PUBLICACIONES MATEMATICAS DEL URUGUAY

VOLUMEN 5

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UNE GÉNÉRALISATION DE L'INVARIANT DE MALGRANGE

Marcos Sebastiani

Dédié à Monsieur le Professeur J. L. Massera

Soit M une variété différentiable (C^∞) de dimension $n \geq 2$ orientée et soit X un champ de vecteurs (C^∞) tangent à M. Soit c une sous-variété compacte à bord de M de la même dimension. Le flot H_t associé à X est bien défini pour $t \in \mathbf{R}$ au voisinage de c. Dans cet article on considére l'action de H_t sur les variétés c (et, plus généralement, sur les n-chaînes) et on applique ceci à l'étude des rapports entre germes de champs de vecteurs et de fonctions, ce qui conduit à une généralisation de l'invariant de Malgrange d'une singularité isolée.

Cet article a été motivé par [2] et par des entretiens avec Jean-Paul Brasselet à qui je suis heureux d'exprimer ici ma reconnaissance.

Une approche possible à l'action de H_t sur c est de considérer la variation des integrales $\int_{H_t(c)} \omega$ oú ω est une n-forme différentielle (C^∞) sur M. Pour ce faire, on applique le lemme suivant à un cycle fondamental de c (on travaille en homologie singulière différentiable; si d est une chaîne, ∂d denote le bord et $Sup(d)$, son support).

LEMME. Soit η une q-forme fermée sur M et soit d une q-chaîne de $M (q \geq 1)$. Alors,

$$\frac{d}{dt} \int_{H_t(d)} \eta \,|_{t=0} = q \int_{\partial d} \eta_X$$

ou η_X est la $(q-1)$ - forme définie por:

$$\eta_X(Y_1, \ldots, Y_{q-1}) = \eta(X, Y_1, \ldots, Y_{q-1})$$

Preuve

$$\frac{d}{dt}\int_{H_t(d)} \eta|_{t=0} = \int_d L_X\eta$$

oú L_X est la dérivée de Lie ([3]$chap.1ex.(3.3)$). Comme η est fermée, $L_X\eta = qd\eta_X$([3]$pg.35$) et on conclut par Stokes.

Supposons maintenant que M est un voisinage de $O \in \mathbf{R}^n$ (qu'on prendra plus petit au besoin). Soit $f : M \longrightarrow \mathbf{R}$ (C^∞) telle que $f(0) = 0, f(x) > 0$ se $x \neq 0$ et $df(x) \neq 0$ si $x \neq 0$. Soit ω une n-forme sur M. Soit $c_s = \{f \leq s\}$. Alors

$$V_\omega(s) = \frac{d}{dt}\int_{H_t(c_s)} \omega\Big|_{t=0} \quad (0 < s < \epsilon)$$

définit un germe de fonction V_ω. Si X est transverse au variétés de niveau de f (fonction de Lyapounoff) alors $V_\omega \neq 0$ quand $\omega = dx_1 \wedge \ldots \wedge dx_n$.

Considerons maintenant le cas analytique complexe c'est à dire, M est un voisinage de $O \in \mathbf{C}^n$ ($n \geq 2$), X est un champ de vecteurs analytique sur M et $f : M \longrightarrow \mathbf{C}$ est une fonction analytique telle que $f(0) = 0$, $df(0) = 0$ et $df(x) \neq 0$ si $x \neq 0$.

DEFINITION Une famille c (de n-cycles relatifs à f) est la donnée d'une n-chaîne c_s pour chaque $s > 0$ assez petit, qui dépend continûment de s, telle que $f(x) = s$ pour tout $x \in Sup(\partial c_s)$ et avec la propriété que pour chaque voisinage U de 0 il existe $\rho > 0$ tel que $Sup(c_s) \subset U$ si $0 < s < \rho$.

La famille est appelée triviale si pour chaque voisinage U de O il existe $\rho > 0$ tel que ∂c_s soit un bord dans $f^{-1}(s) \cap U$ si $0 < s < \rho$. (∂c_s est un "cycle évanescent": voir [1]$ch.\ I$). (Ces familles de chaîines remplacent les variétés c_s du cas réel).

À une famille c et à une n-forme ω holomorphe au voisinage de 0 on peut associer un germe de fonction V_ω^c défini par:

$$V_\omega^c(s) = \frac{d}{dt}\int_{H_t(c_s)} \omega\Big|_{t=0} \quad (0 < s < \epsilon)$$

D'aprés le lemme et [4]§4 on a un développement:
(*)

$$V_\omega^c(s) = \sum_{\alpha,q} a_{\alpha,q} s^\alpha \log^q s \qquad a_{\alpha,q} \in \mathbf{C}$$

où $0 \leq q \leq n-1$ et où $\alpha > 0$ et $e^{2\pi i\alpha}$

est valeur propre de la monodromie de f en 0.

THÉORÈME. a) Si la famille c est triviale, $V^c_\omega = 0$ pour toute ω.

b) Si $V^c_\omega = 0$ pour toute famille c, alors $\omega_X = df \wedge \alpha + d\beta$ au voisinage de 0, où α, β, sont des $(n-2)$-formes analytiques, et réciproquement.

c) Si f est intégrale première de X, $V^c_\omega = 0$ pour toute famille c et tonte forme ω.

d) Si X est transverse aux variétés de niveau non singulières $f(z) = w, w \neq 0$ de f, alors $V^c_\omega = 0$ pour toute forme ω implique que la famille est triviale.

Preuve (a) D'écoule du lemme et du fait que $\omega_X | f^{-1}(s), s > 0$, est fermée par des raisons de dimension.

(b) A été demontré dans [2] pour $\omega = dx_1 \wedge \ldots \wedge dx_n$, mais la preuve est la même por ω quelconque.

(c) Si f est intégrale première de X,

$$Sup(H_t(\partial c_s)) \subset f^{-1}(s)$$

pour tout t. Soit $\omega = d\eta$. Alors

$$\int_{H_t(c_s)} \omega = \int_{H_t(\partial c_s)} \eta = \int_{H_t(\partial c_s)} \eta|_{f^{-1}(s)}$$

et cette dernière intégrale est constante parce que

$$\eta | f^{-1}(s)$$

est fermée.

(d) La condition de transversalité s'exprime par:

$$\sum X_i(\partial f / \partial x_i) = g f^m$$

au voisinage de 0, avec g analytique, $g(0) \neq 0$.

Supposons la famille c non triviale. Alors, il existe une $(n-1)$ forme η holomorphe au voisinage de 0 telle que

$$\int_{\partial c_s} \eta$$

n'est identiquement nulle dans aucun intervalle $(0, \epsilon)$. ([1]ch.III§12.1). Soit

$$df \wedge \eta = h dx_1 \wedge \ldots \wedge dx_n$$

. Soit

$$\theta = hg^{-1}dx_1\wedge...\wedge dx_n$$

. Alors,

(**)

$$df\wedge(f^m\eta) = ndf\wedge\theta_X$$

En effet,

$$\theta_X = \frac{1}{n}hg^{-1}\sum(-1)^{i+1}X_i dx_1\wedge...\overset{i}{\vee}...\wedge dx_n$$

Donc,

$$ndf\wedge\theta_x = hg^{-1}\sum X_i(\partial f/\partial x_i)dx_1\wedge...\wedge dx_n = hf^m dx_1\wedge...\wedge dx_n = f^m df\wedge\eta = df\wedge(f^m\eta)$$

L'égalité (**) implique que pour tout $s > 0$ assez petit:

$$f^m\eta|f^{-1}(s) = n\theta_X|f^{-1}(s)$$

D'aprés le lemme et tenant compte du fait que $Sup(\partial c_s) \subset f^{-1}(s)$,

$$V_\theta^c(s) = n\int_{\partial c_s}\theta_x = \int_{\partial c_s} f^m\eta = s^m\int_{\partial c_s}\eta$$

Donc,

$$V_\theta^c \neq 0$$

EXEMPLE. Soit $f(x,y)$ analytique réelle au voisinage de $(0,0) \in \mathbf{R}^2$. Supposons:

$$f(0,0) = 0, f(x,y) > 0 \text{ si } (x,y) \neq (0,0)$$

et $(0,0)$ point critique isolé de l'extension analytique de f au voisinage de $(0,0) \in \mathbf{C}^2$. Soit $c = \{c_s\}$ oú c_s est un cycle fondamental de $\{f \leq s\}$. Soit $X = (f_x, f_y)$ et $\omega = dx\wedge dy$.

Alors, d'aprés le lemme

$$V_\omega^c(s) = \int_{\partial c_s}(-f_y d_x + f_x d_y) > 0$$

parce que $(-f_y, f_x)$ est tangent à ∂c_s. Donc, c n'est pas triviale.

Dans le cas particulier où f est un polynôme homogéne de degré $m \geq 2$ on a

$$V_\omega^c(s) = ks, k > 0$$

En effet,

$$V_\omega^c(s) = \int_{f \leq s} \Delta f \, dx dy$$

On calcule cette intégrale en coordonnées polaires.

Soit $f = \rho^m g(\varphi)$ où $g > 0$ est périodique de période 2π. Alors,

$$\Delta f = m(m-1)\rho^{m-2} g(\varphi) + \rho^{m-2} g''(\varphi) + m\rho^{m-2} g(\varphi)$$

Donc,

$$\begin{aligned} V_\omega^c(s) &= \int_0^{2\pi} d\varphi \int_0^{\sqrt[m]{s/g(\varphi)}} (m^2 g(\varphi) + g''(\varphi))\rho^{m-1} d\rho = \\ &= 2\pi m s + \frac{s}{m} \int_0^{2\pi} \frac{g''(\varphi)}{g(\varphi)} d\varphi = (2\pi m + \frac{1}{m} \int_0^{2\pi} \left(\frac{g'(\varphi)}{g(\varphi)} \right)^2 d\varphi) s. \end{aligned}$$

(En intégrant par parties et utilisant la périodicité de g). Dans le cas général, on peut conjecturer que

$$\lim_{s \to 0^+} V_\omega^c(s)/s$$

existe et est positif.

OBSÉRVATIONS FINALES. Les développements (*) permettent d'étendre l'invariant de Malgrange $\sigma(f)$ ([4] Remarque 4,8) aux couples (X, f). On définit $\sigma(X, f)$ comme étant la borne inférieure des α tels qu'il existe une n-forme ω et une famille c pour les-quelles $a_{\alpha,q} \neq 0$ pour un q dans le développement de $V_\omega^c(s)$. Si il n'exite pas de tel α, on pose $\sigma(X, f) = +\infty$. Naturellement, $\sigma(X, f) \geq \sigma(f)$. Si $X(0) \neq 0$, $\sigma(X, f) = \sigma(f)$. En effet, si $X(0) \neq 0$ toute $(n-1)$ - forme est localement du type ω_X, et on applique le lemme. Si 0 est un point critique isolé de X, l'égalité $g\omega_X = \omega_{gX}$ et le lemme montrent que $\sigma(X, f)$ ne dépend que de f et du feuilletage défini par X.

Dans le cas particulier de l' exemple précédent, $\sigma(X, f) \leq 1$. En particulier, $\sigma(X, f) = 1$ si $f = x^2 + y^2$ et $X = (x, y)$.

Finalement, si f est une intégrale premiére, $\sigma(X, f) = +\infty$ (Théoréme (c)). On peut conjecturer que la réciproque est vraie (comparer avec [2] et théoréme (d)).

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SUR LES ÉQUATIONS DES SÉPARATRICES ANALYTIQUES

Marcos Sebastiani

On se dome un champ de vecteurs analytique X au voisinage de $0 \in \mathbf{C}^2$ tel que $X(0) = 0$, $X(x) \neq 0$ si $x \neq 0$. Toutes les considérations qui suivent sont de nature locale en 0 et on identifie souvent un germe (d'ensemble ou de fonction) en 0 avec un représentant. On note $\mathcal{O}$ l'anneau local de $\mathbf{C}^2$ en 0.

Une *séparatrice* (analytique) est un germe Z d'ensemble analytique irréductible de dimension 1, tel que si $x \in Z$, $x \neq 0$, alors $X(x) \in T_x(Z)$. L'éxistence d'un tel object est démontrée dans [1]. Toute séparatrice Z est défine par une équation analytique $g = 0$. Par example, si $X = (xy, x + y)$, $x = 0$ définit une séparatrice analytique Z. Mais on peut aussi définir Z par l'équation $e^{-y}x = 0$. Cette derniére équation a l'avantage suivant: X est transverse aux variétés de niveau $e^{-y}x = cte. \neq 0$, tandis qu'il ne l'est pas aux variétés $x = cte. \neq 0$. (Rappelons que X n'est pas transverse en x à $f = cte. = f(x)$ si et seulement si $df(X)(x) = 0$). Ces considérations conduisent aux notions de L-équation et L-séparatrice (définition 1). Dans cet article on étudie les propriétés des L- séparatrices. On introduit une notion de "multiplicité" d'une L- séparatrice et on établit son rapport avec l'invariant de Malgrange d'une L-équation et l'invariant de Malgrange généralisé dans [5] (théoréme 2). On montre qu'une séparatrice analytique qui est l'ensemble des zéros d'une intégrale premierè n'est pas une L-séparatrice (corollaire 2).

1- DÉFINITION 1. Soit Z une séparatrice analytique. Une équation analytique $g = 0$ définissant Z est une *L-équation* si X est transverse aux variétes de niveau $g = cte. \neq 0$. Si Z admet une L-équation, on dit que Z est une *L-séparatrice.*

EXEMPLES: a) Si $X = (x, y)$, alors $x = 0$ est une L-équation. X ne posséde pas d'intégrale premiére holomorphe. Si $X = (x, -y)$, alors $x = 0$ est une L-équation. X posséde l'intégrale première xy.

b) Si $X = (2x, 3y)$ et Z est défini par $x^3 - y^2 = 0$, alors Z est une L-séparatrice.

DÉFINITION 2. Soit Y un germe analytique irréductible de dimension 1 défini pas $g = 0$ où $g \in \mathcal{O}$ est irréductible. Si $h \in \mathcal{O}$ on note $ord_Y h$ le plus grand entier m tel que g^m divise h dans $\mathcal{O}$. (Si $h = 0$ on pose $ord_Y h = +\infty$).

DÉFINITION 3. Si Z est une séparatrice analytique on définit

$$\nu_Z = Sup\{ord_Z df(X) : f = 0$$

est une équation analytique irréductible définissant $Z\}$.

OBSÉRVATIONS: a) On a $0 < \nu_Z \leq +\infty$. Si Z est l'ensemble des zéros d'une intégrale première holomorphe, alors $\nu_Z = +\infty$.

b) Les notions de L-équation, L-séparatrice et l'entier ν_Z *ne dépendent que du feuilletage* défini par X.

THÉOREME 1. Soit Z uné L-séparatrice . Alors:

a) $0 < \nu_Z < +\infty$;

b) si $g = 0$ est une équation analytique irréductible définissant Z alors $ord_Z dg(X) = \nu_Z$ si et seulement si $g = 0$ est une L-équation.

c) $\nu_Z = min\{ord_Z dh(X) : h \in \mathcal{O}$ et $dh(X) = 0$ est une équation définissant $Z\}$.

COROLLAIRE 1. Soit Z une L-séparatrice. Alors,

a) il exist une L-équation irréductibledéfinissant Z;

b) si $\nu_Z = 1$, tonte équation analytique irréductible définissant Z est une L-équation.

COROLLAIRE 2. Si Z est une séparatrice analytique qui est l'ensemble des zéros d'une intégrale première holomorphe, alors Z n'est par une L-séparatrice.

Preuve. Soit f une intégrale premiére, $f \in \mathcal{O}$, telle que $f = 0$ définit Z. Supposons qu'il exist une L-équation $g = 0$ définissant Z, g irréductible dans

$\mathcal{O}$ (corollaire 1). Alors,

$$f = ug^m, m \geq 1,$$

$$u \in \mathcal{O}, u(0) \neq 0.$$

Soit

$$dg(X) = vg^\nu, \nu = \nu_Z,$$

$$v \in \mathcal{O}, v(0) \neq 0.$$

Alors,

$$0 = df(X) = g^m du(X) + muvg^{m+\nu-1}$$

Si $\nu = 1, du(X) + muv = 0$,
ce qui est impossible car $X(0) = 0$.
Si $\nu > 1, du(X) = -muvg^{\nu-1}$. Alors $du(X) = 0$ définit Z et $ord_Z du(X) = \nu - 1$, ce qui contredit la partie (c) du Théoréme 1.

EXEMPLE. Soit $X = (2y, 3x^2)$ qui admet $f = x^3 - y^2$ comme intégrale première. Alors $Z = \{x^3 - y^2 = 0\}$ est une sépaaratrice analytique qui n'est par une L-séparatrice.

COROLLAIRE 3. Si il existe une L-séparatrice Z telle que $\nu_Z > 1$,alors X n'admet pas d'intégrale première holomorphe.

Preuve. Soit $g = 0$ une L-équation irréductible de la L-séparatrice Z. Supposons que $\nu_Z > 1$ et qu'il existe une intégrale première f. Alors, $f = hg^m$, $h \in \mathcal{O}$, $m \geq 1$ et g ne divise pas h dans $\mathcal{O}$ (car on peut supposer $f(0) = 0$).

D'après le corollaire 2, $h(0) = 0$.

D'autre part,

$$0 = df(X) = g^m dh(X) + mg^{m-1}hdg(X).$$

D'après le théorème 1 et $\nu_Z > 1$, on a que g^2 divise $dg(X)$. Donc, g divise $dh(X)$. Cela implique que h/Z est constante. Comme $h(0) = 0$, on a $h/Z = 0$. Alors g divise h, ce qui est une contradiction.

EXEMPLES: a) Si $X = (xy, x + y)$ on a vu que $Z = \{x = 0\}$ est une L-séparatrice définie par la L-équation $e^{-y}x = 0$. Comme $d(e^{-y}x)(X) = -e^{-y}x^2$ on a $\nu_Z = 2$. Alors X n'admet par d'intégrale première holomorphe.

b) Soit $X = (2xy^2+xy-x^2, y^3+y^2-xy)$. Alors X n'admet par d'intégrale première ([2] ch. II et [6]) et $Z = \{x = 0\}$ est une séparatrice analytique qui n'est par une L-séparatrice. En effet, supposons que $g = 0$ est une L-équation irréductible définissant Z. Alors $g = ux$, $u \in \mathcal{O}$, $u(0) \neq 0$. En plus,

$$dg(X) = (u + xu_x)(2xy^2 + xy - x^2) + xu_y((y^3 + y^2 - xy).$$

Puisque $u(0) \neq 0, xy$ a un coefficient non-nul dans la série de Taylor de $dg(X)$. Ceci implique

$$\nu_Z = ord_Z dg(X) = 1$$

(Théorème 1).

Mais alors, par le corollaire 1(b), $x = 0$ serait aussi une L-équation définissant Z. Mais comme

$$dx(X) = 2xy^2 + xy - x^2 = x(2y^2 + y - x),$$

$dx(X) = 0$ ne définit pas Z, ce qui est une contradiction.

Démonstration du théorème 1. Soit $g = 0$ une L-équation définissant Z. Comme Z est irréductible, $g = \tilde{g}^m$, $\tilde{g} \in \mathcal{O}$ irréductible, $m \geq 1$. Alors,

$$dg(X) = m\tilde{g}^{m-1}d\tilde{g}(X).$$

Donc, $d\tilde{g}(X)$ divise $dg(X)$. Alors $d\tilde{g}(X) = 0$ définit Z et $\tilde{g} = 0$ est une L-équation irréductible définissant Z.

Fixons une L-équation irréductible $g = 0$ définissant Z. Alors $dg(X) = 0$ définit Z. Donc, (c) entraîne (a).

Pour prouver (c), soit $h \in \mathcal{O}$, avec $h(0) = 0$, tel que $dh(X) = 0$ définit Z et

$$ord_Z dh(X) \leq ord_Z dk(X)$$

pour tout $k \in \mathcal{O}$ tel que $dk(X) = 0$ définit Z.

Soit $f = 0$ une équation irréductible définissant Z. Comme $dh(X)/Z = 0$ et $T_x(Z) = \mathbb{C}X(x)$ pour tout $x \in Z, x \neq 0, h/Z$ est constante. Donc, $h/Z = 0$.

Alors, $h = h'f, h' \in \mathcal{O}$. Donc,

$$dh(X) = fdh'(X) + h'df(X).$$

Soient $df(X) = uf^m, dh(X) = vf^n, m, n \geq 1$, $u, v \in \mathcal{O}$, $v(0) \neq 0$. Alors,

$$vf^n = fdh'(X) + uh'f^m.$$

Supposens $m > n$. Si $n = 1$ on a:

$$v = dh'(X) + uh'f^{m-1}$$

ce qui est impossible parce que $X(0) = 0$. Si $n > 1$,

$$dh'(X) = (v - uh'f^{m-n})f^{n-1}$$

Donc, $dh'(X) = 0$ définit Z et

$$ord_Z dh'(X) = n - 1 = ord_Z dh(X) - 1$$

ce qui est une contradiction.

On a prouvé $m \leq n$. En particulier,

$$ord_Z dg(X) \leq n \leq ord_Z dg(X).$$

Donc,

$$ord_Z df(X) = m \leq n = ord_Z dg(X)$$

ce qui prouve (c) et aussi que $ord_Z dg(X) = \nu_Z$.

Soit $f = 0$ une équation irréductible définissant Z telle que $ord_Z df(X) = \nu_Z$. Alors,

$$df(X) = kg^m, dg(X) = ug^m, f = vg$$

où $k, u, v \in \mathcal{O}$, $m = \nu_Z, u(0)v(0) \neq 0$. Alors,

$$df(X) = gdv(X) + vdg(X).$$

D'où,

$$dv(X) = g^{m-1}(k - vu).$$

Si $m = 1$, comme $X(0) = 0$, on a $k(0) \neq 0$.

Si $m > 1$ et $h(0) = 0, dv(X) = 0$ définit Z et

$$ord_Z dv(X) = m - 1 < \nu_Z$$

ce qui contredit (c). Donc $k(0) \neq 0$. Alors, $f = 0$ est une L-équation définissant Z.

2- THÉORÈME 2. Soit $f = 0$ une L-équation irréductible définissant la L-séparatrice Z. Alors

$$\tau(X, f) = \tau(f) + \nu_Z - 1$$

orí $\tau(f)$ est l'invariant de Malgrange ([4] remarque 4.8) et $\tau(X, f)$ en est la généralisation définie dans [5].

Preuve. Un a $df(X) = uf^m, u \in \mathcal{O}$, $u(0) \neq 0, m = \nu_Z$.

En divisant X par u on peut supposer $df(X) = f^m$. Donc, si H_t est le flot engendré par X, on a;

$$f^0 H_t(x) = g(t, f)$$

où $g \in \mathcal{O}$ et $g(0, s) = s$ et $\frac{\partial g}{\partial t}(0, s) = s^m$.

Soit η un germe de 1-forme holomorphe. Soient α, ω des germes de formes holomorphes tels que:

$$d\alpha = df \wedge \eta, \omega = d\eta.$$

Soit $c_s (0 < s < \varepsilon)$ une famille de 2-cycles relatifs à f (au seus de [5]; alors ∂c_s est un cycle évanescent). Considérons les fonctions:

$$\varphi_\eta(t, s) = \int_{H_t(c_s)} df \wedge \eta = \int_{\partial c_s} H_t^*(\alpha)$$

$$\psi_\eta(t, s) = \int_{H_t(c_s)} \omega = \int_{\alpha c_s} H_t^*(\eta).$$

Ces fonctions sont analytiques pour $|t| < \rho, 0 < |s| < \varepsilon, \frac{-\pi}{2} < \arg s < \frac{\pi}{2}$.

Puisque $dH_t^*(\alpha) = df \wedge \frac{\partial g}{\partial s}(t, f) H_t^*(\eta)$, on a que

$$\frac{\partial \varphi_\eta}{\partial s}(t, s) = \frac{\partial g}{\partial s}(t, s) \psi_\eta(t, s)$$

([4],(4.3)).

D'autre part (voir [5]),

$$\frac{\partial\varphi_\eta}{\partial t}(0,s) = 2\int_{\partial_{\epsilon,s}}(df\wedge\eta)_X = \int_{\partial_{\epsilon,s}} f^m\eta = s^m\psi_\eta(0,s)$$

(si ω est une 2-forme, ω_X est la 1-forme: $\omega_X(Y) = \omega(X,Y)$, voeir [3] pg. 35).

Alors, $\frac{\partial}{\partial s}\frac{\partial}{\partial t}\varphi_\eta(0,s) = ms^{m-1}\psi_\eta(0,s) + s^m\frac{\partial\psi_\eta}{\partial s}(0,s)$

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s}\varphi_\eta(0,s) = ms^{m-1}\psi_\eta(0,s) + \frac{\partial\psi_\eta}{\partial t}(0,s).$$

Donc,

$$s^m\frac{\partial\psi_\eta}{\partial s}(0,s) = \frac{\partial\psi_\eta}{\partial t}(0,s).$$

D'après les définitions, ceci entraine

$$m + \tau(f) - 1 = \tau(X,f).$$

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RECOUVREMENTS ALÉATOIRES (*)

Jean-Pierre KAHANE (Université Paris-Sud, Orsay)

Quelle est la probabilité pour que des objets aléatoires recouvrent un objet donné ? C'est un sujet qui a une longue histoire, et qui n'avance pas très vite. Je saisis volontiers l'occasion que me donne la Gazette de raconter ce que j'en sais, parce que l'histoire en est intéressante, et que les problèmes en suspens méritent d'être connus.

De Borel à Dvoretzky

L'histoire remonte à Emile Borel. Dans les années 1890, le prolongement analytique des fonctions données par une série de Taylor était à l'ordre du jour. En 1892, Poincaré et Hadamard avaient découvert des êtres étranges : des séries de Taylor non prolongeables hors du cercle de convergence. Juste après sa thèse (1895), Borel publie une note aux Comptes-Rendus (1896) et un article dans Acta Mathematica (1897) avec un énoncé provocateur : *en général, une série de Taylor admet son cercle de convergence comme coupure*. Il faut entendre par là que, si les modules des coefficients sont donnés et que les arguments sont des variables aléatoires indépendantes équidistribuées sur $(0, 2\pi)$, la probabilité d'avoir un point régulier sur le cercle de convergence est nulle. Sous cette forme, l'énoncé et la démonstration sont dus à Steinhaus (1929). Borel attribuait une grande importance à son résultat. Comme il le dit dans sa Notice de 1912, *la difficulté principale était d'en préciser le sens avant d'en donner la démonstration*. En vérité, *préciser le sens* a été une oeuvre de longue haleine, et ce n'est pas étonnant parce que l'énoncé implique des concepts fondamentaux (la probabilité nulle, l'indépendance) qui étaient encore loin de toute formalisation. Les principales étapes, avant Kolmogorov 1933, en ont été *les probabilités dénombrables* de Borel, et la réduction de la probabilité à la mesure de Lebesgue par Steinhaus. Par contre, *donner la démonstration*, en se fiant à l'intuition pour les concepts fondamentaux, était à la portée de Borel dès 1897. La démonstration de Borel consiste à partager la série en blocs de termes consécutifs, et à associer à chaque bloc un intervalle (aléatoire) du cercle de convergence. Ensuite, je cite, *on a donc sur un cercle une infinité d'arcs indépendants, dont la somme dépasse tout nombre donné, donc, en général, tout point du cercle appartiendra à une infinité d'arcs*. Et de là résulte que tout point du cercle est singulier.

Répétons l'argument de Borel. On a sur le cercle (disons maintenant le cercle $\mathbf{T}$, de longueur 1) des arcs $I_n(\omega)$ $(n = 1, 2, \cdots)$ dont les longueurs ℓ_n sont données, et dont les centres ω_n sont aléatoires, indépendants, uniformément distribués sur $\mathbf{T}$. Ainsi la probabilité qu'un point donné du cercle appartienne à $I_n(\omega)$ est ℓ_n. Si $\sum \ell_n = \infty$, ce point appartient presque

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sûrement à $\overline{\lim}\, I_n(\omega)$ (Borel-Cantelli). On voit que Borel-Cantelli est utile (et de plus parfaitement énoncé pour sa partie non évidente, sur un cas particulier équivalent au cas général) par Borel dès 1897.

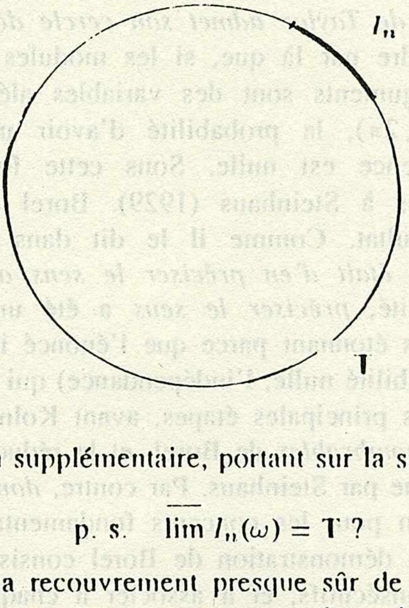
Poursuivons. Supposons toujours $\sum \ell_n = \infty$; alors

$$\forall t \in \mathbf{T} \qquad \text{p. s. } t \in \overline{\lim}\, I_n(\omega).$$

Désignons par λ la mesure de Lebesgue sur T et par P la probabilité sur Ω, et appliquons Fubini. L'ensemble des $(t, \omega) \in \mathbf{T} \times \Omega$ pour lesquels $t \in \overline{\lim}\, I_n(\omega)$ est de mesure pleine pour la mesure $\lambda \otimes P$. Par conséquent

$$\text{p. s.} \quad \lambda(\overline{\lim}\, I_n(\omega)) = 1.$$

Presque sûrement presque tout le cercle est recouvert.



A quelle condition supplémentaire, portant sur la suite ℓ_n, a-t-on

$$\text{p. s.} \quad \overline{\lim}\, I_n(\omega) = \mathbf{T}\ ?$$

On dit alors qu'il y a recouvrement presque sûr de T par les $I_n(\omega)$ (il est facile de voir que le recouvrement presque sûr et le recouvrement presque sûr par une infinité de $I_n(\omega)$ sont équivalents). La suite ℓ_n étant donnée, ne vérifiant pas cette condition supplémentaire, peut-on déterminer quelles parties A de T sont recouvertes presque sûrement :

$$\text{p. s.} \quad A \subset \overline{\lim}\, I_n(\omega)\ ?$$

La première question est posée par A. Dvoretzky en 1957. La seconde est abordée dans la première édition de mon livre *Some random series of functions* (1968), en vue de la détermination de la dimension de Hausdorff de l'ensemble aléatoire

$$F = \mathbf{T} \setminus \overline{\lim}\, I_n(\omega).$$

De Dvoretzky à Shepp

La question de Dvoretzky intéressa Paul Lévy, qui me la communiqua. Ma première contribution fut d'observer qu'il y a recouvrement quand $\ell_n = \frac{1+\epsilon}{n}$ ($\epsilon > 0$, $n \geq n_0$), et que la condition de recouvrement n'est pas stable par changement de (ℓ_n) en $(\lambda \ell_n)$ ($\lambda < 1$ fixe) (1959). Ensuite Pierre Billard introduisit une méthode, dont je dirai un mot tout à l'heure, qui permet de montrer le non recouvrement pour $\ell_n = \frac{1-\epsilon}{n}$ (1965). Le cas $\ell_n = \frac{1}{n}$ restait en suspens. P. Erdös avait annoncé que c'est un cas de recouvrement (1961), mais n'avait pas donné de preuve. Billard fut seulement capable de prouver que, dans ce cas, l'ensemble non recouvert est au plus dénombrable (le lecteur comprend bien que je sous-entends désormais *presque sûrement).*

Voici la situation telle qu'elle était connue en 1968. Prenons d'abord $\ell_n = \frac{a}{n}$. T est recouvert si $a > 1$, non recouvert si $a < 1$. Lorsque $a < 1$, l'ensemble non recouvert a pour dimension $1 - a$. Une partie donnée A de T est recouverte si $\dim A < a$, et non recouverte si $\dim A > a$. Le cas $\ell_n = \frac{1}{n}$ est ouvert, et le cas $\dim A = a$ nécessite une investigation plus poussée.

La méthode de Billard est le premier modèle d'une théorie qui s'est ensuite développée dans diverses directions, celle des produits de poids aléatoires indépendants. Ici les poids sont

$$P_n(t, \omega) = \frac{1 - \chi_n(t - \omega_n)}{1 - \ell_n}$$

où $\chi_n(t - \omega_n)$ est la fonction indicatrice de $I_n(\omega)$. Les produits

$$Q_n(t, \omega) = \prod_{m=1}^{n} P_m(t, \omega)$$

forment une martingale positive pour chaque t fixé, et il en est de même pour les intégrales

$$J_n(\omega) = \int_{\mathrm{T}} Q_n(t, \omega)\,dt.$$

Si les $I_n(\omega)$ recouvrent T, $J_n(\omega)$ est nulle à partir d'un certain rang : la martingale est dégénérée. Si au contraire la martingale est uniformément intégrable dans $L^1(\Omega)$, elle n'est pas dégénérée, donc les $I_n(\omega)$ ne recouvrent pas T. Une condition simple d'intégralité uniforme est que les $J_n(\omega)$ soient bornées dans $L^2(\Omega)$, soit

$$EJ_n^2(\cdot) = \iint_{\mathrm{T}^2} E(Q_n(t, \cdot)Q_n(s, \cdot))\,dt\,ds = O(1),$$

c'est-à-dire, en posant

$$k_n(t - s) = E(Q_n(t, \cdot)Q_n(s, \cdot)),$$

$$\int_{\mathrm{T}} k_n(t)\,dt = O(1).$$

Cette condition s'écrit encore

$$k \in L^1(\mathrm{T})$$

avec

$$k(t) = \exp \sum_1^{\infty} (\ell_n - |t|)^+, \quad -\frac{1}{2} < t < \frac{1}{2}$$

(par exemple, si $\ell_n = \frac{a}{n}$, $k(t) \approx |t|^{-a}$). Billard en donne une forme affaiblie plus lisible : pour que T ne soit pas recouvert (au sens $\mathrm{T} \neq \overline{\lim}\, I_n(\omega)$), il suffit que la série

$$(B) \qquad \sum_1^{\infty} \ell_n^2 \exp(\ell_1 + \ell_2 + \cdots \ell_n)$$

converge (on suppose ici, et désormais, $\ell_1 \geq \ell_2 \geq \ell_3 \geq \cdots$).

La méthode est flexible. S'il s'agit de recouvrir un borélien A au lieu de T, on prend une mesure positive σ portée par A, et on considère la martingale

$$J_n(\omega; \sigma) = \int_{\mathrm{T}} Q_n(t, \omega) d\sigma(t).$$

La condition qu'elle soit bornée dans $L^2(\Omega)$ s'écrit maintenant

$$\int\int_{\mathrm{T}^2} k(t - s) d\sigma(t) d\sigma(s) < \infty,$$

ce qu'on exprime en disant que l'énergie de σ par rapport au noyau $k(\cdot)$ est finie. Or, la condition pour que A porte une telle mesure, de k-énergie finie (et non nulle, bien sûr), est que la capacité de A par rapport à $k(\cdot)$ soit positive :

$$\mathrm{Cap}_k A > 0.$$

Tout cela, avec l'application à une formule explicite pour la dimension de l'ensemble non recouvert, se trouve dans mon livre de 1968 (sauf que le terme de martingale n'est pas utilisé). Nous étions alors loin de penser que les conditions obtenues par la méthode de Billard étaient nécessaires et suffisantes.

Le livre attira l'attention sur le sujet, et en particulier sur le cas en suspens : $\ell_n = \frac{1}{n}$. En 1971, ce cas fut traité par Steven Orey et, indépendamment, par Benoît Mandelbrot : il y a recouvrement, comme Erdös l'avait prévu. L'étude d'Orey ne fut jamais publiée, parce que, quelques semaines après qu'il l'ait faite, Lawrence Shepp obtint la solution complète : T est recouvert si la série

$$(S) \qquad \sum_{-\infty}^{\infty} \frac{1}{n^2} \exp(\ell_1 + \ell_2 + \cdots + \ell_n)$$

diverge, et n'est pas recouvert si elle converge. En fait, la convergence de la série (S) équivaut à

$$k \in L^1(\mathbb{T}),$$

que la méthode de Billard nous a fait voir comme condition suffisante de non recouvrement. La convergence de (B) entraîne celle de (S), mais ne lui est pas équivalente - l'écart n'est pas bien grand, puisque les deux séries sont simultanément convergentes ou divergentes lorsque $\inf(n\ell_n) > 0$. La partie difficile et nouvelle, dans le théorème de Shepp - outre l'introduction élégante de la série (S) - est de montrer qu'il y a recouvrement lorsque $k \notin L^1$.

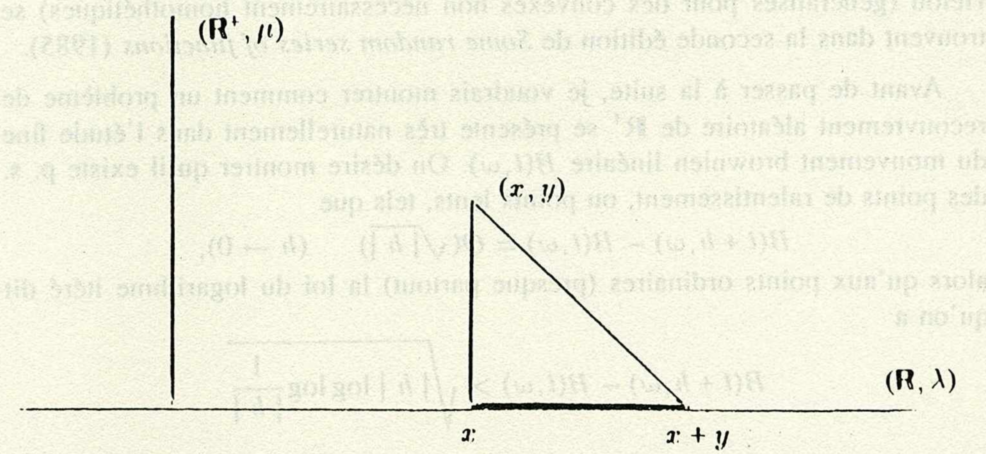
Avant de s'occuper du recouvrement à la Dvoretzky, B. Mandelbrot avait introduit, toujours en 1972, un autre type de *random cutouts*, sur la droite cette fois. On donne (au lieu des longueurs ℓ_n) une mesure μ localement bornée sur $(0, \infty)$, mais, dans les cas intéressants, non bornée au voisinage de 0 (on peut penser à $\mu = \sum \delta_{\ell_n}$). On considère le processus de Poisson ponctuel dans $\mathbb{R} \times \mathbb{R}^+$ dont l'intensité est $\lambda \otimes \mu$ (λ est la mesure de Lebesgue sur $\mathbb{R}$). A chaque point (x, y) de ce processus ponctuel on associe l'intervalle $(x, x + y)$, et on demande à quelle condition sur μ la droite $\mathbb{R}$ est recouvert p. s. par ces intervalles. La réponse, toujours due à Shepp (1972), s'exprime sous la forme

$$k \notin L^1(e^{-t}dt),$$

avec maintenant

$$k(t) = \exp \int_t^\infty \mu(y, \infty) dy.$$

Il est clair qu'il y a un rapport entre les deux questions. Néanmoins, ce n'est qu'en 1987, à l'occasion d'un cours que j'ai donné à Urbana, que le premier théorème de Shepp est apparu comme conséquence du second. Je dirai tout à l'heure comment le second se démontre facilement, à l'aide d'une méthode de temps d'arrêt, due à Svante Janson (1983), que j'ai beaucoup exploitée dans mon cours d'Urbana et ensuite.



Thème et variations

Revenons à 1973. Les théorèmes de Shepp ont rendu le sujet relativement populaire. Mario Wschebor, alors mon élève, étudie le recouvrement du cercle par des ensembles translatés au hasard, et la question analogue pour la droite et les processus de Poisson; essentiellement, le résultat exprime que, parmi les ensembles E_n dont les mesures de Lebesgue ℓ_n sont données, ceux qui recouvrent le mieux sont les intervalles de longueurs ℓ_n (le résultat est intuitif, si l'on songe au cas des parfaits totalement discontinus, dont la réunion ne recouvre jamais le cercle). John Hawkes s'intéresse au recouvrement d'une partie A de $\mathbb{T}$ par les intervalles I_n, et aux propriétés de l'ensemble non recouvert F - par exemple, dans le cas $\ell_n = \frac{a}{n}$, $0 < a < 1$, F est p. s. un ensemble $\mathcal{D}_a$ mais pas un ensemble $\mathcal{D}$, en désignant par $\mathcal{D}$ la classe des ensembles dont la dimension d'entropie et la dimension de Hausdorff sont égales. Jorgen Hoffmann-Jørgensen élargit le cadre, en considérant le recouvrement d'un compact métrique par des boules aléatoires; c'est en effet le cadre naturel pour la plupart des applications.

Un peu plus tard, en 1978, Youssef El Hélou étudie le recouvrement de $\mathbb{T}^d$ par des convexes homothétiques aléatoires : la forme des convexes et la suite de leurs volumes v_n sont données; les translations sont aléatoires, indépendantes, distribuées selon la mesure de Haar sur $\mathbb{T}^d$. El Hélou généralise les résultats de Billard et les miens :

1) il n'y a pas recouvrement lorsque

$$\sum v_n^2 \exp(v_1 + \cdots + v_n) < \infty.$$

2) si $v_n = \frac{a}{n}$ avec $a < 1$, une partie donnée A de $\mathbb{T}^d$, borélienne, est recouverte si $\dim A < ad$, et non recouverte si $\dim A > ad$; l'ensemble non recouvert a pour dimension $(1-a)d$.

3) si $v_n = \frac{1}{n}$, l'ensemble non recouvert est au plus dénombrable.

Le théorème de Shepp sur le recouvrement de $\mathbb{T}$ et les résultats de El Hélou (généralisés pour des convexes non nécessairement homothétiques) se trouvent dans la seconde édition de *Some random series of functions* (1985).

Avant de passer à la suite, je voudrais montrer comment un problème de recouvrement aléatoire de $\mathbb{R}^+$ se présente très naturellement dans l'étude fine du mouvement brownien linéaire $B(t,\omega)$. On désire montrer qu'il existe p. s. des points de ralentissement, ou points lents, tels que

$$B(t+h,\omega) - B(t,\omega) = O(\sqrt{|h|}) \qquad (h \to 0),$$

alors qu'aux points ordinaires (presque partout) la loi du logarithme itéré dit qu'on a

$$B(t+h,\omega) - B(t,\omega) > \sqrt{|h| \log\log \frac{1}{|h|}}$$

pour une infinité de valeurs de h tendant vers 0. Pour cela, cherchons à montrer un peu plus, c'est-à-dire l'existence de points t tels que

$$\begin{cases} B(t,\omega) = 0 \\ B^2(t+h,\omega) \leq \lambda \mid h \mid \quad \text{pour tout } h \end{cases}$$

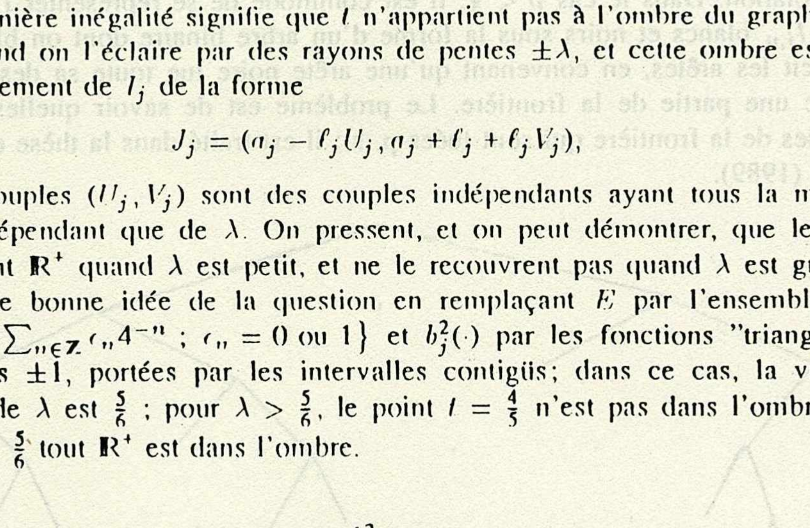
λ étant une constante positive à déterminer. Ici intervient une construction de Paul Lévy : pour obtenir $B(\cdot,\omega)$, on peut d'abord mettre en place l'ensemble E des zéros, qui est un ensemble aléatoire assez bien connu, de dimension $\frac{1}{2}$, puis, sur chacun des intervalles $I_j = (a_j, a_j + \ell_j)$ contigüs à E, placer, indépendamment les unes des autres, des "excursions" $b_j(\cdot,\omega)$, dont la loi est également bien connue. Sur E, on prend $B(\cdot,\omega) = 0$, et sur I_j, $B(\cdot,\omega) = b_j(\cdot,\omega)$. Les points t cherchés sont donc les $t \in E$ tels que, pour tout j,

$$b_j^2(t+h,\omega) \leq \lambda \mid h \mid .$$

Cette dernière inégalité signifie que t n'appartient pas à l'ombre du graphe de $b_j^2(\cdot)$ quand on l'éclaire par des rayons de pentes $\pm\lambda$, et cette ombre est un agrandissement de I_j de la forme

$$J_j = (a_j - \ell_j U_j, a_j + \ell_j + \ell_j V_j),$$

où les couples (U_j, V_j) sont des couples indépendants ayant tous la même loi, ne dépendant que de λ. On pressent, et on peut démontrer, que les J_j recouvrent $\mathbb{R}^+$ quand λ est petit, et ne le recouvrent pas quand λ est grand. On a une bonne idée de la question en remplaçant E par l'ensemble de Cantor $\left\{\sum_{n\in\mathbb{Z}} \epsilon_n 4^{-n} \; ; \; \epsilon_n = 0 \text{ ou } 1\right\}$ et $b_j^2(\cdot)$ par les fonctions "triangles", de pentes ± 1, portées par les intervalles contigüs; dans ce cas, la valeur critique de λ est $\frac{5}{6}$; pour $\lambda > \frac{5}{6}$, le point $t = \frac{4}{5}$ n'est pas dans l'ombre, et pour $\lambda < \frac{5}{6}$ tout $\mathbb{R}^+$ est dans l'ombre.

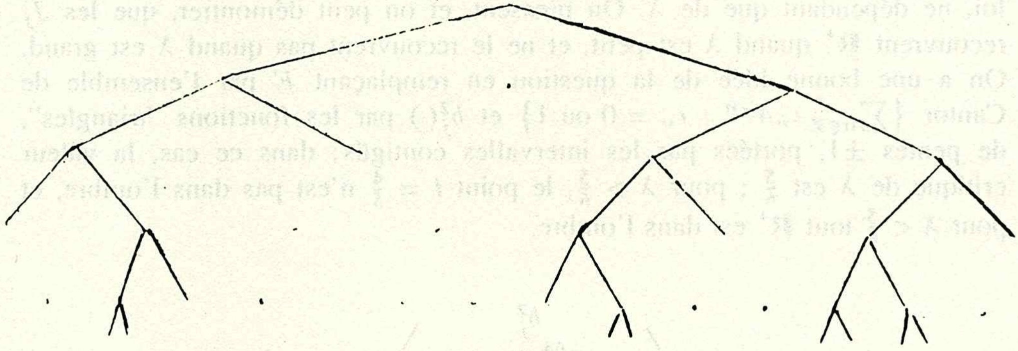


D'ailleurs, il n'existe pas de points t tels que

$$(*) \qquad B(t+h,\omega) - B(t,\omega) = o(\sqrt{\mid h \mid}) \qquad (h \to 0) \; ;$$

c'est la forme forte du théorème de non-dérivabilité donnée par Dvoretzky, et c'est encore une affaire de recouvrement aléatoire. Ici le principe est d'associer à tout intervalle dyadique $I_{j,n} = \left[\frac{n}{2^j}, \frac{n+1}{2^j}\right]$ ($j = 0, 1, \cdots$; $n = 0, 1, \cdots 2^j - 1$) l'écart $X_{j,n}$ entre la valeur de $B(\cdot, \omega)$ en son milieu et la moyenne de ses valeurs aux extrémités. Les variables aléatoires $X_{j,n}$ sont gaussiennes centrées, de variances 2^{-j-2}, et indépendantes. Ainsi, pour un λ donné, les évènements $X_{j,n}^2 > \lambda 2^{-j}$ sont indépendants et de même probabilité $p = p(\lambda)$. Appelons blancs les intervalles $I_{j,n}$ correspondants, et noirs les autres. S'il existait un point t vérifiant (*), les $I_{j,n}$ contenant t seraient tous blancs à partir d'un certain rang $j_0(\omega)$, donc les $I_{j,n}$ noirs ne recouvriraient pas l'intervalle $[0, 1]$ une infinité de fois. Or on reconnaît un processus de naissance et de mort, où la valeur critique est $p = \frac{1}{2}$: pour $p \leq \frac{1}{2}$, il ne reste aucun point blanc. Donc il n'existe pas de t vérifiant (*).

Cette preuve est l'occasion de voir les processus de naissance et de mort comme des problèmes de recouvrement. Il en est de même pour la percolation. Dans le cas $p < \frac{1}{2}$, il est commode de se représenter l'évolution des $I_{j,n}$ blancs et noirs sous la forme d'un arbre binaire dont on blanchit ou noircit les arêtes, en convenant qu'une arête noire tue toute sa descendance, donc une partie de la frontière. Le problème est de savoir quelles sont les parties de la frontière qui sont tuées p. s.; il est traité dans la thèse de Fan Ai Hua (1989).



Il y a bien d'autres aspects des problèmes de recouvrement. Ils semblent pertinents pour analyser les cratères de la lune, la vision à travers les galaxies, les interruptions de trafic routier ou la saturation des lignes téléphoniques. Au cours des années 1980 ils ont donné lieu à un grand nombre de travaux, que je ne m'efforcerai pas de résumer. Le lecteur intéressé peut consulter les livres de Mandelbrot (1985) et de Hall (1988).

Je me bornerai à signaler deux articles de Svante Janson (1983,1985). Le premier introduit la méthode de temps d'arrêt que j'appliquerai tout à l'heure à la démonstration du théorème de Shepp. Le second considère la question du recouvrement de T^d par des translatés aléatoires $K + \omega_n$ d'un convexe K ; très curieusement, la loi du nombre minimum de translatés $K + \omega_1$,

$K+\omega_2, \cdots K+\omega_n$ dont la réunion recouvre $\mathbb{T}^d$ dépend non seulement du volume de K, qui donne le terme principal, mais de sa forme : pour $d \geq 3$, les boules couvrent mieux que les cubes!

Actualité

J'en viens à l'état actuel de la question, tel du moins que je le connais.

Voici le cadre général. On donne un espace de probabilité, des objets aléatoires indépendants $G_n(\omega)$, et un objet fixe A, et on cherche la probabilité pour que A soit recouvert une infinité de fois par les $G_n(\omega)$. D'après la loi du zéro-un, c'est nécessairement 0 ou 1. On suppose naturellement que les objets $G_n(\omega)$ et A sont des parties d'un espace T ; il est bon de supposer T localement compact (pratiquement, ce sera $\mathbb{R}^d$ ou $\mathbb{T}^d$), A fermé et les $G_n(\omega)$ ouverts. On pose

$$k_N(t,s)=\prod_{n=1}^{N}\frac{1-P(s\in G_n)-P(t\in G_n)+P(\{s,t\}\subset G_n)}{(1-P(s\in G_n))(1-P(t\in G_n))}$$

La méthode de Billard donne le résultat le plus facile de la théorie : *si A porte une mesure de probabilité σ d'énergie bornée par rapport au noyau $k_N(t,s)$ quand $N\to\infty$, A est p. s. non recouvert une infinité de fois par les $G_n(\omega)$.*

On voit, sur des exemples, que c'est loin d'être une condition nécessaire et suffisante. Cependant, curieusement, la condition est bien nécessaire et suffisante dans certains cas très naturels.

Exemple 1 (recouvrement à la Dvoretzky) : $T=\mathbb{T}$, $G_n(\omega)=I_n(\omega)=\left(-\frac{\ell_n}{2}+\omega_n, \frac{\ell_n}{2}+\omega_n\right)$. La condition, comme on l'a vu, s'écrit $\mathrm{Cap}_k A>0$ avec

$$k(t)=\exp\sum_{1}^{\infty}(\ell_n-|t|)^+$$

(par exemple, $\mathrm{Cap}_\alpha A>0$ si $\ell_n=\frac{\alpha}{n}$; par exemple encore, $k\in L^1(\mathbb{T})$ si la mesure de Lebesgue de A est positive). *C'est une condition nécessaire et suffisante de non recouvrement de* A (généralisation du théorème de Shepp).

Exemple 2 (recouvrement de $\mathbb{T}^d$ par des convexes homothétiques aléatoires) : $T=\mathbb{T}^d$, $G_n(\omega)=g_n+\omega_n$, où les g_n sont des convexes homothétiques donnés de volumes v_n $(1>v_1\geq v_2\geq\cdots)$, (ω_n) une suite de variables aléatoires indépendantes uniformément distribuées sur $\mathbb{T}^d$, et $A=\mathbb{T}^d$. La condition est

$$(B_d):\quad \int_0^1 \exp\sum_{n=1}^{\infty}v_n\left(1-\left(\frac{s}{v_n}\right)^{1/d}\right)^+ ds<\infty.$$

Si les g_n sont des simplexes, c'est une condition nécessaire et suffisante de non recouvrement de $\mathbb{T}^d$ (autre généralisation du théorème de Shepp).

Ecrivons (B_∞) pour la convergence de la série de Billard :

$$(B_\infty) \quad : \quad \sum_1^\infty v_n^2 \exp(v_1 + v_2 + \cdots + v_n) < \infty.$$

On a alors

$$(B_\infty) \Rightarrow (B_{d+1}) \Rightarrow (B_d) \Rightarrow (B_1) \qquad (d = 1, 2, \cdots)$$

et on sait que (B_1) équivaut à la convergence de la série de Shepp :

$$(B_1) \quad : \quad \sum_1^\infty \frac{1}{n^2} \exp(v_1 + v_2 + \cdots + v_n) < \infty.$$

Quand $\inf(n v_n) > 0$, toutes ces conditions sont équivalentes. Cependant les implications ci-dessus sont strictes : pour chaque d, il existe des suites v_n telles qu'il y ait recouvrement par des simplexes homothétiques de volumes v_n en dimension $> d$, et non recouvrement en dimension $\leq d$. Il est facile de voir que (B_∞) équivaut au fait que les intégrales (B_d) sont uniformément majorées; c'est donc une condition un peu plus stricte que la conjonction de toutes les (B_d).

Exemple 3 (recouvrement à la Mandelbrot). Enoncé analogue, avec maintenant

$$k(t) = \exp \int_t^\infty \mu(y, \infty) dy.$$

Exemple 4 (recouvrement poissonnien par des simplexes homothétiques aléatoires). Enoncé analogue, avec une intégrale faisant intervenir une mesure μ au lieu de la suite (v_n).

En fait, ce sont les exemples 3 et 4 qu'on traite d'abord, et on en déduit les exemples 1 et 2. Les exemples 1 et 3 ont été obtenus en 1987, les exemples 2 et 4 en 1990. Dans tous les cas, la clé a été la méthode de temps d'arrêt que je vais expliquer tout à l'heure sur l'exemple 3. Cette méthode est rapide, mais elle cache un peu la relation entre les recouvrements aléatoires et la théorie du potentiel.

Ce qui éclaire la question, c'est une propriété remarquable de l'ensemble non recouvert dans un recouvrement de $\mathbb{R}^+$ par des intervalles $(x, x+y)$ associés à un processus de Poisson ponctuel $\{(x, y)\}$ d'intensité $\lambda \otimes \mu$ dans $\mathbb{R}^+ \times \mathbb{R}^+$ (il est ici très important de prendre $\lambda \otimes \mu$ dans $\mathbb{R}^+ \times \mathbb{R}^+$). On démontre que cet ensemble fermé aléatoire $F(\omega)$ a même loi que l'adhérence de l'ensemble des valeurs d'un certain processus à accroissements positifs, indépendants et stationnaires, partant de 0, que je vais désigner par $L(t, \omega)$. Ainsi, dire qu'un fermé A est recouvert par les intervalles $(x, x+y)$, c'est dire qu'il est disjoint de $F(\omega)$, donc qu'il est polaire pour le processus $L(t, \omega)$: cela traduit bien le fait que A est de capacité nulle par rapport à un certain noyau. L'identification des lois de $F(\omega)$ et de $\overline{L(\mathbb{R}^+,\omega)}$ est un théorème de Fitzsimmons, Fristedt et Shepp (1985) et une démonstration différente, utilisant la théorie des produits de poids aléatoires indépendants, est donnée dans mon article de 1990. Mais ce théorème, au moins dans les cas particuliers les plus importants (tels que $\mu(dy) = a\,dy/y^2$, $0 < a < 1$, et $L(t,\omega)$ =processus de Lévy stable, croissant et d'indice $1-a$), se trouve en germe dans l'article de Benoit Mandelbrot de 1972.

Dans le cas de recouvrement, on peut essayer de mesurer à quel point ce recouvrement est ou n'est pas uniforme; c'est une question suggérée par L. Carleson. Pour fixer les idées, considérons sur T des intervalles $I_n(\omega)$ de longueurs $\ell_n = \frac{a}{n}$ ($a > 1$, $n > n_0$), et de centres ω_n. Leurs fonctions indicatrices sont $\chi_n(t-\omega_n)$, et le problème consiste en l'étude simultanée, pour tous les $t \in \mathrm{T}$, de ces suites aléatoires de 0 et de 1. Voici un énoncé très simple (mais dont la démonstration n'est pas immédiate) : soit (a_n) une suite positive *décroissante* (l'énoncé est faux sans cette condition); alors

$$(\diamond)\quad \begin{cases} \sum \frac{a_n}{n} = \infty \Rightarrow P(\forall t \quad \sum a_n \chi_n(t-\omega_n) = \infty) = 1 \\ \sum \frac{a_n}{n} < \infty \Rightarrow P(\forall t \quad \sum a_n \chi_n(t-\omega_n) < \infty) = 1. \end{cases}$$

Et voici un résultat un peu plus puissant : il existe des nombres positifs $A = A(a)$ et $B = B(a)$ tels que presque sûrement

$$a < \limsup_{N\to\infty} \left(\frac{1}{\log N} \max_t \sum_1^N \chi_n(t-\omega_n) \right) \le A$$

$$B \le \liminf_{N\to\infty} \left(\frac{1}{\log N} \min_t \sum_1^N \chi_n(t-\omega_n) \right) < a\ ;$$

le comportement est presque le même en tous les points, mais les inégalités strictes montrent qu'il y a néanmoins des points plus ou moins recouverts (Fan Ai hua et J.-P. Kahane, 1992).

J'ajoute que la simplicité de l'énoncé $(\diamond)$ est trompeuse. On ne connait pas la situation lorsque $\ell_n = \frac{1}{n}$. De manière générale, l'étude de la convergence partout et de la divergence partout d'une série aléatoire $\sum_1^\infty f_n(t-\omega_n)$, où les f_n sont des fonctions données, définies sur T et à valeurs dans $[0,1]$, ne paraît pas une question facile.

Voici quelques problèmes en suspens.

1. Trouver une condition nécessaire et suffisante de recouvrement de T^d par des convexes homothétiques $G_n(\omega) = g_n + \omega_n$ de volumes v_n. Dépend-elle de la forme des convexes? En particulier, démontrer que des boules aléatoires de volumes $\frac{1}{n}$ ($n \geq n_0$) recouvrent $\mathbb{T}^d$ (on sait seulement que l'ensemble non recouvert est au plus dénombrable).

2. Dans le cas de boules de volumes $\frac{a}{n}$ dans $\mathbb{T}^d$ ($a < 1$, $n = 1, 2, \cdots$), l'ensemble fermé non recouvert a pour dimension $d(1 - a)$ avec probabilité positive. Que peut-on dire de sa dimension topologique? En identifiant $\mathbb{T}^d$ à un cube, pour quelles valeurs de a y a-t-il percolation, c'est-à-dire que deux faces opposées communiquent par une composante connexe de la réunion des boules?

3. Dans le plan hyperbolique, on plante des arbres au hasard suivant un processus de Poisson dont l'intensité est la mesure d'aire hyperbolique, et on suppose qu'au temps t leurs sections horizontales sont des disques de rayon hyperbolique t centrés (au sens hyperbolique) sur le processus de Poisson. Pour un observateur donné, la forêt commence à cacher les arbres au temps $t = (2 + \sqrt{5})^{-1/2} = 0,4858 \cdots$. Est-il vrai qu'à ce moment la forêt cache les arbres pour *tout* observateur?

4. (pour mémoire) On donne des fonctions $f_n \ : \ \mathbb{T} \to [0, 1]$, $n \in \mathbb{N}$. Calculer $P(\forall t \in T \quad \sum f_n(t - \omega_n) < \infty)$ et $P(\forall t \in \mathbb{T} \quad \sum f_n(t - \omega_n) = \infty)$.

La démonstration promise

Pour finir, voici comment on démontre le théorème de Shepp par la méthode de temps d'arrêt. On donne la mesure μ localement bornée sur $(0, \infty)$, et le processus de Poisson $\{(x, y)\}$ d'intensité $\lambda \otimes \mu$ *dans* $\mathbb{R} \times \mathbb{R}^+$ (ici, il est essentiel de prendre $\mathbb{R} \times \mathbb{R}^+$ et non $\mathbb{R}^+ \times \mathbb{R}^+$). On considère qu'il se crée au cours du temps, et que la partie créée avant le temps t se trouve dans le quart de plan $x < t$, $y > 0$. Limitons le d'abord au demi-plan $y < \epsilon$ ($\epsilon > 0$) ; son intensité est alors $\lambda \otimes \mu_\epsilon$, où μ_ϵ est la restriction de μ à (ϵ, ∞). Soit τ_ϵ le premier point non recouvert à droite de 0. Désignons par G_ϵ l'ensemble recouvert et calculons de deux façons

$$I_\epsilon = E \int_0^\infty P(t \in G_\epsilon) e^{-t} dt.$$

D'abord, $P(t \in G_\epsilon)$ ne dépend pas de t, et c'est la probabilité d'avoir un point du processus dans l'angle $t - y < x < t$, soit A_t :

$$P(t \in G_\epsilon) = \exp(-\lambda \otimes \mu_\epsilon(A_t)) = \exp\left(-\int_0^\infty \mu_\epsilon(y, \infty) dy\right)$$

Donc

$$I_\epsilon = \exp\left(-\int_0^\infty \mu_\epsilon(y, \infty) dy\right).$$

D'autre part, faisant intervenir le temps d'arrêt τ_ϵ, et posant $t = \tau_\epsilon + s$,

$$I_\epsilon = E\left(e^{-\tau_\epsilon} \int_0^\infty P(\tau_\epsilon + s \in G \mid \tau_\epsilon) e^{-s} ds\right).$$

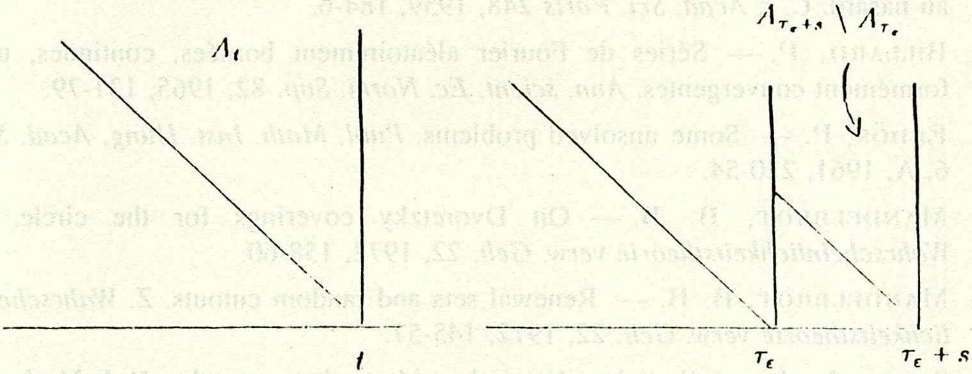
Or la probabilité conditionnelle sous l'intégrale est la probabilité d'avoir un point du processus dans $\Lambda_{\tau_\epsilon+s}$ sachant qu'il n'y en a pas dans Λ_{τ_ϵ} ; c'est

$$\begin{aligned} P(\tau_\epsilon + s \in G \mid \tau_\epsilon) &= \exp(-\lambda \otimes \mu_\epsilon(\Lambda_{\tau_\epsilon+s} \backslash \Lambda_{\tau_\epsilon})) \\ &= \exp\left(-\int_0^s \mu_\epsilon(y,\infty)dy\right) \end{aligned}$$

On voit que

$$E(e^{-\tau_\epsilon}) = \left(\int_0^\infty \exp\left(\int_s^\infty \mu_\epsilon(y,\infty)dy\right) e^{-s} ds\right)^{-1}$$

Pour avoir recouvrement par $G(= G_0)$, il faut et suffit que le second membre tende vers 0 quand $\epsilon \to 0$. C'est la condition de Shepp. On voit aussi que la démonstration donne, via sa transformée de Laplace, la loi du premier point non recouvert à droite de 0.



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THE ZETA FUNCTION
AND NON-DIFFERENTIABILITY OF PRESSURE

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Abstract. We will present a precise result on the lack of Differentiability of Pressure for a certain non-expanding map: the Manneville-Pomeau map.

§0. Introduction

Here we will be interested in presenting Mathematical Models for Phase Transition using Thermodynamic Formalism. We refer the reader to [6] for the mathematical proofs of the results presented here.

First we need to state some definitions and a result of M. Thaller that will be important later:

Definition 1 - The Manneville-Pomeau map f from [0,1] onto [0,1], is given by:
$f(x) = x + x^{1+s}\ (mod 1),\ \ s > 0.$

This map is related with a Poincare section of Lorenz attractor for some special values of the parameters (see[10]).

Theorem 1-(M.Thaller[11])-There exist l(x) density , x in (0,1] such that $du(x) = l(x)\,dx$ is an invariant measure for f . We also have that if
$s < 1$, then u is probability,
and if
$s > 1$, then u is an infinite measure.

We denote as usual
M(f) = {probabilities v on [0,1] such that for all Borel set A, $v(f^{-1}(A)) = v(A)$}

Definition 2- The Pressure P(t) associated with $t \in R$ is

$$P(t) = \sup_{v \in M(f)} \{h(v) + \int (-t log|f'(x)|)\, dv(x)\}$$

Remark- One should think of $-tlog|f'(x)|$ as an external Thermal Potential(see [8]).

Here we will consider t as corresponding $1/T$ where T is temperature in the Physical Problem.

Definition 3 -A probability u_t such that :

$$P(t) = h(u_t) - \int t log|f'(x)| du_t(x)$$

is called an equilibrium state associated with $t \in R$.

It is easy to see in our case, for the Manneville-Pomeau map, that P(0)=log 2 and P(1) = 0.

Definition 4 - We say f is expanding if $|f'(x)| > c > 1$ for all x in the domain of f.

Remark-The map f of Manneville-Pomeau is not expanding because f(0)=0 and f'(0)=1 .

For expanding maps P(t) is a real analytic function on t , and equilibrium states are unique for each $t \in R$ (see [9]). Also u_t changes in a continuous fashion with t in the space M(f) of invariant probabilities.

What can be said about the non-expanding case? Is there any Physical Phenomena where such kind of non-expanding maps can be used to give an interesting model ?

Thermodynamic Formalism is a good model for problems in Statistical Physics in one-dimensional lattices N or Z (see [8]).

The situation on the lattice Z^2 requires to consider actions of Z^2, and this situation will be not consider here.

Phase transition was not very much analyzed in the past in Thermodynamic Formalism until recent years.

Now we will describe some problems in Statistical Mechanics where Phase Transitions occur, in order the reader can understand the main motivation for our main result.

Ising Model- If one decrease the temperature t of a ferromagnetic material (for instance a piece of iron), there exists a value t_0, where suddenly the material have magnetic properties.

One should think in this problem in the following way:
consider the lattice Z (or N if one consider a wall effect in the model) as a model for a long one-dimensional wire, now for each value of Z (that is for each site of the lattice) we consider a spin + or - . In this way one can consider a probability in the Bernoulli Space $\{+,-\}^Z$ as a distribuition of spins in the lattice. For each value of t (that is of temperature) , and due to the interation of the spins, one with each other, it is known in Statistical Physics that there exist a certain distribuition of spins called the Gibbs state u_t that describe precisely what happens in the physical phenomena at time t. The measure u_t is an equilibrium state and depends of the value t. For certain values of t there exist just one equilibrium state, but decreasing t, suddenly appears a value t_0, where two equilibrium states coexist. If one of this states is a Dirac-Delta in the point $\{...,+,+,+,+,+,+,+,...\}$ of the Bernoulli Space, then we say that a spontaneous magnetization appeared at t_0. This is what happen in a First-Order Transition.

This kind of discontinuity of the equilibrium state is also observed at the level of the Free-Energy (in our case will be the Pressure P(t)), by means of a discontinuity of the derivative of the Free-Energy.

Water-Ice Model - At zero degrees water turns into Ice.

This model is consider a second order transition. In this case the Free Energy is differentiable but its derivative is not continuous at the transition value.

Power law singularities appear frequently in First and Second Order Transitions.

§1. Non-Differentiability of Pressure

In analogy with the models of Statistical Mechanics we define:

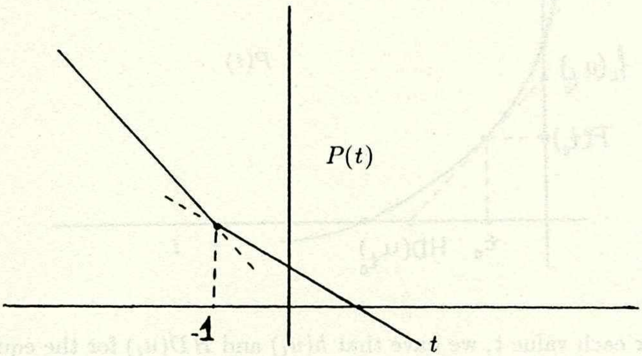
Definition 5 - We have a First-Order Transition at t_0, if P(t) is differentiable for all $t \in R$, different from t_0. In t_0, there exist right and left derivatives but they are not equal.

Definition 6 - We have second-order transition at t_0, if P(t) is differentiable for all $t \in R$, but the second derivative of P(t) is not continuous at t_0.

High-order transitions can be defined in a similar way.

Example 1: The map $h_1(x) = x^2 - 2x$ is such that

P(t) has graph:



In this case we have a First-Order Transition (two equilibrium states) for $t_0 = -1$.

Example 2 [4]: The Lattes rational Map

$$h_2(z) = ((z-2)/z))^2$$

z in the Complex Plane, has First-Order Transition at $t_0 = -2/3$. At the value t_0 there exist two equibrium measures.

Example 3 [5]: The map

$$h_3(z) = z^2 - 2\bar{z}$$

(defined by Hofman-Withers), z in the complex plane, has also a First-Order Transition (three equilibrium states) at $t_0 = -1$. We need here a modification of the definition of the potential, instead of derivative we need to use the Jacobian Derivative of h_3. To be more precise we should say that the set of equilibrium states is a symplex generated by three measures. Antiferromagnetic states appear at the value of transition [5].

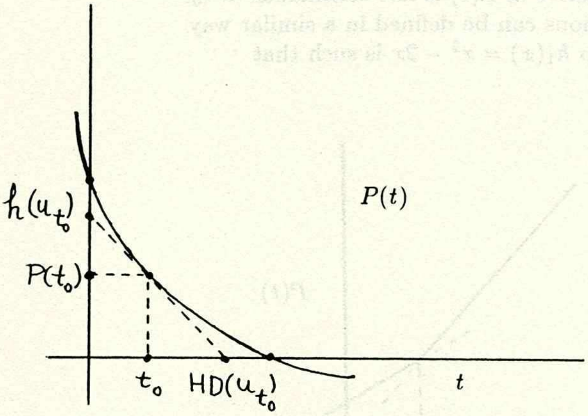
This is a Dynamical System Model for the Potts Model. The analogous of the Yang-Lee zeroes also appears in this dynamical system model.(see [4] ,[5])

As usual we denote by h(v) the entropy of a probability v in M(f). We also denote $HD(v) = inf\{Hausdorff\ Dimension\ of\ sets\ A\ such\ that\ v(A) = 1\}$, the Hausdorff Dimension of the measure v. In general, for almost every x with respect to v:

$$lim_{r->0} \frac{log\ \ v(B(x,r))}{log\ \ r} = HD(v).$$

In simple terms we can say that v(B(x,r)) scales like $r^{HD(v)}$.

Remark - Expanding maps never present Phase Transition because P(t) is real analytic[9]. In this case we have the McCluskey-Manning picture:



Note that for each value t, we have that $h(u_t)$ and $HD(u_t)$ for the equilibrium measure u_t, can be obtained in a geometric way as the intersection of the tangent line to P(t) with y-axis and x-axis.

Therefore, as in a First-Order transition we have different left and right derivatives for P(t) at the transition value t_0, it is reasonable to believe that in general in this case there exists more than one equilibrium state at the value t_0. These probabilities would correspond to the different values of entropy and Hausdorff Dimension of measures obtained from the intersection of the lines corresponding to right and left derivatives of P(t) at t_0 with respectively the y-axis and x-axis.

As we just see in the example 3, we can have three equilibrium states at the transition value t_0.

It is important to point out that there exist one-dimensional maps such that P(t) is not defined or have a very bad behaviour for large sets of t. A lot of results are known for expanding maps and Holder-Continuous Potentials, but is hopeless to try to analyze P(t) for the general non-expanding map.

The Manneville-Pomeau map, nevertheless can be analized in a very precise way. The result we will present here, formalize in a rigorous Mathematical way, results that appear in the literature in Physics by X-J. Wang (see[12]) In other terms we show results of Fisher - Fedelhorf [1] in the context of Thermodynamic Formalism of Bowen - Ruelle - Sinai .

Theorem 2- Suppose f from [0,1] to itself is given by

$$f(x) = x + x^{1+s} \pmod 1 \text{ with } s > 0.$$

Then the pressure

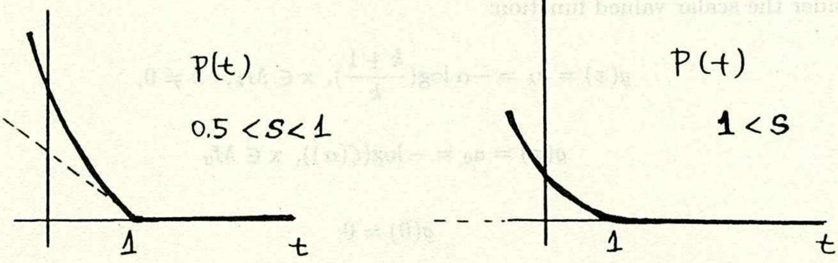
$$P(t) = \sup_{v \in M(f)} \{h(v) - t \int \log |f'(x)| \, dv(x)\}$$

is zero for $t \geq 1$
and for $t < 1$, we have the above expression
for t close to zero:

$$P(t) \approx \begin{cases} A(1-t) + B(1-t)^{\frac{1}{s}}, & \text{for } s < 1, \ s > 0.5 \\ C(1-t)^s, & \text{for } s \geq 1 \end{cases}$$

The values A, B, C above are not zero.

Therefore depending on s, we have the following two kinds of graph of P(t):

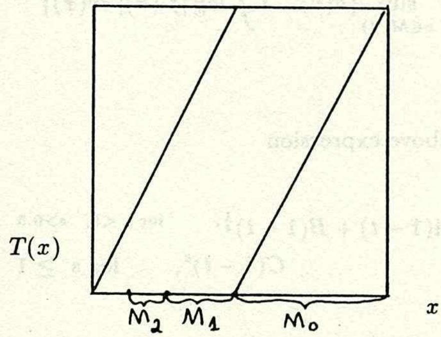


Note that in the case $0.5 < s < 1$, P(t) is not differentiable at t=1, but in the case $s > 1$, P(t) is differentiable at t=1. The case $0.5 < s < 1$ correspond to a first order transition, and the case $s > 1$ to a second order transition.

In order to prove the above result, for the Manneville Pomeau map, we need first to analyze the following map :

$$T(x) = 2x, \quad x \in [0, 1/2]$$

$$T(x) = 2(x - 1/2), \quad x \in (1/2, 1].$$



Consider the Partition M_0, M_1, M_2, M_3, M_4,...,M_n,..., given in the figure below. These sets are defined by $M_0 = [1/2, 1]$, $M_1 = [1/4, 1/2)$, $M_2 = [1/8, 1/4)$, ...,$M_n = [1/2^{n+1}), 1/2^n)$,... and so on.

Now for each value of $\alpha \in (1, \infty)$ define the potential g given in the following way : consider the scalar valued function:

$$g(x) = a_k = -\alpha \log(\frac{k+1}{k}), \text{ x} \in M_k, \quad k \neq 0,$$

$$g(x) = a_0 = -\log(\zeta(\alpha)), \text{ x} \in M_0$$

$$g(0) = 0.$$

In the above ζ denotes the Riemann Zeta-Function.

The potential g is slightly different from the one consider by F.Hofbauer [2] and we use results of this paper in an essential way. The main purpose of F. Hofbauer in the mentioned paper was to show the existence of more then one equilibrium state for some non-Holder Potentials for the shift map in two symbols.

In fact techniques of Ruelle-Perron-Frobenius Operator obtained by Hofbauer are essential to formalize the reasoning of X-J Wang.

In analogous way define:

Definition 7 - For each value $t \in R$, we define

$$p(t) = \sup_{v \in M(T)} \{h(v) + \int tg(x)\, dv(x)\}$$

For the value t=1 there exist two equilibrium measures, the Dirac-Delta on 0 and another measure m. This measure m is not the maximal entropy measure, but one of the equilibrium states for g.

Depending on the value of α, this measure m is a probability or an infinite measure.This is analogous to the result of Thaler mention before. In fact to be precise we should perhaps say that formally an infinite measure should be not consider an equilibrium state, because it is not a probability. Infinite invariant measures are nevertheless very important and are associated with second order transitions.

Note the following fact that can be seen as a model for Phase Transition.

If we begin with a very negative value of t and for each t, we look for the equilibrium state $\{u_t\}$, we will have just one equilibrium state until we reach the value t_0 =1, where we have two equilibrium states. One of them is a Dirac-Delta on 0 and the other is m.

Remark: The value 0 in the binary code is associated with $\{++++++++++....\}$.
Therefore a Delta-Dirac on 0, corresponds to a Delta-Dirac on $\{++++++++++...\}$.

For values of t larger than 1, the unique equilibrium state u_t is the Dirac-Delta on 0. This example shows that equilibrium measures can change in a discontinuous fashion with t. In this case the transition value t_0 is equal 1. The above phenomena can be consider as a mathematical model in Thermodynamis Formalism for a spontaneous magnetization (Phase Transition).

In order to derive the main result we have to show first that the the following functional equation is true:

$$\zeta(\alpha)^t = \sum_{n=1}^{\infty} \frac{e^{-np(t)}}{n^{\alpha t}}.$$

Using the above functional equation and asymptotic expansions used in classical Analytic Number Theory one can show the Theorem 3 (see [6] for the all argument).

Theorem 3: Under the above definitions we have
two possibilities:
(a) $1 < \alpha < 2$, then for $t \leq 1$, $t \to 1$,

$$p(t) = \{\frac{\zeta(\alpha)\log\zeta(\alpha) - \alpha\zeta'(\alpha)}{-\Gamma(1-\alpha)}\}^{\frac{1}{\alpha-1}}(1-t)^{\frac{1}{\alpha-1}} + \text{higher order terms},$$

or

(b) $2 < \alpha < 3$, then for $t \leq 1$, $t \to 1$,

$$p(t) = \frac{\zeta(\alpha)\log\zeta(\alpha) - \alpha\zeta'(\alpha)}{\alpha\zeta'(\alpha-1)}(1-t) + A(1-t)^{\alpha-1}(1+o(1)).$$

In the last case, it will follow that the entropy of m (equilibrium state for g) is

$$\frac{\zeta(\alpha)\log\zeta(\alpha)-\alpha\zeta'(\alpha)}{\alpha\zeta'(\alpha-1)}.$$

Theorem 3 is used to prove Theorem 2(see[6]).

If one wants to relate the above Theorem 3 with the theorem 2 for the Manneville-Pomeau , then one should consider $\alpha = 1 + 1/s$ (see[6]).

Using the specific form, we obtained for the singularity of the Pressure at t=1, we can also analyze the Ruelle Zeta-Function and obtain results analogous to the expanding case for the distribuition of periodic orbits under restrictions related to the norm (mean value of g in the orbit).

This result is analogous to results of Parry and Pollicot[7] for the expanding case.

In a forthcoming paper we will use some of the techniques above to analyze discrete groups of first kind with parabolic elements and also relate the action of the group in the boundary of the hyperbolic disk with the continued fractional expansion with even quotients[3].

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Real analytic convex surfaces with positive topological entropy and rigid body dynamics

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Abstract

Following Knieper and Weiss [9] we exhibit explicit real analytic metrics on S^2 and $\mathbf{RP}^2$ with positive curvature and positive topological entropy using the dynamics of the rigid body.

1 *Introduction*

In [3, 5] Donnay, Burns and Gerber constructed real analytic metrics on S^2 whose geodesic flows are ergodic and have positive metric entropy (in fact they are Bernoulli) showing that the simple topology of the sphere is not an obstruction for having geodesic flows with complicated dynamics. However all their examples required some negative curvature.

In [9], Knieper and Weiss showed the existence of real analytic convex (i.e positively curved) metrics on the two-sphere whose geodesic flow has positive topological entropy. It follows then from a theorem of Katok [8] that the dynamics of the geodesic flow corresponding to those metrics presents a horseshoe. Their examples are obtained from smooth small local perturbations of an ellipsoid with distinct axes. Then using the fact that the

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topological entropy h_{top} is continuous for C^∞ flows on closed three manifolds [8, 11, 14] they obtained also real analytic examples although non-explicitly. Let us mention that the harder question of the existence of a convex metric on S^2 with positive measure entropy remains open.

The aim of the present paper is to describe explicit real analytic convex metric on S^2 and on $\mathbf{RP}^2$ with positive topological entropy arising from the dynamics of the rigid body. Our method of proving that h_{top} is positive follows the strategy in [9] but it differs from the latter on a few aspects. We use the work already done in rigid body dynamics concerning the splitting of heteroclinic connections of hyperbolic closed orbits via the Poincaré-Melnikov- Arnold integral. Using Ziglin's work [15] we show that a perturbation of the Poisson sphere with suitable potential breaks both heteroclinic connections inducing transversal crossing of stable and unstable manifolds and hence the existence of a horseshoe will follow directly. Hence our approach does not need the use of Katok's theorem [8]. Riemannian metrics are obtained thereafter using the Maupertuis principle. Thus we will obtain

Theorem 1.1 *Consider the ellipsoid* E, $\frac{x^2}{a_1}+\frac{y^2}{a_2}+\frac{z^2}{a_3}=1$ *with* $a_3>a_2>a_1>0$ *and let* g_E *denote the canonical metric of* E *induced by* $\mathbf{R}^3$. *Let* r_1, r_2, r_3 *be given real numbers. Then for* $\epsilon \neq 0$ *sufficiently small the geodesic flow of* (E, g) *possesses a horseshoe and hence positive topological entropy where*

$$g = \frac{1-\epsilon(r_1x+r_2y+r_3z)}{a_1a_2a_3(\frac{x^2}{a_1^2}+\frac{y^2}{a_2^2}+\frac{z^2}{a_3^2})}\, g_E \quad (r_2 \neq 0).$$

Also by considering a quadratic potential we exhibit a real analytic convex metric on S^2 with $h_{top}>0$ and such that the antipodal map is an isometry. Thus we also get metrics on $\mathbf{RP}^2$ with the same features. Finally, let us note that at the same time we obtain collective metrics on $SO(3)$ (and thus on S^3) with positive curvature and positive entropy.

We would like to thank Jorge Lewowicz and Miguel Paternain for very useful comments and discussions. We are also grateful for conversations with Howard Weiss that helped considerably to correct and improve the present version of this paper.

2 Euler-Poisson equations and the Poisson sphere

For the material in this section related to the dynamics of the rigid body we refer to [1, 2, 6, 7]. Let $E(3)$ denote the group of euclidean motions on 3-space. It can be viewed as the semidirect product of $SO(3)$ and $\mathbf{R^3}$. The Euler-Poisson equations take place on the dual of the Lie algebra of $E(3)$ that we identify with $\mathbf{R^6} = \mathbf{R^3} \times \mathbf{R^3}$. The orbits of the coadjoint action are symplectic manifolds and in fact they can be described as follows. Let a point in $\mathbf{R^6}$ be denoted by the pair (p, v) with p and v points in $\mathbf{R^3}$. Let $< , >$ denote the usual inner product in three space. Consider the two invariants given by

$$F_1(p,v) = < p,p >, \quad F_2(p,v) = < p,v > .$$

The sets $M_{l_1,l_2} = \{F_1 = l_1, \ F_2 = l_2\}$ are invariant under the coadjoint action of $E(3)$ and generically they are diffeomorphic to T^*S^2, but the symplectic structure on M_{l_1,l_2} is not the same as the canonical symplectic structure on T^*S^2 except when $l_2 = 0$.

Consider parameters $a_3 > a_2 > a_1 > 0$ and let A be the diagonal matrix with entries the a_i. Set $H_0(p,v) = \frac{1}{2} < Av, v >$, and denote by V a function of p only. Finally set $H = H_0 + \epsilon V$. Then H can be viewed as Hamiltonian on each orbit M_{l_1,l_2}. Its Hamiltonian flow is called the Euler-Poisson flow and the corresponding differential equations are called Euler-Poisson differential equations and they describe the motion of a rigid body on a axially symmetric force field. For example when $V =< p,r >$ with r a fixed vector in 3-space we get the motion of a heavy solid body with a fixed point. If $V =< Cp,p >$ for C a diagonal matrix we get the motion of a triaxial ellipsoid in a infinite ideal liquid.

Assume $l_2 = 0$. If $\epsilon = 0$ we have that $H = H_0$ induces a riemannian metric g_P on $S^2$and the sphere equipped with this metric is called the Poisson sphere. It corresponds to a classical reduction of Poisson of the free motion of a solid body (Euler problem). If $p = (p_1, p_2, p_3)$ then the metric g_P on the sphere $< p,p >= 1$ can be written as [13]:

$$g_P = \frac{1}{a_1a_2a_3(\frac{p_1^2}{a_1} + \frac{p_2^2}{a_2} + \frac{p_3^2}{a_3})}(a_1dp_1^2 + a_2dp_2^2 + a_3dp_3^2).$$

When $\epsilon \neq 0$, then H does not induce a riemannian metric anymore but we get a natural mechanical system on T^*S^2 with kinetic energy H_0 and potential ϵV. The substitution $p_1 = x/\sqrt{a_1}$, $p_2 = y/\sqrt{a_2}$, $p_3 = z/\sqrt{a_3}$ takes the sphere $< p,p >= 1$ into the ellipsoid E, $\frac{x^2}{a_1} + \frac{y^2}{a_2} + \frac{z^2}{a_3} = 1$. Under this substitution the metric g_P goes into the metric on E given by

$$\frac{1}{a_1 a_2 a_3(\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} + \frac{z^2}{a_3^2})} g_E,$$

where g_E is the canonical metric on E induced by $\mathbf{R}^3$

The geodesic flow on the Poisson sphere has a very similar dynamics to that of the geodesic flow on the ellipsoid; this will follow from the description of the cross section map in the Depri variables, in Section 3. One has two hyperbolic closed orbits with doubly asymptotic solutions i.e. heteroclinic connections, hence we could ask if the invariant manifolds split when we perturbe H_0 by $H_0 + \epsilon V$ for small ϵ and suitable V. For the classical potentials this has been studied by Kozlov and Ziglin [10, 15]. We will use their results in the next section.

3 *Explicit convex metrics with positive topological entropy*

If f is a diffeomorphism on a surface let $W^{s,u}(p)$ denote the stable and unstable manifolds of the hyperbolic fixed point p. Suppose now f_ϵ denotes a smooth family of diffeomorphisms with $f_0 = f$. Let p_ϵ denote the perturbed hyperbolic point and let $W_\epsilon^{s,u}(p_\epsilon)$ denote the corresponding stable and unstable manifolds for p_ϵ. As it is well known they are injectively immersed curves that vary smoothly with ϵ.

Suppose that f possesses two hyperbolic points p and q. The points p and q are said to have an heteroclinic connection whenever one branch of $W^{u,s}(p)$ coincides with one branch of $W^{s,u}(q)$.

We proceed now to describe explicit convex metrics on S^2 and $\mathbf{RP}^2$ with positive topological entropy. Let us take $V =< p, r >$ where $r = (r_1, r_2, r_3)$ is a fixed vector in 3-space. As we mentioned in the previous section V is the potential associated with the gravitational force.

For fixed values of $l_1 > 0$ and l_2 one can introduce the so called Depri-variables (l, L, g, G) (or Depri-Andoyer-variables, cf. [10] and [15, Section 4]; in the latter reference one can also find what H looks like in these variables).

Now we can consider the symplectomorphisms f_ϵ that arise by considering in the level surface $H = \frac{1}{2}a_2 G_0^2$ (G_0 a fixed positive constant) the cross section $g = const.$ The map $f = f_0$ possesses 2 hyperbolic fixed points $p : (l = 0, L = 0, G = G_0)$ and $q : (l = \pi, L = 0, G = G_0)$. Through these points pass the separatrices (i.e. heteroclinic connections):

$$L = \frac{G_0\sqrt{a_2 - a_1}\,\sin l}{\sqrt{a_3 - a_1\sin^2 l - a_2\cos^2 l}},$$

$$L = -\frac{G_0\sqrt{a_2 - a_1}\,\sin l}{\sqrt{a_3 - a_1\sin^2 l - a_2\cos^2 l}}.$$

It follows directly from Ziglin's analysis of the Poincaré-Melnikov-Arnold integral [15, Remark after Theorem 4] that whenever the area constant $l_2 = 0$ (recall that in this case $M_{l_1,0}$ is symplectomorphic to T^*S^2) the heteroclinic connections split and cross transversally for $\epsilon \neq 0$ sufficiently small and all the values of (r_1, r_2, r_3) except in the so called "Hess-Appelrot" cases. They arise when we have the following relations among the parameters

$$r_1\sqrt{a_3 - a_2} + r_3\sqrt{a_2 - a_1} = 0, \quad r_2 = 0,$$

$$-r_1\sqrt{a_3 - a_2} + r_3\sqrt{a_2 - a_1} = 0, \quad r_2 = 0.$$

These cases are peculiar because of the following reason [15, Theorem 4]:

For $\epsilon \neq 0$ sufficiently small only one pair of the separatrices splits, while the other does not.

Hence if we assume $r_2 \neq 0$ we avoid the Hess-Appelrot cases and we have that for small $\epsilon \neq 0$, $W^s_\epsilon(p_\epsilon)$ intersetcs $W^u_\epsilon(q_\epsilon)$ transversally and $W^u_\epsilon(p_\epsilon)$ also intersects $W^s_\epsilon(q_\epsilon)$ transversally. Hence the existence of a horseshoe follows immediately from the λ-Lemma [12].

Let us summarize the previous discussion in the following

Proposition 3.1 *Consider on T^*S^2 the natural mechanical system $H = H_0 + \epsilon V$ with $V = < p, r >$ and $r_2 \neq 0$. Fix a positive value for the total energy of the system. Then motion of H with the fixed energy has a horseshoe and hence positive topological entropy for all $\epsilon \neq 0$ sufficiently small.*

Remark 3.2 In the "Hess-Appelrot" cases one also has positive topological entropy. This follows from recent results of K. Burns and H. Weiss [4].

But now if we want to construct a riemannian metric with positive topological entropy let us apply the Maupertuis principle [1, 2]. That is consider the riemannian metric on S^2 given by

$$(h - \epsilon V)g_P,$$

where h is a constant bigger than $\epsilon\, max_{S^2} V$ and g_P is the riemannian metric induced on S^2 by H_0. Then the trajectories of the motion of the natural mechanical system H with energy h are geodesics (up to reparametrizations) of the riemannian manifold $(S^2, (h - \epsilon V)g_P)$. Then the previous proposition implies that the geodesic flow on (E, g) has positive topological entropy which proves Theorem 1.1. If we choose the numbers a_i close to 1 then (E, g) clearly has positive curvature.

Suppose now we take $V = < Cp, p >$ where C is a symmetric matrix (that is we are considering now a special case of Kirchoff's equations). In this case the study of the Poincaré-Melnikov-Arnold integral has been carried out by Kozlov [10, p.53]. Once again avoiding Hess-Appelrot cases one obtains a horseshoe and hence metrics of the form $(h - \epsilon V)g_P$ on S^2 with positive topological entropy. Observe that these metrics are invariant under the antipodal map, thus we also obtain real analytic convex metrics on $\mathbf{RP}^2$ with positive h_{top}.

To finish, let us show how to construct collective metrics on $SO(3)$ with positive topological entropy and positive sectional curvature.

Since $E(3)$ is the semidirect product of $SO(3)$ and $\mathbf{R^3}$, there is a Hamiltonian action of $E(3)$ on $T^*SO(3)$. Let us describe the moment map associated with this action. Identify by left translations $T^*SO(3)$ with $SO(3) \times so(3)^*$ and identify $so(3)^*$ with $\mathbf{R^3}$. Fix $q \in \mathbf{R^3}$. Then the moment map ϕ^q is given by [6]

$$\phi^q(a, v) = (-v, a^{-1}q).$$

Consider the collective Hamiltonian $F = H \circ \phi^q$ where H is a Hamiltonian as above. That is F is given by

$$F(a, v) = H_0(v) + \epsilon V(a^{-1}q).$$

Once again we get a natural mechanical system on $T^*SO(3)$, but since it is collective, the orbits of the Hamiltonian flow of F are transformed under ϕ^q into orbits of H [6]. Hence if the topological entropy of the flow of H is positive, so it is the topological entropy of the flow of F. Now apply again the Maupertuis principle as before to get a riemannian metric on $T^*SO(3)$ with positive topological entropy and arbitrarily close to the left invariant metric on $SO(3)$ given by H_0 as we desired.

Remark 3.3 In [7] Holmes and Marsden showed that the motion of a heavy solid body with a fixed point possesses a horseshoe. In spite of the fact that this is acomplished for most of the values of the parameters their proof excludes particularly the value $l_2 = 0$ since they perturbe a Lagrange top and *not* the Euler problem. Therefore their results cannot be combined with the Maupertuis principle and hence they do not yield metrics on S^2 with the desired properties. On the other hand we could have used their results to obtain collective metrics with positive sectional curvature and positive h_{top} on $SO(3)$.

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Closed and prime ideals in ring extensions.

Miguel Ferrero

Dedicated to Prof. José Luis Massera.

Introduction.

Let R be a ring with an identity 1. Recall that if R is commutative, an ideal P of R is said to be prime if:

ab∈P, a∈R, b∈R implies either a∈P or b∈P.

When R is any ring, not necessarily commutative, the above definition takes the following form. The ideal P is said to be prime if:

AB⊆P for ideals A and B of R implies either A⊆P or B⊆P.

An equivalent formulation is as follows:

aRb⊆P, a∈R, b∈R implies either a∈P or b∈P.

The fact that P is a prime ideal of R will be denoted by P◁'R.

The study of the prime ideals of a ring R is a very important subject. Many results and properties justify this affirmation. In particular, it is well known that the prime radical of a ring equals the intersection of all the prime ideals of the ring. Moreover, most of the usual radicals (Jacobson, Brown-McCoy, Levitzki, strongly prime, nil radical, etc.) are intersection of prime ideals. This fact is quite enough to justify the above affirmation.

Let S be an extension of the ring R. The fundamental problem we want to consider here is the relation of the prime ideals of R and the prime ideals of S.

For example, if P is a prime ideal of R, is there a prime ideal P^* of S, such that $P^*\cap R = P$? Conversely, given a prime ideal P^* of S, is necessarily $P^*\cap R$ a prime ideal of R? These questions have an affirmative answer, for example, when R is a commutative ring and S=R[X] is a polynomial ring over R in one indeterminate X.

Another question is the following. Suppose P is a prime ideal of R and P^* is a prime ideal of S with $P^*\cap R = P$. Is it true that P satisfies some property if and only if P^* satisfies the same property?

On the other hand, suppose that R is a commutative integral domain with field of fractions F and P is a non-zero prime ideal of R[X] with $P\cap R = 0$. Then it is well known that there exists a prime ideal P^* of F[X] such that $P^*\cap R[X] = P$. Moreover, $P^* = f(X)F[X]$ for some monic irreducible polynomial $f(X)\in F[X]$. Using this it is easy to prove that the correspondence $P\to P^*$ is a one-to-one correspondence between the set of all the prime ideals P of $R[X]$ with $P \cap R = 0$ and the set of all the prime ideals of $F[X]$

For any prime ideal P of $R[X]$, R commutative yet, $P \cap R$ is a prime ideal of R. So we may factor out the ideals $P \cap R$ and $(P \cap R)[X]$ from R and $R[X]$, respectively. Then we have an integral domain $\bar{R}=R/(P \cap R)$ and a prime ideal $\bar{P}=P/(P \cap R)[X]$ of $\bar{R}[X]$ such that $\bar{P} \cap \bar{R} = 0$. So we may always reduce the study to the former situation.

There are many papers studying prime ideals in ring extensions. Some of the most interesting are listed at the end. Until some years ago we mainly had results on finite extensions. The purpose of this abstract is to give an idea of the results obtained in the series of papers ($[F_1]$, $[F_2]$, $[F_3]$), where we developed a method to study infinite extensions (as, for example, polynomial rings).

The original motivation of the first paper $[F_1]$ was to show that the above results on prime ideals of polynomial rings remain true in the non-commutative case. Actually, a more general class of ideals, rather than the prime ideals, are studied: the so called closed ideals. The results on prime ideals are obtained as particular cases or applications of the more general results.

Hereafter R is a ring with an identity, not necessarily commutative. An ideal I of an extension S of R with $I \cap R = 0$ will be called R-disjoint.

For the proofs of the results the reader should consult the papers ($[F_1]$, $[F_2]$ or $[F_3]$)

1. Polynomial rings.

Let $R[X]$ be the polynomial ring over R in the indeterminate X. If P is a prime ideal of $R[X]$, then $P \cap R$ is a prime ideal of R. So, by factoring out the ideals $P \cap R$ and $(P \cap R)[X]$ from R and $R[X]$, respectively, we may assume that R is a prime ring and $P \cap R = 0$. Hence we assume that R is prime and we study R-disjoint ideals.

If I is a non-zero R-disjoint ideal of $R[X]$, then the minimality of I, $Min(I)$, is defined as the smallest integer number n such that there exists a polynomial $f \in I$ of degree n.

For a non-zero R-disjoint ideal I of $R[X]$ we put

$[I] = \{g \in R[X] :$ there exists $0 \neq H \lhd R$ with $gH \subseteq I\}$

(if $I=0$ we put $[I]=0$).

We may see that $[I]$ is an R-disjoint ideal of $R[X]$ such that $I \subseteq [I]$ and $Min([I]) = Min(I)$.

We say that I is closed if $[I] = I$. This terminology is justified because the map

$I \to [\,I\,]$ has some properties which remains a closure operator in a topological space.

Using the above remarks we get the following.

Theorem 1.1. Let R be a prime ring and I an R-disjoint ideal of $R[X]$. Then there exists a smallest closed ideal $\bar{I}$ of $R[X]$ which contains I. Moreover, $\bar{I} = [\,I\,]$ and it is the largest ideal J of $R[X]$ which contains I and satisfies $\mathrm{Min}(J) = \mathrm{Min}(I)$.

From this theorem we get, in particular, that $[\,I\,]$ is the unique closed ideal of $R[X]$ which contains I and satisfies $\mathrm{Min}([\,I\,]) = \mathrm{Min}(I)$.

Since R is prime, as in the commutative case there exists the complete ring of right quotients Q of R (see [L], [S]). The center C of Q is a field which is called the extended centroid of R. A subring T of Q which contains R is said to be a ring of right quotients of R.

The following fundamental lemma can be proved.

Lemma 1.2. Let I be a T-disjoint ideal of $T[X]$, where T is a ring of right quotients of R. Then I is closed if and only if there exists a monic polynomial $f_0 \in C[X]$ such that $I = Q[X]f_0 \cap T[X]$.

Using this lemma we can prove one of the main results of this section.

Theorem 1.3. Let R be a prime ring and T a ring of right quotients of R. Then there is a one-to-one correspondence between the following

i) The set of all the closed ideals of $R[X]$.

ii) The set of all the closed ideals of $T[X]$.

iii) The set of all the monic polynomials of $C[X]$, where C is the extended centroid of R.

Moreover, this correspondence associates the ideal I of $R[X]$ with I^* of $T[X]$ and $f_0 \in C[X]$ if $I^* \cap R[X] = I$ and $Q[X]f_0 \cap T[X] = I^*$.

It is easy to prove that every R-disjoint prime ideal of $R[X]$ is closed. We actually have

Corollary 1.4. A T-disjoint ideal P of $T[X]$ is prime if and only if $P = Q[X]f_0 \cap T[X]$, for some monic irreducible polynomial $f_0 \in C[X]$.

As an application we get that the correspondence of theorem 1.3 preserves prime ideals. Hence we can prove the following.

Corollary 1.5. Let T be a ring of right quotients of R. Then the correspondence of Theorem 1.3 is a one-to-one correspondence between the following

i)The set of all the R-disjoint prime ideal of R[X].

ii) The set of all the T-disjoint prime ideal of T[X].

iii) The set of all the maximal ideals of C[X].

There are two nice applications of the former results. First, an ideal P of R is said to be (right) strongly prime if for every ideal I properly containing P, there exists a finite subset F of R such that $F\subseteq I$ and $Fa\subseteq P$, $a\in R$, implies $a\in P$ [PPS]. Every strongly prime ideal is prime and the converse is not true. The first application is the following:

Theorem 1.6. Let R be a ring. Then every prime ideal of R is strongly prime if and only if the same is true of R[X].

On the other hand, for $a\in R$ we denote by r(a) the right annihilator of a in R. Then r(a) is a right ideal of R.

A right ideal H of R is said to be essential if $H\cap J\neq 0$ for every non-zero right ideal J of R.

The right singular ideal Z(R) of R is defined as the set of all the elements $a\in R$ such that r(a) is an essential right ideal of R ([G], p.30). A prime ideal P of R is said to be (right) nonsingular if Z(R/P)=0.

The second application is as follows:

Theorem 1.7. Let R be a ring. Then every prime ideal of R is nonsingular if and only if the same is true of R[X].

All the results mentioned in this section are obtained in [F_1]. With appropiate adaptations the results can be generalized, as we have done in [F_2] and [F_3]. We explain this is the next sections.

2. Centred Extensions.

The ring S is said to be a centred extensions of R if S is generated, as an R-module, by a set of R-centralizing elements (which contains 1). When S has a finite set of generators of this type, then S is said to be a liberal extensions of R [RS].

In the case that there exists a basis of R-centralizing elements containing 1, then S is called a free centred extension. That is, S is a free centred extension of R if there exists a basis $E=(e_i)_{i\in\Omega}$ of S over R such that $ae_i = e_i a$, for every $a\in R$, $i\in\Omega$, and there exists $i_0\in\Omega$ with $e_{i_0}=1$. We write $S=R[E]$.

There are several usual examples of free centred extensions. Namely, a group ring or even a semigroup ring RG. In particular, a polynomial ring in any set of, either commuting or non-commuting, indeterminates. Also, a matrix ring, even a ring of infinite matrices provided that every matrix in the ring has a finite number of non-zero entries and the identity is adjoined. Finally, a tensor product $S=R\otimes_L K$, where L is a field and R and K are L-algebras.

As in the case of section 1, we may always reduce to the prime case. So we may assume that R is prime and P is a prime ideal of S which is R-disjoint.

In this section we summarize the results in $[F_2]$ on closed and prime ideals in free centred extensions.

Almost all the results given in §1 can be extended to free centred extensions. Excluding technical complications, the maint point is to give an adequate definition of the notion of minimality of an ideal.

Any element $a\in S=R[E]$ can be uniquely written as a finite sum $a=\sum_{i\in\Omega} a_i e_i$, where $a_i\in R$. The e-coefficient of a will be sometimes denoted by $a(e)$, i.e., for a given above $a(e_i) = a_i$, for all $i\in\Omega$. The support of a is defined as usual by $\mathrm{supp}(a) = \{e\in E : a(e) \neq 0\}$.

If I is an R-disjoint ideal of S, a non-zero element $a\in I$ is said to be of minimal support in I if for every $b\in I$ with $\mathrm{supp}(b) \subsetneq \mathrm{supp}(a)$ we have $b=0$. We denote by $M(I)$ the set of all the elements of minimal support in I. The minimality of I is defined by $\mathrm{Min}(I) = \{\mathrm{supp}(a) : a\in M(I)\}$.

The definition of $[I]$ is given as in §1:

$$[I] = \{b\in S : \text{there exists } 0 \neq H \lhd R \text{ such that } bH \subseteq I\}.$$

The ideal I is said to be closed if $[I] = I$.

We have

Theorem 2.1. For any R-disjoint ideal I of S, [I] is the largest ideal J of S which contains I and satisfies Min (J) = Min (I). Also [I] is closed and, moreover, it is the smallest closed ideal of S which contains I. In particular, [I] is the unique closed ideal of S which contains I and satirfies Min ([I]) = Min (I).

It is convenient to point out that the above Theorem is very important in the rest of the paper [F_2].

Let Q be the complete ring of right quotients of R and T an intermediate subring. Then it is easy to see that the ring Q [E] is well defined and we have $R[E] \subseteq T[E] \subseteq Q[E]$ and $C[E] \subseteq Q[E]$.

There is a way to generalize Lemma 1.2. Instead of taking just one polynomial $f_0 \in C[X]$ we have to define certain subset $M_C(I) \subseteq C[E]$. This has been done in the first part of ([F_2], §2) and so we can get.

Theorem 2.2. Let R be a prime ring and T a ring of right quotients of R. Then there is a one-to-one correspondence between the following:

i) The set of all the closed ideals of R [E].

ii) The set of all the closed ideals of T [E].

iii) The set of all the ideals of C [E].

Moreover, this correspondence associates the closed ideal I of R [E] with the closed ideal I^* of T [E] and the ideal K of C [E] if $I^* \cap R[E] = I$ and $I^* = Q[E]K \cap T[E]$.

It is possible to prove that R [E] is prime if and only if T [E] (C [E]) is prime. Also, the correspondence of Theorem 2.2 for prime ideals, which gives a generalization of Corollary 1.5.

Aplications corresponding to theorems 1.6 and 1.7 can also obtained. We have:

Theorem 2.3. Let R be a strongly prime (resp. nonsingular prime) ring and let P be an ideal of R [E] which is maximal with respect to $P \cap R = 0$. Then P is a strongly prime (resp. nonsingular prime) ideal.

When the prime ideal is not maximal R-disjoint, P is not always strongly prime (resp. nonsingular prime). We have to assume something. We can prove the following:

Theorem 2.4. Let R be a ring, $S = R[E]$ with E either a finite or a commuting set. Then every prime ideal of R is strongly prime (resp. nonsingular) if and only if the same is true of S.

3. Centred bimodules over prime rings.

In $[F_3]$ we generalize the results given in §2 by considering centred bimodules over prime rings. We briefly relate these results here.

Let R be a prime ring and M an R-bimodule. We say that M is a centred bimodule if there exists a set $(x_i)_{i\in\Omega}$ of elements of M such that $M = \Sigma Rx_i$ and $rx_i = x_i r$ for every $r \in R$, $i \in \Omega$.

In the paper mentioned above we first define the notion of closed submodules. We also show that there exists a Q-module M^* which is an extension of M. Using this and the restriction of M^* to C (if $M^* = \sum_{i\in\Lambda} Qe_i$, then $M^*_C = \sum_{i\in\Lambda} Ce_i$) we can get the generalization of Theorem 2.2.

In this paper we also study about nonsingular submodules, i.e., submodules N such that M/N is nonsingular as an R-module. When R is a prime nonsingular ring this submodules are just the closed submodules.

We also define the class of strongly closed submodules of M and we relate this notion with the notion of strongly prime rings.

An application of the results concerning closed submodules is given to study the torsion free rank of a submodule N (see [R]).

Finally, if we assume that S is an extension of R which is a centred bimodule over R we can apply the former results. Thus all the results given in §2 can be obtained without the assumption that S is free as a centred extension.

Remark. Closed ideals have also been used to study prime ideals in Ore extensions (see [CFG], [FM] and [LM]).

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On some extensions of the commutant lifting theorem

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Abstract. We give some extensions to $\mathbf{Z}^2$ and to general groups of the theorem on the lifting of the commutant due to Nagy and Foias.

Introduction

Our aim is to give some extensions of the celebrated Nagy-Foias commutant lifting theorem [N-F]. In section I we recall some basic definitions related to lifting problems in general groups. In section II we define three sets of liftings. Then, in section III, we associate with each commutant a family of isometries and establish a bijection between the set of the minimal unitary extensions of that family and one of the sets of liftings. As a consequence, a lifting result in general groups - theorem (IV.3) - is proved in section IV. Related results are stated in section V. The general framework developed in section III is applied, in sections VI and VII, to commutants in $\mathbf{Z}^2$.

I.- Basic definitions and properties

In this paper, Γ will always be an abelian group with neutral element e and Γ_1 a sub-semigroup of Γ such that $e \in \Gamma_1$ and every $u \in \Gamma$ can be written as $u = s - t$, with $s,t \in \Gamma_1$. A semigroup of contractions on Γ_1 is a set $T = \{T(s): s \in \Gamma_1\} \subset \mathcal{L}(E)$ of contractive operators on a Hilbert space E such that T(e) equals the identity I_E on E and $T(s+t) = T(s)T(t)$, $\forall$ $s,t \in \Gamma_1$.

A minimal unitary dilation of such a semigroup T is a group $U = \{U(s): s \in \Gamma\} \subset \mathcal{L}(F)$ of unitary operators on a Hilbert space $F \supset E$ such that:

i) $T(s) = P^F{}_E U(s)_{|E}$ ($= P^F{}_E U(s) i^F{}_E$) for every $s \in \Gamma_1$

($P^F{}_E$ denotes the orthogonal projection of F onto E and $i^F{}_E$ the inclusion of E in F).

ii) $F = \bigvee\{U(s)E: s \in \Gamma\}$ (minimality condition),

i.e., F is the closed linear span of $\{U(s)E: s \in \Gamma\}$.

To each minimal unitary dilation $U \subset \mathcal{L}(F)$ of T a minimal isometric dilation of T is naturally associated as follows. Set $M = \bigvee\{U(s)E: s \in \Gamma_1\}$ and $W(s) = U(s)_{|M}$ for every $s \in \Gamma_1$; then $W := \{W(s): s \in \Gamma_1\} \subset \mathcal{L}(M)$ is a semigroup of isometries on Γ_1 such that $T(s) = P^M{}_E W(s)_{|E}$ and $M = \bigvee\{W(s)E: s \in \Gamma_1\}$. From these two properties it follows that $P^M{}_E$

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intertwines T and W, i.e., that

$$T(s)P^M{}_E = P^M{}_E W(s)\ ,\ \forall s \in \Gamma_1 . \tag{I.1}$$

For $s \in \Gamma_1$ let $\Lambda(s)$ be the closure in M of $\{[W(s) - T(s)]a: a \in E\}$. Since $P^M{}_E W(s)_{|E} = T(s)$, $\Lambda(s)$ is orthogonal to E. Let $(M \ominus E)$ denote the orthogonal complement of E in M; from $W(s)(M \ominus E) \subset (M \ominus E)$ it follows that $\Lambda(s)$ is also orthogonal to $W(s)(M \ominus E)$. Since $E \oplus \Lambda(s) = E \vee W(s)E$ it follows that

$$E \vee W(s)M = E \oplus \Lambda(s) \oplus W(s)(M \ominus E)\ . \tag{I.2}$$

If $\Lambda_*(s)$ is the closure in M of $\{[I_M - W(s)T(s)^*]a: a \in E\}$ then

$$E \vee W(s)M = \Lambda_*(s) \oplus W(s)M\ ; \tag{I.3}$$

in fact, it stems from (I.1) that $T(s)^* = W(s)^*{}_{|E}$, so $\Lambda_*(s)$ is orthogonal to $W(s)M$.

A <u>commutant</u> on the semigroup Γ_1 is a set $\{T_1, T_2, X\}$ such that:

i) $T_j = \{T_j(s): s \in \Gamma_1\} \subset \mathcal{L}(E_j)$ is a semigroup of contractions on Γ_1, $j = 1,2$;

ii) $X \in \mathcal{L}(E_1, E_2)$ intertwines T_1 and T_2, i.e., $X\, T_1(s) = T_2(s)\, X$ for every $s \in \Gamma_1$.

Assume that, for j = 1,2, minimal unitary dilations of T_j exist and fix one, $U_j \subset \mathcal{L}(F_j)$. Set $M_j = \vee\{U_j(s)E_j: s \in \Gamma_1\}$ and $\tilde{M}_j = \vee\{U_j(-s)E_j: s \in \Gamma_1\}$. Then, a <u>lifting</u> of X is an operator $\tau \in \mathcal{L}(F_1, F_2)$ such that

$$P^{F_2}{}_{E_2}\, \tau_{|M_1} = X\, P^{M_1}{}_{E_1}\ ,\ \|\tau\| = \|X\|\ ,\ \tau M_1 \subset M_2\ ,\ \tau^* \tilde{M}_2 \subset \tilde{M}_1 .$$

The set of all liftings of X will be denoted by LIF(X).

When Γ is the group of integers $\mathbb{Z}$ and Γ_1 the semigroup $\mathbb{Z}_1$ of positive integers, a contractive semigroup is given by a contractive operator and a well-known theorem of Nagy [N-F] says that it always has an essentially unique minimal unitary dilation, while the commutant lifting theorem asserts that LIF(X) is never empty.

II.- Lifting problems

We shall say that τ belongs to the set LIF(X)" when:

i) $\tau \in \mathcal{L}(F_1, F_2)$ and intertwines U_1 and U_2, (ii) $P^{F_2}{}_{E_2}\, \tau_{|M_1} = X\, P^{M_1}{}_{E_1}$, (iii) $\|\tau\| = \|X\|$.

Set LIF(X)' = $\{\tau \in \text{LIF(X)}": \tau M_1 \subset M_2\}$. Thus, LIF(X) = $\{\tau \in \text{LIF(X)}': \tau^* \tilde{M}_2 \subset \tilde{M}_1\}$ and LIF(X) $\subset$ LIF(X)' $\subset$ LIF(X)". In this paper we consider the <u>lifting problems</u> of giving conditions for these sets to be non void.

Set $W_j(s) = U_j(s)_{|M_j}$ for every $s \in \Gamma_1$, so $W_j := \{W_j(s): s \in \Gamma_1\}$ is a minimal isometric dilation of T_j, j = 1,2. The corresponding set of contractive intertwining dilations is

CID(X) = $\{\gamma \in \mathcal{L}(M_1, M_2): \gamma W_1(s) = W_2(s)\gamma, \forall\ s \in \Gamma_1;\ P^{M_2}{}_{E_2}\, \gamma = X\, P^{M_1}{}_{E_1};\ \|\gamma\| = \|X\|\}$.

The usual version of the Nagy-Foias theorem is that, for $\Gamma = \mathbb{Z}$ and $\Gamma_1 = \mathbb{Z}_1$, CID(X) is non void.

Now, LIF(X)' can be identified with CID(X); in fact, if $\gamma \in \mathcal{L}(M_1,M_2)$ intertwines W_1 and W_2, there exists only one operator $\tau \in \mathcal{L}(F_1,F_2)$ such that $\gamma = \tau_{|M_1}$ and τ intertwines U_1 and U_2; that operator is given by $\tau U_1(u)v = U_2(u)\gamma v$, $u \in \Gamma$, $v \in E_1$; $\|\tau\| = \|\gamma\|$ holds. Thus:

(II.1) <u>PROPOSITION</u> **A bijection from LIF(X)' onto CID(X) that preserves norms is given by associating to each $\tau \in$ LIF(X)' its restriction $\gamma = \tau_{|M_1}$.**

Moreover, if $\tau \in \mathcal{L}(F_1,F_2)$ intertwines U_1 and U_2 and $\gamma := P^{F_2}{}_{M_2}\tau_{|M_1}$ intertwines W_1 and W_2, then $\tau M_1 \subset M_2$: in fact, since there exists $\tau_1 \in \mathcal{L}(F_1,F_2)$ that intertwines U_1 and U_2 and such that $\gamma = \tau_{1|M_1}$, it is enough to show that $\tau = \tau_1$; now, for any $u,u' \in \Gamma$, $v \in E_1$ and $w \in E_2$, let $s \in \Gamma_1$ be such that $s+u$, $s+u' \in \Gamma_1$; then $\langle \tau_1 U_1(u)v, U_2(u')w\rangle = \langle \tau_1 W_1(s+u)v, W_2(s+u')w\rangle = \langle \tau U_1(u)v, U_2(u')w\rangle$. Consequently:

(II.2) <u>PROPOSITION</u> **$\tau \in$ LIF(X)' iff $\tau \in$ LIF(X)" and $P^{F_2}{}_{M_2}\tau_{|M_1}$ intertwines W_1 and W_2.**

If $\gamma \in \mathcal{L}(M_1,M_2)$ intertwines W_1 and W_2 and $P^{M_2}{}_{E_2}\gamma_{|E_1} = X$, then $P^{M_2}{}_{E_2}\gamma = X P^{M_1}{}_{E_1}$; thus, LIF(X)' =

$$\{\tau \in \mathcal{L}(F_1,F_2): \tau \text{ intertwines } U_1 \text{ and } U_2,\ P^{F_2}{}_{E_2}\tau_{|E_1} = X,\ \|\tau\| = \|X\|, \tau M_1 \subset M_2\}.$$

<u>Dual lifting problems</u> are naturally defined by setting $\tilde{U}_j(s) = U_j(-s)$ for every $s \in \Gamma$, $\tilde{T}_j(s) = T_j(s)^*$, $\tilde{W}_j(s) = \tilde{U}_j(s)_{|M'_j}$ for every $s \in \Gamma_1$, and considering the dual commutant $\{\tilde{T}_2, \tilde{T}_1, X^*\}$. Then:

$$\text{LIF}(X^*)" = \{\tau^*: \tau \in \text{LIF}(X)"\}$$

because, if $\tau \in$ LIF(X)" and $s \in \Gamma_1$, then $P^{F_2}{}_{E_2}U_2(s)\tau_{|E_1} = P^{F_2}{}_{E_2}\tau U_1(s)_{|E_1} = X P^{M_1}{}_{E_1}U_1(s)_{|E_1} = X\,T_1(s) = T_2(s)\,X = P^{F_2}{}_{E_2}U_2(s)_{|E_2}X$, so $P^{F_1}{}_{E_1}\tau^* U_2(s)^* i^{F_2}{}_{E_2} = X^* P^{F_2}{}_{E_2}U_2(s)^* i^{F_2}{}_{E_2}$, and consequently $\tau^* \in$ LIF(X*)". Since LIF(X) = $\{\tau \in \mathcal{L}(F_1,F_2)$: τ intertwines U_1 and U_2, $P^{F_2}{}_{E_2}\tau_{|E_1} = X$, $\|\tau\| = \|X\|, \tau M_1 \subset M_2$, $\tau^*\tilde{M}_2 \subset \tilde{M}_1\}$, we see that:

$$\text{LIF}(X^*) = \{\tau^*: \tau \in \text{LIF}(X)\}.$$

III.- The family of isometries associated to a lifting problem

To the above posed lifting problems we shall now associate a Hilbert space H and a family $V = \{V(s): s \in \Gamma_1\}$ of isometries with domains and ranges contained in H, such that those problems can be solved only if V can be extended to a unitary representation of Γ, in which case the elements of LIF(X)" are given in a natural way.

Leaving aside the trivial case $X = 0$ we assume that $\|X\| = 1$. Let $P_+ = P^{M_1}{}_{E_1}$, $P_- = P^{\tilde{M}_2}{}_{E_2}$. For any $(h_1,h_2), (h'_1,h'_2) \in M_1 \times \tilde{M}_2$ we set $\langle (h_1,h_2),(h'_1,h'_2)\rangle = \langle h_1,h'_1\rangle_{M_1} + \langle XP_+h_1,h'_2\rangle_{\tilde{M}_2} + \langle h_2, XP_+h'_1\rangle_{\tilde{M}_2} + \langle h_2,h'_2\rangle_{\tilde{M}_2}$,

thus getting a positive semidefinite scalar product and generating in the usual way a Hilbert space H such that M_1 and $\tilde{M}_2$ can be considered as closed subspaces of H and $H = M_1 \vee \tilde{M}_2$.

Remark that $X = P^H{}_{E_2|E_1}$ and, moreover, $X P^{M_1}{}_{E_1} = P^H{}_{\tilde{M}_2|M_1}$. Thus, $\tilde{M}_2 \perp (M_1 \ominus E_1)$ and $(\tilde{M}_2 \ominus E_2) \perp M_1$, so

$$H = (\tilde{M}_2 \ominus E_2) \oplus (E_2 \vee E_1) \oplus (M_1 \ominus E_1).$$

Let $\mathfrak{D}'_X$ be the closure in H of $\{(I - X)v: v \in E_1\}$; associating $D_X v$ to each $(I - X)v$ an isometry from $\mathfrak{D}'_X$ onto $\mathfrak{D}_X$ is defined (where, as usual, $D_X = (I - X^*X)^{1/2}$ and $\mathfrak{D}_X$ denotes the closure of $\{D_X v: v \in E_1\}$) . From $X = P^H{}_{E_2|E_1}$ it follows that $E_2 \perp \mathfrak{D}'_X$; since $E_1 \subset E_2 \oplus \mathfrak{D}'_X$, we see that $E_2 \vee E_1 = E_2 \oplus \mathfrak{D}'_X$. Summing up:

$$H = \tilde{M}_2 \oplus \mathfrak{D}'_X \oplus (M_1 \ominus E_1) \approx \tilde{M}_2 \oplus \mathfrak{D}_X \oplus (M_1 \ominus E_1).$$

For each $s \in \Gamma_1$ set $D(s) = [\tilde{W}_2(s)\tilde{M}_2] \vee M_1$ and define the isometry $V(s): D(s) \to H$ by $V(s)[\tilde{W}_2(s)v + u] = v + W_1(s)u$, $\forall\ v \in \tilde{M}_2$, $u \in M_1$. In fact,

$\|\tilde{W}_2(s)v + u\|_H{}^2 = \|\tilde{W}_2(s)v\|_{\tilde{M}_2}{}^2 + 2\,\mathrm{Re}\,\langle XP_+u, \tilde{W}_2(s)v\rangle_{\tilde{M}_2} + \|u\|_{M_1}{}^2 =$

$\|v\|_{\tilde{M}_2}{}^2 + 2\,\mathrm{Re}\,\langle XP_+W_1(s)u,v\rangle_{\tilde{M}_2} + \|W_1(s)u\|_{M_1}{}^2 = \|v + W_1(s)u\|_H{}^2$, because $\langle XP_+u,\tilde{W}_2(s)v\rangle_{\tilde{M}_2} = \langle \tilde{W}_2(s)^*XP_+u,v\rangle_{\tilde{M}_2} = \langle T_2(s)XP_+u,v\rangle_{\tilde{M}_2} = \langle XT_1(s)P_+u,v\rangle_{\tilde{M}_2} = \langle XP_+W_1(s)u,v\rangle_{\tilde{M}_2}$.

Thus, the family V of isometries associated with lifting problems is well defined.

Remark that, if $s,t \in \Gamma_1$, then $D(s+t) \subset D(s)$, $V(s)D(s+t) \subset D(t)$ and $V(s+t) = V(t)V(s)_{|D(s+t)}$.

We shall now see that any $\tau \in$ LIF(X)" generates a unitary representation U of Γ on a space F that contains H, such that U extends V and τ can be represented as a restriction of an orthogonal projection on F.

For any $(h_1,h_2), (h'_1,h'_2) \in F_1 \times F_2$ we set $\langle (h_1,h_2),(h'_1,h'_2)\rangle := \langle h_1,h'_1\rangle_{F_1} + \langle \tau h_1,h'_2\rangle_{F_2} + \langle h_2, \tau h'_1\rangle_{F_2} + \langle h_2,h'_2\rangle_{F_2}$; since $\|\tau\| = \|X\| = 1$, we get a positive semidefinite scalar product that generates a Hilbert space $F = F_1 \vee F_2$. Obviously, $\tau =$

$P^F{}_{F_2|F_1}$.

Now, H can be considered as a subspace of F because $\langle \tau h_1, h'_2\rangle_{F_2} = \langle XP_+ h_1, h'_2\rangle_{\tilde{M}_2}$ holds for every $h_1 = W_1(s)v$, $h'_2 = \widetilde{W}_2(t)u$, $s,t \in \Gamma_1$, $(v,u) \in E_1 \times E_2$. In fact, since $P^{F_2}{}_{E_2}\tau_{|M_1} = XP_+$, $\langle \tau W_1(s)v, \widetilde{W}_2(t)u\rangle_{F_2} = \langle \tau U_1(s)v, U_2(-t)u\rangle_{F_2} = \langle \tau U_1(s+t)v, u\rangle_{F_2} = \langle XP_+U_1(s+t)v, u\rangle_{E_2} = \langle XT_1(s+t)v, u\rangle_{E_2} = \langle T_2(t)XP_+U_1(s)v, u\rangle_{E_2} = \langle XP_+W_1(s)v, \widetilde{W}_2(t)u\rangle_{E_2}$.

Since τ intertwines U_1 and U_2, a unitary representation U of Γ on F is defined by setting $U(s)[f_1+f_2] := U_1(s)f_1 + U_2(s)f_2$, $\forall\ s \in \Gamma$, $f_1 \in F_1$, $f_2 \in F_2$. For every $s \in \Gamma_1$, $v \in \tilde{M}_2$, $u \in M_1$ we get $U(s)[\widetilde{W}_2(s)v + u] = U_2(s)[U_2(-s)v] + U_1(s)u = v + W_1(s)u$, so $U(s)_{|D(s)} = V(s)$. Also, $F = \bigvee\{U(s)H: s \in \Gamma\}$, because $\bigvee\{U(s)H: s \in \Gamma\} \supset \bigvee\{U(s)E_j: s \in \Gamma\} = F_j$, $j = 1,2$.

Let $\mathcal{U}$ be the (eventually empty) family of couples (U,F), with F a Hilbert space that contains H and U a unitary representation of Γ on F that extends V, such that the minimality condition $F = \bigvee\{U(s)H: s \in \Gamma\}$ holds. We say that $(U,F) \approx (U',F')$ in $\mathcal{U}$ if there exists a unitary operator $R \in \mathcal{L}(F,F')$ such that $R_{|H} = I_H$ and $R\,U(s) = U'(s)\,R$ for every $s \in \Gamma$. With this notation, the above construction shows that , if $\tau \in LIF(X)''$, there exists $(U,F) \in \mathcal{U}$ such that $F = F_1 \vee F_2$ and $\tau = P^F{}_{F_2|F_1}$.

We shall now prove that, if $(U,F) \in \mathcal{U}$, it may always be assumed that $F = F_1 \vee F_2$ and that $U_{|F_j} = U_j$, $j = 1,2$. Recall that $F \supset H \supset M_1$ and $U(s)_{|M_1} = V(s)_{|M_1} = U_1(s)_{|M_1}$, $\forall\ s \in \Gamma_1$; also, $F_1 = \bigvee\{U_1(-t)M_1 : t \in \Gamma_1\}$. Set $F'_1 = \bigvee\{U(-t)M_1 : t \in \Gamma_1\}$ and $U'_1(s) \equiv U(s)_{|F'_1}$; then $F \supset F'_1$ and a unitary operator $R_1 \in \mathcal{L}(F_1,F'_1)$ such that $R_1U_1(s) \equiv U'(s)R_1$ is defined by $R_1[U_1(-t)h] = U'_1(-t)h$, $\forall\ t \in \Gamma_1$, $h \in M_1$. Thus, F_1 may be identified with the subspace F'_1 of F and U_1 with the restriction U'_1 of U to F'_1. In the same way we can prove the assertion concerning F_2 and U_2. Moreover, $F_1 \vee F_2 = \bigvee\{U(s)[M_1 + \tilde{M}_2]: s \in \Gamma\} = \bigvee\{U(s)H: s \in \Gamma\} = F$.

Thus, a correspondence $\mathcal{T}: \mathcal{U} \to \mathcal{L}(F_1,F_2)$ is defined by associating to each $(U,F) \in \mathcal{U}$ the operator $\tau = P^{F_2}{}_{F_2|F_1}$. We have seen that for every $\tau \in LIF(X)''$ there exists $(U,F) \in \mathcal{U}$ such that $\tau = \mathcal{T}(U,F)$. Now we shall prove the converse, i.e., that for every $(U,F) \in \mathcal{U}$, $\tau := \mathcal{T}(U,F) \in LIF(X)''$.

Let $v_1 \in E_1$, $v_2 \in E_2$ and $s \in \Gamma_1$; then $\langle (P^{F_2}{}_{E_2}\tau_{|M_1})U_1(s)v_1, v_2\rangle_{E_2} = \langle P^{F_2}{}_{E_2}U_1(s)v_1, v_2\rangle_F = \langle U_1(s)v_1, v_2\rangle_H = \langle XP_1U_1(s)v_1, v_2\rangle_{E_2}$, so $P^{F_2}{}_{E_2}\tau_{|M_1} = X P^{M_1}{}_{E_1}$. Then $1 \geq \|\tau\| \geq \|X\| = 1$, so $\|\tau\| = 1$. If $s,t_1,t_2 \in \Gamma$, $v_1 \in E_1$, $v_2 \in E_2$, then

$\langle \tau U_1(s)U_1(t_1)v_1, U_2(t_2)v_2\rangle_{F_2} = \langle P^F{}_{F_2}U(s+t_1)v_1, U(t_2)v_2\rangle_F = \langle U_1(t_1)v_1, U_2(t_2-s)v_2\rangle_F = \langle P^F{}_{F_2}U(t_1)v_1, U(t_2-s)v_2\rangle_F = \langle \tau U_1(t_1)v_1, U_2(s)^* U_2(t_2)v_2\rangle_{F_2}$; thus, $\tau U_1(s) \equiv U_2(s)\tau$.

Finally, if $(U,F), (U',F') \in \mathcal{U}$ are such that $P^{F_2}{}_{F_2|F_1} = P^{F'_2}{}_{F_2|F_1}$, then setting $R[U_1(s_1)v_1 + U_2(s_2)v_2] \equiv U'_1(s_1)v_1 + U'_2(s_2)v_2$, it is easy to see that $(U,F) \approx (U',F')$, so $\mathcal{J}$ is injective. Summing up we have the following, which is our main technical result.

(III.1) <u>THEOREM</u> **The correspondence $\mathcal{J}$ is a bijection between the sets $\mathcal{U}$ and LIF(X)''.**

As a first application, we point out a sufficient condition for the three sets of liftings we have been considering to be the same one.

(III2) <u>PROPOSITION</u> **If $\Gamma = \Gamma_1 \cup (-\Gamma_1)$, then LIF(X) = LIF(X)' = LIF(X)''** .

PROOF. Let $\tau \in$ LIF(X)"; in order to show that it belongs to LIF(X)' it is enough, by proposition (II.2), to see that $\gamma := P^{F_2}{}_{M_2}\tau_{|M_1}$ intertwines W_1 and W_2, i.e., that

(#) $\langle \gamma W_1(s)W_1(t_1)v_1, W_2(t_2)v_2\rangle_{M_2} = \langle W_2(s)\gamma W_1(t_1)v_1, W_2(t_2)v_2\rangle_{M_2}$

holds for every $s,t_1,t_2 \in \Gamma_1$, $v_1 \in E_1$, $v_2 \in E_2$. Call A the left side of equality (#), B the right one, let $(U,F) \in \mathcal{U}$ be such that $\tau = \mathcal{J}(U,F)$ and consider the following two cases.

(i) $t_2 - s \in \Gamma_1$: $A = \langle U(s+t_1)v_1, U(t_2)v_2\rangle_F = \langle P^{F_2}{}_{M_2}W_1(t_1)v_1, W_2(t_2-s)v_2\rangle_{M_2} = B$.

(ii) $s - t_2 \in \Gamma_1$: $A = \langle U(s-t_2+t_1)v_1, v_2\rangle_F = \langle XP^{M_1}{}_{E_1}W_1(s-t_2)W_1(t_1)v_1, v_2\rangle_{E_2} = \langle T_2(s-t_2)XP^{M_1}{}_{E_1}W_1(t_1)v_1, v_2\rangle_{E_2} = \langle T_2(s-t_2)P^{M_2}{}_{E_2}\gamma W_1(t_1)v_1, v_2\rangle_{E_2} = \langle W_2(s-t_2)\gamma W_1(t_1)v_1, v_2\rangle_{E_2} = B$.

Thus, $\tau \in$ LIF(X)'. Since $\tau^* \in$ LIF(X^*)", it follows that $\tau^* \in$ LIF(X^*)' and in particular that $\tau^*\tilde{M}_2 \subset \tilde{M}_1$, so $\tau \in$ LIF(X). The proof is over.

We now turno to a closer study of the family $V = \{V(s): s \in \Gamma_1\}$. Set $R(s) = V(s)D(s)$, $\forall s \in \Gamma_1$. Thus,

$R(s) = \tilde{M}_2 \vee [W_1(s)M_1] = (\tilde{M}_2 \ominus E_2) \oplus (E_2 \vee W_1(s)E_1) \oplus W_1(s)(M_1 \ominus E_1)$,

because $W_1(s)(M_1 \ominus E_1) \subset (M_1 \ominus E_1)$, since it is orthogonal to E_1. As before we see that

$E_2 \vee W_1(s)E_1 = E_2 \oplus \{(I - XP^{M_1}{}_{E_1})W_1(s)v: v \in E_1\}^-$, so:

(III.3) $R(s) = \tilde{M}_2 \oplus \{(I - XP^{M_1}{}_{E_1})W_1(s)v: v \in E_1\}^- \oplus W_1(s)(M_1 \ominus E_1)$, $\forall s \in \Gamma_1$.

From $V(s)^{-1}[f+W_1(s)g] = \tilde{W}_2(s)f+g$, $\forall f \in \tilde{M}_2, g \in M_1$, we get

(III.4) $D(s) = \widetilde{W}_2(s)\widetilde{M}_2 \oplus \{(I - \widetilde{W}_2(s)XT_1(s))v: v \in E_1\}^- \oplus (M_1 \ominus E_1), \forall s \in \Gamma_1$.

And $V(s): D(s) \to R(s)$ is given by

(III.5) $V(s)\{\widetilde{W}_2(s)f \oplus [I-\widetilde{W}_2(s)XT_1(s)]v \oplus g\} = f \oplus [W_1(s)-XT_1(s)]v \oplus W_1(s)g$
$\forall s \in \Gamma_1, f \in \widetilde{M}_2, v \in E_1$ and $g \in (M_1 \ominus E_1)$.

IV.- An extension of the Nagy-Foias theorem to general groups

Let $T = \{T(s): s \in \Gamma_1\} \subset \mathcal{L}(E)$ be a semigroup of unitary operators on Γ_1 and $\gamma \in \mathcal{L}(E)$ a contraction that commutes with every element of T; then a semigroup of contractions $T^\# = \{T^\#(s,n): (s,n) \in \Gamma_1 \times \mathbb{Z}_1\} \subset \mathcal{L}(E)$ is defined by $T^\#(s,n) \equiv T(s)\gamma^n$. Working as in proposition (I.6.3) of [N-F] it can be seen that $T^\#$ has a minimal unitary dilation $U^\# = \{U^\#(s,n) = U(s)Q^n: (s,n) \in \Gamma \times \mathbb{Z}\} \subset \mathcal{L}(F)$. Let $W^\# = \{W^\#(s,n) = W(s)R^n: (s,n) \in \Gamma_1 \times \mathbb{Z}_1\} \subset \mathcal{L}(M)$ be the corresponding minimal isometric dilation of $T^\#$, i.e., $M = \bigvee\{U^\#(s,n)E: (s,n) \in \Gamma_1 \times \mathbb{Z}_1\}$ and $W^\#(s,n) = U^\#(s,n)_{|M}$. Then $M = \bigvee\{Q^nE: n \in \mathbb{Z}_1\}$ and $W(s)Q^nE = Q^nE, \forall (s,n) \in \Gamma_1 \times \mathbb{Z}_1$, so $W(s)M \equiv M$ and $W = \{W(s): s \in \Gamma_1\} \subset \mathcal{L}(M)$ is a semigroup of unitary operators.

We shall now see that for a commutant $\{T^\#_1, T^\#_2, X\}$ on a semigroup $\Gamma_1 \times \mathbb{Z}_1$, with $T^\#_1 \equiv T^\#$ as above, LIF(X)" is non empty. By duality, the same holds when $T^\#_2$ is given by a unitary representation of Γ_1 and a contraction commuting with it.

(IV.3) <u>THEOREM</u> **Let the following conditions hold:**

a) Γ is an abelian group with neutral element e and Γ_1 is a sub-semigroup of Γ such that $e \in \Gamma_1$ and every $u \in \Gamma$ can be written as $u = s - t$, with $s,t \in \Gamma_1$.

b) $T_1 = \{T_1(s): s \in \Gamma_1\} \subset \mathcal{L}(E_1)$ is a semigroup of unitary operators on Γ_1 and $Y_1 \in \mathcal{L}(E_1)$ is a contraction that commutes with every element of T_1.

c) $U_1 \subset \mathcal{L}(F_1)$ is a unitary representation of Γ, $Q_1 \in \mathcal{L}(F_1)$ is a unitary operator that commutes with every element of U_1 and $U^\#_1(s,n) \equiv U_1(s)Q_1^n$ gives a minimal unitary dilation on $\Gamma \times \mathbb{Z}$ of the semigroup on $\Gamma_1 \times \mathbb{Z}_1$ defined by $T^\#_1(s,n) \equiv T_1(s)Y_1^n$.

d) $T_2 = \{T_2(s): s \in \Gamma_1\} \subset \mathcal{L}(E_2)$ is a semigroup of contractions on Γ_1 and $Y_2 \in \mathcal{L}(E_2)$ is a contraction that commutes with every element of T_2.

e) There exist $U_2 \subset \mathcal{L}(F_2)$, a unitary representation of Γ, and $Q_2 \in \mathcal{L}(F_2)$, a unitary operator that commutes with every element of U_2, such that $U^\#_2(s,n) \equiv U_2(s)Q_2^n$ gives a minimal unitary dilation on $\Gamma \times \mathbb{Z}$ of the

semigroup on $\Gamma_1 \times \mathbb{Z}_1$ defined by $T^{\#}_2(s,n) = T_2(s)Y_2^{\,n}$.

f) $X \in \mathcal{L}(E_1,E_2)$ intertwines T_1 and T_2 as well as Y_1 and Y_2.

Then there exists an operator $\tau \in \mathcal{L}(F_1,F_2)$ intertwining U_1 and U_2 as well as Q_1 and Q_2, such that $P^{F_2}{}_{E_2}\tau_{|E_1} = X$ and $\|\tau\| = \|X\|$.

REMARK. We have already observed that, for T_1, Y_1 as in (b), there always exist U_1, Q_1 as in (c). On the contrary, for T_2, Y_2 as in (d), (e) need not hold, so it has to be explicitely assumed.

PROOF. We consider the family $V^{\#} = \{V^{\#}(s,n): (s,n) \in \Gamma_1 \times \mathbb{Z}_1\}$ of isometries associated to $\{T^{\#}_1, T^{\#}_2, X\}$. Each element of $V^{\#}$ has its domain $D(s,n)$ and its range contained in the space $H = M_1 \vee \tilde{M}_2$, with $M_1 = \vee\{U^{\#}_1(s,n)E_1: (s,n) \in \Gamma_1 \times \mathbb{Z}_1\}$ and $\tilde{M}_2 = \vee\{U^{\#}_2(-s,-n)E_2: (s,n) \in \Gamma_1 \times \mathbb{Z}_1\}$. For each $(s,n) \in \Gamma_1 \times \mathbb{Z}_1$, $D(s,n) = [\tilde{W}^{\#}_2(s,n)\tilde{M}_2] \vee M_1 = [\tilde{W}_2(s)\tilde{R}_2^{\,n}\tilde{M}_2] \vee M_1$ and $V^{\#}(s,n)[\tilde{W}_2(s)\tilde{R}_2^{\,n}v+u] = v+W_1(s)R_1^{\,n}u$, $\forall\, v \in \tilde{M}_2$, $u \in M_1$, with $W'^{\#}_2(s,n) = \tilde{W}_2(s)\tilde{R}_2^{\,n}$, $\tilde{W}_2(s) = U_2(-s)_{|\tilde{M}_2}$, $\tilde{R}_2 = Q_2^{-1}{}_{|\tilde{M}_2}$, $W_1(s) = U_1(s)_{|M_1}$, $R_1 = Q_{1|M_1}$. Since each $W_1(s)$ is a unitary operator, $V^{\#}(s,n)D(s,n) = \tilde{M}_2 \vee R_1^{\,n}M_1$, so $V(s) := V^{\#}(s,0)$ is an isometry with domain $[\tilde{W}_2(s)\tilde{M}_2] \vee M_1$ and range H, $\forall\, s \in \Gamma_1$. Thus, $V' := \{V(s)^{-1}: s \in \Gamma_1\}$ is an isometric representation of Γ_1. Call B the isometry $V^{\#}(e,1)$ from $\tilde{R}_2\tilde{M}_2 \vee M_1$ onto $\tilde{M}_2 \vee R_1M_1$. By theorem (III.1) in [A.1], in order to see that there exist a unitary representation $U' \subset \mathcal{L}(F)$ of Γ that extends V' to a space F that contains H and a unitary operator $A' \in \mathcal{L}(F)$ that extends B^{-1} and commutes with every element of U', it is enough to prove that $\langle V(s)^{-1}B^{-1}f, V(t)^{-1}B^{-1}g\rangle_H = \langle V(s)^{-1}f, V(t)^{-1}g\rangle_H$ holds for every f,g in the domain of B^{-1} and every $s,t \in \Gamma_1$, i.e., that

$$(*)\quad \langle V(s)^{-1}B^{-1}(u+R_1v), V(t)^{-1}B^{-1}(u'+R_1v')\rangle_H =$$

$$\langle V(s)^{-1}(u+R_1v), V(t)^{-1}(u'+R_1v')\rangle_H,\quad \forall\, u,u' \in \tilde{M}_2,\ v,v' \in M_1,\ s,t \in \Gamma_1.$$

Now, the left side of $(*)$ equals $\langle V(s)^{-1}(\tilde{R}_2u+v), V(t)^{-1}(\tilde{R}_2u'+v')\rangle_H =$

$\langle \tilde{W}_2(s)\tilde{R}_2u+W_1(s)^{-1}v, \tilde{W}_2(t)\tilde{R}_2u'+W_1(t)^{-1}v'\rangle_H =$

$\langle \tilde{R}_2\tilde{W}_2(s)u+W_1(s)^{-1}v, \tilde{R}_2\tilde{W}_2(t)u'+W_1(t)^{-1}v'\rangle_H =$

$\langle B[\tilde{R}_2\tilde{W}_2(s)u+W_1(s)^{-1}v], B[\tilde{R}_2\tilde{W}_2(t)u'+W_1(t)^{-1}v']\rangle_H =$

$\langle \tilde{W}_2(s)u+R_1W_1(s)^{-1}v, \tilde{W}_2(t)u'+R_1W_1(t)^{-1}v'\rangle_H$, which is equal to the right side of $(*)$.

Since $V^{\#}(s,n)_{|D(s,n)} = V(s)B^n{}_{|D(s,n)}$, it is clear that in this way we get a unitary extension of $V^{\#}$ to the space F that contains H. From (III.1) the result follows.

Theorem (IV.1) for $\Gamma = \{e\}$ and proposition (III.2) give the Nagy-Foias theorem.

V.- Complementary remarks

As a direct consequence of theorem (IV.1) and proposition (III.2) we also obtain the following.

(V.1) THEOREM **Let $\{T_1, T_2, X\}$ be a commutant on a semigroup $\Gamma_1 = \Gamma_o \times \mathbb{Z}_1$, where Γ_o is an abelian group; set $\Gamma = \Gamma_o \times \mathbb{Z}$. Then LIF(X) is non empty.**

PROOF. $\{T_j(s,0): s \in \Gamma_1\}$ is a unitary group for $j = 1,2$, so theorem (IV.1) shows that LIF(X)" is non empty; since $\Gamma = \Gamma_1 \cup (\Gamma_1\text{-})$, LIF(X)" = LIF(X).

This result was essentially proved in [A.1] by a more complicated method that was also applied to the study of commutants in $\mathbb{Z}^2$. The last problem will be considered in the next sections, using the simpler approach we have presented here, which is an extension of the proof of Nagy-Foias theorem given in [A.2]. In [F-F] several proofs of the same theorem are considered. The one we are extending can be seen as a "scattering proof", in the A-A-K sense, as it is stressed in Sarason's presentation of it [S].

A continuous version of the commutant lifting theorem was proved in [A.3] and - with a simpler and more conceptual proof - in [B]. Combining the basic result of the last paper with our approach here a still simpler proof is obtained, as we now sketch.

(V.2) THEOREM **For $j = 1,2$ let T_j be a weakly continuous semigroup of contractions in the Hilbert space E_j and U_j its minimal unitary dilation to the space F_j, and set $M_j = \bigvee \{U_j(s)E_j: s \geq 0\}$, $\tilde{M}_j = \bigvee \{U_j(-s)E_j: s \geq 0\}$. Let $X \in \mathcal{L}(E_1, E_2)$ intertwine T_1 and T_2. Then there exists an operator $\tau \in \mathcal{L}(F_1, F_2)$ intertwining U_1 and U_2 such that:**

$$P^{F_2}_{E_2}\tau_{|M_1} = X P^{M_1}_{E_1}, \quad \|\tau\| = \|X\|, \quad \tau M_1 \subset M_2, \quad \tau^* \tilde{M}_2 \subset \tilde{M}_1 .$$

PROOF. Set $\mathbb{R}_1 = \{s \in \mathbb{R}: s \geq 0\}$ and consider the commutant $\{T_1, T_2, X\}$ on $\mathbb{R}_1$. Then the associated family of isometries V is a local semigroup of isometries in the sense of [B], where it is proved that its naturally defined generator can be extendet to a self adjoint operator, so $\mathcal{U}$ is non empty. The result follows.

The last statement is precisely the continuous version of the commutant lifting theorem as it is stated in [F], where it is applied to the solution of an interpolation problem of Dym and Gohberg [D-G].

VI.- Commutants in $\mathbb{Z}^2$

We consider the case $\Gamma = \mathbb{Z}^2$, $\Gamma_1 = \mathbb{Z}^2{}_1 := \{(m,n) \in \mathbb{Z}^2: m,n \geq 0\}$. If $T \subset \mathcal{L}(E)$ is a semigroup of contractions on $\mathbb{Z}^2{}_1$, a well-known theorem due to Ando (se [N-F], p. 20) shows that a unitary dilation of T exists. Thus, if $\{T_1, T_2, X\}$ is a commutant on $\mathbb{Z}^2{}_1$, we can consider the corresponding lifting problems, with the same dfinitions and notations as in sections I and II.

In this case, for $j = 1,2$, $T_j(m,n) = T'_j{}^m\ T''_j{}^n$, for every $m,n \geq 0$, where T'_j and T''_j are commuting contractions in the Hilbert space E_j, while $W_j(m,n) = W'_j{}^m\ W''_j{}^n$, $\forall\ m,n \geq 0$, and $U_j(m,n) = U'_j{}^m\ U''_j{}^n$, $\forall\ m,n \in \mathbb{Z}$, with W'_j, W''_j commuting isometries in M_j and U'_j, U''_j commuting unitary operators in F_j. Also, $\widetilde{W}_2(m,n) = \widetilde{W}'_2{}^m \widetilde{W}''_2{}^n$, where $\widetilde{W}'_2$ and $\widetilde{W}''_2$ are commuting isometries in $\widetilde{M}_2$.

Then, for each $(m,n) \in \mathbb{Z}^2{}_1$, we have $D(m,n) = [\widetilde{W}'_2{}^m \widetilde{W}''_2{}^n \widetilde{M}_2] \vee M_1$ and $R(m,n) = \widetilde{M}_2 \vee [W'_1{}^m\ W''_1{}^n M_1]$, while the isometry $V(m,n)$: $D(m,n) \to R(m,n)$ is defined by $V(m,n)\ [\widetilde{W}'_2{}^m \widetilde{W}''_2{}^n v + u] = v + W'_1{}^m\ W''_1{}^n u$, $\forall\ v \in \widetilde{M}_2$, $u \in M_1$. Set $V' = V(1,0)$, $V'' = V(0,1)$, $D' = D(1,0)$, $D'' = D(0,1)$, $R' = R(1,0)$, $R'' = R(0,1)$. Since $V(m,n) = V'^m V''^n{}_{|D(m,n)}$, in this case $\mathcal{U}$ can be identified with the family of all the (equivalence classes of) couples (U',U") of commuting unitary operators that extend (V',V"), respectively, to a Hilbert space $F = \vee\{U'_j{}^m\ U''_j{}^n H: (m,n) \in \mathbb{Z}^2\}$ that contains H. We know that there is a bijection between $\mathcal{U}$ and LIF(X)". Thus:

(VI.1) <u>THEOREM</u> For $j = 1,2$ let T'_j and T''_j be commuting contractions in a Hilbert space E_j, U'_j and U''_j commuting unitary operators in a Hilbert space F_j such that $\{U'_j{}^m\ U''_j{}^n : m,n \in \mathbb{Z}\}$ is a unitary dilation of the semigroup $\{T'_j{}^m\ T''_j{}^n : m,n \geq 0\}$. Let $X \in \mathcal{L}(E_1,E_2)$ be such that $X\ T'_1 = T'_2\ X$, $X\ T''_1 = T''_2\ X$ and $||X|| = 1$. Set $M_1 = \vee\{U'_1{}^m\ U''_1{}^n E_1 : (m,n) \geq 0\}$, $\widetilde{M}_2 = \vee\{U'_2{}^{-m}\ U''_2{}^{-n} E_1 : (m,n) \geq 0\}$. Let H be the Hilbert space defined by $H = M_1 \vee \widetilde{M}_2$ and $P^H{}_{\widetilde{M}_2|M_1} = X\ P^{M_1}{}_{E_1}$. Let the subspaces D', D", R', R" of H and the isometries V', V", with domains D', D" and ranges R', R", respectively, be given by: $D' = U'_2{}^*\widetilde{M}_2 \vee M_1$, $D'' = U''_2{}^*\widetilde{M}_2 \vee M_1$, $R' = \widetilde{M}_2 \vee U'_1 M_1$, $R'' = \widetilde{M}_2 \vee U''_1 M_1$ and $V'[U'_2 v+u] = v+U'_1 v$, $V''[U''_2 v+u] = v+U''_1 v$, $\forall\ v \in \widetilde{M}_2$, $u \in M_1$. Then the following conditions are equivalent:

a) There exists $\tau \in \mathcal{L}(F_1,F_2)$ such that $\tau U'_1 = U'_2 \tau$, $\tau U''_1 = U''_2 \tau$, $P^{F_2}{}_{E_2} \tau_{|M_1} = X\ P^{M_1}{}_{E_1}$ and $||\tau|| = ||X||$.

b) There exist two commuting unitary operators U', U" that extend V', V", respectively, to a Hilbert space F containing H.

In order to apply this theorem we shall need the following remarks. From the results of section III we know that $H = \widetilde{M}_2 \oplus \{(I-X)a: a \in E_1\}^- \oplus (M_1 \ominus E_1) \approx \widetilde{M}_2 \oplus \mathcal{D}_X \oplus (M_1 \ominus E_1)$. Also, (III.4) and (III.3) show that, for every $(m,n) \in \mathbb{Z}^2{}_1$,

$$D(m,n) = \widetilde{W}_2(m,n)\widetilde{M}_2 \oplus \{[I-\widetilde{W}_2(m,n)XT_1(m,n)]a: a \in E_1\}^- \oplus (M_1 \theta E_1),$$
$$R(m,n) = \widetilde{M}_2 \oplus \{(I- XP^{M_1}{}_{E_1})W_1(m,n)a: a \in E_1\}^- \oplus W_1(m,n)(M_1 \theta E_1).$$

Set $D = D(1,1)$ and $R = R(1,1)$. Thus

$$D' = \widetilde{W}'_2\widetilde{M}_2 \oplus \{[I-\widetilde{W}'_2XT'_1]a: a \in E_1\}^- \oplus (M_1\theta E_1),$$
$$D = \widetilde{W}'_2\widetilde{W}''_2\widetilde{M}_2 \oplus \{[I-\widetilde{W}'_2\widetilde{W}''_2XT'_1T''_1]a: a \in E_1\}^- \oplus (M_1\theta E_1),$$
$$R' = \widetilde{M}_2 \oplus \{(I- XP^{M_1}{}_{E_1})W'_1a: a \in E_1\}^- \oplus W'_1(M_1\theta E_1) \text{ and}$$
$$R = \widetilde{M}_2 \oplus \{(I- XP^{M_1}{}_{E_1})W'_1W''_1a: a \in E_1\}^- \oplus W'_1W''_1(M_1\theta E_1).$$

From formula (I.2) we obtain $E_1 \vee (W''_1M_1) = E_1 \oplus \Lambda(0,1) \oplus W''_1(M_1\theta E_1)$; from $M_1 = E_1 \vee (W''_1M_1)$ it follows that $(M_1\theta E_1) = \Lambda(0,1) \oplus W''_1(M_1\theta E_1)$, so $W'_1(M_1\theta E_1) = W'_1W''_1(M_1\theta E_1) \oplus \{W'_1(W''_1-T''_1)b: b \in E_1\}^-$. Thus,

$$(R'\theta R) = \left[\{(I- XP^{M_1}{}_{E_1})W'_1a: a\in E_1\}^- \oplus \{W'_1(W''_1-T''_1)b: b\in E_1\}^-\right]$$
$$\theta \{(I- XP^{M_1}{}_{E_1})W'_1W''_1c: c\in E_1\}^-.$$

Now, the correspondence given, for any $a \in E_1$ and $b \in E_1$, by

$(I- XP^{M_1}{}_{E_1})W'_1a \oplus W'_1(W''_1-T''_1)b \rightarrow D_{XT'_1}a \oplus D_{T''_1}b$ is an isometry of $\left[\{(I- XP^{M_1}{}_{E_1})W'_1a: a\in E_1\}^- \oplus \{W'_1(W''_1-T''_1)b: b\in E_1\}^-\right]$ onto $\mathfrak{D}_{XT'_1} \oplus \mathfrak{D}_{T''_1}$ that, for every $c \in E_1$, takes $(I- XP^{M_1}{}_{E_1})W'_1W''_1c = (I- XP^{M_1}{}_{E_1})W'_1T''_1c \oplus W'_1(W''_1-T''_1)c$ to $D_{XT'_1}T''_1c \oplus D_{T''_1}c$. Consequently:

$$\text{(VI.2)} \qquad (R'\theta R) \approx [\mathfrak{D}_{XT'_1} \oplus \mathfrak{D}_{T''_1}]\ \theta\ \{D_{XT'_1}T''_1c \oplus D_{T''_1}c: c \in E_1\}^-.$$

Working in a similar way, from formula (I.3) we obtain

$$(D'\theta D) = \left[\{\widetilde{W}'_2(I- \widetilde{W}''_2T''_2)a: a\in E_2\}^- \oplus \{(I-\widetilde{W}'_2XT'_1)b: b\in E_1\}^-\right]$$
$$\theta \{(I- \widetilde{W}'_2\widetilde{W}''_2XT'_1T''_1)c: c\in E_1\}^-,$$

and we see that the correspondence $\widetilde{W}'_2(I- \widetilde{W}''_2T''_2)a \oplus (I-\widetilde{W}'_2XT'_1)b \rightarrow D_{T''_2}a \oplus D_{T'_2X}b$ shows that

$$\text{(VI.3)} \quad (D'\theta D) \approx [\mathfrak{D}_{T''_2} \oplus \mathfrak{D}_{T'_2X}]\ \theta\ \{D_{T''_2}T'_2Xc \oplus D_{T'_2X}c: c \in E_1\}^-.$$

We shall now state a reformulation of theorem (VI.1). We saw that

$D' = U'_2{}^*\widetilde{M}_2 \oplus \{(I-U'_2{}^*XT'_1)a: a \in E_1\}^- \oplus (M_1\theta E_1)$ and

$D'' = U''_2{}^*\widetilde{M}_2 \oplus \{(I-U''_2{}^*XT''_1)a: a \in E_1\}^- \oplus (M_1\theta E_1)$, while (III.5) shows that, for every $(m,n) \in \mathbf{Z}^2{}_1$, $v \in \widetilde{M}_2$, $a \in E_1$ and $u \in (M_1\theta E_1)$,

$V(m,n)\{\widetilde{W}_2(m,n)v \oplus [I-\widetilde{W}_2(m,n)XT_1(m,n)]a \oplus u\} =$
$v \oplus [W_1(m,n)-XT_1(m,n)]a \oplus W_1(m,n)u$,
so $V'\{U'_2{}^*v+(I-U'_2{}^*XT'_1)a+u\} = v+(U'_1-XT'_1)a+U'_1u$, $V''\{U''_2{}^*v+(I-U''_2{}^*XT''_1)a+u\} = v+(U''_1-XT''_1)a+U''_1u$.

We know that $X\, P^{M_1}{}_{E_1} = P^H{}_{\tilde{M}_2|M_1}$ so $X = P^H{}_{\tilde{M}_2|E_1}$; since $(I-U'_2{}^*XT'_1)a =$ $(I-U'_2{}^*T'_2)Xa + (I-X)a$, it follows that
$\{(I-U'_2{}^*XT'_1)a: a \in E_1\}^- \approx \{(I-U'_2{}^*T'_2)Xa \oplus D_Xa: a \in E_1\}^-$. Analagously,
$\{(I-U''_2{}^*XT''_1)a: a \in E_1\}^- \approx \{(I-U''_2{}^*T''_2)Xa \oplus D_Xa: a \in E_1\}^-$.

Summing up, theorem (VI.1) says that:

(VI.1.a) <u>THEOREM</u> **For $j = 1,2$ let T'_j and T''_j be commuting contractions in a Hilbert space E_j, U'_j and U''_j commuting unitary operators in a Hilbert space F_j such that $\{U'_j{}^m U''_j{}^n: m,n \in \mathbb{Z}\}$ is a unitary dilation of the semigroup $\{T'_j{}^m T''_j{}^n: m,n \geq 0\}$. Let $X \in \mathcal{L}(E_1,E_2)$ be such that $X\,T'_1 = T'_2\,X$, $X\,T''_1 = T''_2\,X$ and $||X|| = 1$. Set $M_1 = \bigvee\{U'_1{}^m\, U''_1{}^n E_1: (m,n) \geq 0\}$, $\tilde{M}_2 = \bigvee\{U'_2{}^{-m}\, U''_2{}^{-n}E_1: (m,n) \geq 0\}$. Define the Hilbert space H, the subspaces D',D" of H, and the isometries V',V" with domains D',D", respectively, and ranges contained in H, by:**

$H = \tilde{M}_2 \oplus \mathcal{D}_X \oplus (M_1 \ominus E_1)$;

$D' = U'_2{}^*\tilde{M}_2 \oplus \{(I-U'_2{}^*T'_2)Xa \oplus D_Xa: a \in E_1\}^- \oplus (M_1 \ominus E_1)$,

$D'' = U''_2{}^*\tilde{M}_2 \oplus \{(I-U''_2{}^*T''_2)Xa \oplus D_Xa: a \in E_1\}^- \oplus (M_1 \ominus E_1)$;

$V'\{U'_2{}^*v \oplus (I-U'_2{}^*T'_2)Xa \oplus D_Xa \oplus u\} = v \oplus D_XT'_1a \oplus \{(U'_1-T'_1)a+U'_1u\}$,

$V''\{U''_2{}^*v \oplus (I-U''_2{}^*T''_2)Xa \oplus D_Xa \oplus u\} = v \oplus D_XT''_1a \oplus \{(U''_1-T''_1)a+U''_1u\}$,

$\forall\ v \in \tilde{M}_2$, $a \in E_1$ and $u \in M_1$. Then the following conditions are equivalent:

a) There exists $\tau \in \mathcal{L}(F_1,F_2)$ such that $\tau U'_1 = U'_2\tau$, $\tau U''_1 = U''_2\tau$, $P^{F_2}{}_{E_2}\tau_{|M_1} = X\, P^{M_1}{}_{E_1}$ and $||\tau|| = ||X||$.

b) There exist two commuting unitary operators U', U" that extend V', V", respectively, to a Hilbert space F containing H.

VII.- <u>On the existence of commutative unitary extensions of isometries</u>

Theorem (VI.1) leads us to the consideration of the following problem. We say that the isometry V acts in a Hilbert space H when the domain and the range of V are closed suspaces of H. When V', V" are given isometries acting in H, with domains D',D" and ranges R',R", respectively, we call $\mathcal{U}$ the family of all the (equivalence classes of) couples (U',U") of commuting unitary operatores that extend (V',V"), respectively, to a Hilbert space $F = \bigvee\{U'^m U''^n H: (m,n) \in \mathbb{Z}^2\}$. We need to give conditions that ensure that $\mathcal{U}$ is non void. It was proved in [A-1] that:

(VII.1) If $D' = H$, $\mathcal{U}$ is non void iff $\langle V'^n V''f, V''g\rangle = \langle V'^n f, g\rangle$ holds, $\forall$ f,g $\in$ D", n = 1,2...

(VII.2) THEOREM **Let H be a Hilbert space, D',D",R',R" closed subspaces of H, V',V" isometric operators with domains D',D" and ranges R',R". If $T' \in \mathcal{L}(H)$ is a contractive extension of V' the following conditions are equivalent:**

i) $P_{R''}T'^n V'' P_{D''} = V'' P_{D''} T'^n P_{D''}$, n = 1,2...

ii) $\exists$ U',U", commutative unitary extensions of V',V", respectively, to a Hilbert space $F \supset H$ and U' is a unitary dilation of T'.

iii) $P_{R''}T'^n T'' P_{D''} = P_{R''}T''T'^n P_{D''}$, n = 1,2..., for every contractive extension $T'' \in \mathcal{L}(H)$ of V".

PROOF. For T" as in (iii) it is easy to see that $T''(H\ominus D) \subset (H\ominus R)$, so $T''P_{D''} = V''P_{D''} = P_{R''}T''$; consequently, (i) and (iii) are equivalent. If (ii) holds $P_{R''}T'^n V'' P_{D''} = P^H_{R''}(P^F_H U'^n{}_{|H})U''P^H_{D''} = P^F_{R''}U'^n U''P^H_{D''} = P^F_{R''}U''U'^n P^H_{D''} = P^F_{R''}U''P^F_{D''}U'^n P^H_{D''} = V''P_{D''}T'^n P_{D''}$ and (i) holds. If the last is true, let $W' \in \mathcal{L}(G')$ be a unitary dilation of T'; if f,g $\in$ D" and $n \geq 1$ then $\langle W'^n V''f, V''g\rangle = \langle P_{R''}T'^n V''f, V''g\rangle = \langle V''P_{D''}T'^n f, V''g\rangle = \langle P_{D''}W'^n f, g\rangle = \langle W'^n f, g\rangle$; from (VII.1) the result follows.

When we apply this theorem to the situation considered in (VI.1) we have additional conditions that are considered in the following.

(VII.3) PROPOSITION **Let H,V',V",D',D",R' and R" be as in (VII.2). Set $N'' = H\ominus D''$ and $M'' = H\ominus R''$. Assume that D is a closed subspace of H such that:**

$D \subset D' \cap D''$, $V'D \subset D''$, $V''D \subset D'$, $V'V''_{|D} = V''V'_{|D}$.

a) The following conditions (i) and (ii) are equivalent:

i) $\exists$ $T' \in \mathcal{L}(H)$, a contractive extension of V' such that $T'(D''\ominus D) \subset D''$, $T'V''_{|D''\ominus D} = V''T'_{|D''\ominus D}$;

ii) $\exists$ U',U", commutative unitary extensions of V',V", respectively, to a Hilbert space $F \supset H$, such that $U'(D''\ominus D)$ is orthogonal to N" and $U''U'(D''\ominus D)$ is orthogonal to M".

b) If (a.i) holds $\exists$ U',U" as in (a.ii) and such that U' is a unitary dilation of T'. If (a.ii) holds, $T' := P^F_H U'_{|H}$ is as in (a.i).

PROOF. Assume (a.i); then $T'D'' \subset D''$ and $T'V''P_{D''} = V''P_{D''}T'P_{D''}$, so $P_{R''}T'^n V''P_{D''} =$

$P_{R''}T'^{n-1}V''P_{D''}T'P_{D''} = \dots = V''P_{D''}T'^{n}P_{D''}$, so (VII.2) says that $\exists$ U',U'' $\in \mathcal{U}$ such that U' is a unitary dilation of T'; also, $P^F_{N''}U'_{|D''\ominus D} = P^H_{N''}T'_{|D''\ominus D} = 0$ and $P^F_{M''}U''U'_{|D''\ominus D} = P^F_{M''}U'U''_{|D''\ominus D} = P^H_{M''}T'V''_{|D''\ominus D} = P^H_{M''}V''T'_{|D''\ominus D} = 0$. Now assume (a.ii) and set $T' = P^F_{H}U'_{|H}$; then $P^H_{N''}T'_{|D''\ominus D} = P^F_{N''}U'_{|D''\ominus D} = 0$, so $T'(D''\ominus D) \subset D''$; moreover, $T'V''_{|D''\ominus D} = P^F_{H}U'U''_{|D''\ominus D} = P^F_{H}U''U'_{|D''\ominus D} = P^F_{R''}U''U'_{|D''\ominus D} = V''\,P^F_{D''}U'_{|D''\ominus D} = V''\,P^F_{H}U'_{|D''\ominus D} = V''T'_{|D''\ominus D}$. The proof is over.

(VII.4) <u>COROLLARY</u> **Let H,V',V'',D',D'',R',R'' and D be as in (VII.3). Assume that one of the following equalities hold: (i) D = D'', (ii) D = D', (iii) V'V''D = R'', (iv) V'V''D = R'. Then $\exists$ U',U'', commutative unitary extensions of V',V'', respectively, to a Hilbert space F $\supset$ H. If (i) or (iii) hold an (U',U'') with those properties can be obtained such that U' is a unitary dilation of $V'P_{D'}$.**

PROOF. When (i) holds, we apply (VII.3) with $T' = V'P_{D'}$. When (iii) holds, we consider V'^{-1} and V''^{-1}. The result follows.

VIII.- <u>An extension of the Nagy-Foias theorem to $\mathbb{Z}^2$</u>

Recall that if A $\in \mathcal{L}(G_2,G_3)$ and B $\in \mathcal{L}(G_1,G_2)$ are contractions and C = AB it is said that AB is a regular factorization of C if $\mathfrak{D}_A \oplus \mathfrak{D}_B = \{D_ABv \oplus D_Bv: v \in G_1\}^-$. Thus, if A or B equals the identity, AB is obviously a regular factorization of C. So the next result is an extension of the commutant lifting theorem, which is obtained from the following when T''_j is the identity in E_j, j = 1,2.

(VIII.1) <u>THEOREM</u> **For j = 1,2 let T'_j and T''_j be commuting contractions in a Hilbert space E_j, U'_j and U''_j commuting unitary operators in a Hilbert space F_j such that $\{U'^m_j\, U''^n_j: m,n \in \mathbb{Z}\}$ is a unitary dilation of the semigroup $\{T'^m_j\, T''^n_j: m,n \geq 0\}$. Let X $\in \mathcal{L}(E_1,E_2)$ be such that $X T'_1 = T'_2 X$ and $X T''_1 = T''_2 X$. If one of the factorizations $(XT'_1)T''_1$, $(XT''_1)T'_1$, $T''_2(T'_2X)$, $T'_2(T''_2X)$ of $XT'_1T''_1 = T'_2T''_2X$ is regular then there exists $\tau \in \mathcal{L}(F_1,F_2)$ such that $\tau U'_1 = U'_2\tau$, $\tau U''_1 = U''_2\tau$, $P^{F_2}_{E_2}\tau_{|E_1} = X$ and $\|\tau\| = \|X\|$.**

PROOF. This statement is a consequence fo theorem (VI.1). With its notation and $D = [\tilde{W}'_2\tilde{W}''_2\tilde{M}_2] \vee M_1$, $R = \tilde{M}_2 \vee [W'_1W''_1M_1]$, it is clear that $D \subset D' \cap D''$, $V'D \subset D''$,

$V''D \subset D'$, $V'V''_{|D} = V''V'_{|D}$ and $V'V''D = R$. From corollary (VII.4) and theorem (VI.1) we see that it is enough to prove that:

i) $D = D'$ iff $T'_2(T''_2X)$ is a regular factorization of $XT'_1T''_1 = T'_2T''_2X$,

ii) $R = R'$ iff $(XT'_1)T''_1$ is a regular factorization of $XT'_1T''_1 = T'_2T''_2X$.

Now, formulas (VI.3) and (VI.2) show that

$(D'\theta D) \approx [\mathfrak{D}_{T''_2} \oplus \mathfrak{D}_{T'_2X}]\ \theta\ \{D_{T''_2}T'_2Xc \oplus D_{T'_2X}c: c \in E_1\}^-$,

$(R'\theta R) \approx [\mathfrak{D}_{XT'_1} \oplus \mathfrak{D}_{T''_1}]\ \theta\ \{D_{XT'_1}T''_1c \oplus D_{T''_1}c: c \in E_1\}^-$. The result follows.

The theorem we have just proved can be extended as follows.

(VIII.2) <u>THEOREM</u> **For $j = 1,2$ let T'_j and T''_j be commuting contractions in a Hilbert space E_j, U'_j and U''_j commuting unitary operators in a Hilbert space F_j such that $\{U'^m_j U''^n_j: m,n \in \mathbb{Z}\}$ is a unitary dilation of the semigroup $\{T'^m_j T''^n_j: m,n \geq 0\}$. Let $X \in \mathcal{L}(E_1,E_2)$ be such that $X T'_1 = T'_2 X$, $X T''_1 = T''_2 X$ and $\|X\| = 1$. Set $M_1 = \bigvee \{U'^m_1 U''^n_1 E_1: (m,n) \geq 0\}$, $\tilde{M}_2 = \bigvee \{U'^{-m}_2 U''^{-n}_2 E_1: (m,n) \geq 0\}$. Define the Hilbert space H, the subspaces D',D'' of H, and the isometries V',V'' with domains D',D'', respectively, and ranges contained in H, by:**

$H = \tilde{M}_2 \oplus \mathfrak{D}_X \oplus (M_1 \theta E_1)$;

$D' = U'^*_2\tilde{M}_2 \oplus \{(I-U'^*_2T'_2)Xa \oplus D_Xa: a \in E_1\}^- \oplus (M_1\theta E_1)$,

$D'' = U''^*_2\tilde{M}_2 \oplus \{(I-U''^*_2T''_2)Xa \oplus D_Xa: a \in E_1\}^- \oplus (M_1\theta E_1)$;

$V'\{U'^*_2v \oplus (I-U'^*_2T'_2)Xa \oplus D_Xa \oplus u\} = v \oplus D_XT'_1a \oplus \{(U'_1-T'_1)a + U'_1u\}$,

$V''\{U''^*_2v \oplus (I-U''^*_2T''_2)Xa \oplus D_Xa \oplus u\} = v \oplus D_XT''_1a \oplus \{(U''_1-T''_1)a + U''_1u\}$,

$\forall\ v \in \tilde{M}_2$, $a \in E_1$ and $u \in M_1$. Then the following conditions are equivalent:

a) $P_{R''}(V'P_{D'})^k U''^{-j}_2 e = V''P_{D''}(V'P_{D'})^k U''^{-j-1}_2 e$, $\forall\ k,j \geq 0$, $e \in E_2$.

b) There exist two commuting unitary operators U',U'' that extend V',V'', respectively, to a Hilbert space F containing H and U' is a unitary dilation of $V'P_{D'} \in \mathcal{L}(H)$.

When (a) and (b) hold there exists $\tau \in \mathcal{L}(F_1,F_2)$ such that $\tau U'_1 = U'_2\tau$, $\tau U''_1 = U''_2\tau$, $P^{F_2}_{E_2}\tau_{|M_1} = X P^{M_1}_{E_1}$ and $\|\tau\| = \|X\|$.

PROOF. From theorems (VI.1.a) and (VII.2) it follows that it is enough to show that (a) is the same as

c) $P_{R''}(V'P_{D'})^n V''P_{D''} = V''P_{D''}(V'P_{D'})^n P_{D''}$, $n = 1,2,...$

Since $D'' = [U''^*_2\tilde{M}_2] \vee M_1$ and $R'' = \tilde{M}_2 \vee [U''_1M_1]$, (c) implies (a). Conversely, assume the last. If $u \in M_1$, $P_{R''}(V'P_{D'})^n V''P_{D''}u = P_{R''}U'^n_1 U''_1u = V''P_{D''}(V'P_{D'})^nP_{D''}u$. Set

$u = U'_2{}^{-k} U''_2{}^{-j-1} e$, with $k,j \geq 0$, $e \in E_2$; if $n \leq k$, $V''P_{D''}(V'P_{D'})^n P_{D''}u = U'_2{}^{n-k} U''_2{}^{-j} e =$ $P_{R''}(V'P_{D'})^n V''P_{D''}u$, while, if $n > k$, (a) shows that $P_{R''}(V'P_{D'})^n V''P_{D''}u =$ $P_{R''}(V'P_{D'})^{n-k} U''_2{}^{-j} e = V''P_{D''}(V'P_{D'})^{n-k} U''_2{}^{-j-1} e = V''P_{D''}(V'P_{D'})^n P_{D''}u$.

Thus, (a) implies (c) and the result follows.

REMARK. When $T'_2(T''_2X)$ is a regular factorization of $XT'_1T''_1 = T'_2T''_2X$, we saw that $D = D'$; then corollary (VII.4) says that the above condition (b) holds. Thus, theorem (VIII.2) is in fact an extension of theorem (VIII.1).

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Semigrupos Locales Biparamétricos de Isometrías, Extensiones Autoadjuntas Conmutativas de Parejas de Operadores Simétricos y Aplicaciones

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Abstract

We extend the notion of local semigroup of isometries to the bidimensional case and give conditions for the existence of commutative selfadjoint extensions of a pair of symmetric operators (a problem related to the extension of biparametric local semigroups). The framework developed is applied to the bidimensional Krein's problem.

Se extiende la noción de semigrupo local de isometrías al caso biparamétrico y se dan condiciones para la existencia de extensiones autoadjuntas conmutativas de parejas de operadores simétricos (problema relacionado con la extensión de semigrupos locales biparamétricos). El desarrollo anterior es aplicado al problema de Krein bidimensional.

1 Introducción

Como es bien sabido, varios problemas centrales del Análisis equivalen a determinar si una cierta función puede ser extendida a un conjunto que contenga a su dominio de manera tal que preserve ciertas propiedades. Uno de los resultados de ese tipo que ha recibido más atención figura en una célebre nota de M.G. Krein [11] que establece que toda función continua y de tipo positivo en un intervalo de la recta puede ser extendida a una función continua y de tipo positivo en toda la recta. En la medida en que dicho teorema establece que la función dada puede verse como la restricción de la transformada de Fourier de una medida de Borel positiva, constituye una extensión de ese resultado fundamental del Análisis Armónico que es el teorema de Bochner.

Ahora bien, las diversas demostraciones, por cierto rigurosas y técnicamente elaboradas, que se han ofrecido del precedente teorema de Krein no siempre resultan naturales. A su vez, entre las varias pruebas conocidas del teorema de Bochner hay una que, mediante la introducción natural a partir de las traslaciones de un grupo unitario, lo reduce al teorema de Stone que establece una representación espectral de grupos de ese tipo. Este procedimiento fue extendido por R.Bruzual [6] mediante la introducción de la teoría de " Semigrupos Locales de Operadores " , a partir de lo cual ofreció pruebas de ciertas generalizaciones del teorema de Krein y de otros resultados de interpolación generalizada (ver [6] y [7]).

Cuando se consideran funciones de tipo positivo en un rectángulo del plano el problema se hace más complejo, no siempre tiene respuesta afirmativa, como lo mostró Rudin en [16], y hasta el momento solo se conocen resultados parciales. Este trabajo, que es un resumen de mi tesis de maestría, aspira a colaborar en la extensión a problemas bidimensionales de los métodos basados en la noción de semigrupos locales de operadores.

2 Preliminares

Comenzamos fijando notación , precisando la terminología y repasando algunas definiciones. Si H y K son espacios de Hilbert entonces $\mathcal{L}(H, K)$ es el conjunto de todos los operadores acotados con dominio H y codominio K. Si $H = K$ pondremos $\mathcal{L}(H)$ en lugar de $\mathcal{L}(H, H)$. Un operador $U \in \mathcal{L}(H)$ es *unitario* si conserva las normas y es sobreyectivo. Diremos que V es una *isometría que actúa en* H con dominio D y rango R si D y R son subespacios cerrados de H y $V \in \mathcal{L}(D, R)$ es unitario. Si V es una isometría que actúa en H entonces $D(V)$ y $Ran(V)$ designarán su dominio y rango respectivamente. U es una *extensión unitaria* de la isometría V que actúa en H si $U \in \mathcal{L}(K)$ es unitario siendo K un espacio de Hilbert que contiene a H como subespacio cerrado y U restringido a $D(V)$ coincide con V. Sobre operadores no acotados mantendremos la notación y los resultados dados en [9]. En particular, recordemos que si A es un operador cerrado simétrico (i.e. $A \subset A^*$) densamente definido en H entonces su *transformada de Cayley* $\mathcal{C}(A) = (A - i)(A + i)^{-1}$ es una isometría que actúa en H con dominio $Ran(A + i)$ y rango $Ran(A - i)$. Se tiene además que B es una extensión simétrica de A sii $\mathcal{C}(B)$ es una isometría que extiende a $\mathcal{C}(A)$ y A es autoadjunto (i.e. $A = A^*$) si $\mathcal{C}(A)$ es unitario.

Como dijimos al comienzo, Ramón Bruzual desarrolló en [6] la teoría de *semigrupos locales uniparamétricos de contracciones* y, en particular, de *isometrías* (S.L.U.I. en lo que sigue) . Un S.L.U.I. en un espacio de Hilbert H es una familia: $S =$

$\{ (S(x), H_x) \ / \ 0 \leq x < a \}$, $0 < a \leq \infty$, tal que se satisfacen las siguientes condiciones :

1) $\forall x \in [0,a)$, H_x es un subespacio cerrado de H , $H_x \subset H_y$ si $0 \leq y \leq x < a$ y $S(x)$ es una isometría que actúa en H con domonio H_x .

2) Si $x, y, x+y \in [0,a) entonces S(x)H_{x+y} \subset H_y$ y $S(x+y)f = S(x)S(y)f$, $\forall f \in H_{x+y}$

3) $\lim_{x\to 0} \| S(x)f - f \| = 0$, $\forall f \in \bigcup \{ H_x \ / \ x \in (0,a) \}$.

4) $\bigcup \{ H_x \ / \ x \in (x_0, a) \}$ es denso en H_{x_0} , $\forall x_0 \in [0,a)$.

El generador de S es el operador A definido por : $A f := \lim_{x \longrightarrow 0} \frac{S(x) f - f}{x}$

Las propiedades básicas de semigrupos ordinarios y de sus generadores son extendidas en el mencionado trabajo de Bruzual a semigrupos locales. Damos a continuación un resumen de los resultados fundamentales de [6] (en lo que a S.L.U.I. se refiere) que nosotros usaremos con soltura:

1) El generador A es un operador antisimétrico (i.e. $-iA$ es simétrico) densamente definido en H tal que si $f \in D(A)$ entonces existe $x_0 \in (0,a)$ / $\forall x \in (0,x_0)$ $S(x) f \in D(A)$, $A f \in H_x$ y y $\frac{d}{dx}(S(x) f) = A S(x) f = S(x) A f$. Además, el generador caracteriza al semigrupo.

2) Un grupo fuertemente continuo de operadores unitarios U extiende a S si y solo si el generador de U es una extensión antiautoadjunta del de S

3) (Teorema 1 de [6]) Todo semigrupo local de isometrías en H se puede extender a un grupo fuertemente continuo de operadores unitarios que actúan en un espacio de Hilbert que contiene a H como subespacio cerrado.

3 Semigrupos Locales Biparamétricos de Isometrías en Espacios de Hilbert

En lo que sigue Q designará el rectángulo $[0,a) \times [0,b) \subset R^2$, $0 < a \leq \infty$, $0 < b \leq \infty$ En Q consideraremos la relación de orden parcial dada por $t \leq s$ sii $s - t \in Q$ (i.e, $t_1 \leq s_1$ y $t_2 \leq s_2$ si $t = (t_1, t_2)$ y $s = (s_1, s_2)$).

Definición 3.1 *Sea H un espacio de Hilbert y supongamos que para cada $t \in Q$, H_t es un subespacio cerrado de H y $S(t)$ es una isometría que actúa en H con dominio H_t . Decimos que la familia $S = \{ (S(t), H_t) \ / \ t \in Q \}$ es un semigrupo local biparamétrico de isometrías en H (S.L.B.I. en lo que sigue) sii se verifican las* *condiciones :*

1) $\forall t, s \in Q$ *se tiene que* $H_t \subset H_s$ *si* $s \leq t$. $H_0 = H$ *y* $S(0) = I$ *(identidad en* H*)*

2) Si $t, s, t+s \in Q$ *entonces* $S(t)\ H_{t+s} \subset H_s$ *y* $S(t+s)f = S(t)S(s)f \quad \forall f \in H_{t+s}$

3) $\forall t \in Q$ *se cumple que* $\bigcup_{s>t} H_s$ *es denso en* H_t .

4) $\forall t \in Q$ *y* $\forall f \in H_t$ *la función* $s \longmapsto S(s)f$ *es continua en* $\{ s \in Q \ / \ s \leq t \}$

Sea $S = \{ (S(t), H_t) \ / \ t \in Q \}$ un S.L.B.I. en el espacio de Hilbert H. Designemos con E al subespacio: $E = \bigcup \{ H_{(x,y)} \ / \ (x,y) \in (0,a) \times (0,b) \}$. Es inmediato, a partir de la definición, que E es denso en H y que $\forall f \in E$, $\exists (x_0, y_0) \in (0,a) \times (0,b) \ / \ f \in H_{(x_0,y_0)}$ y, consecuentemente se tendrá que $f \in H_{(x,y)}$ $\forall (x,y) \in [0,x_0] \times [0,y_0]$. Resulta claro también que si ponemos :

$$H_x^1 = H_{(x,0)} \ , \ S_1(x) = S(x,0) \ \forall x \in [0,a) \qquad H_y^2 = H_{(0,y)} \ , \ S_2(y) = S(0,y) \ \forall y \in [0,b)$$

entonces las familias :

$$S_1 = \{ (S_1(x), H_x^1) \ / \ x \in [0,a) \} \qquad S_2 = \{ (S_2(y), H_y^2) \ / \ y \in [0,b) \}$$

constituyen sendos semigrupos locales uniparamétricos de isometrías en H . Además S se factorea como el producto de S_1 por S_2 , en el sentido de que :

$$Si \quad f \in H_{(x,y)} \Longrightarrow f \in H_x^1 \cap H_y^2 \quad y \quad S(x,y)f = S_1(x)\,S_2(y)f = S_2(y)\,S_1(x)f$$

Para $k = 1,2$ designaremos con A_k al generador del semigrupo S_k y con B_k al operador $-\ i\ A_k$. Sabemos que B_k es un operador simétrico densamente definido en H. Es fácil verificar que se pueden extender al caso local las propiedades básicas de los semigrupos bidimensionales no locales (los detalles se encuentran en [14])

El problema fundamental que abordaremos a continuación y que será el objetivo de próximas secciones es el de estudiar la posibilidad de extender el S.L.B.I. S a un grupo de operadores unitarios . El siguiente teorema muestra que tal posibilidad equivale a la existencia de extensiones autoadjuntas conmutativas de los operadores simétricos B_k , $k = 1,2$, en donde $B_k = -\ i\ A_k$. El concepto de conmutatividad para operadores autoadjuntos es el siguiente: se dice que los operadores autoadjuntos C_1 , C_2 conmutan si sus medidas espectrales conmutan en el sentido usual o sea si $E_1(\Delta)\ E_2(\Delta') = E_2(\Delta')\ E_1(\Delta) \quad \forall \Delta\ ,\ \Delta'$ borelianos de R, siendo E_k la medida espectral de C_k $(k = 1, 2)$ y esto es equivalente (ver [15]) a que los grupos unitarios asociados conmuten (i.e. $e^{ixC_1} e^{iyC_2} = e^{iyC_2} e^{ixC_1}$ $\forall x, y \in R$). Si C_1 y C_2 conmutan entonces se puede considerar su medida producto que es la única medida espectral E definida en los borelianos de R^2 tal que $E(\, \Delta \times \Delta' \,) = E_1(\Delta)\, E_2(\Delta')$ $\forall \Delta$, Δ' borelianos de R , lo cual da lugar, por ejemplo, al teorema de Stone bidimensional (ver [15]). El problema de averiguar cuando dos operadores simétricos tienen extensiones autoadjuntas conmutativas es tratado en la sección 4.

Teorema 3.1 *Existe un grupo fuertemente continuo* $U = \{\, U(t) \,/\, t \in R^2 \,\} \subset \mathcal{L}(F)$ *de operadores unitarios actuando en un espacio de Hilbert* F, *que contiene a* H *como subespacio cerrado, que extiende al* S, *esto es:* $S(t) = U(t)\,|_{H}$, $\forall\, t \in Q$ *si y sólo si existen operadores autoadjuntos conmutativos* $\mathcal{B}_1$, $\mathcal{B}_2$ *en* F *que extienden a* B_1 *y* B_2 *respectivamente.*

Demostración :

Si $U = \{\, U(t) \,/\, t \in R^2 \,\} \subset \mathcal{L}(F)$ es un tal grupo entonces es claro que los grupos uniparamétricos de operadores unitarios : $U_1 = \{\, U_1(x) \,/\, x \in R \,\}$, $U_2 = \{\, U_2(y) \,/\, y \in R \,\}$ dados por $U_1(x) = U(x, 0)$, $U_2(y) = U(0, y)$ extienden a S_1 y S_2 respectivamente, y además conmutan (i.e, $U_1(x)\, U_2(y) = U_2(y)\, U_1(x)$ $\forall x, y \in R$). Si $\mathcal{A}_1$ y $\mathcal{A}_2$ son sus respectivos generadores y $\mathcal{B}_k = -\, i\, \mathcal{A}_k$ $(k = 1, 2)$ entonces es $B_k \subset \mathcal{B}_k$ (teorema principal de la sección 1 de [6]) y $\mathcal{B}_1$ conmuta con $\mathcal{B}_2$.

Reciprocamente, si $\mathcal{B}_k$ $k = 1, 2$ son operadores autoadjuntos conmutativos que actúan en F con $H \subset F$ que extienden a B_1 y B_2 respectivamente, entonces los grupos $\{\, e^{ix\mathcal{B}_1} \,/\, x \in R \,\}$, $\{\, e^{iy\mathcal{B}_2} \,/\, y \in R \,\}$ extienden a S_1 y S_2 y conmutan. Resulta entonces que $\{\, U(x, y) = e^{ix\mathcal{B}_1}\, e^{iy\mathcal{B}_2} \,/\, (x, y) \in R^2 \,\}$ es un grupo fuertemente continuo de operadores unitarios en F y es inmediato verificar que extiende a S.

Q.E.D.

4 Sobre Extensiones Autoadjuntas Conmutativas de Parejas de Operadores Simétricos

Una cuestión importante tanto para los problemas de momentos (ver [5]), para la extensión de semigrupos locales, como para la teoría de operadores en general, es poder averiguar cuando es posible encontrar extensiones autoadjuntas conmutativas de familias de operadores simétricos. Este problema nada sencillo presenta rasgos notables desde que Nelson dio en [12] un ejemplo de un espacio de Hilbert H y de dos operadores simétricos B_1 y B_2 densamente definidos en H tales que:

1) $D(B_1) = D(B_2) = D$

2) $B_1\ D \subset D$, $B_2\ D \subset D$

3) Para $k = 1, 2$ es B_k esencialmente autoadjunto (i.e. $\overline{B_k}$ es autoadjunto).

4) $B_1\, B_2\, f \;=\; B_2\, B_1\, f \quad , \quad \forall\, f \in D$

5) $\overline{B_1}$ y $\overline{B_2}$ no conmutan.

o sea, un ejemplo de dos operadores simétricos esencialmente autoadjuntos con el mismo dominio por ellos invariante, que conmutan en el sentido usual pero que no tienen extensiones autoadjuntas conmutativas ni siquiera en un espacio de Hilbert más grande (ver incluso [15]). En esta sección tratamos de enfocar la mencionada cuestión a partir del " *Problema de Extrapolación Bidimensional* " (ver [3]), es decir al problema de encontrar extensiones unitarias conmutativas de parejas de isometrías. El siguiente lema justifica este procedimiento.

Lema 4.1 *Si C_1, C_2 son operadores autoadjuntos que actúan en un espacio de Hilbert H y U_1, U_2 son sus respectivas transformadas de Cayley entonces C_1 conmuta con C_2 si y sólo si $U_1\, U_2 \;=\; U_2\, U_1$*

Demostración :

($\Longrightarrow$) Es una consecuencia inmediata del teorema espectral.

($\Longleftarrow$) Para probar que C_1 conmuta con C_2 alcanza con probar que (ver [15]) $R_1(z)\, R_2(w) \;=\; R_2(w)\, R_1(z) \quad \forall z, w \in C \,/\, Im(z)\,.\,Im(w) \;\neq\; 0$, siendo R_k la resolvente de C_k , $k = 1, 2$, es decir : $R_k(z) \;=\; (\, z\, I \;-\; C_k\,)^{-1}$. Si $Im(z) \;<\; 0$ se tiene que $R_k(z) \;=\; (zI - C_k)^{-1} \;=\; (\, zI \;-\; i(I + U_k)(I - U_k)^{-1}\,)^{-1} \;=$
$= \;(I \;-\; U_k)\; (\; (z \;-\; i)I \;\;-\;\; (z \;+\; i)U_k\;)^{-1} \;\;=\;\; (I \;-\; U_k)\; \frac{1}{z-i}\; (\; I \;\;-\;\; \frac{z+i}{z-i} U_k\;)^{-1} \;\;=$
$(I - U_k)\; \sum_{n=0}^{\infty} (\frac{z+i}{z-i})^n\, U_k^n$
Analogamente se obtiene :$R_k(z) \;=\; (I - U_k)\, \frac{1}{z+i}\, \sum_{n=0}^{\infty} (\frac{z-i}{z+i})^n\, U_k^{-n-1}$, *si* $Im(z) > 0$
Como $U_1\, U_2 \;=\; U_2\, U_1$ y son unitarios se cumple que $U_1^m\, U_2^n \;=\; U_2^n\, U_1^m$ para toda pareja de números enteros m *y* n . Se deduce que $R_1(z)R_2(w) \;=\; R_2(w)R_1(z) \quad \forall z, w \in C \;/\; Im(z)\,.\,Im(w) \;\neq\; 0$.

Q.E.D.

Definición 4.1 *Sean* B_1 , B_2 *operadores simétricos densamente definidos en un espacio de Hilbert* H. *Decimos que* $(\mathcal{B}_1, \mathcal{B}_2, F)$ *es es una extensión autoadjunta conmutativa de la pareja* (B_1, B_2) *sii se satisfacen :*

1) Para $k = 1,2$, $\mathcal{B}_k$ *es un operador autoadjunto que actúa en un espacio de Hilbert* F *que contiene a* H *como subespacio cerrado.*

2) $\mathcal{B}_1$ *conmuta con* $\mathcal{B}_2$.

3) Para $k = 1,2$ $B_k \subset \mathcal{B}_k$.

La extensión es minimal si además se cumple

4) $F = \bigvee \{ U_1(x) U_2(y) H \ / \ x, y \in R \}$ *siendo* U_1 *y* U_2 *los grupos de operadores unitarios dados por :* $U_1(x) = e^{ix\mathcal{B}_1}$, $U_2(y) = e^{iy\mathcal{B}_2}$

Designaremos con $\mathcal{G}$ *a la familia de todas las* $(\mathcal{B}_1, \mathcal{B}_2, F)$ *que verifican las condiciones 1) a 4) , módulo la siguiente relación de equivalencia* $(\mathcal{B}_1, \mathcal{B}_2, F) \sim (\mathcal{B}'_1, \mathcal{B}'_2, F')$ *si existe un operador unitario* $\Lambda \in \mathcal{L}(F, F')$ / $\Lambda \mathcal{B}_k \subset \mathcal{B}'_k \Lambda$ *y* $\Lambda^{-1} \mathcal{B}'_k \subset \mathcal{B}_k \Lambda^{-1}$, $k = 1,2$ *y* $\Lambda |_H = I_H$ *(identidad en H)*

Si B_1 *y* B_2 son operadores simétricos , cerrados, densamente definidos en H sabemos que sus transformadas de Cayley : $V_k = \mathcal{C}(B_k)$, $k = 1,2$, son isometrías que actúan en H con dominios $D(V_k) = Ran\,(B_k + i)$ *y rangos* $Ran(V_k) = Ran\,(B_k - i)$. La pareja de isometrías (V_1, V_2) tiene asociada una familia $\mathcal{U}$ (eventualmente vacía) de extensiones unitarias conmutativas que el próximo teorema relacionará con $\mathcal{G}$. Recordemos que, según [3] , (U_1, U_2, F) es una extensión unitaria conmutativa minimal de la pareja de isometrías (V_1, V_2) que actúan en H sii se verifican :

1) Para cada $k = 1,2$ $U_k \in \mathcal{L}(F)$ es un operador unitario definido en un espacio de Hilbert F que contiene a H como subespacio cerrado .

2) $U_1 U_2 = U_2 U_1$.

3) Para $k = 1,2$ *es* $V_k \subset U_k$.

4) $F = \bigvee \{ U_1^m U_2^n H \ / \ m, n \in Z \}$.

Se designa con $\mathcal{U}$ a la familia de todas las (U_1, U_2, F) que verifican las 4 condiciones mencionadas, módulo la siguiente relación de equivalencia : $(U_1, U_2, F) \sim (U'_1, U'_2, F')$ *si existe un operador unitario* $\Lambda \in \mathcal{L}(F, F')$ / $\Lambda U_k = U'_k \Lambda$ *y* $\Lambda |_H = I_H$

Teorema 4.1 *Sean* B_1 *y* B_2 *operadores simétricos , cerrados, densamente definidos en un espacio de Hilbert* H , $\mathcal{G}$ *la familia de clases de extensiones autoadjuntas conmutativas minimales a ellos asociada y* $\mathcal{U}$ *la familia de clases de extensiones unitarias*

conmutativas minimales de sus transformadas de Cayley : $\mathcal{C}(B_1)$, $\mathcal{C}(B_2)$.
Entonces la correspondencia $\Phi : \mathcal{G} \longrightarrow \mathcal{U} \,/\, [(\mathcal{B}_1, \mathcal{B}_2, F)] \longmapsto [(\mathcal{C}(\mathcal{B}_1), \mathcal{C}(\mathcal{B}_2), F)]$ *es una función biyectiva de* $\mathcal{G}$ *sobre* $\mathcal{U}$.

Demostración :

Observamos, en primer lugar , que el lema anterior y las propiedades de la transformada de Cayley nos permiten asegurar que $(\mathcal{B}_1, \mathcal{B}_2, F)$ es extensión autoadjunta conmutativa de (B_1, B_2, H) sii $(\mathcal{C}(\mathcal{B}_1), \mathcal{C}(\mathcal{B}_2), F)$ es extensión unitaria conmutativa de $(\mathcal{C}(B_1), \mathcal{C}(B_2), H)$. Además la minimalidad se mantiene, en efecto, poniendo :

$$U_k = \mathcal{C}(\mathcal{B}_k) \,,\; U_k(x) = e^{ix\mathcal{B}_k} \,,\; k = 1,2 \,,\; U(x,y) = U_1(x)\, U_2(y) \,,\; \forall x, y \in R$$

$$N = \bigvee \{\, U_1^m U_2^n H \,/\, m, n \in Z \,\} \,,\; N' = \bigvee \{\, U(t)\, H \,/\, t \in R^2 \,\}$$

es claro que $U(t)\, N' = N'\; \forall t \in R^2$ y que $U_1^m\, U_2^n\, N = N\; \forall m, n \in Z$. Ahora bien, N' es invariante por $U(t)\;\; \forall t \in R^2$ de donde resulta que N' es invariante por U_1 *y* U_2 (por ser éstos los cogeneradores de los grupos : $\{U_1(x)\}$ *y* $\{U_2(y)\}$ respectivamente) con lo cual es $U_1^m\, U_2^n\, N' \subset N'$, $\forall m, n \in Z$, de donde se obtiene que $U_1^m\, U_2^n\, N' = N'$, $\forall m, n \in Z$. Razonando de modo análogo se obtiene $U(t)\, N = N$, $\forall t \in R^2$. Se tiene :
$N = \bigvee \{\, U_1^m U_2^n H \,/\, m, n \in Z \,\} \subset \bigvee \{\, U_1^m U_2^n N' \,/\, m, n \in Z \,\} = N' =$
$= \bigvee \{\, U(t)\, H \,/\, t \in R^2 \,\} \subset \bigvee \{\, U(t)\, N \,/\, t \in R^2 \,\} = N$ con lo cual $N = F$ sii $N' = F$.
Es fácil verificar que $(U_1, U_2, F) \sim (U_1', U_2', F')$ si y solo si $(\mathcal{B}_1, \mathcal{B}_2, F) \sim (\mathcal{B}_1', \mathcal{B}_2', F')$ de donde resulta que la correspondencia Φ está bien definida y es uno a uno. Del hecho de que un operador unitario es la transformada de Cayley de un operador autoadjunto si y sólo si no tiene valor propio uno y, de las consideraciones anteriores, se deduce que el recorrido de Φ es $\{\, [(U_1, U_2, F)] \in \mathcal{U} \,/\, U_k$ *no tiene valor propio* $1\;\, k = 1,2 \,\}$.

Ahora bien, supongamos que (U_1, U_2, F) es una extensión unitaria conmutativa (no necesariamente minimal) de (V_1, V_2, H) siendo $V_k = \mathcal{C}(B_k)$, $k = 1,2$. Designemos con S al subespacio $\{\, f \in F \,/\, U_1 f = f \,\}$. Para todo $f \in S$ y para todo $d_1 \in D(V_1)$ se tiene: $\langle d_1 \,,\, f \rangle = \langle d_1 \,,\, U_1^* f \rangle = \langle U_1\, d_1 \,,\, f \rangle = \langle V_1\, d_1 \,,\, f \rangle$ de donde $\langle (I - V_1)\, d_1 \,,\, f \rangle = 0$ y por lo tanto f es ortogonal a $Ran(I - V_1) = D(B_1)$ que es denso en H. Resulta entonces que $H \subset S^{\perp}$. Como S reduce a U_1 y a U_2 podemos considerar los operadores unitarios $\tilde{U_k} = U_k \mid_{S^{\perp}} \in \mathcal{L}(S^{\perp})$. Es claro que $(\tilde{U_1}, \tilde{U_2}, S)$

es una extensión unitaria conmutativa de (V_1, V_2, H) tal que $\tilde{U_1}$ no tiene valor propio 1. Aplicando el mismo procedimiento a esta última extensión obtenemos otra en la cual ninguno de los operadores unitarios tiene valor propio 1. Se deduce que Φ es sobre.

Q.E.D.

Estamos ahora en condiciones de obtener criterios para que $\mathcal{G}$ sea no vacío a partir del teorema A de [1] (resultado de Arocena que da condiciones para que $\mathcal{U}$ sea no vacío). Sean entonces B_1 *y* B_2 operadores simétricos (no necesariamente cerrados), densamente definidos en un espacio de Hilbert H , V_k la transformada de Cayley de $\overline{B_k}$, $D_k = D(V_k) = Ran(\overline{B_k} + i) = \overline{Ran(B_k + i)}$, $R_k = Ran(V_k) = Ran(\overline{B_k} - i) = \overline{Ran(B_k - i)}$, $k = 1, 2$; $\mathcal{G}$ *y* $\mathcal{U}$ las familias asociadas a estos operadores como en la definición y teorema anterior . Manteniendo esta notación hasta el final de la sección tenemos los siguientes resultados :

Teorema 4.2 *Sea* $\mathcal{M} = \{ f \in D(B_1 B_2) \cap D(B_2 B_1) \ / \ B_1 B_2 f = B_2 B_1 f \}$. *Si* $(B_2 + i)(B_1 + i)\mathcal{M}$ *es denso en* $Ran(B_2 + i)$ *entonces* $\mathcal{G}$ *no es vacío.*

Demostración:

Si $\mathcal{D} = (B_1 + i)(B_2 + i)\mathcal{M}$ entonces es fácil verificar las siguientes relaciones: 1) $\overline{\mathcal{D}} \subset D_1 \cap D_2$. 2) $V_1 \overline{\mathcal{D}} \subset D_2$ *y* $V_2 \overline{\mathcal{D}} \subset D_1$. 3) $V_1 V_2 f = V_2 V_1 f \quad \forall f \in \overline{\mathcal{D}}$. Nuestra hipótesis nos asegura que $\overline{\mathcal{D}} = D_2$ y de las relaciones recién mencionadas resulta que $\langle V_1^n V_2 f , V_2 g\rangle = \langle V_1^n f , g\rangle \, \forall f, g \in D_2$ *y* $\forall n = 0, 1, 2, ...$

Del teorema A de [1] se sigue que $\mathcal{U}$ no es vacío y consecuentemente tampoco lo es $\mathcal{G}$.

Q.E.D.

Observación : Como $(\overline{B_2} + i)^{-1} : Ran(\overline{B_2} + i) \longrightarrow D(B_2)$ es continuo (por ser B_2 simétrico) resulta que la hipótesis del teorema anterior implica que $(B_1 + i)\mathcal{M}$ es denso en $D(B_2)$ y consecuentemente $Ran(B_1 + i)$ es denso en H . El recíproco de esto no es cierto (nuevamente el ejemplo de Nelson) .

Teorema 4.3 *Si* B_1 *es esencialmente autoadjunto (* $\overline{B_1}$ *autoadjunto) y* $\{ U_1(x) \ / \ x \in R \}$ *es el grupo de operadores unitarios asociado a* B_1 *(esto es:* $U_1(x) = e^{ix\overline{B_1}}$ *) entonces la condición necesaria y suficiente para que* $\mathcal{G}$ *sea no vacío es que* B_2 *conmute debilmente con la familia* $\{ U_1(x) \ / \ x \in R \}$ *o sea que se verifique:*

(4.4) $\langle U_1(x)\, B_2\, b\, ,\, b' \rangle = \langle U_1(x)\, b\, ,\, B_2\, b' \rangle \quad \forall\, b, b' \in D(B_2) \;\; y \;\; \forall\, x \in R$

Demostración :

Observemos, en primer lugar, que la condición (4.4) del enunciado es equivalente a:

$$(*) \quad \langle U_1(x)\, V_2\, f\, ,\, V_2\, g \rangle = \langle U_1(x)\, f\, ,\, g \rangle \quad \forall\, f, g \in D_2 \;\; y \;\; \forall\, x \in R$$

Ahora bien, si $\mathcal{G}$ es no vacío entonces $\mathcal{U}$ es no vacío y es claro que debe valer $(*)$. Supongamos entonces que $(*)$ se cumple. Se tiene:

$\int_R e^{i\lambda x}\, d\langle E_1(\lambda) V_2\, f\, ,\, V_2\, g \rangle = \langle U_1(x)\, V_2\, f\, ,\, V_2\, g \rangle = \langle U_1(x)\, f\, ,\, g \rangle =$

$= \int_R e^{i\lambda x}\, d\langle E_1(\lambda) f\, ,\, g \rangle\, ,\; \forall\, x \in R \Rightarrow \langle E_1(\lambda)\, V_2\, f\, ,\, V_2\, g \rangle = \langle E_1(\lambda)\, f\, ,\, g \rangle \Rightarrow$

$\langle V_1^n\, V_2\, f\, ,\, V_2\, g \rangle = \int_R \left(\frac{\lambda - i}{\lambda + i}\right)^n d\langle E_1(\lambda) V_2 f\, ,\, V_2 g \rangle = \int_R \left(\frac{\lambda - i}{\lambda + i}\right)^n d\langle E_1(\lambda) f\, ,\, g \rangle =$

$\langle V_1^n\, f\, ,\, g \rangle \;\; \forall\, n = 0, 1, 2, ...\;\; y \;\; \forall\, f, g \in D_2$

Nuevamente, del teorema A de [1] y del teorema 4.1 se sigue el resultado.

Q.E.D.

Observación útil: Es claro que la condición (4.4) es equivalente a:

$$\langle U_1(x)\, B_2\, h\, ,\, h' \rangle = \langle U_1(x)\, h\, ,\, B_2\, h' \rangle \quad \forall\, x \in R\; ,\;\; \forall\, h, h' \in D$$

siendo D un "core" para B_2 , esto es, siendo D un subespacio denso en H tal que la clausura de la restricción de B_2 a D coincida con la clausura de B_2 $(\overline{B_2 \mid_D} = \overline{B_2})$ De las consideraciones anteriores resulta el siguiente corolario:

Corolario 4.1 *Si $U_1(x)\, B_2\, h = B_2\, U_1(x)\, h \quad \forall\, x \in R \;,\;\; \forall\, h \in D$ siendo D un "core" para B_2 (en particular, si $U_1(x)\, B_2 \subset B_2\, U_1(x) \quad \forall\, x \in R$ o sea si B_2 conmuta fuertemente con $U_1(x)\; \forall x \in R$) entonces $\mathcal{G}$ no es vacío.*

5 El Problema de Krein Bidimensional

En lo que sigue consideramos dados los rectángulos $I = (-a, a) \times (-b, b)$, $Q = [0, a) \times [0, b)$, $0 < a \leq \infty$, $0 < b \leq \infty$, y una función $k : 2\, I \longrightarrow C$ continua y de tipo positivo con lo cual verificará la condición :

$$\sum_{t,s\in I} k(t-s)\, h(t)\, \overline{h(s)} \;\geq 0 \;,\; \forall\, h : I \longrightarrow C \;\; con \;\; sop(h) \;\; finito\,.$$

El problema de Krein para la función k consiste en estudiar la existencia de extensiones de k a funciones continuas y de tipo positivo en todo el plano, lo cual es equivalente a la existencia de medidas μ de Borel en R^2, positivas y finitas tales que :

$$k(t) \;=\; \hat{\mu}(t) \;:=\; \int_{R^2} e^{-i\,t\,.\,\lambda}\; d\,\mu(\lambda) \;\;,\;\; \forall\, t \in 2I\,.$$

Designemos con K al núcleo de Toeplitz definido positivo asociado con k , esto es $K \;:\; I \times I \longrightarrow C \;/\; K(t,s) \;=\; k(t-s) \;\;,\; \forall\, t,s \in I$

Para encarar el mencionado problema es conveniente introducir un espacio de Hilbert de funciones (para el cual K será un núcleo reproductor) de la siguiente manera : para cada $t \in I$ designemos con k_t a la función dada por $k_t : I \rightarrow C$ / $k_t(\xi) = k(\xi - t) = K(\xi, t)$ y sea E el espacio de las combinaciones lineales finitas de funciones k_t . Si $f = \sum_{i=1}^{n} \alpha_i\, k_{t_i}$ y $g = \sum_{j=1}^{m} \beta_j\, k_{s_j}$ entonces poniendo : $\langle f,g\rangle := \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i\, \overline{\beta_j}\, k(s_j - t_i)$ resulta claro que $\langle\;,\;\rangle : E \times E \longrightarrow C$ es un producto interno en E valiendo la propiedad reproductora : $\forall\, t \in I \;\; y \;\; \forall\, f \in E$ $\langle f, k_t\rangle = f(t)$. Sea entonces $(H, \langle\;,\;\rangle)$ la completación métrica de E. Los elementos de H son funciones continuas, la convergencia en H implica la convergencia uniforme, y sigue valiendo la propiedad reproductora : $f(t) \;=\; \langle\, f \;,\; k_t\,\rangle \;\;,\;\; \forall t \in I \;\;,\;\; \forall f \in H$. Los detalles de las afirmaciones anteriores y la teoría general de núcleos reproductores se encuentra en [4]. Pasamos a construir a continuación un S.L.B.I. en H asociado de manera natural con k; para cada $t \in Q$ pongamos :

$$E_t \;=\; \{\, f \in E \;/\; f \;=\; \sum_{i=1}^{n} \alpha_i\, k_{t_i} \;\;\; y \;\;\; t_i + t \in I \;,\; \forall i \;\}$$

$$S(t)\, f \;=\; \sum_{i=1}^{n} \alpha_i\, k_{t_i + t} \qquad \text{para} \quad f \;=\; \sum_{i=1}^{n} \alpha_i\, k_{t_i} \;\; \in E_t$$

Es claro que $S(t) \;:\; E_t \longrightarrow E$ es lineal y que $\| \, S(t)\, f \,\| = \| \, f \,\|$, $\forall\, f \in E_t$.
Si H_t es la clausura de E_t en H extendemos $S(t)$ a una isometría de H_t en H

que seguiremos llamando $S(t)$. Como la convergencia en H implica la convergencia uniforme (y por lo tanto la puntual) resulta que si ξ y $\xi - t$ están en I entonces $(S(t)\, f)\,(\xi) = f(\xi - t) \ \ \forall\, f \in H_t$. Lo que estamos definiendo son entonces "traslaciones locales". Es inmediato verificar que $S = \{\, (S(t), H_t\,) \ / \ t \in Q \,\}$ es un S.L.B.I. en H que llamaremos S.L.B.I. asociado a la función de tipo positivo k.

Teorema 5.1 *El problema de Krein tiene solución (k tiene extensiones continuas y de tipo positivo a todo el plano) si y sólo si el S.L.B.I. S asociado a k de acuerdo con la construcción anterior se puede extender a un grupo de operadores unitarios que actúan en un espacio de Hilbert F que contiene a H como subespacio cerrado.*

Demostración :

Sea $U \ = \{\, U(t) \,/\, t \in R^2 \,\} \ \subset \mathcal{L}(F) \ \ , H \subset F$, un grupo de operadores unitarios que extiende a S y E la medida espectral definida en los borelianos de R^2 asociada a U . Por el teorema de Stone bidimensional (ver [15]) tenemos que $\forall\, f, g \in F \ \ y \, \forall\, t \in R^2$ vale :

$$< U(t) f \,, g >= \int_{R^2} e^{i\,t.\lambda} \ d < E(\lambda) f, g >$$

y, consecuentemente, la función $\tilde{k} \ : R^2 \longrightarrow C$ dada por :

$$\tilde{k}(t) \ = < k_0 \,, U(t)\, k_0 \ > = \int_{R^2} e^{-\,i\,t.\lambda} \ d < E(\lambda) k_0, k_0 >$$

es continua y de tipo positivo. Se verifica que para todo t , $s \in I$ vale :

$$k(s-t) \ = \ \langle\, k_t \,,\, k_s \,\rangle \ = \ \langle\, U(t)\, k_0 \,,\, U(s)\, k_0 \,\rangle \ = \ \langle\, k_0 \,,\, U(s-t)\, k_0 \,\rangle \ = \ \tilde{k}(s-t)$$

de donde resulta que $\tilde{k}$ extiende a k.

Reciprocamente, si $g \ : \ R^2 \ \longrightarrow C$ es una función continua y de tipo positivo que extiende a k , sea (F , $\langle \ \ \rangle$) el espacio de Hilbert reproductor para el cual $G \ : \ R^2 \times R^2 \longrightarrow \ C \ \ / \ \ G(t,s) \ = g(t-s)$ es núcleo reproductor de acuerdo con la construcción hecha al comienzo de la sección y sea U el S.L.B.I. asociado a la función g . Como g tiene dominio R^2 entonces $U \ = \ \{\, U(t) \ / \ t \in R^2 \}$ es un grupo de operadores unitarios en F. Si $t_h \in I \ \ y \ \ \alpha_h \in C$, $1 \leq h \leq n$ se tiene $\|\, \sum_{h=1}^{n} \ \alpha_h \ g_{t_h} \,\|_F^2 \ = \ \sum_{i,j=1}^{n} \ \alpha_i \ \overline{\alpha_j} \ \langle g_{t_i}, g_{t_j} \rangle_F \ = \ \sum_{i,j=1}^{n} \ \alpha_i \ \overline{\alpha_j} \ g(t_j - t_i) \ = \sum_{i,j=1}^{n} \ \alpha_i \ \overline{\alpha_j} \ k(t_j - t_i) \ = \|\, \sum_{h=1}^{n} \ \alpha_h \ k_{t_h} \,\|_H^2$, luego, la aplicación $k_t \longmapsto g_t \ \ (t \in I)$

determina una isometría $\Phi : H \longrightarrow F$. Si $t, s, t+s \in I$ entonces: $\Phi\, S(t)\, k_s = \Phi\, k_{t+s} = g_{t+s} = U(t)\, g_s = U(t)\, \Phi\, k_s \Longrightarrow S(t)\, f = \Phi^*\, U(t)\, \Phi\, f \quad , \quad \forall\, f \in H_t$ de donde sesulta que el grupo $\{\, U(t) \,\}$ extiende al $\{\, S(t) \,\}$ (si identificamos H con ΦH)

Q.E.D.

Consideremos la correspondencia que a cada medida espectral E correspondiente a un grupo unitario que extiende a S le hace corresponder la medida de Borel en R^2 positiva y finita μ dada por : $\mu(\,\Delta\,) = \langle\, k\,,\, E(\,\Delta\,)\, k\,\rangle$, que, según el teorema anterior verifica: $k(t) = \hat{\mu}(t) \quad , \quad \forall\, t \in 2I$. Si μ es una medida de Borel en R^2 positiva y finita tal que $\hat{\mu}(t) = k(t)\ , \ \forall\, t \in 2I$, entonces $g : R^2 \longrightarrow C \,/\, g(t) = \hat{\mu}(t)$ es una función continua y de tipo positivo en R^2 que extiende a k . Del teorema anterior se deduce entonces que la correspondencia considerada es sobreyectiva. Si U_1 y U_2 son dos grupos de operadores unitarios con medidas espectrales E_1 y E_2 , respectivamente, que dan lugar a la misma solución del problema de Krein entonces se verifica inmediatamente que $\forall\, X, Y \in I$ vale: $d\,\langle E_1(\lambda)\, k_X\ ,\ k_Y\rangle = e^{-i(X-Y)\lambda}\, d\,\langle E_1(\lambda)\, k\ ,\ k\rangle = e^{-i(X-Y)\lambda}\, d\,\langle E_2(\lambda)\, k\ ,\ k\rangle = d\,\langle E_2(\lambda)\, k_X\ ,\ k_Y\rangle$. Tenemos entonces que la correspondencia mencionada es una biyección entre las medidas espectrales correspondientes a grupos unitarios que extienden al S.L.B.I. S asociado a k y las medidas de Borel en R^2 positivas y finitas cuyas trasformadas de Fourier coinciden con k en $2I$ si identificamos dos medidas espectrales E_1 y E_2 que actúan en F_1 y F_2 cuando: $P_H^{F_1}\, E_1\,|_H = P_H^{F_2}\, E_2\,|_H$.

Manteniendo la notación de las consideraciones anteriores , sean S el S.L.B.I. asociado a la función k , S_1 , S_2 los semigrupos locales uniparamétricos correspondientes a S (como en la sección 3) , A_j el generador de S_j , $B_j = -\, i\, A_j\ ,\ j = 1, 2$. Introduzcamos los operadores diferenciales W_j dados por :

$W_j\ :\ D(W_j) \longrightarrow H \;/\; W_j\, f = i\, \partial_j\, f$ siendo $D(W_j) = \{\, f \in H \,/\, \partial_j\, f \in H\,\}$

En [8] Devinatz estudió estos operadores y mostró su utilidad en el problema de Krein. Con un desarrollo totalmente análogo al realizado por Bruzual en [6] para el caso uniparamétrico se obtiene que ellos están estrechamente vinculados con los operadores B_j mediante la relación:

$$B_j \subset \overline{B_j} = W_j^* \subset W_j = B_j^* \quad , \quad j = 1, 2$$

de modo tal que aparecen de manara natural en nuestro enfoque del problema así como el operador derivada aparece de modo natural cuando se consideran grupos de traslaciones. Observemos entonces que si B_j es esencialmente autoadjunto ($\overline{B_j}$ autoadjunto) será $\overline{B_j} = B_j^*$ y por lo tanto $\overline{B_j} = W_j$. Se tendrá además que si (por ejemplo) $\overline{B_1}$ es autoadjunto y si $\xi - xe_1 \in I$ (en donde $e_1 = (1,0)$ y $e_2 = (0,1)$) entonces $(U_1(x) f)(\xi) = f(\xi - xe_1)$, $\forall f \in H$ *siendo* $U_1(x) = e^{i\,x\,\overline{B_1}}$ y también que si $t + xe_1 \in I$ entonces $U_1(x)\, k_t = k_{t+xe_1}$. Mencionaremos a continuación algunos resultados obtenidos por Devinatz en [8] sobre los operadores W_k que se traducen en resultados sobre los operadores B_k. Como I es abierto entonces para cada $h = \sum_i \alpha_i\, k_{t_i} \in E$ existe un vector (x,y) con $x\,.\,y \neq 0$ tal que si $|\,t.e_1\,| < |\,x\,|$ y $|\,t.e_2\,| < |\,y\,|$ entonces el elemento $h_t := \sum_i \alpha_i\, k_{t_i+t}$ está bien definido y pertenece a E. Como h_t es una función continua de t entonces la siguiente integral existe y es un elemento de H: $\tilde{h}_{(x,y)} := \frac{1}{xy} \int_0^x \int_0^y h_t \; d\,t_1\, d\,t_2$. Es claro que $\tilde{h}_{(x,y)} \to h$ si $(x,y) \to (0,0)$ y consecuentemente el subespacio $\mathcal{D}'$ de las combinaciones lineales finitas de las funciones $\tilde{h}_{(x,y)}$ es denso en H. Con la notación que acabamos de introducir valen entonces los siguientes resultados (ver [8]) :

1) $\mathcal{D}' \subset D(W_1^*) \bigcap D(W_2^*)$, la clausura de la restricción de W_k^* a $\mathcal{D}'$ es W_k^* (k=1,2) $\mathcal{D}' \subset D(W_1^*\, W_2^*) \bigcap D(W_2^*\, W_1^*)$ y vale :

$$W_1^*\, \tilde{h}_{(x,y)} = \frac{i}{xy} \int_0^y (\, h_{(x,\mu)} - h_{(0,\mu)}\,)\, d\mu \ , \ W_2^*\, \tilde{h}(x,y) = \frac{i}{xy} \int_0^x (\, h_{(\lambda,y)} - h_{(\lambda,0)}\,)\, d\lambda$$

$$W_1^*\, W_2^*\, \tilde{h}_{(x,y)} = W_2^*\, W_1^*\, \tilde{h}_{(x,y)} = \frac{-1}{xy} (\, h_{(x,y)} - h_{(x,0)} - h_{(0,y)} - h\,)$$

2) Si las funciones $k_1 : (-2a, 2a) \to C \,/\, k_1(x) = k(x,0)$, $k_2 : (-2b, 2b) \to C \,/\, k_2(y) = k(0,y)$ tienen una única extensión continua y de tipo positivo a toda la recta entonces la clausura de la restricción de W_2^* a $(W_1^* + i)\mathcal{D}'$ es W_2^*

Estamos ahora en condiciones de dar una prueba sencilla de uno de los resultados más importantes de [8]:

Teorema 5.2 *(Teorema 1 de [8]) Sea* $k : 2I \longrightarrow C$ *una función continua y de tipo positivo siendo* I *el rectángulo* $I = (-a,a) \times (-b,b)$, $0 < a \leq \infty$, $0 < b \leq \infty$ *Si ambas funciones:* $k_1 : (-2a, 2a) \longrightarrow C \,/\, k_1(x) = k(x,0)$, $k_2 : (-2b, 2b) \longrightarrow C/$ $k_2(y) = k(0,y)$ *tienen una única extensión continua y de tipo positivo a toda la recta* *entonces la función k tiene también una única extensión continua y de tipo positivo a todo el plano.*

Demostración:

Es fácil ver que en estas hipótesis los operadores B_1 y B_2 son esencialmente autoadjuntos y por lo tanto $\overline{B_1} = W_1^*$ y $\overline{B_2} = W_2^*$. Del resultado 2) mencionado en la página anterior surge que la clausura de la restricción de $(\overline{B_2} + i)$ a $(\overline{B_1} + i)\mathcal{D}'$ es $(\overline{B_2} + i)$.
Si $\mathcal{M} = \{ f \ / \ \bar{B}_1 \bar{B}_2 f \ = \ \bar{B}_2 \bar{B}_1 f \}$ se tiene:

$$Ran(\bar{B}_2 + i) \ = \ \overline{(\bar{B}_2 + i)\,(\bar{B}_1 + i)\,\mathcal{D}'} \subset \overline{(\bar{B}_2 + i)\,(\bar{B}_1 + i)\,\mathcal{M}}$$

Del teorema 4.2 se sigue que B_1 y B_2 tienen extensiones autoadjuntas conmutativas y por lo tanto , de los teoremas 3.1 y 5.1 , resulta que k tiene extensiones continuas y de tipo positivo a todo el plano.

Para la unicidad basta recordar que toda extensión a todo el plano de k continua y de tipo positivo es de la forma :

$$g(t) \ = \ \langle \, k \, , \, U(t) \, k \, \rangle$$

siendo $U \ = \ \{ \, U_1(x) \, U_2(y) \ / \ (x, y) \in R^2 \, \}$ un grupo de operadores unitarios que actúan en algún espacio de Hilbert F que contiene a H como subespacio cerrado tal que los grupos unitarios U_1 y U_2 extienden a los S.L.U.I. S_1 y S_2 correspondientes al S.L.B.I. S asociado a k y que en este caso, como B_1 y B_2 son esencialmente autoadjuntos, los grupos U_1 , U_2 deben extender a los grupos $\{ \, e^{ixB_1} \, \}$ y $\{ \, e^{iyB_2} \, \}$ respectivamente y por lo tanto se tiene:

$$g(x, y) \ = \ \langle \, k \, , \, U_1(x) \, U_2(y) \, k \, \rangle \ = \ \langle \, k \, , \, e^{ixB_1} \, e^{iy\bar{B}_2} \, k \, \rangle \quad \forall \, (x, y) \in R^2$$

lo cual muestra que g está determinada.

Q.E.D.

Finalmente damos un teorema de extensión (que no se menciona en [8]) para el caso particular en que el rectángulo I sea la faja $(-\infty, \infty) \times (-b, b)$

Teorema 5.3 *Toda función* $k : (-\infty,\infty) \times (-2b,2b) \longrightarrow C$ *continua y de tipo positivo admite extensión (no necesariamente única) continua y de tipo positivo a todo el plano.*

Demostración:

Sean H el espacio de Hilbert y S el S.L.B.I. asociados a k como en las discuciones previas. En este caso es claro que el S.L.U.I. S_1 es un grupo de operadores unitarios en H que denotaremos U_1 y que es el grupo de traslaciones derechas en H. Cualquiera que sea el elemento $\tilde{h}_{(r_1,r_2)}$ y $\forall\, x \in R$ se tiene:

$$U_1(x)B_2\tilde{h}_{(r_1,r_2)} = U_1(x)\frac{i}{r_1 r_2}\int_0^{r_1}(h_{(\lambda,r_2)}-h_{(\lambda,0)})d\lambda = \frac{i}{r_1 r_2}\int_0^{r_1}(h_{(\lambda+x,r_2)}-h_{(\lambda+x,0)})d\lambda$$

$$B_2U_1(x)\tilde{h}_{(r_1,r_2)} = B_2\frac{1}{r_1 r_2}\int_0^{r_1}\int_0^{r_2} h_{(x+\lambda,\mu)}\,d\lambda\, d\mu = \frac{i}{r_1 r_2}\int_0^{r_1}(h_{(\lambda+x,r_2)}-h_{(\lambda+x,0)})d\lambda$$

Se deduce que $U_1(x)\,B_2\,h = B_2\,U_1(x)\,h \quad \forall\, h \in \mathcal{D}'$ *y* $\forall\, x \in R$. Como $\mathcal{D}'$ es un core para B_2 del corolario 4.1 y del teorema 5.1 se sigue el resultado.

Q.E.D.

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Un Teorema Central del Límite para campos aleatorios con mezcla de parámetro discreto y su aplicación a la comparación de la media de muestras espaciales.

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ABSTRACT.

In this paper we give a very simple generalization of Bolthausen's Central Limit Theorem for stationary mixing random fields. This result is applied to compare the mean of weakly-dependent spatial data.

RESUMEN.

En este trabajo presentamos una generalización muy simple del Teorema Central del Límite de Bolthausen para campos aleatorios estacionarios con mezcla. Este resultado lo aplicamos para comparar la media de datos espaciales débilmente dependientes.

1. Introducción.

En [7] se construye un F-test asintótico para comparar la media de muestras espaciales débilmente dependientes, basado en la aplicación de un Teorema Central del Límite para campos aleatorios estacionarios con mezcla debido a *Goldie & Greenwood* (cf. [5]). El ejemplo fundamental de aplicación de este test es la comparación de dos muestras, una proveniente de un ángulo sólido de $\mathbb{Z}^d$ y la otra de su complemento. Este test permite abordar algunos problemas concretos , pero es más natural, desde el punto de vista estadístico, formular hipótesis más débiles sobre las regiones a comparar. Una hipótesis natural sobre las regiones de $\mathbb{Z}^d$ a comparar es que ambas sean significativamente voluminosas. Una forma de expresar esto es imponer la siguiente condición:

(V) $\lim_N \dfrac{\text{card}\,(\,B \cap [-N,N]\,)}{(2N+1)^d} = b$, $0< b < 1$, donde B es una de las regiones a comparar y $[-N,N] = \{\, n \in \mathbb{Z}^d : \forall\, 1\leq i \leq d,\ -N \leq n(i) \leq N \,\}$ y donde $N \in \mathbb{N}$.

En este trabajo resolvemos una primera etapa de este problema, en la que se hace la suposición adicional que la frontera entre las regiones es poco voluminosa. Esta suposición se expresará en forma concreta a través de la siguiente condición:

(F) $\lim_N \frac{\text{card}\,(\,\partial B \cap [-N,N]\,)}{(2N+1)^d} = 0$ (el significado exacto de ∂A, $A \subset \mathbb{Z}^d$, se precisará más adelante).

Una segunda etapa del estudio de este problema consistiría en suprimir la hipótesis (F). Sin embargo, como observaremos al final de este trabajo, si (F) no es cierta, aparecen complicaciones que le dan al problema características muy diferentes a las que tiene en el contexto en que nos manejaremos.

2. Preliminares.

En lo que sigue consideraremos $\mathbb{Z}^d$, $d \geq 1$, dotado de la métrica d proveniente de la norma : $\|n\| = \text{máx}\{\,|n(i)| : 1 \leq i \leq d\,\}$.

Si $n \in \mathbb{Z}^d$, $A \subset \mathbb{Z}^d$, $B \subset \mathbb{Z}^d$ emplearemos las siguientes notaciones:

$d(n,A) = \text{mín}\{\, d(n,m) : m \in A\,\}$, $d(A,B) = \text{mín}\{\, d(n,B) : n \in A\,\}$,
$\partial A = \{\, n \in A : d(n,A^c) = 1\,\}$.

A continuación presentamos una serie de notaciones, definiciones y resultados que nos serán de gran utilidad.

2.1. (Cf. [3]) Si $(\Omega, \mathcal{A}, P)$ es un espacio de probabilidad y $\mathcal{F}$, $\mathcal{G}$ son dos sub σ- álgebras de $\mathcal{A}$, definimos los coeficientes de α y ρ - mixing entre $\mathcal{F}$ y $\mathcal{G}$ como:

$\alpha(\mathcal{F}, \mathcal{G}) = \sup\{\, |\, P(A \cap B) - P(A)P(B)\,| : A \in \mathcal{F}, B \in \mathcal{G}\,\}$;

$$\rho(\mathcal{F},\mathcal{G}) = \sup\{\,|\,\mathrm{Corr}(X,Y)\,| : X \in L^2(\Omega,\mathcal{F},P),\ Y \in L^2(\Omega,\mathcal{G},P)\,\}.$$

Un cálculo inmediato muestra que $\alpha(\mathcal{F},\mathcal{G}) \leq \frac{1}{4}\,\rho(\mathcal{F},\mathcal{G})$.

Si $\mathbf{X} = \{X_n : n \in \mathbb{Z}^d\}$ es un proceso de valores reales en $L^2(\Omega,\mathcal{A},P)$ y $A \subset \mathbb{Z}^d$, llamaremos $\sigma^{\mathbf{X}}(A)$ a la σ - álgebra generada por $\{X_n : n \in A\}$. Definimos entonces los coeficientes de α y ρ - mixing de $\mathbf{X}$ como:

$$\alpha^{\mathbf{X}}(m) = \sup\{\alpha(\sigma^{\mathbf{X}}(A), \sigma^{\mathbf{X}}(B)) : A \subset \mathbb{Z}^d, B \subset \mathbb{Z}^d, d(A,B) \geq m\},\ m \in \mathbb{N};$$

$$\rho^{\mathbf{X}}(m) = \sup\{\rho(\sigma^{\mathbf{X}}(A), \sigma^{\mathbf{X}}(B)) : A \subset \mathbb{Z}^d, B \subset \mathbb{Z}^d, d(A,B) \geq m\},\ m \in \mathbb{N}.$$

2.2. (Cf. [1], [6]) Si (S,d) es un espacio métrico separable y $\mathcal{B}(S)$ es su σ - álgebra de Borel, llamaremos $\mathcal{P}(S)$ al conjunto de las medidas de probabilidad en $(S, \mathcal{B}(S))$. Diremos que $\{P_N : N \in \mathbb{N}\} \subset \mathcal{P}(S)$ converge débilmente a $P \in \mathcal{P}(S)$ (y escribiremos $w\text{-}\lim_N P_N = P$) si $A \in \mathcal{B}(S)$ tal que $P(\partial A) = 0$ se tiene que $\lim_N P_N(A) = P(A)$.

(i) Esta convergencia corresponde a la topología inducida en $\mathcal{P}(S)$ por la métrica de Prohorov, definida en $\mathcal{P}(S) \times \mathcal{P}(S)$ como:

$d_{\mathcal{P}}(P,Q) = \inf\{\varepsilon > 0 : P(A) \leq Q(A_\varepsilon) + \varepsilon,\ Q(A) \leq P(A_\varepsilon) + \varepsilon,\ \forall A \in \mathcal{B}(S)\}$, donde $A_\varepsilon = \{x \in S : d(x,A) \leq \varepsilon\}$. Una consecuencia importante de este hecho es que w $-\lim_N P_N = P$ sii para toda subsucesión $\{ P_{N_k} : k \in \mathbb{N} \}$ existe una subsucesión $\{P_{N_{k_m}} : m \in \mathbb{N}\}$ tal que w $-\lim_m P_{N_{k_m}} = P$.

(ii) Decimos que $\Pi \subset \mathcal{P}(S)$ es tenso si dado $\varepsilon>0$ existe un compacto $K(\varepsilon)$ tal que $\forall P \in \Pi$, $P(K(\varepsilon)) > 1-\varepsilon$. El teorema de Prohorov afirma que, si S es completo, Π es tenso sii es relativamente compacto en $(\mathcal{P}(S), d_{\mathcal{P}})$, i. e. sii toda sucesión contenida en Π tiene una subsucesión débilmente convergente a algún elemento de $\mathcal{P}(S)$.

(iii) Si $S= S_1 x S_2$, donde S_1 y S_2 son dos espacios métricos separables, S se dota de la topología producto, y P es un elemento de $\mathcal{P}(S)$ llamaremos P^1 y P^2 a las probabilidades marginales de P, que son elementos de $\mathcal{P}(S_1)$ y $\mathcal{P}(S_2)$, respectivamente. Estas probabilidades se definen del modo siguiente:

$$P^1(A)= P(A x S_2),\ A \in \mathcal{B}(S_1)\ ;\ P^2(B)= P(S_1 x B),\ B \in \mathcal{B}(S_2).$$

Es trivial observar que el conjunto $\Pi \subset \mathcal{P}(S)$ es tenso sii sus dos conjuntos de marginales $\Pi^1 \subset \mathcal{P}(S_1)$ y $\Pi^2 \subset \mathcal{P}(S_2)$ son tensos. Es trivial observar que si $\{ P_N : N \in \mathbb{N} \} \subset \mathcal{P}(S)$ y w $-\lim_N P_N = P$ entonces : w $-\lim_N P_N^1 = P^1$ y w $-\lim_N P_N^2 = P^2$. Por otra parte si P^1 y P^2 son elementos de $\mathcal{P}(S_1)$ y

$\mathcal{P}(S_2)$, respectivamente, entonces $P = P^1 \otimes P^2$ es un elemento de $\mathcal{P}(S)$ y si se tiene $\{ P_N^1 : N \in \mathbb{N} \} \subset \mathcal{P}(S_1)$ y $\{ P_N^2 : N \in \mathbb{N} \} \subset \mathcal{P}(S_2)$ tales que: $w\text{-}\lim_N P_N^1 = P^1$ y $w\text{-}\lim_N P_N^2 = P^2$ entonces $w\text{-}\lim_N P_N^1 \otimes P_N^2 = P^1 \otimes P^2$

(iv) Si X es una variable aleatoria de $(\Omega, \mathcal{A}, P)$ en S_1 , con S_1 como antes, llamaremos P^X a la medida distribución de X, que es el elemento de $\mathcal{P}(S_1)$ definido por : $P^X(B) = P(X^{-1}(B))$, $B \in \mathcal{B}(S_1)$. Si además $\{ X_N : N \in \mathbb{N} \}$ es una sucesión de variables aleatorias de $(\Omega, \mathcal{A}, P)$ en S_1 diremos que $w\text{-}\lim_N X_N = X$ sii $w\text{-}\lim_N P^{X_N} = P^X$. Si h es una función cualquiera de S_1 en S_2 , donde S_2 es como antes, el conjunto de sus puntos de discontinuidad, D_h , es un G_δ (i.e. la intersección de una cantidad numerable de conjuntos abiertos) y por lo tanto, un elemento de $\mathcal{B}(S_1)$. Si h es Borel-medible y X como antes, entonces h(X) es una variable aleatoria de $(\Omega, \mathcal{A}, P)$ en S_2 . Si h es Borel-medible y $P^X(D_h)=0$, entonces $w\text{-}\lim_N X_N = X$ implica $w\text{-}\lim_N h(X_N) = h(X)$.

(v) Un resultado muy importante es el teorema de Skorohod, que establece que si $\{ X_N : N \in \mathbb{N} \}$ y X son como en (iv) y se tiene que w $-\lim_N X_N = X$, entonces existen un cierto espacio de probabilidad $(\Omega^*, \mathcal{A}^*, P^*)$ y variables aleatorias de $(\Omega^*, \mathcal{A}^*, P^*)$ en S_1 , $\{ X_N^* : N \in \mathbb{N} \}$ y X^* tales que: $P^{*X_N^*} = P^{X_N}$; $P^{*X^*} = P^X$; P^*-cs $\lim_N X_N^* = X^*$. Es conveniente destacar que los momentos de una variable aleatoria de valores reales sólo dependen de sus distribuciones , por lo que en el caso en que $S_1 = \mathbb{R}$ los momentos de las variables aleatorias X^* y X, X_N^* y X_N serán iguales.

En muchas de las observaciones anteriores las hipótesis en que fueron enunciadas pueden ser debilitadas, pero resultan ampliamente apropiadas para el contexto en que trabajaremos.

A continuación haremos algunas puntualizaciones sobre la temática anterior en el caso específico en que $S_1 = \mathbb{R}^k$.

(vi) Diremos que $\{ X_N : N \in \mathbb{N} \}$ es uniformemente integrable si $\sup\{ E(\| X_N \| \mathbf{I}_{\{\| X_N \| \geq t\}}) : N \in \mathbb{N} \}$ tiende a cero cuando t tiende a infinito ($\mathbf{I}_A$ es la indicatriz del conjunto A). Si $\{ X_N : N \in \mathbb{N} \}$ converge casi seguramente, la integrabilidad uniforme es condición necesaria y suficiente para la convergencia en L^1. Una condición suficiente para que $\{ X_N : N \in \mathbb{N} \}$ sea uniformemente integrable es $\sup\{ E(\| X_N \|^p) : N \in \mathbb{N} \} < \infty$,

para algún $p > 1$. La integrabilidad uniforme es una condición suficiente para que la sucesión $\{ X_N : N \in \mathbb{N} \}$ sea tensa, i. e., para que la sucesión de sus distribuciones sea tensa.

(vii) Un resultado de utilidad es que si $\{ Y_N : N \in \mathbb{N} \}$ tiende en L^2 al vector nulo y $\{ X_N : N \in \mathbb{N} \}$ está definida en el mismo espacio de probabilidad entonces $\lim_N d_{\mathcal{P}} (P^{X_N}, P^{X_N + Y_N}) = 0$. Por otro lado $w\text{-}\lim_N X_N =$ X sii para cualquier $t \in D^c_{F_X}$ $\lim_N F_{X_N}(t) = F_X(t)$, donde F_X es la función de distribución de X, i.e., $F_X(t) = P^X((-\infty, t])$.

2.3. (Cf. [4]) Es conveniente recordar un resultado muy importante, conocido como Teorema de Lévy - Cramer que establece que si X e Y son dos variables aleatorias independientes tales que X+Y es gaussiana, entonces X e Y son gaussianas.

Por último una observación importante es que a lo largo de este trabajo cuando digamos que un proceso es estacionario entenderemos que es estrictamente estacionario, i.e., que las distribuciones finito - dimensionales del proceso son invariantes por traslaciones.

3. El teorema central del Límite.

Antes de pasar a la demostración de nuestro teorema, probaremos algunos lemas.

3.1. Lema: Sean $\{ X_N : N \in \mathbb{N} \}$ y X variables aleatorias reales tales que:

a) $w\text{-}\lim_N X_N = X$

b) $E(X_N)=0$, $N \in \mathbb{N}$

c) $\lim_N Var(X_N) = V$

Entonces: $E(X)=0$, $Var(X) \leq V$.

Demostración:

Sean $(\Omega^*, \mathcal{A}^*, P^*)$, $\{ X^*_N : N \in \mathbb{N} \}$ y X^* como en 1.2 (v). Por lo allí observado tenemos que $E^*({X^*_N}^2) = E(X_N^2)$, $N \in \mathbb{N}$ y por lo tanto $\sup\{E(X_N^2) : N \in \mathbb{N}\} < \infty$; por lo visto en 1.2. (vi) $\{ X^*_N : N \in \mathbb{N} \}$ es entonces uniformemente integrable y por lo tanto $E^*(X^*) = \lim_N E^*(X^*_N)=0$, de donde resulta que $E(X)=0$. Entonces aplicando el Lema de Fatou se deduce que $Var(X) = E(X^2) = E^*(X^{*2}) \leq V$ ◆

3.2. Lema: Sea $X=\{X_n : n \in Z^d\}$ un proceso de valores reales tal que:

a) $E(X_n)=0$, $n \in Z^d$

b) X es estacionario

c) $0<E(X_n^2)=r(0)<\infty$, $n \in Z^d$.

d) $\sum_{m=0}^{\infty} m^d \rho^X(m) < \infty$.

Si $A \subset Z^d$ y notamos $A_N = A \cap [-N,N]$, $S_N(A)=\sum_{n \in A_N} \frac{X_n}{\sqrt{(2N+1)^d}}$,

entonces se tiene:

(i) $\sum_{n \in Z^d} |r(n)| < \infty$, donde $r(n)=E(X_0 X_n)$, $n \in Z^d$.

(ii) $E(S_N(A)^2) \leq \frac{card(A_N)}{(2N+1)^d} \sum_{n \in A_N - A_N} |r(n)|$,

con $A_N - A_N = \{n \in Z^d : n=m-k, m \in A_N, k \in A_N\}$.

(iii) $E(S_N(A)^2) \leq \sum_{n \in Z^d} |r(n)|$.

(iv) $| E(S_N(A)^2) - \frac{card(A_N)}{(2N+1)^d} \sum_{n \in A_N - A_N} r(n) | \leq$

$$\frac{\operatorname{card}(\partial A_N)}{(2N+1)^d} \quad C \sum_{m=0}^{\infty} m^d \rho^{X}(m) \text{ , con C constante.}$$

Demostración:

Por definición, $|E(X_0 X_n)| \leq r(0)\,\rho^{X}(\|n\|)$, $n \in Z^d$ y

entonces $\sum_{n \in Z^d} |r(n)| \leq r(0) \sum_{n \in Z^d} \rho^{X}(\|n\|) = r(0) \sum_{m=0}^{\infty} \phi(m)\, \rho^{X}(m)$,

donde $\phi(m) = \operatorname{card}\{n \in Z^d : \|n\| = m\} = (m+1)^d - m^d \leq C\, m^{d-1}$, por lo que

resulta que $\sum_{n \in Z^d} |r(n)| \leq C \sum_{m=0}^{\infty} m^{d-1} \rho^{X}(m) < \infty$ y queda probado (i).

Por otra parte, un cálculo inmediato muestra que :

$$E(S_N(A)^2) = \frac{1}{(2N+1)^d} \sum_{m \in A_N} \sum_{k \in A_N} E(X_m X_k) =$$

$$\frac{1}{(2N+1)^d} \sum_{m \in A_N} \sum_{k \in A_N} r(m-k) = \frac{1}{(2N+1)^d} \sum_{n \in A_N - A_N} c(n)\, r(n) \text{ ,}$$

donde $c(n) = \operatorname{card}\{(m,k) \in A_N \times A_N : m-k=n\} = \sum_{m \in A_N} c(n,m)$, con $c(n,m) = 1$

si $m-n \in A_N$ y $c(n,m)=0$ en otro caso. Entonces:

(1) $c(n) \leq \operatorname{card}(A_N)\ \forall n \in A_N$.

Por otra parte sea $m \in A_N$ tal que $c(n,m)=0$; entonces $m-n \notin A_N$.

Consideremos los $g(n)= \sum_{1\le i\le d} |n(i)|+1$ puntos de coordenadas enteras comprendidos en la poligonal coordenada $C(m,n)$ que une a m y m-n ; es obvio que entre ellos debe haber al menos un punto de ∂A_N. A su vez si tenemos $p \in \partial A_N$, existen a lo sumo $g(n)$ posibles $m \in A_N$ tales que $m-n \notin A_N$ y $p \in C(m,n)$; de aquí resulta que $card\{m \in A_N$ tal que $c(n,m)=0\} \le g(n)$ $card(\partial A_N) \le (d\|n\|+1)card(\partial A_N)$. Entonces se tiene :

$$(2)\ c(n) \ge card(A_N) - (d\|n\|+1)card(\partial A_N).$$

Por lo tanto $E(S_N(A)^2) \le \frac{1}{(2N+1)^d} \sum_{n\in A_N - A_N} c(n)|r(n)| \le$

$\frac{card(A_N)}{(2N+1)^d} \sum_{n\in A_N - A_N} |r(n)|$ (por (1)) y (ii) queda demostrado.

La demostración de (iii) es inmediata a partir de (ii).

Por último tenemos que:

$$\left| E(S_N(A)^2) - \frac{card(A_N)}{(2N+1)^d} \sum_{n\in A_N - A_N} r(n) \right| \le$$

$$\sum_{n\in A_N - A_N} |r(n)| \left|c(n) - \frac{card(A_N)}{(2N+1)^d}\right| \le \text{(por (1) y (2))}\ \frac{card(\partial A_N)}{(2N+1)^d}$$

$\Sigma_{n \in A_{\bar{N}} - A_N} [|r(n)| (d \|n\| + 1)] \leq \dfrac{\mathrm{card}(\partial A_N)}{(2N+1)^d} \; C \sum_{m=0}^{\infty} m^d \rho^{X}(m)$ (por

cálculos análogos a los empleados para probar (i)) ◆

3.3. Observaciones: (i) Cuando escribimos $\Sigma_{n \in \mathbf{Z}^d} c(n) = C$, entendemos que para toda sucesión de conjuntos $\{ A_N : N \in \mathbb{N} \}$ que crece a $\mathbf{Z}^d$ se tiene que $\lim_N \Sigma_{m \in A_N} c(m) = C$.

(ii) Nótese que para la parte (i) del Lema anterior era suficiente que $\sum_{m=0}^{\infty} m^{d-1} \rho^{X}(m) < \infty$, lo que se pone en evidencia en el próximo teorema.

El último resultado previo que necesitaremos es el Teorema central del Límite debido a *Bolthausen* (cf.[2]).

3.4. Teorema: Sea $\mathbf{X} = \{ X_n : n \in \mathbf{Z}^d \}$ un proceso de valores reales tal que:

a) $E(X_n) = 0$, $n \in \mathbf{Z}^d$

b) $\mathbf{X}$ es estacionario

c) $0 < E(X_n^2) = r(0) < \infty$, $n \in \mathbf{Z}^d$.

d) $\sum_{m=0}^{\infty} m^{d-1} \rho^{X}(m) < \infty$.

Sea además $\{ A_N : N \in \mathbb{N} \}$ una sucesión de conjuntos que crece a $\mathbb{Z}^d$ tal que:

$$(P) \quad \lim_N \frac{\mathrm{card}(\partial A_N)}{\mathrm{card}(A_N)} = 0$$

Entonces $\sum_{n \in \mathbb{Z}^d} |r(n)| < \infty$, y si $\sum_{n \in \mathbb{Z}^d} r(n) = \sigma^2 > 0$, entonces:

$Z(A_N) = \sum_{n \in A_N} \frac{X_n}{\sqrt{\mathrm{card}(A_N)}}$ converge débilmente a una distribución gaussiana centrada y de varianza σ^2.

A continuación presentamos nuestro teorema.

3.5. Teorema: Sea $\mathbf{X} = \{ X_n : n \in \mathbb{Z}^d \}$ un proceso de valores reales tal que:

a) $E(X_n) = 0$, $n \in \mathbb{Z}^d$

b) $\mathbf{X}$ es estacionario

c) $0 < E(X_n^2) = r(0) < \infty$, $n \in \mathbb{Z}^d$.

d) $\sum_{m=0}^{\infty} m^{d} \rho^{X}(m) < \infty$.

e) $\sum_{n\in Z^d} r(n) = \sigma^2 > 0$

Si además $B \subset Z^d$ cumple con las dos condiciones siguientes:

(V) $\lim_N \dfrac{\text{card}(B \cap [-N,N])}{(2N+1)^d} = b,\ 0 < b < 1.$

(F) $\lim_N \dfrac{\text{card}(\partial B \cap [-N,N])}{(2N+1)^d} = 0$

Entonces el vector $(S_N(B), S_N(B^c))$ converge débilmente a una distribución gaussiana centrada y con matriz de covarianzas: $C = \begin{bmatrix} b\sigma^2 & 0 \\ 0 & (1-b)\sigma^2 \end{bmatrix}$

Demostración:

Por el Lema 3.2. (i), $\sum_{n\in Z^d} |r(n)| < \infty$ y por lo tanto, tiene sentido la hipótesis (e).

Sea $\eta_N = P(S_N(B), S_N(B^c))$; por lo observado en 2.2. (i), bastará ver que para toda subsucesión $\{\eta_{N_k} : k \in \mathbb{N}\}$ existe una subsucesión $\{\eta_{N_{k_m}} : m \in \mathbb{N}\}$ tal que $w\text{-}\lim_m \eta_{N_{k_m}}$ es una probabilidad gaussiana centrada y con matriz de covarianzas **C**.

Por el Lema 3.2. (i) y (iii) se tiene: $\sup\{ E(S_N(A)^2) : N \in \mathbb{N} \} < \infty$, $A \subset \mathbb{Z}^d$ y por lo tanto, por lo observado en 2.2. (vi) y (iii) la sucesión $\{ \eta_{N_k} : k \in \mathbb{N} \}$ es tensa y por lo tanto existe una subsucesión $\{\eta_{N_{k_m}} : m \in \mathbb{N} \}$ tal que $w\text{-}\lim_m \eta_{N_{k_m}} = \eta$, con η una probabilidad en el plano. Para facilitar la escritura nos permitiremos el abuso de notación de llamar $\{ \eta_N : N \in \mathbb{N} \}$ a la subsucesión $\{\eta_{N_{k_m}} : m \in \mathbb{N} \}$. Probaremos que η es gaussiana centrada y con matriz de covarianzas **C**.

Tenemos que:

(1) $E(S_N(A)) = 0$, $N \in \mathbb{N}$.

Por otra parte y por el Lema 3.2. (iv):

$$| E((S_N(B)^2)) - \frac{\text{card}(B_N)}{(2N+1)^d} \sum_{n \in B_N - B_N} r(n) | \leq \frac{\text{card}(\partial B_N)}{(2N+1)^d} C \sum_{m=0}^{\infty} m^d \rho^X(m),$$

que tiende a cero por la condición (F) y por la hipótesis (d). Por otra parte, por la condición (V) $\lim_N \frac{\text{card}(B_N)}{(2N+1)^d} = b$.

Además $B_N - B_N$ crece a $\mathbb{Z}^d$: que crece es trivial y si existiera $m \in \mathbb{Z}^d$ tal que $m \notin B_N - B_N \ \forall N \in \mathbb{N}$ entonces $\forall n \in B_N$, $m+n \notin B_N$. Razonando en forma análoga a la demostración del Lema 3.2. puede verse entonces que en la poligonal coordenada que une a n con m+n debe haber al menos un punto de ∂B_N y que por lo tanto se deduce que: $card(B_N) \leq (d\|m\|+1)$. $(card(\partial B_N))$, lo que contradice la suposición de las condiciones (V) y (F).

Por lo tanto, y por definición, $\lim_N \sum_{n \in B_N - B_N} r(n) = \sigma^2$ y resulta que :

(2) $\lim_N E(S_N(B)^2) = b\sigma^2$.

Sea entonces (X,Y) vector aleatorio en el plano tal que su distribución sea η. Entonces por lo observado en 2.2. (iii), la variable X y la sucesión $\{S_N(B): N \in \mathbb{N}\}$ cumplen las hipótesis del Lema 3.1. y por lo tanto se deduce que:

(3) $E(X)=0$, $Var(X) \leq b\sigma^2$.

Análogamente se deduce que:

(4) $E(Y)=0$, $Var(Y) \leq (1-b)\sigma^2$.

Por otra parte y en virtud de 2.2. (iv), $S_N(B) + S_N(B^c)$ converge débilmente a X+Y. Pero $S_N(B) + S_N(B^c) = Z([-N,N])$ y como [-N,N] cumple la condición (P) de 3.4., se deduce que Z([-N,N]) converge débilmente a una distribución gaussiana centrada y de varianza σ^2 y por lo tanto que :

(5) X+Y es gaussiana centrada y de varianza σ^2.

Consideremos ahora $\{q_N : N \in \mathbb{N}\}$ sucesión que tiende a infinito de forma tal que $\frac{card(\partial B_N) q_N^d}{(2N+1)^d}$ tienda a cero, lo cual es posible por (F). Como se observó en 2.1. $\alpha^X(q_N) \leq \frac{1}{4} \rho^X(q_N)$ que tiende a cero por (d), por lo que $\alpha^X(q_N)$ tiende a cero.

Sea $B^N = \{n \in B_N : d(n, B_N^c) \geq q_N\}$. Sea $D_N(B) = S_N(B) - S_N(B^N)$. Si $n \in B_N$ y $n \notin B^N$ entonces existe $m \in \partial B_N$ tal que $d(n,m) \leq q_N - 1$, por lo que se deduce fácilmente que $card\{n \in B_N : n \notin B^N\} \leq card(\partial B_N)(2q_N - 1)^d$. Como $D_N(B) = S_N[B \cap (B^N)^c]$, por el Lema 3.2. (ii) podemos deducir fácilmente que :

$$E(D_N(B)^2) \leq \left(\sum_{n \in Z^d} |r(n)|\right) \frac{card(\partial B_N)(2q_N - 1)^d}{(2N+1)^d}$$, que tiende a cero.

Por lo tanto, por lo observado en 2.2. (vii), $(S_N(B^N), S_N(B^c))$ converge débilmente a (X,Y).

Sean $x \in D^c_{F_X}$, $y \in D^c_{F_Y}$, entonces es claro que $(x,y) \in D^c_{F_{(X,Y)}}$. Entonces, por lo visto en 2.2. (vii), $\lim_N P(S_N(B^N), S_N(B^c))((-\infty, (x,y)]) = F_{(X,Y)}(x,y)$. Pero por definición:

$|P(S_N(B^N), S_N(B^c))((-\infty, (x,y)]) - P^{S_N(B^N)}((-\infty, x])P^{S_N(B^c)}((-\infty, y])| \leq \alpha^X(q_N)$, por lo que $\lim_N P^{S_N(B^N)}((-\infty, x])P^{S_N(B^c)}((-\infty, y]) = F_{(X,Y)}(x,y)$. Pero por otro lado $\lim_N P^{S_N(B^N)}((-\infty, x,])P^{S_N(B^c)}((-\infty, y,]) = F_X(x)F_Y(y)$, de donde se deduce que si $x \in D^c_{F_X}$, $y \in D^c_{F_Y}$, $F_X(x)F_Y(y) = F_{(X,Y)}(x,y)$. Usando las propiedades características de las funciones de distribución se deduce fácilmente que : $\forall (x,y)\ F_X(x)F_Y(y) = F_{(X,Y)}(x,y)$, lo que implica que:

(6) X e Y son independientes.

De (6) se deduce que $\sigma^2 = Var(X+Y) = Var(X) + Var(Y)$. Teniendo en cuenta (3) y (4) resulta que:

(7) $Var(X) = b\sigma^2$, $Var(Y) = (1-b)\sigma^2$

Además de (5), (6) y 2.3. resulta que:

(8) X e Y son gaussianas.

De(3), (4), (6), (7) y (8) resulta que η es gaussiana centrada y con matriz de covarianzas **C**. ◆

3.6. Observaciones: (i) Este resultado se extiende sin ninguna dificultad al caso en que $\mathbb{Z}^d$ se divide en k regiones que cumplen (V) y (F).

(ii) Dada una región B que cumple con (V) y (F) es muy fácil observar que se puede dividir en dos regiones que cumplen (V) y (F).

4. Comparación de medias de muestras espaciales.

A continuación presentamos una prueba de hipótesis asintótica para comparar la media de dos muestras espaciales débilmente dependientes.

4.1. Teorema: Sea $X=\{X_n : n \in \mathbb{Z}^d\}$ un proceso de valores reales tal que:

a) $\{X_n - E(X_n) : n \in \mathbb{Z}^d\}$ es estacionario

b) $0<Var(X_n)=r(0)<\infty$, $n \in \mathbb{Z}^d$.

c) $\sum_{m=0}^{\infty} m^d \rho^X(m) < \infty$.

d) $\sum_{n \in \mathbb{Z}^d} r(n) = \sigma^2 > 0$, con $r(n)= Cov(X_0, X_n)$, $n \in \mathbb{Z}^d$.

Sea además $B \subset \mathbb{Z}^d$ que cumple con (V) y (F).

Sea B^1, B^2 una partición de B tal que B^1 y B^2 cumplen con (V) y (F).

Sean: $b_N = card(B_N)$, $b_N^c = card(B_N^c)$, $b_N^1 = card(B_N^1)$, $b_N^2 = card(B_N^2)$;

$$E_N = (2N+1)^d \left(\frac{S_N(B^1)}{b_N^1} - \frac{S_N(B^2)}{b_N^2} \right) \sqrt{\left(\frac{1}{b_N^1} + \frac{1}{b_N^2} \right)\left(\frac{1}{b_N} + \frac{1}{b_N^c} \right)} \; ;$$

$$T_N = (2N+1)^d \left(\frac{S_N(B)}{b_N} - \frac{S_N(B^c)}{b_N^c} \right) \; ;$$ entonces si para la prueba de hipótesis:

H_0: existe $\mu \in \mathbb{R}$ tal que $E(X_n)=\mu$, $n \in \mathbb{Z}^d$

H_1: existen $\mu \in \mathbb{R}$ y $\theta \neq 0$ tal que $E(X_n)=\mu$, $n \in B$ y $E(X_n)=\mu+\theta$, $n \in B^c$;

se considera la región crítica:

$$\mathcal{R}_N = \left\{ \frac{(T_N)^2}{(E_N)^2} \geq F_{1-\alpha}(1,1) \right\},$$ y si α_N , β_N son las probabilidades de error de tipo 1 y 2, respectivamente, entonces: $\lim_N \alpha_N = \alpha$, $\lim_N \beta_N = 0$.

Demostración:

Sea $\delta=0$ bajo H_0 y $\delta=\theta$ bajo H_1.

Por 3.5. y 3.6. (i), el vector tridimensional:

$$\left(\frac{(2N+1)^d}{b_N^1} S_N(B^1) - \mu\sqrt{(2N+1)^d},\ \frac{(2N+1)^d}{b_N^2} S_N(B^2) - \mu\sqrt{(2N+1)^d},\ \frac{(2N+1)^d}{b_N^c} S_N(B^c) - (\mu+\delta)\sqrt{(2N+1)^d} \right)$$

converge a una normal centrada con matriz de covarianzas:

$$\begin{pmatrix} \frac{\sigma^2}{b_1} & 0 & 0 \\ 0 & \frac{\sigma^2}{b_2} & 0 \\ 0 & 0 & \frac{\sigma^2}{1-b} \end{pmatrix}$$

donde b_1, b_2, b y 1-b son los límites que aparecen en (V) para B^1, B^2, B y B^c, respectivamente. De allí, aplicando 2.2. (iv) y cálculos triviales se deduce que el vector bidimensional:

(T_N, E_N) converge débilmente a una normal centrada centrada con matriz de covarianzas $\sigma^2(\frac{1}{b} + \frac{1}{1-b})I_{2x2}$, donde I_{2x2} es la matriz identidad de 2x2. Aplicando nuevamente 2.2. (iv) resulta que la distribución de $\frac{(T_N)^2}{(E_N)^2}$ se aproxima a una F de Fischer con excentricidad:

$$e_N(\delta) = \frac{|\delta|\sqrt{(2N+1)^d(1-b)b}}{\sigma}$$

de allí resulta inmediatamente que :

$$\lim_N \alpha_N = \alpha,\ \lim_N \beta_N = 0 \ ◆$$

4.2. Observación:

A los efectos estadísticos, para construír pruebas de hipótesis similares al de 4.1. es necesario que el vector $(S_N(B), S_N(B^c))$ tienda débilmente a una normal no - degenerada. Suprimiendo la hipótesis (F), no cabe esperar que este resultado sea cierto en general. En efecto, tomemos un proceso gaussiano real centrado $\mathbf{X} = \{X_n : n \in \mathbf{Z}\}$ tal que: $\mathbf{X}$ es estacionario, $0 < E(X_n^2) = r(0) < \infty$, $n \in \mathbf{Z}$, $\sum_{m=0}^{\infty} m\rho^{\mathbf{X}}(m) < \infty$ y $\sum_{n \in \mathbf{Z}} r(n) = \sigma^2 > 0$, $\sum_{n \in 2\mathbf{Z}} r(n) = \sum_{n \in 2\mathbf{Z}+1} r(n)$; es muy fácil verificar que se puede construír un tal proceso (más aún, se puede tomar m-dependiente). Empleando nuevamente argumentos como los empleados en 3.1., 3.5. y propiedades básicas de las variables gaussianas, es fácil ver que si se toma $B = 2\mathbf{Z}$, entonces $(S_N(B), S_N(B^c))$ tiende débilmente a (X,X), con X variable normal centrada con varianza σ^2, es decir que el límite es degenerado.

Se puede plantear la pregunta si existe alguna condición más débil que (F) que permita obtener límites gaussianos no - degenerados y consecuentemente construír pruebas de hipótesis similares a 4.1.

Pero más allá de la aplicación estadística pueden plantearse preguntas directamente concernientes al Teorema Central del Límite, como en qué condiciones sobre la frontera se puede garantizar que el límite es gaussiano o que existe el límite débil de $(S_N(B),S_N(B^c))$.

Algunas respuestas primarias a estas preguntas pueden hallarse en [8].

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THREE DIMENSIONAL EXPANSIVE HOMEOMORPHISMS

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Abstract Let f be an expansive homeomorphism of a compact connected three dimensional manifold $\mathbf{M}$. We prove that if the set of topological hyperbolic periodic points of f is dense in $\mathbf{M}$, there exists a local product structure defined on an open invariant dense subset of $\mathbf{M}$

0. Introduction

In this preprint we study dynamical properties of expansive homeomorphisms f of three dimensional smooth connected compact riemannian manifolds. Let $f:\mathbf{M}\to\mathbf{M}$ be a homeomorphism and let d be the distance induced by the riemannian structure; we say that f is expansive if there exist a constant $\alpha>0$ such that $x,y\in\mathbf{M}$, $d(f^n(x),f^n(y))\leq\alpha$, for all $n\in\mathbf{Z}$ implies $x=y$.

This paper is intended to be a contribution to the classification under topological conjugacy (see **1.6**) of all expansive homeomorphisms. We also want to study under which conditions a three-manifold supports an expansive homeomorphism.

We proceed to give a brief description of what is already known on lower dimensions.

Dimension one. It is well known that in the circle S^1 there are no expansive homeomorphisms. An easy proof of this result follows from lemma 2.1 (pp. 117-118) of [8].

Dimension two. For compact orientable surfaces a classification of all expansive homeomorphisms has been obtained independently by J. Lewowicz, [8], and K. Hiraide, [3].

In both papers it is shown that there are no expansive homeomorphisms on the sphere S^2, that any such homeomorphism on the torus T^2 is conjugate to an Anosov diffeomorphism, and that on compact oriented surfaces of genus larger than one, any expansive homeomorphism is topologically equivalent to a pseudo-Anosov map.

Expansivity means, from the topological viewpoint that every point of M has a distinctive dynamical behaviour. Hence, a strong interaction between the dynamics and the topology of M should be expected. Example of this interaction is given, for instance, by the fact that S^2 does not admit expansive homeomorphisms.

In dimension three or more, unless aditional restrictions are imposed, no similar classification is known, even in the case of analitic maps. Moreover, expansivity is a property which may be defined totally in terms of continuity, and it seems that there should be no direct relation between the degree of differentiability imposed to f and the solution to the classification problem.

To obtain the above mentioned classification in dimension two, a crucial role is played (see [8]) by the following nice topological property of $\mathbb{R}^2$: given two non empty compact connected sets A and B, such that $A \cap B$ is not connected then their union $A \cup B$ separates the plane $\mathbb{R}^2$. For instance, such property is in the basis of the proof of the local connectedness of stable sets.

The above mentioned property of $\mathbb{R}^2$ is not true in $\mathbb{R}^n$ with $n>2$. In order to assure separation properties of local stable (or unstable) sets we assume:

HY: <u>The set $Per_H(f)$ (1.3) of periodic topologically hyperbolic points, is dense in M.</u> (For a definition of a periodic topologically hyperbolic point see **1.8**)

About the above hypothesis, we merely say the following. It holds in dimension two, if f is an expansive map. In dimension $n>2$, it can be proved that there exists a residual set $B \subset Diff^1(M)$ such that for every $f \in B$ the periodic points are hyperbolic and they are dense in the non wandering set $\Omega(f)$. (Pugh [11]).

Our results also hold in a more generall context, for instance we could replace periodic points by recurrent points. However we prefer to stay in the above context where the techniques are simpler.

Our main result is the following

5.20 ***Proposition:*** Under the hypothesis that Per_H is a dense set in **M**, there exists an open invariant dense set $A \subset M$ such that there is a local product structure (see **5.19**) in A, all points of A being (topologically) hyperbolic .

We think that the set A can be taken to be M. If so the homeomorphism should be conjugated to an Anosov map and M should be taken to be the 3-torus (see [2], Corollary (6.4)).

We remark here that this conjecture is not true in higher dimensions. It is false even in dimension two (pseudo-Anosov maps are counterexamples). So we believe that dimension three is rather restrictive in this connection.

Although we have obtained some results about M\A, which seem to lead to an affirmative answer to the previous conjecture, in this paper we merely prove proposition **5.20**.

What follows next is a brief sketch of the proof.

Let f be an expansive diffeomorphism.

1-In dimension three expansivity implies for a hyperbolic point that the stable manifold has dimension two or one and the unstable manifold dimension one or two respectively. If p is periodic and hyperbolic, then, assuming dimension two for the local stable manifold and taking an arc contained in it with one of it end-points the point p, it can not occur that this arc always remains in the interior of a small ball (say of diameter less than the constant of expansivity) under negative iterations of f. This implies that there exists $r>0$ independent of the point p such that for every $p \in Per_H(f)$ the local stable manifold separates the ball $B(p,r)$, and the unstable manifold has points in different components.

2- If we assume that $Per_H(f)$ is dense in M, the dimensions of the local stable manifolds of different points $p,p' \in Per_H(f)$ are equal.

3- From 1- and 2- it follows that if p' and p'' are near enough to p, the local stable manifold of p' will intersect the local unstable manifold of p'' at a single point. For x in a suitable neighbourhood V of p in its local stable manifold, we construct a compact connected set $C(x)$ contained in the local unstable set of x. Analogously we obtain, for y in a suitable neighbourhood V of p in its local unstable manifold, a set $D(y)$ contained in the local stable set of y.

4- We prove that $C(x)$ intersects $D(y)$ at a single point z which varies continuously with x and y. The map that sends $(x,y) \mapsto z$ is injective due to the expansivity of f.

5- Using the Theorem of Invariance of Domain, the previous result allows us to conclude that there exists a local product structure in an open dense set A, invariant under f.

1. Preliminary Definitions

1.1 ***Definition*** An homeomorphism $f:M \to M$ is said to be expansive with a constant of expansivity α, if for $x,y \in M$, $d(f^n(x),f^n(y)) \leq \alpha$, for all $n \in \mathbb{Z}$ then $x=y$.

1.2 ***Definition*** A point $p \in M$ is a periodic point iff $\exists m \in \mathbb{N}^+$: $f^m(p)=p$. The number m is said to be the period of p if it is the least positive number with this property.

If $f:M\to M$ is a diffeomorphism we shall denote the tangent map of f by $Tf:TM\to TM$.

Let $p\in M$ be a periodic point of period k. We say that p is a hyperbolic point if the tangent map $Tf^k(p): TM_p\to TM_p$ has no eigenvalues of absolute value one.

$W^s(x)=W^s(x,f)=\{y\in M/ \lim_{n\to+\infty} d(f^n(x),f^n(y))=0\}$ is called the stable set of p with respect to f.

$W^u(x)=W^u(x,f)=\{y\in M/ \lim_{n\to-\infty} d(f^n(x),f^n(y))=0\}$ is called the unstable set of p with respect to f.

1.3 ***Definition*** $W^s_\varepsilon(x)=W^s_\varepsilon(x,f)=\{y\in M/\forall n\geq 0: d(f^n(x),f^n(y))\leq\varepsilon\}$ is called the local ε-stable set of p with respect to f.

$W^u_\varepsilon(x)=W^u_\varepsilon(x,f)=\{y\in M/\forall n\leq 0: d(f^n(x),f^n(y))\leq\varepsilon\}$ is called the local ε-unstable set of p with respect to f.

1.4 ***Lemma*** If $\varepsilon\leq\alpha$, where α is a constant of expansivity of f, then $W^s_\varepsilon(x)\subset W^s(x)$ and $W^u_\varepsilon(x)\subset W^u(x)$.

Proof: Assume this is not the case. Then there exists $y\in W^s_\varepsilon(x)$ such that $\exists\delta:\forall n'\in\mathbb{N}:\exists n\in\mathbb{N}:(n\geq n')$ and $(d(f^n(x),f^n(y))\geq\delta)$, then there exists a convergent sequence $\{n_k\}_{k\in\mathbb{N}}$ such that $\lim_{k\to+\infty} f^{n_k}(y)=z$ and $\lim_{k\to+\infty} f^{n_k}(x)=w$; $z\neq w$ and $\forall n\in\mathbb{Z}: d(f^n(x),f^n(y))\leq\varepsilon\leq\alpha$ which is absurd.□ It is well known that if f is a diffeomorphism and p is a periodic hyperbolic point of period m then $W^s(p)$ and $W^u(p)$ are manifolds injectively immersed in

M, of the same degree of differentiability of f, $W^s(p)$ is tangent to $V_p \subset TM_p$, and $W^u(x)$ is tangent to $W_p \subset TM_p$ where V_p and W_p are linear subspaces of TM_p such that $TM_p = V_p \oplus W_p$; the tangent map of f^m, Tf^m, restricted to V_p has eigenvalues of absolute value less than one while the eigenvalues of Tf^m restricted to W_p have absolute value larger than one. It is also known that for $\varepsilon < \alpha$ suficiently small $W^s_\varepsilon(x)$ and $W^u_\varepsilon(x)$ are imbedded submanifolds of M such that $\dim(W^s_\varepsilon(x)) = \dim(W^s(p)) = \dim(V_p) = r$ and $W^s_\varepsilon(x)$ is diffeomorphic to D^r the canonical r-dimensional disk; analogous considerations also hold for $W^u_\varepsilon(x)$.

1.5 *Definition* A point $x \in M$ is said to be stable for $f: M \to M$, if $\forall \varepsilon > 0 : \exists \delta > 0 : \forall y \in M : (d(x,y) \le \delta \Rightarrow \forall n \ge 0 : d(f^n(x), f^n(y)) \le \varepsilon)$.

1.6 *Lemma* An expansive homeomorphism $f: M \to M$ admits no stable points.

Proof: See [7]. □

If x is a periodic hyperbolic point for an expansive diffeomorphism and $\dim(M) = 3$ we have only two possibilities: $\dim(W^s_\varepsilon(x)) = 2$ (which implies that $\dim(W^u_\varepsilon(x)) = 1$) or $\dim(W^s_\varepsilon(x)) = 1$ (which implies that $\dim(W^u_\varepsilon(x)) = 2$).

1.7 *Definition:* Let M,N be topological manifolds. Let $f: M \to M$; $g: N \to N$ be homeomorphisms. We say that f and g are topologically conjugate if there exists a homeomorphism $h:M\to N$ such that $hf=gh$.

It is easily seen that the property of being expansive is preserved by topological conjugacy, and that the stable (unstable) set of f at a point x, $W^s_\varepsilon(x,f)$, ($W^u_\varepsilon(x,f)$) coincides with $h^{-1}W^s_\varepsilon(h(x),g)$ (respectively $h^{-1}W^u_\varepsilon(h(x),g)$).

If f is a homeomorphism we define (topological) hyperbolicity for periodic points of f as follows:

1.8 *Definition:* If $p\in M$ is a periodic point with period m, we say that p is topologically hyperbolic if there is a local conjugacy between f^m and a linear hyperbolic map $T:TM_p\to TM_p$.

Let us call $Per_H(f)=Per_H=\{x\in M$ / x is periodic and topologically hyperbolic for $f:M\to M\}$. In the next section we prove some properties of the stable and unstable sets of $x\in Per_H$.

We only remark here that under our hypothesis, for every $x\in Per_H$, there exists a local stable topological manifold which is locally topologically conjugated to the subspace E of TM_p associated to the eigenvalues of T with absolute value smaller than 1, and a local unstable topological manifold which is locally topologically conjugated to the subspace I of TM_p associated to the eigenvalues of T with absolute value greater than 1 and that $TM_p=E\oplus I$, this implies that $W^s_\varepsilon(p,f)$ contains a topological disk of the same dimension of E.

2. Separation Properties

2.1 Let f be a homeomorphism $f:M\to M$ of the compact connected riemannian manifold M, and let $\hat{M}=\hat{M}_f$, with the flow φ, the suspension of (M,f) under the constant function 1; we assume also that $\hat{M}$ is endowed with some riemannian metric. We may identify M with $\pi(M\times\{0\})$, π being the suspension projection of $M\times\mathbb{R}$ onto $\hat{M}$, and $f(x)$ with $\varphi_1(x)=\varphi(x,1)$; call M_t the manifold $M_t=\varphi_t(M)$, so $M=M_0=\varphi_0(M)$. It is obvious that $\cup_{t\in\mathbb{R}}M_t=\hat{M}$. It is easy to show that if f is expansive then $\varphi|_A$ is expansive; here A denotes the invariant set of the flow φ defined by $A=\{(p,q)\in\hat{M}\times\hat{M}/\exists t\in\mathbb{R}:p,q\in M_t\}$. We may also assume that α is a constant of expansivity for f and φ.

Following Lewowicz (see [6] and [8]) expansivity implies that there are Lyapunov functions $\mathcal{U},\dot{\mathcal{U}},\ddot{\mathcal{U}}: \{(p,q)\in A/d(p,q)<\gamma\}\to\mathbb{R}$ such that:

1) $\mathcal{U}(p,q)\geq 0$ and $\mathcal{U}(p,q)=0$ iff $p=q$;

2) the Lyapunov derivative $\dot{\mathcal{U}}(p,q)<0$ (>0) if $(p,q)\in A$; $p\neq q$ and $q\in W^s_\varepsilon(x,\varphi)$ (respectively $q\in W^u_\varepsilon(x,\varphi)$), where $\varepsilon\leq\min\{\alpha,\gamma\}$;

3) $\ddot{\mathcal{U}}(p,q)>0$ if $(p,q)\in A$, $p\neq q$; $\ddot{\mathcal{U}}(p,p)=0$.

2.2 Let σ be a positive real number such that if

$\mathcal{U}(p,q)\leq\sigma\Rightarrow d(p,q)\leq\varepsilon'\leq\min\{\gamma,\frac{\alpha}{2}\}$;

and let ρ,ξ be positive real numbers such that if

$d(p,q)\leq\rho\Rightarrow\mathcal{U}(p,q)\leq\sigma-\xi>0$.

2.3 *Definition*: $T_p(\sigma)=\bigcup_{t\in\mathbb{R}}\{q\in\hat{M}/(\varphi_t(p),q)\in A)$ and $\mathcal{U}(\varphi_t(p),q)\le\sigma\}$ is called the "tube centered in the orbit of p with radius σ".

The t_0-section of the tube is the set $T_p(\sigma,t_0)=\{q\in\hat{M}/((\varphi_{t_0}(p),q)\in A)$ and $\mathcal{U}(\varphi_{t_0}(p),q)\le\sigma\}$

The boundary of the tube $T_p(\sigma)$ is the set $\partial T_p(\sigma)=$ $=\{q\in T_p(\sigma)/\exists t\in\mathbb{R}:\mathcal{U}(\varphi_t(p),q)=\sigma\}$.

The interior of the tube $T_p(\sigma)$ is the set $Int(T_p(\sigma))=$ $=\{q\in T_p(\sigma)/\nexists t\in\mathbb{R}:\mathcal{U}(\varphi_t(p),q)<\sigma\}$.

2.4 *Lemma*: Let $p\in Per_H$ and let $\varepsilon>0$ be such that $\frac{\rho}{2}<\varepsilon<\rho$. Then $W^s(p)$ contains an r-disk H of the same dimension than $W^s(p)$ with the following property.

Let $\phi:D^r\to H$ be a homeomorphism of the canonical closed disk D^r onto H with $\phi(0)=p$. Then given any arc $\gamma:[0,1]\to D^r$ such that $\gamma(0)=0$ and $\gamma(1)\in\partial D^r$ there exists $s\in(0,1]$ such that $\phi\circ\gamma(s)\in\partial T_p(\sigma,0)$; i.e.: $\phi\circ\gamma(s)$ lies in the boundary of the 0-section of the tube.

***Proof*:** Assume that $\dim(W^s(p))=r$ and let D^r be the closed unit disk in $\mathbb{R}^r$. There exists $g:D^r\to W^s_\varepsilon(p)$ such that $g(D^r)$ is homeomorphic to D^r. We may suppose that $g(0)=p$. Let $v\in\partial D^r$ and $q=g(v)$. Identifying in the suspension $M_0=\pi(M\times\{0\})$ with M, there exists $t_q\in(-\infty,0)$ such that $\varphi_{t_q}(q)\notin T_p(\sigma)$; otherwise for every $t\le0$ we would have $d(\varphi_t(q),\varphi_t(p))<\alpha$ and this contradicts the expansivity of f (see [7] , lemma 2.2 pp. 570-) . By continuity there exists a neighbourhood U(v) in D^r such that for every

$w\in U(v):\varphi_{t_q}(g(w))\notin T_p(\sigma)$. Using the compactness of ∂D^r, we conclude that there exists $T<0$ such that $\forall q\in g(\partial D^r):\exists t\in[T,0):\varphi_{t_q}(q)\notin T_p(\sigma)$. Let k be the period of p. There exists a minimum $m\in\mathbb{N}, m>0$, such that $-mk\le T$.

Let $\gamma:[0,1]\to D^r$ be a continuous curve joining 0 with ∂D^r ; i.e. $\gamma(0)=0$ and $\gamma(1)\in\partial D^r$. We prove that there exists $s_0\in(0,1]$ such that $f^{-mk}g\circ\gamma(s_0)\in\partial T_p(\sigma)$ in the section $T_p(\sigma,0)=T_p(\sigma,-mk)$ $(\varphi_{-mk}(p)=p)$.

Assume that this is not true. Then it holds that :

a) There is $\lambda>0$ such that for every $s\in[0,\lambda)$ it holds that $\varphi_{-mk}\circ g\circ\gamma(s)=f^{-mk}\circ g\circ\gamma(s)\in Int(T_p(\sigma))$. Arguing by contradiction we conclude that for every $s\in[0,1]:f^{-mk}\circ g\circ\gamma(s)\in Int(T_p(\sigma))$. If not there would exist $s_1\in(0,1]$: $f^{-mk}\circ g\circ\gamma(s_1)\notin Int(T_p(\sigma))$ which implies, using the fact that it should not belong to the boundary of the tube $\partial T_p(\sigma)$, that $\mathcal{U}(p,f^{-mk}\circ g\circ\gamma(s_1))>\sigma$, but $\mathcal{U}(p,p)=0$ and $p=\varphi_{-mk}\circ g\circ\gamma(0)=$ $=f^{-mk}\circ g\circ\gamma(0)$ hence there exists $s_0\in(0,s_1)$ such that $\mathcal{U}(p,f^{-mk}\circ g\circ\gamma(s_0))=\sigma$ and we arrive to a contradiction. We conclude that $f^{-mk}\circ g\circ\gamma([0,1])\subset Int(T_p(\sigma))$.

b) There is $\lambda'>0$ such that for every $s\in[0,\lambda')$ it holds that for every $t\in[-mk,0]$: $\varphi_t\circ g\circ\gamma(s)\in T_p(\sigma)$ (continuity).

c) There exists $\lambda'':1>\lambda''>0$ such that for every $s\in(1-\lambda'',1]$ there exists $t_s\in[-mk,0]$ such that $\varphi_{t_s}\circ g\circ\gamma(s)\notin T_p(\sigma)$.

d) Let $s^*=\sup\{s\in[0,1]/\ \forall t\in[-mk,0]: \mathcal{U}(\varphi_t(p),\varphi_t\circ g\circ\gamma(s))<\sigma\}$, then there exists t^* belonging to $(-mk,0)$ such that

$\mathcal{U}(\varphi_{t^*}(p),\varphi_{t^*}\circ g\circ\gamma(s^*))=\sigma$ and $\mathcal{U}(\varphi_t(p),\varphi_t\circ g\circ\gamma(s^*))\leq\sigma$ for all $t\in[-mk,0]$. But this means that t^* is a maximum for $\mathcal{U}$ of the trayectory through $x=f^{-mk}\circ g\circ\gamma(s^*)$; this is not possible because $\ddot{\mathcal{U}}>0$. So we arrive to a contradiction.

Therefore $H=f^{-mk}\circ g(D^r)$ satisfies the thesis. □

We recall here some results on Algebraic Topology that will be used in what follows, see ([1], [9] or [10]).

Let M be an orientable differentiable C^∞ manifold, not necessarily compact of dimension n and let $F\subset M$, F a closed topological submanifold of M; $U=M\setminus F$ is open in M so it inherits a natural structure of a C^∞ manifold.

Consider the simplicial chain and cochain complex of M, (denoted as $C_*(M)$ and $C^*(M)$ respectively) and $C_c^*(M)\subset C^*(M)$ the cochain complex with compact support. $C_c^*(M)$ is a subcomplex of $C^*(M)$ and may be defined as follows:

$\xi\in C^m(M)$ is in $C_c^*(M)$ iff $\text{support}(\xi)=\bigcup_{\sigma\in C_m(M)}\{|\sigma|/\xi(\sigma)\neq 0\}$ is compact, here C_m stays for the module of the m-chains.

We define the cohomology with compact support as the cohomology of the cochain complex $C_c^*(M)$ and denote it as $H_c^*(M)$.

Using Alexander- Pontrjaguin's Duality Theorem ([9], numèro 20) we deduce that $H_q(M,U)=H_c^{n-q}(F)$.

The following lemma is a generalization of Jordan-Brouwer Separation Theorem.

2.5 ***Lemma:*** Let B be a topological space homeomorphic to an open ball in $\mathbb{R}^3$, and $F\subset B$, F closed in B, homeomorphic to an open set of $\mathbb{R}^2$. Then F separates B. Moreover if F has k connected components then $B\setminus F$ has k+1 connected components.

Proof: It is clear from the hypothesis that F is an oriented topological manifold of dimension 2. Let $\mathbb{K}$ be an arbitrary field, and take homology with coefficients in $\mathbb{K}$. Let U be $B\setminus F$.

Using the Exact Homology Sequence we conclude that the sequence

$$\to H_1(B)\to H_1(B,U)\to H_0(U)\to H_0(B)\to H_0(B,U)\to \quad \text{is exact.}$$

As B is homeomorphic to a ball in $\mathbb{R}^3$ we deduce that $H_1(B)=0$ and being B path connected and $U\neq\emptyset$ we conclude that $H_0(B,U)=0$ (see [1], Proposition (13.10)).

We want to compute $\dim(H_0(U))=\#$(connected components of $U=B\setminus F$).

We have that $H_1(B,U)=H_c^2(F)$. and by duality $H_c^2(F)=H_0(F,\emptyset)=H_0(F)$.

Hence $\dim(H_1(B,U))=\dim(H_0(F))=\#\{\text{connected components of } F\}=$ $=k>0$. Thus $H_1(B,U)\cong\mathbb{K}^k$ and by hypothesis $H_0(B)\cong\mathbb{K}$.

From the short exact sequence

$$0\to H_1(B,U)\to H_0(U)\to H_0(B)\to 0$$

we conclude that $H_0(U)\cong\mathbb{K}^{k+1}$. Hence U has k+1 connected components.□

2.6 ***Corollary:*** If F is a connected set, then B\F has two connected components.□

2.7 ***Lemma:*** Let B be an open ball in $\mathbb{R}^3$, D^2 the cannonical open disk in $\mathbb{R}^2$ and ϕ a continuous function $\phi:D^2\to\mathbb{R}^3$ which is a homeomorphism of D^2 onto its image. Assume moreover that $\phi(0)=p\in int(B)$ and that the following condition holds:

if $\gamma:[0,1]\to D^2$ is a continuous curve joining $\gamma(0)=0\in D^2$ with a point $\gamma(1)\in\partial D^2$ then $\phi(\gamma([0,1])\not\subset B$ (i.e.: there exists $t\in(0,1]$ such that $\phi(\gamma(t))\notin B$).

Then $B\backslash\phi(D^2)$ is not connected, and if $\mathbb{X}$ denotes the connected component of $\phi^{-1}(B)$ that contains $0\in\mathbb{R}^2$ then $\phi(\mathbb{X})$ disconnects B in two connected components.

Proof: (suggested by M. Sebastiani)

Let $F=\phi(\mathbb{X})$, where $\mathbb{X}$ is the connected component of 0 in $\phi^{-1}(B)\cap D^2$. B is an open set in $\mathbb{R}^3$ so $\phi^{-1}(B)$ is open in D^2. D^2 is locally arcwise connected, then $\mathbb{X}$ is locally arcwise connected and open in D^2. Hence $\mathbb{X}$ is arcwise connected. It follows that $\mathbb{X}\cap\partial D^2=\emptyset$. Otherwise there exists $x\in\mathbb{X}\cap\partial D^2$ and a curve $\gamma\subset\mathbb{X}\subset D^2$ joining 0 with x such that $\phi(\gamma([0,1])\subset B$ contradicting the hypothesis. Then $\mathbb{X}$ is an open set in D^2 and consequently in $\mathbb{R}^2$. So $F=\phi(\mathbb{X})$ is homeomorphic to an open set of $\mathbb{R}^2$.

As $\mathbb{X}$ is a connected component it is closed in $\phi^{-1}(B)$. Now, ϕ is a homeomorphism between D^2 and its image and we have a homeomorphism $\phi:\phi^{-1}(B)\to B\cap\phi(D^2)$. Then $F=\phi(\mathbb{X})$ is a closed set in $B\cap\phi(D^2)$ which is closed in B and so F is closed in

B. Aplying **2.6** we conclude that $B\setminus F$ has two connected components.

In order to see that $B\setminus\phi(D^2)$ is not connected we observe that the Theorem of Invariance of Domain implies that $\phi(D^2)$ has empty interior in $\mathbb{R}^3$. So $B\setminus\phi(D^2)$ is a dense set in $B\setminus F$. Then $B\setminus\phi(D^2)$ is not connected.□

If $f:M\to M$ is an α-expansive homeomorphism in the same hypothesis and same notation of **2.4** the following result follows:

2.8 ***Lemma***: Let $H=f^{-mk}\circ g(D^2)$. Then, if r is as in **2.2**, H separates the ball $B(p,r)=\{q\in M/d(p,q)<r\}$.

Proof: It follows from **2.4** and **2.7**.□

2.10 ***Proposition***: In the same hypothesis of **2.4**, if $S(p)$ is the connected component of $H=f^{-mk}\circ g(D^2)$ that contains p in $B(p,r)$, and $\phi=f^{-mk}\circ g$, then it follows that $S(p)=\phi(X)\subset W^s_{\varepsilon'}(p)$. Here X is as in **2.7** and ε' is as in **2.2**.

Proof: It is obvious that $S(p)=\phi(X)$. All we need to prove is that $S(p)\subset W^s_{\varepsilon'}(p)$. We prove that for every $q\in S(p)$ and for every $t\geq 0$ we have that $\varphi_t(q)\in T_p(\sigma)$. By construction $f^{mk}(S(p))\subset g(D^2)\subset W^s_{\varepsilon'}(p)$, so we only have to prove that for all $t\in(0,mk)$ $\varphi_t(q)\in T_p(\sigma)$.

The set $B=\{q\in S(p)/\exists t\in(0,mk): \varphi_t(q)\notin T_p(\sigma)\}$ and $A=\{q\in S(p)/\forall t\in(0,mk): \varphi_t(q)\in Int(T_p(\sigma))\}$ are open in $S(p)$. The point $p\in A$ so $A\neq\emptyset$; it is obvious that $A\cap B=\emptyset$, and also it holds that $S(p)=A\cup B$ because if it were not then there would exist

$q \in S(p) \setminus (A \cup B)$ and that means that there exist $q \in S(p)$ such that there exists $t \in (0, mk)$ depending on q, $t = t(q)$, such that $\varphi_{t(q)}(q) \in \partial(T_p(\sigma))$) and (for all $t \in (0, mk): \varphi_t(q) \in T_p(\sigma)$). But this means that at $t(q)$ the function $\mathcal{U}$ would have a maximum, which contradicts the fact that $\ddot{\mathcal{U}} > 0$ in $T_p(\sigma)$. As $S(p)$ is a connected set we have that $B = \emptyset$, the empty set.□

2.11 *Remarks*:

1) Using lemma 2.2 of [8] we may choose $r > 0$ such that $W^s_{\varepsilon'}(p) \cap B(p,r) \subset W^s_\varepsilon(p)$. So we assume that $S(p) \subset W^s_\varepsilon(p)$.

2) $S(p)$ separates $B(p,r)$. If f is a homeomorphism and p a topologically hyperbolic point for f, it follows from **(1.8)** that $W^u_\varepsilon(p)$ has points in both connected components of $B(p,r) \setminus S(p)$

3) By the compactness of M, there is an $r > 0$ such that for every $p \in M$ $B(p,r)$ is contained in the domain of a coordinate map of M. Is enough to take r equal or less than a Lebesgue number of an open covering by local maps of M. Hence $B(p,r)$ is homeomorphic to an euclidean ball in $\mathbb{R}^3$.

We have an analogous result for the local unstable set of $p \in Per_H$.

2.12 *Proposition*: $W^u_\varepsilon(p)$ contains in $B(p,r)$ a connected component $U(p)$ such that $p \in U(p)$ and $U(p)$ reachs the boundary $\partial B(p,r)$ of $B(p,r)$ in two points q and q' which are in different connected components with respect to $S(p)$.

Proof It follows from **2.4** and remark **2.11.2**.□

3. INVARIANCE OF THE DIMENSIONS OF $W^s_\varepsilon(p)$ AND $W^u_\varepsilon(p)$.

We want to show that if $p,p'\in Per_H$ and their distance $d(p,p')$ is small enough then $\dim(W^u_\varepsilon(p))=\dim(W^u_\varepsilon(p'))$ and $\dim(W^s_\varepsilon(p))=\dim(W^s_\varepsilon(p'))$.

3.1 ***Lemma:*** Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in M with limit x. Then for every $\delta>0$ there exists $N\in\mathbb{N}$ such that for every $n\geq N$ and for every $y\in W^s_\varepsilon(x_n)$ the distance $d(y,W^s_\varepsilon(x))$ is smaller than δ. (Note that $W^s_\varepsilon(x)$ and $W^s_\varepsilon(x_n)$ are considered only as subsets of M.)

Proof: Let us suppose the conclusion is false. Then there exists $\delta_0>0$ and $y_{n_k}\in W^s_\varepsilon(x_{n_k})$ such that $d(y_{n_k},W^s_\varepsilon(x))\geq\delta_0$. We may assume that y_{n_k} converges to a point z.

From the fact that $y_{n_k}\in W^s_\varepsilon(x_{n_k})$ we conclude that for every $m\geq 0$: $d(f^m(y_{n_k}),f^m(x_{n_k}))\leq\varepsilon$. This and the continuity of f implies that $d(f^m(z),f^m(x))\leq\varepsilon$. Hence $z\in W^s_\varepsilon(x)$ and at the same time $d(z,W^s_\varepsilon(x))\geq\delta_0$. This is absurd. □

3.2 ***Remark:*** Let as call Z to $(S(p)\setminus W^s_\varepsilon(p))\cap B(p,r)$ where $S(p)$ is the connected component of $W^s_\varepsilon(p)$ that contains p in $B(p,r)$. If f is an expensive homeomorphism and $p \in Per_H$, we have that there is a positive number r_0 such that in $clos(B(p,r_0))$ it holds that the infimum of the distance between $S(p)$ and Z, $\inf\{d(S(p),Z)\}$, is a number r_p greater than zero. This is true because certainly $S(p)$ contains $W^S_r(p)$ and by lemma 2.2 of [8] we may choose $r_0>0$ such that $W^s_\varepsilon(p) \cap clos(B(p,r_0)) \subset W^S_r(p)$.

Thus $(W^s_\varepsilon(p)\setminus S(p)) \cap clos(B(p,r_0))=\emptyset$ and this implies our assertion. A similar statement holds for the unstable manifold of a topological periodic hyperbolic point.

Fix a positive number r_p, smaller than ρ (with ρ chosen as in **2.2**), such that $\min\{d(S(p),W^s_\varepsilon(p)\setminus S(p))\} \geq r_p$ and $\min\{d(U(p),W^u_\varepsilon(p)\setminus U(p))\} \geq r_p$ in $clos(B(p,r_0))$.

3.3 ***Lemma:*** Given $r>0$, there is $\delta>0$ such that for all $x \in M$ there exist $\delta>0$ such that the distance $d(W^s_\varepsilon(x)\setminus B(x,r),W^u_\varepsilon(x)\setminus B(x,r))$ is greater than δ.

Proof: Fix $x \in M$, if there is not a $\delta_x>0$ such that the distance $d(W^s_\varepsilon(x)\setminus B(x,r),W^u_\varepsilon(x)\setminus B(x,r)) \geq \delta_x$ then there exist convergent sequences $\{q_n\} \subset W^s_\varepsilon(x)$ and $\{v_n\} \subset W^u_\varepsilon(x)$ such that both converge to the same point q, $q_n \xrightarrow[n]{} q$ and $v_n \xrightarrow[n]{} q$. But this implies that $q \in W^s_\varepsilon(x) \cap W^u_\varepsilon(x)$. This violates the expansivity of f. Suppose that there is no $\delta>0$ which serves for all the points $x\in M$. Thus there is a sequence $\{x_n\}\subset M$ such that the distance $d(W^s_\varepsilon(x_n)\backslash B(x_n,r), W^u_\varepsilon(x_n)\backslash B(x_n,r)) \leq 1/n$. As M is compact, we may suppose that $\{x_n\}$ converges to a point x and again we have $\{q_n\}\subset W^s_\varepsilon(x)$ and $\{v_n\}\subset W^u_\varepsilon(x)$ both converging to the same point $q\in W^s_\varepsilon(x)\cap W^u_\varepsilon(x)$. □

3.4 *Remarks*: 1) Given $p\in Per_H$ and $\{p_n\}_{n\in N}\subset Per_H$ such that $\lim_{n\to+\infty} p_n=p$, if we take $0<\delta'<\min\{r_p,\delta\}$ then **3.1** reads: $\forall n\geq N$: $\forall q\in S(p_n)$: $d(q,S(p))<\delta'$. The same is true for the connected components $U(p)$ and $U(p_n)$ of p and p_n of the unstable manifolds.

2) Aplying **3.1** and **3.3** we have that given a point $x\in M$ there is a positive number $\lambda>0$ such that for every $y\in M$ such that the distance $d(x,y)<\lambda$, it holds that $d(W^s_\varepsilon(x)\backslash B(x,r), W^u_\varepsilon(y)\backslash B(x,r)) > \delta/2$ and at the same time $d(W^s_\varepsilon(y)\backslash B(x,r), W^u_\varepsilon(x)\backslash B(x,r)) > \delta/2$.

3.5 *Lemma*: If $p,p'\in Per_H$, there exists $\delta>0$ such that $d(p,p')<\delta$ implies at the same time $\dim(W^u_\varepsilon(p))=\dim(W^u_\varepsilon(p'))$ and $\dim(W^s_\varepsilon(p))=\dim(W^s_\varepsilon(p')$.

Proof : Suppose this is false. Assume that $\dim(W^s_\varepsilon(p))=2$ and that there exists a sequence $\{p_n\}_{n\in N}\subset Per_H$ such that $\lim_{n\to+\infty} p_n=p$ and

$\dim(W^s_\varepsilon(p_n))=1$. This implies that $\dim(W^u_\varepsilon(p_n))=2$. Using **3.4** we know that for every $\delta>0$ there is $N\in\mathbb{N}$ such that for all

$$n\geq N\begin{cases}\forall q\in U(p_n)\colon\ d(q,U(p))<\delta\\ \forall q\in S(p_n)\colon\ d(q,S(p))<\delta\end{cases}.$$

Let us suppose that there are points $q,q'\in U(p_n)$ in different regions with respect to $S(p)$ in $B(p,r)$. As $U(p_n)$ is an arc-connected set we may join q and q' by an arc contained in it. It will intersect $S(p)$ in a point $x\in U(p_n)\cap S(p)$. But locally in x, $U(p_n)$ is homeomorphic to a 2-disk and then it could not be disconnected substracting x to it, so it must exist $y\in U(p_n)\cap S(p)$, $y\neq x$. Hence as $y,x\in S(p)$ for all $m\geq 0$: $d(f^m(x),f^m(y))<\alpha$, and, as $y,x\in U(p_n)$, for every $m\leq 0$: $d(f^m(x),f^m(y))<\alpha$, contradicting the expansivity.

So we conclude that $U(p_n)$ has points in only one of the regions determined by $S(p)$ in $B(p,r)$. This and the above argument shows that $U(p_n)\cap S(p)$ is empty or reduces to a single point (*).

Let r be smaller or equal than ρ and greater or equal than r_p (r_p as in **3.2**); and let $\delta<\min\left\{\frac{\delta_p}{2},\frac{r_p}{2}\right\}$. Then $S(p_n)$ intersects $\partial B(p,r)$ in two points x_1,x_2. If we join both points by an arc in $B(p,r)$, this arc will intersect $U(p_n)$ because $U(p_n)$ separates in $B(p,r)$. But we may join x_1 with a point $y_1\in S(p)$ with a geodesic arc without intersections with $U(p_n)$: otherwise we would have a point $w\in U(p_n)$ at a distance less than δ from $S(p)$, but at the same time $d(w,U(p))<\delta$ contradicting the fact that $d(U(p),S(p))\geq\delta$

outside $B(p,r_p)$. We also may join x_2 with a point $y_2 \in S(p)$ with a geodesic arc without intersections with $U(p_n)$. Hence we can construct an arc contained in $S(p)$ joining y_1 with y_2 without intersections with $U(p_n)$. This is posible due to remark (*). So we arrive to a contradiction with the separation property of $U(p_n)$ and the proof is complete.□

3.6 ***Corollary:*** If Per_H is a dense set in M then all periodic hyperbolic points have stable (unstable) manifolds of same dimension.□

3.7 We will assume on the sequel that for every $p \in Per_H$ $\dim(W^u_\varepsilon(p))=1$ and $\dim(W^s_\varepsilon(p))=2$.

4. INTERSECTION OF STABLE AND UNSTABLE MANIFOLDS.

We want to prove that given any point $p \in Per_H$ there exists a neighbourhood $V(p) \subset B(p,r)$ such that if $p' \in Per_H \cap V(p)$ the stable (unstable) manifold of p will intersect the unstable (stable) manifold of p' in one and only one point. Via a local map we may assume that we are working in a ball in $\mathbb{R}^3$.

4.1 ***Lemma:*** Given $p \in Per_H$ and $\delta_p > 0$, as in **3.3**, there exist a neighbourhood V(p) of p included in B(p,r) such that if $p' \in Per_H \cap V(p)$ U(p') intersects the boundary $\partial B(p,r)$ of the ball B(p,r) in two points each of which is at a distance smaller than $\frac{\delta_p}{2}$ from one of the points of intersection of U(p) and $\partial B(p,r)$.

Proof: Write $\{q_1, q_{-1}\} = U(p) \cap \partial B(p,r)$ and $U(p') \cap \partial B(p,r) = \{q'_1$ and $q'_{-1}\}$. It is clear from **2.11** that if p' is close enough to p, U(p') intersects $\partial B(p,r)$. We can get that $d(q'_i, U(p)) < \delta$ $i=1,-1$.

This implies that $d(q_i,q'_i)<\frac{\delta_p}{2}$, $i=1,-1$ (interchanging subindices if necessary). If this were not true, there would exist $p'_n \xrightarrow[n]{} p$ such that - being $\{q^n_{-1},q^n_1\}=U(p'_n)\cap\partial B(p,r)$ - $d(q^n_1,U(p))<\frac{1}{n}$ and $d(q^n_1,q_1)\geq\frac{\delta_p}{2}$. Taking limits of a convergent subsequence we obtain $q^\infty_1\in\partial B(p,r)\cap U(p)$ and $q^\infty_1\neq q_1, q_{-1}$. This is absurd. Suppose now that at the same time it holds $d(q_1,q'_1)<\frac{\delta_p}{2}$ and $d(q_1,q'_{-1})<\frac{\delta_p}{2}$, so both points q'_1 and q'_{-1} are at a distance less than $\frac{\delta_p}{2}$ of the same point q_1. Then q'_1 and q'_{-1} are in the same connected component of $B(p,r)\setminus S(p)$ but in different components of $B(p,r)\setminus S(p')$. We may join q'_1 and q'_{-1} with an arc contained in the component of $\partial B(p,r)\setminus S(p)$ that contains q_1 such that all points in the arc will be at a distance less than $\frac{\delta_p}{2}$. To see this take a local map of q_1 containing the images of both q'_1 and q'_{-1}. This arc will intersect $S(p')$ in a point x. Then $d(x,U(p))<\frac{\delta_p}{2}$. But using **3.1** and the fact that $x\in S(p')$, we may choose $V(p)$ in such a way that $d(x,S(p))<\frac{\delta_p}{2}$. It is clear now that $d(S(p)\setminus B(p,r_p),U(p)\setminus B(p,r_p))<\delta_p$ but this contradicts **3.3**.□

4.2 *Proposition*: Under the same hypothesis of **4.1** for all $p'\in Per_H\cap V(p)$ it holds that $U(p')\cap S(p)$ consists of only one point.

Proof: By **4.1** q'_1 and q'_{-1} are in different connected components of $B(p,r)\setminus S(p)$ so $U(p')$ intersects $S(p)$. The expansivity hypothesis implies that the intersection is only one point.□

4.3 ***Proposition:*** For any p there exists a neighbourhood $V(p)$ such that if $p'\in V(p)\cap Per_H$ then $S(p')\cap U(p)$ consists of only one point.

Proof: $S(p')$ separates $B(p,r)$ and we may choose $V(p)$ such that for every $p'\in V(p)$ it holds that for all $q\in S(p')$: $d(q,U(p))<\delta$ and for all $q\in S(p')$: $d(q,S(p))<\delta$. Choose $\delta<\frac{\delta_p}{2}$ and suppose that $U(p)$ joins q_1 with q_{-1} without intersections with $S(p')$. As in **4.1** we may assume that $d(q_1,q'_1)<\frac{\delta_p}{2}$ and $d(q_{-1},q'_{-1})<\frac{\delta_p}{2}$. Joining q_i with q'_i, with arcs Γ_i, $i=1,-1$, on $\partial B(p,r)$, one of them (for instance Γ_1) intersects $S(p')$. Let $x\in\Gamma_1\cap S(p')$ hence $d(x,S(p))<\frac{\delta_p}{2}$. We may choose the arc Γ_1 in such a way that all its points would be at a distance less than $\frac{\delta_p}{2}$ from q_1. So $d(x,U(p))<\frac{\delta_p}{2}$.

Then $d(S(p)\setminus B(p,r_p),U(p)\setminus B(p,r_p))<\delta_p$, contradicting **3.3**.□

5. LOCAL PRODUCT STRUCTURE

We prove now that, under the hypothesis that Per_H is dense in M, there exists an open invariant dense subset $A\subset M$ of (topologically) hyperbolic points.

5.1 ***Definition:*** $C\subset M$ a subset is said to be a continuum iff it is non empty compact and connected, it is a non degenerate continuum if it does not reduce to a single point.

5.2 Let M be a complete metric space. Let $\mathcal{A}\neq\emptyset$ be a directed set with order $\geq$ and let $\{C_\alpha\}_{\alpha\in\mathcal{A}}$ be a family of continua such that $\alpha\geq\alpha' \Rightarrow C_\alpha\subset C_{\alpha'}$.

Then a very well known result says that $C_\infty=\bigcap_{\alpha\in\mathcal{A}} C_\alpha$ is also a continuum (see [4]).

Let $p\in Per_H$, $\overline{Per_H}=M$, and $V(p)$ be as in **4.2**, **4.3**. Let $x\in V(p)\cap S(p)$ and $V_s(x)$ a connected neighbourhood of x in $V(p)\cap S(p)$ such that $V_s(x)=V(x)\cap S(p)$, $V(x)$ being a neighbourhood of x in M. Define $\mathcal{A}_{V_s(x)}=\{p'\in Per_H\cap V(p)\cap V(x)/\ U(p')\cap V_s(x)\neq\emptyset\}$.

5.3 ***Lemma***: In the above hypothesis $\mathcal{A}_{V_s(x)}$ is not empty for every $V_s(x)$.

Proof: If not, there would exist $V_s(x)$ such that $U(p')$ would not intersect $V_s(x)$. By **4.2**, $U(p')$ cuts $S(p)$ in $B(p,r)$. Let $\{p_n\}_{n\in N}\subset Per_H\cap V(p)$, $p_n \xrightarrow[n]{} x$. Then $U(p_n)$ cuts $S(p)$ in a point q_n. Taking a convergent subsequence if it is necessary we may suppose that $q_n \xrightarrow[n]{} q_\infty$, then $q_\infty\in S(p)$ and $q_\infty\notin V_s(x)$ so $q_\infty\neq x$.

Now we have that for every $m\geq 0$: $(d(f^m(x),f^m(p))\leq\varepsilon$ and $d(f^m(q_\infty),f^m(p))\leq\varepsilon)$

So for every $m\geq 0$: $d(f^m(x),f^m(q_\infty))\leq 2\varepsilon$.

As $\varepsilon<\frac{\alpha}{2}$ we have that for all $m\geq 0$: $d(f^m(x),f^m(q_\infty))\leq\alpha$.

Also, as q_n is in $U(p_n)$, it holds that for all negative m $(d(f^m(q_n),f^m(p_n))\leq\varepsilon$. Taking limits on n we obtain $\forall m\leq 0$: $d(f^m(x),f^m(q_\infty))\leq\varepsilon$.

So we obtain that for every $m\in\mathbf{Z}$: $d(f^m(x),f^m(q_\infty))\leq\alpha$. Using expansivity we have that this implies that $q_\infty=x$. This means that the thesis must hold.□

5.4 Let us define $\mathcal{V}_{V_s(x)}=\bigcup_{p'\in\mathcal{A}_{V_s(x)}} U(p')$; the previous lemma guarantees that $\mathcal{V}_{V_s(x)}$ is not empty for every $V_s(x)$. Define $\mathcal{K}_{V_s(x)}=\mathrm{clos}(V_s(x)\cup\mathcal{V}_{V_s(x)})$ where the prefix clos stays for the topological clousure of $V_s(x)\cup\mathcal{V}_{V_s(x)}$. It is easily seen that $V_s(x)\cup\mathcal{V}_{V_s(x)}$ is a non empty arcwise connected set, so $\mathcal{K}_{V_s(x)}$ is a continuum.

Let $\{V_s^{(i)}(x)\}_{i\in\mathbb{N}}$ be a basis of connected neighbourhoods of x relative to $S(p)$ such that $i>i' \Rightarrow V_s^{(i)}(x)\subset V_s^{(i')}(x)$ and $\mathrm{diam}(V^{(i)})\to 0$ when $i\to\infty$, where $V_s^{(i)}=V^{(i)}\cap S(p)$. Define $C(x)=\bigcap_{i\in\mathbb{N}} \mathcal{K}_{V_s^{(i)}}$. Then using **5.2** and **5.3** we conclude that $C(x)$ is a continuum.

5.5 ***Lemma:*** $C(x)$ intersects $\partial B(p,r)$.

Proof: It is clear that for every $i\in\mathbb{N}$: $\mathcal{K}_{V_s^{(i)}}$ has the above property. Let y_i and y'_i be the intersection points of $U(p_i)$ with $\partial B(p,r)$ where $p_i\in \mathcal{A}_{V_s^{(i)}}$; then y_i and y'_i are in $\mathcal{K}_{V_s^{(i)}}$.

Take a convergent subsequence such that $y_i \underset{i}{\to} y_\infty$ and $y'_i \underset{i}{\to} y'_\infty$.

If we choose $V(p)$ small enough we have that $\forall i\in\mathbb{N}$: y_i and y'_i are in different regions of $B(p,r)\setminus S(p)$ and at a distance less than $\frac{\delta_p}{2}$ from $U(p)$. This implies that y_∞ and y'_∞ are in $\partial B(p,r)$ and in different regions of $B(p,r)\setminus S(p)$ (they couldn't be in $S(p)$

because in that case it would result $d(S(p),U(p))<\delta_p$ in $B(p,r)\setminus B(p,r_p))$.□

5.6 ***Lemma:*** If $W^u_\varepsilon(x)$ is the unstable set of x then $C(x)\subset W^u_\varepsilon(x)$.

Proof: Let $x'\in C(x)$, $x'\neq x$, then $x'\notin S(p)$, otherwise as $x'\neq x$ there exists $n_0\in\mathbb{N}$ such that x' does not belong to $V_s^{(i)}(x)$. But $x'\in \mathcal{K}_{V_s^{(i)}}$, so there exists $p_i\in \mathcal{A}_{V_s^{(i)}}$ and $x_i\in \mathcal{K}_{V_s^{(i)}}$ is such that $x_i\in U(p_i)$, and $p_i \underset{i}{\rightarrow} x$, $x_i \underset{i}{\rightarrow} x'$. Then for all $n\geq 0$: $d(f^{-n}(x_i),f^{-n}(p_i))\leq\varepsilon$. Thus for every $n\geq 0$: $d(f^{-n}(x'),f^{-n}(x))\leq\varepsilon$, but also $\forall n\geq 0$: $d(f^{n}(x'),f^{n}(x))\leq\varepsilon$ because we are suposing that $x'\in S(p)$. This violates expansivity. A similar argument shows that $x'\in W^u_\varepsilon(x)$.□

5.7 ***Lemma:*** For all $p'\in V(p)$: $C(x)\cap S(p')$ consists of a single point that we call z.

Proof: Uniqueness follows from expansivity. We only have to prove that $C(x)\cap S(p')\neq\emptyset$. As in **5.5** we may prove that for every $i\in\mathbb{N}$: $\mathcal{K}_{V_s^{(i)}}\cap S(p')\neq\emptyset$ (because an arc $U(p_i)$ is contained in $\mathcal{K}_{V_s^{(i)}}$ which intersects $S(p')$). But $\mathcal{K}_{V_s^{(i)}}\cap S(p') \subset \mathcal{K}_{V_s^{(j)}}\cap S(p')$ if $i>j$. Aplying **5.2** we obtain a continuum in $C(x)\cap S(p')$.□

5.8 Let $y\in U(p)\cap V(p)$ and let $V_u(y)$ be a connected neighbourhood in $U(p)$ of the point y of the form $V_u(y)=V(y)\cap U(p)$ with $V(y)$ a neighbourhood of y in M. We define $\mathcal{B}_{V_u(y)}=\{p'\in Per_H\cap V(p)\cap V(y)/\ S(p')\cap V_u(y)\neq\emptyset\}$. The following analogous to **5.3** follows.

5.9 ***Lemma:*** For every $V_u(y)$ we have that $\mathcal{B}_{V_u(y)}\neq\emptyset$.

Proof: Same proof as that of **5.3**.□

5.10 As in **5.4** we define:

$\mathcal{V}_{V_u(y)}=\bigcup_{p'\in \partial V_u(y)} S(p')$ and $\mathcal{K}_{V_u(y)}=\overline{V_u(y)\cup\mathcal{V}_{V_u(y)}}$ It follows that $\mathcal{K}_{V_u(y)}$ is a continuum.

Let $\{V_u^{(i)}\}_{i\in\mathbb{N}}$ be a fundamental system of connected neighbourhoods of y in U(p) such that $i>i' \Rightarrow V_u^{(i)}\subset V_u^{(i')}$ and $\operatorname{diam}(V^{(i)})\to 0$ when $i\to\infty$ where $V_u^{(i)}=V^{(i)}\cap U(p)$.

Define $D(y)=\bigcap_{i\in\mathbb{N}} \mathcal{K}_{V_u^{(i)}}$. Then using **5.2** and **5.9** we conclude that D(y) is a continuum.

5.11 ***Lemma***: If $W_\varepsilon^s(y)$ is the stable set of y then $D(y)\subset W_\varepsilon^s(y)$.

Proof: The same of **5.6**.□

5.12 ***Lemma***: For every hyperbolic point $p'\in V(p)$: $D(y)\cap U(p')$ consists of a single point that we call w.

Proof: The same of **5.7**.□

5.13 ***Lemma***: D(y) separates B(p,r).

Proof: D(y) is a compact connected set and $D(y)\neq B(p,r)$ (otherwise y should be a stable point). Consider the set $D(y)+\lambda=\{z\in B(p,r)/d(z,D(y))\leq\lambda\}$. Then for every $\lambda\in\mathbb{R}$, $\lambda\geq 0$, $D(y)+\lambda$ is compact and connected. Using **5.11** and the expansivity of f we conclude that U(p) intersects D(y) only in y. Fix $\lambda>0$ and let $q\in U(p)$ be a point in a different side than p with respect to y in U(p), $d(q,D(y))>\lambda$. Let us prove that for every continuous arc

$\gamma:[0,1]\to B(p,r)$ joining $p=\gamma(0)$ with $q=\gamma(1)$, γ intersects $D(y)$. Suppose this is not true. Then there exists a continuous arc γ such that $p=\gamma(0)$, $q=\gamma(1)$ and $\gamma(t)\notin D(y)$ for every $t\in[0,1]$. Let $\lambda_0\leq\min\{d(\gamma(t),D(y)) \,/\, t\in[0,1]\}$. We may assume that λ_0 is positive. Then for every $\lambda<\lambda_0$: $D(y)+\lambda$ does not separate $B(p,r)$ and also due to **1.6** $D(y)+\lambda\neq B(p,r)$ if we take λ_0 small enough. Let $p_n\underset{n}{\to}y$, consider $S(p_n)$ such that $\forall q\in S(p_n)$: $d(q,D(y))<\lambda/3$. Then $S(p_n)$ does not intersect γ so it could not separate p from q in $B(p,r)$. Now, $S(p_n)$ intersects $U(p)$ in a point that we may assume to be between p and q. This means that p and q are in different connected components of $B(p,r)\setminus S(p_n)$. This is a contradiction.□

5.14 ***Proposition:*** Given $x\in S(p)\cap V(p)$, $y\in U(p)\cap V(p)$ there exists a unique point z such that $z\in C(x)\cap D(y)$.

Proof: Let us consider $\{p_n\}_{n\in\mathbb{N}}$ with $p_n\underset{n}{\to}y$; then we may suppose that $C(x)$ intersects $S(p_n) \subset \mathcal{K}_{V_u}(n)$ in a unique point z_n (lemma 5.7). Considering the compact sets $C_n = \mathcal{K}_{V_u}(n)\cap C(x)$ and applying 5.2 to $C_\infty=\bigcap_{n\in\mathbb{N}} C_n$ we see that there is $z\in C(x)\cap D(y)$. The uniqueness ifollows from expansivity.□

5.15 Now we define the function $h:V_s(p)\times V_u(p)\longrightarrow M$; where $V_s(p)=S(p)\cap V(p)$ and $V_u(p)=U(p)\cap V(p)$ as $h(x,y)=z$ if $\{z\}=C(x)\cap D(y)$.

5.16 ***Lemma:*** The map h is continuous and injective.

Proof: Let $(x,y)\neq(x',y')$ and suppose that $h(x,y)=h(x',y')=z$. Then $C(x)\cap D(y)=C(x')\cap D(y')=\{z\}$, and also $C(x)\cap D(y')=C(x')\cap D(y)=\{z\}$. Hence we may assume that $y\neq y'$ and that $z=h(x,y)=h(x,y')$.

As $z \in D(y)$ it follows that for every $n \geq 0$: $d(f^n(z), f^n(y)) \leq \varepsilon < \frac{\alpha}{2}$ Also $z \in D(y')$ implies that for every $n \geq 0$: $d(f^n(z), f^n(y')) \leq \varepsilon < \frac{\alpha}{2}$ We conclude that for all $n \geq 0$ $d(f^n(y), f^n(y')) < \alpha$. As $y \in U(p)$ we also have that for all $n \leq 0$: $d(f^n(p), f^n(y)) \leq \varepsilon < \frac{\alpha}{2}$ Also $y' \in U(p)$ so for every $n \leq 0$: $d(f^n(p), f^n(y)) \leq \varepsilon < \frac{\alpha}{2}$. Then for $n \leq 0$: $d(f^n(y), f^n(y')) < \alpha$

So that for every $n \in \mathbf{Z}$: $d(f^n(y), f^n(y')) < \alpha$. Hence $y = y'$ which is absurd. This proves injectivity.

In order to prove continuity let $x_n \xrightarrow[n]{} x$; $y_n \xrightarrow[n]{} y$; $z_n = h(x_n, y_n)$.

Then $\{z_n\} = C(x_n) \cap D(y_n)$. We may take a convergent subsequence $\{z_{n_k}\}$ such that $z_{n_k} \xrightarrow[k]{} z_\infty$. As $z_n \in C(x_n) \subset W^u_\varepsilon(x_n)$ it follows that for every $\nu \in \mathbf{N}$: $d(f^\nu(z_n), f^\nu(x_n)) \leq \varepsilon$. This implies that for every $\nu \in \mathbf{N}$: $d(f^\nu(z_\infty), f^\nu(x)) \leq \varepsilon$ and so that $z_\infty \in W^u_\varepsilon(x)$.

In the same way we prove that $z_\infty \in W^s_\varepsilon(y)$. As $C(x) \subset W^u_\varepsilon(x)$ and $D(y) \subset W^s_\varepsilon(y)$ we have that $C(x) \cap D(y) = \{z\}$. Expansivity implies that $z = z_\infty$ so $h(x_{n_k}, y_{n_k}) \xrightarrow[k]{} h(x,y) = z$. As the previous fact holds for every convergent subsequence $\{z_{n_k}\}$ we conclude that h is continuous and the proof is complete.□

5.17 ***Lemma:*** $C(x) \cap h(V_s(p) \times V_u(p)) = W^u_\varepsilon(x) \cap h(V_s(p) \times V_u(p))$.

Proof: Suppose it is false. There exists $z' \in W^u_\varepsilon(x)$, $z' \notin C(x)$.

As $z' \in h(V_s(p) \times V_u(p))$ so there exist (x',y') such that $h(x',y')=z'$. This means that $\{z'\}=C(x') \cap D(y')$ with $x' \neq x$, hence $z' \in W^u_\varepsilon(x')$ and also $z' \in W^u_\varepsilon(x)$. This and the fact that $x,x' \in S(p)$ implies that for all $n \in Z$: $d(f^n(x),f^n(x')) < \alpha$ hence $x=x'$ and this is a contradiction.□

5.18 ***Lemma***: $D(y) \cap h(V_s(p) \times V_u(p)) = W^s_\varepsilon(y) \cap h(V_s(p) \times V_u(p))$.

Proof: The same of 5.17.□

5.19 ***Definition***: We say that $y \in M$ has a local product structure if there is a homeomorphism of $\mathbb{R}^3$ onto an open neighbourhood of y such that it maps horizontal planes onto open subsets of local stable (unstable) sets and vertical lines onto open subsets of local unstable (resp. stable) sets.

5.20 ***Proposition***: If Per_H is dense in M, there exists an open invariant dense set $A \subset M$ consisting of (topologically) hyperbolic points such that in A there is a local product structure.

Proof: By the Theorem of Invariance of Domain being $h: V_s(p) \times V_u(p) \to h(V_s(p) \times V_u(p)) \subset M$ an injective continuous map from an open set in M into another open set in M it is a homeomorphism.

Let A be the set $A=\{y \in M \ / \ y$ has a local product structure$\}$. Then $A \supset \bigcup_{p \in Per_H} h_p(V_s(p) \times V_u(p)) \supset Per_H$ so density follows from $\overline{Per_H}=M$.

It is also trivial that A is open.

Let $z \in A$ then there exist $B(z,2\nu) \subset A$. For every $w \in W^s_\varepsilon(z)$ it holds that $\lim_{n\to\infty} d(f^n(z),f^n(w))=0$. This implies that there is a positive integer N such that for all $n \geq N$: $f^n(w) \in W^s_\nu(f^n(z))$

As Per_H is dense in M and is included in the non-wandering set, M itself is the non wandering set, so there is $n \geq N$ such that $d(f^n(z),z) < \nu$. We conclude that $W^s_\nu(f^n(z)) \subset A$ and $f^n(W^s_\varepsilon(z)) \subset W^s_\nu(f^n(z))$, hence $f^{-n}(W^s_\nu(f^n(z))) \supset \supset W^s_\varepsilon(z)$ and $W^s_\varepsilon(z)$ is a topological manifold.□

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Twisting Products in Hopf algebras and the construction of the Quantum double

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Abstract

Let H be a finite dimensional Hopf algebra and B an (H, H^*)-comodule algebra. The purpose of this paper is to present a construction in which the product of B is twisted by the actions of H and H^*. The constructions of the smash product and of the Quantum double appear as special cases.

1 Introduction

We start with a brief summary of the contents of the paper.

1. Introduction.

In this section we fix the basic notations. We introduce the concept of paired bialgebras –generalizing the situation of a finite-dimensional Hopf algebra and its dual–. If H and K are paired bialgebras and ψ and ϕ are left(right) H (K)-comodule structures on a vector space V, to the triple (V, ψ, ϕ) we associate a map $\psi \wedge \phi : V \otimes V \to V \otimes V$ that will later be used to

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twist the products. Also for ψ and ϕ as above, the concept of compatibility is introduced.

2. Twisting Products.

Assume that we have a triple (V, ψ, ϕ) as above and a product μ in V in such a way that the comodule structures are compatible with the product. Using the map $\psi \wedge \phi$ we twist the product μ in order to obtain a new associative product -that we call the twisted product- that has the same neutral element than the original one. We show that the well known construction of the smash product $H^* \sharp A$ can be considered as the twisting of the usual convolution product with a pair of natural coactions on $H^* \otimes A$.

3. Iterated Twisting.

In this section we iterate the construction of twisting. We show that under natural hypothesis, if we have a finite family of paired bialgebras and a corresponding family of structures ψ_i, ϕ_i on a vector space V with a product μ, we can twist μ with <u>all</u> the structures, and obtain a new associative product on V.

4. The Quantum double as a twist of the convolution product.

Let K be a finite dimensional Hopf algebra, and consider in $K^* \otimes K$ the natural component-wise (or convolution) product. In this section we show that if we consider a convenient double twist of the convolution product, we obtain in $K^* \otimes K$ the product introduced by Drinfeld to define the Quantum double $D(K)$. In this way the Quantum double appears as the outcome of a two step twisting process. We start with the convolution product, in the first step we obtain the smash product and in the second the product defined by Drinfeld in $D(K)$.

Suppose that k is a fixed field and that H is Hopf algebra defined over k. The Hopf Algebra structure of H is given by Δ, ε (comultiplication and counit) μ, u (multiplication and unit) and σ (antipode). In case H is finite dimensional H^* (the linear dual of H) is also a Hopf algebra and its structure is given by maps denoted as Δ^*, ε^* (comultiplication and counit) μ^*, u^* (multiplication and unit) and σ^* (antipode). Notice that (for example) Δ^* is not the dual of the comultiplication Δ but the comultiplication of H^* that is the dual of the multiplication μ of H.

We refer the reader to [8] for the relevant concepts on the general theory of Hopf Algebras.

The following notations will be in force along this paper :

1. $\Delta^2 : H \to H \otimes H \otimes H$ is the map $\Delta^2 = (\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$ and in general $\Delta^n : H \to H^{\otimes^{n+1}}$ is the map : $\Delta^n = (\Delta \otimes id \otimes ... \otimes id)\Delta^{n-1} = (id \otimes \Delta \otimes ... \otimes id)\Delta^{n-1} = ... = (id \otimes id \otimes ... \otimes \Delta)\Delta^{n-1}$.

2. Given an arbitrary bialgebra K we call K^{cop} and K^{op} the same bialgebra with opposite comultiplication and multiplication respectively. Observe that if K has an invertible antipode – called σ –, then K^{cop} and K^{op} have σ^{-1} as antipode.

3. If H is a colagebra, the comultiplication $\Delta : H \to H \otimes H$ is at the same time a left H-comodule structure and a right H-comodule structure on H. When viewed as such, it will be denoted as Δ_l and Δ_r respectively and H, when equipped with one or the other, will be denoted as H_l and H_r respectively.

4. If H is a bialgebra and M and N are right H-comodules with structures χ_M and χ_N respectively, we denote as $M \boxtimes N$ the vector space $M \otimes N$ equipped with the so called tensor product right H-comodule structure : $\chi_{M} \boxtimes \chi_N = (id \otimes id \otimes \mu)(id \otimes s \otimes id)(\chi_M \otimes \chi_N)$ (see for example [3] for the definition and properties of $\chi_M \boxtimes \chi_N$) .

5. The symbol $s : A \otimes B \to B \otimes A$ denotes the map that switches the tensor factors. More generally if τ is a permutation of n elements, and V_i, $i = 1, .., n$ are arbitrary k-spaces, we call $s_\tau : V_1 \otimes V_2 \otimes .. \otimes V_n \to V_{\tau(1)} \otimes V_{\tau(2)} \otimes .. \otimes V_{\tau(n)}$ the map that permutes the tensor factors in the same fashion than τ. In particular $s = s_{(1,2)}$.

Definition 1.1 *Suppose that H and K are arbitrary k–bialgebras. We say that H and K are paired if there exists a k-linear map $p : H \otimes K \to k$ (called the pairing) such that the diagrams below commute:*

1.

$$
\begin{array}{ccccc}
H \otimes K \otimes K & \xrightarrow{id \otimes \mu} & H \otimes K & \xrightarrow{p} & k \\
\Big\downarrow {\scriptstyle \Delta \otimes id \otimes id} & & & & \Big\uparrow \\
H \otimes H \otimes K \otimes K & & & & {\scriptstyle m_k} \\
\Big\downarrow {\scriptstyle id \otimes s \otimes id} & & & & \Big\uparrow \\
H \otimes K \otimes H \otimes K & & \xrightarrow{\qquad p \otimes p \qquad} & & k \otimes k
\end{array}
$$

2.

$$
\begin{array}{ccccc}
H \otimes H \otimes K & \xrightarrow{\mu \otimes id} & H \otimes K & \xrightarrow{p} & k \\
\Big\downarrow {\scriptstyle id \otimes id \otimes \Delta} & & & & \Big\uparrow \\
H \otimes H \otimes K \otimes K & & & & {\scriptstyle m_k} \\
\Big\downarrow {\scriptstyle id \otimes s \otimes id} & & & & \Big\uparrow \\
H \otimes K \otimes H \otimes K & & \xrightarrow{\qquad p \otimes p \qquad} & & k \otimes k
\end{array}
$$

3.

$$
\begin{array}{ccccc}
 & & H \otimes K & & \\
 & \swarrow {\scriptstyle id \otimes u} & \Big\downarrow {\scriptstyle p} & {\scriptstyle u \otimes id} \searrow & \\
H \otimes k & \xrightarrow{\varepsilon \otimes id} & k & \xleftarrow{id \otimes \varepsilon} & k \otimes K
\end{array}
$$

If there is no danger of confusion we abbreviate $p(h\otimes k)$ *as* $p(h\otimes k) = \langle h,k\rangle$.

The commutativity of the diagrams above can be expressed equationally in the following way: (we use for the following formulae the notation of Sweedler)

1. $\langle h, k.k'\rangle = \sum\langle h_{(1)},k\rangle\langle h_{(2)},k'\rangle$

2. $\langle h.h', k\rangle = \sum\langle h, k_{(1)}\rangle\langle h', k_{(2)}\rangle$

3. $\langle h, 1\rangle = \varepsilon(h)$, $\langle 1, k\rangle = \varepsilon(k)$

It is very easy to check that if H is a finite-dimensional bialgebra the evaluation map is a pairing between H and H^*.

It is well known –see for example [8]– that if H is a finite dimensional coalgebra, any right H-comodule structure in a vector space V produces a left H^*-module structure in the same vector space V.

In the same way if H and K are paired bialgebras and V is an arbitrary vector space, there is a biyective correspondence between left (right) H-comodule structures on V and right (left) K-module structures on the same vector space V. We write explicitly the definitions for the two cases that are needed along this paper.

1. Given $\psi : V \to H\otimes V$ a left comodule structure on V the map: $\overline{\psi} = (id\otimes p)(s\otimes id)(\psi\otimes id) : V\otimes K \to H\otimes V\otimes K \to V\otimes H\otimes K \to V$ is a right K-module structure on V.

   We abbreviate sometimes $\overline{\psi}(v\otimes k)$ as $v \;_{\psi}\!\leftarrow k$ and if there is no danger of confusion we write $v \;_{\psi}\!\leftarrow k$ simply as $v \leftarrow k$.

2. Given $\phi : V \to V\otimes K$ a right comodule structure on V the map: $\overline{\phi} = (p\otimes id)(id\otimes s)(id\otimes \phi) : H\otimes V \to H\otimes V\otimes K \to H\otimes K\otimes V \to V$ is a left H-module structure on V. As before we abbreviate $\overline{\phi}(h\otimes v)$ as $h \rightarrow_{\phi} v$ or simply $h \rightarrow v$.

Next definition will play a very important technical role in what follows.

Definition 1.2 *Assume that* H *and* K *are paired bialgebras. Let be* V *a vector space ,* $\psi : V \to H\otimes V$ *a left* H*-comodule structure on* V *and* $\phi : V \to V\otimes K$ *a right* K*-comodule structure on* V*. We define the map* $\psi\wedge\phi : V\otimes V \to V\otimes V$ *by the following diagram:*

$$
\begin{array}{ccc}
V \otimes V & \xrightarrow{\quad \psi \wedge \phi \quad} & V \otimes V \\
\Big\downarrow {\scriptstyle \psi \otimes \phi} & & \Big\downarrow {\scriptstyle id \otimes p \otimes id} \\
H \otimes V \otimes V \otimes K & \xrightarrow{\quad s \otimes s \quad} & V \otimes H \otimes K \otimes V
\end{array}
$$

In explicit terms one can write that $\psi \wedge \phi = (id \otimes p \otimes id)(s \otimes s)(\psi \otimes \phi)$ and in terms of elements if $\psi(b) = \sum h_i \otimes b_i$ and $\phi(c) = \sum c_j \otimes k_j$: $(\psi \wedge \phi)(b \otimes c) = \sum \langle h_i, k_j \rangle b_i \otimes c_j = \sum b_i \otimes (h_i \rightharpoonup c) = \sum (b \leftharpoonup k_j) \otimes c_j$.

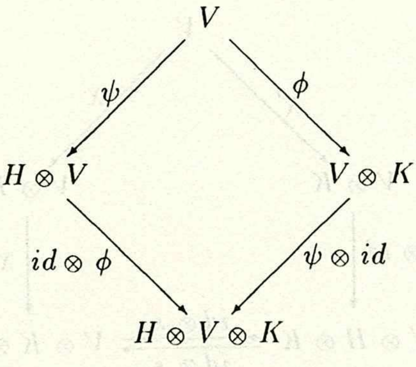
From the last two equalities one sees that the diagrams below commute:

$$
\begin{array}{ccc}
V \otimes V \otimes K & \xrightarrow{\quad id \otimes s \quad} & V \otimes K \otimes V \\
\Big\uparrow {\scriptstyle id \otimes \phi} & & \Big\downarrow {\scriptstyle \overline{\psi} \otimes id} \\
V \otimes V & \xrightarrow{\quad \psi \wedge \phi \quad} & V \otimes V \\
\Big\downarrow {\scriptstyle \psi \otimes id} & & \Big\uparrow {\scriptstyle id \otimes \overline{\phi}} \\
H \otimes V \otimes V & \xrightarrow{\quad s \otimes id \quad} & V \otimes H \otimes V
\end{array}
$$

In other words one has that : $\psi \wedge \phi = (\overline{\psi} \otimes id)(id \otimes s)(id \otimes \phi) = (id \otimes \overline{\phi})(s \otimes id)(\psi \otimes id)$.

For the definition that follows we do not need H and K to be paired.

Definition 1.3 *Assume that V is a vector space and that H and K are bialgebras. Suppose that $\psi : V \to H \otimes V$ and $\phi : V \to V \otimes K$ are left and right comodule structures. We say that ψ and ϕ are compatible if the diagram below is commutative:*



notice that the definition above is equivalent to ask that the map ϕ is a morphism of left H-comodules when we endow V with the structure ψ and $V \otimes K$ with the structure $\psi \otimes id$.

In the case that H and K are paired, the formulae that follow are consequences of the compatibility. Take $b \in V$, $h \in H$, $k \in K$ and write $\psi(b) = \sum h_i \otimes b_i$ and $\phi(b) = \sum B_j \otimes k_j$. Then:

1. $\sum(B_j \leftarrow k_j) = \sum(h_i \rightarrow b_i)$
2. $(h \rightarrow b) \leftarrow k = h \rightarrow (b \leftarrow k)$
3. $\phi(b \leftarrow k) = \sum(B_j \leftarrow k) \otimes k_j$
4. $\psi(h \rightarrow b) = \sum h_i \otimes (h \rightarrow b_i)$

notice that equality 2. says that V is an (H, K)-bimodule, equality 3. that $\overline{\psi}$ is a morphism of K-comodules (when we endow $V \otimes K$ with the structure $(id \otimes s)(\phi \otimes id)$) and equality 4. that $\overline{\phi}$ is a morphism of H-comodules (when we endow $H \otimes V$ with the structure $(s \otimes id)(id \otimes \psi)$).

We will need the notion of compatibility in the case of (for example) two right comodule structures. We give it below.

Definition 1.4 *Assume that V be is a vector space , and that H and K are bialgebras. Suppose $\chi : V \rightarrow V \otimes H$ and $\chi' : V \rightarrow V \otimes K$ are right H and K-comodule structures. We say that χ and χ' are compatible if the following diagram commutes.*

$$
\begin{array}{ccccc}
 & & V & & \\
 & \swarrow^{\chi'} & & \searrow^{\chi} & \\
V \otimes K & & & & V \otimes H \\
\downarrow{\scriptstyle \chi \otimes id} & & & & \downarrow{\scriptstyle \chi' \otimes id} \\
V \otimes H \otimes K & & \underset{id \otimes s}{\overset{id \otimes s}{\rightleftarrows}} & & V \otimes K \otimes H
\end{array}
$$

In a similar way than before if we endow $V \otimes H$ with the structure of right K-comodule given by $(id \otimes s)(\chi' \otimes id) : V \otimes H \to V \otimes H \otimes K$, the condition of compatibility is equivalent to the condition that the map $\chi : V \to V \otimes H$ is a morphism of K-comodules. From the symmetry of the diagram we see that the definition of compatibility is symmetric, i.e. if χ and χ' are compatible so are χ' and χ.

In the case in which H and K are paired, the condition of compatibility implies the same type of relationships than before between the coactions and the actions of H and K. Assume that $b \in V$, $h \in H$, $k \in K$ and write $\chi(b) = \sum b_i \otimes h_i$ and $\chi'(b) = \sum B_j \otimes k_j$. Then:

1. $\sum(k_j \rightharpoonup B_j) = \sum(h_i \rightharpoonup b_i)$
2. $h \rightharpoonup (k \rightharpoonup b) = k \rightharpoonup (h \rightharpoonup b)$.
3. $\chi'(k \rightharpoonup b) = \sum(k \rightharpoonup B_j) \otimes k_j$
4. $\chi(h \rightharpoonup b) = \sum(h \rightharpoonup b_i) \otimes h_i$

Next, we recall the definition of K-comodule algebra, or more generally the definition of compatibility of a "product" and a coaction.

Definition 1.5 *Let K be a bialgebra and V be a right K-comodule with structure ϕ. A k-bilinear map $\mu : V \otimes V \to V$ is said to be compatible with ϕ, or that μ is a ϕ-multiplication if the map $\mu : V \boxtimes V \to V$ is a right K-comodule morphism. If μ is a structure of (associative) k-algebra on V, we say that V is a right K-comodule algebra.*

In explicit terms, μ is a ϕ-multiplication if for $v, w \in V$ and $\phi(v) = \sum v_i \otimes k_i$, $\phi(w) = \sum w_j \otimes k'_j$ it follows that $\phi(v.w) = \sum v_i.w_j \otimes k_i k'_j$. Here we abbreviate $\mu(v \otimes w) = v.w$.

I follows easily from the definition above that if H and K are paired bialgebras, and V is a right K-comodule with a multiplication compatible with ϕ, the associated structure of left H- module on V, that we call $\overline{\phi}$: $H \otimes V \to V$, is compatible with the multiplication μ in the sense that: $\overline{\phi}(h \otimes v.w) = \sum \overline{\phi}(h'_i \otimes v).\overline{\phi}(h''_i \otimes w)$ (or $h \rightharpoonup (v.w) = \sum (h'_i \rightharpoonup v).(h''_i \rightharpoonup w)$) if $\Delta(h) = \sum h'_i \otimes h''_i$.

In a completely similar way one defines the concepts above in the case of left comodule structures. It is also clear that if we take $V = K$ and μ the multiplication of K, then μ is a Δ_r and a Δ_l multiplication.

2 Twisting Products

In this section we consider the following situation. Let H and K be paired bialgebras and suppose that B is an arbitrary k-space equipped with an associative multiplication μ_B and with maps verifying the conditions that follow:

a) A left H-comodule structure $\psi : B \to H \otimes B$.

b) A right K-comodule structure $\phi : B \to B \otimes K$.

c) The map μ_B is a ψ-multiplication.

d) The map μ_B is a ϕ-multiplication.

e) The structures ψ and ϕ are compatible.

Once the maps **a)** and **b)** are given, one can define the twist of μ_B with ψ and ϕ.

Definition 2.1 *The map $\mu_B(\psi \wedge \phi) = \mu_B(id \otimes \overline{\phi})(s \otimes id)(\psi \otimes id) = \mu_B(\overline{\psi} \otimes id)(id \otimes s)(id \otimes \phi)$ is a k-bilinear map from $B \otimes B \to B$ and is called the twist of the multiplication μ_B with the coactions ψ and ϕ and is denoted as* $\mu_{B,\psi,\phi}$

If there is no danger of confusion we write $\mu_B(b\otimes c)=b.c$ and $\mu_{B,\psi,\phi}(b\otimes c)=b\star c$.

In explicit terms one has that if $\psi(b)=\sum_j h_j\otimes \hat{b}_j$ and $\phi(c)=\sum_r \bar{c}_r\otimes k_r$, then: $b\star c=\sum_{j,r}\langle h_j,k_r\rangle \hat{b}_j.\bar{c}_r=\sum_r(b\leftarrow k_r).\bar{c}_r=\sum_j \hat{b}_j.(h_j\rightarrow c)$.

Theorem 2.1 *In the situation above the twisted product $\mu_{B,\psi,\phi}$ is associative.*

Proof : Write $\phi(c)=\sum_r \bar{c}_r\otimes k_r$, $\phi(d)=\sum_x \bar{d}_x\otimes K_x$, $\phi(\bar{d}_x)=\sum_{x'}\bar{d}_{x,x'}\otimes K_{x,x'}$ and $\Delta(K_x)=\sum_{x''}K'_{x,x''}\otimes K''_{x,x''}$. One has that

$$\sum_{x,x''}\bar{d}_x\otimes K'_{x,x''}\otimes K''_{x,x''}=\sum_{x,x'}\bar{d}_{x,x'}\otimes K_{x,x'}\otimes K_x \qquad (\natural)$$

Also $b\star c=\sum_r(b\leftarrow k_r).\bar{c}_r$, $c\star d=\sum_x(c\leftarrow K_x).\bar{d}_x$

Using the compatiblity of both structures we deduce, (see equation 3. after **Definition 1.3**) that $\phi(c\leftarrow K_x)=\sum_r(\bar{c}_r\leftarrow K_x)\otimes k_r$ and so that $\phi(c\star d)=\sum_{x,r,x'}(\bar{c}_r\leftarrow K_x).\bar{d}_{x,x'}\otimes k_rK_{x,x'}$. Hence:

$$b\star(c\star d)=\sum_{x,r,x'}(b\leftarrow k_rK_{x,x'})(\bar{c}_r\leftarrow K_x)\bar{d}_{x,x'}$$

Now, $(b\star c)\star d=\sum_{r,x}(((b\leftarrow k_r)\bar{c}_r)\leftarrow K_x)\bar{d}_x=\sum_{r,x,x''}((b\leftarrow k_r)\leftarrow K'_{x,x'})(\bar{c}_r\leftarrow K''_{x,x''})\bar{d}_x=\sum_{r,x,x''}(b\leftarrow k_rK'_{x,x'})(\bar{c}_r\leftarrow K''_{x,x''})\bar{d}_x$.
So that :

$$(b\star c)\star d=\sum_{r,x,x''}(b\leftarrow k_rK'_{x,x'})(\bar{c}_r\leftarrow K''_{x,x''})\bar{d}_x$$

From equation ($\natural$) it follows immediately that: $(b\star c)\star d=b\star(c\star d)$.

Q.E.D.

Suppose there exists an identity element 1_B for the product μ_B. Assume moreover that $\psi(1_B)=1_H\otimes 1_B$ and $\phi(1_B)=1_B\otimes 1_K$.

In these hypothesis we have that if $b,c\in B$ and $\psi(b)=\sum_j h_j\otimes \hat{b}_j$ and $\phi(c)=\sum_r \bar{c}_r\otimes k_r$, then : $1_B\star c=\sum_r\langle 1_H,k_r\rangle 1_B.\bar{c}_r=\sum_r\varepsilon(k_r)1_B.\bar{c}_r=1_B.(\sum_r\varepsilon(k_r)\bar{c}_r)=1_B.c=c$ and $b\star 1_B=\sum_j\langle h_j,1_K\rangle \hat{b}_j.1_B=\sum_j\varepsilon(h_j)\hat{b}_j.1_B=$

$(\sum_j \varepsilon(h_j)\hat{b}_j).1_B = b.1_B = b$. So that 1_B is also the identity element for the twisted product.

Now we show that the well known construction of the smash product (see [1]) can be viewed as a particular case of the construction above.

Let H and K be as above and let A be an arbitrary right K-comodule algebra with structure χ_A. Consider on the vector space $B = H \otimes A$ the left H-comodule structure $\psi = \Delta \otimes id : H \otimes A \to H \otimes H \otimes A$ and the right K-comodule structure $\phi = id \otimes \chi_A : H \otimes A \to H \otimes A \otimes K$. We endow $H \otimes A$ with the usual tensor product multiplication , i.e., if $h \otimes a, h' \otimes b \in H \otimes A$ we define $(h \otimes a)(h' \otimes b) = hh' \otimes ab$. This component-wise multiplication will be (sometimes) called the convolution product. This is because in the case that the pairing p puts H and K in duality , there is a natural identification of $H \otimes A$ with $Hom_k(K, A)$. Via this identification the product considered above becomes the convolution product in $Hom_k(K, A)$.

It is an easy exercise to show that $H \otimes A$ together with the comodule structures and the product considered above, verify conditions **a**) – **e**) given at the beginning of this section.

If we take $h \otimes a, \overline{h} \otimes b \in H \otimes A$ and consider: $\Delta(h) = \sum h'_i \otimes h''_i$ and $\chi_A(b) = \sum b_j \otimes k_j$ using the formulae that appear immediately before Theorem 2.1. , we see that $(h \otimes a) \star (\overline{h} \otimes b) = \sum \langle h'_i, k_j \rangle h''_i \overline{h} \otimes ab_j$.

If we recall now that the right K-coaction χ_A induces a left H-action on A and that the left H-coaction Δ_l induces a right K-action on H, we can write that:

$$(h\otimes a)\star(\overline{h}\otimes b) = \sum\langle h'_i, k_j\rangle h''_i\overline{h}\otimes ab_j = \sum h''_i\overline{h}\otimes a(h'_i \rightharpoonup b) = \sum(h \leftharpoonup k_j)\overline{h}\otimes ab_j$$

In the case in which we take H to be K^* for some finite dimensional Hopf algebra K, the associative product $\star$ obtained above by the natural twisting of the convolution product, concides –up to a switching of the tensor factors– with what in the literature is known as the smash product. In other words if $\sharp$ denotes the smash product in $K^* \otimes A$ the algebra $(B, \star)$ is isomorphic with $(K^* \otimes A, \sharp)$ (see for example [1] , [4] or [5] for the standard definition and basic properties of the smash product).

3 Iterated Twisting

We start assuming that we are in the situation established at the beginning of **Section 2**. Next theorem guarantees that –under very natural hypothesis– the compatibility of a comodule structure and a multiplication is preserved if we twist the multiplication.

Theorem 3.1 *Let B be as before and assume that L is another Hopf Algebra and $\chi : B \to L\otimes B$ is an additional left L-comodule structure on B compatible with ψ and ϕ. Assume also that the map μ_B is a χ-multiplication. Then $\mu_{B,\psi,\phi}$ is also a χ-multiplication.*

Proof : Write $\chi(b) = \sum_i l_i\otimes b_i$, $\chi(c) = \sum_m L_m\otimes c_m$, $\psi(b) = \sum_j h_j\otimes \hat{b}_j$, $\psi(b_i) = \sum_{i'} h_{i,i'} \otimes \hat{b}_{i,i'}$ and $\chi(\hat{b}_j) = \sum_{j'} l_{j,j'}\otimes b_{j,j'}$. From the compatiblity of ψ and χ we conclude that

$$\sum_{i,i'} h_{i,i'} \otimes l_i \otimes \hat{b}_{i,i'} = \sum_{j,j'} h_j \otimes l_{j,j'} \otimes b_{j,j'} \qquad (\sharp)$$

We have that $\chi(b \star c) = \sum_j \chi(\hat{b}_j)\chi(h_j \rightharpoonup c)$.

By the compatiblity hypothesis we have that : $\chi(h_j \rightharpoonup c) = \sum_m L_m \otimes (h_j \rightharpoonup c_m)$.

Hence: $\chi(b\star c) = \sum_{j,j',m} l_{j,j'}L_m \otimes b_{j,j'}(h_j \rightharpoonup c_m)$.

Now, $\chi(b)\star\chi(c) = \sum_{i,m} l_iL_m \otimes (b_i \star c_m) = \sum_{i,m,i'} l_iL_m \otimes \hat{b}_{i,i'}(h_{i,i'} \rightharpoonup c_m)$.

The equality $\chi(b\star c) = \chi(b)\star\chi(c)$ follows immediately from $(\sharp)$.

Q.E.D.

Hence, we can iterate the constructions of **Section 2** and obtain the Theorem that follows.

Theorem 3.2 *Let B be an associative k-algebra with multiplication μ_B and identity 1_B and let (H_i, K_i), $i = 1,..,n$ be a family of paired Hopf algebras. Assume that B is equipped with:*

a) *A family of left H_i-comodule structures, $\psi_i : B \to H_i \otimes B, i = 1,..,n$*

b) *A family of right K_i-comodule structures $\phi_i : B \to B \otimes K_i, i = 1,..,n$*

such that:

c) *The map μ_B is a ψ_i multiplication for $i = 1, .., n$.*

d) *The map μ_B is a ϕ_i multiplication for $i = 1, .., n$.*

e) *The structures $\{\psi_1, .., \psi_n, \phi_1, .., \phi_n\}$ are compatible.*

If τ is an arbitrary permutation of n elements call μ_τ the following bilinear map on B.

$$\mu_\tau = (...((\mu_{B,\psi_{\tau(1)},\phi_{\tau(1)}})_{\psi_{\tau(2)},\phi_{\tau(2)}})...)_{\psi_{\tau(n)},\phi_{\tau(n)}}$$

Then:

i) *If τ and τ' are two permutations then $\mu_\tau = \mu_{\tau'}$. We call $\tilde{\mu}$ the map defined as the common value of all μ_τ.*

ii) *The map $\tilde{\mu}$ is an structure of k-algebra on B with identity 1_B.*

Proof : The conclusion **ii)** can easily be proved by induction using the results of Section 2. Theorem 3.1. is used in order to guarantee that hypothesis c) and d) are verified at all the stages of the twisting process[1] Hence, only **i)** has to be proved. As an arbitrary permutation is the product of transpositions it is enough to prove the case $n = 2$. We show that

$$(\mu_{B,\psi_1,\phi_1})_{\psi_2,\phi_2} = (\mu_{B,\psi_2,\phi_2})_{\psi_1,\phi_1}$$

Explicitly, the equation above means that

$$\mu_B(\psi_1 \wedge \phi_1)(\psi_2 \wedge \phi_2) = \mu_B(\psi_2 \wedge \phi_2)(\psi_1 \wedge \phi_1)$$

We prove that

$$(\psi_1 \wedge \phi_1)(\psi_2 \wedge \phi_2) = (\psi_2 \wedge \phi_2)(\psi_1 \wedge \phi_1)$$

Substituting $(\psi_i \wedge \phi_i)(\psi_j \wedge \phi_j)$ by their explicit expressions we see that all we have to prove is:

$$(id \otimes p \otimes id)(s \otimes s)(\psi_1 \otimes \phi_1)(id \otimes p \otimes id)(s \otimes s)(\psi_2 \otimes \phi_2) =$$

[1] We used here a right version of Theorem 3.1. that can be proved as before.

$$(id\otimes p\otimes id)(s\otimes s)(\psi_2\otimes\phi_2)(id\otimes p\otimes id)(s\otimes s)(\psi_1\otimes\phi_1) \qquad (\dagger)$$

Call P the map

$$P=(id\otimes p\otimes id)(s\otimes s)(id\otimes id\otimes p\otimes id\otimes id)(id\otimes s\otimes s\otimes id)$$

The left hand side of equality ($\dagger$) can be transformed into:

$$LHS=P(s\otimes id\otimes id\otimes s)(id\otimes\psi_1\otimes\phi_1\otimes id)(\psi_2\otimes\phi_2)$$

The right hand side of equality ($\dagger$) can be transformed into:

$$RHS=P(s\otimes id\otimes id\otimes s)(id\otimes\psi_2\otimes\phi_2\otimes id)(\psi_1\otimes\phi_1)$$

Now, from the compatibility hypothesis we conclude that:

$$(s\otimes id\otimes id\otimes s)(id\otimes\psi_2\otimes\phi_2\otimes id)(\psi_1\otimes\phi_1)=(id\otimes\psi_1\otimes\phi_1\otimes id)(\psi_2\otimes\phi_2)$$

In order to finish the proof of ($\dagger$) all we have to do is to see that:

$$P=P(s\otimes id\otimes id\otimes s)$$

This last equality can be easily verified by a direct computation.
This finishes the proof of part i) and of the Theorem.

Q.E.D.

As an illustration and for future use we write explicitly the value of $\tilde{\mu}(b\otimes c)$ for the case of $n=2$.
Write :
$\psi_1(b)=\sum_{j_1} h^1_{j_1}\otimes b^1_{j_1}$, $\psi_2(b)=\sum_{j_2} h^2_{j_2}\otimes b^2_{j_2}$, $\psi_1(b^2_{j_2})=\sum_{j_{21}} h^{21}_{j_{21}}\otimes b^{21}_{j_{21}}$,
$\psi_2(b^1_{j_1})=\sum_{j_{12}} h^{12}_{j_{12}}\otimes b^{12}_{j_{12}}$, $\phi_1(c)=\sum_{m_1} c^1_{m_1}\otimes k^1_{m_1}$, $\phi_2(c)=\sum_{m_2} c^2_{m_2}\otimes k^2_{m_2}$,
$\phi_1(c^2_{m_2})=\sum_{m_{21}} c^{21}_{m_{21}}\otimes k^{21}_{m_{21}}$, $\phi_2(c^1_{m_1})=\sum_{m_{12}} c^{12}_{m_{12}}\otimes k^{12}_{m_{12}}$.

$$\tilde{\mu}(b\otimes c)=$$

$$\sum_{m_2,m_{21}}((b\;{}_2\!\leftarrow k^2_{m_2})\;{}_1\!\leftarrow k^{21}_{m_{21}}).c^{21}_{m_{21}}=\sum_{m_1,m_{12}}((b\;{}_2\!\leftarrow k^{12}_{m_{12}})\;{}_1\!\leftarrow k^1_{m_1}).c^{12}_{m_{12}}=$$

$$\sum_{j_2,j_{21}} b^{21}_{j_{21}}.(h^{21}_{j_{21}}\rightarrow_1(h^2_{j_2}\rightarrow_2 c))=\sum_{j_1,j_{12}} b^{12}_{j_{12}}.(h^1_{j_1}\rightarrow_1(h^{12}_{j_{12}}\rightarrow_2 c))=$$

$$\sum_{m_1,j_2} (b^2_{j_2}\ {}_1\leftarrow k^1_{m_1})(h^2_{j_2} \rightarrow_2 c^1_{m_1}) = \sum_{j_1,m_2} (b^1_{j_1}\ {}_2\leftarrow k^2_{m_2})(h^1_{j_1} \rightarrow_1 c^2_{m_2})$$

notice that, as we observed before, in the first four of the expressions above we can change the order of the iterated actions.

Moreover any of these expresions coincide with the value :

$$\tilde{\mu}(b\otimes c) = \sum_{j_2,j_{21},m_2,m_{21}} \langle h^{21}_{j_{21}}, k^{21}_{m_{21}}\rangle \langle h^2_{j_2}, k^2_{m_2}\rangle b^{21}_{j_{21}} c^{21}_{m_{21}}$$

and similar ones that are deduced from the compatibility conditions.

The mechanics of the formula that yields the doubly twisted product becomes clearer if we introduce the maps $\psi_{12} = (id\otimes\psi_1)\psi_2 : B \rightarrow H_2\otimes B \rightarrow H_2\otimes H_1\otimes B$ and $\phi_{12} = (\phi_1\otimes id)\phi_2 : B \rightarrow B\otimes K_2 \rightarrow B\otimes K_1\otimes K_2$ and write in a manner that mimics Sweedler's sigma notation: $\psi_{12}(b) = \sum h_{(2)}\otimes h_{(1)}\otimes b_{(0)}$, $\phi_{12}(c) = \sum c_{(0)}\otimes k_{(1)}\otimes k_{(2)}$. Then:

$$b\star c = \sum \langle h_{(2)}, k_{(2)}\rangle \langle h_{(1)}, k_{(1)}\rangle b_{(0)}c_{(0)}$$

4 The Quantum double as a twist of the convolution product

As it is explained in [2] , if K is a finite dimensional Hopf algebra, a new Hopf algebra based on the vector space $K^*\otimes K$ can be constructed in terms of the structure maps of K. This new Hopf algebra is called $D(K)$ and it is proved in the mentioned paper that $D(K)$ is a quasi-triangular Hopf algebra (see [2] for the definition of these concepts) and in particular that there exists an element $R\in D(K)\otimes D(K)$ that is a solution of the Quantum Yang Baxter Equation.

In this section we show that the multiplication defined by Drinfeld in $K^*\otimes K = D(K)$ can be considered as a double twist of the convolution product in $End(K)$ or equivalently of the componentwise product in $K^*\otimes K$.

We start by recalling Drinfeld's definition. Consider in the vector space $K^*\otimes K$ the following multiplication. If $f\otimes a$ and $g\otimes b$ are two elements of $K\otimes K^*$ write : $\Delta^2(b) = \sum_l b_{1,l}\otimes b_{2,l}\otimes b_{3,l}$, $\Delta^{*2}(f) = \sum_m f_{1,m}\otimes f_{2,m}\otimes f_{3,m}$. Define then :

$$(a\otimes f)\flat(b\otimes g) = \sum_{l,m} f_{3,m}(\sigma^{-1}(b_{1,l}))f_{1,m}(b_{3,l})f_{2,m}g\otimes ab_{2,l}$$

Endowed with this product $K^* \otimes K$ becomes an associative algebra (and a bialgebra with respect to the comultiplication given by the tensor product of the comultiplications Δ on K and $s\Delta^*$ on K^*). This bialgebra is in fact a Hopf algebra and is the one that in [2] Drinfeld denotes as $D(K)$ and calls the Quantum Double (and later by other people the Drinfeld double) of K.

The formula above is not the one used by Drinfeld to define the multiplication. He defined the multiplication in terms of the products of the elements of a cannonical basis $e_i^* \otimes e_j$ of $K^* \otimes K$.

It is an easy exercise to show that the formula above coincides with the one given in [2].

notice also that Drinfeld worked with $K \otimes K^*$ instead of $K^* \otimes K$. We adopted the order given secondly only because some of the coalgebra structures are easier to write.

Before proving that the product just defined coincides with a double twist of the convolution we make some general comments.

Suppose that B is al algebra and L is a bialgebra. Assume that $\delta : B \to B \otimes L$ is a right L-comodule structure and that with respect to this structure B is an L-comodule algebra. If we consider $s\delta : B \to L^{cop} \otimes B$ one sees that with respect to this structure B is an L^{cop}-comodule algebra. Assume moreover that there exists a k-map $\alpha : L \to L$ that is simultaneosly an antimorphism of algebras and of coalgebras. If we consider $(id \otimes \alpha)\delta : B \to B \otimes L^{op,cop}$, one sees that $(id \otimes \alpha)\delta$ us a right $L^{op,cop}$- comodule structure and that with respect to this structure B is an $L^{op,cop}$-comodule algebra.

Of course all these considerations -with the evident changes- are valid if δ is a left L-comodule structure.

Observe that if (H, K) is a couple of paired bialgebras, then (H^{op}, K^{cop}), (H^{cop}, K^{op}) and $(H^{op,cop}, K^{op,cop})$ are also paired. If L is a bialgebra we write $L^{op,cop}$ as L'

Assume now, that K is a bialgebra and that a A is a k-algebra. Suppose that there is a pair of compatible comodule structures $\xi : A \to K' \otimes A$ and $\chi : A \to A \otimes K$. Assume also that the multiplication of A is compatible with both ξ and χ.

Since ξ and χ are compatible we have that $(\xi \otimes id)\chi = (id \otimes \chi)\xi : A \to K' \otimes A \otimes K$. We call this map (ξ, χ).

Now, assume that we also have a bialgebra H such that H and K are paired, and endow $H \otimes A$ with the comodule structures given below.

$$\psi = (\Delta_l \otimes id) : H \otimes A \rightarrow H \otimes H \otimes A$$

$$\phi = id \otimes \chi : H \otimes A \rightarrow H \otimes A \otimes K$$

$$\psi' = (s \otimes id)(\Delta_r \otimes id) : H \otimes A \xrightarrow{\Delta_r \otimes id} H \otimes H \otimes A \xrightarrow{s \otimes id} H^{cop} \otimes H \otimes A$$

$$\phi' = (id \otimes s)(id \otimes \xi) : H \otimes A \xrightarrow{id \otimes \xi} H \otimes K' \otimes A \xrightarrow{id \otimes s} H \otimes A \otimes K^{op}$$

If we endow $H \otimes A$ with the componentwise multiplication and call it μ, the system consisting of $(H \otimes A, \mu, \psi, \phi, \psi', \phi')$ is in the situation of the hypothesis of Theorem 3.2. The compatibility of (ψ, ψ') follows from the coassociativity of Δ and the compatibility of (ϕ, ϕ') follows from the compatibility of ξ and χ. The rest of the verification is trivial.

By Theorem 3.2. twisting μ with the two pair of structures given above, we obtain a new associative product $\star$ on $H \otimes A$ that has as unit the element $1 \otimes 1$. If we write :

$$\Delta^2(h) = \sum h_{(1)} \otimes h_{(2)} \otimes h_{(3)} \in H \otimes H \otimes H$$

$$(\xi, \chi)(b) = \sum k_{(1)} \otimes b_{(2)} \otimes k_{(3)} \in K' \otimes A \otimes K$$

using the formulae of Section 3 we obtain that :

$$(h \otimes a) \star (k' \otimes b) = \sum \langle h_{(1)}, k_{(3)} \rangle \langle h_{(3)}, k_{(1)} \rangle h_{(2)} k' \otimes a b_{(2)}$$

We particularize now to the case that $A = K$ and that K is a Hopf algebra with invertible antipode σ. Assume also that $\chi : A \rightarrow A \otimes K$ is $\Delta : K \rightarrow K \otimes K$ and that $\xi : A \rightarrow K' \otimes A$ is $(\sigma^{-1} \otimes id)\Delta : K \rightarrow K' \otimes K$. It is immediate to verify that Δ and $(\sigma^{-1} \otimes id)\Delta$ are compatible comodule structures and that K equipped with its multiplication is a Δ-comodule algebra and also a $(\sigma^{-1} \otimes id)\Delta$-comodule algebra. For the verification of the last assertion one has to recall that σ^{-1} is an antimorphism of algebras and of coalgebras.

In accordance with the general situation if we consider the structures:

$$\psi = \Delta \otimes id : H \otimes K \rightarrow H \otimes H \otimes K$$

$$\phi = id \otimes \Delta : H \otimes K \to H \otimes K \otimes K$$

$$\psi' = (s \otimes id)(\Delta \otimes id) : H \otimes K \overset{\Delta \otimes id}{\to} H \otimes H \otimes K \overset{s \otimes id}{\to} H^{cop} \otimes H \otimes K$$

$$\phi' = (id \otimes s)(id \otimes (\sigma^{-1} \otimes id)\Delta) : H \otimes K \overset{id \otimes \Delta}{\to} H \otimes K \otimes K \overset{id \otimes \sigma^{-1} \otimes id}{\to} H \otimes K' \otimes K \overset{id \otimes s}{\to} H \otimes K \otimes K^{op}$$

and twist the component wise product in $H \otimes K$ with these structures we obtain a new associative product $\star$.

Observe that $(\xi, \chi) = (\xi \otimes id)\chi = (\sigma^{-1} \otimes id \otimes id)(\Delta \otimes id)\Delta = (\sigma^{-1} \otimes id \otimes id)\Delta^2$.

Hence:

$$(h \otimes k) \star (h' \otimes k') = \sum \langle h_{(1)}, k'_{(3)} \rangle \langle h_{(3)}, \sigma^{-1}(k'_{(1)}) \rangle h_{(2)} h' \otimes k k'_{(2)}$$

if $\Delta^2(h) = \sum h_{(1)} \otimes h_{(2)} \otimes h_{(3)}$ and $\Delta^2(k') = \sum k'_{(1)} \otimes k'_{(2)} \otimes k'_{(3)}$.

Suppose now that K is a finite dimensional Hopf algebra and that $H = K^*$. Consider the pairing given by the evaluation. It is obvious that in this case the formula for $\star$ coincides with the formula for $\flat$. We have in fact proved the following theorem.

Theorem 4.1 *Assume that K is a finite dimensional Hopf algebra. If $\star$ is the product on $K^* \otimes K$ obtained by twisting the convolution product with respect to the structures $\{\Delta^* \otimes id, id \otimes \Delta, (s \otimes id)(\Delta^* \otimes id), (id \otimes s)(id \otimes \sigma^{-1} \otimes id)(id \otimes \Delta)\}$, then $D(K) \cong (K^* \otimes K, \star)$.*

We finish this section with a couple of observations.

a) At the end of **Section 2.** we showed that the smash product $K^* \sharp K$ is the twist of the convolution product with respect to the structures $\{\Delta^* \otimes id, id \otimes \Delta\}$. Above we showed that if we twist the smash product with respect to $\{(s \otimes id)(\Delta^* \otimes id), (id \otimes s)(id \otimes \sigma^{-1} \otimes id)(id \otimes \Delta)\}$ we obtain the product defined by Drinfeld in $K^* \otimes K$. In this sense, one can think of the construction of the Quantum double as the result of a two step iteration of the same twisting process . At the zero step we obtain the convolution product, at the first the smash product and at the second the product defined by Drinfeld in $D(K)$.

b) In the papers [6] and [7], Majid presented the Quantum Double as a particular case of the general construction of the "double cross product of Hopf algebras". The generalization presented above goes along diferent lines and responds to the following preocupation: to put some order into the multiplicity of products that were appearing on $K \otimes K^*$ (convolution, smash, and Drinfeld's).

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