 + K(\gamma_\zeta(t)) = 0.$$

We say that u is the slope of $(J,\dot{J})$.

As in [7] we define:

$\Phi_\zeta(m,n,\varphi) = e^{i(\varphi-\pi/2)}\phi \quad (i\phi(\zeta,m),n)$

for (m,n,φ) close to $0 \in \mathbb{R}^3$; moreover it can be shown (see [7]) that Φ_ζ is a local diffeomorphism at $0 \in \mathbb{R}^3$ and that if $F_\zeta\ (n,\varphi) = \Phi_\zeta(o,n,\varphi)$, then $\frac{\partial F}{\partial n}\zeta(0,0)$ and $\frac{\partial F}{\partial \varphi}\zeta(0,0)$ are vectors of $H(\zeta)$ and $V(\zeta)$ respectively.

Let $B_\varepsilon(0)$ be the open ball of centre 0 and radius ε on $\mathbb{R}^2$. Then, we can choose ε such that $F_\zeta\ (B_\varepsilon(0)) = N_\varepsilon(\zeta)$ is transversal to the orbits of ϕ_t for every $\zeta \in UM$.

Define $m: \mathbb{C} - \{0\} \to S^1$ as $m(z) = \frac{\bar{z}}{z}$ and $g(t) = m(J(t)+i\dot{J}(t))$, where J is a solution of the Jacobi equation.

Then, $\frac{g'(t)}{ig\ (t)} = 2\ \frac{\dot{J}^2+KJ^2}{\dot{J}^2+J^2}$. This implies that there is $R > 0$ such that if $|\dot{J}(t)| > R|J(t)|$, then $\frac{g'(t)}{ig(t)} > 1$. This yields

<u>Remark 1</u>

There is $t_0 = t_0(R)$ such that if $|\dot{J}(0)| > R|J(0)|$, then $J(t) = 0$ for some t, $|t| \leq t_0$. □

For suitable $\delta > 0$ and $\theta \in N_\delta(\zeta)$ we define $P_{\zeta,\theta} : N_\delta(\zeta) \to N_{\varepsilon_{-1}}(\theta)$ as the projection along the geodesic flow, and set $G_\theta = F_\theta \circ P_{\zeta,\theta} \circ F_\zeta$. (Observe that the map $(\theta,n,\varphi) \to G_\theta(n,\varphi)$ is differentiable and that $G_\zeta = \mathrm{id}$).

Remark 2

There is $\delta > 0$ such that $(G_\theta)'_p(1,a)$ has slope $> R$ if $|p| < \delta$ and $|a| > 2R$. □

Take $\zeta \in \tilde{U}M$. For $|n| < \delta$ and $t \geq 0$, define $\zeta_t^-(n) = \zeta_t^-$ and $\zeta_t^+(n) = \zeta_t^+$ as the unit vectors for which

$$\begin{cases} \gamma_{\zeta_t^-}(0) = \pi\tilde{\phi}(i\zeta,n) \\ \gamma_{\zeta_t^-}(\tau) = \gamma_\zeta(t) \end{cases} \quad \text{for some } \tau = \tau^-(n,t) \tag{1}$$

and

$$\begin{cases} \gamma_{-\zeta_t^+}(0) = \pi\tilde{\phi}(i\zeta,n) \\ \gamma_{-\zeta_t^+}(\tau) = \gamma_\zeta(-t) \end{cases} \quad \text{for some } \tau = \tau^+(n,t)$$

Notice that, for fixed n, $\tau^\pm(n,t) \to +\infty$ as $t \to +\infty$. Define $\alpha_t^\pm(n) = (n, f_t^\pm(n))$ in such a way that

$$F_\zeta \, \alpha_t^\pm = \zeta_t^\pm$$

Remark 3

$J_t(t) = 0$ for the Jacobi field J_t with initial conditions

$$\frac{d}{dn}\bigg|_{n=0} \zeta_t^-(n) = \left(J_t(0), \dot{J}_t(0)\right)$$

Analogously, for $\tau = \tau^-(n_0,t)$ as in (1) we have $J_\tau(\tau) = 0$ for the Jacobi field J_τ with initial conditions

$$\left(J_\tau(0),\ \dot{J}_\tau(0)\right) = \frac{d}{dn}\Big|_{n=n_0} \left(G_{\zeta_t^-(n_0)}(\alpha_t(n))\right) \qquad \square$$

Now define $u_t^-(\zeta,s) = \dot{J}_t(s)/J_t(s)$, for J_t as in the preceeding remark. It is well known (see [2]) that $u_t^-(\zeta,s)$ converges as $t \to \infty$ to a function $u^-(\zeta,s)$ which is also a solution of the Riccati equation. We will write $u^-(\zeta,0) = u^-(\zeta)$. Reversing time it is easy to find another limit solution u^+such that $u^+(-\zeta) = -u^-(\zeta)$.

Remark 4

(See proposition 2.12 of [2]).

Let $0 \neq (V,W) \in S(\zeta)$. Then

$$\lim_{t\to+\infty} |\phi_t'(V,W)| = \infty \text{ if } W \neq u^-(\zeta)V \qquad \square$$

On account of remarks 1,2 and 3 we can obtain $t_0 = t_0(R,\delta)$ such that

$|(f_t^\pm)'(n)| < 2R$ for $|n| < \delta$ and $t > t_0$.

Then $\zeta_t^\pm$ are uniformly lipschitz. This and Arzela-Ascoli theorem permit to find $\zeta^\pm(n)$, limit functions of $\zeta_t^\pm(n)$ for $t \to +\infty$, that are uniformly lipschitz.

Define $\eta_t^-(n) = \tilde{\phi}\left(\zeta_t^-(n),\ \tau^-(n,t)\right)$ for τ^- as in (1). As $(\eta_t^-)'(n_0) \in S(\eta_t^-(n_0))$ we get $(\zeta_t^-)'(n_0) + \tilde{\phi}(\zeta_t^-(n_0),\ 0)\ \frac{\partial\tau^-}{\partial n}(n_0,t) \in S(\zeta_t^-(n_0))$.

Therefore, $\tau^-(n,t)$ is uniformly lipschitz in n and consequently there is $\tau_\zeta^-(n)$, a limit function of $\tau^-(n,t) - t$, which is uniformly lipschitz in n, Therefore, for suitable δ, we obtain small $\tau_\zeta^-(n)$.

Set $C_t^-(\zeta,n) = \tilde{\phi}(\zeta_t^-(n),\ \tau^-(n,t) - t)$

and $C^-(\zeta,n) = \tilde{\phi}(\zeta^-(n),\ \tau_\zeta^-(n))$.

(Obviously a similar construction holds for the past).

<u>Remark 5</u>

Assume that

$\theta_t(n) = \tilde{\phi}(\zeta^-(n),\ \tau_n(t)) \in N_\delta(\phi_t(\zeta))$, for $t \geq 0$ and smooth increasing surjective functions

$$\tau_n : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$$

As the foliation $C^-(\zeta,\cdot)$ is $\tilde{\phi}$ -invariant (i.e.$\tilde{\phi}_t C^-(\zeta,n) = C^-(\tilde{\phi}_t(\zeta),m)$ for suitable m), we get, for small δ,

$\tau_n(t) = T_t(n) + t$ where T_t are uniformly lipschitz. □

Now, we need the following lemma due to Green (see [5])

<u>Lemma 1</u>

Assume that γ_1, and γ_2 are geodesics of $\tilde{M}$, the universal covering of M, such that $\gamma_1(0) = \gamma_2(0)$ and $\dot{\gamma}_1(0) \neq \dot{\gamma}_2(0)$, then

$$\sup_{t\geq 0} \text{dist}\ (\gamma_1(t),\ \gamma_2(\mathbb{R}^+)) = \infty \qquad \text{and}$$

$$\sup_{t\leq 0} \text{dist}\ (\gamma_1(t),\ \gamma_2(\mathbb{R}^-)) = \infty$$

Let γ_1 and γ_2 be two different geodesics of $\tilde{M}$ (i.e. γ_2 is not a reparametrization of γ_1).

We say that they are asymptotic for the future if

$$\sup_{t\geq 0} \text{dist}\ (\gamma_1(t),\ \gamma_2(\mathbb{R}^+)) < \infty$$

and that they are asymptotic for the past if

$$\sup_{t\leq 0} \text{dist}\ (\gamma_1(t),\ \gamma_2(\mathbb{R}^-)) < \infty$$

and that they are bi-asymptotic if they are asymptotic both for the future and for the past.

It is easy to see that ϕ_t is expansive if there is $\alpha > 0$ such that

if there is a smooth surjective increasing function $\tau : \mathbb{R} \to \mathbb{R}$, with $\tau(0) = 0$ for which $\phi_{\tau(t)}(\eta) \in N_\alpha(\phi_t(\zeta))$ for every $t \in \mathbb{R}$ then $\zeta = \eta$..

III. PROOF OF THE THEOREMS

Proof of theorem 2:

If the geodesic flow is not expansive on the manifold the same holds for the universal covering $\tilde{M}$ and then we can find $\eta \in N_\delta(\zeta)$, $\zeta \neq \eta$ vectors of $U\tilde{M}$, and a smooth increasing surjective function τ with $\tau(0) = 0$, such that

$$\phi_{\tau(t)}(\eta) \in N_\delta(\phi_t\ \ (\zeta)),\ \text{for } t \in \mathbb{R} \qquad (2)$$

this means that the geodesics γ_ζ and γ_η bound a strip, because they cannot cross on account of lemma 1.

Define $I = \{n/\pi\phi(i\zeta,n)$ is between γ_ζ and $\gamma_\eta\}$ and fix some $n_0 \in I$.

The geodesic segments $\left\{\gamma_{\zeta_t^-(n_0)}(u),\ 0 \le u \le \tau^-(n_0,t)\right\}$ (τ^- is the same as in the definition of ζ_t^-) stay between γ_η and γ_ζ, again, according to lemma 1. Therefore $\gamma_{\zeta^-(n_0)}$ is asymptotic to both γ_ζ and γ_η for the future.

Analogously $\gamma_{\zeta^+(n_0)}$ is asymptotic to both γ_ζ and γ_η for the past.

But then, $\zeta^+(n_0) = \zeta^-(n_0)$, because if this is false lemma 1 again implies

$$\sup_{t\ge 0} \text{dist}\,(\gamma_{\zeta^+_{(n_0)}}(t),\ \gamma_{\zeta^-_{(n_0)}}(t)) = \infty$$

and then for some $t_0 > 0$, $\gamma_{\zeta^+_{(n_0)}}(t_0)$ belongs to γ_ζ (or γ_η), and then $\sup_{t\le 0}$ dist $(\gamma_{\tilde{\phi}_{t_0}\zeta^+(n_0)}(t),\ \gamma_\zeta(\bar{\mathbb{R}})) = \infty$

wich is a contradiction. Then we get that $\zeta^+ = \zeta^-$ on I.

Notice that (2) can be written as

$\theta_t(n) = \tilde{\phi}(\zeta^-(n),\ T_t(n) + t) \in N_\delta(\phi_t(\zeta))$

for $n \in I$ and $t \in \mathbb{R}$, where T_t are as in remark 5. Then we can obtain positive numbers A, B and L for which

$$B \ge l(\theta_t) \ge A \int_I |\tilde{\phi}_t'(\zeta^-)'(n)|\,dn - L.$$

We claim that the slope of $\left[P_{\zeta,\zeta^-(n)}\right]'(\zeta^-)'(n)$ equals $u^-(\zeta^-(n))$ for a.e. $n \in I$.

If this were not true we could find a positive measure set $E \subset I$, so that

$\lim_{t\to+\infty} |\tilde{\phi}_t'(\zeta^-)'(n)| = \infty$, for $n \in E$, according to remark 4.

We may assume, via Egorov theorem, that the limit is uniform in E and then

$\lim_{t\to+\infty} \int_E |\tilde{\phi}_t'(\zeta^-)'(n)|dn = \infty$, which is absurd.

An analogous argument shows that the slope of $\left[P_{\zeta,\zeta^+(n)}\right]'(\zeta^+)'(n)$ equals $u^+(\zeta^+(n))$ for a.e.$n\in I$.Define $\xi: I\to UM$ as

$$\xi(n) = \zeta^-(n) = \zeta^+(n) .$$

Then $u^+(\xi(n)) = u^-(\xi(n))$ for a.e.$n \in I$ and this proves theorem 2.

Proposition 1.

Let ϕ be the geodesic flow of a compact surface without conjugate points.Then if ϕ is length expansive it is expansive.

Proof :

If ϕ is not expansive,the arguments of the proof of theorem 2 show the existence of a curve ξ such that

$$\sup_{t\in\mathbb{R}} \text{ length } \phi_t(\xi) < \infty ,$$

actualy

$$\sup_{t\in\mathbb{R}} \text{ length } \phi_t(\xi) < \frac{B+A}{L}$$

for A B and L as in the proof of thorem 2.

Now we need the following lemmas

Lemma 2

Assume that ϕ_t is expansive and α is as in the preliminaries.

For every $0 < \varepsilon < \alpha$ there is $T > 0$ such that if for some $t > 2T$ we have that

$\phi_{\tau(u)}(\eta) \in N_\varepsilon\left(\phi_u(\zeta)\right)$ for $0 \le u \le t$ and some smooth increasing function τ with $\tau(0)=0$

then $\phi_{\tau(u)}(\eta) \in N_{\varepsilon/2}\left(\phi_u(\zeta)\right)$ for $T \le u \le t-T$

Proof

If the lemma is false, for every $K > 0$ there are points ζ_K and η_K, numbers $K \le u_K \le t_K - K$ and functions $\tau_K(u) = \tau(u,\zeta_K,\eta_K)$,

such that

$$\phi(\eta_K, \tau_K(u)) \in N_\varepsilon\left(\phi_u(\zeta_K)\right) \text{ for } 0 \le u \le t_K$$

and $\text{dist}\left(\phi_{\tau_K(u_K)}(\eta_K), \phi_{u_K}(\zeta_K)\right) \ge \beta$, for suitable $\beta > 0$

Let ζ and η be limit points of the sequences $\phi_{u_K}(\zeta_K)$ and $\phi_{\tau_K(u_K)}(\eta_K)$, respectively. Then $\text{dist}(\eta,\zeta) \ge \beta > 0$. We also have that

$$\phi_{\tau_K(u+u_K)}(\eta_K) \in N_\varepsilon\left(\phi_{(u+u_K)}(\zeta_K)\right) \quad \text{for } -u_K \le u \le t_K - u_K.$$

As $t_K \to \infty$ and $t_K - u_K \to \infty$ as $K \to \infty$, we get $\phi_{\tau(t)}(\eta) \in N_\varepsilon\left(\phi_t(\zeta)\right)$ for every $t \in \mathbb{R}$ and some smooth increasing and surjective function $\tau : \mathbb{R} \to \mathbb{R}$, $\tau(0) = 0$, which is absurd.

Lemma 3.

Assume that ϕ_t and α are as in lemma 2.

Given $0 < \varepsilon < \alpha$, there is $\delta > 0$ such that if $|n| < \delta$, then $\tilde{\phi}_{\tau(u)}\left(\zeta^-(n)\right) \in N_\varepsilon\left(\tilde{\phi}_u(\zeta)\right)$ for $u \geq 0$.

Proof

The lemma is an immediate consequence of the following claim: there is $\delta > 0$ such that if $|n| < \delta$, then $\tilde{\phi}_{\tau(u)}\left(\zeta_t^-(n)\right) \in N_\varepsilon\left(\tilde{\phi}_u(\zeta)\right)$ for $0 \leq u \leq t$ and some smooth increasing function τ such that $\tau(0)=0$.

If this were not true we could find sequences n_K, t_K, u_K such that $t_K - T \leq u_K \leq t_K$, $n_K \to 0$, $t_K \to \infty$ (where T is as in the previous lemma) and

$\tilde{\phi}_{\tau_K(u)}\left(\zeta_{t_K}^-(n_K)\right) \in N_\varepsilon\left(\tilde{\phi}_u(\zeta)\right)$ for $0 \leq u \leq t_K$

and $\text{dist}\left(\tilde{\phi}_{\tau_K(u_K)}(\zeta_{t_K}^-(n_K)), \tilde{\phi}_{u_K}(\zeta)\right) = \varepsilon$

If x and y are limit points of the projections (onto UM) of $\tilde{\phi}_{t_K}(\zeta)$ and $\tilde{\phi}_{\tau_K(t_K)}\left(\zeta_{t_K}^-(n_K)\right)$ respectively, then $x \neq y$, $\pi x = \pi y$

and $\phi_{\tau(t)}(y) \in N_\varepsilon(\phi_t(x))$ for $t \leq 0$ and some smooth increasing, surjective function $\tau: \mathbb{R}^- \to \mathbb{R}^-, \tau(0) = 0$, which contradicts lemma 1.

Proposition 2

Let ϕ be the geodesic flow of a surface with no conjugate points. Then ϕ is expansive if and only if there are no bi-asymptotic geodesics on the universal covering of M.

Proof

Assume that ϕ_t is expansive. If there are two bi-asymptotic geodesics on the universal covering we can find, as in theorem 2, a point ζ for which $\zeta^+ = \zeta^-$ on some interval, but this and lemma 3 contradict expansivity. The converse is obviously true.

Proof of theorem 1:

We showed in [9] that if the geodesic flow of a compact surface is expansive then the surface has no conjugate points;this and lemma 5 of section IV prove that a) implies b).On the other hand proposition 1 says that b) implies a) and proposition 2 gives the equivalence of a) and c).This completes the proof of theorem 1.

Corollary

The geodesic flow of the example in [1] is topologically equivalent to an Anosov flow.

Proof

In [4], Ghys proves that a geodesic flow on a manifold with no conjugate points and no bi-asymptotic geodesics on the universal covering is topologically equivalent to a geodesic flow on a surface of constant negative curvature. Then the corollary follows from this and proposition 2.

IV. EXPANSIVITY IMPLIES LENGTH EXPANSIVITY

Let $\phi : M \times \mathbb{R} \to M$ be a non - singular flow on a compact riemannian manifold M. For suitable $\varepsilon > 0$ define

$$H_\varepsilon(q) = \{ \exp_q v \text{ such that } <\dot{\phi}(q,0), v> = 0 \text{ and } |v| < \varepsilon \}$$

Define

$$N(\varepsilon) = \{ (x,y) \in M\times M \text{ such that } y \in H_\varepsilon(x) \} \quad \text{and}$$

$$B(\varepsilon) = \{ (x,y) \in M\times M \text{ such that } \mathrm{dist}\,(x,y) < \varepsilon \}$$

Choose $\delta > 0$ and $c > 0$ small numbers such that there is a unique smooth function $\tau : B(\delta) \times [-c,c] \to \mathbb{R}$ such that

$$\phi(y,\tau(x,y,t)) \in H_\varepsilon(\phi(x,t) \quad \text{for} \quad t \in [-c,c] \text{ and } (x,y) \in B(\delta).$$

For $U:N(\delta) \to \mathbb{R}$ define $\dot{U}:N(\delta)\to\mathbb{R}$, the derivative of U, as

$$\dot{U}(x,y) = \frac{d}{dt}\Big|_{t=0} \{ U(\phi(x,t)), \phi(y,\tau(x,y,t))) \}.$$

Let $\ddot{U}$ stand for the derivative of $\dot{U}$.

The following lemma, which is based on techniques of [6], is proved in [9].

Lemma 4.

If ϕ is expansive there is $\sigma>0$ and a continuous function $U:N(\sigma) \to \mathbb{R}$ such that

$$U(x,y) \geq 0 \quad \text{and} \quad U(x,y) = 0 \quad \text{iff} \quad x = y$$

$$\ddot{U}(x,y) \geq 0 \quad \text{and} \quad \ddot{U}(x,y) = 0 \quad \text{iff} \quad x = y .$$

With this lemma we can prove

Lemma 5.

If ϕ is expansive it is length expansive.

Proof:

Take $0 < r < r_1$ such that if $U(x,y) \leq r$ and $U(x,z) \geq r_1$ then

$$U(y,\phi(z,\tau(y,z,0))) \geq r .$$

Choose $r_2 < r$ such that if $U(x,y) \leq r_2$ and $U(x,z) \leq r_2$ then $U(y,\phi(z,\tau(y,z,0))) < r$. Take $\rho > 0$ such that $\mathrm{dist}(x,y) \geq \rho$ if $U(x,y) \geq r_2$.

The expansivity of ϕ and the condition $\ddot{U} > 0$ permit to find $T^* > 0$ such that if $r_2 \leq U(x,y) \leq r$ and $\dot{U}(x,y) \geq 0$ ($\dot{U}(x,y) \leq 0$), then

$$U(\phi(x,t),\phi(y,\tau(x,y,t))) \geq r_1$$

for some t, $0 \leq t \leq T^*$, (resp., $-T^* \leq t \leq 0$).

Set

$$T = \sup \{ \ |\tau| \text{ such that } U(\phi(x,t),\phi(y,\tau)) \le r_1 \text{ and } |t| \le T^* \}.$$

Consider a curve $\alpha : [0,1] \to M$ not contained in an orbit of ϕ. We may assume, without loss of generality, that the image of α lies in $B(\sigma)$ in such a way that we can define α_1 as

$\alpha_1(s) = \phi\ (\alpha(s),\tau(\alpha(0),\alpha(s),0))$ for $s \in [0,1]$.

We may also assume that $U(\alpha_1(0),\alpha_1(s)) \le r_2$ for $s \in [0,1]$ and $U(\alpha_1(0),\alpha_1(1)) = r_2$.

If $\dot{U}(\alpha_1(0),\alpha_1(1) \ge 0$ we next show how the length of α_1 duplicates for the future i.e. when it is positively translated by the flow.

If $\dot{U}(\alpha_1(0),\alpha_1(1)) \le 0$ a similar procedure holds for the past.

$$\text{Set} \qquad \xi_t(s) = \phi(\alpha_1(s),\tau(\alpha_1(0),\alpha_1(s),t)) \quad .$$

Take $0 < s_0 < s_1$ such that for some t, $0 \le t \le T^*$, we have

$$U(\xi_t(0),\xi_t(s_0)) = r \quad \text{and} \quad U(\xi_t(0),\xi_t(s_1)) \ge r_1 .$$

Then

$$U(\xi_t(s_0),\phi(\xi_t(s_1),\tau(\xi_t(s_0),\xi_t(s_1),0))) \geq r$$

and therefore

$$\dot{U}(\xi_t(0),\xi_t(s_0)) > 0 \qquad \text{and}$$

$$\dot{U}(\xi_t(s_0),\ \phi(\xi_t(s_1),\tau(\xi_t(s_0),\xi_t(s_1),0))) > 0 \ .$$

On account of this we can find numbers $0 \leq t_1 \leq T^*$

and $0 \leq s_2^1 < u_2^1 \leq s_2^2 < u_2^2 < 1$ such that we can

define functions $\alpha_2^i : [\ s_2^i\ ,\ u_2^i\ ] \to M$, $i=1,2$

with the following properties

- $\alpha_2^1(s) = \phi\ (\ \alpha_1(s),\tau(\alpha_2^1(s_2^1),\alpha_1(s),t_1))$
- $\dot{U}(\alpha_2^1(s_2^1)\ ,\ \alpha_2^1(u_2^1)\ ) \geq 0$
- $U(\alpha_2^1(s_2^1)\ ,\ \alpha_2^1(u_2^1)\ ) = r_2$.
- $U(\alpha_2^1(s_2^1\ ),\alpha_2^1(s)) \leq r_2$ for $s \in [s_2^1,\ u_2^1\ ]$.

This procedure of duplication can be carried out inductively in the following way:

Suppose that for some k > 1 we have defined curves

$$\alpha_k^{i} : [s_k^{i}, u_k^{i}] \to M , \; u_k^{i} \le s_k^{i+1} , \; 1 \le i \le 2^{k-1}$$

with the following properties:

(a) $\alpha_k^{i}(s) = \phi(\alpha_1(s), \tau_{k,i}(s))$ with $|\tau_{k,i}(s)| \le (k-1)T$.

(b) $\dot{U}(\alpha_k^{i}(s_k^{i}), \alpha_k^{i}(u_k^{i})) \ge 0$.

(c) $U(\alpha_k^{i}(s_k^{i}), \alpha_k^{i}(u_k^{i})) = r_2$.

(d) $U(\alpha_k^{i}(s_k^{i}), \alpha_k^{i}(s)) \le r_2$ for $s \in [s_k^{i}, u_k^{i}]$.

Using the same procedure that we used to construct α_2^{i}, we can find numbers

$$s_k^{i} \le s_{k+1}^{2i-1} < u_{k+1}^{2i-1} \le s_{k+1}^{2i} < u_{k+1}^{2i} \le u_k^{i}$$

and curves

$$\alpha_{k+1}^{2i-1} : [s_{k+1}^{2i-1}, u_{k+1}^{2i-1}] \to M$$

$$\alpha_{k+1}^{2i} : [s_{k+1}^{2i}, u_{k+1}^{2i}] \to M$$

such that conditions (a), (b), (c) and (d) hold for k+1 instead of k.

Now define

$$\beta_k(s) = \phi\ (\ \alpha_1(s)\ , (k-1)T\ )\ .$$

On account of the previous arguments there is ρ_0 >0, close to ρ,

such that

$$\text{dist}\ (\ \beta_k(s_k^i)\ ,\ \beta_k(u_k^i)\ )\ \geq \rho_0 \quad \text{for } 1 \leq i \leq 2^{k-1}\ .$$

Then length $\beta_k \geq \rho_0\ 2^{k-1}$ and hence there is R >0

such that length $\phi_{kT}(\alpha) \geq R\ 2^k$ for $k \geq 0$.

This completes the proof of the lemma.

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FACULTAD DE INGENIERIA
IMERL
Av. J. HERRERA Y REISSIG 565,CC30
MONTEVIDEO
URUGUAY

FACULTAD DE CIENCIAS
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INDICE

INSTRUCCIONES PARA LA PRE-SENTACION DE TRABAJOS EN LAS PMU.

Dado que los articulos serán fotocopiados directamente del original, se agradece tener en cuenta las recomendaciones que siguen:

Tamaño de página: Tamaño del texto escrito: ancho, 13cm. y largo, 16.5 cm.
Datos del autor: A continuación del título conviene anotar, además del nombre del autor, su lugar de trabajo, asi como consignar al final del trabajo su dirección de contacto.
Abstract: Se debe incluir un resumen, en inglés y en el idioma del trabajo.

Numeración de las páginas: Se numerará a lapiz, para que luego se numere en función de la ubicación en el volumen.
REFERENCIAS Se sugiere presentar-las como a continuación se indica:

INSTRUCTIONS FOR THE PRESEN-TATION OF ARTICLES FOR THE PMU

Due to the fact that the articles will be photocopied directly from the originals, we will appreciate the consideration of the following recommendations:

Page size: The size of the written text is: 13cm. width, and 16,5cm. length.
Author's information: After the title it is convenient to write the author's name, his place of work, as well as at the end of the article, his contact address.
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Pages numbering: Pages should be numbered in pencil. They will be renumbered according to their place in the volume.
REFERENCES: The following presen-tation is suggested:

[9] PARANJAPE, S. R. and PARK, C. Distribution of the supremum of the two parameter Yeh-Wiener process on the boundary. J. Appl. Probability 10 (1973), 875-88.
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TEXTO: En cuanto a la presentacion del texto, el siguiente modelo es recomendable:

TEXT: The following model is re-commended for the presentation of the text:

1 Introduction. The aim of this paper is to give upper and lower estimates for the probability density. . . . .

3 Two "Kolmogorov forward inequations" for p. In what follows, D denotes the differential heat operator . . . . .

THEOREM 2. The density p satisfies the inequalities

(i) $Dp \geq 0$

and

For the proof, we shall

PROOF OF THEOREM 2. Given any

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