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CLUSTERS, PROXIMITY INEQUALITIES AND ZARISKI-LIPMAN COMPLETE IDEAL THEORY*

by Antonio CAMPILLO, Gérard GONZALEZ-SPRINBERG et Monique LEJEUNE-JALABERT

Introduction

The starting point of the topic is a result concerning linear systems of plane algebraic curves with assigned base points which was published in 1915 in [E.C]. In Chap. 2, book 4, F. Enriques and O. Chisini discuss the following problem : does there exist plane algebraic curves which pass through an assigned set of infinitely near points of the plane with assigned multiplicities ? They prove that some numerical inequalities, the so called "proximity inequalities" are necessary and sufficient for the existence of curves with the required property (at least if no condition is imposed on their degree).

Almost twenty years later, O. Zariski begins a systematic study of "complete ideals". In dimension two, these ideals adequately describe complete linear systems defined by infinitely near base conditions. One of the main results of the theory is that any complete ideal in a regular two-dimensional local ring has a unique factorization into simple complete ideals [Z.S]. It turns out that the exponents which appear in the factorization are easily computed from the proximity inequalities for the corresponding linear system (see [L3], [LJ]).

These results do not extend directly to higher dimensional case and actually no substantial progress was achieved during fifty years. It is only recently that, by allowing factors with negative exponents, J. Lipman was able to recover a unique factorization statement [L2]. The result holds for finitely supported complete ideals in a regular local ring R of any dimension. This condition means that the ideal is supported at the closed point and that there exists a finite succession of point blowing-ups $\sigma_1, \ldots, \sigma_n$ such that its inverse image by $\sigma_1 \circ \cdots \circ \sigma_n$ is locally principal. Roughly speaking the only infinitely near fixed sub-varieties of the corresponding linear system are closed points. As in dimension two, the special $*$-simple ideals admitted as factors are in one to one

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* Preliminary version 1993.

correspondence with finite chains of infinitely near points. This paper is also devoted to the study of finitely supported complete ideals. It may be considered as an introduction to Zariski-Lipman's theory. Following the original view point of the italian school, we put the emphasis on the geometrical side. In fact, we only consider ideals in the local ring of a point O on a non singular algebraic variety X defined over an algebraically closed field. This allows us to make use of intersection theory and makes appear connections with more modern developments on the study of birational morphisms and the minimal model program.

In § 1, first we introduce some terminology. The given set of assigned base points (a "constellation") with assigned multiplicities $\underline{m}$ is called a cluster. Now the question is to characterize those clusters $\mathcal{A}$ for which the set of hypersurfaces which pass through the given base points with the given multiplicities (for short, which pass effectively through $\mathcal{A}$) is not empty and has no other base points. In dimension greater than two, imposing isolated base points may force the linear system to have fixed subvarieties of positive dimension. Next we express some previous results given in [L1] and [L2] as a dictionary between finitely supported complete ideals and clusters with the above property.

Actually, to each cluster $\mathcal{A}$, we associate a proper birational morphism $\pi : Z \to X$ and a Cartier divisor $D(\mathcal{A})$ on Z ; in proposition 1.2.7, we alternatively characterize them by saying that $-D(\mathcal{A})$ is π-generated *i.e.* is generated by its global section on a neighborhood of $\pi^{-1}(O)$. As a consequence, we get some polynomial inequalities on $\underline{m}$ which hold for these special clusters. By a theorem of Kleiman the linear ones imply those of greater degree. An equivalent formulation is that $-D(\mathcal{A})$ is π-nef (*i.e.* $D(\mathcal{A}) \cdot V \leq 0$ for any irreducible curve V contracted by π).

If the dimension of X is two, these inequalities are nothing but the proximity inequalities of [E.C] and actually they provide the wanted characterization. If the dimension of X is at least three, usually this is no longer true (example 1.3.9). Nevertheless, it remains true if the cluster is provided with a toric action. This is applied to discuss the factorization of finitely supported complete monomial ideals into special $*$-simple factors in another paper. The cone of effective projective curves contracted by π, $NE(Z/X)$, appears to play an essential role in this discussion.

In § 2, we fix a cluster coming from a finitely supported ideal and we analyze further the geometry of the complete linear system $\mathfrak{s}(\mathcal{A})$ defined on the germ (X, O). For each base point Q, we get a linear system with assigned base points of hypersurfaces of a given degree, $\mathfrak{b}_Q(\mathcal{A})$, on the exceptional divisor of the blowing-up of Q. This linear system of projective hypersurfaces may not be complete. From this construction, we derive some conditions on $\mathcal{A}$ which are strictly stronger than the proximity inequalities when the dimension of X is at least three (2.4.1). We also derive various explicit examples and counter-examples (2.4.2–2.4.4).

The main application of this analysis is given in § 3. If the characteristic of the ground field is zero, we prove as a corollary of Bertini's theorem, that the canonical process of eliminating base points of $\mathfrak{s}(\mathcal{A})$ by successive point blowing-ups is an embedded resolution of any complete intersection defined by r, $1 \leq r < \dim X$, "general enough" hypersurface germs in $\mathfrak{s}(\mathcal{A})$ (*i.e.* at the last step, its total transform is a scheme having only normal crossings). This is a partial generalization of the desingularization process of a hypersurface singularity which is non-degenerate with respect to its Newton polyhedron $\mathcal{N}$ (e.g. [V]). The complete ideal to consider here is the one generated by all the monomials in $\mathcal{N}$. It may not be finitely supported. On the other hand, there exist surface singularities in C^3 for which this process provides an embedded resolution but which are degenerate with respect to their Newton polyhedron in any coordinate system. A further study of these surface singularities seems attractive.

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1. Constellations, clusters, proximity inequalities and finitely supported complete ideals

Throughout this paper, an algebraic variety will mean a reduced and irreducible scheme of finite type over an algebraically closed field K. A point will mean a closed point.

From now on, X will denote a non singular algebraic variety of dimension $d \geq 2$ and O will be a point on X. In the sequel, we consider various birational morphisms. The subset of the source where the morphism is not an isomorphism will be called its exceptional locus. An exceptional subvariety will be a subvariety of the exceptional locus.

1.1.1. Definition. — Any point Q on any variety Z, obtained from X by a finite succession of point blowing-ups, is called an infinitely near point of X. If O is the image of Q, we say that *Q is infinitely near O.*

1.1.2. DEFINITION. — *A constellation (of infinitely near points of X) with origin at O* consists of a finite set of points infinitely near O, $C = \{Q_0, Q_1, \ldots, Q_n\}$, where $n \geq 0$, $Q_0 = O$ and each Q_i, $i = 1, \ldots, n$ is a point on the variety Z_i obtained from Z_{i-1} by blowing up Q_{i-1}, ($Z_0 = X$).

We call the variety $Z := Z_{n+1}$ the sky of C.

Let $\pi : Z \to X$ be the composition $\sigma_1 \circ \sigma_2 \circ \ldots \circ \sigma_{n+1}$ where $\sigma_i : Z_i \to Z_{i-1}$ denotes the blowing-up with center Q_{i-1} and let $\pi_1 : Z \to Z_1 =: X_1$ be the composition $\sigma_2 \circ \ldots \circ \sigma_{n+1}$. The image by π_1 of its exceptional locus is a finite subset C_1 of points of X_1. Obviously $Q_1 \in C_1$. Let $p_2 : X_2 \to X_1$ denote the blowing-up with center C_1. By the universal property of blowing-ups, there exists a unique morphism $\pi_2 : Z \to X_2$ factoring π_1. For each $Q \in C_1$ distinct from Q_1, there exists a unique i, $2 \leq i \leq n$, such that Q_i is a point going to Q in the open set where $\sigma_2 \circ \ldots \circ \sigma_i : Z_i \to Z_1$ is an isomorphism. By identifying Q and Q_i, we may view C_1 as a subset of C.

If $C \neq C_1 \cup \{Q_0\}$, let $p_3 : X_3 \to X_2$ be the blowing-up with center the image C_2 by π_2 of its exceptional locus. As above C_2 is a finite non empty subset of X_2. Let $Q \in C_2$, then $p_2(Q)$ is a point of C_1 which is identified with some $Q_i \in Z_i$. The varieties X_1 and Z_i are locally canonically identified at these points, hence there exists a unique Q_j with $i < j$ corresponding to Q such that Z_j and X_2 are locally canonically identified at these points ; thus the set C_2 may be viewed as a subset of C disjoint from $C_1 \cup \{Q_0\}$. Finally, by induction and with $C_0 = \{Q_0\}$, we get a partition $C = C_0 \cup C_1 \cup \ldots \cup C_t$.

$$
\begin{tikzcd}
X_{t+1} \arrow[d] & & & & Z_{n+1} = Z \arrow[llll, "\sim"'] \arrow[dddlll, "\pi_1"] \arrow[ddlll, "\pi_2"'] \arrow[dd, dashed, "\pi"] \\
X_t \arrow[dr, dashed] & & & & \\
& C_2 \subset X_2 \arrow[d] & & & Z_1 \arrow[dl] \\
& C_1 \subset X_1 \arrow[rr, equal] \arrow[dr] & & Z_1 \arrow[dl] & \\
& & X = Z_0 & &
\end{tikzcd}
$$

If $Q_i \in C_l$, we call the integer $l = l(Q_i)$ the *level* of Q_i. Using classical language, we also say that Q_i belongs to the l^{th} infinitesimal neighborhood of O. After relabeling the Q_i if necessary, one may assume that $l(Q_j) > l(Q_i)$ implies that $j > i$. (Note that by doing so, one may modify the σ_i, hence the factorization of π).

If $\ell(Q_n) = n$, i.e. if $\sigma_i(Q_i) = Q_{i-1}$, $1 \leq i \leq n$, we say that $C = \{Q_0, \ldots, Q_n\}$ is a chain.

For each $Q = Q_i \in C$, we denote by B_Q (or B_i) the exceptional divisor of the blowing-up σ_{i+1} of Q_i, and by E_Q (or E_i) its strict transform on any of the Z_h, $i + 1 < h \leq n + 1$ as well as B_Q (or B_i).

1.1.3. — We associate a tree Γ (with Q_0 as the root) to the constellation C in the following way : the vertices of Γ are in one to one correspondence with the points $Q_0, \ldots, Q_n$ and the edges with the pairs (Q_i, Q_j) such that $l(Q_j) = l(Q_i)+1$ and $Q_j \in B_i$. Clearly Q_j is infinitely near Q_i if and only if either $Q_j = Q_i$ or $l(Q_j) > l(Q_i)$ and the corresponding points on Γ are connected by a going down sequence of edges. If this is so, we write $Q_j \geq Q_i$.

1.1.4. — We say that Q_j is *proximate to* Q_i, and write $Q_j \to Q_i$, (or $j \to i$) if $Q_j \in E_i$. If $d = \dim X \geq 3$, one has $Q_j \to Q_i$ if and only if $j > i$ and $E_i \cap E_j \neq \emptyset$. Furthermore, if $\mathcal{J} = \{i_1 < \cdots < i_k\}$ is a subset of $\{0, \ldots, n\}$ with $1 \leq k \leq d - 1$, the following conditions are equivalent :

(i) $E_{i_1} \cap \cdots \cap E_{i_k} \neq \emptyset$

(ii) $i_j \to i_\ell$, $1 \leq \ell < j \leq k$

(iii) $i_k \to i_\ell$, $1 \leq \ell < k$.

Indeed, if (iii) holds, then $Q_{i_k} \in E_{i_1} \cap \cdots \cap E_{i_{k-1}}$ (in Z_{i_k}), therefore $E_{i_1} \cap \cdots \cap E_{i_k} \neq \emptyset$ in Z_{i_k+1} and in Z. If $\mathcal{J}$ satisfies any of these conditions, we say that $\{Q_{i_1}, \ldots, Q_{i_k}\}$ or simply $\mathcal{J}$ *is completely self-proximate.* We may extend this definition without change to the case $d = 2$ since for $k = 1$, conditions (i), (ii) and (iii) hold trivially. In any case, if $\mathcal{J}$ is completely self-proximate, the constellation $\{Q_{i_1}, \ldots, Q_{i_k}\}$ is a chain originated at Q_{i_1} of at most $d - 1$ points.

The proximity relations among points of C are conveniently represented by mean of the $(n + 1) \times (n + 1)$ matrix $M = (\mu_{ij})$ given by $\mu_{ii} = 1$, $\mu_{ij} = -1$ if $i \to j$ and $\mu_{ij} = 0$ otherwise. We call it the *proximity matrix* of C. It was first introduced by Du Val in [DV] and it appeared further in [D], [Ca], [LJ], [L3].

1.1.5. — Let $\mathbb{E} := \bigoplus \mathbb{Z}E_Q$ be the free group of divisors with exceptional support on Z and for each $Q = Q_i \in C$, let E_Q^* (or E_i^*) be the total transform of B_Q (or B_i) in Z ; obviously $E_i^* = E_i \bmod \left(\sum_{j>i} \mathbb{Z}_{\geq 0} E_j\right)$, where $\mathbb{Z}_{\geq 0} = \{n \in \mathbb{Z} \mid n \geq 0\}$. Hence both $E := (E_0, \ldots, E_n)$ and $E^* := (E_0^*, \ldots, E_n^*)$ are $\mathbb{Z}$-basis of $\mathbb{E}$. In fact, we have

1.1.6. PROPOSITION. — *For each $Q \in C$, $E_Q = E_Q^* - \sum_{P \to Q} E_P^*$ in $\mathbb{E}$. In other words, viewing formally E and E^* as row matrices, $E = E^* M$.*

Proof. — We proceed by induction on the number of points in C. Let $Q = Q_i$, $0 \leq i \leq n$. If $n = 0$ or if $i = n$, it is obvious since in both cases $E_Q = E_Q^*$ and no points in C are proximate to Q. If $0 \leq i < n$, let $\widetilde{C} = \{Q_0, \ldots, Q_{n-1}\}$. By the inductive hypothesis, one has

$$E_Q = E_Q^* - \sum_{P \to Q,\ P \neq Q_n} E_P^*$$

in the free group of divisors with exceptional support on Z_n. Here E_Q, E_Q^*, E_P^* mean respectively the strict and total transform of B_Q and the total transform of B_P on Z_n. Now the total transform of E_Q on $Z = Z_{n+1}$ is E_Q if $Q_n \notin E_Q$, or $E_Q + B_{Q_n}$ if $Q_n \in E_Q$, i.e., $Q_n \to Q$, where now E_Q means the strict transform of B_Q on Z. This completes the proof.

1.1.7. DEFINITION. — *A cluster of infinitely near points of X with origin at O* consists of a constellation $C = \{Q_0, \ldots, Q_n\}$ together with a "column vector" of non-negative integers $\underline{m} = {}^t(m_0, \ldots, m_n)$. The integer m_j is called the *weight* (or *virtual multiplicity*) of Q_j in the cluster.

1.1.8. — We associate to each cluster $\mathcal{A} = (C, \underline{m})$ the divisor with exceptional support on Z, $D(\mathcal{A}) = \sum m_i E_i^*$. From 1.1.6, it follows that $D(\mathcal{A}) = \sum d_i E_i$ with $\underline{d} = {}^t(d_0, \ldots, d_n) = M^{-1}\underline{m}$.

1.2.1. DEFINITION. — An ideal I in $R := \mathcal{O}_{X,O}$ is finitely supported if I is primary for the maximal ideal M of R and if there exists a constellation C of infinitely near points of X with origin at O such that $I\mathcal{O}_Z$ is an invertible sheaf, where Z is the sky of C.

Before giving the next definition, we need to introduce some additional notations. For any point Q infinitely near O, let R_Q be the local ring of Q on the space on which

Q lies and let M_Q be its maximal ideal. For $0 \neq f \in R_Q$, we set

$$\operatorname{ord}_Q f = \max \{ n \mid f \in M_Q^n \}$$

Now if I is a non-zero ideal in R_Q, we set

$$\operatorname{ord}_Q I = \min \{ \operatorname{ord}_Q f \mid f \in I \setminus (0) \}.$$

1.2.2. DEFINITION. — To each finitely supported ideal I in $\mathcal{O}_{X,O}$ we associate a cluster of infinitely near points with origin at O, $\mathcal{A}_I = (C_I, \underline{m})$ in the following way :

(1) $C_I = \{Q_0, \ldots, Q_n\}$ is the minimal constellation (i.e., with the minimal number of points) such that $I\mathcal{O}_Z$ is invertible ; $Q_0, \ldots, Q_n$ are called the *base points* of I and C_I is the constellation of base points of I.

(2) The weights of $\mathcal{A}_I$ and the weak transforms $I_{Q_i} = I_i$ of I at Q_i, $0 \leq i \leq n$, are defined simultaneously by induction on i by setting :

(i) $I_0 = I$, $m_0 = \operatorname{ord}_{Q_0} I_0$

(ii) for $Q_i \in X_l$ going to $Q_j \in X_{l-1}$ $(1 \leq l \leq t)$,

$I_i = (x)^{-m_j} I_j \, \mathcal{O}_{X_l, Q_i}$, $m_i = \operatorname{ord}_{Q_i} I_i$, where $x = 0$ is a local equation of B_j at Q_i.

(Recall that $\pi_l : X_l \to X_{l-1}$ coincides with the blowing-up with center Q_j on a neighborhood of the exceptional fiber of Q_j). Clearly the weights in $\mathcal{A}_I$ are strictly positive integers and the weak transforms $I_0, \ldots, I_n$ are finitely supported ideals. The notion of finitely supported ideal was introduced by Lipman in [L2], def. 1.8, 1.20.

1.2.3. DEFINITION. — Let I be a finitely supported ideal, C_I its associated constellation and Z the sky of C_I. We associate to I an effective divisor with exceptional support on Z :

$$D_I = \sum_Q d_Q \, E_Q = \sum_{0 \leq i \leq n} d_i \, E_i$$

by setting $I\mathcal{O}_Z = \mathcal{O}_Z(-D_I)$.

Recall that by definition of C_I and Z, $I\mathcal{O}_Z$ is an invertible sheaf. It follows immediately from 1.1.8 and from definitions 1.2.2 and 1.2.3 that

1.2.4. LEMMA. — *One has* $D(\mathcal{A}_I) = D_I$.

Proof. — Indeed, $I\mathcal{O}_Z = \mathcal{O}_Z(-\sum m_Q \, E_Q^*)$ on the sky Z of C_I.

1.2.5. DEFINITION. — R being any commutative ring and I being an ideal in R, an element $f \in R$ is integral over R if f satisfies a condition of the form

$$f^s + g_1 f^{s-1} + \ldots + g_s = 0 \quad , \quad g_j \in I^j \quad , \quad 1 \leq j \leq s$$

The set of all such f, denoted $\overline{I}$, is called the integral closure or completion of I ; the completion $\overline{I}$ is itself an ideal. An ideal I is integrally closed or *complete* if $I = \overline{I}$ ([Z.S], [L2]).

Recall that if $d = \dim X = 2$, any product of complete ideals in $\mathcal{O}_{X,O}$ is again complete ([Z.S]). This is no longer true for $d \geq 3$. For any two ideals, I, J in $\mathcal{O}_{X,O}$, the $*$-*product* of I and J, denoted $I * J$, is defined to be the completion of IJ ([L2], Def.1.13). A complete ideal K in $\mathcal{O}_{X,O}$ is said to be $*$-*simple* (simple if $d = 2$) if whenever $K = I * J$ with ideals I and J either $1 \in I$ or $1 \in J$.

1.2.6. DEFINITION. — Let $\pi : Z \to X$ be a proper morphism onto a variety X. A divisor D on Z is said to be π-*generated* if the natural homomorphism $\pi^*\pi_*\mathcal{O}_Z(D) \to \mathcal{O}_Z(D)$ is surjective.

For instance, if π is birational and D is exceptional, this condition means that $\mathcal{O}_Z(D)$ is generated by its global sections on a neighborhood of the support of D.

1.2.7. PROPOSITION. — *Let α be the map of sets*

$$\alpha : \left\{ \begin{array}{l} \textit{finitely supported} \\ \textit{complete ideals in } \mathcal{O}_{X,O} \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \textit{clusters of infinitely near} \\ \textit{points with origin at } O \end{array} \right\}$$

which takes I to $\mathcal{A}_I$

i) *The map α is injective.*

ii) *The image of α consists of those clusters $\mathcal{A} = (C, \underline{m})$ for which $\underline{m} > 0$ (i.e. $m_Q \neq 0$, $\forall Q \in C$) and $-D(\mathcal{A})$ is π-generated, where $\pi : Z \to X$ is the canonical map from the sky of C to X.*

Proof. — The first assertion is proposition (1.10) of [L2]. Here is an alternative proof. In fact, one can recover I from $\mathcal{A}_I$. Indeed, since the canonical map $\pi : Z \to X$ from the sky of C_I to X is a proper birational map , X is non singular and I is complete, then I is the stalk at O of $\pi_*(I\mathcal{O}_Z)$ [L1], prop 6.2. From definition 1.2.3, and lemma 1.2.4, we get that

$$I\mathcal{O}_Z = \mathcal{O}_Z(-D_I) = \mathcal{O}_Z(-D(\mathcal{A}_I))$$

hence $-D(\mathcal{A}_I)$ is π-generated.

The proof of the second assertion follows [L1], § 18. Let $\mathcal{A} = (C, \underline{m})$ be a cluster and assume that $-D := -D(\mathcal{A})$ is π-generated. This means that I being the stalk of $\pi_*(\mathcal{O}_Z(-D))$ at O, $I\mathcal{O}_Z = \mathcal{O}_Z(-D)$. Hence by 1.2.1, I is finitely supported and by 1.2.2 the constellation of base points C_I of I is contained in C. The variety Z dominates the sky of C_I and $D = \sum m_Q E_Q^*$ is the total transform of $D_I = D(\mathcal{A}_I)$ on Z. Therefore by 1.1.8, $\underline{m}$ is obtained from the weight vector $\underline{m}_I$ of $\mathcal{A}_I$ by adding 0 for those $Q \in C \setminus C_I$. Since $m_I > 0$, $\underline{m} > 0$ implies that $\mathcal{A} = \mathcal{A}_I$. Now, by [L1] lemma 5.3, I is complete. Indeed Z is non singular and $\mathcal{O}_Z(-D)$ being invertible, is complete. So $\mathcal{A}$ is in the image of α.

1.2.8. Definition. — Given a constellation C with origin at O, the set of clusters whose constellation is contained in C and belong to the image of α is called the *galaxy* of C.

1.2.9. Remark. — Note that given C, its galaxy $\mathcal{G}$ has a natural structure of commutative monoid. Indeed $\mathcal{A} = \alpha(I)$ belongs to $\mathcal{G}$ if and only if, Z being the sky of C, $I\mathcal{O}_Z$ is invertible. Therefore, if $\mathcal{A}_i = \alpha(I_i)$, $i = 1, 2$, are in $\mathcal{G}$, $\alpha(I_1 * I_2)$, still belongs to it, since $I_1 I_2 \mathcal{O}_Z$ is invertible and by [L2], prop. 1.10, the inverse image on Z of $I_1 I_2$ and of its completion $I_1 * I_2$ coincide and we may set $\mathcal{A}_1 + \mathcal{A}_2 := \alpha(I_1 * I_2)$. The weight of Q in $\mathcal{A}_1 + \mathcal{A}_2$ is the sum of its weights in $\mathcal{A}_1$ and $\mathcal{A}_2$. As a consequence, there exist clusters $\mathcal{A} = (C, \underline{m}) \in \mathcal{G}$ for which $m_Q > 0$ for any $Q \in C$. For each maximal point P of C, consider the special $*$-simple ideal I_P associated to the descending chain from P to O ([L2], prop. 2.1). and let I be the $*$-product of the I_p's for all such maximal P. By 1.2.2, the weights in each $\alpha(I_p)$ are strictly positive integers, hence for any $Q \in C$, the weight of Q in $\alpha(I)$ is non zero. Finally observe that the map which takes $\mathcal{A}$ to $D(\mathcal{A})$ identifies the galaxy of C with the set $\mathbb{E}^\#$ of those effective exceptional divisors $D \neq 0$ on the sky Z of C such that $-D$ is π-generated, where $\pi : Z \longrightarrow X$ is the canonical map. The structure of monoid is given by the addition of divisors.

We proceed now to generalize the proximity inequalities of [E.C] to higher dimensional case. To do so, we make use of intersection theory.

1.3.1. — In this subsection, we fix a *constellation* $C = \{Q_0, \ldots, Q_n\}$ with origin at O ; using the same notations as in subsection 1.1, we denote by $|D|$ the exceptional fiber $\bigcup_Q E_Q$ of the canonical map $\pi : Z \longrightarrow X$ from the sky of C to X. Note that the support of any exceptional divisor on Z is contained in $|D|$. For each k, $0 \leq k \leq d-1$, $A_k(|D|)$ denotes the group of k-cycles on $|D|$ modulo rational equivalence. Since $|D|$ is a projective variety, one can associate to any 0-cycle $\alpha \in A_0(|D|)$ a rational integer $\deg(\alpha)$.

Recall that the intersection product $V \bullet W$ of two irreducible subvarieties of a non singular variety Z is defined in the Chow group $A_m(V \cap W)$ where $m = \dim V + \dim W - \dim Z$ ([F], chap 8). Here, if $D_1, \ldots, D_s$, $(1 \leq s \leq d)$ are s effective exceptional divisors on Z and if V is a k-cycle on Z, $s \leq k \leq d$, we consider $D_1 \bullet \cdots \bullet D_s \bullet V$ as a class in $A_{k-s}(|D|)$. If $k = s$, we write for simplicity $D_1 \bullet \cdots \bullet D_s \bullet V$ instead of $\deg(D_1 \bullet \cdots \bullet D_s \bullet V)$. If $k = s = d$ and $V = Z$, we write $D_1 \bullet \cdots \bullet D_d$ in place of $D_1 \bullet \cdots \bullet D_d \bullet Z$.

1.3.2. Proposition. — *Let I be a finitely supported complete ideal whose constellation C_I is contained in C and let D be the exceptional divisor such that $I\mathcal{O}_Z = \mathcal{O}_Z(-D)$. For any k-dimensional irreducible subvariety V of Z contained in $|D|$ with $1 \leq k \leq d-1$, the inequality*

$$(-D)^k \bullet V \geq 0$$

holds.

Proof. — Let $Y \to X$ be the map obtained by blowing-up I and let $q : Z \to Y$ be the morphism factoring π ; the morphism q is proper. There exists a X-closed immersion $i : Y \hookrightarrow X \underset{K}{\times} \mathbb{P}^n_K$ such that $I\mathcal{O}_Y = i^*(\mathcal{O}(1))$ where $\mathcal{O}(1)$ is the canonical twisting sheaf on $X \underset{K}{\times} \mathbb{P}^n_K$. Hence, one has a commutative diagram

$$\begin{array}{ccccc} |D| & & \xrightarrow{\quad f \quad} & & \{O\} \underset{K}{\times} \mathbb{P}^n_K \\ \downarrow & & & & \downarrow \\ Z & \xrightarrow{q} & Y & \xrightarrow{i} & X \underset{K}{\times} \mathbb{P}^n_K \end{array}$$

with f proper and such that $\mathcal{O}_Z(-D) \otimes \mathcal{O}_{|D|} = f^*(\mathcal{O}_{\mathbb{P}^n}(1))$. Since $V \subset |D|$, $(-D)^k \bullet V = c_1\left(\mathcal{O}_X(-D) \otimes \mathcal{O}_{|D|}\right)^k \bullet V$ in $A_0(|D|)$ where c_1 denotes the first Chern class. Now, applying the projection formula and taking degrees on both sides, one has in $\mathbb{Z}$

$$(-D)^k \bullet V = \deg f_*\left(c_1\left(f^*\mathcal{O}_{\mathbb{P}^n}(1)\right)^k \bullet V\right) = \deg c_1\left(\mathcal{O}_{\mathbb{P}^n}(1)\right)^k \bullet f_* V.$$

If the dimension of $f(V)$ is less than k, by definition $f_* V = 0$ in $A_k(\mathbb{P}^n_K)$. Thus $(-D)^k \bullet V = 0$.

If not, $(-D)^k \bullet V > 0$, since it is the product of the degree of the projective variety $f(V)$ in $\mathbb{P}^n_K$ by the ramification index of the induced morphism $V \to f(V)$; i.e., one has

$$(-D)^k \bullet V = \deg f(V) \cdot \left[K(V) : K\left(f(V)\right)\right]$$

where $K(V)$ (resp. $K(f(V))$ is the function field on V (resp. $f(V)$).

1.3.3. — Now, we will reformulate the above inequalities in terms of the weights $\underline{m}$ of the cluster $\mathcal{A}_I$ associated to the finitely supported complete ideal I and geometrical invariants of V. Note that setting $m_i = 0$ for $Q_i \in C \setminus C_I$, one has $D = \sum_{0 \leq i \leq n} m_i E_i^*$.

Before going further, we need to compute the intersection product of any two exceptional divisors on Z respectively in the basis E^* and E of $\mathbb{E}$ (1.1.5). First, we fix some notations. For any $Q = Q_i \in C$, let τ_Q (or τ_i) : $E_Q \to B_Q$ be the morphism induced by the canonical projection $Z \to Z_{i+1}$. The morphism τ_Q is a finite composition of point blowing-ups. Indeed, the set of points $Q_j \to Q$ is the disjoint union of a finite number of constellations of infinitely near points of B_Q whose respective origins are those Q_j in B_Q. The space E_Q patches the skies of these various constellations in an obvious sense.

1.3.4. Lemma. — *If $j \to i$, the exceptional divisors E_j^* and E_i intersect properly. More precisely, $E_j^* \bullet E_i$ is the class of the total transform on $E_i \subset Z$ of the exceptional divisor of the blowing-up of Q_j in $E_i \subset Z_j$.*

If $j = i$, the intersection product $E_i^* \bullet E_i = -\tau_i^*(H_i)$ where H_i is the class of a hyperplane in the projective $(d-1)$ space B_i.

In all other cases $E_j^* \bullet E_i = 0$.

Proof. — By definition, $\mathcal{O}_Z(E_j^*) \otimes \mathcal{O}_{E_i}$ is the $\mathcal{O}_{E_i}$-invertible sheaf corresponding to $E_j^* \bullet E_i$. In any case, $\mathcal{O}_Z(E_j^*)$ is the inverse image of $\mathcal{O}_{Z_{j+1}}(B_j)$ by the canonical map $Z \to Z_{j+1}$.

If $i > j$, this morphism factors through Z_i. Now, E_i is contracted to Q_i in Z_i. Therefore there exists an affine neighborhood of E_i on Z on which $\mathcal{O}_Z(E_j^*)$ is free ; so $E_j^* \bullet E_i = 0$.

If $i = j$, $\mathcal{O}_Z(E_i^*) \otimes \mathcal{O}_{E_i} = \tau_i^* \left[\mathcal{O}_{Z_{i+1}}(B_i) \otimes \mathcal{O}_{B_i}\right] = \tau_i^* \mathcal{O}_{B_i}(-1)$.

If $i < j$, $\mathcal{O}_Z(E_j^*) \otimes \mathcal{O}_{E_i}$ is the inverse image on $E_i \subset Z$ of $\mathcal{O}_{Z_{j+1}}(B_j) \otimes \mathcal{O}_{E_i}$ with now $E_i \subset Z_{j+1}$. The trace of B_j on E_i is nothing but the exceptional divisor of the blowing-up of Q_j in E_i understood as a subspace of Z_j. It is not empty if and only if $j \to i$.

1.3.4.1. Corollary. — *One has*

$$\begin{aligned} E_{i_1}^* \bullet \cdots \bullet E_{i_d}^* &= (-1)^{d-1} && \text{if } i_1 = \cdots = i_d \\ &= 0 && \text{otherwise.} \end{aligned}$$

Proof. — It follows immediately from 1.3.4 that $E_j^* \bullet E_i^* = 0$ if $i \neq j$. Indeed, by the symmetry of the intersection product, it is enough to consider the case where $i > j$. Now $E_i^* \in \bigoplus_{\ell \geq i} \mathbb{Z} E_\ell$. In the same way, we get that :

$$E_i^* \bullet E_i^* = E_i^* \bullet E_i = -\tau_i^*(H_i) \ .$$

Hence by applying the projection formula, we get

$$\begin{aligned}(E_i^*)^d &= (E_i)^{d-1} \bullet E_i^* = -(E_i)^{d-2} \bullet \tau_i^*(H_i) \\ &= -(B_i)^{d-2} \bullet H_i = -c_1(\mathcal{O}_{Z_{i+1}}(B_i) \otimes \mathcal{O}_{B_i})^{d-2} \bullet H_i \\ &= -c_1(\mathcal{O}(-1))^{d-2} \bullet H_i = (-1)^{d-1}.\end{aligned}$$

Now, let us come back to the exceptional k-subvariety V. Let $\mathcal{J}_V := \{Q \in C \mid V \subset E_Q\}$. Since V is irreducible, $\mathcal{J}_V$ is not empty. If $d = 2$, there exists a unique Q in $\mathcal{J}_V$ and $V = E_Q$. If $d \geq 3$, $\mathcal{J}_V$ is completely self-proximate (1.1.4) and contains at most $d - k$ points. In any case $\mathcal{J}_V$ is a chain and there exists a maximum Q_V for the ordering $\geq$ in $\mathcal{J}_V$ (Q if no confusion is likely). The point $Q = Q_V$in $\mathcal{J}_V$ is alternatively characterized by the following two facts :

(i) $\tau_P : E_P \to B_P$ contracts V to a point if $P \neq Q$.

(ii) $W := \tau_Q(V)$ is a k-dimensional projective sub-variety of B_Q.

In particular V is the strict transform of W in E_Q. We are now able to state and prove the avatar of 1.3.2 announced in 1.3.3.

1.3.5. THEOREM. — *Let I and V be as in 1.3.2 and let $Q = Q_V$ and $W \subset B_Q = \mathbb{P}_K^{d-1}$ be defined from V as above. Then one has*

$$\deg(W)\ m_Q^k \geq \sum_{R \to Q} e_R(W)\ m_R^k$$

where $\deg(W)$ *is the degree of W in B_Q, $e_R(W)$ is the multiplicity of the strict transform of W at R and $\underline{m}$ is the weight vector of the cluster $\mathcal{A}_I$ associated to I completed by 0 for those $R \in C \setminus C_I$.*

Proof. — Set $Q = Q_i$. Since $D = \sum_{0 \leq j \leq n} m_j E_j^*$ and $E_j^* \bullet E_{j'}^* = 0$ if $j \neq j'$, it is enough to prove that

$$\begin{aligned}(-E_j^*)^k \bullet V \quad &= -e_R(W) \quad && \text{if } j \to i \text{ and } R = Q_j \\ &= \deg(W) && \text{if } j = i \\ &= 0 && \text{otherwise .}\end{aligned}$$

Now, if $i > j$, $E_j^* \bullet V = 0$ since V is a subvariety of E_i and $\mathcal{O}_Z(E_j^*)$ is free on an affine neighborhood of E_i (1.3.4). If $i = j$, applying the projection formula, we get :

$$(-E_i^*)^k \bullet V = (-B_i)^k \bullet W = c_1(\mathcal{O}_{Z_{i+1}}(-B_i) \otimes \mathcal{O}_{B_i})^k \bullet W$$
$$= c_1(\mathcal{O}(1))^k \bullet W = \deg(W)$$

Finally, assume $i < j$. Then by definition of $Q = Q_i$, V is not contained in $\bigcup_{\ell \geq j} E_\ell$. Consequently, its image in Z_{j+1} is not contained in B_j. As it coincides with the strict transform of W, we denote it by W_{j+1}. As above by the projection formula, one has :

$$(-E_j^*)^k \bullet W = (-B_j)^k \bullet W_{j+1} = c_1(\mathcal{O}_{Z_{j+1}}(-B_j) \otimes \mathcal{O}_{B_j})^{k-1} \bullet (-B_j \bullet W_{j+1})$$
$$= -c_1(\mathcal{O}(1))^{k-1} \bullet (B_j \bullet W_{j+1}) = -\deg(B_j \bullet W_{j+1}) .$$

Now since $j \geq i+1$, $B_j \bullet W_{j+1}$ is the exceptional divisor of the blowing-up of Q_j in the strict transform W_j of W in Z_j, namely the projective tangent cone $\operatorname{Proj} C_R W_j$ of W_j at $R = Q_j$. Its degree is nothing but the multiplicity $e_R(W)$ since both coincide with the degree of the Hilbert Samuel polynomial of the local ring $\mathcal{O}_{W_j,R}$. Note that if $j \not\to i$, then $Q_j \notin W_j$. Indeed by definition, this means that $Q_j \notin E_i$. But recall that $W \subset B_i$, therefore the same inclusion holds for their respective strict transform W_j and E_i in Z_j.

1.3.5.1. Remark. — If $e_R(W) \neq 0$ in 1.3.5, then $R \to P$ for each $P \in \mathcal{J}_V$: Indeed this is obvious if $d = 2$. If not, set $P = Q_\ell$. With $R = Q_j$ and $Q = Q_i$ as above, we have that $V \subset E_\ell \cap E_i$. Now, since $j \to i$, one has $j > i$ and by definition of Q, one has $i > \ell$. So $W_j \subset E_\ell \cap E_i$ in Z_j. Hence if $e_R(W) \neq 0$, then $R = Q_j \in W_j$, so $R \in E_\ell$ and $R \to Q_\ell = P$.

Theorem 1.3.5 motivates the following definition :

1.3.6. Definition. — Let $\mathcal{A} = (C, \underline{m})$ be a cluster and W a k-dimensional subvariety of B_Q for some $Q \in C$ with $k \geq 1$. We say that $\mathcal{A}$ satisfies the *proximity inequality with respect to* W if one has

$$\deg(W)\, m_Q^k \geq \sum_{R \to Q} e_R(W)\, m_R^k .$$

The integers k and $s = \deg(W)$ are called, respectively, the *degree* and the *class* of the proximity inequality.

In other words, this theorem expresses that the cluster associated to a finitely supported complete ideal I satisfies the proximity inequalities with respect to any subvariety of B_Q for each $Q \in C_I$.

1.3.7. — The following proximity inequalities are intrinsically associated to the cluster : take a subset $\mathcal{J} = \{i_1 < \ldots < i_l\}$, $1 \leq l \leq d-1$, of $\{0, 1, \ldots, n\}$

such that $\{Q_{i_1}, \dots, Q_{i_l}\}$ is completely self-proximate and set $Q = Q_{i_l}$ and $W = E_{i_1} \cap \dots \cap E_{i_{l-1}} \cap B_{i_l} \subset B_Q$. Here $k = \dim W = d - l$ and $s = \deg W = 1$, so one has the following class one and degree $d - l$ proximity inequalities

$$m_{i_l}^{d-l} \geq \sum_{j \to i, \forall i \in \mathcal{J}} m_j^{d-l}$$

Note that these special (proximity) inequalities only depend on combinatorial data associated to the cluster, namely the tree Γ of the constellation (1.1.3), the proximity matrix and the weight vector $\underline{m}$. For this reason, we call them *combinatorial proximity inequalities*. The combinatorial proximity inequalities correspond intersection theoretically to the conditions

$$r_{\mathcal{J}} := (-D)^{d-l} \bullet E_{i_1} \bullet \cdots \bullet E_{i_l} \geq 0.$$

Finally note that although for $d = 3$, $r_{\mathcal{J}}$ has a meaning if $\mathcal{J}$ is not completely self-proximate, the condition $r_{\mathcal{J}} \geq 0$ is superfluous since $E_{i_1} \cap \dots \cap E_{i_l} = \emptyset$ obviously implies $r_{\mathcal{J}} = 0$.

For $d = 2$, there is only one proximity inequality for each $Q \in C$, namely that corresponding to $W = B_Q$. It is a degree one and class one combinatorial inequality given by

$$m_Q \geq \sum_{R \to Q} m_R.$$

It is known that the galaxy $\mathcal{G}$ of a given constellation C is the set of clusters $\mathcal{A} = (C, \underline{m})$ for which the set of proximity inequalities on $\underline{m}$ for Q ranging in C hold ([E.C], [Ca], [LJ],[L3]). As a consequence and the intersection matrix $((E_Q \bullet E_R))$ being negative definite with determinant -1 (1.6.6), one has :

1.3.8. THEOREM. — *For $d = 2$, $\mathcal{G}$ is a regular cone in the following sense : the map D which takes $\mathcal{A}$ to $D(\mathcal{A})$ identifies $\mathcal{G}$ with*

$$E^{\#} = \sum_{Q \in C} \mathbb{Z}_{\geq 0}(-E_Q^{\vee})$$

where $(E_Q^{\vee})$ is the basis of $\mathbb{E}$, identified with its dual $\mathbb{E}^{\vee} = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{E}, \mathbb{Z})$ via the bilinear form defined by the intersection matrix, dual to $(E_Q)_{Q \in C}$, i.e. such that $(E_Q^{\vee} \bullet E_R) = 0$ if $Q \neq R$ and 1 otherwise.

Proof. — Indeed, any $D \in \mathbb{E}$ such that $(D \bullet E_Q) \leq 0$ for every $Q \in C$ is effective (cf. [L1], § 18) and it follows from 1.2.9 and the above remarks that $D(\mathcal{G}) = E^{\#} = \{D \in \mathbb{E} \mid (D \bullet E_Q) \leq 0 \text{ for every } Q \in C\}$.

For $d = 3$, the example 1.3.9 shows that this is no longer true.

1.3.9. EXAMPLE. — Consider a point O in a three-dimensional non singular variety X and a non singular cubic curve W_0 in the exceptional divisor B_0 obtained by blowing-up O. Consider the constellation C consisting of O and nine points $P_1, \dots, P_9$ in general position on W_0 (*i.e.* such that no other cubic curve in B_0 goes through all of them). Let $\mathcal{A} = (C, \underline{m})$ with $m_O = 3$ and $m_{P_i} = 1$, $1 \leq i \leq 9$.

Since W_0 is the only cubic in B_0 going through $P_1, \dots, P_9$, it follows that the cluster $\mathcal{A}$ may not be associated to a finitely supported ideal I. In fact, if otherwise, M being the maximal ideal of $\mathcal{O}_{X,O}$, I should be included in M^3 and the class mod M^4 of any $f \in I$ should be a scalar multiple of an homogeneous polynomial defining W_0. Consequently, W_0 should be a base curve of I and I would not be finitely supported.

Nevertheless, $\mathcal{A}$ satisfies all the proximity inequalities. First, this is obvious for those corresponding to $W \subset B_{P_i}$, $1 \leq i \leq 9$. If $W = B_O$, one has

$$m_O^2 = 3^2 = 9 \geq \sum_{R \to O} m_R^2 = 9.1 \ .$$

Finally, if W is a curve in B_0 distinct from W_0, then by Bézout's theorem

$$3\deg(W) \geq \sum_{R \in W_0 \cap W} e_R(W) \geq \sum_{1 \leq i \leq 9} e_{P_i}(W) \ .$$

If $W = W_0$,

$$3\deg(W_0) = 9 \geq \sum_{1 \leq i \leq 9} e_{P_i}(W_0) = 9 \ .$$

Further relationship between Bézout's theorem and proximity inequalities will appear in 2.4.1.

1.3.10. REMARK. — A cluster $\mathcal{A} = (C, m)$ which satisfies the proximity inequalities of degree 1 (*i.e.* with respect to any curve of B_Q for each $Q \in C$) satisfies the proximity inequalities of any degree.

This can be rephrased by saying that $\mathcal{A}$ satisfies the proximity inequalities if and only if $-D(\mathcal{A})$ is *π-nef (numerically effective)*.

Indeed recall that a divisor D on the sky Z of C is said to be π-nef if $D \bullet V \geq 0$ for any exceptional irreducible curve. Then apply Kleiman's theorem ([K] p. 320) to $\mathcal{O}_Z(-D(\mathcal{A})) \otimes \mathcal{O}_{E_i}$ for each E_i.

2. Linear systems with infinitely near base conditions

2.1.1. — In this section, we fix a *cluster* $\mathcal{A} = (C, \underline{m})$ of infinitely near points of X with origin at O. As in 1.1.8, $D = \sum_{Q \in C} m_Q E_Q^* = \sum_{Q \in C} d_Q E_Q$ denotes its associated exceptional divisor on the sky Z of C and $\pi : Z \to X$ is the canonical map. We maintain the notation of § 1. The complete linear system on the germ of X at O associated to $\mathcal{A}$ will be defined by valuative conditions.

For each $Q \in C$, the valuation v_Q of the function field $K(X)$ centered at B_Q is the valuation given by the order function ord_Q of the local ring R_Q of Q on the variety on which it lies (1.2).

For $0 \neq f \in R := \mathcal{O}_{X,O}$, the strict transform f'_Q (resp. the virtual transform $f_{Q,\mathcal{A}}$) of f at Q is defined inductively as follows :

i) $f'_O = f_{O,\mathcal{A}} = f$.

ii) for $Q \in X_\ell$ going to $P \in X_{\ell-1}$, $1 \leq \ell \leq t$, (see 1.1.2)

$f'_Q = (x)^{-v_P(f'_P)} f'_P \mathcal{O}_{X_\ell,Q}$ (resp. $f_{Q,\mathcal{A}} = (x)^{-m_P} f_{P,\mathcal{A}} \mathcal{O}_{X_\ell,Q}$) where $x = 0$ is a local equation of the exceptional divisor B_P at Q. Unlike f'_Q, $f_{Q,\mathcal{A}}$ may belong to $K(X) \setminus R_Q$.

Set $e_Q(f) = v_Q(f'_Q)$ and $e_{Q,\mathcal{A}}(f) = v_Q(f_{Q,\mathcal{A}})$. Note that both $e_Q(f)$ and $e_{Q,\mathcal{A}}(f)$ depend only on f and $\mathcal{A}$.

For simplicity, here we call "hypersurface" an effective Cartier divisor on the germ (X, O) of X at O. In particular, it need not be reduced.

2.1.2. Definition. — Let $\mathcal{A} = (C, \underline{m})$ be a cluster as above. An hypersurface H on (X, O) is said to pass through C with virtual multiplicities $\underline{m}$, or simply to pass through $\mathcal{A}$, (resp. to pass effectively through $\mathcal{A}$), if for each $Q \in C$, one has

$$e_{Q,\mathcal{A}}(f) \geq \text{(resp. } =) m_Q$$

where $f = 0$ is any equation defining H.

Let $\mathfrak{s}(\mathcal{A})$ be the complete linear system of hypersurfaces passing through $\mathcal{A}$.

The following lemma characterizes the hypersurfaces in $\mathfrak{s}(\mathcal{A})$.

2.1.3. LEMMA. — *With the preceding notations, let I be the stalk of $\pi_* \mathcal{O}_Z(-D)$ at O. For $0 \neq f \in R$, the following conditions are equivalent :*

(i) *The hypersurface H_f defined by f in (X, O) passes through $\mathcal{A}$, i.e. $e_{Q,\mathcal{A}}(f) \geq m_Q$, for each $Q \in C$*

(ii) $f \in I$

(iii) $v_Q(f) \geq d_Q$, *for each $Q \in C$.*

Proof. — The equivalences between (ii) and (iii) are obvious. Now, from the definition of $f_{Q,\mathcal{A}}$, one has in $K(X)$

$$f_{Q,\mathcal{A}} = f \prod_{P|Q \to P} x_P^{-m_P}$$

where $x_P = 0$ is a local equation for the exceptional divisor B_P on a neighborhood of the image of Q.

On the other hand, by definition of E_Q^*, one has

$$E_Q^* = E_Q + \sum_{P|P \to Q} v_P(x_Q) E_P \qquad (*)$$

where, as above, $x_Q = 0$ is a local equation of B_Q on a neighborhood of the image of P.

But by 1.1.6, ${}^tE^* = {}^tM^{-1} \cdot {}^tE$ and by 1.1.8, $\underline{d} = M^{-1}\underline{m}$. Using the explicit expression for M^{-1} provided by $(*)$, it follows that

$$d_Q - m_Q = \sum_{Q \to P} v_Q(x_P) m_P \ .$$

Hence :

$$e_{Q,\mathcal{A}}(f) - m_Q = v_Q(f_{Q,\mathcal{A}}) - m_Q = v_Q(f) - d_Q$$

and the inequalities (ii) and (iv) are equivalent.

2.1.4. REMARK – DEFINITION. — Let $\pi^*(H_f) = \mathrm{div}(f)$ be the total transform of H_f on Z ; H_f passes through $\mathcal{A}$ if and only $\pi^*(H_f) - D$ is an effective divisor. We call it the *virtual transform* of H_f on Z and denote it by $\overline{\pi}(H_f)$. It follows immediately from the previous equality that H_f passes effectively through $\mathcal{A}$ if and only if $v_Q(f) = d_Q$ for each $Q \in C$, *i.e.* if D is the exceptional part of $\pi^*(H_f)$. Since this last divisor is $\sum_Q e_Q(f) E_Q^*$, another equivalent condition is that $e_Q(f) = m_Q$ for each $Q \in C$.

2.1.5. REMARK. — Recall that the ideal I in 2.1.3 is complete. When $d > 2$, it may not be finitely supported. If $\underline{m} > 0$, I is finitely supported and its associated cluster is $\mathcal{A}$, if and only if $-D$ is π-generated (1.2.7). Geometrically speaking, this last condition means that $\mathfrak{s}(\mathcal{A})$ has no other base points than those in C, and that there exist hypersurfaces which pass effectively through $\mathcal{A}$. In example 1.3.9, I is not finitely supported.

We derive from 2.1.3 the following characterizations of the elements in a finitely supported complete ideal.

2.1.6. THEOREM. — *Let I be a finitely supported complete ideal in $\mathcal{O}_{X,O}$ and let $\mathcal{A}$ be its associated cluster (cf. 1.2.2). For $0 \neq f \in \mathcal{O}_{X,O}$, the following conditions are equivalent :*

(i) $f \in I$

(ii) *The hypersurface H_f defined by f passes through $\mathcal{A}$*

(iii) $v_Q(f) \geq d_Q$, *for each $Q \in C$*

(iv) $v_Q(f) \geq d_Q$, *for each $Q \in C$ such that :*

$$r_Q = (-D)^{d-1} \bullet E_Q = m_Q^{d-1} - \sum_{R \to Q} m_R^{d-1} > 0 \; .$$

Proof. — We know that I is the stalk of $\pi_* \mathcal{O}_Z(-D(\mathcal{A}_I))$ at O (1.2.7) and, by definition, $\mathcal{A} = \mathcal{A}_I$. Hence in view of 2.1.3, the only assertion which remains to prove is that (iv) implies (i). Let $\overline{\sigma} : \overline{Y} \to X$ denote the blowing-up with respect to I followed by normalization and let $\overline{q} : Z \to \overline{Y}$ be the map factoring π. By [L1], prop. 6.2, I is the stalk of $\overline{\sigma}_*(I\mathcal{O}_{\overline{Y}})$ at O. Thus, it will be sufficient to show that the E_Q with $r_Q \neq 0$ are in one to one correspondence with the prime divisors F on $\overline{Y}$ for which $v_F(I\mathcal{O}_{\overline{Y}}) \neq 0$ and check that for F_Q corresponding to E_Q, one has $v_{F_Q}(I\mathcal{O}_{\overline{Y}}) = d_Q$.

Now, the normalization being a finite morphism, it is clear from the proof of 1.3.2 that $r_Q = 0$ if and only if the dimension of $\overline{q}(E_Q)$ is less than $d-1$. On the contrary, $F_Q := \overline{q}(E_Q)$ is the center of the valuation v_Q on $\overline{Y}$; indeed π induces an isomorphism from $\mathcal{O}_{\overline{Y},F_Q}$ to the discrete valuation ring $\mathcal{O}_{Z,E_Q}$ of v_Q. Hence $v_{F_Q}(I\mathcal{O}_{\overline{Y}}) = v_Q(I\mathcal{O}_{\overline{Y}}) = v_Q(I\mathcal{O}_Z) = d_Q$.

2.1.7. COROLLARY. — *The Rees valuations of a finitely supported complete ideal (i.e. the valuations centered at the irreducible components of the exceptional divisor of the normalized blowing-up of I) are those v_Q for which the* *corresponding combinatorial* $(d-1)$ *proximity inequality is strict, i.e.*

$$m_Q^{d-1} > \sum_{R \to Q} m_R^{d-1} .$$

2.2.1. — Finitely supported ideals behave well under restriction. As a corollary, we get an interpretation of the class one proximity inequalities (1.3.6) which generalizes 2.1.7.

Let $\mathcal{A} = (C, \underline{m})$ be a cluster of infinitely near points of X with origin at O and let Y be a non singular algebraic subvariety of X passing through O of dimension $k+1 \geq 2$. Then the points of C which lie on the strict transform of Y may be viewed as infinitely near points of Y. By attaching to each one of them its weight in $\mathcal{A}$, one gets a cluster $\mathcal{A}_Y = (C_Y, \underline{m}_Y)$ of infinitely near points of Y with origin at O ; we call it the *cluster induced by* $\mathcal{A}$ *on* Y.

2.2.2. PROPOSITION. — *Let* I *be a finitely supported ideal in* $\mathcal{O}_{X,O}$ *and let* $\mathcal{A} = (C, \underline{m})$ *be its associated cluster. For each* $Q \in C_Y$, *let* $I_{Q|Y_Q}$ *be the image of the weak transform* I_Q *of* I *at* Q *in the local ring of* Q *on the strict transform* Y_Q *of* Y.

(i) *For each* $Q \in C_Y$, *the ideal* $I_{Q|Y_Q}$ *is finitely supported and its associated cluster is the cluster induced by* $\mathcal{A}_{I_Q}$ *on* Y_Q.

(ii) *For each pair of points* P, Q *in* C_Y *with* $P \geq Q$, $I_{P|Y_P}$ *is the weak transform of* $I_{Q|Y_Q}$ *at* P.

Proof. — First we prove that for any $Q \in C_Y$, one has $\mathrm{ord}_Q\, I_Q = \mathrm{ord}_Q\, I_{Q|Y_Q}$. In fact, if otherwise, the tangent directions to Y_Q at Q would determine a subvariety of base points of I in B_Q of dimension $k \geq 1$. This may not be, since I is finitely supported. Note that if Q is an infinitely near point of Y above O and if $Q \notin C_Y$, the same equality holds since both members vanish.

Now consider $P \in C_Y$ such that $\ell(P) = \ell(Q)+1$. The exceptional divisor $B_Q(Y)$ of the blowing-up of Q in Y_Q is, locally at P, the trace of B_Q on Y_P ; i.e. if $x = 0$ is a local equation of B_Q at P, then $x_{|Y_P} = 0$ is a local equation of $B_Q(Y)$ at P where $x_{|Y_P}$ is the image of x in $\mathcal{O}_{Y_P,P}$. The assertions (i) and (ii) follow directly from the above observations and definitions 1.2.1 and 1.2.2.

2.2.3. COROLLARY. — *With* Y, I *and* $\mathcal{A}$ *as above, the Rees valuations of the image* $I_{|Y}$ *of* I *in* $\mathcal{O}_{Y,O}$ *are the order functions* v_Q *on the function field* $K(Y)$ *of*

Y *for which the class one proximity inequality for* $\mathcal{A}$ *with respect to the projective tangent space* $\operatorname{Proj} T_Q Y_Q$ *of* Y_Q *at* Q *is strict i.e.*

$$m_Q^k > \sum_{R \in C_Y, R \to Q} m_R^k .$$

2.2.4. REMARK. — Note that given $Q \in C$ and a k-dimensional linear subvariety W of B_Q, there exists a least $P \nleq Q$ such that W is the projective tangent space at Q of the strict transform of some non singular Y going through P. The proximity inequality with respect to W is strict if and only if the order function v_Q on $K(Y)$ is a Rees valuation of $I_{P|Y}$.

2.3.1. — Various global linear systems are naturally attached to clusters and finitely supported ideals (Basic definitions and properties of linear systems are given in [H], chap. II.7).

Let $\mathcal{A} = (C, \underline{m})$ be a cluster and let $D = D(\mathcal{A})$. First observe that by 2.1.3 and 2.1.4, the *complete linear system* $\mathfrak{d}(\mathcal{A})$ on the germ $(Z, |D|)$ of Z along the exceptional fiber $|D|$ of $\pi : Z \to X$ corresponding to $\mathcal{O}_Z(-D)$, is the set of virtual transforms of those hypersurfaces in (X, O) passing through $\mathcal{A}$. Indeed the stalk I of $\pi_*(\mathcal{O}_Z(-D))$ at O is the set of global sections of the inverse image of $\mathcal{O}_Z(-D)$ on $(Z, |D|)$ and, by definition, $\mathfrak{d}(\mathcal{A}) = \{ \operatorname{div}(f) - D \mid f \in I \}$.

For each completely self-proximate set $\mathcal{J} = \{i_1 < \cdots < i_k\}$ with $1 \le k \le d-1$, we denote by $\mathfrak{d}_{\mathcal{J}}(\mathcal{A})$ *the trace of* $\mathfrak{d}(\mathcal{A})$ *on* $E_{\mathcal{J}} := E_{i_1} \cap \cdots \cap E_{i_k}$ (1.1.4).

Recall that, by definition, $\mathfrak{d}_{\mathcal{J}}(\mathcal{A})$ is the linear system on $E_{\mathcal{J}}$ corresponding to the image of I under the natural map

$$I \longrightarrow H^0(E_{\mathcal{J}}, \mathcal{O}_Z(-D) \otimes \mathcal{O}_{E_{\mathcal{J}}}) .$$

It consists of all divisors $\tilde{\pi}(H_f) \bullet E_{\mathcal{J}}$ with $f \in I$ such that the virtual transform $\tilde{\pi}(H_f)$ of the hypersurface H_f defined by f intersects $E_{\mathcal{J}}$ properly. Even if $\mathcal{A}$ is associated to a finitely supported complete ideal I, $\mathfrak{d}_{\mathcal{J}}(\mathcal{A})$ may be a proper subsystem of the complete linear system $\mathfrak{c}_{\mathcal{J}}(\mathcal{A})$ on $E_{\mathcal{J}}$ associated to $\mathcal{O}_Z(-D) \otimes \mathcal{O}_{E_{\mathcal{J}}}$ (cf. example 2.4.3 below).

The following notations generalize those which have been introduced in 1.3.3 and 1.3.4. Let $r_{\mathcal{J}} : E_{\mathcal{J}} \to B_{\mathcal{J}} := E_{i_1} \cap \cdots \cap B_{i_k} \subset Z_{i_k+1}$ be the map induced by $Z \to Z_{i_k+1}$, let $L_{\mathcal{J}} := r_{\mathcal{J}}^*(H_{\mathcal{J}})$ be the total transform of a general hyperplane $H_{\mathcal{J}}$ in the $(d-k)$ projective space $B_{\mathcal{J}}$ and for $i \to \mathcal{J}$ (i.e. $i \to i_\ell$, $1 \le \ell \le k$) let $E_{i,\mathcal{J}}^* := E_i^* \bullet E_{\mathcal{J}}$ (this is also the total transform on $E_{\mathcal{J}} \subset Z$ of the exceptional divisor of the blowing-up of Q_i in $E_{\mathcal{J}} = E_{i_1} \cap \cdots \cap E_{i_k} \subset Z_{i+1}$). Finally let $m_{\mathcal{J}} := m_{i_k}$.

2.3.2. LEMMA. — *With the preceding notation, there is a natural $\mathcal{O}_{E_{\mathcal{J}}}$-isomorphism*

$$\mathcal{O}_Z(-D)\otimes\mathcal{O}_{E_{\mathcal{J}}}\simeq\mathcal{O}_{E_{\mathcal{J}}}\Big(m_{\mathcal{J}}L_{\mathcal{J}}-\sum_{i\to\mathcal{J}}m_iE^*_{i,\mathcal{J}}\Big)\,.$$

Proof. — Since $D=\sum m_iE_i^*$, this is a straightforward consequence of 1.3.4.

This last computation suggests to extend the notions of constellations, clusters and associated linear systems from local to projective context.

2.3.3. DEFINITIONS. — *A constellation (resp. cluster) of points of $B\simeq\mathbb{P}^{\ell}_K$, $\ell\geq 2$,* consists of finitely many constellations (resp. clusters) with distinct origins in B.

The *sky* of such a constellation is the variety obtained by blowing-up its points.

Given such a cluster, its *associated exceptional divisor* is the sum of the exceptional divisors of each one of the clusters originated at a point of B in it. (1.1.8).

Given an integer $m\geq 1$ and a cluster $\mathfrak{A}$, the *linear system* $\mathfrak{s}_{\mathfrak{A}}(m)$ is the set of those hypersurfaces of degree m in B passing through each one of the clusters originated at a point of B in $\mathfrak{A}$ (2.1.2).

We extend these definitions to $B\simeq\mathbb{P}^1_K$ by identifying infinitely near and proper points of $\mathbb{P}^1_K$.

2.3.4. PROPOSITION. — *Let $D=D(\mathfrak{A})$ be the exceptional divisor of a cluster $\mathfrak{A}$ of points of B and let $\tau:E\to B$ be the canonical map from its sky E to B. For any projective hypersurface W in B, let $\tau^*(W)$ denote its total transform on E and set $\tilde{\tau}(W)=\tau^*(W)-D$.*

The map $\tilde{\tau}$ is a projective isomorphism from $\mathfrak{s}_{\mathfrak{A}}(m)$ to the complete linear system $\mathfrak{c}_{\mathfrak{A}}(m)$ corresponding to $\mathcal{O}_E(mL-D)$, where $L:=\tau^*(H)$ is the total transform of a general hyperplane H in B. We call $\tilde{\tau}(W)$ the virtual transform *of W on E (with respect to $\mathfrak{A}$).*

Proof. — It follows immediately from 2.1.3 that $\mathfrak{s}_{\mathfrak{A}}(m)$ corresponds to the subspace $H^0(\mathfrak{I}\otimes\mathcal{O}_B(m))$ of $H^0(\mathcal{O}_B(m))$ where $\mathfrak{I}:=\tau_*(\mathcal{O}_E(-D))$. Therefore for $W\in\mathfrak{s}_{\mathfrak{A}}(m)$, there exists a unique $F\in H^0(\mathfrak{I}\otimes\mathcal{O}_B(m))$ such that $W=\operatorname{div}(F)+mH$; since $\tilde{\tau}(W)=\operatorname{div}(F)+mL-D$, it is enough to verify that the natural isomorphism $K(B)\simeq K(E)$ of function fields respectively of B and E induces an isomorphism from $H^0(\mathfrak{I}\otimes\mathcal{O}_B(m))$ with $H^0(\mathcal{O}_E(mL-D))$. It is a direct consequence of the definitions of $\mathfrak{I}$, L and of the projection formula.

2.3.5. — Note that, $\mathcal{J} = \{i_1 < \cdots < i_k\}$ being a completely self-proximate set of indices as in 2.3.1, $\mathfrak{C}(\mathcal{J}) := \{Q_i \in C \mid i \to \mathcal{J}\}$ is a constellation of points of $B_{\mathcal{J}}$ whose sky is $E_{\mathcal{J}}$. By restricting to $\mathfrak{C}(\mathcal{J})$ the weights in $\mathcal{A}$, we get a cluster $\mathfrak{A}(\mathcal{J})$ whose associated exceptional divisor is $\sum_{i \to \mathcal{J}} m_i E^*_{i,\mathcal{J}}$. Lemma 2.3.2 expresses that the complete linear system $\mathfrak{c}_{\mathcal{J}}(\mathcal{A})$ on $E_{\mathcal{J}}$ is nothing but $\mathfrak{c}_{\mathfrak{A}(\mathcal{J})}(m_{\mathcal{J}})$. Similarly, we write $\mathfrak{a}_{\mathcal{J}}(\mathcal{A})$ instead of $\mathfrak{s}_{\mathfrak{A}(\mathcal{J})}(m_{\mathcal{J}})$; the linear system $\mathfrak{a}_{\mathcal{J}}(\mathcal{A})$ is the set of those hypersurfaces of degree $m_{\mathcal{J}}$ in $B_{\mathcal{J}}$ passing through each one of the constellations originated at a point of $B_{\mathcal{J}}$ contained in $\{Q_i \in C \mid i \to \mathcal{J}\}$ with virtual multiplicities induced by $\underline{m}$. By 2.3.4, the operation of taking virtual transform $\tilde{\tau}_{\mathcal{J}}$ is an isomorphism from $\mathfrak{a}_{\mathcal{J}}(\mathcal{A})$ to $\mathfrak{c}_{\mathcal{J}}(\mathcal{A})$. We denote by $\mathfrak{b}_{\mathcal{J}}(\mathcal{A})$ the linear subsystem of $\mathfrak{a}_{\mathcal{J}}(\mathcal{A})$ corresponding to $\mathfrak{d}_{\mathcal{J}}(\mathcal{A})$.

More generally, if I is a finitely supported ideal with associated cluster $\mathcal{A}$, we consider the linear subsystem $\mathfrak{d}(I)$ of $\mathfrak{d}(\mathcal{A})$ defined by $\mathfrak{d}(I) := \{\operatorname{div}(f) - D \mid f \in I\}$ and its trace $\mathfrak{d}_{\mathcal{J}}(I)$ on $E_{\mathcal{J}}$. We denote by $\mathfrak{b}_{\mathcal{J}}(I)$ the corresponding subsystem of $\mathfrak{a}_{\mathcal{J}}(\mathcal{A})$.

If I is complete, then $\mathfrak{d}_{\mathcal{J}}(I) = \mathfrak{d}_{\mathcal{J}}(\mathcal{A})$ and $\mathfrak{b}_{\mathcal{J}}(I) = \mathfrak{b}_{\mathcal{J}}(\mathcal{A})$. The linear system $\mathfrak{b}_{\mathcal{J}}(I)$ can also be described directly from the weak transform of I at $Q_{\mathcal{J}} := Q_{i_k}$ as follows :

2.3.6. THEOREM. — *Let I be a finitely supported ideal and let $\mathfrak{b}_{\mathcal{J}}(I)$ be the linear system of hypersurfaces of degree $m_{\mathcal{J}}$ in $B_{\mathcal{J}}$ just defined.*

(i) *The image $I_{|\mathcal{J}}$ of the weak transform I_Q of I at $Q = Q_{i_k}$ in the local ring $S_{\mathcal{J}}$ of Q on $E_{i_1} \cap \cdots \cap E_{i_{k-1}} \subset Z_{i_k}$ has order $m_{\mathcal{J}} (= m_Q)$ with respect to $M_{\mathcal{J}} := \operatorname{Max} S_{\mathcal{J}}$.*

(ii) *Let $\operatorname{In} I_{|\mathcal{J}}$ be the image of $I_{|\mathcal{J}}$ under the canonical map*

$$M_{\mathcal{J}}^{m_{\mathcal{J}}} \longrightarrow M_{\mathcal{J}}^{m_{\mathcal{J}}} / M_{\mathcal{J}}^{m_{\mathcal{J}}+1} = H^0(\mathcal{O}_{B_{\mathcal{J}}}(m_{\mathcal{J}})) \ .$$

Then $\mathfrak{b}_{\mathcal{J}}(I)$ is the linear system corresponding to $\operatorname{In} I_{|\mathcal{J}}$.

In particular, for each base point $Q = Q_i$ of I, $\mathfrak{b}_Q(I) = \mathfrak{b}_{\{i\}}(I)$ is the linear system given by $\operatorname{In} I_Q$.

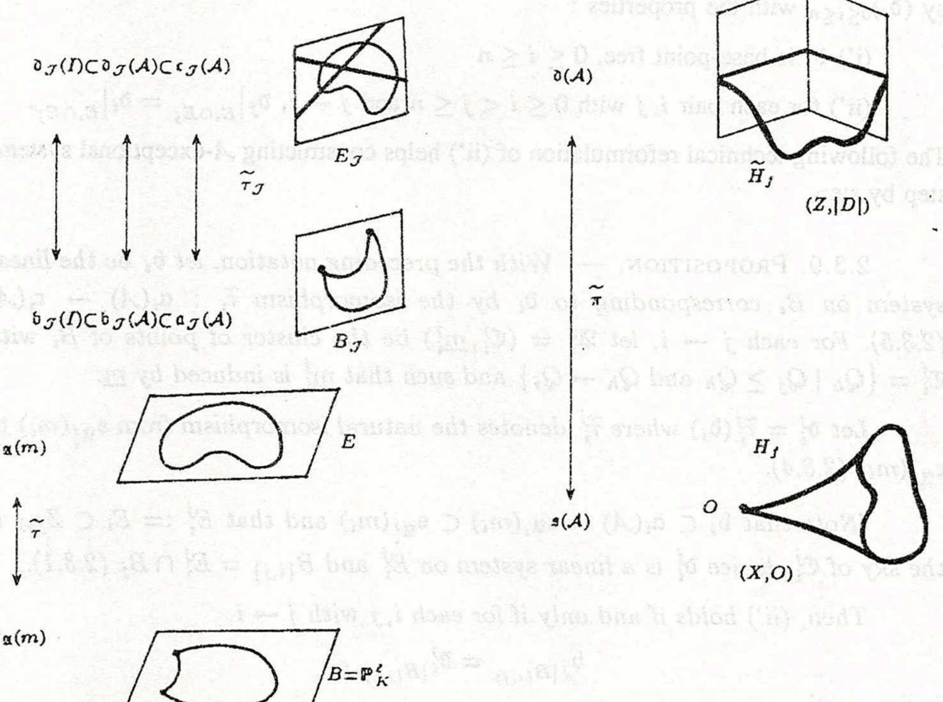
Proof. — First, we observe that by definition, $\mathfrak{d}(I)$ coincides with $\mathfrak{d}(I_Q)$ on a neighborhood of $E_Q = E_{i_k}$. Now, it follows from 2.2.2 applied to I_Q and $Y = E_{i_1} \cap \cdots \cap E_{i_{k-1}} \subset Z_{i_k}$ and from the definition of the trace of a linear system that $\operatorname{ord}_Q(I_{|\mathcal{J}}) = \operatorname{ord}_Q I_Q = m_Q$ and that $\mathfrak{d}_{\mathcal{J}}(I) = \mathfrak{d}_Q(I_{|\mathcal{J}})$. Indeed, 2.2.2 ii) expresses that the operations of taking virtual transforms and intersecting properly commute. Consequently, replacing $(E_{i_1} \cap \cdots \cap E_{i_{k-1}}, Q)$ by (Z, O) and $I_{|\mathcal{J}}$ by I, we are reduced to prove (ii) in the case where $k = 1$ and $\mathcal{J} = \{i_1\} = \{0\}$. So consider $f \in I$. The virtual transform $\widetilde{\pi}(H_f)$ of H_f intersects E_0 properly if and only if $\operatorname{ord}_O f = m_0$ and if this is so by 2.1.4, 1.3.4, 2.3.4, 2.3.5

$$\widetilde{\pi}(H_f)\bullet E_0 = \pi^*(H_f)\bullet E_0 - \sum m_i E_i^* \bullet E_0$$

$$\tau_0^*(H_{\operatorname{In} f}) - \sum_{i\to 0} m_i E_{i,0}^* = \widetilde{\tau}_0(H_{\operatorname{In} f})$$

where $\tau_0 : E_0 \to B_0$ is the canonical map and $\widetilde{\tau}_0(H_{\operatorname{In} f})$ is the virtual transform on E_0 of the hypersurface of degree m_0 in B_0 defined by $\operatorname{In} f = f \bmod M_0^{m_0+1}$ (with respect to $\mathfrak{A}(0)$). This completes the proof.

The following diagrams gather all linear systems previously introduced and bijections between them are represented by $\updownarrow$



2.3.7. Remark. — Let Λ be the set of completely self-proximate sets. If I is a finitely supported ideal, then

(i) for each $\mathcal{J} \in \Lambda$, $\mathfrak{d}_{\mathcal{J}}(I)$ is base-point free

(ii) for each pair $\mathcal{J}'$, $\mathcal{J}$ of sets in Λ with $\mathcal{J} \subset \mathcal{J}'$, $\mathfrak{d}_{\mathcal{J}}(I)_{|\mathcal{J}'} = \mathfrak{d}_{\mathcal{J}'}(I)$, where $\mathfrak{d}_{\mathcal{J}}(I)_{|\mathcal{J}'}$ is the trace of $\mathfrak{d}_{\mathcal{J}}(I)$ on $E_{\mathcal{J}'}$.

Proof. — Indeed, it follows from 1.2.3 and 1.2.4 that, I being finitely supported, $I\mathcal{O}_Z = \mathcal{O}_Z(-D)$ where D is the exceptional divisor of $\mathcal{A} = \mathcal{A}_I$; this implies that $\mathfrak{d}(I)$ is base-point free hence (i). Condition (ii) follows immediately from the definition of trace since both linear systems correspond to the image of I under the natural map $\overline{I} \to H^0(E_{\mathcal{J}'}, \mathcal{O}_Z(-D) \otimes \mathcal{O}_{E_{\mathcal{J}'}})$.

This motivates the following definition :

2.3.8. Definition. — Let Λ be as above and for each $\mathcal{J} \in \Lambda$, let $\mathfrak{d}_{\mathcal{J}}$ be a linear subsystem of $\mathfrak{c}_{\mathcal{J}}(\mathcal{A})$ (see 2.3.1) ; we say that $\underline{\mathfrak{d}} = (\mathfrak{d}_{\mathcal{J}})_{\mathcal{J}\in\Lambda}$ is an $\mathcal{A}$-exceptional system if conditions (i) and (ii) of 2.3.7 hold for $\underline{\mathfrak{d}}$.

Note that, if $C = \{Q_0, \ldots, Q_n\}$, an $\mathcal{A}$-exceptional system is uniquely determined by $(\mathfrak{d}_i)_{0\le i\le n}$ with the properties :

(i') $\mathfrak{d}_i$ is base-point free, $0 \le i \le n$

(ii') for each pair i, j with $0 \le i < j \le n$ and $j \to i$, $\mathfrak{d}_j\big|_{E_i\cap E_j} = \mathfrak{d}_i\big|_{E_i\cap E_j}$.

The following technical reformulation of (ii') helps constructing $\mathcal{A}$-exceptional systems step by step.

2.3.9. Proposition. — *With the preceding notation, let $\mathfrak{b}_i$ be the linear system on B_i corresponding to $\mathfrak{d}_i$ by the isomorphism $\tilde{\tau}_i : \mathfrak{a}_i(\mathcal{A}) \to \mathfrak{c}_i(\mathcal{A})$ (2.3.5). For each $j \to i$, let $\mathfrak{A}_i^j = (\mathfrak{C}_i^j, \underline{m}_i^j)$ be the cluster of points of B_i with $\mathfrak{C}_i^j = \{Q_h \mid Q_j \ge Q_h \text{ and } Q_h \to Q_i\}$ and such that $\underline{m}_i^j$ is induced by $\underline{m}$.*

Let $\mathfrak{d}_i^j = \tilde{\tau}_i^j(\mathfrak{b}_i)$ where $\tilde{\tau}_i^j$ denotes the natural isomorphism from $\mathfrak{s}_{\mathfrak{A}_i^j}(m_i)$ to $\mathfrak{c}_{\mathfrak{A}_i^j}(m_i)$ (2.3.4).

(Note that $\mathfrak{b}_i \subset \mathfrak{a}_i(\mathcal{A}) = \mathfrak{s}_{\mathfrak{A}_i}(m_i) \subset \mathfrak{s}_{\mathfrak{A}_i^j}(m_i)$ and that $E_i^j := E_i \subset Z_{j+1}$ is the sky of $\mathfrak{C}_i^j$, hence $\mathfrak{d}_i^j$ is a linear system on E_i^j and $B_{\{i,j\}} = E_i^j \cap B_j$ (2.3.1).)

Then, (ii') holds if and only if for each i, j with $j \to i$,

$$\mathfrak{b}_{j\,|B_{\{i,j\}}} = \mathfrak{d}^j_{i\,|B_{\{i,j\}}} .$$

Proof. — Set $\mathcal{J} = \{i<j\}$ and consider the isomorphism $\tilde{\tau}_{\mathcal{J}}:\mathfrak{a}_{\mathcal{J}}(\mathcal{A}) \to \mathfrak{c}_{\mathcal{J}}(\mathcal{A})$ (2.3.5). Then it is enough to prove that :

$$\tilde{\tau}_{\mathcal{J}}\big(\mathfrak{b}_{j|B_{\mathcal{J}}}\big) = \mathfrak{d}_{j|E_{\mathcal{J}}} \quad\text{and}\quad \tilde{\tau}_{\mathcal{J}}\big(\mathfrak{d}^j_{i|B_{\mathcal{J}}}\big) = \mathfrak{d}_{i|E_{\mathcal{J}}} .$$

If H is an hypersurface in $\mathfrak{b}_j$, H contains $B_{\mathcal{J}}$ if and only if its virtual transform $\tilde{\tau}_j(H)$ on E_j contains $E_{\mathcal{J}}$ and by 1.3.4, $\tilde{\tau}_j(H){\bullet}E_{\mathcal{J}} = \tilde{\tau}_{\mathcal{J}}(H{\bullet}B_{\mathcal{J}})$. Similarly, if H is an hypersurface in $\mathfrak{b}_i$, $\tilde{\tau}_i^j(H)$ contains $B_{\mathcal{J}}$ if and only if $\tilde{\tau}_i(H)$ contains $E_{\mathcal{J}}$ and $\tilde{\tau}_i(H){\bullet}E_{\mathcal{J}} = \tilde{\tau}_{\mathcal{J}}\big(\tilde{\tau}_i^j(H){\bullet}B_{\mathcal{J}}\big)$. This completes the proof.

From the above geometric discussion, we derive a series of examples and counter-examples. We begin by a remark.

2.4.1. REMARK. — Let $\mathcal{A}$ be a cluster. For any $d \geq 2$, we have the following implications between the following five conditions :

(i) There exists a finitely supported complete ideal I such that $\mathcal{A} = \mathcal{A}_I$.

(ii) $-D(\mathcal{A})$ is π-generated and no weight in $\mathcal{A}$ is equal to zero.

(iii) There exists an $\mathcal{A}$-exceptional system (2.3.8).

(iv) For each i, $0 \leq i \leq n$, $\mathfrak{c}_i(\mathcal{A})$ is base-point free (2.3.1).

(PI) For each i, $0 \leq i \leq n$, $\mathcal{A}$ satisfies the proximity inequalities with respect to any projective subvariety (resp. curve)of B_i (1.3.6, 1.3.10).

(NEF) $-D(\mathcal{A})$ is π-nef

(i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (PI) $\Leftrightarrow$ (NEF)

If $d = 2$, all conditions are equivalent (1.3.7).

Proof. — The equivalence (i) $\Leftrightarrow$ (ii) has been proved in 1.2.7. According to 2.3.8, (i) $\Rightarrow$ (iii) is remark 2.3.7. Since $\mathfrak{d}_i \subset \mathfrak{c}_i(\mathcal{A})$, (iii) $\Rightarrow$ (iv). Now, with $D = D(\mathcal{A})$, (iv) is equivalent to saying that $\mathcal{O}_Z(-D) \otimes \mathcal{O}_{E_i}$ is generated by its global sections. Hence, by [F], th. 12.1, if V is the strict transform on E_i of a k-dimensional subvariety W of B_i,

$$\deg(W)\, m_i^k - \sum_{j \to i} e_{Q_j}(W)\, m_j^k = (-D)^k \bullet V = (-D \bullet E_i)^k \bullet V \geq 0 \,.$$

A geometric proof of this inequality can also be derived from Bézout's theorem applied to W and the intersection of k general hypersurfaces in $\mathfrak{a}_i(\mathcal{A})$.

2.4.2. REMARK. — If $d > 2$, then (PI) $\not\Rightarrow$ (iv). In example 1.3.9, (PI) holds but W_0 is the only cubic in B_0 passing through $P_1, \ldots, P_9$, hence $\mathfrak{a}_0(\mathcal{A}) = \{W_0\}$ and the strict transform of W_0 on E_0 is a base curve in $\mathfrak{c}_0(\mathcal{A})$.

(iv) $\not\Rightarrow$ (iii). Assume $d = 3$ and let C be the constellation consisting of a point $O \in X$, six points $Q = Q_0, Q_1, \ldots, Q_5$ in general position in B_O and four points $P_1, \ldots, P_4$ in general position in B_Q. Condition (iv) holds for $\mathcal{A} = (C, \underline{m})$ with $m_O = 3$, $m_Q = 2$, $m_{Q_i} = m_{P_j} = 1$, $1 \leq i \leq 5$, $1 \leq j \leq 4$. Indeed, $\mathfrak{a}_O(\mathcal{A})$ is the pencil of cubics in B_O having a double point at Q and passing through Q_i, $1 \leq i \leq 5$ and $\mathfrak{c}_O(\mathcal{A})$ is the set of their strict transform by blowing-up Q_i, $0 \leq i \leq 5$. By Bézout's theorem, $Q_0, \ldots, Q_5$ are the only proper or infinitely near points of B_O in the intersection of any two curves in $\mathfrak{a}_O(\mathcal{A})$. Hence $\mathfrak{c}_O(\mathcal{A})$ is base-point free. The linear system $\mathfrak{a}_Q(\mathcal{A})$ is the pencil of conics passing through P_j, $1 \leq j \leq 4$, and as above by Bézout's theorem, $\mathfrak{c}_Q(\mathcal{A})$ is base-point free.

Now $\mathfrak{a}_O(\mathcal{A})$ and $\mathfrak{a}_Q(\mathcal{A})$ being pencils, for any exceptional system $\underline{\mathfrak{d}}$, one has $\mathfrak{d}_O = \mathfrak{c}_O(\mathcal{A})$ and $\mathfrak{d}_Q = \mathfrak{c}_Q(\mathcal{A})$ hence $\mathfrak{b}_O = \mathfrak{a}_O(\mathcal{A})$ and $\mathfrak{b}_Q = \mathfrak{a}_Q(\mathcal{A})$. By 2.3.9, this is a contradiction, since P_j, $1 \leq j \leq 4$, being in general position, the trace of $\mathfrak{d}_O^Q$ on $E_O \cap B_Q \simeq \mathbb{P}^1$ which coincides with that of $\mathfrak{c}_O(\mathcal{A})$ and the trace of $\mathfrak{c}_Q(\mathcal{A})$ are distinct pencils in the complete linear system $|\mathcal{O}_{\mathbb{P}^1}(2)|$.

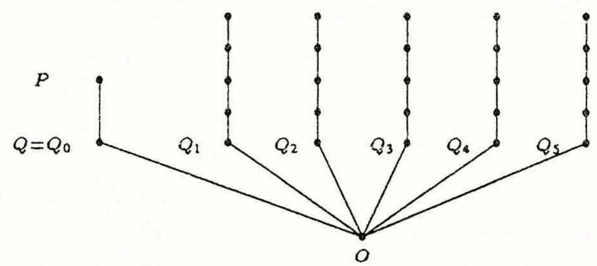
In example 2.4.3 (resp 2.4.4) below, I is a finitely supported complete ideal and there exists a point Q in the constellation of base points of I such that the linear system $\mathfrak{d}_Q(I)$ (resp. the weak transform I_Q) is not complete.

2.4.3. Example. — Here again $d = 3$. Let C be the constellation consisting of a point $O \in X$, a point Q in B_O and four points $P_1, \ldots, P_4$ in general position in B_Q. Let $\mathcal{A} = (C, \underline{m})$ with $m_O = m_Q = 2$ and $m_{P_i} = 1$, $1 \leq i \leq 4$. As usual, let Z be the sky of C and let $\pi : Z \to X$ be the canonical map. Set $D = D(\mathcal{A})$. One can check that the stalk I of $\pi_*(\mathcal{O}_Z(-D))$ at O is a finitely supported ideal and that $\mathcal{A}$ is its associated cluster.

Now by 2.3.5, $\mathfrak{a}_O(\mathcal{A})$ is the net (two dimensional linear system) of conics in B_O having a double point at Q and $\mathfrak{a}_Q(\mathcal{A})$ is the pencil of conics in B_Q passing through $P_1, \ldots, P_4$. Therefore, its linear subsystem $\mathfrak{b}_Q(I)$ coincides with $\mathfrak{a}_Q(\mathcal{A})$; this implies that the trace of $\mathfrak{b}_Q(I)$ on $E_O \cap B_Q$ is a one-dimensional subsystem of $|\mathcal{O}_{\mathbb{P}^1}(2)|$.

On the other hand, assume $\mathfrak{d}_O(I)$ is complete (*i.e.* $= \mathfrak{c}_O(\mathcal{A})$) ; then $\mathfrak{b}_O(I) = \mathfrak{a}_O(\mathcal{A})$ and $\mathfrak{d}_O^Q(I)$ is the set of the strict transforms on E_O of conics in $\mathfrak{a}_O(\mathcal{A})$. Its trace on $E_O \cap B_Q \simeq \mathbb{P}^1$ is $|\mathcal{O}_{\mathbb{P}^1}(2)|$. This is a contradiction by 2.3.7 and 2.3.9. Hence $\mathfrak{d}_O(I)$ is not complete.

2.4.4. Example. — The ideal $J := (x^2z - y^3, y^2z - x^3, z^7)$ is finitely supported. The tree of its constellation of base points is



The weights in its associated cluster $\mathcal{A}$ are equal to one except $m_O = 3$ and $m_Q = m_P = 2$. Its completion I has the same associated cluster ([L2], prop. 1.10). We will show that the weak transform I_Q of I is not complete. First, we observe that is enough to check that $\mathfrak{b}_Q(I) \neq \mathfrak{b}_Q(\overline{I_Q})$ where $\overline{I_Q}$ is the completion of I_Q. Indeed by 2.3.6, one has $\mathfrak{b}_Q(I) = \mathfrak{b}_Q(I_Q)$. We proceed now to compute $\mathfrak{b}_Q(\overline{I_Q})$ and its trace on $E_O \cap B_Q$. The cluster of points of X_1 with origin at Q associated to I_Q is $\mathcal{A}_Q := (\{Q, P\}, m_Q = m_P = 2)$ and by lemma 2.1.3 and remark 2.1.5, $f \in \overline{I_Q}$ if and only if the hypersurface defined by f in (X_1, Q) passes through $\mathcal{A}_Q$. Here, with the notation of 1.2.1, $p_1 : X_1 \to X$ is the blowing-up with center O and $p_2 : X_2 \to X_1$ is the blowing-up with center $Q_0, \ldots, Q_5$. In the chart of X_1 given by $x' = x/z$, $y' = y/z$, $z' = z$, the point Q is $(0,0,0)$. In the chart of X_2 given by $x'' = x'/z'$, $y'' = y'/z'$, $z'' = z'$, the point P is $(0,0,0)$. Therefore $\overline{I_Q}$ is the ideal in $\mathcal{O}_{X_1,Q}$ generated by $x'^{\alpha} y'^{\beta} z'^{\gamma}$ with $2(\alpha+\beta)+\gamma \geq 4$ and, by 2.3.6, $\mathfrak{b}_Q(\overline{I_Q})$ is the linear system of conics in B_Q identified with the projective tangent space of B_O at Q given by the vector space generated by $x'^2, x'y', y'^2$. So, $E_O \cap B_Q$ being the projective line given by $z' = 0$, $\mathfrak{b}_Q(\overline{I_Q})_{|E_O \cap B_Q}$ is identified with the complete linear system $|\mathcal{O}_{\mathbb{P}^1}(2)|$; its dimension is 2.

To complete the proof, it is enough to check that $\mathfrak{b}_Q(I)_{|E_O \cap B_Q}$ has dimension 1. Since $Q \to O$, it coincides with $\mathfrak{d}_O^Q(I)_{|E_O \cap B_Q}$ where $\mathfrak{d}_O^Q(I)$ is the set of virtual transforms on E_O (with respect to the cluster of points of $B_O(\{Q\}, 2)$) of curves on B_O in $\mathfrak{b}_O(I)$.

From 2.3.6, it is obvious that $\mathfrak{b}_O(J)$ is the linear system of cubics in B_O (identified with the projective tangent space of X at O) given by the vector space generated by $x^2z - y^3$, $y^2z - x^3$. But, by definition, $\mathfrak{a}_O(\mathcal{A})$ consists of the cubics in B_O going through $Q_1, \ldots, Q_5$ and having multiplicity at least 2 at Q. Hence $\mathfrak{a}_O(\mathcal{A})$ is a pencil. Since $\mathfrak{b}_O(J) \subset \mathfrak{b}_O(I) \subset \mathfrak{a}_O(\mathcal{A})$ the inclusions are equality, $\mathfrak{d}_O^Q(I)$ consists of the strict transforms of curves in $\mathfrak{b}_O(J)$ and its trace on $E_O \cap B_Q$ is the pencil in $|\mathcal{O}_{\mathbb{P}^1}(2)|$ defined by the vector space generated by x'^2, y'^2.

3. Some embedded resolutions

In this last section, we will be concerned with embedded resolutions of complete intersections defined by general enough elements in a finitely supported ideal I of $\mathcal{O}_{X,O}$.

First we recall what an embedded resolution is.

3.1.1. DEFINITION. — Let V be a reduced subscheme of a non singular algebraic variety X with an isolated singular point at O. A projective and birational morphism $\pi : Z \to X$ is called an *embedded resolution* of V if :

a) Z is non singular and π induces an isomorphism of $Z \setminus \pi^{-1}(O)$ on $X \setminus O$.

b) the inverse image $\pi^{-1}(V)$ of V on Z is a normal crossings subscheme.

Condition b) means that each irreducible component of $\pi^{-1}(V)$ is non singular and that for any $Q \in \pi^{-1}(O)$, there exists a regular system of parameters $(u_1, \ldots, u_r)$ of $\mathcal{O}_{Z,Q}$ and non negative integers r, $\alpha_{r+1}, \ldots, \alpha_d$ such that $(u_1, \ldots, u_r)$ (resp. $(u_1, \ldots, u_r)u_{r+1}^{\alpha_{r+1}} \cdots u_d^{\alpha_d}$) is the ideal of $\mathcal{O}_{Z,Q}$ defining the strict transform V' (resp. the total transform) of V at Q. Note that $r, \alpha_{r+1}, \ldots, \alpha_d$ depend on Q and that $\alpha_{r+1}, \ldots, \alpha_d$ may not all vanish. The restriction $\pi_{|V'} : V' \to V$ is a desingularization of V.

Now we make precise what we mean by saying that some property holds for a general r-uple of elements of I. We maintain the notation of § 1 and § 2.

3.1.2. DEFINITION. — Let I be a finitely supported ideal of $\mathcal{O}_{X,O}$ and let $\mathcal{A} = (C, \underline{m})$ be its associated cluster. We say that $f \in I$ is *proper* if for any $\mathcal{J} = \{i_1 < \cdots < i_k\}$ with $1 \leq k \leq d = \dim(X, O)$ such that $E_{\mathcal{J}} := E_{i_1} \cap \cdots \cap E_{i_k} \neq \emptyset$ (in the sky Z of C), the image $f_{\mathcal{J}}$ of the virtual transform $f_{Q,\mathcal{A}}$ of f at $Q = Q_{i_k}$ in the local ring $S_{\mathcal{J}}$ of Q on $E_{i_1} \cap \cdots \cap E_{i_{k-1}} \subset Z_{i_k}$ has order m_Q with respect to the maximal ideal $M_{\mathcal{J}}$ of $S_{\mathcal{J}}$.

Recall that in view of definitions 1.2.2 and 2.1.1, the order of $f_{Q,\mathcal{A}}$ at Q is at least m_Q. Moreover, observe that for any $\mathcal{J}$ as above, the image $I_{|\mathcal{J}}$ of the weak transform I_Q of I at Q in $S_{\mathcal{J}}$ has order m_Q with respect to $M_{\mathcal{J}}$. Indeed, according to 2.3.6, this is so if $k < d$, *i.e.* if $\mathcal{J}$ is completely self-proximate. Now if $k = d$, $Q = Q_{i_d}$ is a point on the line $E_{i_1} \cap \cdots \cap E_{i_{d-1}} \subset Z_{i_d}$ and by 1.2.2, $\mathrm{ord}_Q\, I_Q = m_Q$; hence it is equivalent to say that $I_{|\mathcal{J}}$ has order m_Q and that $B_{\mathcal{J}} := E_{i_1} \cap \cdots \cap E_{i_{d-1}} \cap B_{i_d} \subset Z_{i_d+1}$ is not a base point of I (*i.e.* $B_{\mathcal{J}} \notin C$), where, following 1.1.2, B_{i_d} is the exceptional divisor of the blowing-up of Q_{i_d}. But since the line $E_{i_1} \cap \cdots \cap E_{i_{d-1}}$ and the divisor B_{i_d} intersect transversally at $B_{\mathcal{J}}$, and their respective strict transforms on the sky Z of C intersect transversally at some point, this implies that the canonical map $Z \to Z_{i_d+1}$ does not factor through the blowing-up of $B_{\mathcal{J}}$, therefore $B_{\mathcal{J}} \notin C$. It follows immediately from the above remark, that if $I = (f_0, \ldots, f_n)$, there exists a non empty Zariski open set Ω in $\mathbf{P}^n_K$ such that if $(\lambda_0 : \cdots : \lambda_n) \in \Omega$, then $f = \Sigma \lambda_i f_i$ is proper. Therefore

3.1.3. LEMMA. — *Any finitely supported ideal can be generated by proper elements.*

The following proposition expresses the geometrical significance of definition 3.1.2.

3.1.4. PROPOSITION. — *Let $f \in I$ be proper. Then,*

i) the hypersurface H_f defined by f in (X, O) passes effectively through $\mathcal{A}$.

ii) for any completely self-proximate set $\mathcal{J} = \{i_1 < \cdots < i_k\}$ with $1 \leq k < d-1$ (resp. $k = d-1$) the hypersurface $W_{\mathcal{J}}(f)$ of degree $m_{\mathcal{J}} := m_{i_k}$ in the linear system $\mathfrak{b}_{\mathcal{J}}(I)$, defined by the homogeneous polynomial $\operatorname{In} f_{\mathcal{J}} \bmod M_{\mathcal{J}}^{m_{\mathcal{J}}+1}$ *passes effectively (resp. passes) through the cluster $\mathfrak{A}(\mathcal{J})$ of points of $B_{\mathcal{J}} := E_{i_1} \cap \cdots \cap B_{i_k} \subset Z_{i_k+1}$ obtained by restricting $\underline{m}$ to $\mathfrak{C}(\mathcal{J}) = \{Q_i \in C \mid i \rightarrow \mathcal{J}\}$.*

The strict transform H'_f of H_f on the sky Z of C intersects $E_{\mathcal{J}}$ properly and $H'_f \bullet E_{\mathcal{J}}$ is the strict transform of $W_{\mathcal{J}}(f)$ (resp. its virtual transform with respect to $\mathfrak{A}(\mathcal{J})$).

iii) for any $\mathcal{J} = \{i_1 < \cdots < i_d\}$ such that $E_{\mathcal{J}} \neq \emptyset$, H'_f does not contain the point $E_{\mathcal{J}}$.

Proof. — Using 2.2.2, the assertions i) and ii) are essentially a reformulation of 2.1.2, 2.3.6, 2.3.4 and 2.3.5. The assertion iii) follows from the fact that f being proper, the tangent line to $E_{i_1} \cap \cdots \cap E_{i_d - 1} \subset Z_{i_d}$ at Q_{i_d} is not contained in the tangent cone of the strict transform of H_f.

Now consider $r < d$ proper elements $f_1, \ldots, f_r \in I$ and let $\mathcal{J}$ be any completely self-proximate set. Extending the notation of 2.3.6, we denote by $\operatorname{In}[f_1, \ldots, f_r]_{|\mathcal{J}}$ the K-subvector space of $\operatorname{In} I_{|\mathcal{J}}$ generated by $\operatorname{In} f_{i,\mathcal{J}}$, $1 \leq i \leq r$. Our assumption on $f_1, \ldots, f_r$ implies that each one of these r forms is different from zero but they need not be linearly independent.

3.1.5. DEFINITION. — With I and r as above, we say that a *property holds for a general r-uple of elements of I*, if for any completely self-proximate $\mathcal{J} = \{i_1 < \cdots < i_k\}$ with $1 \leq k < d$, there exists a non empty Zariski open set $\Omega_{\mathcal{J}}$ in the Grassmann variety, $G_r(\operatorname{In} I_{|\mathcal{J}})$, of vector-subspaces of rank r of $\operatorname{In} I_{|\mathcal{J}}$ such that the property holds for those proper $f_1, \ldots, f_r$ in I such that $\operatorname{In}[f_1, \ldots, f_r]_{|\mathcal{J}} \in \Omega_{\mathcal{J}}$.

3.1.6. THEOREM. — *Assume that the characteristic of the ground field K is zero. Then, for a general r-uple of element $(f_1, \ldots, f_r)$ in a finitely supported ideal I of $\mathcal{O}_{X,O}$ with $1 \leq r < \dim(X, O)$,*

i) the subscheme $H_{f_1, \ldots, f_r}$ of (X, O) defined by $f_1, \ldots, f_r$ is a reduced complete intersection.

ii) *the canonical map $\pi : Z \to X$ from the sky Z of the constellation of base points of I to X is an embedded resolution of $H_{f_1,\dots,f_r}$.*

Proof. — For any subvariety V of X, the exceptional part of $\pi^{-1}(V)$ is a normal crossing divisor. Let V' denote the strict transform of V on Z. If V is the intersection of r hypersurfaces $H_1,\dots,H_r$ in (X,O), then V is a reduced complete intersection and π is an embedded resolution of V if and only if for any set $\mathcal{J} = \{i_1 < \cdots < i_k\}$ with $1 \le k \le d$ such that $E_{\mathcal{J}} \neq \emptyset$ as in 3.1.2, either V' does not intersect $E_{\mathcal{J}}$ or $V' \cap E_{\mathcal{J}}$ is a non singular variety of codimension r in $E_{\mathcal{J}}$; in particular, this last condition implies that for $k > d-r$, $V' \cap E_{\mathcal{J}} = \emptyset$. Here $V' \cap E_{\mathcal{J}}$ denotes the scheme theoretic intersection.

In addition assume that H_i is defined by a proper $f_i \in I$ and let H_i' be its strict transform on Z.

First consider $\mathcal{J}$ as above with $k = d$. Then $f_1,\dots,f_r$ being proper, by 3.1.4, iii), for any i, $1 \le i \le r$, the hypersurface H_i' does not contain the point $E_{\mathcal{J}}$. Hence a fortiori $H_1' \cap \cdots \cap H_r' \cap E_{\mathcal{J}} = \emptyset$.

Now assume that $\mathcal{J}$ is completely self-proximate. Then there exists a morphism $\varphi_{\mathcal{J}} : E_{\mathcal{J}} \to \mathbb{P}^n_K$ and a linear subspace L of codimension r in $\mathbb{P}^n_K$ such that the scheme-theoretic intersection $H_1' \cap \cdots \cap H_r' \cap E_{\mathcal{J}}$ coincides with the inverse image $\varphi_{\mathcal{J}}^{-1}(L)$ of L on $E_{\mathcal{J}}$. The construction is standard (cf. 1.3.2). Choose a system of generators $g_0,\dots,g_n$ of I consisting of proper elements. Let $Y \subset X \underset{K}{\times} \mathbb{P}^n_K$ be the closure (in the sense of schemes) of the graph of the morphism $X \setminus \{O\} \to \mathbb{P}^n_K$ given by $x \mapsto (g_0(x) : \cdots : g_n(x))$; recall that I is primary for the maximal ideal of $\mathcal{O}_{X,O}$. The map $Y \to X$ induced by the projection on X is the blowing-up of I, the map $G : Y \to \mathbb{P}^n_K$ is the so-called Gauss map. Now $I\mathcal{O}_Z$ being invertible, Z dominates Y. Set $\varphi_{\mathcal{J}} := G \circ q_{|E_{\mathcal{J}}}$ where $q : Z \to Y$ is the morphism factoring π. Finally let L be the linear subvariety of $\mathbb{P}^n_K$ defined by the r linear equations $(X_0,\dots,X_n)\lambda = 0$ where λ is the image in K, canonically identified with the residue field of O on X, of the $n \times r$ matrix A with entries in $\mathcal{O}_{X,O}$ such that $(f_1,\dots,f_r) = (g_0,\dots,g_n)A$. Because $f_1,\dots,f_r$; $g_0,\dots,g_n$ are proper one has $H_1' \cap \cdots \cap H_r' \cap E_{\mathcal{J}} = \varphi_{\mathcal{J}}^{-1}(L)$.

By applying Bertini's theorem to $\varphi_{\mathcal{J}}$ ([J], cor. 6.11), one gets a non empty Zariski open set $\Omega^*_{\mathcal{J}}$ in the Grassmann variety, $G(r,n)$, of linear subvarieties of codimension r in $\mathbb{P}^n_K$ such that for $L \in \Omega^*_{\mathcal{J}}$, $\varphi_{\mathcal{J}}^{-1}(L)$ is empty if the dimension of $\varphi_{\mathcal{J}}(E_{\mathcal{J}})$ is less than r (in particular if $k > d-r$) and is a smooth equidimensional subvariety of codimension r of $E_{\mathcal{J}}$ otherwise.

Therefore if, for any completely self-proximate set $\mathcal{J}$, the r-uple of proper elements $f_1,\dots,f_r$ of I gives rise to a linear subspace $L \in \Omega^*_{\mathcal{J}}$, then $H_1' \cap \cdots \cap H_r'$

is a non singular subvariety of codimension r of Z ; hence it coincides with the strict transform of $H_1 \cap \cdots \cap H_r$ in Z and by the above criterion $H_1 \cap \cdots \cap H_r$ is a reduced complete intersection having π as an embedded resolution.

Finally the open set $\Omega_{\mathcal{J}}$ in $G_r(\operatorname{In} I_{|\mathcal{J}})$ is obtained from $\Omega^*_{\mathcal{J}}$ by identifying the one dimensional space generated by $\operatorname{In} g_{i,\mathcal{J}}$, $0 \leq i \leq n$, with the coordinate hyperplane $X_i = 0$ in $\mathbb{P}^n_K$.

3.1.7. Remark. — The above genericity condition on a r-uple $f_1, \ldots, f_r$ of proper elements in I amounts to requiring that for any completely self-proximate set $\mathcal{J}$ and for any proper point Q of $B_{\mathcal{J}}$ in $W_{\mathcal{J}}(f_1) \cap \cdots \cap W_{\mathcal{J}}(f_r)$ which is not a base point of I, the hypersurfaces $W_{\mathcal{J}}(f_i)$, $1 \leq i \leq r$ are non singular and intersect transversally at Q.

By analogy with the notion of non degeneracy of a polynomial or a formal series (resp. r Laurent polynomials) with respect to its Newton polyhedron given in [V] (resp. [Kh]), we will say that such an r-uple is *non-degenerate with respect to the cluster* $\mathcal{A}_I$ associated to I. The cluster extends the role played by the Newton polyhedron, the condition of properness of $f \in I$ replaces that of having a given polyhedron as Newton polyhedron, finally the previous transversality condition is the adaptation to this context of that of [Kh]. We could reformulate 3.1.6 by saying that the non-degeneracy condition is open.

3.1.8 Corollary. — *For each base point Q of I, the multiplicity $e_Q(H_{f_1,\ldots,f_r})$ at Q of the strict transform of the complete intersection $H_{f_1,\ldots,f_r}$ defined by a general r-uple of elements as in 3.1.6 is m_Q^r where m_Q is the weight of Q in the cluster associated to I.*

Proof. — By definition 3.1.2, for any proper $f \in I$, the virtual transform $f_{Q,\mathcal{A}}$ of f at Q coincides with its strict transform f'_Q and it is a proper element of the weak transform I_Q of I at Q. In particular the assertion holds for $r = 1$. Moreover, if $f_1, \ldots, f_r$ is a general r-uple of elements of I, by remark 3.1.7, $f'_{1,Q}, \ldots, f'_{r,Q}$ is a general r-uple of elements of I_Q. Hence the subscheme defined by these elements is a reduced complete intersection ; therefore it coincides with the strict transform of $H_{f_1,\ldots,f_r}$ at Q. Now $e_Q(H_{f_1,\ldots,f_r})$ is the degree of the exceptional divisor of the blowing-up of Q in this last scheme. By the above remarks, this divisor is noting but $W_Q(f_1) \cap \cdots \cap W_Q(f_r)$. Each one of these hypersurfaces of B_Q has degree m_Q. This completes the proof.

3.1.9. Corollary. — *Let Y be a non singular subvariety of X passing* *through O of dimension at least two. Then, under the assumptions of 3.1.6, for a general r-uple $(f_1, \ldots, f_r)$ of elements in I with $1 \leq r < \dim(Y, O)$,*

i) *the subscheme $H_{f_1,\ldots,f_r} \cap Y$ is a reduced complete intersection of (Y, O).*

ii) *the canonical map $\pi_Y : Z_Y \to Y$ from the strict transform $Z_Y$of Y in Z to Y induced by π is an embedded resolution of $H_{f_1,\ldots,f_r} \cap Y$.*

Proof. — We maintain the notation of2.3.6, 3.1.2 and 3.1.5. Let I_Y be the image of I in the local ring of O on Y. By 2.2.2, I_Y is a finitely supported ideal ; its associated cluster is the cluster $\mathcal{A}_Y = (C_Y, \underline{m}_Y)$ induced by $\mathcal{A}$ on Y, hence Z_Y is the sky of its constellation of base points C_Y and π_Y is the canonical map $Z_Y \to Y$.

In addition, the restriction f_Y of a proper $f \in I$ is a proper element of I_Y if and only if for any completely self-proximate set $\mathcal{J} = \{i_1 < \cdots < i_k\}$ with $1 \leq k \leq \dim(Y, O)$ such that $Z_Y \cap E_{i_1} \cap \cdots \cap E_{i_k} \neq \emptyset$ (i.e. $Q_{i_k} \in C_Y$), $\mathrm{In}\, f_{\mathcal{J}}$ does not belong to the kernel $K_{\mathcal{J}}$ of the canonical surjection $\mathrm{In}\, I_{|\mathcal{J}} \to \mathrm{In}\, I_{Y|\mathcal{J}}$. In particular, if $f_1, \ldots, f_r$ is a r-uple of proper elements in I and if for any such $\mathcal{J}$, $\mathrm{In}[f_1, \ldots, f_r]_{|\mathcal{J}} \cap K_Y = \{0\}$ then $f_{1,Y}, \ldots, f_{r,Y}$ is a r-uple of proper elements in I_Y and the K-vector spaces $\mathrm{In}[f_1, \ldots, f_r]_{|\mathcal{J}}$ and $\mathrm{In}[f_{1,Y}, \ldots, f_{r,Y}]_{|\mathcal{J}}$, which is nothing but its image in $\mathrm{In}\, I_{|\mathcal{J}}$, have the same rank. Now let $\Sigma_{\mathcal{J}} = \{V \in G_r(\mathrm{In}\, I_{|\mathcal{J}}) \mid V \cap K_{\mathcal{J}} \neq \{0\}\}$ and let $\widetilde{\Omega}_{\mathcal{J}}$ be the non empty Zariski open set of $G_r(\mathrm{In}\, I_{Y|\mathcal{J}})$ provided by theorem 3.1.6 applied to I_Y for $k < \dim(Y, O)$. Its inverse $\Omega_{\mathcal{J}}$ under the canonical map :

$$G_r(\mathrm{In}\, I_{|\mathcal{J}}) \setminus \Sigma_{\mathcal{J}} \longrightarrow G_r(\mathrm{In}\, I_{Y|\mathcal{J}})$$

which sends V to its image in $\mathrm{In}\, I_{Y|\mathcal{J}}$ is a non empty Zariski open set in $G_r(\mathrm{In}\, I_{|\mathcal{J}})$. The properties stated in 3.1.9 hold for those proper $f_1, \ldots, f_r$ in I such that $\mathrm{In}[f_1, \ldots, f_r]_{|\mathcal{J}} \in \Omega_Y$ (resp. $\notin \Sigma_{\mathcal{J}}$) for any completely self-proximate set $\mathcal{J}$ such that $Q_{i_k} \in C_Y$ with $k < \dim(Y, O)$ (resp. $k = \dim(Y, O)$).

From these results, we derive another geometric interpretation of the combinatorial proximity inequalities $r_{\mathcal{J}} = (-D)^{d-k} \bullet E_{i_1} \bullet \cdots \bullet E_{i_k} \geq 0$ (see 1.3.7).

3.1.10. Corollary. — *Let $C = (Q_0, \ldots, Q_n)$ be the constellation of base points of I and let $\mathcal{J} = \{i_1 < \cdots < i_k\}$, $1 \leq k \leq d-1$, be a completely self-proximate set. Then under the assumptions of 3.1.6, for a general $(d-k)$-uple $f_1, \ldots, f_{d-k}$ of elements of I,*

i) *the intersection of the strict transform of $H_{f_1,\ldots,f_{d-k}}$ at Q_{i_k} with $E_{i_1} \cap \cdots \cap E_{i_{k-1}}$ (in the ambient space Z_{i_k} containing Q_{i_k}) is a reduced complete intersection curve.*

ii) *the non negative integer* $r_{\mathcal{J}}$ *is the number of its branches, i.e. analytically irreducible components, whose tangent direction at* Q_{i_k} *(identified with a point of* $B_{\mathcal{J}}$*) is not a base point of* I*. These branches are non singular and have distinct tangents.*

In particular, $H_{f_1,\dots,f_{d-1}}$ *has* $\sum_{0\le i\le n} r_i$ *branches and it can be formally decomposed as :*

$$\bigcup_{Q_i\in C}\ \bigcup_{1\le j\le r_i} \Gamma_{i,j}$$

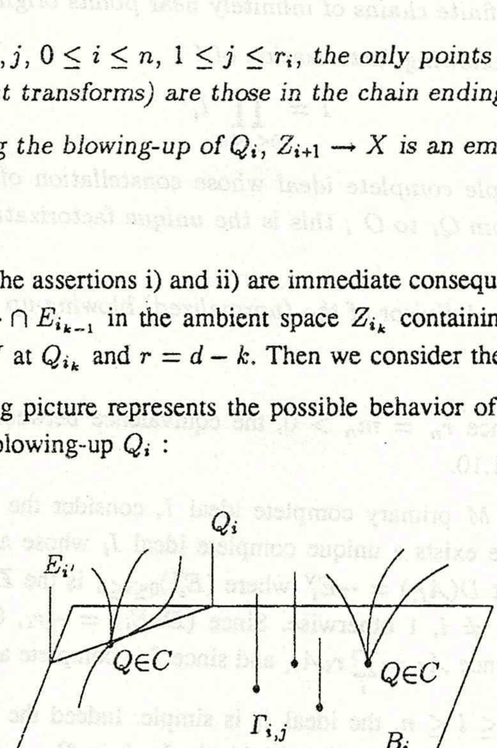
where

i') *for each* i,j, $0\le i\le n$, $1\le j\le r_i$, *the only points of* C *on* $\Gamma_{i,j}$ *(i.e. on its convenient strict transforms) are those in the chain ending at* Q_i.

ii') Z_{i+1} *being the blowing-up of* Q_i, $Z_{i+1}\to X$ *is an embedded resolution of* $\bigcup_{1\le j\le r_i} \Gamma_{i,j}$.

Proof. — The assertions i) and ii) are immediate consequences of 3.1.9 applied with $Y = E_{i_1}\cap\cdots\cap E_{i_{k-1}}$ in the ambient space Z_{i_k} containing Q_{i_k}, $I = I_{Q_{i_k}}$ the weak transform of I at Q_{i_k} and $r = d-k$. Then we consider the special case $k=1$.

The following picture represents the possible behavior of the various branches of $H_{f_1,\dots,f_{d-1}}$ after blowing-up Q_i :



When $d=2$, any primary ideal I for the maximal ideal M of $\mathcal{O}_{X,O}$ is finitely supported. If I is complete, its factorization into a product of simple complete ideals is given by the decomposition into branches of the curve in (X,O) defined by a general $f\in I$ (see [Z2], th. 11.2).

3.1.11. THEOREM (Zariski). — *Suppose* $d=2$. *Let* I *be a* M*-primary complete ideal of* $\mathcal{O}_{X,O}$ *and let* $D=\sum_{0\le i\le n} m_i E_i^*$ *be the exceptional divisor on* *the sky of the constellation of base points $C_I = \{Q_0, \ldots, Q_n\}$ of I associated to I (1.2.3).*

The following conditions are equivalent :

i) *I is simple.*

ii) *For a general $f \in I$, the curve H_f defined by f is analytically irreducible.*

iii) *$r_i = (-D)\bullet E_i = m_i - \sum_{j \to i} m_j = 1$ if $i = n$ and 0 otherwise.*

The map which takes I to C_I is a one to one correspondence between simple complete ideals and finite chains of infinitely near points originated at O.

One has the following factorization of I :

$$I = \prod_{0 \leq i \leq n} I_i^{r_i}$$

where I_i is the simple complete ideal whose constellation of base points is the descending chain from Q_i to O ; this is the unique factorization of I into simple complete ideals.

The exceptional divisor of the (normalized) blowing-up of I is irreducible if and only if $I = I_n^{r_n}$.

Proof. — Since $r_n = m_n > 0$, the equivalence between ii) and iii) follows immediately from 3.1.10.

Now for any M primary complete ideal I, consider the galaxy $\mathcal{G}$ of C_I. By 1.2.9 and 1.3.8, there exists a unique complete ideal I_i whose associated cluster $\mathcal{A}_{I_i}$ is in $\mathcal{G}$ and such that $D(\mathcal{A}_{I_i}) = -E_i^\vee$ where $(E_i^\vee)_{0 \leq i \leq n}$ is the $\mathbb{Z}$-basis of $\mathbb{E}$ such that $(E_i^\vee \bullet E_j) = 0$ for $j \neq i$, 1 otherwise. Since $(D \bullet E_i) = -r_i$, $0 \leq i \leq n$, one has $D = \sum_i r_i(-E_i^\vee)$, hence $\mathcal{A}_I = \sum_i r_i \mathcal{A}_{I_i}$ and since I is complete and $d = 2$, $I = \prod_i I_i^{r_i}$.

For any i, $0 \leq i \leq n$, the ideal I_i is simple. Indeed the base points of I_i are contained in C_I. Assume $I_i = J_1 \cdot J_2$ with ideals J_1, J_2 in $\mathcal{O}_{X,O}$. One also may assume J_1 and J_2 to be complete. The base points of J_1 and J_2 are among those of I_i, hence they are contained in C_I. Therefore there exist $D_h \in D(\mathcal{G})$ such that $D(\mathcal{A}_{J_i}) = D_h$, $h = 1, 2$ and $D(\mathcal{A}_{I_i}) = -E_i^\vee = D_1 + D_2$. From the characterization of $D(\mathcal{G})$ given in 1.3.8, it follows that either D_1 or D_2 is 0. Hence either J_1 or J_2 is $\mathcal{O}_{X,O}$.

Note also that $I_0, \ldots, I_n$ are the only simple complete ideals whose base points are contained in C_I ; indeed, if J is such an ideal, then $\mathcal{A}_J \in \mathcal{G}$, hence there exists $s_i \in \mathbb{Z}_{\geq 0}$, $0 \leq i \leq n$, such that $D(\mathcal{A}_J) = \sum_i s_i(-E_i^\vee)$, $J = \prod_i I_i^{s_i}$ and J being simple, is one of the I_i's. Therefore the above factorization of I is the unique one.

Finally, since $r_n > 0$, I itself is simple if and only if $r_n = 1$ and $r_i = 0$, $0 \leq i < n$ i.e. i) $\Leftrightarrow$ iii). Moreover, if this is so, for any general, hence proper $f \in I$, the curve H_f defined by f is a branch which passes effectively through the cluster $\mathcal{A}_I = (C_I, \underline{m})$ associated to I (3.1.4) ; in other words, for any i, $0 \leq i \leq n$, Q_i is a point on the strict transform of H_f. This forces $C_I = \{Q_0, \ldots, Q_n\}$ to be the descending chain from Q_n to O.

Now one can recover the weights $\underline{m}$ from C_I ; $\underline{m}$ is the only solution of the triangular system $m_n = 1$, $m_i - \sum_{j \to i} m_j = 0$, $0 \leq i < n$ and this system depends only on the proximity relations in C_I. Note that $E_n^{\vee} = \sum m_i E_i^*$.

On the other hand, if $C = \{Q_0, \ldots, Q_n\}$ is a chain, this same system produces a cluster for which the proximity inequalities hold trivially . Since $d = 2$, this cluster is associated to a complete ideal which is simple by the equivalence of iii) and i). Hence any chain originated at O is the constellation of one and only one simple complete ideal.

Finally, from 2.1.7, the exceptional divisor of the normalized blowing-up of I is irreducible if and only if $r_i = 0$, $0 \leq i < n$, i.e. if $I = I_n^{r_n}$.

The above results are used as an essential tool to understand the isolated singularities of surfaces obtained by blowing-up complete ideals in dimension 2. These singularities are the so-called "sandwiched singularities" ; cf. [S], [GS].

None of the previous assertions extends in higher dimension. In particular I may be simple and C_I may not be a chain. In [L2], §2, Lipman recovers a one to one correspondence between the set of finite chains of infinitely near points with origin at O and a class of special $*$-simple ideals in $\mathcal{O}_{X,O}$. However, the exceptional divisor of the normalized blowing-up of such ideals may still be reducible. Monomial examples are easily obtained from the combinatorial formula computing the weights in the associated cluster from the chain of points given in [C.G.L], th. 10.

We end this section by some more remarks on the singularities of hypersurfaces or more generally of complete intersections defined by a general r-uple of elements in a finitely supported ideal.

3.2.1. Remark. — Let (V, O) be a reduced curve singularity given by $f = 0$ in a non singular surface X. Then there exists a complete ideal I in $\mathcal{O}_{X,O}$ such that $f \in I$ and is non-degenerate with respect to the cluster $\mathcal{A}_I$ (3.1.7). Such an I is not uniquely determined.

Proof. — Consider any embedded resolution $\pi : Z \to X$ of V ; π is a finite composition of point blowing-ups and we get a cluster $\mathcal{A}$ by giving for weight to each blown-up point Q, the multiplicity $e_Q(V)$ of the strict transform of V at Q. The proximity inequalities hold trivially for $\mathcal{A}$ since its associated exceptional divisor $D(\mathcal{A})$ is the exceptional part of the total transform π^*V of V, hence for any irreducible component E_i of $\pi^{-1}(O)$, $r_i = (-D)\bullet E_i$ is the number of branches of V whose strict transforms on Z meet E_i and $r_i \geq 0$. Let I be the stalk of $\pi_*\mathcal{O}_Z(-D)$ at O ; I is complete and since $d = 2$, $\mathcal{A} = \mathcal{A}_I$ (1.2.8, 1.3.7, 2.1.5). By definition, V passes through $\mathcal{A}$, so $f \in I$; finally f is non-degenerated with respect to $\mathcal{A}_I$ because π is an embedded resolution of $V = H_f$. ∎

Following the comparison between clusters and Newton polygons introduced in 3.1.7 recall that, on the contrary, in general there does not exist any coordinate system for which f is non degenerate with respect to its Newton polygon. If the characteristic of K is zero and if (V, O) is analytically irreducible, this is so if and only if (V, O) has only one Puiseux characteristic exponent. In higher dimension, these singularities are quite special, since they admit an embedded resolution and hence a desingularization which is a finite composition of point blowing-ups (*). As for curves, one may find among them, hypersurface singularities which are defined by degenerate functions with respect to their Newton polyhedron in any coordinate system.

3.2.2. Example. — Let (S, O) be the surface in $\mathbb{C}^3$ defined by :

$$f = z^4 + z^2x^2 + zx^3y - (y^2 - x^3)^2 + yx^6 .$$

The ideal $I = (z^4, z^2x^2, zx^3y, (y^2 - x^3)^2, yx^6)$ is finitely supported and f is non-degenerate with respect to $\mathcal{A}_I$.

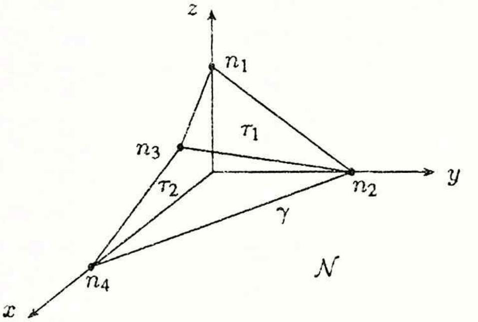
There does not exist any coordinate system in which f becomes non-degenerate with respect to its Newton polyhedron.

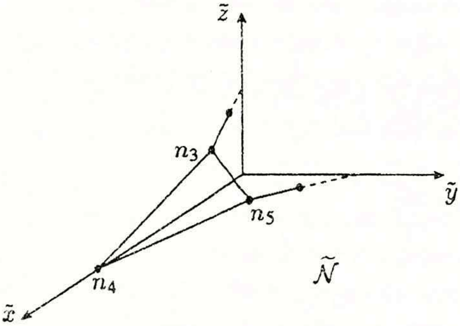
Proof. — Let Γ be the intersection of S with the plane $z = 0$; Γ has two Puiseux characteristic pairs 3/2 and 9/2. One can check that $\mathcal{A}_I = (C_I, \underline{m})$ where $C_I = \{Q_0 = O, \dots, Q_5\}$ is the chain of infinitely near points of O of level at most 5 lying on Γ and $\underline{m} = \{4, 2, 2, 2, 1, 1\} = \{e_{Q_i}(\Gamma)\}_{0 \leq i \leq 5}$.

The Newton polyhedron $\mathcal{N}$ of f with respect to (x, y, z) has four vertices $n_1 = (0, 0, 4)$, $n_2 = (0, 4, 0)$, $n_3 = (2, 0, 2)$, $n_4 = (6, 0, 0)$, five edges and two 2-dimensional faces $\tau_1 = \langle n_1, n_2, n_3 \rangle$ and $\tau_2 = \langle n_2, n_3, n_4 \rangle$ respectively normal to (1,1,1)

(*) Nevertheless, this property is not enough to characterize them. For example, an embedded resolution of the singularity of type $T_{5,5,5}$ at the origin O of the surface defined by $f = xyz + x^5 + y^5 + z^5$ in $\mathbb{C}^3$, is obtained by successively blowing-up O and the three double-points of its strict transform ; but there exists no finitely supported ideal I such that f is non-degenerate with respect to $\mathcal{A}_I$.

and (2,3,4).





The polynomial f is degenerate with respect to $\mathcal{N}$ because the sum of its monomials f_γ on its edge $\gamma = \langle n_2, n_4 \rangle$ is $(y^2 - x^3)^2$, hence the scheme defined by f_γ has singular points in $\mathbb{C}^{*3} = \{(x, y, z) \in \mathbb{C}^3, xyz \neq 0\}$.

Now we discuss the effect of a coordinate change on $\mathcal{N}$.

First we observe that, up to permutation and scalar multiplication, a change of coordinates which would make f non-degenerate with respect to its Newton polygon should be of the form :

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} \tag{3.2.2.1}$$

where $*$ denotes any complex number. (This is also equivalent to saying that Q_0, Q_1, Q_2 remain 0-orbits of the natural action of $\mathbb{C}^{*3} = \{(\tilde{x}, \tilde{y}, \tilde{z}) \in \mathbb{C}^3 \mid \tilde{x}\tilde{y}\tilde{z} \neq 0\}$ on $\mathbb{C}^3$.) Indeed the singular locus of the tangent cone of S at O is the line $y = z = 0$ and it should remain the intersection of two coordinate hyperplanes, say $\tilde{y} = \tilde{z} = 0$, hence the first column of the matrix. Now the coefficient λ of $\tilde{y}$ in z must vanish, otherwise in addition to n_3, the point $n_5 = (2, 2, 0)$ should be a vertex of the Newton polyhedron $\widetilde{\mathcal{N}}$ of f with respect to $(\tilde{x}, \tilde{y}, \tilde{z})$. The sum of its monomials on $\langle n_3, n_5 \rangle$, namely $(\tilde{z} + \lambda\tilde{y})^2\tilde{x}^2$ would define a scheme having singular points in $\tilde{x}\tilde{y}\tilde{z} \neq 0$, contradicting the fact that f is non-degenerate with respect to $\widetilde{\mathcal{N}}$.

This computation implies in particular that $\langle n_2, n_3 \rangle$ remains the intersection of the faces $\tilde{\tau}_1$ and $\tilde{\tau}_2$ of $\widetilde{\mathcal{N}}$ respectively normal to (1,1,1) and (2,3,4). Now let f_{τ_2} (resp. $f_{\tilde{\tau}_2}$) denote the sum of monomials of f in (x, y, z) on τ_2, (resp. $(\tilde{x}, \tilde{y}, \tilde{z})$ on $\tilde{\tau}_2$). One has :

$$f_{\tau_2} = z^2x^2 + (y^2 - x^3)^2$$

and it follows from 3.2.2.1 that there exists $\mu \in \mathbb{C}$ such that

$$f_{\tilde{\tau}_2} = f_{\tau_2}(\tilde{x}, \tilde{y}, \tilde{z} - \mu\tilde{x}^2).$$

Indeed, the homogeneous polynomial of lowest degree for the weights (2,3,4) in $(\tilde{x},\tilde{y},\tilde{z})$ appearing respectively in x,y,z, is $\tilde{x},\tilde{y},\tilde{z}-\mu\tilde{x}^2$ for some $\mu \in \mathbb{C}$. If $\mu \neq 0$, the singular locus of the surface defined by $f_{\tilde{\tau}_2}$ is the curve given by $\tilde{x}=t^2$, $\tilde{y}=t^3$, $\tilde{z}=\mu t^4$. Since this curve has points in $\tilde{x}\tilde{y}\tilde{z}\neq 0$, f is degenerate with respect to $\widetilde{\mathcal{N}}$.

If $\mu = 0$, then $f_{\tilde{\tau}_2} = f_{\tau_2}$, $\tau_2 = \tilde{\tau}_2$, γ remains an edge of $\tilde{\tau}_2$ and f remains degenerate for $\widetilde{\mathcal{N}}$. This completes the proof. ■

Note that f being proper in I, Q_3 is a double point of the intersection of the strict transform of S with the exceptional divisor of the blowing-up of Q_2. The last computation implies that Q_3 lies on a 1 or 2-dimensional orbit of any $\mathbb{C}^{*3}$-action on $\mathbb{C}^3$ leaving $Q_0,\ldots,Q_2$ fixed. The points $Q_0,\ldots,Q_3$ are intrinsically characterized from S, since they are its infinitely near singular points.

Finally we observe that :

3.2.3. PROPOSITION. — *The minimal desingularization of a complete intersection surface defined by general elements in a finitely supported ideal I is a composition of point blowing-ups, namely those $Q \in C_I$ such that $m_Q \neq 1$, where $\mathcal{A}_I = (C_I, \underline{m})$ is the cluster associated to I.*

Proof. — We have already noticed in proving 3.1.8 that if $f_1,\ldots,f_r$ is non-degenerate with respect to $\mathcal{A}_I$ and if $f'_{1Q},\ldots,f'_{rQ}$ and I_Q denote respectively the strict and the weak transform of $f_1,\ldots,f_r$ and I at $Q \in C_I$, then $f'_{1Q},\ldots,f'_{rQ}$ is non-degenerate with respect to $\mathcal{A}_{I_Q}$. Therefore if $r = \dim(X,O)-2$ and S denotes the complete intersection $H_{f_1,\ldots,f_r}$, it is enough to prove that the minimal desingularization $\pi : V \to S$ factors through the blowing-up $\sigma_1 : S_1 \to S$ with center O, provided O is a singular point of S, i.e. $m_O \neq 1$ by 3.1.8.

Let $\pi_1 : V_1 \to S_1$ be the minimal desingularization of S_1. Because of the minimality property of π, there exists a commutative diagram

$$\begin{array}{ccc} V_1 & \xrightarrow{\tau_1} & V \\ {\scriptstyle\pi_1}\downarrow & & \downarrow{\scriptstyle\pi} \\ S_1 & \xrightarrow[\sigma_1]{} & S \end{array}$$

and since the morphism τ_1 is birational and V_1 and V are non singular surfaces, τ_1 is a composition of point blowing-ups. Actually, M being the maximal ideal of $\mathcal{O}_{X,O}$, τ_1 is the minimal sequence of point blowing-ups which makes the inverse image of M invertible.

If τ_1 is not an isomorphism, τ_1 contracts an exceptional curve of the first kind E of V_1 on a point $P \in V$ such that $M\mathcal{O}_V$ is not invertible locally at P, while, π_1 being the minimal desingularization of S_1, π_1 may not contract E on a point of S_1. Hence there exists an irreducible component F of $\sigma^{-1}(O)$ such that $\tau(E) = F$.

We will now get a contradiction by analyzing the geometric behavior of a general hyperplane section Δ of S. Here general means that Δ is the schematic intersection of S and of a non singular hypersurface Y in (X, O) whose projective tangent hyperplane $H = \mathrm{Proj}\, T_{Y,O}$ at O does not contain any $Q \in C_I$ of level 1 and intersects $\sigma_1^{-1}(O)$ identified with the projective tangent cone of S at O transversally.

Recall that $f_1, \ldots, f_r$ being non-degenerate and $W_O(f_i)$ denoting the exceptional divisor of the blowing-up of O in the hypersurface H_{f_i}, $1 \leq i \leq r$, one has $\sigma_1^{-1}(O) = W_O(f_1) \cap \cdots \cap W_0(f_r)$ and that any Q on $\sigma_1^{-1}(O)$, which is not in C_I is a non singular point of $\sigma_1^{-1}(O)$ and S_1 (3.1.7–3.1.8). This implies in particular that the exceptional divisor $\sigma_1^{-1}(O)$ is reduced and that Δ is a reduced complete intersection curve.

Now since $F \cap H \neq \emptyset$, on the one hand Δ has a branch Γ whose strict transform on S_1 (resp. V_1) meets F (resp. E). Therefore the exceptional point of the strict transform of Γ on V is P.

On the other hand, one gets an embedded resolution of S by blowing-up the points $Q \in C_I$ and the conditions imposed on Y imply that the blowing-up with center O in X is an embedded resolution of Δ. Therefore the branches of Δ are non singular and intersect transversally ; in particular Γ is non singular.

If O is a singular point of S, we have got our contradiction. Indeed S is normal at O because S is a complete intersection and O is an isolated singular point. The exceptional fiber of $\pi(O)$ has no isolated points. An easy computation shows that $M\mathcal{O}_V$ is invertible on a neighborhood of the exceptional point of the strict transform of any non singular curve on S.

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Geometry of $\mathbb{Z}^d$ and when does the Central Limit Theorem hold for weakly dependent random fields. *

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ABSTRACT.

We study the asymptotic distribution of the sequence $S_N(A,X)= \sqrt{(2N+1)^{-d}} \left(\sum_{n \in A_N} X_n \right)$, where A is a subset of $\mathbb{Z}^d$, $A_N = A \cap [-N,N]^d$, $v(A)= \lim_N \operatorname{card}(A_N) (2N+1)^{-d} \in (0,1)$ and X is a stationary weakly dependent random field. We show that the geometry of A has a relevant influence on the problem. More specifically, $S_N(A,X)$ is asymptotically normal for each X that satisfies certain mixing hipotheses if and only if A verifies that $F_N(n;A)= \operatorname{card}\{A_N^c \cap (n+A_N)\}(2N+1)^{-d}$ has a limit $F(n;A)$ as $N\to\infty$, for each $n \in \mathbb{Z}^d$. We also study the class of sets A that satisfy this condition. As an application, we develop an asymptotic test for the comparison of the mean of two weakly dependent spatial samples.

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1. Introduction, Notation and Statement of the Main Result.

A problem common to many disciplines is that of comparing samples of dependent spatial data. Spatial Statistics arise naturally in environmental science, geostatical analysis, epidemiological studies and other areas (for an expert account see [13], [27]). A probabilistic model that takes into account dependence is reasonable in many cases, in particular when there are transference phenomena that "mix" the information of the different sample points. We will consider here a very simple model (the sample corresponds to a stationary mixing random field with finite second moment) and a very simple problem: a test for the homogeneity of the mean against the assumption that there is a difference in the mean in two given subsets of the sample space.We will consider the case of both discrete and continuous data and develop asymptotic methods to solve it. Since the basic ideas of the method are the same in both cases, we will concentrate our presentation in the discrete case, that is when the random field is indexed by the lattice $\mathbb{Z}^d$.

If we consider an iid sample, then, via the Central Limit Theorem (CLT, for short), we can construct an statistic whose asymptotic distribution is an F of Fischer, with excentricity 0 under the hypothesis of homogeneity, and with an excentricity that goes to infinity under the alternative. This give us an asymptotic F-test with a given level of significance and consistent for each fixed alternative (we will call it F-test, for short). If the sample is dependent, we show that the viability of an F-test depends strongly on the geometry of the subsets that we are compairing. If the sample has a mixing property (even if it is m-dependent) and if the border between this subsets is "regular", we have a suitable CLT and hence, an F-test; but, if that border is "very irregular", CLT can fail to hold, and thus, we do not have an F-test anymore. Therefore, we will discuss results related to the CLT, showing the influence of the geometrical factor, and as an application, we will give an F-test when it is possible. In what follows, we will precise the problem and the notations.

Consider a real valued random field $X = \{ X_n : n \in \mathbb{Z}^d \}$. Given a subset A of $\mathbb{Z}^d$ denote: $A_N = A \cap [-N;N]^d$. Denote $G(\mathbb{Z}^d)$ the class of subsets for wich the limit :

$$v(A) = \lim_N \operatorname{card}(A_N)\,(2N+1)^{-d} \quad \text{exists and } 0 < v(A) < 1. \quad (1.1)$$

In $\mathbb{Z}^d$ we will take the distance d induced by the restriction of the norm: $\|n\| = \max\{|n(i)| : 1 \le i \le d\}$,and use the following notations for $A \subset \mathbb{Z}^d$, $B \subset \mathbb{Z}^d$:

$$d(A,B) = \min\{d(n,m) : n \in A,\ m \in B\},\ \partial A = \{n \in A : d(n,A^c) = 1\}.$$

Define:

$$M_N(A;X) = \sqrt{(2N+1)^{-d}}\,\Big(\sum_{n \in A_N} X_n;\ \sum_{n \in (A^c)_N} X_n\Big)$$

and

$$S_N(A;X) = \sqrt{(2N+1)^{-d}}\,\Big(\sum_{n \in A_N} X_n\Big)$$

<u>**Definition 1.1.:**</u> We will say that a subset A of $\mathbb{Z}^d$ belongs to the class M ($\mathbb{Z}^d$) if it belongs to G ($\mathbb{Z}^d$) and satisfies the following condition:

The sequence $F_N(n;A) = \operatorname{card}\{A^c_N \cap (n+A_N)\}(2N+1)^{-d}$ has a limit $F(n;A)$ as $N \to \infty$, for each $n \in \mathbb{Z}^d$. (1.2)
(We will say that F(.;A) is the ***border function*** of the set A and that A has an ***asymptotically measurable border***).

In the case d=1, for the construction of invariant means via Nonstandard Analysis, a definition than can be considered related to Definition 1.1 appears(cf. [28], page 86). We will also extend our definition to the continuous case.

We will deal with mixing random fields. Under the assumption of stationarity and mixing it is possible to extend most of the asymptotic results for iid sequences (for a detailed study, see [8], [19], [21] and [22]; for examples of statistical models satisfying mixing conditions see [16]). Given a probability space $(\Omega,\mathbf{A},P)$ and $\mathbf{F}$, $\mathbf{G}$ sub σ- fields of $\mathbf{A}$, the α and ρ - mixing coefficients between $\mathbf{F}$ and $\mathbf{G}$ are defined by :

$$\alpha(\mathbf{F},\mathbf{G})=\sup\{\,|P(A\cap B)-P(A)P(B)|:\ A\in\mathbf{F},\ B\in\mathbf{G}\,\}. \tag{1.3}$$

$$\rho(\mathbf{F},\mathbf{G})=\sup\{\,|\mathrm{Corr}(X,Y)|:\ X\in L^2(\mathbf{F}),\ Y\in L^2(\mathbf{G})\,\}. \tag{1.4}$$

If we have a real valued random field $X=\{X_t : t\in T\}$ with $T=\mathbb{Z}^d$ or $\mathbb{R}^d$ and $A\subset T$, we will denote by $\sigma^X(A)$ the σ - field generated by $\{X_t : t\in A\}$. We have several alternatives for the definition of the α - mixing coefficients of the field X: (here $m\in\mathbb{N}$)

$$\alpha^X(\Phi,m)=\sup\{\alpha(\sigma^X(A),\sigma^X(B)):\ A\subset T,\ B\subset T,\ d(A,B)\geq m\}$$
$$\alpha^X(\Sigma,m)=\sup\{\alpha(\sigma^X(A),\sigma^X(B)):\ A\subset T,\ B\subset T,\ A,B\in\Sigma,\ d(A,B)\geq m\}$$
$$\alpha^X(\Pi,m)=\sup\{\alpha(\sigma^X(A),\sigma^X(B)):A\subset T,\ B\subset T,\ A\in\Pi,\ d(A,B)\geq m\}\ (1.5),$$

where Σ stands for the half-spaces of T and Π for the rectangles of T with sides paralel to the coordinated axes . It follows from the definitions that $\alpha^X(\Phi,m)\geq\alpha^X(\Sigma,m)\geq\alpha^X(\Pi,m)$. In a similar way, we can define the ρ- mixing coefficients of X . If $\xi^X(\mathcal{X},m)$ goes to zero as m goes to infinity (where $\xi=\alpha$ or ρ and $\mathcal{X}=\Phi$, Σ or Π) , we will say that X is $\xi^X(\mathcal{X})$ mixing . In general $\rho^X(\mathcal{X})$ mixing implies $\alpha^X(\mathcal{X})$ mixing (cf. [8], [16]).

Bradley has given an extension of Kolmogorov-Rozanov inequality that implies the following result (cf. [11]):

If X is strictly stationary then:

$\alpha^X(\Phi)$ and $\rho^X(\Phi)$ mixing are equivalent ; If d> 1 $\alpha^X(\Sigma)$ and $\rho^X(\Sigma)$ mixing are equivalent. (1.6)

This result, based on a nice application of the CLT for mixing sequences, gives the idea that $\alpha^X(\Phi)$ (and $\alpha^X(\Sigma)$ for d> 1) mixing is a strong assumption. The coefficient $\alpha^X(\Pi;m)$ was introduced by Bulinskii, who has given examples showing that $\alpha^X(\Pi)$ mixing is actually weaker than $\alpha^X(\Sigma)$ mixing (cf. [4], [5], [6]).

In this paper we will also consider the coefficients $\alpha^X(\Pi;m;a,b)$, with a, b, $m\in\mathbb{N}$. They are defined as in (1.5), but with the additional restriction:

$$\mathrm{card}(A) \le a,\ \mathrm{card}(B) \le b. \quad (1.7)$$

We will use the following well-known covariance inequalities (cf. [8], [16], [21]).

If X is $\mathbf{F}$ - measurable random variable, Y is $\mathbf{G}$ - measurable and they are both a.s. bounded by 1, then $|\mathrm{Cov}\,(X,Y)| \le 4\ \alpha(\mathbf{F}, \mathbf{G})$. (1.8)

More in general, if p,q, r ≥ 1, $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$, $X\in L^p(\mathbf{F})$, $Y\in L^q(\mathbf{G})$,

then: $|\mathrm{Cov}\,(X,Y)| \le 8\ (\alpha(\mathbf{F}, \mathbf{G}))^{\frac{1}{r}}\ \|X\|_p\ \|Y\|_q$. (1.9)

We will consider two sets of assumptions for our field X, we will list in what follows all the hypotheses we will use.

(H1) $E(X_n)=0$, for each $n\in\mathbb{Z}^d$.

(H2) X is strictly stationary

(H3) $E(|X_0|^2) < \infty$.

(H4) $\lim_m \alpha^X(\Pi;m;\infty,\infty)=0$.

(H5) $\sum_{m=1}^{\infty} m^{d-1}\alpha^X(\Pi;m;a,b) < \infty$ for $a+b\leq 4$

(H6) $\lim_m m^d\alpha^X(\Pi;m;1,\infty)=0.$

(H7) $\sum_{n\in\mathbb{Z}^d} |r^X(n)|<\infty$, where $r^X(n)=E(X_0X_n)$.

(H8) $\sum_{n\in\mathbb{Z}^d} r^X(n)=\sigma^2(X)>0$

(H9) For each $A\subset\mathbb{Z}^d$, $\lim_J \limsup_N E\{(S_N(A, X-X^J))^2\}=0$,

where X^J is the truncation by J of the random field X, defined by:

$$X_n^J=X_n 1_{\{|X_n|\leq J\}} - E\{X_n 1_{\{|X_n|\leq J\}}\}.$$

(H10) There is a real number $C(J)>0$, depending only on X and J such that $\sup_{A\subset[-N,N]^d} E\{(S_N(A, X^J))^4\}\leq C(J)\left[\frac{\mathrm{card}(A_N)}{((2N+1)^d)}\right]^2$.

<u>Definition 1.2.:</u> We will call **B** to the class of real valued random fields $X=\{X_n : n\in\mathbb{Z}^d\}$ that satisfy (H1), (H2), (H3), (H4), (H7), (H8), (H9) and (H10). We will call **S** to the class of random fields that satisfy (H1), (H2), (H3), (H5),(H6), (H8) and (H9). Finally, let us call **F** to the union of **B** and **S**.

<u>Remarks 1.1. :</u>

a) The basic idea is that the class B allow us to derive a CLT using Bernshtein's "big blocks" method (cf. [1]), while for the class S we can obtain a CLT by using Stein's methods (cf. [31]). It should be noticed that in (H4) no rate is assumed, but since "big blocks" are involved, it is sometimes difficult to verify this kind of conditions. In certain applications, like Gibbs fields , conditions of asymptotic independence between very big blocks are quite useless (cf. [15]). Stein's method requires conditions on "small blocks", like (H5), and asymptotic independence between one-point sets and arbitrary sets, like (H6). On the other hand, in (H5) and (H6) rates are required. This is the reason why we prefer to include both classes as alternatives.

b) We will see later that we can replace (H9) and (H10) by the condition:

(M) $\rho^X(\Pi;1;\infty,\infty)<1$, that is if X satisfies (H1), (H2), (H3), (H4), (H7), (H8) and (M), X belongs to $\mathbb{B}$.

c) We will also see that we can replace (H9) by the condition:

$$\text{(R)} \int_0^1 \alpha^{-1}(u)\ (Q(2u))^2\ du<\infty\ ;$$ where Q is the quantile function defined by $Q(u)=\inf\{t>0: P(|X_0|>t)\leq u\}$ and α^{-1} depends on the mixing coefficients between one-point sets:

$$\alpha^{-1}(u)=\text{card}\{n\in\mathbb{Z}^d : \alpha(\sigma^X(\{0\}),\sigma^X(\{n\}))>u\}.$$

As an example, if X satisfy (H1), (H2), (H3), (H5), (H6), (H8) and (R), then X belongs to $\mathbb{S}$.

We can present now our main result.

<u>**Theorem 1.1:**</u> Let $A\in G(\mathbb{Z}^d)$. Then the following statements are equivalent:

(i) For each $X\in F$, the random variable: $S_N(A,X)$ has a weak limit $S(A,X)$.

(ii) For each $X\in F$, the random vector $M_N(A,X)$ has a weak limit $M(A,X)$.

(iii) For each $X\in F$, the random vector $M_N(A,X)$ has a weak limit $M(A,X)$, whose distribution is gaussian, centered and with variance matrix $C(X)$ given by: $C(X)_{11}=v(A)\sigma^2(X)-\gamma(X)$, $C(X)_{12}=C(X)_{21}=\gamma(X)$, $C(X)_{22}=(1-v(A))\sigma^2(X)-\gamma(X)$, where

$$\gamma(X)=\sum_{n\in\mathbb{Z}^d} r^X(n)F(n;A), \text{ with } F(.;A): \mathbb{Z}^d \to [0,1]\,. \quad (1.10)$$

(iv) $A\in M(\mathbb{Z}^d)$.

__Remarks 1.2.:__

a) The funcion $F(.;A)$ that appears in (1.10) is the border function of A.

b) In fact, we are going to show that if $A\notin M(\mathbb{Z}^d)$ there is a stationary, centered and m-dependent gaussian field X such that $S_N(A,X)$ has no weak limit .

c) From (H7) It follows that $\gamma(X)$ is well-defined.

The paper is organized as follows: in section 2 we pesent the proof of Theorem 1.1, together with some preliminary results. In section 3 we study the class $M(\mathbb{Z}^d)$, giving examples of sets that are not included in this class. We also include in this section an extension of Theorem 1.1., called Proposition 3.1: we consider there a k-valued random vector, corresponding to the normalized mean of X over k disjoint subsets whose union is $\mathbb{Z}^d$ (Theorem 1.1 corresponds to the case k=2). As an application of Proposition 3.1, we develop in section 4 an asymptotic F-test for the comparison of means. The result of the application of this test to some simulated samples is shown in the appendix.

2. Central Limit Theorems.

Theorem 1.1 follows as an inmediate consequence of Propositions 2.1, 2.3 and 2.4. The proofs that we will present use standard methods for mixing processes (cf. [7], [14], [19], [22]).

Proposition 2.1:

a) Let A be a subset of $\mathbb{Z}^d$. Then: $E\{(S_N(A,X))^2\} = \sum_{n\in\mathbb{Z}^d} r^X(n)H_N(n;A)$, with

$H_N(n;A) = \text{card}\{A_N \cap (n+A_N)\}(2N+1)^{-d}$.

b) If X satisfies (H7) then there is a constant C depending only on X such that, for any A subset of $\mathbb{Z}^d$, $E\{(S_N(A,X))^2\} \leq C \dfrac{\text{card}(A_N)}{(2N+1)^d}$.

c) If $A\in G(\mathbb{Z}^d)$, then : $\lim_N (H_N(n;A) + F_N(n;A)) = v(A)$.

d) If $A\in G(\mathbb{Z}^d)$ and $A\notin M(\mathbb{Z}^d)$ there is a stationary, centered, m-dependent gaussian field X such that $S_N(A,X)$ has no weak limit.

Proof: We have: $E\{(S_N(A,X))^2\} = (2N+1)^{-d}\left(\sum_{k,m\in A_N} E(X_k\ X_m)\right) =$

$\sum_{n\in\mathbb{Z}^d} r^X(n)C_N(n;A)$, with $C_N(n;A) = \text{card}\{(k,m)\in A_N \times A_N : k-m = n\}(2N+1)^{-d} =$

$H_N(n;A)$, and a) follows.

To prove b), note that $H_N(n;A)$ is bounded by $\text{card}(A_N)$ and use (H7).

Consider $A\in G(\mathbb{Z}^d)$; we have that:

$H_N(n;A) + F_N(n;A) = \text{card}\{[-N,N]^d) \cap (n+A_N)\}(2N+1)^{-d}$ (2.1) ;

If $m\in A_{N-\|n\|}$ then $m+n\in[-N,N]^d \cap (n+A_N)$ and it is obvious that

$\lim_N(\mathrm{card}(A_N) - \mathrm{card}(A_{N-\|n\|}))(2N+1)^{-d}=0$; from this and (2.1), c) follows.

Consider $A \notin M(\mathbb{Z}^d)$; pick $n^* \in \mathbb{Z}^d$ such that the sequence: $\{F_N(n^*;A): N \in \mathbb{N}\} \subset [0;1]$ has no limit (it is obvious that we can take $n^* \neq 0$) .

Then there are two subsequences $F_{N_{(m,1)}}(n^*;A)$ and $F_{N_{(m,2)}}(n^*;A)$ that converges to $\phi(1)$ and $\phi(2)$, respectively. Let us consider a centered, stationary, gaussian field X with covariances: $r^X(0)=1$, $r^X(n)=\rho \in (0,\frac{1}{2})$ for $n=n^*,-n^*$ and $r^X(n)=0$ for $n \neq 0, n^*, -n^*$. It follows that X is $\|n^*\|$- dependent.

From a) and c) we have that for i=1,2, $E\{(S_{N_{(m,i)}}(A;X))^2\}$ converges to $v(A)(1+\rho)-\phi(i)\rho$; since X is gaussian, it follows that $S_{N_{(m,1)}}(A;X)$ converges to a $N(0,v(A)(1+\rho)-\phi(1)\rho)$ distribution and that $S_{N_{(m,2)}}(A;X)$ converges to a $N(0,v(A)(1+\rho)-\phi(2)\rho)$ distribution. Hence, $S_N(A;X)$ has no weak limit ♦

The following result, whose proof is elementary, characterize the possible limits under some of the hypotheses considered above.

Proposition 2.2: Let X be a random field that satisfy (H1), (H2), (H3), (H4), (H7) and A a subset of $\mathbb{Z}^d$. If $S_N(A,X)$ converges weakly to a random variable Z, then Z is gaussian.

Proof: Take 0<s<1 and N(s) the integer part of sN. Then we get:

$$S_N(A,X)= S_{N(s)}(A,X)\left[\sqrt{\left(\frac{2N(s)+1}{2N+1}\right)^d}\right] + S_N(A-A_{N(s)},X) \quad (2.2).$$

The first term on the right side of (2.2) converges weakly to $\sqrt{s^{-d}}\, Z$; by Proposition 2.1., part b), $S_N(A - A_{N(s)},X)$ is uniformly integrable and hence, tight. Choose a convergent subsequence of $S_N(A - A_{N(s)},X)$ and call Y(s) to its limit. Both Z and Y(s) are centered and with finite second moment.

Take q(N) increasing to infinity such that $\lim_N \frac{q(N)}{N} = 0$.

Applying again Proposition 2.1., part b), it follows that we can replace $S_{N(s)}(A,X)\ [\sqrt{(\frac{2N(s)+1}{2N+1})^d}\ ]$ by $S_{N(s)-\ q(N(s))}(A,X)\ [\sqrt{(\frac{2N(s)+1}{2N+1})^d}\ ]$ in (2.2.) , without changing asymptotic distributions.

It follows that if ψ_N stands for the characteristic function of $S_N(A,X)$ and ν_N stands for the characteristic function of $S_N(A - A_{N(s)},X)$, then, by (1.9):

$|\psi_N(t) - \psi_{N(s)-\ q(N(s))}(\sqrt{(\frac{2N(s)+1}{2N+1})^d}\ t)\ \nu_N(t)\ | \le 8\alpha^X(q(N))$ and then, by (H4), we deduce that if ψ is the characteristic function of Z and $\nu^{(s)}$ the characteristic function of Y(s), then:

$$\psi(t) = \psi([\sqrt{s^{-d}}]\ t)\nu^{(s)}(t)\ , \text{ for each real } t,\ 0<s<1. \quad (2.3)$$

This implies that Z belongs to the class L of Kintchine (see [18], page 553); what follows is just the proof that the only distribution with finite second moment that belongs to this class is the gaussian.

Iterating (2.3) we obtain that Z is the weak limit of a triangular array of independent random variables (centered and with bounded second moment); therefore Z is infinitely divisible and its characteristic never vanishes.

From (2.3), we obtain:

$$\nu^{(s)}([\sqrt{s^{-d}}]t)\nu^{(s)}(t)= \frac{\psi([\sqrt{s^{-d}}]\ t)}{\psi([s^{-d}]\ t)}\nu^{(s)}(t) =$$

$$\frac{\psi(t)}{\psi([s^{-d}]\ t)} = \nu^{(s^2)}(t). \quad (2.4)$$

Iterating (2.4) we deduce that for each $0<s<1$, $Y(s)$ is infinitely divisible.

Then, we can use the following representation (see [2], pages 384-388)

$$\psi(t)=\exp\{\int_{-\infty}^{\infty} \frac{\exp(itx)-1-itx}{x^2}\, d\xi(x)\},$$

$$\nu^{(s)}(t)= \exp\{\int_{-\infty}^{\infty} \frac{\exp(itx)-1-itx}{x^2}\, d\theta^{(s)}(x)\},$$

where $\xi,\theta^{(s)}$, are positive finite measures that characterize ψ and $\nu^{(s)}$. (2.5)

From (2.3) and (2.5) it follows that:

$$d\xi(x)=[\sqrt{s^{d}}]d\xi([\sqrt{s^{-d}}]x) + d\theta^{(s)}(x) \text{ for all real } x, 0<s<1. \quad (2.6)$$

It follows from (2.6) that if:

$M^{(s)}(a) = \int_{[a,\frac{a}{s}]} \frac{1}{x^2}\, d\xi(x) + \int_{[-\frac{a}{s},-a]} \frac{1}{x^2}\, d\xi(x)$ for $a>0$, $M^{(s)}$ is an increasing function of a for each $0<s<1$.

If ξ is not concentrated in the origin, then there is an $0<s_0<1$ and an $a_0>0$, such that $M^{(s_0)}(a_0)>0$; but for $a \geq a_0$ we have that:

$\xi([a,\frac{a}{s}]U[-\frac{a}{s},-a]) \geq a^2M^{(s_0)}(a) \geq a^2M^{(s_0)}(a_0)$, that goes to infinity with a, what contradicts the finitess of ξ◆

In what follows, we give the proof of the CLT using

Bernshtein's method.

<u>Proposition 2.3.:</u> Let X be a real valued random field such that $X \in B$, and consider $A \in M(\mathbb{Z}^d)$; then the random vector $M_N(A,X)$ has a weak limit $M(A,X)$, whose distribution is gaussian, centered and with variance matrix $C(X)$ given by:

$C(X)_{11} = v(A)\sigma^2(X) - \gamma(X)$, $C(X)_{12} = C(X)_{21} = \gamma(X)$, $C(X)_{22} = (1-v(A))\sigma^2(X) - \gamma(X)$, (where $\gamma(X)$ is defined as in (1.10) taking as $F(.;A)$ the border function of the set A).

<u>Proof:</u> Since (H9) holds, it is enough to prove the result for X bounded.

Take λ, μ two real numbers such that:

$\tau(\lambda,\mu) = \lambda^2[v(A)\sigma^2(X) - \gamma(X)] + 2\lambda\mu\gamma(X) + \mu^2[(1-v(A)\sigma^2(X) - \gamma(X)] > 0$ (2.7), where $\gamma(X)$ is defined as in (1.10) taking as $F(.;A)$ the border function of the set A.

It is enough to prove that :

$W_N(\lambda,\mu) = \sqrt{(2N+1)^{-d}}\,(\lambda \sum_{n \in A_N} X_n + \mu \sum_{n \in (A^c)_N} X_n)$ converges weakly to a

$N(0,\tau(\lambda,\mu))$ distribution, where $A_N = A \cap [-N;N]^d$ (λ,μ will be assumed fixed).

Consider the real valued random field:

$X(\lambda,\mu) = \{X_n(\lambda,\mu) : n \in \mathbb{Z}^d\}$, where $X_n(\lambda,\mu) = \lambda X_n$ if $n \in A$, and $X_n(\lambda,\mu) = \mu X_n$ in other case.

Set: $S_N(\lambda,\mu)(C) = \sqrt{(2N+1)^{-d}} \sum_{n \in C_N} X_n(\lambda,\mu)$. (2.8)

Consider two non-decreasing sequences of natural numbers $p(N)$, $q(N)$ such that:

$\lim_N p(N)=\lim_N q(N)=\infty;$

$$\lim_N \frac{q(N)}{p(N)} = \lim_N \frac{p(N)}{N} = \lim_N k(N)\alpha^X(\Pi;q(N);\infty,\infty)=0;$$

where $k(N)=\text{int}(\frac{2N}{p(N)+q(N)})^d$ and "int" stands for the integer part (the existence of such sequences fllows from (H4)).

A bit of notation: for $i=0;1;...;\text{int}(\frac{2N}{p(N)+q(N)})-1$, call $J_N(i)$ to the interval $[-N+ip(N)+iq(N),-N+(i+1)p(N)+iq(N)]$; consider their union, $J_N = \cup\{J_N(i): i=0,1,...,\text{int}(\frac{2N}{p(N)+q(N)})-1\}$ and $\Delta_N=(J_N)^d$.

Δ_N is the union of $k(N)$ disjoints d-cubes of side $p(N)$:

$\Delta_N = \cup\{\Delta_N(i): i=1,2,..., K(N)\}$; hence $\text{card}(\Delta_N)= K(N)(p(N)+1)^d$. Even more, if $i\neq h, \boldsymbol{d}(\Delta_N(i),\Delta_N(i))\geq q(N)$. (2.9)

Using the computations made in Proposition 2.1 parts a) and c) we get:

$$E\{(S_N(\lambda,\mu)(C))^2\}=(2N+1)^{-d}\{\lambda^2 \sum_{n\in\mathbb{Z}^d} \text{card}\{(C_N\cap A)\cap(n+C_N\cap A)\, r^X(n)$$

$$+2\lambda\mu \sum_{n\in\mathbb{Z}^d} \text{card}\{(C_N\cap A^c)\cap(n+C_N\cap A)\}\, r^X(n)$$

$$+\mu^2 \sum_{n\in\mathbb{Z}^d} \text{card}\{(C_N\cap A^c)\cap(n+C_N\cap A^c)\}\, r^X(n)\} \quad (2.10);$$

$\lim_N E\{(S_N(\lambda,\mu)(\mathbb{Z}^d)^2\}=\tau(\lambda,\mu)$ (2.11), and

$S_N(\lambda,\mu)(\Delta_N^c)^2$ converges in L^2 to 0. (2.12)

Therefore, it will be be enough to prove that $S_N(\lambda,\mu)(\Delta_N)$ converges weakly to a $N(0,\tau(\lambda,\mu))$ distribution.

We have that:

$$S_N(\lambda,\mu)(\Delta_N)= \sum_{i=1}^{i=k(N)} S_N(\lambda,\mu)(\Delta_N(i)) \quad (2.13);\ \text{then,}$$

using(1.9), we obtain:

$$|E\{\exp(itS_N(\lambda,\mu)(\Delta_N)\}- \prod_{m=1}^{m=k(N)} E\{\exp(itS_N(\lambda,\mu)(\Delta_N(m)))\}| \leq$$

$4K(N)\alpha^X(q(N))(2.14)$;

hence, $S_N(\lambda,\mu)(\Delta_N)$ has the same asymptotic distribution than:

$$Z_N(\lambda,\mu)= \sum_{i=1}^{i=k(N)} Z^i_N(\lambda,\mu);\ \text{where } \{ Z^i_N(\lambda,\mu): i=1,..,K(N)\} \text{ is}$$

a triangular array of independent copies of $\{ S_N(\lambda,\mu)(\Delta_N(i)) : i=1,...,K(N)\}$.

Using (2.10) we have:

$$E\{(Z_N(\lambda,\mu))^2\}= \sum_{i=1}^{i=k(N)} E\{S_N(\lambda,\mu)(\Delta_N(i)))^2\}=$$

$$(2N+1)^{-d} \{\lambda^2 \sum_{n\in\mathbb{Z}^d} \sum_{i=1}^{i=k(N)} \text{card}\{ (\Delta_N(i)\cap A)\cap (n+\Delta_N(i)\cap A)\ r^X(n)$$

$$+2\lambda\mu \sum_{n\in\mathbb{Z}^d} \sum_{i=1}^{i=k(N)} \text{card}\{ (\Delta_N(i)\cap A^c)\cap (n+\Delta_N(i)\cap A)\}\ r^X(n)$$

$$+\mu^2 \sum_{n\in\mathbb{Z}^d} \sum_{i=1}^{i=k(N)} \text{card}\{ (\Delta_N(i)\cap A^c)\cap (n+\Delta_N(i)\cap A^c)\}\ r^X(n)\}. \quad (2.15)$$

Fix $n\in\mathbb{Z}^d$ and pick $N(n)$ large enough such that $q(N)\geq \|n\|$ for

$N \geq N(n)$. It follows that for $N \geq N(n)$, $\Delta_N(i)$ and $n+\Delta_N(h)$ are disjoint sets if $i \neq h$. Using this and the fact that $\lim_N \text{card}([-N;N]^d - \Delta_N)(2N+1)^{-d} = 0$, we have that:

$$\lim_N \text{card}(A_N \cap (n+A_N))(2N+1)^{-d} =$$

$$\lim_N \sum_{i=1}^{i=k(N)} \text{card}\{(\Delta_N(i) \cap A) \cap (n+\Delta_N(i) \cap A)\}(2N+1)^{-d}$$; using the same idea in the other terms of (2.15) it follows from (2.11) that:

$$\lim_N E\{(Z_N(\lambda,\mu))^2\} = \tau(\lambda,\mu) > 0. \quad (2.16)$$

To conclude the proof it is enough to prove that the triangular array $\{ Z_N^i(\lambda,\mu): i=1,\ldots,K(N)\}$ satisfies Lyapunov´s condition; i.e., that $L_N(\lambda,\mu)$ goes to 0 with N ; where:

$$L_N(\lambda,\mu) = \sum_{i=1}^{i=k(N)} E\{(Z_N^i(\lambda,\mu))^4\}. \quad (2.17)$$

From (H10) we have:

$$L_N(\lambda,\mu) \leq k(N) \max_{1 \leq i \leq k(N)} E\{(Z_N^i(\lambda,\mu))^4\} \leq$$

$$C(J)k(N)\left[\frac{(p(N)+1)^d}{(2N+1)^d}\right]^2 = C(J)\left(\frac{2N}{p(N)+q(N)}\right)^d \left[\frac{(p(N)+1)^d}{(2N+1)^d}\right]^2,$$ that goes to 0 with N.

Therefore, $L_N(\lambda,\mu)$ goes to 0 with N ♦

The following proposition give sufficient conditions to have asymptotic normality, based on the ρ - mixing coefficient.

<u>**Proposition 2.4.:**</u> Let X be a random field that satisfies (H1), (H2), (H3), (H4), (H7), (H8) and:

(M) $\rho^X(\Pi;1;\infty,\infty) < 1$.

Then X belongs to B, and therefore, $M_N(A,X)$ has a weak limit $M(A,X)$, whose distribution is gaussian, centered and with variance matrix $C(X)$ given by:

$$C(X)_{11}= v(A)\sigma^2(X)-\gamma(X),\ C(X)_{12}= C(X)_{21} = \gamma(X),\ C(X)_{22}=(1-v(A))\sigma^2(X)-\gamma(X)\ .$$

Proof: It is enough to prove that (H9) and (H10) hold.

The basic arguments are the following inequallities:

Consider F a finite set, and a set of centered random variables $\{ X_t : t\in F \}$ with moments of order $q\geq 2$ finite and:

$\rho = \max \{\rho(\sigma^X(A), \sigma^X(B)): A\subset F, B\subset F, A \text{ and } B \text{ disjoint}\}$,

$R = \sup \{ |Corr (V,W)| : V\in S^X(A), W\in S^X(F-A): A\subset F \}$ and

$S^X(A)$ stands for the set of linear combinations of $\{ X_t : t\in A \}$.

Then we have that:

$$\text{If } q=2,\ R< 1: \left(\frac{1-R}{1+R}\right)\sum_{t\in F} E\{(X_t)^2\} \leq E\{ (\sum_{t\in F} X_t)^2\} \leq \left(\frac{1+R}{1-R}\right)\sum_{t\in F} E\{(X_t)^2\}\ . \tag{2.18}$$

If $q=4$, $\rho < 1$, there is a constant C depending only on ρ such that:

$$E\{ (\sum_{t\in F} X_t)^4\}\leq C\{ (\sum_{t\in F} E\{(X_t)^2\})^2 + \sum_{t\in F} E\{(X_t)^2\}\}.\tag{2.19}$$

(For the proof of (2.18) see cf. [9], Lemma 1; for the proof of (2.19) cf. [12], Lemma 3)

Applying (2.18) and the trivial observation that $R\leq\rho$, we get:

$$E\{(S_N(A, X-X^J))^2\} \leq \left(\frac{1+\rho^X(\Pi;1;\infty, \infty)}{1-\rho^X(\Pi;1;\infty, \infty)}\right) (2N+1)^{-d} \sum_{n\in A_N} E\{(X_n - X_n^J)^2\}=$$

$$\left(\frac{1+\rho^X(\Pi;1;\infty,\ \infty)}{1-\rho^X(\Pi;1;\infty,\ \infty)}\right)(2N+1)^{-d}\,\mathrm{card}(A_N)\,E\{(X_0-X_0^J)^2\}\leq$$

$$\left(\frac{1+\rho^X(\Pi;1;\infty,\ \infty)}{1-\rho^X(\Pi;1;\infty,\ \infty)}\right)E\{(X_0-X_0^J)^2\},\text{ and (H9) follows.}$$

In order to prove (H10), we apply (2.19) to X^J and obtain:

$$E\{(S_N(A,X^J))^4\}\leq C(2N+1)^{-2d}\{[\sum_{n\in A_N}E(X_n^{J\,2})]^2+\sum_{n\in A_N}E(X_n^{J\,2})\}.\quad(2.20)$$

Using the fact that $|X_n^J|\leq 2J$ we obtain:

$$E\{(S_N(A,X^J))^4\}\leq C(2N+1)^{-2d}(16J^4\mathrm{card}(A_N)^2+4J^2\mathrm{card}(A_N)]\leq$$

$C(J)(2N+1)^{-2d}\mathrm{card}(A_N)^2$, where $C(J)$ is a constant depending only on X and J, and (H10) follows♦

Remark 2.1.:

a) Using Bradley's results, condition (M) can be restated in the following way:

$$\alpha^X(\Phi;1;\infty,\infty)<\frac{1}{4}\quad\text{(cf. [11], Theorem 1).}$$

b) It should be noticed that condition (M) is weaker than $\alpha^X(\Phi)$- mixing (cf. [10], Theorem 1 or [12], Remark 2).

We can also get the CLT using Stein's method.

Proposition 2.4.: Let X be a real valued random field such that $X\in S$, and consider $A\in M(\mathbb{Z}^d)$; then the random vector $M_N(A,X)$ has a weak limit $M(A,X)$, whose distribution is gaussian, centered and with variance matrix $\mathbf{C}(X)$ given by:

$C(X)_{11} = v(A)\sigma^2(X) - \gamma(X)$, $C(X)_{12} = C(X)_{21} = \gamma(X)$, $C(X)_{22} = (1-v(A))\sigma^2(X) - \gamma(X)$.

<u>Proof:</u> Since (H9) holds, it is again enough to prove the result for X bounded.
In that case, one can follow very closely the arguments given by Bolthausen in [3]. Details are left to the reader ♦

In what follows we give another sufficient condition to have asymptotic normality, under adittional assumptions on the quantile function and on the mixing coefficients. The basic tool is the following covariance inequallity, due to Rio (cf. [26], Theorem 1.2), wich has been used to obtain functional CLT´s (cf. [17]):

If X is a random field that satisfies (H2) and (H3), then:

$$\sum_{n\in\mathbb{Z}^d} |Cov(X_0, X_n)| \le \int_0^1 \alpha^{-1}(u)\ (Q(2u))^2\ du$$;where Q is the quantile

function, $Q(u)=\inf\{t>0: P(|X_0|>t)\le u\}$ and α^{-1} depends on the mixing coefficients between one-pont sets and is defined by:

$$\alpha^{-1}(u)=\mathrm{card}\{n\in\mathbb{Z}^d: \alpha(\sigma^X(\{0\});\sigma^X(\{n\}))> u\}. \quad (2.21)$$

<u>Proposition 2.6.:</u> Let X be a random field that satisfies (H1), (H2), (H3) and the following condition:

$$\text{(R)} \quad \int_0^1 \alpha^{-1}(u)\ (Q(2u))^2\ du < \infty.$$

Then X satisfies (H7) and (H9).
In particular, if X satisfies (H1), (H2), (H3), (H5), (H6), (R) and (H8), then X belongs to S.

<u>Proof:</u> (H7) follows inmediately from (2.21).
Applying (2.21) to $X\text{-}X^J$ it is clear that is enough to show

that:

$$\lim_{J} \int_0^1 \alpha^{-1}(u)\ (Q^J(2u))^2\ du = 0,$$ where Q^J stands for the quantile function of the random variable $X_0 1_{\{|X_0|>J\}}$. (2.22)

But it is easy to see that $Q^J(u) = Q(u) 1_{\{u \le P(|X_0|>J)\}}$ and (2.22) follows from (R) and Dominated Convergence Theorem♦

<u>**Remarks 2.2.:**</u>

a) It is important to study under what kind of conditions (H8) holds. If we assume:

(U) $E\{(S_N(\mathbb{Z}^d, X)^2)\}(2N+1)^d$ is unbounded ;

and that one of the following conditions holds:

(i) $\lim_m R^X(m)=0$, where:

$R^X(m) = \sup\{ |\mathrm{Corr}(V,W)| : V\in H^X(A),\ W\in H^X(B): A\subset\mathbb{Z}^d,\ B\subset\mathbb{Z}^d,\ d(A,B) \ge m \}$

and $H^X(A)$ stands for the L^2 -closure of the set of linear combinations of $\{ X_n : n\in A \}$;

(ii) $\sum_{n\in\mathbb{Z}^d} \|n\|^d |r^X(n)| < \infty$;

then (H8) holds.

The proof that (U) and (ii) imply (H8) is elementary; the fact that (U) and (i) imply (H8) follows from Bradley (cf. [9], Theorem 3).

In addition, if we have that:

(C) $R^X(1) < 1$,

then, by (2.18), (U) holds. In particular, (M) implies (U).

If the random field is positively associated ($r^X(n) \geq 0$ for all n), then it is obvious that (H8) is satisfied if X is non-null (For CLT's for associated random fields, cf. [30]).

b) Consider a linear field X given by $X_n = \sum_{m \in \mathbb{Z}^d} \psi_{n-m} W_m$,

where the random field $W = \{ W_n : n \in \mathbb{Z}^d \} \in F$, and the kernel $\{ \psi_n : n \in \mathbb{Z}^d \}$

belongs to $L^1(\mathbb{Z}^d)$, i.e., $\sum_{m \in \mathbb{Z}^d} |\psi_n| < \infty$. This kind of fields appear in many applications, even in the case W iid. Assume that (H7) and (H8) are satisfied. It is very easy to see that Theorem 1.1 holds in this case, using standard argument (for recent results for linear fields, cf. [23]). It is well known that X is not necessarily mixing. More precisely, if d=1 and X is the solution of the AR (1) equation $X_n = \frac{1}{2} X_{n-1} + \frac{1}{2} W_n$, where $W = \{ W_n : n \in \mathbb{Z} \}$ are iid such that $P(W_n = \frac{1}{2}) = \frac{1}{2} = P(W_n = -\frac{1}{2})$, then X fails to be mixing (cf. [29]), eventhough it is a linear field as above. But X can be suitable approximated by moving averages on W (truncating the summation), and it is easy to verify that we can apply theorem 2.1 to these approximations; consequently, we get the Central Limit Theorem for X itself.

c) We can consider an increasing sequence of finite subsets of $\mathbb{Z}^d$, $\{A_N : N \in \mathbb{N}\}$, and say that it is *M-convergent* if:

$$H_N(n) = \text{card}\{A_N \cap (n + A_N)\}(2N+1)^{-d} \text{ has a limit } H(n) \text{ as } N \to \infty,$$

for each $n \in \mathbb{Z}^d$, and $0 < H(0) < 1$. Following the ideas that we have used, it can be proved that if X satisfies(H1), (H2), (H3), (H4), (H7), (H8) and (M), then CLT holds for $S_N(A;X) = \sqrt{(2N+1)^{-d}} \left(\sum_{n \in A_N} X_n \right)$. This extends the notion of convergence in the sense of Van´t Hove (cf.[6], [21]), that has been used in the development of CLT´s for random fields.

d) Condition (R) includes as a particular case hypotheses of the Davydov´s type:

(i) There is a $\delta>0$ such that $E(X_0{}^{2+\delta})<\infty$.

$$\text{(ii)} \quad \sum_{m=1}^{\infty} m^{d-1}(\alpha^X(\Phi;m;1,1))^{\frac{\delta}{2+\delta}}<\infty.$$

3. Examples.

We will study in this section examples of sets that belong to $M(\mathbb{Z}^d)$, and examples of sets that do not belong to this class. All the proofs are elementary, and we will frequently set d=1 for the sake of simplicity; the extension of the examples to greater dimensions is trivial.

We will begin showing that the class $M(\mathbb{Z}^d)$ contains all the sets with "regular" border, in the sense of the following definition.

Definition 3.1: Let A be an element of $G(\mathbb{Z}^d)$. If $v(\partial A)=0$, we will say that A has ***null border***. We will cal $NB(\mathbb{Z}^d)$ to the class of sets that have null border.

Lemma 3.1: Let A be a subset of $\mathbb{Z}^d$. Then for each $n\in\mathbb{Z}^d$, $N\in\mathbb{N}$, we have that: $F_N(n;A)\le (d\|n\|+1)\, \mathrm{card}(\partial A_N)](2N+1)^{-d}$ (3.1).

In particular, $NB(\mathbb{Z}^d)\subset M(\mathbb{Z}^d)$.

Proof: If n=0, (3.1.) is trivial, because $F_N(0;A)=0$.

If $n \neq 0$; we have $\operatorname{card}\{A_N^c \cap (n+A_N)\} = \sum_{m \in A_N^c} 1_{\{m-n \in A_N\}}$.

If $m \in A_N^c$ and $m-n \in A_N$, consider the $g(n) = \sum_{i=1}^{i=d} |n(i)|+1$ points with integer coordinates included in the oriented poligonal $C(m,n)$ that joins m and $m-n$; it is obvious that at least one of this points must belong to ∂A_N. If $p \in \partial A_N$, there are no more than $g(n)$ points $m \in A_N^c$ such that $m-n \in \partial A_N$ and $p \in C(m,n)$; from this we have that:

$$\operatorname{card}\{A_N^c \cap (n+A_N)\} \leq g(n) \operatorname{card}(\partial A_N) \leq (d\,\|n\|+1) \operatorname{card}(\partial A_N), \text{ and (3.1)}$$

follows.

If $A \in NB(\mathbb{Z}^d)$, then, applying (3.1) we get:

$\lim_N F_N(n;A)=0 \ \forall n \in \mathbb{Z}^d$, $A \in M(\mathbb{Z}^d)$ and $F(.;A) \equiv 0$ ♦

<u>**Remark 3.1.:**</u>

a) If A belongs to $NB(\mathbb{Z}^d)$ and X is a random field that satisfies (H1), (H2), (H4), (H7) , (H8) and such that $S_N(\mathbb{Z}^d, X)$ is asymptotically normal, then $M_N(A, X)$ converges weakly to a gaussian with independent coordinates.

This can be proved as follows: (H7) implies tighness of $M_N(A,X)$. Pick any weakly convergent subsequence. Let us call (V,W) to a random variable with the distribution of the limit. (H4) implies that V and W are independent. Since $S_N(\mathbb{Z}^d, X)$ is asymptotically normal, then V+W is gaussian; then, from the well-known theorem of Cramer (cf. [18], page 498), V and W are gaussian.

Therefore, it is easy to develop an F-test for the comparison of the means If the set A has null border (cf. [24]).

b) The inclusion is strict: take $A = 2\mathbb{Z}$, d=1. We have that $F(n;A)=0$ if n is even and $F(n;A)=\frac{1}{2}$ if n is odd. Then, for each $X\in F$, $M_N(A;X)$ has a weak limit $M(A;X)$, distributed as a centered gaussian random vectorwhose coordinates have variance $\frac{1}{2}\sum_{n\in 2\mathbb{Z}} r^X(n)$ and covariance

$\frac{1}{2}\sum_{n\in 2\mathbb{Z}+1} r^X(n)$. In particular we can obtain, for particular processes X, a degenerate limit. It is very easy to extend this example to d>1.

The following result shows that there is an uncountable family of sets that do not belong to $M(\mathbb{Z}^d)$ (in fact this family has the cardinal of the continuum).

<u>Lemma 3.2:</u> Define $A(0)=[0,100)$, $A(n)=[100^{2^{n-1}}, 100^{2^n})$ for $n\geq 1$. Take: $A(n,0)=A(n)\cap(5\mathbb{Z})$, $A(n,1)=A(n)\cap\{(10\mathbb{Z})\cup(10\mathbb{Z}+1)\}$, $n\in\mathbb{N}$. Given $x\in[0,1]$, write $x=\sum_{n=0}^{\infty}(x(n)2^{-n})$, where $x(n)=\mathrm{int}(2^n x - \sum_{i=0}^{i=n-1} x(i)2^{n-i})\in\{0,1\}$ for each n, and define $\phi(x)=\cup\{A(n,x(n)) : n\in\mathbb{N}\}$.

Then, if D stands for the set of diadic numbers (countable), $\phi(x)$ belongs to $G(\mathbb{Z}) - M(\mathbb{Z})$ for each $x\in[0,1] - D$.

<u>Proof:</u> Let $x\in[0,1] - D$. Then there is pair of subsequences n(k) and m(k) such that $x(n(k))=1$ for each k and $x(m(k))=0$ for each k.

Consider $N(k)=100^{2^{n(k)-1}}$, $M(k)=100^{2^{m(k)-1}}$; then:

$$F_{N(k)}(1;A)\leq\frac{1}{10}\left\{\frac{100^{2^{n(k)}}-100^{2^{n(k)-1}}}{2\ 100^{2^{n(k)}}+1}\right\}+\frac{1}{5}\left\{\frac{100^{2^{n(k)-1}}}{2\ 100^{2^{n(k)}}+1}\right\}$$

$$F_{M(k)}(1;A) \geq \frac{1}{5}\left\{\frac{100^{2^{n(k)}} - 100^{2^{n(k)-1}}}{2\ 100^{2^{n(k)}}+1}\right\} + \frac{1}{10}\left\{\frac{100^{2^{n(k)-1}}}{2\ 100^{2^{n(k)}}+1}\right\};$$

Hence:

$\limsup_k F_{N(k)}(1;A) \leq \frac{10.001}{200.000} < \frac{19.999}{200.000} \leq \limsup_k F_{M(k)}(1;A)$, and A does not belong to M ($\mathbb{Z}$), while it is obvious that A belongs to G ($\mathbb{Z}$) and $v(A)=\frac{1}{10}$ ♦

Remarks 3.2.:

a) The following example is very illustrative. It follows from the previous Lemma that if we consider $A=\Phi(\frac{2}{3})$ and the centered, stationary, gaussian and 1-dependent process $X=\{X_n;\ n\in\mathbb{Z}\}$, with $r^X(0)=1$, $r^X(1)=r^X(-1)=\rho\in(0;1/2)$, then $S_N(A;X)$ has no weak limit . However it is obvious that $S_N(B;X)$ has a gaussian weak limit for any B of NB($\mathbb{Z}$). The same random field, with a very simple structure (gaussain and 1-dependent) has essentially different asymptotic behaviours for different sets.

b) $M(\mathbb{Z}^d)$ is not an algebra. In fact, if d=1, using the notation of Lemma 3.2, define $B(n,0)=A(n)\cap(5\mathbb{Z}-10\mathbb{Z})$; $B(n,1)=A(n)\cap(10\mathbb{Z}+1)$ and $\eta(x)=\cup\{B(n,x(n)) : n\in\mathbb{N}\}$. It is easy to see that $\eta(x)$ belongs to M($\mathbb{Z}$) and it is obvious that 10 $\mathbb{Z}$ belongs to M($\mathbb{Z}$) , while $\Phi(x)=\eta(x)\cup(10\mathbb{Z})$ does not belong to M($\mathbb{Z}$) .

Taking into account this last remark, we will introduce an additional definition that allow us to extend Theorem 1.1 to the case in wich we consider a k-valued random vector, correponding to the normalized mean of X over the k sets of a partition of $\mathbb{Z}^d$.

Definition 3.2: Let A and B be elements of G ($\mathbb{Z}^d$), we will say that the pair $\{A,B\}$ has ***asymptotically measurable relative border*** if:

For each $n\in\mathbb{Z}^d$; $F_N(n;A,B)) = \text{card}\{A_N \cap (n+B_N\}(2N+1)^{-d}$

has a limit $F(n;A;B)$ as $N \to \infty$; $F(.;A;B)$ is the ***relative border function*** of the sets A and B. (3.2)

Definition 3.3: We will say that the subsets $A^{(1)},\dots,A^{(k)}$ of $\mathbb{Z}^d$ define an ***asymptotically measurable partition of*** $\mathbb{Z}^d$ if:

$A^{(1)},\dots,A^{(k)}$ is a partition of $\mathbb{Z}^d$. (3.3)

$A^{(i)}\in G(\mathbb{Z}^d)$, $i=0,1,\dots,k$. (3.4)

Each one of the pairs $\{A^{(i)},A^{(h)}\}$; $i,h= 1,\dots,k$ has asymptotically measurable relative border. (3.5)

(Observe that (3.3), (3.4) and (3.5) imply that $A^{(i)}\in M(\mathbb{Z}^d)$, $i=0,1,\dots,k$.

Given $A^{(1)},\dots,A^{(k)}$ subsets of $\mathbb{Z}^d$ such that (3.3) and (3.4) hold, we are going to consider:

$$M_N(A^{(1)},\dots,A^{(k)};X) = \sqrt{(2N+1)^{-d}}\,\Big(\sum_{n\in(A^{(1)})_N} X_n,\dots,\sum_{n\in(A^{(k)})_N} X_n\Big). \quad (3.6)$$

Then we have:

Proposition 3.1: Let $A^{(1)},\dots,A^{(k)}$ be subsets of $\mathbb{Z}^d$ such that (3.3) and (3.4) hold. Then the following statements are equivalent:

(i) For each $X\in F$, the random vector $M_N(A^{(1)},\dots,A^{(k)};X)$ has a weak limit $M(A^{(1)},\dots,A^{(k)};X)$

(ii) For each $X\in F$, the random vector $M_N(A^{(1)},\dots,A^{(k)};X)$ has a weak limit $M(A^{(1)},\dots,A^{(k)};X)$; whose distribution is gaussian, centered and with covariance matrix $C(X)$ given by:

$$C(X)_{ih} = \sum_{n \in \mathbb{Z}^d} r^X(n) F(n; A^{(i)}, A^{(h)}); \text{ with } F(.; A^{(i)}, A^{(h)}): \mathbb{Z}^d \to [0;1]$$

for i,h=1,..,k. (3.7)

(iii) $A^{(1)},\ldots,A^{(k)}$ define an asymptotically mesurable partition of $\mathbb{Z}^d$.

Remarks 3.3: a) The funcion $F(.;A^{(i)},A^{(h)})$ that appears in (iii) is the relative border function of the sets $A^{(i)}$ and $A^{(h)}$.

b) As before, if (iii) does not hold there is a stationary, centered, m-dependent gaussian field X such that $M_N(A^{(1)},\ldots,A^{(k)};X)$ has no weak limit.

c) An important question for the statistical applications we will consider later is the following: given $A \in M(\mathbb{Z}^d)$, is it possible to divide A^c in two pieces, B and C, such that A, B,C define an asymptotically mesurable partition of $\mathbb{Z}^d$?

We will see now that we can generate a random set and obtain almost surely an asymptotically mesurable border. In addition, we will answer to the last question.

Consider a random set $A(\omega)$ constructed as follows:

$U = \{ U_n : n \in \mathbb{Z}^d \}$ is a $\alpha^X(\Pi)$- mixing strictly stationary random field and such that each coordinate U_n has a Bernoulli(p) distribution, where $0<p<1$; set $A(\omega) = \{ n \in \mathbb{Z}^d : U_n(\omega) = 1 \}$.

Lemma 3.3: Let A be defined as above, then:

(i) $A \in M(\mathbb{Z}^d)$ a.s.

(ii) $F(0;A)=0$; $F(n;A) = p - E(U_0 U_n)$, for each $n \neq 0$; a.s.

Proof: Given $n \in \mathbb{Z}^d$ let us consider the random field $Y^n = \{ Y^n_m : m \in \mathbb{Z}^d \}$ defined by $Y^n_m = U_{n+m}(1 - U_m)$, then Y^n is stationary, ergodic, bounded (indeed, each coordinate is Bernoulli) and from the Strong Law of Large Numbers it follows that:

$F_N(n,A) = (2N+1)^{-d} \sum_{m \in [-N,N]^d} Y^n_m$ converges almost surely to $F(n;A)$ ♦

Lemma 3.4: Given $A \in M(\mathbb{Z}^d)$, let us take the stationary random field $U = \{U_n : n \in \mathbb{Z}^d\}$ as before. If we define $B(\omega) = \{n \in A^c : U_n(\omega) = 1\}$ and $C(\omega) = A^c - B(\omega)$, then:

(i) A,B,C define a.s. an asymptoticaly measurable partition of $\mathbb{Z}^d$.

(ii) $v(B) = p(-v(A))$; $v(C) = (1-p)(1-v(A))$; $F(n;B,C) = (p - E(U_0 U_n))(1 - V(A))$ if $n \neq 0$ (0 if $n=0$); $F(n;A,B) = p\, F(n;A)$ if $n \neq 0$ (0 if $n=0$); $F(n;A,C) = (1-p)\, F(n;A)$ if $n \neq 0$ (0 if $n=0$), a.s.

Proof: Completely similar to Lemma 3.3 ♦

Remark 3.4: The continuous case. The definition of $M(\mathbb{R}^d)$ is completely similar to that of $M(\mathbb{Z}^d)$, replacing "cardinal" by "Lebesgue measure". We can give results similar to the previous for $M(\mathbb{R}^d)$. We can also derive a "continuous" version of Theorem 1.1, replacing "sum" by "integral with respect to Lebesgue measure", but the ideas involved are the same. It is interesting to observe some connections that the class $M(\mathbb{R}^d)$ has with some ideas of Harmonic Analysis.

Consider d=1. Wiener called "regular" to a pair of real valued measurable functions f,g such that the correlation:

$$C(f,g;t) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t+s)g(s)ds$$ exists for each t real (cf. [32]).

It is well known that the class of regular functions in the sense of Wiener is not a vector space and that it includes strictly the class of quasiperiodic functions, i.e., the clausure of the linear space generated by $\{ \exp(i\lambda t) : \lambda \in \mathbb{R} \}$ with respect to the Marcinkiewicz's norm:

$$\|f\| = \{\limsup_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} | f(s)|^2 ds \}^{1/2}$$ (cf. [20]).

It should be noticed that the class of quasiperiodic functions is an algebra. It is obvious that $A \in M(\mathbb{R})$ iff the indicators of A and A^c are regular in the sense of Wiener. We will call quasiperiodic to a set a such that its indicator is a quasiperiodic function. Thus, quasiperiodic sets are asymptotically measurable. But inclusion is strict: there are sets $A \in M(\mathbb{R})$ that are not quasiperiodic. Indeed, the class of quasiperiodic sets is an algebra, while $M(\mathbb{R})$ is not.

4. A test for the comparison of the means.

Let us consider a random field $X = \{ X_n : n \in \mathbb{Z}^d \}$ such that $X(c) \in F$, where $X(c) = \{ X_n(c) : n \in \mathbb{Z}^d \}$ is given by $X_n(c) = X_n - E(X_n)$. Assume that there exists $A \in M(\mathbb{Z}^d)$ such that $E(X_n) = \mu$ if $n \in A$ and $E(X_n) = \mu + \delta$ if $n \in A^c$ (μ and $\mu+\delta$ are unknown but A is known).

Consider the following test:

$\{H_0: \delta = 0;\ H_1: \delta = \delta^* \neq 0\ (\delta^* \text{unknown})\}$.

Take B, C known such that A,B,C define an asymptotically measurable partition of $\mathbb{Z}^d$ (there are such sets by Lemma 3.4.

If V, W are random variables with finite second moment such that Var (W)> 0, denote $K(V,W) = V - (\frac{Corr(V,W)}{Var(W)})\, W$ (4.1)

It is obvious that if (V,W) is gaussian, then (K(V,W),W) is gaussian and its coordinates are independent.

If N, Q, Y are random variables with finite second moment such that $E(N)= a\mu$, $E(Q)=b(\mu+\delta)$, $E(Y)=c(\mu+\delta)$ with a, b, c> 0, we will denote

$V(N,Q,Y;a,b,c)= N - (\frac{a}{c}\, Y)$ and $W(N,Q,Y;a,b,c)=Y - (\frac{c}{b}\, Q)$. (4.2)

Obviously, $E(\, W(N,Q,Y;a,b,c))=0$ and $E(\, V(N,Q,Y;a,b,c))=-a\delta$.

If $C_1(N,Q,Y;a,b,c)=Var(Y) - (\frac{c}{b}Cov(Y,Q))+[(\frac{c}{b})^2 Var(Q)]>0$,

$C_2(N,Q,Y;a,b,c)=Var(N) - (\frac{a}{c}Cov(N,Y))+[(\frac{a}{c})^2 Var(Y)]>0$, the distribution of the random vector (N,Q,Y) is gaussian and we denote:

$$\tau(N,Q,Y;a,b,c)=\{\frac{C_1(N,Q,Y;a,b,c)}{C_2(N,Q,Y;a,b,c)}\}^2$$ then:

$\zeta(N,Q,Y;a,b,c)=\{\frac{K(V(N,Q,Y;a,b,c),W(N,Q,Y;a,b,c))}{W(N,Q,Y;a,b,c)}\}^2\, \tau\,(N,Q,Y;a,b,c)$ follows a Fischer´s F distribution $F(1,1;e(N,Q,Y;a,b,c))$, with 1 and 1 degrees of freedom, where:

$e(N,Q,Y;a,b,c)= |a\delta|\sqrt{\tau(N,Q,Y;a,b,c)}$ is the excentricity. (4.3)

As in proposition (3.3), we will use the following notation:

$$C(X)_{ih}= \sum_{n\in\mathbb{Z}^d} Cov(X_0X_n)F(n;A^{(i)},A^{(h)}), \text{ where } A^{(1)}=A,\ A^{(2)}=B,\ A^{(3)}=C. \quad (4.4)$$

We will make the following assumptions:

(A1) $X(c)\in \mathbf{F}$

(A2) $C(X)_{22} - (\frac{v(C)}{v(B)})\, C(X)_{23} + (\frac{v(C)}{v(B)})^2\, C(X)_{33} > 0$, and

$$(A3)\ C(X)_{11} - \left(\frac{v(A)}{v(C)}\right) C(X)_{13} + \left(\frac{v(A)}{v(C)}\right)^2 C(X)_{33} > 0.$$

Given the significance level α, we are going to consider the critical region:

$$R_N(\alpha)=\{\zeta_N(A;B;C;X)>F_{1-\alpha}(1,1)\} \text{ ; where:}$$

$$\zeta_N(A;B;C;X)=\zeta(M_N(A;B;C;X);v(A_N)\sqrt{(2N+1)^d},v(B_N)\sqrt{(2N+1)^d},v(C_N)\sqrt{(2N+1)^d})$$

and $F_{1-\alpha}(m,n)$ stands for the 1-α percentile of a Fischer´s F distribution with m, n degrees of freedom. (4.5)

Finally, we denote α_N and $\beta_N(\delta^*)$ the probabilities of error of first and second kind, respectively, corresponding to the critical region (4.5).Then, we have:

__Theorem 4.1:__ Under the assumptions (A1), (A2), (A3):

$$\lim_N \alpha_N = \alpha \text{ , } \lim_N \beta_N(\delta^*)= 0, \text{ for each } \delta^* \neq 0.$$

__Proof:__ We have that:

$$\lim_N C_1(M_N(A;B;C;X);\ v(A_N)\sqrt{(2N+1)^d},\ v(B_N)\sqrt{(2N+1)^d},v(C_N)\sqrt{(2N+1)^d}) =$$
$$C(X)_{22} - \left(\frac{v(C)}{v(B)}\right) C(X)_{23} + \left(\frac{v(C)}{v(B)}\right)^2 C(X)_{33} \quad (4.6),$$

and that: $\lim_N C_2(M_N(A;B;C;X);\ v(A_N)\sqrt{(2N+1)^d},\ v(B_N)\sqrt{(2N+1)^d}) =$
$$C(X)_{11} - \left(\frac{v(A)}{v(C)}\right) C(X)_{13} + \left(\frac{v(A)}{v(C)}\right)^2 C(X)_{33} \quad (4.7).$$

Applying Theorem 3.3 to the random field X(c) , using (4.7) and the basic poperties of weak convergence, we conclude that the distribution of $\zeta_N(A;B;C;X)$ approaches a $F(1,1;e_N(\delta))$; where $e_N(\delta)$ is the excentricity :

$$e_N(\delta) \sqrt{(2N+1)^d}|v(A)\delta|\{\frac{C(X)_{22} - (\frac{v(C)}{v(B)}) C(X)_{23} + (\frac{v(C)}{v(B)})^2 C(X)_{33}}{C(X)_{11} - (\frac{v(A)}{v(C)}) C(X)_{13} + (\frac{v(A)}{v(C)})^2 C(X)_{33}}\}$$

.(4.8)

Since $\alpha_N = P(R_N(\alpha);\delta=0)$ and $\beta_N(\delta^*) = P(R_N(\alpha);\delta=\delta^*)$, it follows inmediately that $\lim_N \alpha_N = \alpha$, $\lim_N \beta_N(\delta^*) = 0$ ♦

<u>Remarks 4.1:</u>

a) This is an F-test. In a similar way, we can present another kind of tests (χ^2, for instance), but there is no essential difference between these proposals. The number of degrees of freedom is (1,1) just to present the result as clear as posible. It is obvious that it could be modified to get (k,l) dehgrees of freedom.

b) This test involves parameters like $C(X)_{11}$ that in general, are unknown. However, it is easy to show that they can be consistently estimated under some aditional assumptions. For instance, if we assume that the coordinates of X have bounded moments of order $4+\delta$, we assume stronger mixing conditions (of the Davydov´s type), we replace in the formulae of the parameters the covariances of X by its standard estimates, and the series by a sum of m_N terms, it is easy to show that m_N can be choosen properly in order to obtain consistent estimates of the parameters (cf. [25]).

c) It is natural to ask how different is the behaviour of this F-test with respect to the behaviour of another F-test. For instance, the classical F -test assume an iid sample, and the F-test presented in [24] assume dependent samples but "thin" border (in the sense of Definition 3.1).Roughly speaking, the comparison of these three F-tests could be taken as an evaluation of the relevance of the dependence and the geometry on the statistical problem. More precisely, take dependent data and an asymptotically measurable geometry, but with "thick" border. The performance of the classical F-test will show what happens if you ignore completely the dependence, the performance of the F-test given previously in [24] will show what happens if you take into account the dependence of the data but ignore the complexity of the geometry, and, at last, our F-test will show what happens if you take both problems into account. This comparison is presented in the appendix, using simulations.

d) If we consider a sequence of alternatives

$H_1(N)$: $\theta = \sqrt{(2N+1)}^{-d}\,\delta^*$ ($\delta^* \neq 0$; unknown), then following the arguments of the proof of Theorem 4.1, it can be shown that:

$$\lim_N \beta_N\left(\sqrt{(2N+1)}^{-d}\,\theta^*\right) = F(1,1;e(\theta^*))\,(F_{1-\alpha}(1,1)),$$ where

$$e(\theta^*) = |v(A)\theta^*|\left\{\frac{C(X)_{22} - \left(\frac{v(C)}{v(B)}\right)C(X)_{23} + \left(\frac{v(C)}{v(B)}\right)^2 C(X)_{33}}{C(X)_{11} - \left(\frac{v(A)}{v(C)}\right)C(X)_{13} + \left(\frac{v(A)}{v(C)}\right)^2 C(X)_{33}}\right\}.$$

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Appendix.

The aim of this appendix is to show the behaviour of the method given in Theorem 4.1 and to enphasize that the geometry is relevant in the comparison of dependent data. We are not trying to make an exhaustive study , taking into account any other alternative method, or a variety of different specific problems, but only consider some simple cases.

At first,we are going to take d=1 and compare the method given in Theorem 4.1 (we will call it M) with the classical F- test for the comparison of two iid samples (CM, in what follows) and the F-test given in [24], wich assumes weakly dependent data but "null border" geometry (we will call it NBGM). This test has as critical region the following set:

$$C_N(\alpha)=\{\omega_N(A,B,C;X)>F_{1-\alpha}(1,1)\},\text{ where:}$$

$$\omega_N(A,B,C;X)=\left(\frac{v(B)}{v(A)}\right)\left\{\frac{S_N(A;X)-\left(\frac{v(A)}{1-v(A)}\right)S_N(A^c;X)}{S_N(B;X)-\left(\frac{v(B)}{v(C)}\right)S_N(C;X)}\right\}^2$$, and B, C are disjoint sets with null border whose union is A^c.

We will take A=2ℤ. Our process will be a stationary and m-dependent gaussian process (indeed, a moving average of a gaussian white noise). We consider different values of m (5, 10, 20) in order to evaluate the reponse to different dependence structures, $\alpha = 5\%$, N=100 and we give estimates for α_N and $\beta_N(\delta^*)$ for moderate values of δ^* (0.25, 0.1). Each estimate will be based on1000 simulations. The results are the following (in percentages):

	α_N			$\beta_N(\delta^*)$					
					$\delta^*=0.1$			$\delta^*=0.25$	
m	CM	NBGM	M	CM	NBGM	M	CM	NBGM	M
5	0	2.2	5.9	99	90.9	84	99.8	95.9	91.7
10	0	0.1	4.5	98.8	92.3	77.3	100	96.2	89.8
20	0	0.1	5.8	97.4	95.6	79.8	100	98.7	88.6

We can observe that there is no relevant difference between CM and NBGM; although NBGM seems to behave better. It seems to be a significant difference between M and the others. To study this more deeply; we are going to compare NBGM and M for big, moderate and small values of δ^* (1, 0.1, 0.01), and taking m= 5, 10, 20, 40, 60, 80 and 100. All the characteristics of the simulations are the same as above.Then, we obtain:

	α_N		β_N					
δ^*			1		0.1		0.01	
m	NBGM	M	NBGM	M	NBGM	M	NBGM	M
5	2.2	5.9	60.2	42.8	95.9	91.7	96.6	92.8
10	0.1	4.5	71.8	28.4	96.2	89.8	99.1	95.8
20	0.1	5.8	81.6	29	98.7	88.6	99.7	92.7
40	0.4	11.1	85.2	31.8	98.9	87.6	99.9	86.8
60	0.1	14.7	88.3	33.6	98.1	79.3	99.8	84.4
80	0.2	16.9	89.8	34.4	98.7	74.6	99.9	83.1
100	0.2	19.5	90	37.8	98.4	68.6	99.9	81.4

We can observe that M behaves in a more powerful way. In the sequel; we are going to consider only $\delta^*=1$, and observe the reponse of NBGM and M to changes in N:

		α_N		β_N	
m	N	NBGM	M	NBGM	M
10	50	0.4	4	80.6	40.1
10	100	0.1	4.5	71.8	28.4
10	200	0.2	2.3	61.4	12.8
20	50	0.3	6	83.8	46.2
20	100	0.1	5.8	81.6	29
20	200	0.1	5.4	71.3	12.9
20	2500	0	2	22.5	0
40	50	0.1	11.7	88	50.2
40	100	0.4	11.1	85.2	31.8
40	200	0.3	10.8	77.4	14.1
40	1000	0	4.5	62	1.5

		α_N		β_N	
m	N	NBGM	M	NBGM	M
60	50	0.1	14.1	90.9	55.4
60	100	0.1	14.7	88.3	33.6
100	100	0.2	19.5	90	37.8
100	5100	0	12.5	43	0

Finally , we take d=2 and generate a random field using analogous methods to those used before for d=1. We indicate the sample size $S=(2N+1)^d$ and the order of dependence $D=m^d$ to show that the behaviour of M in both dimensions is similar.

					α_N		β_N	
d	m	N	S	D	NBGM	M	NBGM	M
1	20	200	401	20	0.1	5.4	71.3	12.9
2	4	10	441	16	0.1	9.18	66.45	6.73
1	20	2500	5001	20	0	2	22.5	0
2	4	34	4761	16	0	4.33	20.67	0
1	100	100	201	100	0.2	19.5	90	37.8
2	10	6	169	100	0	34	90.5	37.5
1	100	5100	10201	100	0	12.5	43	0
2	10	50	10201	100	0	23.5	49.5	0

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El Agente y el Principal *

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Abstract

This paper survey some know results in the theory of Moral Hazard, when from asymmetries of information the agent effort cannot be monitored perfectly. We show some sufficient condition for the existence of the second best solution, we prove that it is not a Pareto optimal solution, and show some sufficient condition for the validity of the first order approach. Most of this results are well know, the main thing of this paper is that it show these results as a unity in a particular but expressive case.

Introducción

La Economía de la Información se propone estudiar situaciones en la que una parte de los agentes económicos no disponen de toda la información, ya sea referido a lo que los demás están haciendo, o saben, o en relación a las oportunidades de transacciones óptimas. Entre las áreas de investigación que tratan del problema de la asimetría de la información se destacan: la teoría del Perjuicio Moral, de la Selección Adversa, Búsqueda Optima, y la teoría de Expectativas Racionales.

En el caso de la Teoría del Prejuicio Moral, el problema consiste en que una parte de los agentes toman decisiones que afectan a los retornos de los demás sin que estos sean capaces de monitorear totalmente estas decisiones en provecho propio. La solución para este problema consiste en elaborar un programa de incentivos, pautado en un contrato, en el que será establecido el pago del agente por el principal, una vez que sean observados determinados resultados, dependientes del esfuerzo del primero.

El problema central de la teoría será entonces el de establecer un contrato óptimo en el sentido de que beneficios y riesgos sean distribuídos de forma tal que el agente tenga incentivos para elegir aquellas acciones que maximicen las utilidades de uno y otro.

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*This paper is in final form and no version of it will be submitted for publication elsewhere.

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Existe abundante literatura referido al tema, son referencias clásicas, [G-H], [H]. El presente trabajo pretende dar una visión sintética de algunos resultados conocidos pero publicados en lugares diferentes, probando la existencia de solución en un caso particular pero esclarecedor de las técnicas utilizadas. Se muestra también que como resultado de la información incompleta se llega a un resultado que no es un Optimo de Pareto.

En el presente trabajo, en la primera sección presentamos el modelo, en la sección dos, a través de la llamada aproximación de primer orden, veremos que bajo determinadas hipótesis es posible hallar un programa óptimo de incentivos; en la sección tercera veremos que este óptimo no es Optimo de Pareto, lo que es debido precisamente a la asimetría de la información, pues el agente estará mejor informado que el principal respecto a la elección del tipo de esfuerzo que será desarrollado para alcanzar determinados resultados. El costo de la desinformación es precisamente la desviación de la regla socialmente óptima en el sentido de Pareto. En la sección cuarta, presentaremos algunos aplicaciones del modelo y finalmente en la última sección haremos la demostración de la existencia de la solución del llamado problema débil.

1 El Modelo

Un individuo, el agente, tiene que tomar decisiones que afectan a su propio bienestar y al de otro(s) individuo(s), el principal, a cambio de cierto pago, la forma del cual será establecida en un contrato. El conjunto de los contratos posibles será indicado como S.

El agente eligirá una acción a dentro de un cierto conjunto $\mathcal{A}$ de acciones posibles. Supondremos que la acción elegida corresponde a un determinado tipo de esfuerzo a cuya intensidad a corresponde un determinado número real positivo. Asi entonces $\mathcal{A} \subset \mathcal{R}_+$.

A cada elección de a está asociada una determinada distribución de probabilidad $F(x/a)$ sobre los posibles retornos monetarios brutos los que serán representados por x. Dichos retornos serán función de los estados de la naturaleza así como de la acción elegida por el agente. Los estados de la naturaleza forman un espacio de probabilidad el que será representado por $(\Omega, \mathcal{B}, \mu)$. Luego $x : \Omega \times \mathcal{A} \to \mathcal{R}_+$.

El contrato establecido entre las dos partes puede representarse por una función $s \in \mathcal{S}; s : X \to \mathcal{R}_+$, siendo X el conjunto de retornos monetarios brutos posibles, el que supondremos real positivo y compacto. Es decir $s(x), x \in X$ representará el pago que recibirá el agente, una vez conocido x.

En nuestro modelo $x \in X$, tendrá para el principal el valor de un señal para el esfuerzo desplegado por el agente, al que el principal no puede medir directamente. Combinaremos en que a un mayor x corresponde un mayor a, esto es, mayores x corresponden a elecciones más eficientes de a por parte del agente.

Definición 1 *Decimos que una señal x es más favorable que otra y, lo que escribiremos como $x \succ y$ si para toda distribución a priori G para a, la distribución a posteriori $G(\cdot/x)$ domina estocasticamente en el sentido de primer orden a $G(\cdot/y)$.*

Recordemos que una distribución F, para la variable aleatoria θ domina estocasticamente a otra G, para la referida variable cuando para toda función U creciente se tiene que: $\int_A U(\theta)dF(\theta) > \int_A U(\theta)dG(\theta)$, siendo A el dominio de definición de la función U. Ver [R].

Sea $f(x/a)$ la densidad condicional de x cuando a toma un valor particular y sea $g(a/x)$ la función de densidad a priori para G.

Por el teorema de Bayes tenemos que:

$$\frac{g(a'/x)}{g(a/x)} = \frac{g(a')f(x/a')}{g(a)f(x/a)}.$$

Se tiene la siguiente proposición:

Proposición 1 *Una señal x es más favorable que otra y, si y sólo si para toda $a' > a$ se tiene que $f(x/a')f(y/a) - f(x/a)f(y/a') > 0$.*

La demostración puede verse en [M].

El principal posee una función de utilidad $u : \mathcal{S} \times \mathcal{R} \to \mathcal{R}$,

$$u(s,a) = E[U(x - s(x))] = \int U(x - s(x))dF(x/a),$$

siendo U creciente en $x - s(x)$, con $U_{xx} \leq 0$.

El agente tiene una función de utilidad $v : \mathcal{S} \times \mathcal{A} \to \mathcal{R}$,

$$v(s,a) = \int V(s(x))dF(x/a),$$

siendo $V_s > 0$ y $V_{ss} \leq 0$, esto es el agente es adverso al riesgo.

Una vez que el agente elige un determinado $a \in \mathcal{A}$, concomitantemente elige una cierta función de distribución $F(x/a)$ común para el agente y el principal, tal que el principal resolverá el siguiente programa:

$$Max_{[s,a]} \int_X U(x - s(x))f(x/a)dx \tag{1}$$

sujeto a las condiciones:

$$\int_X V(s(x))f(x/a)dx - c(a) \geq K \tag{2}$$

$$a \in argmax \int_X V(s(x))f(x/a)dx - c(a), \tag{3}$$

siendo $c(a)$ el costo del agente para implementar $a \in \mathcal{A}$, que supondremos real y convexa con dominio en A, $f(x/a)$ es la densidad correspondiente a la distribución $F(x/a)$.

En el apéndice 1, discutiremos condiciones que garantizan la existencia de la solución, en las condiciones de nuestro modelo alcanza con la existencia de un $\bar{s}$ y de un $\bar{a}$ para los que $v(\bar{s}, \bar{a}) > K$

En el programa es el principal quien decide la acción que va a ser implementada y elige un programa de incentivos acorde con esa finalidad, el principal conoce las preferencias del agente.

La primera de las restricciones que figuran en el programa, tiene por objetivo asegurar al agente un mínimo en la utilidad esperada de forma de garantizar su participción en el programa, la última asegura que dado un contrato, $s \in \mathcal{S}$, la acción elegida por el principal maximiza la utilidad del agente.

Si el principal tiene como observar directamente la acción elegida por el agente entonces la última restricción es superflua, basta en este caso poner en el contrato una cláusula que obligue al agente a implementar una determinada acción. En este caso el contrato solución del problema es conocido como la "solución de primera vez". En el caso en que el principal no pueda observar directamente la acción elegida por el agente, la segunda restricción tiene sentido, la solución obtenida se llamará entonces la "solución de segunda vez".

2 La Aproximación De Primer Orden

En el caso en que la condición 3) pueda ser reemplazada por la condición más débil

$$v_a(x, a) = \int v(s(x))f_a(x/a)dx - c'(a) = 0, \tag{4}$$

el programa que se obtiene sustituyendo 3) por 4) es llamado problema débil, o aproximación de primer orden.

El problema débil (P.D.), queda caracterizado entonces por:

$$Max_{[s,a]} \int_X U(x - s(x))f(x/a)dx, \tag{5}$$

sujeto a las condiciones:

$$\int_X V(s(x))f(x/a)dx - c(a) \geq k, \tag{6}$$

$$\int_X V(s(x))f(x/a)dx - c'(a) = 0. \tag{7}$$

Mirrlees [Mi], fue quien primero observó que las soluciones obtenidas para el problema débil no son necesariamente soluciones para el problema original, obsérvese que el hecho de que $a \in \mathcal{A}$

verifique la ecuación (7), no implica que deba verificar (3). Mirrless, prueba que para que una solución del P.D. sea solución del problema original, o no relajado, las siguientes dos condiciones son suficientes:

1) Condición de monotonía en la razón de verosimilitud. CMRV.

2) Condición de convexidad en la función de distribución. CCFD

Definición 2 *Una familia de densidades $\{f(x/a)\}_{a\in\mathcal{A}}$ satisface la propiedad de monotonía estricta en la razón de verosimilitud, CMRV, si para todo $x > y$ y para todo $a' > a$ vale:*

$$\frac{f(x/a')}{f(x/a)} > \frac{(f(y/a')}{f(y/a)}.$$

Definición 3 *Una familia de distribuciones $\{F(x/a)\}_{a\in\mathcal{A}}$ satisface la condición de convexidad en la función de distribución, CCFD, si para todo $a, b \in \mathcal{A}$ y $\lambda \in [0,1]$ vale:*

$$F(x/\lambda a + (1-\lambda)b) \leq \lambda F(x/a) + (1-\lambda)F(x/b).$$

Proposición 2 *Si CMRV y CDFC son satisfechas se tiene que para que $s^*(x)$ y a^* sean solución para 1) con las restricciones 2)y 3) es necesario y suficiente que satisfagan las siguientes dos ecuaciones:*

$$\frac{U'(x - s^*(x))}{V'(s^*(x))} = \lambda + \nu\frac{f_a(x/a^*)}{f(x/a^*)}. \tag{8}$$

$$\int U(x - s^*(x))f_a(x/a^*)dx + \nu\left\{\int V(s^*(x))f_{aa}(x/a^*)dx - c''(a)\right\} = 0, \tag{9}$$

siendo λ y ν multiplicadores de Lagrange.

Prueba Se demuestra considerando el método de Lagrange para el problema de maximización sujeto a restricciones y que bajo CMRV y CCFD toda solución del problema débil, es solución para el problema original. Ver apéndice 2.

Nota 1 *De la proposición 1 y de la propia definición de CMRV se tiene que si $x > y$ entonces x es más favorable que y.*

Lema 1 *La densidad $f(x/a)$ satisface CMRV Si y solamente si para todo $a \in \mathcal{A}$, $\frac{f_a(\cdot/a)}{f(\cdot/a)}$ es creciente.*

Prueba: Obsérvese que: $\frac{f_a(x/a)}{f(x/a)} = \frac{\partial ln f(x/a)}{\partial a}$ de donde se sigue que: $\frac{f(x/a)}{f(x/a')} = exp\{\int_a^{a'} \frac{f_a(x/a)}{f(x/a)}da\}$. Luego por el hecho de ser $\frac{f_a(x/a)}{f(x/a)}$ función creciente con la señal obtenemos el resultado.

Lema 2 *Con las hipótesis CMRV y CCFD tenemos que $s'(x) > 0$, esto es el pago del agente aumenta con el crecimiento de la señal.*

Prueba A partir de la ecuación (8) obtenemos que si $\frac{f_a(x/a)}{f(x/a)}$ crece con x entonces $\frac{U'(x-s^*(x))}{V'(s^*(x))}$ es creciente, derivando obtenemos que $s'(x) > 0$.

Nota 2 *En la medida en que una señal mayor, supone la realización de un mayor esfuerzo por parte del agente, el lema dice que, el contrato óptimo establecerá un mecanismo que asegure que la retribución del agente aumentará con el esfuerzo por él desplegado lo que será monitoreado através de la señal x.*

3 Optimalidad

Será mostrado en esta sección que la regla $s(\cdot)$ no es óptimo de Pareto en casos en que como el presentado en la sección anterior se buscan incentivos para que el agente realice un determinado esfuerzo, con asimetría en la información. El costo de la desinformación será precisamente el alejamiento de la regla óptima en el sentido de Pareto.

Recordamos que una distribución de recursos es óptimo de Pareto cuando no es posible obtener una redistribución de los mismos sin perjudicar a por lo menos uno de los agentes económicos.

Lo anteriormente dicho, será corolario de los siguientes dos lemas.

Lema 3 *Sean U y V funciones de utilidad estrictamente cuasi-cóncavas, derivables y estrictamente crecientes, entonces la solución del siguiente problema existe, es única y es un óptimo de Pareto,*

$$Max_{[z]}U(z),$$

sujeto a las condiciones:

$$v(y) \geq c$$
$$z + y = r$$

Prueba: Sea $C = \{(z,y) \in R^2_+ : z + y = r \; ; \; V(y) \geq c\}$, suponemos que es no vacío. Sea $\{z', y'\} \in C$ y consideremos el siguiente problema equivalente:

$$Max_{[z]}U(z),$$

Sujeto a las condiciones:

$$V(y) \geq c$$
$$V(y) \geq V(y')$$
$$z + y = r$$

Sea C' $= \{(z,y) \in R^2_+$: satisfacen las nuevas restricciones.$\}$, C' es compacto, como V es continua existe la solución, la unicidad proviene de la estricta cuasi-concavidad. El hecho de que la solución es un óptimo de Pareto se verifica facilmente.

Veremos que nuestra afirmación resulta de considerar la equivalencia entre el problema original y el presentado anteriormente. Para esto considere: $z = x - s(x), y = s(x)$, y suponga que la regla $s(\cdot)$ es un óptimo de Pareto. En este caso, bajo las hipótesis $U' > 0, U'' \leq 0, V' > 0,$ *y* $V'' < 0$ tenemos por el lema anterior que $\frac{U'(x-s(x))}{V'(s(x))} = K$, donde K es el multiplicador de Lagrange, para el problema de primera vez. De acuerdo con esto y con la ecuación (8) tenemos la siguiente identidad: $\lambda + \nu \frac{f_a(x/a)}{f(x/a)} = K$.

Como $\frac{\partial \int f(x/a)dx}{\partial a} = \int f_a(x/a)dx = 0$, se sigue que $\lambda = K$ y que $\nu \frac{f_a(x/a)}{f(x/a)} = 0$.

El siguiente lema probará que entonces $f_a(x/a) = 0$.

Lema 4 *Si la regla $s^*(\cdot)$ es un óptimo de Pareto con $c'(a) > 0$ y $F_a(x/a) < 0$ entonces $\nu > 0$.*

Nota 3 *Observe que la hipótesis $F_a(a/x)$ es coherente con el hecho de que a una señal mejor corresponde un esfuerzo mayor, esto es: $P_{a'}(x \leq h) \leq P_a(x \leq h)$ con $a' > a$; $a', a \in \mathcal{A}$.*

Prueba: Del lema anterior resulta que $\lambda = K > 0$, se obtiene entonces que:

$$\frac{U'(x - s(x))}{V'(s(x))} = \lambda + \nu \frac{f_a(x/a)}{f(x/a)} \leq \lambda = \frac{U'(x - s^\lambda(x))}{V'(s^\lambda(x))},$$

donde $s^\lambda(x)$ es una solución de primera vez.

Sea $X^+ = \{x \in X : f_a(x/a) \geq 0\}$ y $X^- = \{x \in X : f_a(x/a) \leq 0\}$.

Definamos $r(x) = x - s(x)$ consecuentemente, $r^\lambda(x) = x - s^\lambda(x)$.

Como $\frac{U'(r(x)}{V'(x-r(x))}$ es decreciente con respecto a $r(x)$ tenemos, $r(x) \geq r^\lambda(x)\ \forall x \in X^+$ y que $r(x) \leq r^\lambda(x)\ \forall x \in X^-$. En cualquiera de los dos casos valen las siguientes desigualdades:

$$\int_X U(r(x))f_a(x/a)dx \geq \int_X U(r^\lambda(x))f_a(x/a)dx > 0, \tag{10}$$

la última desigualdad sale de la dominancia estocástica de primer orden y de la hipótesis $F_a(x/a) < 0$.

En la ecuación (9) tenemos que el factor que multiplica a ν es negativo, (Mirrless [Mi], prueba Que bajo CCFd, la utilidad del agente es cóncava en la acción). Considerando (10) resulta que ν debe ser positivo.

Ahora bien, si $\nu > 0$ tenemos que $\frac{f_a(x/a)}{f(x/a)}$ debe ser igual a cero, esto es $f_a(x/a) = 0$. Pero por hipótesis tenemos que $F_a(x/a) < 0$ lo que supone $\int f_a(x/a)dx < 0$ lo que contradice a $f_a(x/a) = 0$.

Resulta entonces que $\frac{U'(x-s(x))}{V'(s(x))}$ debe ser diferente de λ y por lo tanto no es óptimo de Pareto.

4 Algunas Aplicaciones

1) Bajo las hipótesis del modelo se observa que la regla $s(\cdot)$ se desvía de la siguiente forma con respecto a la regla $s^\lambda(\cdot)$ óptimo de Pareto:

$$\begin{cases} s(x) \geq s^\lambda(x) & en\ X^+ \\ s(x) \leq s^\lambda(x) & en\ X^- \end{cases}$$

La prueba se desprende del hecho de ser la función $S(x) = \frac{U'(x-s(x))}{V'(s(x))}$ creciente en $s(x)$. La prueba puede verse en [H].

2) Suponga ahora que $A = \{l, h\}$, esto es el agente puede elegir entre una de dos posibles opciones: trabajar debilmente l, o trabajar fuertemente h. Supongamos además que $U(w) = w$ representa la utilidad del principal. La ecuación (8) tendrá la siguiente forma:

$$\frac{1}{v'(s(x))} = a + b\left[1 - \frac{f(x/l)}{f(x/h)}\right].$$

Sea q la probabilidad apriori para h y q' la probabilidad a posteriori, esto es $q = P(h)$ y $q' = P(h/x)$, entonces $q' = \frac{qf(x/h)}{qf(x/h)+(1-q)f(x/l)}$.

Luego: $\frac{1}{v'(s(x))} = a + b\frac{(q'-q)}{q(1-q')}$.

Se desprende de aqui que el agente recibirá una penalización si los resultados obtenidos, esto es el valor del producto x, revisa las creencias en sentido negativo, mientras que será premiado en caso contrario.

3) Puede considerarse la siguiente extensión para el caso de dos acciones, el agente puede elegir en $A = \{a : a = \lambda h + (1-\lambda)l \ \ \lambda \in [0,1]\}$, suponiendo que $f(x/a) = \lambda f(x/l) + (1-\lambda)f(x/h)$.

La condición (3) quedará así : $\int V(s(x))(f(x/h) - f(x/l))dx \geq \frac{c(a)-c(a')}{a-a'}$.

Por más aplicaciones ver [H] o [L].

5 Apéndices

En esta sección discutiremos brevemente algunas condiciones para la existencia de la solución de segunda vez.

5.1 Apéndice 1. Prueba de la Existencia de La Solución.

Llamamos $F(s)$ al conjunto de las $a \in \mathcal{A}$ que verifican la condición (2). Supondremos que dicho conjunto es no vacío, de lo contrario el problema no tiene sentido, pues el agente preferiría quedar al margen de la actividad. En las condiciones del modelo, como se verifica facilmente, $F(s)$ es una aplicación semicontinua superiormente.

Sea ahora $G(s)$ el conjunto de las $a \in \mathcal{A}$ que verifican la ecuación (3), por el hecho de ser A compacto, las condiciones impuestas sobre las funciones en el integrando aseguran que $G(s)$ es no vacío. Considerada $G(s)$ como aplicación del espacio de los contratos S en el conjunto de las acciones cal A resulta ser semicontinua superiormente.

Luego $\Gamma(s) = F(s) \cap G(s)$ es una aplicación semicontinua superiormente de S en A. Entonces siendo $\mathcal{M}(s) = Max_{a \in \Gamma(s)} \int U(x - s(x)) f(x/a) dx$ obtenemos por el teorema del máximo, ver [B], que $M(s)$ es una función semicontinua superiormente. Si elegimos el espacio de los contratos S como un conjunto compacto el problema tiene solución.

Admitiendo que X es compacto, un posible conjunto con esta propiedad es $\mathcal{S} = \{s : X \rightarrow \mathcal{R}$ Hölder- continuas$\}$. El teorema de Arzelá Ascoli permite concluir que este conjunto es relativamente compacto. Ver [K].

Condiciones más generales para la existencia de la solución pueden encontrarse en [Y]. Por ejemplo, para modelos con propiedades análogas a las del nuestro, alcanza con que el conjunto $\mathcal{S}$ sea debilmente compacto.

5.2 Apéndice 2. Validez de la Aproximación de Primer Orden.

Las ecuaciones (8) y (9) pueden obtenerse también a partir de la regla de Euler para el cálculo variacional, el hecho de que caracterizan a un máximo provienen de la concavidad de las funciones del programa. Ver por ejemplo [C]. Obsérvese que el conjunto B de los pares (s, a) que satisfacen las restricciones del problema original, está contenido en C, este es el conjunto de los pares (s, a) que satisfacen las condiciones del P.D.. Mirrless en [Mi], prueba que CMRV y CCFD aseguran la concavidad en la acción, para la función de utilidad del agente. Esto permite afirmar que si (s^*, a^*) maximiza u sobre C, entonces (s^*, a^*) es un elemento de B.

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ON THE LIKELIHOOD RATIO AND THE KULLBACK-LEIBLER DISTANCE

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Abstract

In this note the LR is seen in the perspective of the generalized Kullback-Leibler distance. The generalized K-L distance permits to obtain a unified vision of the Maximum Likelihood Estimate, mainly in the case of Models, i.e. when the underlying distribution in the sampling does not correspond to any parameter value. This is specially interesting when dealing with dependent observations, and shows some robustness of the ML method. Likewise the generalized K-L distance leads to extend to the dependent case several well-known properties of the likelihood ratio in the independent case.

1. Introduction.

There is an extensive literature about consistency and other properties of the ML Estimates. This is also the case of several connected concepts as the likelihood ratio test. However the results

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following the Wald's classical proof of consistency of MLE are often established only in the independent case.

Nevertheless the concept of generalized K-L distance permits precisely a unified approach without discriminating between the independent and the dependent case.

The next section contains some notations and summarizes well-known facts about the K-L distance. Section 3 deals with the independent case and shows some related examples of the "robustness" of the method. Section 4 examines discrete time stochastic processes, generalizes the K-L distance and establishes some results for the MLE. Finally Section 5 provides some applications and examples of estimation of parameters and testing hypotheses.

Though the principal purpose of this note is to show a unified vision of some statistical applications through the generalized K-L distance, we point out that Propositions 3.1 and 4.1 and Theorems 4.1 and 5.1 apparently are not in the literature. Theorem 5.2 is well known since Pinsker ([14]) but here is used in applications. Theorems 5.3 and 5.4 are versions of theorems in Basawa and Scott ([5]).

2. Definition, notations.

We will use very often along this paper the K-L "distance", also called K-L divergence or relative entropy, which is however a distance only in broad sense, (see Kullback ([12])).

DEFINITION 1.

If P, Q are probability measures on a measurable space $(\mathcal{W},\mathcal{A})$ the (classical) K-L distance is :

$$\rho(P,Q) = \rho_K(P,Q) = \int p \log(p/q)\, d\nu = E_P(\log (p/q)) \qquad (2.1)$$

if $P \ll Q \ll \nu$,

$\rho(P,Q) = \infty$ if P is not absolutely continuous with respect to Q, where ν is a σ-finite measure on $(\mathcal{W},\mathcal{A})$ and $p=dP/d\nu$, $q=dQ/d\nu$ are the corresponding Radon-Nikodym derivatives.

In general $\rho(P,Q) \neq \rho(Q,P)$, i.e. the K-L distance privileges the first argument, which will be always, in this paper, the "true probability", the "true density function", etc.

We remind now some elementary results about the K-L distance:

PROPERTIES

(i) $\rho(P, Q) \geq 0$ and $\rho(P, Q)=0$ if and only if $P=Q$ (ν).

If the Hellinger distance between P and Q is defined by:

$$\rho_H(P, Q) = \tfrac{1}{2} \int (p^{1/2} - q^{1/2})^2 d\nu = 1 - \int (pq)^{1/2} d\nu .$$

then, it is known that

(ii) $$0 \leq 2\rho_H(P, Q) \leq \rho(P, Q) .$$

(It is obvious that the Hellinger distance is a distance in the proper sense).

If we denote P^n, Q^n the respective product measures, then

(iii) $$\rho(P^n, Q^n) = n\, \rho(P, Q).$$

(Cf. for instance, Borovkov ([8])).

3. Some results and comments in the independent case.

Let $X_1, X_2, X_3, \ldots, X_n, \ldots$ be independently distributed random variables with a common density $p(x)$ with respect to a σ-finite measure ν, and let $q(x,\theta) = q^\theta(x)$, $\theta \in \Theta$ be a family of densities for some parametric space Θ.

Assume that $P \ll Q^\theta$ for all $\theta \in \Theta$. In this situation, if $P = Q^{\theta_0}$ for some $\theta_0 \in \Theta$ and some regularity conditions are verified it is known that the MLE $\hat{\theta}_n$, (or any approximate MLE in the sense of A.Wald, ([17])), converges a.s. (P) to the true parameter value. It is also known that, under similar conditions, if there exists a θ_0^* such that for every $\theta \in \Theta$, $\theta \neq \theta_0^*$,

$$E_P(\log(q^\theta / q^{\theta_0^*})) < 0 \tag{3.1}$$

(and this last condition implies uniqueness of θ_0^*), then

$$\hat{\theta}_n \to \theta_0^* \quad \text{a.s. (P).}$$

(Vid.Huber ([10])). Note that the condition (3.1) may be fullfilled even though $P \neq Q^\theta$ for all $\theta \in \Theta$.

When dealing with the MLE along this paper we impose the following conditions:

3.1. *Assumptions.*

a) ("Identifiability") For all $\theta \in \Theta$, $\theta' \in \Theta$, $P(q^\theta(x) \neq q^{\theta'}(x)) > 0$ if $\theta \neq \theta'$.

b) The parametric space Θ is a compact topological space with a countable basis for the topology.

The simple form of b) is assumed only for the sake of simplicity.

Condition (3.1) is easily derived from the following one:

$$\rho(P,Q^{\theta_0^*}) < \rho(P,Q^{\theta}),\ \theta\in\Theta,\quad \theta \neq \theta_0^* \qquad (3.2)$$

i.e. θ_0^* minimizes strictly the Kullback-Leibler distance $\rho(P,Q^{\theta})$.

It is necessary however some care about finiteness of involved integrals. In fact it is possible that θ_0^* satisfies 3.1 and $\rho(P,Q^{\theta_0^*}) = \infty$. (Cf. example 4.1, case (3)).

We point out that L. Kullback in his book "Information Theory and Statistics", (Kullback ([12])), minimizes this distance, but the optimization is performed in the first argument, i.e. in the measure P.

A simple example will be illustrative.

Example 3.1.

Suppose we have a uniform density, (the true density), in the rectangle

$$[-c/2,\ c/2] \times [-d/2,\ d/2] \quad \text{, with } c \geq d$$

and we consider the family of centered bivariate Gaussian densities with covariance matrix Σ :

$$\Sigma = O^t A O$$

where the matrices A and O are, respectively :

$$A = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix} \qquad O = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

The parameter space is taken, for simplicity, as

$$\Theta = \{(a,b,\phi)/\ a \geq b > 0,\ \phi\in [-\delta,\pi-\delta]\}.$$

Straightforward calculations show that the best fitting is obtained when

$$a/b = c/d\ ,\ \phi=0\ .$$

On the other hand, it is possible to check directly that the MLE actually converges with P=1. (See also Examples 4.1, 2 and 5.1, 2 in fine).

In fact

$$\theta_o^* = (c/(12)^{1/2}, d/(12)^{1/2}, 0).$$

The following proposition will also be illustrative.

PROPOSITION 3.1.

Let P,Q be probability measures on $(\mathbf{R}^d, \mathcal{B}^d)$, $P \ll Q \ll \lambda$, with λ the Lebesgue measure, (for instance). Suppose that $q = dQ/d\lambda$ is a symmetric density and that $-\log(q(x))$ is a convex function.

Let $p = dP/d\lambda$ be another symmetric density and consider the family $q(x-\theta)$, $\theta \in R^k$ which verifies $E_p(|\log(p(x)/q(x-\theta)|) < \infty$ at least for some value θ_1.

Then, the best fitting of $q(.-\theta)$ to $p(.)$ is obtained at $\theta_o^*=0$.

PROOF:

It is a simple application of convexity of $-\log(q)$ which implies that

$$\int_0^\infty (\log(p(x)/q(x-\theta)+\log(p(x)/q(x+\theta))\, p(x)\, dx \geq$$

$$\int_0^\infty (\log(p^2(x)/q(x)^2)\, p(x) dx \qquad .$$

Examples 3.2

$q(x-\theta) = \frac{1}{2}\exp(-|x-\theta|)$ verifies the assumption of convexity for the $-\log(q)$ function. Let $p(x) = 1/(2\pi)^{1/2} x^2 \exp(-x^2/2)$ be the true density. The fitting occurs at $\theta_o^*=0$, as expected.

A slight change in the latter example gives rise to

$$p(\mathbf{x})=1/(2\pi)(x^2+y^2)\exp(-(x^2+y^2)/2),$$

and the Model

$$q(\mathbf{x})=1/(2\pi\sigma^2)\exp(-((x-\mu_1)^2+(y-\mu_2)^2)/(2\sigma^2)),\ \theta=(\boldsymbol{\mu},\sigma)\in R^2xR^+.$$

In this case the model fits at the value $\theta_0^*=(\mu_0^*, \sigma_0^*) = (\mathbf{0},3^{1/2})$.

REMARK 3.1. In both cases clearly the MLE converges a.s. (P) to θ_0^*.

4. Discrete time stochastic processes.

4.1 *Some previous ideas.*

In paragraph 1 we pointed out some properties of the K-L distance in the independent case.

However there is another important property to be considered.

(iv) Let $(\mathcal{W},\mathcal{A})$ be a measurable space and $\mathcal{B}\subset\mathcal{A}$ a σ-algebra. If P and Q, (P«Q) are probability measures on $(\mathcal{W},\mathcal{A})$ then

$$\rho(P/\mathcal{B},Q/\mathcal{B})\leq\rho(P,Q) \tag{4.1}$$

with equality if and only if dP/dQ is $\mathcal{B}$-measurable (a.e. Q). (Here $P/\mathcal{B}, Q/\mathcal{B}$, denote the restrictions to $\mathcal{B}$ of the probability measures P,Q).

This fact is obvious if $\rho(P,Q)=\infty$.

On the other hand, when P « Q it is an easy consequence of the convexity of the function

$$\phi(x) = x\log x+1-x$$

with x=dP/dQ. In fact Jensen's inequality yields

$$E_Q(\phi(X)/\mathcal{B})\geq\phi(E_Q(X/\mathcal{B}))$$

which implies (4.1).

Hence, if there is a filtration, $\mathfrak{F}_1 \subset \mathfrak{F}_2 \subset ... \subset \mathfrak{F}_n \subset ... \subset \mathfrak{F}_\infty = \mathcal{A}$ we will have that

$$\rho_n = \rho(P/\mathfrak{F}_n, Q/\mathfrak{F}_n) \nearrow a \leq \infty$$

In particular, if we consider $\mathfrak{F}_n$, the σ-algebra generated by $X_1, X_2, X_3, ..., X_n$ we will have

$$\lim \rho_n \leq \infty.$$

Suppose now that Q^θ is a family of distributions as a model for a process $\{X_n\}$, $n \geq 1$, which has true distribution P, and $P \ll Q \ll \nu$, where ν is a σ-finite measure on $(\mathcal{W}, \mathcal{A})$.

We consider

$$\rho_n(\theta) = \rho(P_n, Q_n^\theta)$$

If $P_n \ll Q_n^\theta \ll \nu$ and we denote by p_n, q_n^θ the corresponding Radon-Nikodym derivatives, we have that $\rho_n(\theta)$ is increasing for every $\theta \in \Theta$.

Then we summarize the possibilities for the behavior of $\rho_n(\theta)$ in the following.

PROPOSITION 4.1.

There are only three possibilities as $n \to +\infty$:

(1) $0 < \rho_n(\theta) \nearrow h(\theta)$, finite for some $\theta \in \Theta$.

(2) $0 < \rho_n(\theta) \nearrow +\infty$ for all $\theta \in \Theta$, but there is a sequence $\{b_n\}_{n \geq 1}$, $b_n \to +\infty$ such as $b_n^{-1} . \rho_n(\theta)$ converges to a strictly positive value $g(\theta)$ for all $\theta \in \Theta$, which at least is finite for some value of the parameter.

(3) There is not such a sequence b_n, (for instance, for all θ, $\rho_n(\theta) = +\infty$ for some value of n, which can be dependent of θ).

Therefore, the value $g(\theta)$, (or $h(\theta)$) may be interpreted as a "generalized" K-L distance.

We will see for instance, that if some additional conditions are satisfied, the MLE converges to the value θ_0^* which strictly minimizes $g(\theta)$. Also, by using the "generalized" K-L distance, we will prove some propositions which extend known statements for tests of hypotheses in the independent case. These theorems are exposed in paragraph 5.2.

Examples 4.1.

For (1):

If $Q^\theta = P$ for some value of $\theta \in \Theta$, say θ_0, $\rho_n(\theta_0)=0$ for all n. Then, if the outcomes are i.i.d., the MLE converges a.s. to the true value when well-kown conditions are fullfilled.

A different example:

Consider a random variable which is the same for all values of the index n: $X_n=X_1$.

Suppose that X_1 has unknown density p(.), and consider the density $p(.-\theta)$, $\theta \in R$, as a model and add some condition for integrability.

In this example the information remains the same for $n \geq 1$, and the MLE is not consistent except for trivial situations.

The following is another example. Assume a process of independent random variables with density

$$p_i(x)= 1/(2\pi)^{1/2}\exp(-(x+a_i)^2/2),\ i=1,2,...n,...,\ \text{with } \sum_{i=1}^{\infty} a_i^2<\infty,$$

and a model

$$q(x,\theta)=1/(2\pi)^{1/2}\exp(-(x+\theta)^2/2).$$

Then $h(\theta)=\infty$ for $\theta\neq 0$ and $h(0)<\infty$.

Observe that in this example $\hat{\theta}_n \to 0$ c.s. (P), i.e. the MLE converges to the parameter value that minimizes $h(\theta)$, (it is a "consistent " estimate).

For (2):

It suffices to consider any independent and identically distributed sequence of r.v. with density p(.) and a model q^θ (.) with $\rho_1(\theta)<\infty$ for some θ. In this case, $b_n= 1/n$.

Theorems 4.1 and 5.1 establish important examples for the dependent case.

The case (3) in Proposition 2 is illustrated by taking a process of i.i.d. r. v. with

$$p(x)=\frac{1}{\pi}\frac{1}{1+x^2},$$

and the model

$$q^\theta(x)=\tfrac{1}{2}\exp(-|x-\theta|),\ \theta\in R$$

We obtain $\rho_n(\theta)=\rho(P_n, Q^\theta_n)=\infty$ for all $\theta\in R$. However, $\hat{\theta}_n$ is the sample median, a consistent estimate of the median of the distribution. Actually this example shows that it can happen that (3.2) has not sense but (3.1) is verified.

In what follows we consider mainly the case (2) and $b_n = n$. In paragraph 5.1 below, the example 3 shows that 1/n cannot be the "unique" usual sequence in case (2). For stationary sequences however, it seems that the "canonical" sequence is $1/b_n = 1/n$.

Actually, under mild assumptions it is easy to prove the following proposition:

" In the stationary case there exists (in general)

$$\lim \frac{1}{n} \rho_n \text{"} \tag{4.2}$$

If the restrictions P_n, Q_n of the probability measures P, Q to $\mathcal{F}_n = \sigma(X_1, X_2, X_3, ..., X_n)$ verify $P_n \ll Q_n$ for each n, and both are invariant for the shift, let us denote:

$$p_n = \frac{dP_n}{dQ_n} \quad \text{and} \quad p_{1,n} = p_n(X_1, X_2\, X_3, ..., X_n)\,,$$

$$p_{h,k} = p_{h,k}(X_h, X_{h+1}, ..., X_k) = p_{k-h+1}(X_h, X_{h+1}, ..., X_k).$$

Assume that

$$\int |\log p_{1,n}|\, dP < \infty \text{ for every n.}$$

Then, we have

$$\int \log p_{1,n}\, dP \geq \int \log p_{1,k} \cdot p_{k+1,2k} \cdots p_{(r-1)k,\, rk} \cdot p_{rk,\, n} dP,$$

provided that

$$P_{1,n} \ll P_{1,k} \otimes P_{k+1,\, 2k} \otimes \ldots \otimes P_{(r-1)k,\, rk} \otimes P_{rk,\, n} \tag{4.3}$$

for every n, where n= r.k+s with $o \leq s < k$, (k fixed).

Therefore, by stationarity we have:

$$\frac{1}{n} \int \log p_{1,n}\, dP \geq \frac{(n-s)/k}{n} \int \log p_{1,k}\, dP + \frac{1}{n} \int \log p_{1,s}\, dP$$

and then, again by stationarity if

$$\int \log p_{1,m}\, dP < +\infty \text{ for } o \leq m \leq k,$$

(4.4)

$$\underline{\lim} \frac{1}{n} \int \log p_{1,n}\, dP \geq \frac{1}{k} \int \log p_{1,k}\, dP \tag{4.5}$$

Now, if (4.3) and (4.4) are verified for all k, we can take

$$\overline{\lim}\frac{1}{k}\int \log p_{1,k}\, dP$$

and the limit in (4.2) exists. .

4.2 *Applications of generalized K-L distance: consistency of MLE.*

We denote q^o, q_n^o the densities corresponding to Q^θ, Q_n^θ for $\theta = \theta_o^*$.

The remaining notations are the ones established above.

THEOREM 4.1.

Suppose that the assumptions a) and b) of paragraph 2.1 are satisfied, that $P_n \ll Q_n^\theta$ for each $\theta \in \Theta$ and all n, that P, $Q^\theta \ll \nu$ where ν is a σ-finite measure in the underlying space, (v.g. the canonical ($\mathbf{R}^\infty$, $\mathbf{B}_\infty$)), and that:

(i) $\lim \frac{1}{n} E_P(\log(p_n / q_n^o)) = g(\theta_o^*)$, where θ_o^* is the point where the function $g(\theta)$ attains its strict minimum.

(ii) $\lim \frac{1}{n} \log(p_n(X_1,X_2,X_3,\ldots,X_n)/ q_n^\theta(X_1,X_2,X_3,\ldots,X_n))=g(\theta)$ with P=1 for each $\theta \in \Theta$,

(iii)Denoting $\check{q}_n = \check{q}_n(X_1,X_2,X_3,\ldots X_n) = \sup_{\theta' \in V} q_n^{\theta'}(X_1,X_2,X_3,\ldots X_n)$,

we suppose that, for each $\theta \in \Theta$

$$\lim_{V \downarrow \{\theta\}} \lim \frac{1}{n} \log (\check{q}_n / q_n^o) \leq g(\theta_o^*)-g(\theta) \text{ with } P=1.$$

The limits are taken in the indicated order: first in n, then when V shrinks to θ.

Then, if $\hat{\theta}_n$ is an approximate ML estimator,

$$\hat{\theta}_n \to \theta_o^* \text{ a.s. } (P).$$

REMARKS 4.1.The most important assumption is (iii), that can weakened as follows

$$\lim_{V\downarrow\{\theta\}} \overline{\lim} \frac{1}{n} \log (q_n^V / \quad q_n^\theta) \leq f(\theta) \ < 0$$

for each $\theta\in\Theta$, $\theta \neq \theta_0^*$ a.s. (P). It is possible to prove the Theorem 4.1 with this assumption only. Anyway (i) and (ii) are often verified (Cf. Barron ([3]), Theorem 1)).

In Leroux ([13]) is proved the consistency of MLE for hidden Markov Chains following the scheme of Theorem 4.1. We point out that B.G. Leroux arrives in his paper at the same concept of generalized K-L distance for hidden Markov models (p.136).

Obviously if $P = Q^\theta$ for some θ_0, the assumption (i) is fulfilled with $\theta_0^* = \theta_0$. In this case, $g(\theta_0)=0$.

In general, there is a θ_0^* which minimizes the generalized K-L distance. This happens when Θ is compact and

$$\lim \frac{1}{n} E_p(\log p_n / \quad q_n^\theta) = g(\theta)$$

is a continuous function of θ.

Theorem 4.1. generalizes the independent case, where (i) appears in the form of the expression (3.1) and (ii) is a consequence of the Law of Large Numbers:

$$\lim_n \frac{1}{n} \log(p_1(X_1)p_1(X_2)...p_1(X_n)/q_1{}^\theta(X_1)\, q_1{}^\theta(X_2)...q_1{}^\theta(X_n)=$$

$$E_p(\log(p(X_1)/q_n{}^\theta(X_1)) = g(\theta),$$

where the convergence is a.s.(P).

There are some hypotheses that can be weakened in Theorem 4.1. For example, it is possible that the limits in (ii) and

(iii) should be verified only in probability. The proof is easily modified to cover this situation. (Cf. Theorem 5.2. and related examples).

PROOF OF THEOREM 4.1:

We omit the proof because it is very easy and can be performed following the method of Wald ([17]). .

5. Applications and examples.

With the same notations that we have been using up to here, let $\{X_n\}_{n\geq 0}$ be a Markov process defined on $(\mathcal{W}, \mathcal{A}, P)$. We assume that the measurable space is the canonical $(\mathbf{R}^\infty, \mathbf{B}_\infty)$, and $\{X_n\}_{n\geq 0}$ the coordinate process.

Let us suppose that the true probability P is absolutely continuous with respect to a σ-finite measure ν, and that under P the process has a stationary transition probability p(.,.) with a unique invariant density p(.). Under P the process is stationary and $p(x, .) \ll p(.)$. (Cf. condition 2.1 (i) in Billinsley ([6])). These conditions assure the ergodicity of the process .

We consider a model Q^θ, $\theta\in\Theta$, with stationary density $q^\theta(.)$ and (stationary) transition probability $q^\theta(.,.)$, $P\ll Q^\theta\ll\nu$ for every $\theta\in\Theta$, and we assume the hypotheses a) and b) of paragraph 3.1.

Then,we have

THEOREM 5.1.

If $E_P(\log (p(x_1,x_2)/q^{\theta}(x_1,x_2)))< +\infty$ for some $\theta\in\Theta$ (1)

and there is a θ_o^* such that

$$E_P(\log (p(x_1)/q^{\theta_o^*}(x_1)))<+\infty$$

$$E_P(\log (q^{\theta}(x_1,x_2)/ q^{\theta_o^*}(x_1,x_2)))< 0 \qquad \text{for all } \theta\neq\theta_o^*; \qquad (2)$$

Denote $q^{V}(x_1,x_2) = \sup\limits_{\theta'\in V} q_n^{\theta'}(x_1,x_2)$ where $V=V_\theta$ is a neighborhood of θ and suppose that

$$E_P(\log^+(q^{V}(x_1)/q^{\theta_o^*}(x_1)))<\infty$$

$$E_P(\log^+(q^{V}(x_1,x_2)/q^{\theta_o^*}(x_1,x_2)))<\infty$$

(3)

for some V for each θ. (It is assumed measurability of the involved functions). Suppose also that when $\theta_n\to\theta$,

$$\overline{\lim}\, q^{\theta_n}(x_1,x_2) \leq q^{\theta}(x_1,x_2) \text{ a.s.(P)}, \qquad (4)$$

i. e. $q^{\theta}(x_1,x_2)$ is upper semicontinuous in θ a.s.(P).

Finally if $\hat{\theta}_n$ verifies

$$\prod_{i=1}^{n} \frac{q^{\hat{\theta}_n}(x_i, x_{i+1})}{q^{\theta}(x_i, x_{i+1})} \geq c>0 \text{ for all } \theta\in\Theta, \quad \text{then}$$

$$\hat{\theta}_n\to\theta_o^* \text{ a.s. (P)}.$$

PROOF:

We omit details. The keys of the proof are ergodicity and Theorem 4.1.

REMARK 5.1.

We observe that G. Roussas, ([16]) , and B.L.S Prakasa Rao, ([15]), state theorems close to ours in the case

$Q^{\theta}=P$.The former reference also provides a proof of asymptotic normality.

The observation of P.Billingsley, ([7]), that the initial density is negligeable for Markov Chains has a complete justification through the generalized K-L distance: in our hypotheses the function $g(\theta)$ does not depend on this first term.

Clearly, it is possible (and straightforward) to generalize the precedent result for p-steps backward. (Cf. the following example).

Examples 5.1.

1.Consider an AR (1) stationary process $x_n=\rho_0\, x_{n-1}+u_n$ with

$|\rho_0|<1$ and u_n independent from the past with a centered and symmetric distribution with variance σ_0^2. (For instance, a density with compact support and $E_p(|\log(p)|)<\infty$).

For the model we take also an AR (1) process with u_n -density

$$q(x)=1/(2\pi\sigma^2)^{1/2}\exp(-x^2/2\sigma^2).$$

We take the parameter space $\Theta=\{(\sigma^2,\rho)\,/\,\sigma^2>0, |\rho|<1\}$.

It is a simple matter to solve the equation (3.1) and to determine $\theta_0^*=(\sigma_0^2,\rho_0)$.

The conditions of Theorem 5.1 are easily verified. The ML estimators are, in this case:

$$\hat{\rho}_n=\sum_{i=1}^{n}x_ix_{i-1}/\sum_{i=1}^{n}x_{i-1}^2,\qquad \hat{\sigma}_n^2=\frac{1}{n}\Big[\sum_{i=1}^{n}x_i^2-\Big(\sum_{i=1}^{n}x_ix_{i-1}\Big)^2/\sum_{i=1}^{n}x_{i-1}^2\Big]$$

which are clearly consistent, (See, for instance, Anderson ([2])).

2. To show a situation in which Theorem 5.1 is not applicable but the same conclusion is obtained, consider a random walk.

Let $p_n(x_1,x_2,...,x_n) = p(x_1).p(x_2-x_1)...p(x_n-x_{n-1})$ be the underlying distribution and consider the model:

$q_n(x_1,x_2,...,x_n)=q^\theta(x_1)\, q^\theta(x_2-x_1)...q^\theta(x_n-x_{n-1})$ with $\theta\in\Theta\subset R^k$, $p \ll q$. Also suppose:

(a) $g(\theta)=\int p(x)\log(p(x)/q^{\theta}(x))dx<\infty$ for some parameter value θ_o^* where $g(\theta)$ attains its strict minimum.

(b) $\sup_{\theta'\in V} q_n^{\theta'}(x) \downarrow q_n^{\theta}(x)$ when $V\downarrow\{\theta\}$, V neighborhood of θ, a.s. (P), (i.e, the function $q_n^\theta(x)$ is - at least - upper semicontinuous in θ).

In fact, we may take the true distribution and the same model as in example 3.1 and we will obtain the same estimate and limit a.s.(P). (In this case X is a bidimensional random vector).

3. A slight modification of the underlying distribution provides an example of a sequence $\{1/b_n\}$ different from $\{1/n\}$. (See paragraph 4.1).

Suppose that p(.) changes in each step. For instance, let p_n be

$$p_n(x) = \mathbf{1}_{(-n,n)^c}(x)\frac{c_n}{\pi(1+x^2)} \quad \text{with}$$

$$c_n = \frac{1}{1-\frac{2}{\pi}\arctan(n)} = \frac{\pi}{2}n + o(n) \quad \text{and } q = \frac{1}{\pi(1+x^2)}$$

where $\mathbf{1}_A(.)$ denotes the indicator function of A, A^c symbolizes the complementary set of A, and $o(n)/n \to 0$.

Then,

$$\int(\log(p_n(x)/q(x))p_n(x)dx = 2\int_n^\infty c_n\log(c_n)\frac{dx}{\pi(1+x^2)} = 2\log(c_n)$$

and for the generalized K-L distance we obtain:

$$\rho_n = \sum_{i=1}^{n} 2\log(c_i) = 2n\log(\pi/2) + 2\log(n!) + O(\log(n!)).$$

But

$$\log(n!) \cong \frac{1}{2}\log(2\pi) + n\log n - n + \frac{1}{2}\log n.$$

Therefore, in this case, b_n^{-1} can be chosen as $\frac{1}{n\log n}$. (The model can be taken as $\frac{1}{\pi(1+(x-\theta)^2)}$, but it has very limited interest).

5.2. *Stationary Gaussian process.*

Let $X_1, X_2, \ldots, X_n, \ldots$ be a discrete time stationary Gaussian process with spectral density f^o and f^θ a stationary Gaussian model for the process.

Suppose that $0<m\leq f^o$, $f^\theta \leq M<\infty$. Then, if P, Q^θ are the true distribution and the model distribution and P_n, Q_n^θ the respective finite dimensional distributions, we have:

THEOREM 5.2.

$$\lim_n \frac{1}{n}\rho(P_n,\ Q_n^\theta) = \frac{1}{4\pi}\int_{-\pi}^{\pi}(f^o/f^\theta - 1 - \log(f^o/f^\theta))d\lambda$$

being λ the Lebesgue measure.

This limit is strictly positive (a.e. (λ)) unless that $f^o = f^\theta$ a.e. (λ).

Moreover

$$\frac{1}{n}\log p_n(X_1X_2,\ldots,X_n)/q_n^\theta(X_1,X_2,\ldots,X_n) \to \frac{1}{4\pi}\int_{-\pi}^{\pi}(f^o/f^\theta - 1 - \log(f^o/f^\theta))d\lambda$$

the convergence being in $L^2(P)$.

For a proof, see for example Dacunha Castelle - Duflo ([9]), Vol. II, pp. 70-72 .

REMARK 5.2.

Theorem 5.2 assures the applicability of Theorem 4.1 to Gaussian stationary process with spectral densities satisfying the convenient hypotheses.

In fact, if there exists a parameter value θ_0^* such that

$$g(\theta)=\frac{1}{4\pi}\int_{-\pi}^{\pi}(f^{o}/f^{\theta}-1-\log(f^{o}/f^{\theta}))d\lambda$$

attains a strict minimum, assumption (i) of Theorem 4.1 is verified. On the other hand, the second statement of Theorem 5.2 assures that assumption (ii) of Theorem 4.1 is fulfilled in probability, so that it is only necessary the verification of (iii) in probability, in order to apply Theorem 4.1.

The hypothesis $0<m\leq f^{o}, f^{\theta}\leq M<\infty$ can be weakened to the following one $f^{o}/f^{\theta}\leq M<\infty$ and still the convergence in probability to a positive constant stated in the Theorem 5.2. is true.

Examples 5.3.

1.Assume a moving average process

$$X_t=Z_t+\theta Z_{t-1}$$

with Z_t a Gaussian white noise with variance 1 and $|\theta|<1$. Let the model be an AR(1) process

$$X_t-\rho X_{t-1}=V_t$$

with $|\rho|\leq 1-\delta$ and V_t a Gaussian white noise independent of the past with variance σ^2.

For the respective spectral densities, we have:

$$f^{o}(\lambda)=(1+\theta^2+2\theta\cos\lambda)/2\pi,\quad f^{\rho}(\lambda)=\frac{1}{2\pi}\,\sigma^2.(1+\rho^2-2\rho\cos\lambda)^{-1}$$

Then, as

$$\frac{f^{o}(\lambda)}{f^{\rho}(\lambda)}=(1+\theta^2+2\theta\cos\lambda)(1+\rho^2-2\rho\cos\lambda)/\sigma^2$$

we have for the generalized K-L distance:

$$g(\rho,\theta)=[(1+\rho^2)(1+\theta^2-2\theta\rho)]/2\sigma^2-1/2+1/2\log(\sigma^2)$$

and for θ_o^* we obtain the value:

$$\rho_o^*=\theta/(1+\theta^2),\quad \sigma_o^{*2}=1+\theta^2-\theta^2/(1+\theta^2)^2$$

$(\int_{-\pi}^{\pi}\log((1+\theta^2+2\theta\cos\lambda)d\lambda=0$, by Poisson-Jensen formula. Cf. for instance, L. Ahlfors ([1])).

It is quite elementary to verify the hypothesis (iii) of Theorem 4.1. Therefore, the conclusion of T.1 follows.

2. Suppose now the same model, but the true process is a Gaussian ARMA(1,1):

$$X_t-\theta X_{t-1}=Z_t+\theta Z_{t-1},\quad |\theta|<1.$$

In this case,

$$\frac{f^{o}(\lambda)}{f^{\rho}(l)}=\frac{(1+\theta^2+2\theta\cos\lambda)}{\sigma^2}\cdot\frac{(1+\rho^2-2\rho\cos\lambda)}{(1+\theta^2-2\theta\cos\lambda)}\quad\text{and}$$

$$g(\rho,\sigma^2)=1/2\frac{(1+\theta^2)(1+\rho^2)}{\sigma^2(1-\theta^2)}-\frac{\theta\rho}{\sigma^2(1-\theta^2)}+\frac{\theta^2(1+\rho^2)-\theta\rho(1+\theta^2)}{\sigma^2(1-\theta^2)}+$$

$$(-1/2)+1/2\log(\sigma^2)$$

We obtain $\rho_o^*=\dfrac{(2+\theta^2)\theta}{(1+3\theta^2)}$, $\sigma_o^{*2}=\dfrac{1+2\theta^2+5\theta^4-\theta^6}{(1+3\theta^2)(1-\theta^2)}$.

We point out that the variance of the X_t of the Model is exactly the variance of X_t for the true process, and the true correlation coefficient is: $\dfrac{2(1+\theta^2)\theta}{(1+3\theta^2)}$.

3. Consider a p-moving average and a model AR(1).

For instance, we have $X_n = \frac{1}{p}\sum_{i=1}^{p} \xi_{n-i+1}$ where $\{\xi_i\}_{i\geq 2-p}$ is a Gaussian white noise with variance σ_o^2, and for the model we have $Y_n = \sum_{i=1}^{\infty} \rho^{i-1}\zeta_{n-i+1}$ where $\{\zeta_j\}_{j\geq -\infty}$ is also a Gaussian white noise with (unknown) variance σ^2.

For the quotient of densities we have :

$$\frac{f^o(\lambda)}{f^\theta(\lambda)} = \sigma_o^2 \cdot \frac{1}{p^2} \frac{(1-\cos p\lambda).(1-2\rho\cos\lambda+\rho^2)}{\sigma^2(1-\cos\lambda)}.$$

A few computations show that:

$\rho_o^* = \dfrac{p-1}{p}$ and $\sigma_o^{*2} = \sigma_o^2 \cdot \dfrac{p^3}{2p-1}$ and it is easy to see that the MLE converges a.s.(P) to these values.

Observe that Theorem 5.2 is not applicable because the true density vanishes for instance in $\lambda = \frac{2\pi}{p}$ (p>1), but the condition

$$\frac{f^o}{f^\theta} \leq M < \infty \quad \text{is verified}.$$

Observe also that Y_n has the same variance that the moving average.

4. Finally, let $X_t=(Z_t+Z_{t-1})/2$ be the true process with Z_t a Gaussian white noise with variance 1 and a Model $Y_t=V_t+\theta V_{t-1}$

with $\theta > 0$, fixed, and V_t a Gaussian white noise with unknown variance σ^2.

The quotient of densities is:

$$\frac{f^o}{f^{\sigma^2}} = 1/4\ \frac{(1-\cos 2\lambda)}{(1-\cos\lambda)\sigma^2}\ \frac{1}{(1+\theta^2+2\theta\cos\lambda)}$$ and we obtain for the variance of the model the estimate $\frac{1}{2(1+\theta)}$.

We observe that the Model is non-Markovian. (Compare with Barron ([3])).

As an example of application of the *generalized Kullback-Leibler distance* in a different direction we have, in the context of the previous paragraphs, the following statements (Cf. Basawa & Scott ([5])):

THEOREM 5.3. (Probability ratio test).

(i) If for a sequence $\{b_n\}_{n\geq 1}$, $b_n \to +\infty$ we have $1/b_n \log(p_n/q_n) \to c > 0$ a.s. (P) and we consider a sequence of Neyman-Pearson tests with size $\alpha_n \to \alpha \in (0,1)$, then $\overline{\lim}\ 1/b_n \log(\beta_n) \leq -c$, where β_n is the error of second kind probability.

(ii) A similar result is verified if $1/b_n \log(p_n/q_n) \to Z$, where Z is (a.s. (P)) a positive random variable .The convergence being a.s. (P). In this case however

$$\overline{\lim}\ 1/b_n \log(\beta_n) \leq -t_\alpha \text{ where } t_\alpha \text{ is defined by } P(Z \leq t_\alpha) = t_\alpha.$$

PROOF OF THEOREM 5.3:

(i) Let $A_n = \{q_n/p_n \geq k_n\}$ be the critical region. Since

$$1/b_n \log(\beta_n) = 1/b_n \log\left(\int_{A_n^c} dQ_n\right) = 1/b_n \log\left(\int_{A_n^c} (q_n/p_n)\, dP_n\right) \leq$$

$\leq 1/b_n \log(k_n.P(A_n^c)) = 1/b_n \log(k_n) + 1/b_n \log(1-\alpha_n)$,

by using that $1/b_n \log(p_n/q_n) \to c$ a.s.(P), that $b_n \to +\infty$ and that $\alpha_n \to \alpha \in (0,1)$, we will have that

$$\overline{\lim}\ 1/b_n \log(\beta_n) \leq \overline{\lim}\ 1/b_n \log(k_n) \leq -c.$$

For (ii), we start with the inequality

$$\overline{\lim}\ 1/b_n \log(\beta_n) \leq \overline{\lim}\ 1/b_n \log(k_n).$$

Now

$$P(\{1/b_n \log(q_n/p_n) \leq -t_\alpha + \varepsilon\}) \geq$$

$$P(\{1/b_n \log(q_n/p_n) + Z \leq \varepsilon\} \cap \{-Z \leq -t_\alpha\}) \to P(\{Z \geq t_\alpha\}) = 1-\alpha,$$

and if we choose α' such as $P(\{Z \geq t_{\alpha'}\}) = 1-\alpha' > 1-\alpha$, we will have simultaneously that

$$P(\{1/b_n \log(k_n)\} \leq 1/b_n \log(q_n/p_n)\}) \to \alpha \quad \text{and}$$

$$P(\{1/b_n \log(q_n/p_n) \leq -t_{\alpha'} + \varepsilon\}) \to 1-\alpha' > 1-\alpha, \quad \text{a.s. (P)}$$

Hence

$$\overline{\lim}\ 1/b_n \log(\beta_n) \leq -t_{\alpha'} + \varepsilon \text{ for all } \varepsilon > 0 \text{ and all } \alpha' < \alpha, \text{ that is}$$

$$\overline{\lim}\ 1/b_n \log(\beta_n) \leq -t_\alpha.$$

THEOREM 5.4.

For all test ϕ_n with size $\alpha_n \leq \alpha \in (0,1)$,

$$\underline{\lim}\ 1/b_n \log(\beta_n^{\phi_n}) \geq -t_\alpha.$$

PROOF :

Without loss of generality we can choose $k_n = \exp(-b_n t_\alpha)$.

Let B_n^c be a critical region with $P(B_n^c) = \gamma_n \leq \gamma < \alpha$ and

$\tilde{A}_n = \{q_n/p_n \geq k_n^{1+\varepsilon}\}$. We have :

$$\beta_n{}^{\phi_n} = Q_n(B_n) \geq \int_{B_n \cap \tilde{A}_n} dQ_n \geq \int_{B_n \cap \tilde{A}_n} k_n^{1+\varepsilon} \, dP_n =$$

$$= k_n^{1+\varepsilon} . P(B_n \cap \tilde{A}_n) \geq k_n^{1+\varepsilon} (P(B_n) - P(\tilde{A}_n^c))$$

Since

$$P(\tilde{A}_n^c) = P(\{1/bn \log(q_n/p_n) \leq (1+\varepsilon)1/bn \log k_n\}) = P(\{1/bn \log(q_n/p_n) \leq (1+\varepsilon)(-t_\alpha\}) \to P(\{ Z \geq t_\alpha (1+\varepsilon)\}) \leq 1-\alpha$$

and

$$P(B_n) = 1-\gamma_n \geq 1-\gamma > 1-\alpha,$$

we find that

$$\underline{\lim}\ 1/b_n \log(\beta_n{}^{\phi_n}) \geq -t_\alpha.(1+\varepsilon), \text{ for all } \varepsilon > 0.$$

REMARK 5.3.

(Asymptotic optimality of the Neyman-Pearson test).

If ϕ_n^* stands for the Neyman-Pearson test, we obtain as an immediate consequence of the last Theorem that

$\underline{\lim}\ 1/b_n \log(\beta_n{}^{\phi_n^*}) \geq -t_\alpha \geq \overline{\lim}\ 1/b_n \log(\beta_n{}^{\phi_n^*})$, i.e. there is equality .

Examples 5.4.

As a source of examples for the Theorems 5.3 and 5.4 we mention only the case of exchangeable variables. Concretely we may consider the following distributions:

$$p_n = \int_R 1/(2\pi)^{n/2} \exp(- \sum_{i=1}^{\infty}(x_i-\mu)^2/2). \exp(-\mu^2/2)\, d\mu =$$

$$1/(2\pi)^{n/2}\ 1/(n+1)^{1/2} \exp(-\sum_{i=1}^{\infty}(x_i-\overline{x})^2/2).\ \exp(-\overline{x}^2/2(n+1))$$

and for the model q_n the multivariate density function corresponding to n independent standard Gaussian variables. In this case we have clearly

$$1/n \log(q_n/p_n) \to Z > 0 \text{ a.s } (P).$$

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Convex Delay Endomorphisms

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Abstract

In this paper delay equations $x_{n+k} = f(x_n, ..., x_{n+k-1})$ are considered, where the function f is supposed to be convex, having a unique point of maximum. It is proved that if there are no stationary solutions then all solutions must diverge. Considering the one parameter family $f_\mu = \mu + f$ and associating to it a family of two dimensional maps F_μ it is shown that the set of points having bounded orbit under F_μ is homemorphic to the product of a Cantor set and a circle, and is hyperbolic and stable.

1 Introduction

Any delay equation of order k:

$$x_{n+k} = f(x_n, ..., x_{n+k-1}) \tag{1}$$

can be associated with a transformation of R^k given by

$$F(x_1, ..., x_k) = (x_2, ..., x_k, f(x_1, ...x_k)) \tag{2}$$

Any orbit of the map F is in one to one correspondence with a solution of the delay equation (1). Here we will deal with delay equations where the function f is *convex*, in the sense that f is a C^2 function such that the quadratic form associated with the second derivative is definite at every point. In this case equation (1) is called a convex delay equation and the map F defined in (2) is called a convex delay endomorphism. In the rest of this work, we will take this quadratic form negatively definite, so that f could have at most one critical point that should be a maximum. A stationary solution of the delay equation (1) is a constant solution $x_n = x$ for every n; the existence of such an x is equivalent to have a solution of the equation $f(x, ..., x) = x$; Moreover, the fixed points of F are the points $(x, ..., x)$, where x is a solution of $f(x, ..., x) = x$. So when f is convex the delay equation associated would have at most two stationary solutions, or, wich is the same, the endomorphism F would have at most two fixed points. We will prove the following result:

Theorem 1.1 *Let f be convex and suppose that F has no fixed points. Then the ω limit set under F of any point in R^k is empty.*

*The final version of this paper will be submitted for publication elsewhere

In terms of delay equations this says that if f is convex and there are no stationary solutions, then all the solutions must diverge.
Consider a convex first order equation given by $f : R \to R$, and suppose that f is not only convex but there is a negative constant such that f'' is less than this constant. If we push up the graph of f vertically, we will obtain a one parameter family $f_\mu = \mu + f$; for this one dimensional map it is easy to see that for every large parameters the function f_μ will have two fixed repelling points and that the set of preimages of any one of these points accumulates in a Cantor hyperbolic set which is the complement in the line of the basin of attraction of ∞ (or, what is the same, the set of points with empty ω limit set). Under some new conditions on the function f that will be defined in section 3, this result remains true for second order equations; these are open conditions, define a set $\mathcal{U}$, and imply that F is convex.

Theorem 1.2 *There exists an open set $\mathcal{U}$ in $C^2(R^2)$ such that for any $f \in \mathcal{U}$ the family of endomorphisms $F_\mu(x,y) = (y, \mu + f(x,y))$ has the following properties, for every μ sufficiently large:*
a) F_μ has two fixed saddle points
b) The closure of the stable manifold of one of these points is diffeomorphic to the product of a Cantor set K with a circle S^1
c) The basin of ∞ is the complementary set in R^2 of the closure of the stable manifold.

As a corollary of the proof of this theorem it can also be obtained a description of the dynamics of F_μ restricted to the closure of the stable manifold ($= K \times S^1$). Each circle of $K \times S^1$ is mapped into a not closed curve contained in other circle, so this defines a one dimensional map on K, that becomes equivalent to a shift:

Theorem 1.3 *Let W^s_μ be the stable manifold of one of the fixed points of F_μ, and $\overline{W^s_\mu}$ its closure. Consider the set: $\Lambda = \bigcap_{n\geq 0} F^n_\mu(\overline{W^s_\mu})$. Then Λ is compact, F_μ − invariant, hyperbolic and coincides with the closure of the periodic points of F. Two different cases can occur: Either Λ is a horseshoe and F/Λ is a homeomorphism, or it is contained in the unstable manifold of each one of the fixed points, which in this case are equal.*

The second alternative of the last theorem it is not generic: the usual case is the first. Now the dynamics of the maps F_μ are completely described for every large parameter values.
For a particular family of quadratic delay endomorphisms, the first theorem was proved by Whitley, in [W], where it is also described the maximal invariant set, however these proofs cannot be extended to this general case, and the stable manifolds are not studied.

A very interesting reference on the subject of delay equations is the book of P. Montel, [Mon], where the theory of delay maps is treated from a general viewpoint.

We aknowledge R. Mañé and P. Duarte for useful suggestions. We are also indebted with IMPA, Rio de Janeiro, where we find the hospitality that encourage us to carry out this work.

2 Absence of fixed points

As was explained in the introduction the hypthesis of theorem one is equivalent to the non existence of solutions of the equation $f(x,...,x) = x$ or, which is the same, the graph of f does not intersects the diagonal of R^{k+1}. Let $f''(x)$ be the Hessian matrix of f at the point x. By hypothesis, f is convex, which means that if Q_x is the quadratic form associated with $f''(x)$, then $Q_x(v) = vf''(x)v^t < 0$ for each vector v not zero.

PROOF OF THEOREM 1.1

As the graph of f doesn't intersects the diagonal of R^{k+1}, there is a positive number α and a unique point $x_0 \in R^n$ such that the graph of $f + \alpha$ intersects the diagonal of R^{k+1} at $(x_0,...,x_0)$. Without loss of generality it can be assumed that $x_0 = 0$; then, using Taylor's expansion around 0, it is obtained:

$$f(x) = -\alpha + v.x + xHx + Rx \tag{3}$$

where $v = f'(0)$, $H = f''(0)$ and $R : R^k \to R$ is a C^2 function such that $lim_{x\to 0}R(x)/|x|^2 = 0$ Denoting $v = (v_1,...,v_k)$ observe that the vector $(v_1,...,v_k,-1)$ is orthogonal to the tangent space of the graph of f at 0, which by assumption contains the diagonal of R^{k+1}, so that $\sum_{i=1}^k v_i = 1$. Now define the following Lyapunov function:

$$L(x_1,...,x_k) = v_1x_1 + (v_1+v_2)x_2 + ... + (v_1 + ... + v_{k-1})x_{k-1} + x_k \tag{4}$$

As it is well known, to prove the theorem it is sufficient to show that for every $x \in R^2$, $L(F(x)) - L(x) < 0$. Then, using (3), (4) and that $\sum v_i = 1$, it is obtained:

$$\begin{aligned} L(F(x)) - L(x) &= v_1x_2 + (v_1+v_2)x_3 + ... + (v_1 + ... + v_{k-1})x_k + f(x) - L(x) \\ &= -\alpha + xHx + R(x) \end{aligned} \tag{5}$$

Now define the function $\varphi : R^k \to R$ by $\varphi(x) = xHx + R(x)$ and observe that $\varphi(0) = 0$, $\varphi'(0) = 0$ and $\varphi''(x) = f''(x)$. So φ'' is negative definite from which it follows that $\varphi(x) < 0$ for every $x \in R^k$, x not zero. This implies that $L(F(x)) - L(x) \le -\alpha < 0$ in (5), and the theorem is proved.

3 Dynamics for large parameter values

We will begin by describing the C^2-open set $\mathcal{U}$ for which the theorems are valid;

Let

$$B = -sup\{\partial_{22}f(x,y) : (x,y) \in R^2\}$$

$$A = -inf\{\partial_{11}f(x,y) : (x,y) \in R^2\}$$

$$A' = -sup\{\partial_{11}f(x,y) : (x,y) \in R^2\}$$

Definition 3.1 *Let $\mathcal{U}$ be the set of C^2 functions $f : R^2 \to R$ such that the following conditions hold:*

$$(P1) \qquad B \geq KA \qquad ;$$

where K is a positive number to be defined later

$$(P2) \qquad -\partial_{11}f(x,y) \geq |\partial_{12}f(x,y)| \quad \forall(x,y) \in R^2$$

$$(P3) \qquad A' > 0$$

Remarks:

1. (P1) and (P2) together imply that f is convex. Using also (P3) it follows that $lim_{|(x,y)| \to \infty} f(x,y) = -\infty$.

2. It is clear that this set $\mathcal{U}$ is open in the C^2 topology.

3. The theorems 1.2 and 1.3 are not valid in general if $B < A$: take for example $f(x,y) = -Ax^2 - By^2$ with $A > B$, calculate the eigenvalues of the fixed points of F_μ, and observe that it are not saddles.

4. The number K is an absolute constant independent also of $f \in \mathcal{U}$.

Now define the one parameter family to be considered: take $f \in \mathcal{U}$, and define: $f_\mu(x,y) = \mu + f(x,y)$ and $F_\mu : R^2 \to R^2$ by $F_\mu(x,y) = (y, f_\mu(x,y))$.

Now let's introduce some elementary curves that will play an important role. The critical curves of f_μ are:

$$l_1 = \{(x,y) \; : \; \partial_1 f_\mu(x,y) = 0\}$$

$$l_2 = \{(x,y) \; : \; \partial_2 f_\mu(x,y) = 0\}$$

These curves are in fact independent of μ; l_1 is the graph of a function of y, so that $l_1 = \{(\tilde{x}(y), y) : y \in R\}$, with

$$\tilde{x}'(y) = -\frac{\partial_{12}f(\tilde{x}(y),y)}{\partial_{11}f(\tilde{x}(y),y)}$$

l_2 is the graph of a function of x, so that $l_2 = \{(x, \tilde{y}(x)) : x \in R\}$, with

$$\tilde{y}'(x) = -\frac{\partial_{12}f(x,\tilde{y}(x))}{\partial_{22}f(x,\tilde{y}(x))}$$

By properties (P1) and (P2) we have that:

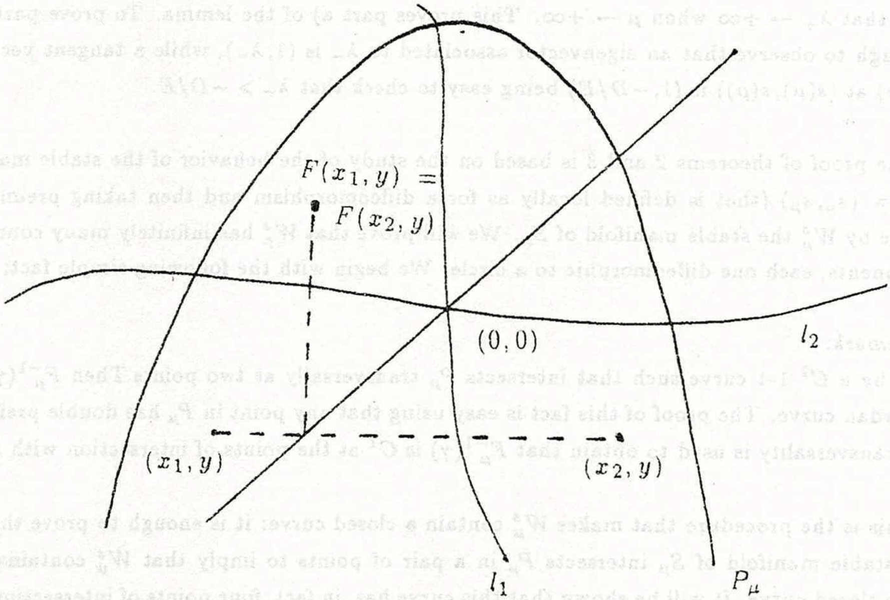
$$|\tilde{x}'(y)| < 1/K \;\; \forall y \;\; \textit{and} \;\; |\tilde{y}'(x)| < 1/K^2 \;\; \forall x$$

So $K > 1$ implies that l_1 and l_2 have one and only one point of intersection that will be supposed to be (0,0) by making a translation. From this it follows that f_μ takes its maximum at (0,0).

Also observe that l_1 is the set of critical points of F_μ.

The image P_μ of l_1 under F_μ is the graph of a function $\tilde{z}_\mu(x) = f_\mu(\tilde{x}(x), x)$, that has negative second derivative as it is easy to check using (P1) and (P2). So the complementary set of P_μ contains two connected components, one of which, $\tilde{P}_\mu$, is convex; actually, $F_\mu(R^2) = P_\mu \cup \tilde{P}_\mu$. Any point outside $P_\mu \cup \tilde{P}_\mu$ has no preimages under F_μ; a point in P_μ has only one preimage lying on l_1; and points in $\tilde{P}_\mu$ have two preimages, having the same second coordinate and located one at each side of l_1.

Denote by $\xi_\alpha(\mu)$ the α-level curve of f_μ, that is, $\xi_\alpha(\mu) = \{(x, y) : f_\mu(x, y) = \alpha\}$



Lemma 3.1 *For every μ sufficiently large it is defined a function s of μ such that:*

a) $(s(\mu), s(\mu))$ is a fixed saddle point of f_μ

b) $s(\mu) \to -\infty$ as $\mu \to +\infty$

$s'(\mu) \to 0$ as $\mu \to +\infty$

c) A local stable manifold of $(s(\mu), s(\mu))$ is transversal to $\xi(\mu)$, the family of level curves of f_μ.

Proof: As was explained before, the fixed points of F_μ are the points (x, x) for which $f_\mu(x, x) = x$. Let $g(x) = f(x, x)$. Using (P1), (P2) and (P3) it is easy to see that g has negative second derivative bounded below from zero which implies that the graph of g intersects any line $y = x - \mu$ for μ large enough. As g has its maximum at zero, one of this points will have negative coordinates; let's denote this point by $(s(\mu), s(\mu))$. It is clear that $s(\mu) \to -\infty$ as $\mu \to +\infty$ and that $s'(\mu) = (1 - g'(s_\mu))^{-1}$, which implies part b. Let's prove that (s_μ, s_μ) is a saddle point. The eigenvalues are given by

$$\lambda_\pm = 1/2(E \pm \sqrt{E^2 + 4D})$$

where $E = E_\mu = \partial_2 f(s_\mu, s_\mu)$ and $D = D_\mu = \partial_1 f(s_\mu, s_\mu)$

Now observe that:

$$D_\mu = \int_{s_\mu}^{0} -\partial_{12} f(x,x) - \partial_{11} f(x,x)\, dx = \int_{s_\mu}^{0} -\partial_{11} f(x,x)\left(1 + \frac{\partial_{12} f(x,x)}{\partial_{11} f(x,x)}\right) dx \le A(1+K^{-1})(-s_\mu)$$

where (P2) was used. Similarly, using (P1) and (P2) it can be obtained that

$$E_\mu = \int_{s_\mu}^{0} -\partial_{22} f(x,x)\left(1 + \frac{\partial_{12} f(x,x)}{\partial_{22} f(x,x)}\right) dx \ \ge\ B(1 - 1/K^2)(-s_\mu)$$

Therefore $E_\mu / D_\mu > 1$ which implies that $\lambda_- \in (-1, 0)$. In addition it follows from the facts above that $\lambda_+ \to +\infty$ when $\mu \to +\infty$. This proves part a) of the lemma. To prove part c) it is enough to observe that an eigenvector associated to λ_- is $(1, \lambda_-)$, while a tangent vector to $\xi_{s(\mu)}(\mu)$ at $(s(\mu), s(\mu))$ is $(1, -D/E)$ being easy to check that $\lambda_- > -D/E$.

The proof of theorems 2 and 3 is based on the study of the behavior of the stable manifold of $S_\mu = (s_\mu, s_\mu)$ (that is defined locally as for a diffeomorphism and then taking preimages). Denote by W^s_μ the stable manifold of S_μ. We will prove that W^s_μ has infinitely many connected components, each one diffeomorphic to a circle. We begin with the following simple fact:

Remark:
Let γ be a C^1 1-1 curve such that intersects P_μ transversally at two points.Then $F_\mu^{-1}(\gamma)$ is a C^1 Jordan curve. The proof of this fact is easy using that any point in P_μ has double preimage. The transversality is used to obtain that $F_\mu^{-1}(\gamma)$ is C^1 at the points of intersection with l_1.

This is the procedure that makes W^s_μ contain a closed curve: it is enough to prove that the local stable manifold of S_μ intersects P_μ in a pair of points to imply that W^s_μ contains a C^1 simple closed curve. It will be shown that this curve has, in fact, four points of intersection with P_μ; taking the preimage under F_μ of this curve it will be obtained another closed simple curve, which will also intersect P_μ at four points. Automatically, the following preimages under F_μ, give a sequence of closed curves each one having four points of intersection with P_μ. To prove these facts we will first show that W^s_μ is transversal to $\xi(\mu)$ before its intersection whith l_1 or l_2, this, as we will see, implies that these intersections actually occur. And secondly, a technique will be developed permitting us to study the set W^s_μ as it was a level curve of f_μ.
As f is convex, every level curve $\xi_\alpha(\mu)$ is a Jordan C^2 curve that enclose a convex region. In general, if ξ is a Jordan curve then $i(\xi)$ will denote the bounded component and $e(\xi)$ the unbounded component of $R^2 \setminus \xi$. As the maximum of each f_μ is taken at (0,0) we have that $\xi_\alpha(\mu) = \phi$ for $\alpha > \mu + f(0,0)$, and that $(0,0) \in i(\xi_\alpha(\mu))$ for $\alpha < \mu + f(0,0)$; in this case, $\xi_\alpha(\mu)$ intersects both l_1 and l_2, the intersections with l_1 correspond to the horizontal tangents of $\xi_\alpha(\mu)$ and those with l_2 to the vertical tangents of $\xi_\alpha(\mu)$. For any fixed μ, the level curves $\xi_\alpha(\mu)$ form a foliation of $R^2 \setminus (0,0)$, that we have denote by $\xi(\mu)$.Let γ be any C^1 curve that is transversal to the family $\xi(\mu)$; then we will say that γ is *entering* $\xi(\mu)$ at t if $(f \circ \gamma)'(t) > 0$ and that is *leaving* $\xi(\mu)$ at t if $(f \circ \gamma),(t) < 0$.
Let's denote by Q_1 the connected component of $R^2 \setminus l_1 \cup l_2$ wich contains S_μ. Let $\alpha = \alpha_\mu$ be a

curve parametrizing the connected component of $W^s_\mu \cap Q_1$ wich contains the point S_μ, and with the following properties, where we take μ large and drop the subindex:

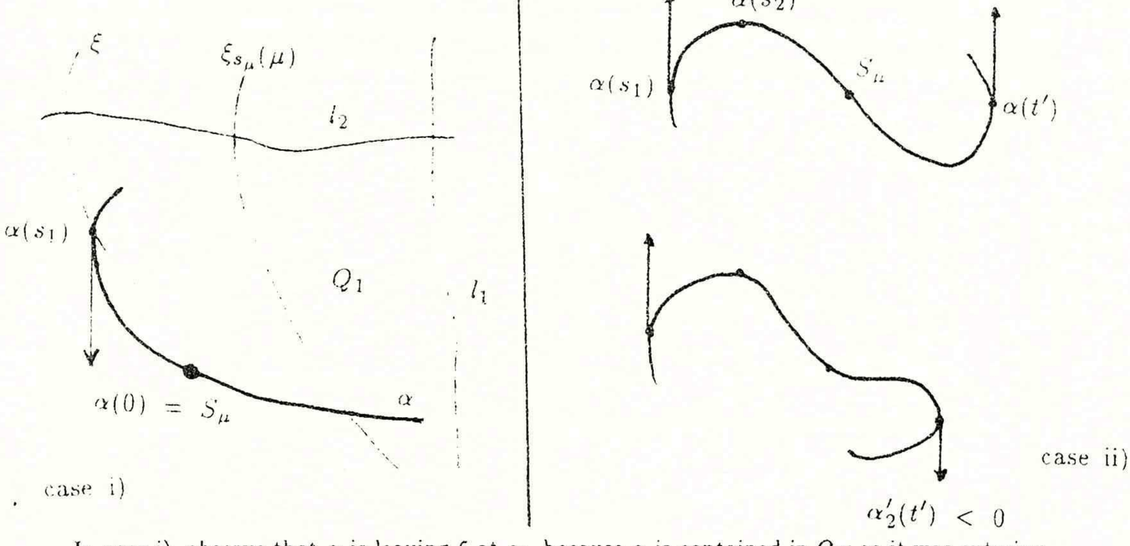
- $\alpha(0) = S_\mu$.
- $\alpha(t) = (\alpha_1(t), \alpha_2(t))$ with $\alpha_1(t) > 0$ *for t small.*

It follows from lemma 3.1 that α is entering $\xi(\mu)$ at $t = 0$.

Lemma 3.2 α_μ *is transversal to* $\xi(\mu)$

Proof: .Observe first that if at a point t, α is tangent to ξ, then $f \circ \gamma$ has a critical point at t, so that $F \circ \gamma$ has horizontal tangent at t, and this implies that $F^2 \circ \gamma$ has vertical tangent at t. Reasoning by contradiction, suppose that at a point $s < 0$, α is tangent to some curve of ξ; let $s_0 = max\{s < 0 : \alpha \text{ is tangent to } \xi \text{ at } s\}$. Then, at s_0, $F \circ \alpha$ has horizontal tangent and $F^2 \circ \gamma$ has vertical tangent. Now, as α is part of W^s, wich is invariant, it follows that there exists $s_1 \in (s_0, 0)$, such that α has a vertical tangent at s_1 (that is, $\alpha_1'(s_1) = 0$). Redefine, if necessary s_1 as maximum with this property. Obviously $s_0 < s_1 < 0$, and we have to distinguish between two cases:

i) $\alpha_2'(s_1) < 0$ and *ii)* $\alpha_2'(s_1) > 0$



In case i), observe that α is leaving ξ at s_1, because α is contained in Q_1; as it was entering ξ at zero there must occur a tangency between α and ξ in the interval $(s_1, 0)$, which is a contradiction with the definition of s_0.

In case ii), there must exist a point s_2, $s_1 < s_2 < 0$, such that $\alpha_2'(s_2) = 0$. Take s_2 maximum with this property. If $\alpha_1'(s_2) < 0$, we conclude that α is leaving ξ at s_2, so as in case 1 a contradiction appears. If $\alpha_1'(s_2) > 0$ define $t' > 0$ such that $F(\alpha(s_2)) = \alpha(t')$ (so $\alpha_1'(t') = 0$). Now $\alpha_2'(t') > 0$ implies that there exists $t'' \in (0, t')$ such that $\alpha_2'(t'') = 0$; thus, taking the image of $\alpha(t')$ we find a point of vertical tangency between α and ξ which corresponds to an $s \in (s_1, 0)$, in contradiction with the definition of s_1. Therefore $\alpha_2'(t') < 0$, so there exists

$t''' \in (0, t')$ such that ξ and α are tangent at t'''; it follows that α has horizontal tangent at a point in $(s_2, 0)$, which contradicts the definition of s_2.

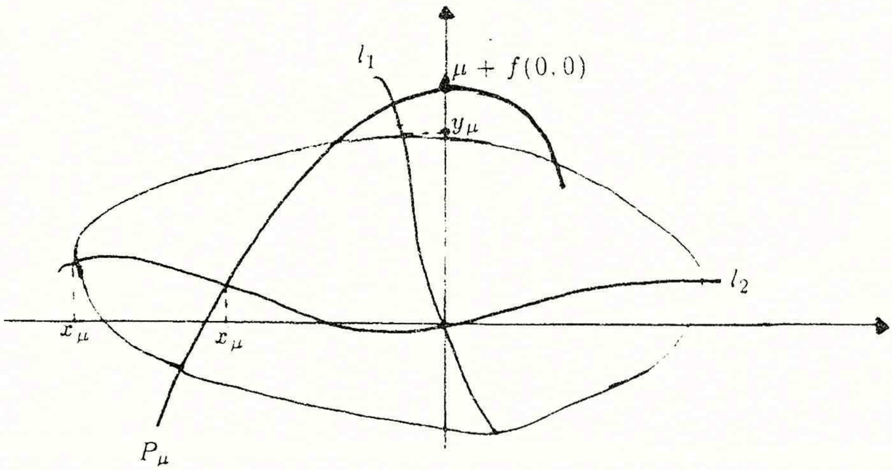
The following two lemmas, that will be used often later, imply that the level curve of f_μ passing throught the fixed point S_μ must intersect the set P_μ; this, together with the previous result will imply that also W^s_μ intersects P_μ; then, using the remark above lemma 1 forces W^s_μ to contain a C^1 Jordan curve.

Lemma 3.3 *Let τ be a C^1 function of μ such that $\tau'(\mu) \to 0$ as $\mu \to \infty$. then for all μ sufficiently large $\xi_{\tau(\mu)}(\mu)$ has four points of intersection with P_μ.*

Proof: Let's first calculate $y_\mu = max\{y : (x, y) \in \xi_{\tau(\mu)}(\mu)\}$. As it is easy to see, this maximum must be taken at a point of intersection of $\xi_{\tau(\mu)}(\mu)$ with l_1 so that y_μ satisfies: $f_\mu(\tilde{x}(y_\mu), y_\mu) = \tau(\mu)$. This implies that $y_\mu \to \infty$ as $\mu \to \infty$ because $f(\tilde{x}(y_\mu), y_\mu) = \tau(\mu) - \mu$ which tends to $-\infty$ as $\mu \to \infty$ by hypothesis. Therefore, as $\partial_1 f_\mu(\tilde{x}(y_\mu), y_\mu) = 0$, it follows that:

$$y'_\mu = \frac{\tau'(\mu) - 1}{\partial_2 f(\tilde{x}(y_\mu), y_\mu)}$$

From this it can be obtained that $y'_\mu \to 0$ as $\mu \to \infty$ because $\partial_2 f(\tilde{x}(y_\mu), y_\mu) \to +\infty$. In addition, the maximum second coordinate of points in P_μ is $\mu + f(0, 0)$, which results to be greater than y_μ for every μ large, because $y'_\mu \to 0$. This shows that P_μ crosses $\xi_{\tau(\mu)}(\mu)$ vertically.



Now let x_μ be the first coordinate of the left point of intersection of l_2 with $\xi_{\tau(\mu)}(\mu)$ and $\hat{x}_\mu$ the first coordinate of the left point of intersection of l_2 with P_μ. We claim that $| x_\mu | > | \hat{x}_\mu |$. Observe that x_μ satisfies the equation:

$$f_\mu(x_\mu, \tilde{y}(x_\mu)) = \tau_\mu$$

so that $x_\mu \to -\infty$ as $\mu \longrightarrow +\infty$,which can be proved as above.

Using (P3) it follows that:

$$f(x_\mu,\tilde{y}(x_\mu)) = \int_0^{x_\mu} \partial_1 f(t,\tilde{y}(t))dt + f(0,0)$$

$$\partial_1 f(t,\tilde{y}(t)) = \int_0^t \partial_{11} f(s,\tilde{y}(s)) - \frac{(\partial_{12} f(s,\tilde{y}(s)))^2}{\partial_{22} f(s,\tilde{y}(s))} ds \le -A'(1-1/K^3)t$$

similarly, but now using (P2), it follows that:

$$\partial_1 f(t,\tilde{y}(t)) \ge -A(1+1/K^3)t$$

and this implies that:

$$\frac{A'}{2}(1-1/K^3)x_\mu^2 \le \mu - \tau(\mu) \le \frac{A}{2}(1+1/K^3)x_\mu^2$$

and therefore:

$$liminf_{\mu\to\infty}\frac{|x_\mu|}{\sqrt{A_0^{-1}\mu}} \ge 1 \tag{6}$$

where $A_0 = \frac{A}{2}(1+1/K^3)$.

Now let's estimate the point $\hat{x}_\mu$. It is easy to see that $\tilde{z}_\mu(x) \le -B_0x^2+\mu$, where $B_0 = \frac{B}{2}(1-1/K^3)$ from which it follows that P_μ can be substituted by the parabola $y=-B_0x^2+\mu$. This, together with the fact that l_2 is contained in the cone $|y| \le x/K^2$, imply that:

$$|\hat{x}_\mu| \le \frac{1/K^2+\sqrt{1/K^2+4B_0\mu}}{2B_0}$$

from wich it follows that:

$$limsup_{\mu\to+\infty}\frac{|\hat{x}_\mu|}{\sqrt{B_0^{-1}\mu}} \le 1 \tag{7}$$

As $B_0 > A_0$, (6) and (7) imply the claim. Observe that this should be repeated for right intersections. So this shows that P_μ crosses $\xi_{\tau(\mu)}(\mu)$ also horizontally. This finishes the proof of the lemma.

Let τ be a C^1 function of μ such that $\tau'(\mu)\to 0$ as $\mu\to\infty$. Then the lemma just proved implies that for any point in $\tilde{P}\setminus i(\xi_{\tau(\mu)}(\mu))$ the partial derivative with respect to the second variable is not zero. We will need now to find a lower bound for this derivative and , more than this, we will show that a relation between the partial derivative with respect to the first and second variables exists. This will be used later to obtain stable foliations in $\tilde{P}_\mu\setminus i(\xi_{\tau(\mu)}(\mu))$.

Lemma 3.4 *There exists λ (for example, $\lambda=10$) such that, if $(x,y)\in e(\xi_{\tau(\mu)}(\mu))\cap\tilde{P}_\mu$ and μ is sufficiently large then:*

$$\left|\frac{\partial_2 f_\mu(x,y)}{\partial_1 f_\mu(x,y)}\right| \ge \lambda$$

Proof: . Firstly observe that:

$$| \partial_2 f(x,y) |=| \partial_2 f(x,\tilde{y}(x)) + \int_{\tilde{y}(x)}^{y} \partial_{22} f(x,s)ds | \geq B \mid y - \tilde{y}(x) \mid$$

And in the same manner:

$$| \partial_1 f(x,y) | \leq A \mid \tilde{x}(y) - y \mid$$

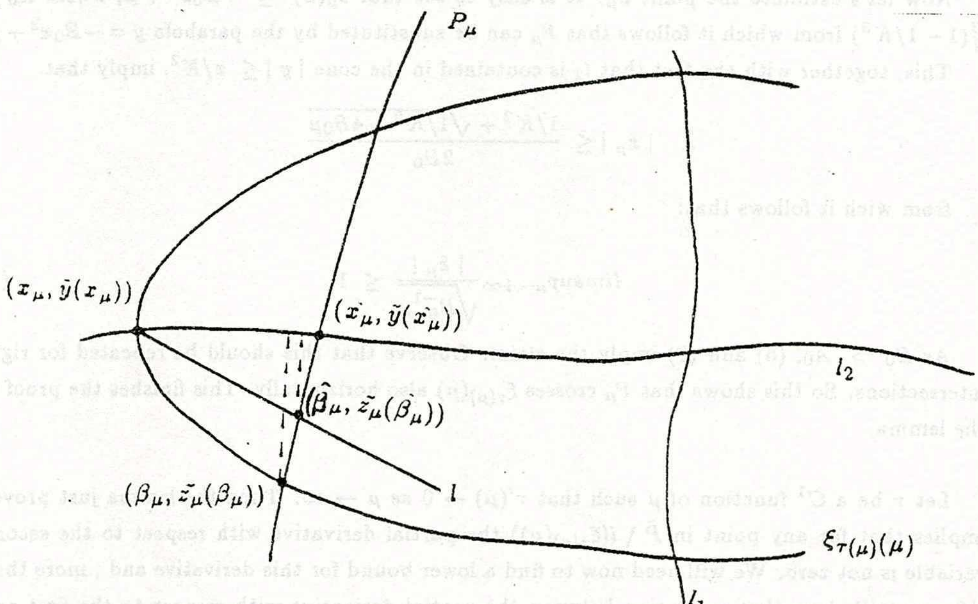
From this it is obtained that:

$$\left|\frac{\partial_2 f(x,y)}{\partial_1 f(x,y)}\right| \geq \frac{B}{A}\frac{| y - \tilde{y}(x) |}{| \tilde{x}(y) - x |} \tag{8}$$

Now suppose that a constant λ independent of μ was found such that:

$$\left|\frac{\tilde{y}(x) - y}{\tilde{x}(y) - x}\right| \geq \frac{A\lambda}{B} \tag{9}$$

for any point (x,y) of intersection of P_μ with $\xi_{\tau(\mu)}(\mu)$. It follows that the same estimate is valid for any other point in $P_\mu \cap \xi_{\tau(\mu)}(\mu)$ (this can easily be seen using that the tangent vector to P_μ is almost vertical at points not approaching l_1, see the figure). In fact, what we will show is that (9) is valid for $(x,y) = (\beta_\mu, \tilde{z}_\mu(\beta_\mu))$, the point of intersection of P_μ with $\xi_{\tau(\mu)}(\mu)$ located at Q_1. For the other points in $P_\mu \cap \xi_{\tau(\mu)}(\mu)$ the reasoning is similar.



Let's begin estimating the numerator of (9): The level curve $\xi_{\tau(\mu)}(\mu)$ is given by the equation $f_\mu(x,y) = \tau(\mu)$ wich defines a function $X(y)$ in a neighborhood of the point $(x_\mu, \tilde{y}(x_\mu))$ such that: $X(\tilde{y}(x_\mu)) = x_\mu$, $f_\mu(X(y), y) = \tau(\mu)$ and therefore:

$$X'(y) = -\frac{\partial_2 f(X(y), y)}{\partial_1 f(X(y), y)} \tag{10}$$

Derivating once more it can easily be obtained that $X''(y) < 0$; thus, we can assume that:

$$\left|\frac{\partial_2 f(X(y), y)}{\partial_1 f(X(y), y)}\right| \leq \lambda \tag{11}$$

because the contrary assumption trivially implies the lemma. As $X''(y) > 0$ equations (10) and (11) imply that $X'(y) \leq \lambda$, for every $| y - \tilde{y}(x_\mu) | \leq X^{-1}(\hat{x}_\mu)$, where for $X^{-1}(\hat{x}_\mu)$ we denote that preimage of $\hat{x}_\mu$ contained in Q_1. Now this implies that for $y \in (\tilde{y}(x_\mu), X^{-1}(x_\mu))$:

$$|X(y) - x_\mu| \leq \lambda |y - \tilde{y}(x_\mu)| \tag{12}$$

Let l be the line $x - x_\mu = -\lambda(y - \tilde{y}(x_\mu))$. It follows that the vertical distance from $(\hat{x}_\mu, \tilde{y}(\hat{x}_\mu))$ to l is:

$$\tilde{y}(\hat{x}_\mu) - y = \frac{\hat{x}_\mu - x_\mu}{\lambda} \tag{13}$$

Now, if $(\hat{\beta}_\mu, \tilde{z}_\mu(\hat{\beta}_\mu))$ is the point of intersection of P_μ with l, then it follows from (12) that:

$$\tilde{y}(\beta_\mu) - \tilde{z}_\mu(\beta_\mu) \geq \tilde{y}(\hat{\beta}_\mu) - \tilde{z}_\mu(\hat{\beta}_\mu) \tag{14}$$

But $\hat{\beta}_\mu$ can be estimated easily, because P_μ can be substituted by the line $y - \tilde{y}(\hat{x}_\mu) = -2B_0\hat{x}_\mu(x - \hat{x}_\mu)$ (this follows from the fact that $| \tilde{z}'_\mu(x) | > -2B_0\hat{x}_\mu$ for $x < \hat{x}_\mu$), and this gives, just intersecting this line with l:

$$\hat{\beta}_\mu - \hat{x}_\mu \leq \frac{y - \tilde{y}(\hat{x}_\mu)}{-2B_0\hat{x}_\mu} = \frac{\tilde{y}(x_\mu) - \tilde{y}(\hat{x}_\mu) - 1/\lambda(\hat{\beta}_\mu - x_\mu)}{-2B_0\hat{x}_\mu}$$

and following:

$$| \hat{\beta}_\mu - \hat{x}_\mu | = \left|\frac{\tilde{y}(x_\mu) - \tilde{y}(\hat{x}_\mu) + 1/\lambda(x_\mu - \hat{x}_\mu)}{2B_0\hat{x}_\mu(1 + 1/\lambda)}\right| \leq \frac{(1/\lambda + 1/K^2) \, | x_\mu - \hat{x}_\mu |}{| 2B_0\hat{x}_\mu(1 + 1/\lambda) |} \tag{15}$$

Finally, using (13) and (15) it can be obtained that:

$$\tilde{y}(\hat{\beta}_\mu) - \tilde{z}_\mu(\hat{\beta}_\mu) \geq 1/\lambda(\hat{x}_\mu - x_\mu) - (1/K^2 + 1/\lambda)(\hat{x}_\mu - \hat{\beta}_\mu) \geq \left(1/\lambda - \frac{(1/K^2 + 1/\lambda)^2}{| 2B_0\hat{x}_\mu(1 + 1/\lambda) |}\right)(\hat{x}_\mu - x_\mu)$$

Therefore we can take μ large in such a way that:

$$\tilde{y}(\hat{\beta}_\mu) - \tilde{z}_\mu(\hat{\beta}_\mu) \geq \frac{\hat{x}_\mu - x_\mu}{2\lambda}$$

This provides, using also (14), an estimative for $\tilde{y}(\beta_\mu) - \tilde{z}(\beta_\mu)$.

Now join this with (8) and the fact that the horizontal distance from $(\beta_\mu, \tilde{z}_\mu(\beta_\mu)$ to l_1 is less than $|x_\mu|$ to obtain that:

$$\left|\frac{\partial_2 f(\beta_\mu, \tilde{z}_\mu(\beta_\mu))}{\partial_1 f(\beta_\mu, \tilde{z}_\mu(\beta_\mu))}\right| \geq \frac{B}{2A\lambda}\frac{\hat{x}_\mu - x_\mu}{-x_\mu} = \frac{B}{2A\lambda}(1 - \frac{\hat{x}_\mu}{x_\mu})$$

Thus, using the estimatives for x_μ and $\hat{x}_\mu$ obtained in the previous lemma it follows that, for μ sufficiently large,

$$\left|\frac{\partial_2 f(\beta_\mu, \tilde{z}_\mu(\beta_\mu))}{\partial_1 f(\beta_\mu, \tilde{z}_\mu(\beta_\mu))}\right| \geq \frac{B}{2A\lambda}(1-\sqrt{B_0/A_0}) \geq \frac{B}{4A\lambda} > K/4\lambda > \sqrt{K}/4$$

To do the last step works, we make $\lambda < \sqrt{K}$, so for any λ satisfying this, the lemma is proved (recall (9)). In particular, we can take $\lambda = 10$ if K is large enough.

This provides the necessary techniques to obtain stable foliations.

Lemma 3.5 *Let τ be a C^1 function of μ such that $\tau'(\mu) \to 0$ at infinity. Let $R_\mu = \tilde{P}_\mu \cap$: $e(\xi_{\tau(\mu)}(\mu))$ and define $G_\mu = \bigcap_{n\geq 0} F_\mu^{-n}(R_\mu)$. Then, if μ is sufficiently large, there exists a C^1 stable foliation of G_μ invariant under F_μ.*

Proof: . Fix any μ large enough and drop the index μ. Observe first that $F(G) \subset G$. Define, for each $x \in G$ a cone $C_x = \{(u,v) \; : \mid v/u \mid < \epsilon\}$ where ϵ is a positive number to be chosen. Now, for $(u,v) \in C_{F(x)}$ we have:

$$DF^{-1}_{F(x)}(u,v) = \frac{-1}{\partial_1 f}(u\partial_2 f - v, -u\partial_1 f) = (u_1, v_1) \tag{16}$$

where the derivatives are calculated at $F(x)$. Furthermore:

$$\left|\frac{v_1}{u_1}\right| = \left|\frac{u\partial_1 f}{u\partial_2 f - v}\right| = \left|\frac{\partial_1 f}{\partial_2 f - v/u}\right| \leq \left|\frac{\partial_1 f}{\partial_2 f/2}\right|$$

if $\epsilon < \mid \partial_2 f \mid /2$. But $F(x) \in G \subset e(\xi_{\tau(\mu)}(\mu))$ so that the previous lemma can be applied to obtain:

$$\left|\frac{v_1}{u_1}\right| \leq 2/\lambda < \epsilon$$

if $\epsilon = 3/\lambda$. This ϵ satisfies also $\epsilon < \mid \partial_2 f \mid /2$ if μ is sufficiently large, because $\lambda(=10)$ is independent of μ, while $\mid \partial_2 f \mid \to \infty$ for points in $e(\xi_{\tau(\mu)}(\mu))$. This proves that $(u_1, v_1) \in C_x$ if $(u,v) \in C_F(x)$. In addition, using (16):

$$\begin{aligned}
\mid (u_1, v_1) \mid &= \mid u_1 \mid + \mid v_1 \mid = \frac{\mid u\partial_2 f - v \mid + \mid u\partial_1 f \mid}{\mid \partial_1 f \mid} \geq \\
&\geq \frac{\mid u \mid (\mid \partial_2 f \mid - \mid u/v \mid + \mid \partial_1 f \mid)}{\mid \partial_1 f \mid} \geq \frac{\mid u \mid}{2}\left|\frac{\partial_2 f}{\partial_1 f}\right| \geq \\
&\geq \frac{\lambda}{2}\mid u \mid \geq \frac{\lambda}{2}\frac{\mid u \mid + \mid v \mid}{1+\epsilon} = \frac{\lambda}{2(1+\epsilon)}\mid (u,v) \mid > 2 \mid (u,v) \mid
\end{aligned}$$

This proves that DF^{-1} leaves the family of cones invariant and expands length. As it is known this implies the existence of the foliation (see [HPS]), thus proving the lemma.

PROOF OF THEOREM 1.2

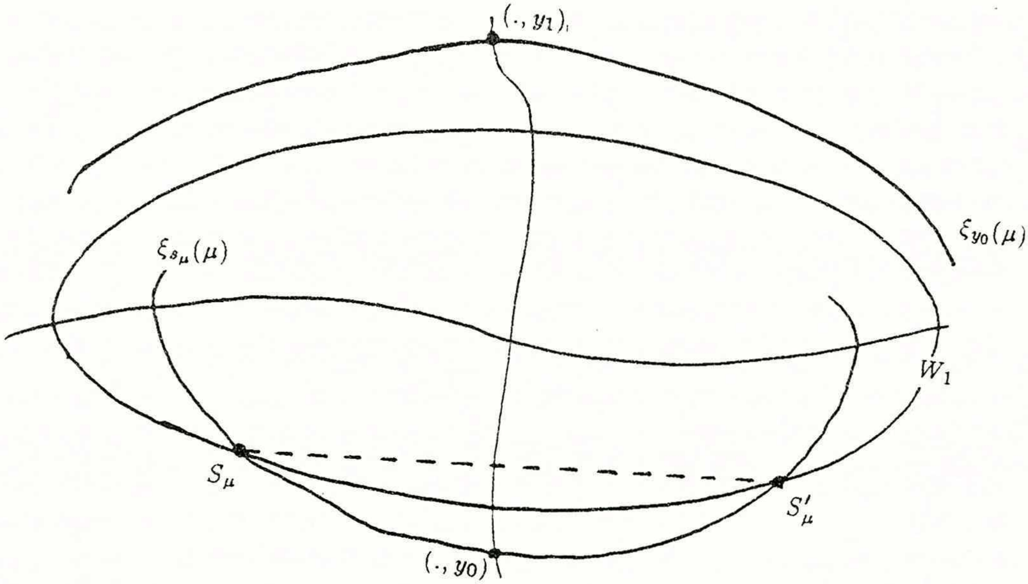
STEP 1: W^s_μ has infinitely many connected components.

It is known, by lemma 3.2, that the connected component of $W^s_\mu \cap Q_1$ containing S_μ (parametrized by the curve α), is transversal to the family of level curves ξ. This means that $\alpha(t) \in e(\xi_{s_\mu}(\mu))$ for $t < 0$, because $f(\alpha(0)) = s_\mu$. In addition, by lemma 3.1, it follows that $lim_{\mu\to\infty} s'_\mu = 0$, and thus lemma 3.3 (with s_μ in place of r), can be applied to obtain that $\xi_{s_\mu}(\mu)$ intersects P_μ in Q_1. Joining these facts it follows that α also intersects P_μ unless it doesn't reach l_2 nor P_μ. But in this latter case we will find a contradiction: firstly, this implies that there is a two periodic orbit $\{p_1, p_2\}$ such that p_1 and p_2 are the extreme points of α. Now it follows that the direction given by the tangent to α at p_1, is non contracting. Also observe that:

$$\left|\frac{\alpha_2'(t_1)}{\alpha_1'(t_1)}\right| < \left|\frac{\partial_1 f}{\partial_2 f}\right| < \lambda^{-1}$$

where t_1 is such that $\alpha(t_1) = p_1$ and the last inequality follows from lemma 3.4. Now the equation above implies that the tangent direction to α at p_1 is contained in the stable cones as defined in the previous lemma: thus we have found a contradiction because this direction must be an invariant non contracting direction.

Until now we have thus proved that α (and so also W^s_μ) intersect P_μ at one point. Let's denote by α_1 the curve $F^{-1}(\alpha) \setminus \alpha$ and let's show that it also intersects P_μ : in fact, let S'_μ be the preimage of S_μ which is not S_μ. The image of that part of α_1 that lies between l_1 and S'_μ, is located above S_μ, and this implies that α_1 is outside $\xi_{s_\mu}(\mu)$ between l_1 and S'_μ. At S'_μ, α_1 intersects $\xi_{s_\mu}(\mu)$, and after this, α_1 is contained in $e(\xi_{s_\mu}(\mu))$, so that lemmas 3.3 and 3.4 can be used as before to obtain that α_1 also intersects P_μ. Therefore, we have proved that W^s_μ contains a C^1 curve intersecting P_μ transversally at a pair of points, which implies that W^s_μ contains a closed simple C^1 curve that contains the point S_μ, and that will be denoted by W_1.



Let y_0 be the second coordinate of the intersection of $\xi_{s_\mu}(\mu)$ with l_1. It is clear by lemma 3.2 that W_1 is contained in $\{(x,y) : y > y_0\}$. As the image of W_1 is contained in W_1, it follows that $W_1 \subset i(\xi_{y_0}(\mu))$. Now let's calculate the dependence of y_0 on μ: y_0 must satisfy the equation $f_\mu(\tilde{x}(y_0), y_0) = s_\mu$, hence it follows that:

$$y_0'(\mu) = -\frac{1}{\partial_2 f_\mu(\tilde{x}(y_0), y_0)},$$

This implies, as in the proof of lemma 4.2, that $y_0'(\mu) \to 0$ as $\mu \to \infty$. Therefore lemma 4.2 can be applied to y_0 in place of τ to obtain that $\xi_{y_0}(\mu)$ intersects P_μ at four points and so W_1 also intersects P_μ in four points. This means that the preimage $F^{-1}(W_1)$ contains another closed simple C^1 curve that will be denoted by W_2. Now we will prove that also W_2 intersects P_μ at four points. To do this apply the same idea as before: first observe that $W_1 \subset \{(x,y) : y < y_1\}$, where y_1 is the maximum of the second coordinates of points in $\xi_{y_0}(\mu)$, then it follows that W_2 has to be contained in $e(\xi_{y_1}(\mu))$, so it suffices to show that $y_1' \to 0$ and use lemma 3.3. In fact y_1 satisfies the equation $f_\mu(\tilde{x}(y_1), y_1) = y_0$ so that $1 + \partial_2 f(\tilde{x}(y_1), y_1) y_1' = y_0'$, which implies that $y_1'(\mu) \to 0$ as $\mu \to \infty$, thus lemma 3.3 says that $\xi_{y_1}(\mu)$ (and so also W_2) intersects P_μ at four points. Thus the preimage of W_2 has also two simple closed C^1 curves as preimages, which, by simple inspection of the location of preimages must be both contained in $e(W_2)$ and $i(W_1)$. Furthermore each one of these new curves must intersect P_μ at four points, and so each one has a pair of curves as preimage, and so on. This implies that W_μ^s has infinitely many components, each one of which is a closed C^1 curve.

STEP 2: The complementary set of the closure of W_μ^s is the basin of ∞, that is, the set of points with empty ω limit set.

If we prove that $e(W_1)$ is contained in the basin of ∞ then it will follow that $i(W_2) = F^{-1}(e(W_1))$ is also contained in the basin of ∞. Now the preimage of this open disc is an annulus whose boundary is the preimage of W_2.It follows that W_μ^s accumulates on the complementary set of the basin of ∞, as this is an open set, the step 2 it is proved; so what we must show, is that $e(W_1)$ is contained in the basin of ∞. Every point in $e(W_1)$ must also lie in $e(\xi_{y_0}(\mu))$ so that lemma 3.5 can be applied to obtain a stable foliation each of which leaves intersect P_μ. This induces a one dimensional map from P_μ into itself, that has a fixed point corresponding to S_μ, and either carries every point to ∞ or has another fixed point. But the latter case is impossible because it would imply the existence of another fixed point of F_μ with negative coordinates.

To finish the proof of theorem 1.2 it remains to show that the closure of W_μ^s is a Cantor set of closed curves. To do this we will need an unstable foliation defined outside the curve W_2.

Lemma 3.6 *Let μ be sufficiently large and define $H = \bigcap_{n\geq 0} F_\mu^n(\tilde{P}_\mu) \setminus \bigcup_{n\geq 0} F^{-n}(i(W2))$. Then there exists an unstable, almost vertical, C^1 foliation defined on H and invariant under F.*

Proof: : First observe that if $x \in H$ then a preimage of x is contained in H. For each point in H define a cone $C = \{(u,v) : u/v < \epsilon\}$, where ϵ is a small number to be defined. Take $(u,v) \in C$ and $x \in H$; then, calculating $DF_x(u,v) = (u_1, v_1)$, it can be obtained that:

$$
\begin{aligned}
|u_1/v_1| &= \left|\frac{v}{u\partial_1 f + v\partial_2 f}\right| \le \frac{1}{|\partial_2 f| - |\partial_1 f||u/v|} \le \frac{1}{|\partial_2 f| - \epsilon\lambda^{-1}|\partial_2 f|} \le \\
&\le \frac{1}{|\partial_2 f|/2} \le \epsilon \qquad (17)
\end{aligned}
$$

where it was used the lemma 3.4, and $\epsilon = 3/B$ This proves that $(u_1, v_1) \in C_{F(x)}$ for $(u, v) \in C_x$. Furthermore:

$$
\begin{aligned}
|(u_1, v_1)| &= |u_1| + |v_1| = |v| + |u\partial_1 f + v\partial_2 f| \ge |v|(1 + |\partial_2 f| - |\partial_1 f||u/v|) \\
&\ge \frac{|v||\partial_2 f|}{2} > \frac{|\partial_2 f|}{2(1+\epsilon)}|(u, v)| \qquad (18)
\end{aligned}
$$

It follows that DF expands lenght of vectors in the cones and the lemma follows by the results of [HPS].

Define $I_1 = \overline{i(W_1)} \cap P_\mu$ and $I_2 = F(I_1) \cap \overline{i(W_1)}$, ($\bar{A}$ denotes the closure of A). I_1 is the union of two curves and I_2 is the union of at most four curves. What we must show is that $\overline{W^s_\mu} \cap I_1$ is a Cantor set.

Observe that the stable foliation obtained in lemma 3.5 can be extended to $\tilde{P_\mu} \setminus \bigcup_{n \ge 0} F_\mu^{-n}(i(W_2)) = \tilde{P_\mu} \cap \overline{W^s_\mu}$ because $i(W_2) \supset i(\xi_{y_1}(\mu))$ and $y_1'(\mu) \to 0$ as $\mu \to \infty$ which was shown in step one. This defines a map π which carries points in $\overline{W^s_\mu} \cap I_2$ to I_1 along the leaves of the stable foliation. Now the proof will be completed by observing the three following facts:

1. The map F restricted to $I_1 \cap F^{-1}(I_2)$ is an expansive map because I_1 and I_2 are almost vertical lines and lemma 3.6 can be applied. This implies that this restriction of F satisfies bounded distortion properties and so it preserves cross ratios of intervals (this is a well known fact, for the definitions see [M]).

2. The map π has been defined as induced by a stable foliation of a C^2 map, F_μ. This implies that π also has to satisfy bounded distortion properties (this is an observation of Newhouse that can be found in [PT]). Now, as above, the map π also preserves cross ratios.

3. Maps which preserves cross ratios of intervals define Cantor sets (this is a simple fact).

The proof of the theorem 1.2 is complete.

PROOF OF THEOREM 1.3

Fix any large value of μ. Suppose first that there exists some integer $n > 0$ such that F restricted to $F^n(R^2)$ is one to one. Then obviously F/Λ is a homeomorphism, (recall that $\Lambda = \bigcap_{n \ge 0} F^n(\overline{W^s_\mu})$). To prove that F/Λ is a shift we proceed as for a horseshoe: first give an itinerary $j(x) \in 2^Z$ to each x in Λ and then prove that j conjugates F/λ with the shift. To obtain the hyperbolicity just use the foliations shown to exist in lemmas 3.5 and 3.6.

If there is no $n > 0$ such that $F/F^n(R^2)$ is one to one then it follows that the unstable manifolds of the fixed points must coincide because there is a contraction in the horizontal direction. Now Λ is contained in the unstable manifold of S_μ (and of the other fixed point). Finally, the hyperbolicity follows from lemma 3.6 and the fact that these unstable manifolds have to be contained in the unstable foliation.

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UNITARY EXTENSIONS OF ISOMETRIES AND INTERPOLATION PROBLEMS: DILATION AND LIFTING THEOREMS

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Our aim is to give a rapid introduction to the use of operator theoretic methods in interpolation and extension problems. Here, the fundamental dilation and lifting theorems are established. Only basic notions concerning Hilbert spaces, measure theory and complex functions are assumed. The emphasis is on the unifying and geometric features of the method of unitary extensions of isometries. Nevertheless, some proofs and even some statements are perhaps new.[1]

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I. BASIC CONSTRUCTIONS

Naimark´s dilation theorem

Unitary operators in Hilbert spaces are very nice objects. Suitable constructions of unitary operators give fundamental results concerning interpolation problems. One of those constructions is the unitary dilation of a function of positive type. Roughly speaking, such functions are the Fourier transforms of positive measures. We start giving a basic example and the definition of that notion.

Let H be a Hilbert space, $\mathcal{L}(H)$ the set of bounded operators in H, $U \in \mathcal{L}(H)$ a unitary operator and E a closed subspace of H. If P_E denotes the orthogonal projection of H onto E, let the function $k: \mathbb{Z} \to \mathcal{L}(E)$ be given

[1]This paper is in final form and no version of it will be submitted for publication elsewhere.

by $k(n) = P_E U^n{}_{|E}$. The support of any function h is the set $\operatorname{supp} h = \{h \neq 0\}$. Then:

(1) $$\sum\{\langle k(m-n)h(m),h(n)\rangle : m,n \in \mathbb{Z}\} \geq 0$$

holds for every function of finite support $h: \mathbb{Z} \to E$.

Whenever (1) is verified, k is said a function of positive type.

The content of the following Naimark's dilation theorem is that the converse of the previous example is always essentially true.

For any Hilbert spaces E and F, $\mathbf{L}(E,F)$ is the set of bounded operators from E to F; if E is a closed subspace of F, $i_E = (P_E)^*$ is the inclusion of E in F; I is the identity in the space under consideration; if $\{G_t\}$ is a family of subsets of E, $\vee\{G_t\}$ denotes the smallest closed subspace of E that contains every set G_t. Then:

(2) <u>THEOREM</u> Let E be a Hilbert space and $k: \mathbb{Z} \to \mathbf{L}(E)$ a function of positive type. There exists a Hilbert space H, a unitary operator $U \in \mathbf{L}(H)$ and an operator $\rho \in \mathbf{L}(E,H)$ such that $k(n) = \rho^* U^n \rho$, $\forall\, n \in \mathbb{Z}$. It may be assumed that $H = \vee\{U^n \rho E: n \in \mathbb{Z}\}$, and then H, U and ρ are unique up to unitary isomorphisms. If $k(0) = I$, E may be considered as a closed subspace of H and $\rho = i_E$.

<u>Sketch of the proof</u>

Set $H' = \{h: \mathbb{Z} \to E, \operatorname{supp} h \text{ is finite}\}$ and $\langle h,h'\rangle = \sum\{\langle k(m-n)h(m),h'(n)\rangle : m,n \in \mathbb{Z}\}$ for any $h,h' \in H'$. Then $H'_0 := \{h \in H: \langle h,h'\rangle = 0\}$ is a subspace of the vector space H'; let π be the projection of H' onto the quotient H'/H'_0; setting $\langle \pi h,\pi h'\rangle = \langle h,h'\rangle$, we obtain a scalar product in H'/H'_0, so the corresponding completion H is a Hilbert space and π can be considered as a map from H' onto a dense subspace of H.

Let S be the shift in H', i.e., the operator given by $Sh(n) \equiv h(n-1)$; an isometry V with domain and range $\pi H'$ is defined by $V\pi = \pi S$, so V can be uniquely extended to a unitary operator $U \in \mathbf{L}(H)$. If any $v \in E$ is identified with $h \in H'$ such that $\operatorname{supp} h = \{0\}$ and $h(0) = v$, and ρ denotes the restriction of π to E, then $\|\rho v\|^2 = \langle k(0)v,v\rangle$, so $\rho \in \mathbf{L}(E,H)$. From $\langle k(n)v,w\rangle \equiv \langle \pi S^n v,\pi w\rangle$ and $\pi S^n{}_{|E} \equiv U^n \rho$, it follows that $k(n) \equiv$

$\rho^* U^n \rho$. Since any $h \in H'$ is the sum of vectors like $S^n v$, $H = \bigvee\{U^n \rho E: n \in \mathbb{Z}\}$.

If H_1, U_1 and ρ_1 are as H, U and ρ, by setting $A(U^n \rho v) \equiv U_1{}^n \rho_1 v$ a unitary operator $A \in \mathcal{L}(H,H_1)$ is defined in such a way that $AU = U_1 A$ and $A\rho = \rho_1$; that is the unicity statement.

(3) Exercise Naimark´s dilation theorem holds when $\mathbb{Z}$ is replaced by any group Γ. State and prove it.

Unitary extensions of an isometry

The above proof produces the fundamental operator U as the unitary extension of an isometry V. In order to extend the scope of Naimark´s method, we shall see that any isometry V acting in a Hilbert space H (i.e., such that its domain D and its range R are subspaces of H) can be extended to a unitary operator U in a Hilbert space F containing H (as a closed subspace).

We may assume that D and R are closed subspaces; let N and M be their orthogonal complements in H, respectively (the so-called defect subspaces of V): $N = H \ominus D$, $M = H \ominus R$. Set $F = H \oplus H \equiv D \oplus N \oplus R \oplus M$ and define U by $U(d,n,r,m) \equiv (V^{-1}r,n,Vd,m)$; the assertion follows.

In several problems the following notations will be useful: $(U,F) \in \mathcal{U}$ if U is a unitary extension of V to a Hilbert space $F \supset H$ such that $F = \bigvee\{U^n H: n \in \mathbb{Z}\}$; (U,F) and (U',F') define the same element in $\mathcal{U}$ if there exists a unitary operator $A \in \mathcal{L}(F,F')$ such that $AU = U'A$ and the restriction of A to H equals the identity in H, in which case we write $(U,F) \approx (U',F')$. Each $(U,F) \in \mathcal{U}$ will be called a minimal unitary extension of V.

(4) Exercise Prove that $\#(\mathcal{U}) = 1$ if $N = \{0\}$ or $M = \{0\}$ and that $\#(\mathcal{U}) = \infty$ in any other case.

Extending a function of positive type

Let a be a positive integer and E a Hilbert space. Call H'_a the space of functions $h: \mathbb{Z} \to E$ such that $\operatorname{supp} h \subset \{n \in \mathbb{Z}: 0 \le n \le a\}$. Then

$$k: \{n \in \mathbb{Z}: |n| \le a\} \to \mathcal{L}(E)$$

is said of positive type if $\sum\{\langle k(m-n)h(m),h(n)\rangle: m,n \in \mathbb{Z}\} \ge 0$ holds for every $h \in H'_a$.

Set $D' = \{h \in H'_a: Sh \in H'_a\}$ and $R' = SD'$. Applying Naimark's dilation method we obtain a Hilbert space H_a, an operator π from H'_a onto a dense subspace of H_a, an isometry V with domain and range equal to (the closure of) $\pi D'$ and $\pi R'$, respectively, and an operator $\rho \in \mathcal{L}(E,H_a)$ such that $k(n) \equiv \rho^* V^n \rho$ if $0 \leq n \leq a$. For any $(U,F) \in \mathcal{U}$ set $K(n) = \rho^* P_E U^n i_E \rho$, $\forall\, n \in \mathbb{Z}$. Thus:

(5) PROPOSITION Any function of positive type $k: \{n \in \mathbb{Z}: |n| \leq a\} \to \mathcal{L}(E)$ can be extended to a function of positive type $K: \mathbb{Z} \to \mathcal{L}(E)$.

Call $\mathcal{K}$ the set of all positive extensions K of k to $\mathbb{Z}$. Then:

(6) Exercise There exists a bijection between $\mathcal{K}$ and $\mathcal{U}$.

Complements: Some remarks on Fourier transforms

If $\mathbb{C}$ is the field of complex numbers and $\mathbb{T} = \{z \in \mathbb{C}: |z| = 1\}$, $C(\mathbb{T})$ is the Banach space of complex continuous functions on $\mathbb{T}$ with the $\|.\|_\infty$ norm and $M(\mathbb{T})$ is its dual, i.e., the space of complex Borel measures on $\mathbb{T}$. Set $e_n(z) = z^n$ for every $z \in \mathbb{T}$ and $n \in \mathbb{Z}$. If $v \in M(\mathbb{T})$ its Fourier transform $\hat{v}: \mathbb{Z} \to \mathbb{C}$ is given by $\hat{v}(n) \equiv \int_{\mathbb{T}} e_{-n} dv$.

(7) Exercise $\hat{v}: \mathbb{Z} \to \mathbb{C}$ is of positive type iff $v \geq 0$.

(8) Exercise Naimark's dilation theorem and the spectral theorem for a unitary operator give a proof of the Herglotz-Bochner theorem: a function $k: \mathbb{Z} \to \mathbb{C}$ is of positive type iff there exists a positive measure $v \in M(\mathbb{T})$ such that $k = \hat{v}$.

• A fundamental reference for dilation theory is [NF]. References concerning unitary extensions of isometries and functions of positive type can be found in [AF].

II. AN EXTENSION OF SARASON'S INTERPOLATION THEOREM

In the previous chapter some results were established by using the data of a given problem, stated in a Hilbert space, to construct a new Hilbert space and an isometry acting in it, in such a way that the unitary extensions of that isometry give the solutions of the problem. This method can be applied to several interpolation problems. An economic way of doing it is to prove by that method the following abstract interpolation theorem, by means of which such problems can be solved.

(1) THEOREM Let $U_1 \in \mathbf{L}(G_1)$ and $U_2 \in \mathbf{L}(G_2)$ be unitary operators in Hilbert spaces, $B_1 \subset G_1$ and $B_2 \subset G_2$ closed subspaces such that $U_1 B_1 \subset B_1$ and $U_2^{-1} B_2 \subset B_2$, $\vee\{U_1{}^n B_1 : n \leq 0\} = G_1$, $\vee\{U_2{}^n B_2 : n \geq 0\} = G_2$. If $A \in \mathbf{L}(B_1, B_2)$ is such that $AU_{1|B_1} = P_{B_2} U_2 A$ then there exists $\tilde{A} \in \mathbf{L}(G_1, G_2)$ such that: $\tilde{A}U_1 = U_2\tilde{A}$, $A = P_{B_2}\tilde{A}_{|B_1}$ and $\|A\| = \|\tilde{A}\|$.

A proof can be based in the following fact.

(2) Representation of contractions Let $A \in \mathbf{L}(B_1, B_2)$ be a contraction between two Hilbert spaces (i.e., $\|A\| \leq 1$). There exists a Hilbert space H and two isometries $u_j \in \mathbf{L}(B_j, H)$, $j = 1,2$, such that $H = (u_1 B_1) \vee (u_2 B_2)$ and $A = u_2{}^* u_1$.

In the vector space $E := B_1 \times B_2$ we set $<(b_1,b_2),(b'_1,b'_2)> \equiv <b_1,b'_1>_{B_1} + <b_2,Ab'_1>_{B_2} + <Ab_1,b'>_{B_2} + <b_2,b'_2>_{B_2}$, thus obtaining "nearly" a scalar product, because $<(b_1,b_2),(b_1,b_2)> = 0$ does not imply $(b_1,b_2) = 0$. As before, a quotient and a completion give a Hilbert space H and a natural operator π from E onto a dense subspace of H. Setting $u_1 b_1 \equiv \pi(b_1,0)$ and $u_2 b_2 \equiv \pi(0,b_2)$, the result follows.

Sketch of the proof of theorem (1)

a) If $A = 0$, the result is obvious; thus, we may assume that $\|A\| = 1$.

b) With notation as in (2), set $D = (u_1 B_1) \vee (u_2 U_2{}^{-1} B_2)$ and $R = (u_1 U_1 B_1) \vee (u_2 B_2)$; since $AU_{1|B_1} = P_{B_2} U_2 A$, an isometry V with domain D and range R is defined by $V(u_1 b_1 + u_2 U_2{}^{-1} b_2) \equiv u_1 U_1 b_1 + u_2 b_2$.

c) Let $(U,F) \in \mathcal{U}$. An isometric extension $\tilde{u}_1 \in \mathbf{L}(G_1, F)$ of u_1 is defined by $\tilde{u}_1(U_1{}^n b) = U^n u_1 b$, for every $n \in \mathbb{Z}$ and $b \in B_1$. In fact, let

$n_1,\dots,n_k \in \mathbb{Z}$ and $b_1,\dots,b_k \in B_1$; if $s \in \mathbb{Z}$ is such that $s+n_1,\dots,s+n_k \geq 0$ then $\|U_1^{n_1}b_1 + \dots + U_1^{n_k}b_k\|_{G_1} = \|U_1^{s+n_1}b_1 + \dots + U_1^{s+n_k}b_k\|_{B_1} =$

$\|V^{s+n_1}u_1b_1 + \dots + V^{s+n_k}u_1b_k\|_H = \|U^{n_1}u_1b_1 + \dots + U^{n_k}u_1b_k\|_F$.

Analogously, an isometric extension $\tilde{u}_2 \in \mathcal{L}(G_2,F)$ of u_2 is defined by $\tilde{u}_2(U_2{}^n b) = U^n u_2 b$, for every $n \in \mathbb{Z}$ and $b \in B_2$. Clearly, $\tilde{u}_j U_j = U u_j$, $j = 1,2$.

d) Set $\tilde{A} = \tilde{u}_2{}^* \tilde{u}_1 \in \mathcal{L}(G_1,G_2)$. Then $\|\tilde{A}\| \leq 1$ and $\tilde{A}U_1 = U_2\tilde{A}$. For any $b_1 \in B_1$ and $b_2 \in B_2$ $\langle (P_{B_2}\tilde{A}_{|B_1})b_1,b_2\rangle_{B_2} = \langle \tilde{u}_1 b_1, \tilde{u}_2 b_2\rangle_F =$ $\langle Ab_1,b_2\rangle_{B_2}$, so $P_{B_2}\tilde{A}_{|B_1} = A$ and $\|\tilde{A}\| = \|A\|$.

(3) <u>Exercise</u> With notation as in theorem (1) and its proof set:

$$\mathcal{A} = \{\tilde{A} \in \mathcal{L}(G_1,G_2): \tilde{A}U_1 = U_2\tilde{A},\ A = P_{B_2}\tilde{A}_{|B_1},\ \|A\| = \|\tilde{A}\|\}.$$

Prove that there exists a bijection between $\mathcal{A}$ and $\mathcal{U}$.

In order to obtain a concrete version of the above theorem, set $L^p \equiv L^p(\mathbb{T},m)$, with $1 \leq p \leq \infty$ and m the normalized Lebesgue measure in $\mathbb{T}$, and let H^p be the closed linear span in L^p of $\{e_n: n \geq 0\}$. The Fourier transform $\hat{f}$ of $f \in L^1$ is the Fourier transform of the measure $(f\,dm) \in M(\mathbb{T})$. If $f \in H^1$, an analytic function in $\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}$ which we also call f is given by $f(z) \equiv \sum\{\hat{f}(n)z^n: n \in \mathbb{Z}\}$.

The shift is the operator $S \in \mathcal{L}(L^2)$ given by $(Sf)(z) \equiv zf(z)$.

(4) <u>Exercise</u> Let $X \in \mathcal{L}(L^2)$ be such that $XS = SX$ and $h = Xe_0$. Set $M_h f \equiv hf$, i.e., M_h is the multiplication by h. Then $X = M_h$, $h \in L^\infty$ and $\|h\|_\infty = \|X\|$.

Sarason's generalized interpolation theorem says that:

(5) <u>THEOREM</u> Let K be a closed subspace of H^2 such that $S(H^2 \ominus K) \subset H^2 \ominus K$. If $T = P_K S_{|K}$ and $A' \in \mathcal{L}(K)$ commutes with T then there exists $h \in H^\infty$ such that $A'g = P_K(hg)$, $\forall\, g \in K$, and

$\|A'\| = \|h\|_\infty$.

Sketch of the proof

Set $H^2_- = L^2 \ominus H^2$. Set $G_1 = G_2 = L^2$, $U_1 = U_2 = S$, $B_1 = H^2$, $B_2 = K \oplus H^2_-$, and consider $A := A'P_K$ as an operator from H^2 to $K \oplus H^2_-$. Thus $AU_{1|B_1} = A'TP_K = TA'P_K = P_{B_2}U_2A$. Let $\tilde{A} \in \mathcal{L}(L^2)$ be given by theorem (1). Let $h \in L^\infty$ be such that $\tilde{A}$ is the multiplication by h, $\|h\|_\infty = \|\tilde{A}\| = \|A'\|$. From $A = P_{B_2}\tilde{A}_{|B_1}$ it follows that $\langle A'P_K u,v\rangle = \langle hu,v\rangle$ for every $u \in H^2$ and $v \in K \oplus H^2_-$. Since $A'K \subset K \subset H^2$, $h \in H^\infty$ and $A'g = P_K(hg)$, $\forall\, g \in K$.

Complementary remarks on Toeplitz and Hankel operators, their symbols and Nehari's theorem

Let E be the linear span of $\{e_n: n \geq 0\}$; a linear operator $T: E \to H^2$ is called a Toeplitz operator if $\langle Tv,w\rangle = \langle TSv,Sw\rangle$ for every $v,w \in E$. A finite or infinite square matriz $[t_{ij}]_{i,j\geq 0}$ is called a Toeplitz matrix if there exists $\{a_j\}$ such that $t_{ij} \equiv a_{i-j}$. Thus, T is a Toeplitz operator iff $[\langle Te_i,e_j\rangle]_{i,j\geq 0}$ is a Toeplitz matrix.

(6) Exercise A Toeplitz operator T defines a bounded operator in H^2 iff there exists $g \in L^\infty(\mathbb{T})$ such that $T = P_{H^2}M_g$; in such case, $\|T\| = \|g\|_\infty$ $\hat{g}(i-j) \equiv \langle Te_i,e_j\rangle$ and g is called the symbol of T.
(Show that the sesquilinear form $B: E \times E \to \mathbb{C}$ given by $B(v,w) \equiv \langle Tv,w\rangle$ can be extended to a bounded sesquilinear form $B': L^2 \times L^2 \to \mathbb{C}$ such that $B'(v,w) \equiv B'(Sv,Sw)$ and consider the operator T' associated to B'.

A linear operator $H: E \to H^2_-$ such that $P_{H^2_-} SH = HS_{|H^2}$ is called a Hankel operator. A finite or infinite square matriz $[h_{ij}]_{i,j\geq 0}$ is called a Hankel matrix if there exists $\{a_j\}$ such that $h_{ij} \equiv a_{i+j}$. Thus, H is a Hankel operator iff $[\langle He_i,e_{-j-1}\rangle]_{i,j\geq 0}$ is a Hankel matrix.

Nehari's theorem says that

(7) THEOREM A Hankel operator H defines a bounded operator

from H^2 to H^2_- iff there exists $g \in L^\infty(\mathbb{T})$ such that $H = P_{H^2_-} M_g$; in such case $\|H\| = \mathrm{dist}_{L^\infty}(g, H^\infty)$ and $\hat{g}(-i-j-1) = \langle He_i, e_{-j-1} \rangle$ for every $i,j \geq 0$.

When the above conditions hold, g is called the symbol of H.

Exercise: prove (7) by applying theorem (1) with $G_1 = G_2 = L^2$, $U_1 = U_2 = S$, $B_1 = H^2$ and $B_2 = H^2_-$.

• Sarason's general interpolation method is presented in [S]. Fundamental references concerning this subject are [FF] and [N].

III. APPLICATIONS TO CLASSIC PROBLEMS

On the Nevanlinna-Pick problem

Given any set J, a function $k: J \times J \to \mathbb{C}$ is positive definite (p.d.) if for every $n \geq 1$, $t_1, \ldots, t_n \in J$ and $c_1, \ldots, c_n \in \mathbb{C}$ it is true that $\sum\{k(t_i, t_j)c_i\bar{c}_j : 1 \leq i,j \leq n\} \geq 0$. Then:

(1) THEOREM Let J be any set, $\{z_t : t \in J\}$ a set of different points in $\mathbb{D}$ and $\{w_t : t \in J\} \subset \mathbb{C}$. Set $\mathcal{H} = \{h \in H^\infty : h(z_t) = w_t, \forall\, t \in J, \|h\|_\infty \leq 1\}$. Define $k: J \times J \to \mathbb{C}$ by $k(s,t) = [1 - w_s \bar{w}_t]/[1 - z_s \bar{z}_t]$. Then $\mathcal{H}$ is non empty iff k is positive definite.

Sketch of the proof

For any $u \in \mathbb{D}$ set $\psi_u(z) = (1 - \bar{u}z)^{-1}$; then $f(u) = \langle f, \psi_u \rangle$ for every $f \in H^2$. The set $\{\psi_{z_t} : t \in J\}$ is linearly independent; in its linear span K' define a linear operator X by $X(\psi_{z_t}) = \bar{w}_t \psi_{z_t}$, $\forall\, t \in J$; it can be seen that $\|X\| \leq 1$ is the same as k being p.d. Call K the closed linear span in L^2 of K'. If there exists $h \in \mathcal{H}$, $X = (P_K M_{h|K})^*$, so k is p.d.

Conversely, assume the last; thus $X \in \mathcal{L}(K)$ and $\|X\| \leq 1$. Apply theorem (II.1) with $U_1 = U_2 = S$, $B_1 = H^2$, $B_2 = K \oplus H^2_-$ and

$A \in \mathbb{L}(B_1,B_2)$ such that $Af \equiv X^*P_K f$. For any $n \geq 0$ and $t \in J$ we have $<ASe_n,\psi_{z_t}> = <Se_n,X\psi_{z_t}> = w_t<e_{n+1},\psi_{z_t}> = w_t(z_t)^{n+1} = z_t w_t<e_n,\psi_{z_t}> = z_t<e_n,X\psi_{z_t}> = z_t(Ae_n)(z_t) = <SAe_n,\psi_{z_t}>$; since AB_1 is orthogonal to H^2_- it follows that $AS_{|B_1} = P_{B_2}SA$. Consequently $\mathcal{A}$ is non empty. For any $\tilde{A} \in \mathcal{A}$ set $h = \tilde{A}e_o$, so $\tilde{A} = M_h$. Then $\|h\|_\infty = \|A\| \leq 1$. If $g \in H^2_-$, $<h,g> = <P_{B_2}\tilde{A}e_o,g> = <Ae_o,g> = 0$, so $h \in H^\infty$. For any $t \in J$, $h(z_t) = <h,\psi_{z_t}> = <P_{B_2}\tilde{A}e_o,\psi_{z_t}> = <Ae_o,\psi_{z_t}> = <e_o,X\psi_{z_t}> = w_t$. Summing up, $h \in \mathcal{H}$.

Note that a bijection from $\mathcal{A}$ onto $\mathcal{H}$ is given by $\tilde{A} \to h = \tilde{A}e_o$.

On the Carathéodory-Féjer problem

(2) THEOREM Let $c_o,\dots,c_n \in \mathbb{C}$. Set $\mathcal{H} = \{g \in H^\infty: \hat{g}(j) = c_j\ , j = 0,\dots,n\ ,\ \|g\|_\infty \leq 1\}$. Define $T \in \mathbb{L}(\mathbb{C}^{n+1})$ by the matrix $[t_{ij}]_{0\leq i,j\leq n}$ such that $t_{ij} = c_{i-j}$ if $i \geq j$ and $t_{ij} = 0$ is $i < j$. Then $\mathcal{H}$ is non empty iff $\|T\| \leq 1$.

Exercise Let K be the span of e_j, $j = 0,\dots,n$ and $f(z) = c_o + c_1 z + \dots c_n z^n$. Note that $[t_{ij}]$ is the matrix of $(P_K M_{f|K})$. Prove the above theorem by means of (II.1) as before.

Complements: On the Nudelmann and Rosenblum-Rovnyak interpolation theorem

This theorem has several applications. It says that:

(3) THEOREM Let E be a complex vector space, E' its algebraic dual, ρ a linear operator in E, ρ' its dual, $b,c \in E$. Let F be a subspace of E' such that $\rho'F \subset F$ and $\sum\{|(\rho^j c,x')|^2: j \geq 0\} < \infty$, $\forall\ x' \in F$. The following conditions are equivalent:

i) $\exists\ f \in H^\infty(\mathbb{T})$ such that $\|f\|_\infty \leq 1$ and

$(b,x') = \sum\{\hat{f}(j)(\rho^j c,x'): j \geq 0\}$, $\forall\ x' \in F$;

ii) $\sum\{|(\rho^j b,x')|^2: j \geq 0\} \leq \sum\{|(\rho^j c,x')|^2: j \geq 0\}$, $\forall\ x' \in F$.

Note that the equality in (i) can be seen as an extension of $b = f(\rho)c$, which makes sense for example when f is a polynomial.

Exercise

Set $G_1 = G_2 = L^2$, $U_1 = U_2 = S$, $B_1 = H^2$, $B_2 = K \oplus H^2_-$, with K the closure in H^2 of $\{\Sigma_{j\geq 0}(\rho^j c,x')e_j: x' \in F\}$. Show that $S^{-1}B_2 \subset B_2$.

Assume (ii). Show that the contraction $X \in \mathcal{L}(K,H^2)$ given by $X[\Sigma_{j\geq 0}(\rho^j c,x')e_j] \equiv \Sigma_{j\geq 0}(\rho^j b,x')e_j$ is such that $XP_{H^2}S^{-1}{}_{|K} = P_{H^2}S^{-1}X$. By applying theorem with $A \in \mathcal{L}(B_1,B_2)$ such that $A^* = XP_K$ obtain $g \in L^\infty(\mathbb{T})$ such that:

(iii) $<gb_1,b_2> = <Ab_1,b_2>$, $\forall\, b_1 \in B_1$, $b_2 \in B_2$.

Show that $g \in H^\infty(\mathbb{T})$ and complete the proof of (i) by setting $f(z) \equiv \overline{g(\bar{z})}$.

Conversely, assume (i). Show that (iii) defines a contraction $A \in \mathcal{L}(B_1,B_2)$ such that $A^*[\Sigma_{j\geq 0}(\rho^j c,x')e_j] \equiv \Sigma_{j\geq 0}(\rho^j b,x')e_j$, so (ii) holds.

- Concerning this chapter see [FF], [N] and [RR].

IV. UNITARY DILATIONS OF CONTRACTIONS AND THE NAGY-FOIAS THEOREM

A special unitary extension of an isometry

Consider an isometry V acting in a Hilbert space H with domain D, range R, and defect subspaces $N = H \ominus D$, $M = H \ominus R$. Let G be the Hilbert space of sequences $f = \{f_j: j \in \mathbb{Z}\}$ such that $f_j \in M$ if $j < 0$, $f_o \in H$, $f_j \in N$ if $j > 0$ and $\Sigma\{\|f_j\|^2: j \in \mathbb{Z}\} < \infty$, with the scalar product $<f,g> = \Sigma\{<f_j,g_j>: j \in \mathbb{Z}\}$. Identify H with $\{f \in G: f_j = 0 \text{ if } j \neq 0\}$ and define $g = Uf$ by $g_j = f_{j-1}$ if $j < 0$, $g_o = f_{-1} + VP_Df_o$, $g_1 = P_Nf_o$ and $g_j = f_{j-1}$ if $j > 1$.

(1) Exercise (U,G) is a minimal unitary extension of V and $(VP_D)^n =$

$P_H U^n{}_{|H}$ holds for every $n \geq 0$.

Isometric and unitary dilations of contractions

An operator $T \in \mathcal{L}(E)$ in a Hilbert space is a contraction if $\|T\| \leq 1$. It is said that $W \in \mathcal{L}(M)$ is a minimal isometric dilation of T if M is a Hilbert space that contains E as a closed subspace and W is an isometry such that $T^n = P_E W^n{}_{|E}$, $\forall\, n \geq 0$, and $M = \vee\{W^n E: n \geq 0\}$. Analogously, $U \in \mathcal{L}(G)$ is a minimal unitary dilation of T if G is a Hilbert space that contains E as a closed subspace and U is a unitary operator such that $T^n = P_E U^n{}_{|E}$, $\forall\, n \geq 0$, and $G = \vee\{U^n E: n \in \mathbb{Z}\}$. The following fundamental result is due to Nagy.

(2) THEOREM Every contraction in a Hilbert space $T \in \mathcal{L}(E)$ has a minimal isometric dilation $W \in \mathcal{L}(M)$ and a minimal unitary dilation $U \in \mathcal{L}(G)$ which are unique up to unitary isomorphisms.

Sketch of the proof

Given a contraction $T \in \mathcal{L}(E)$ set $D_T = (I - T^*T)^{1/2}$ and let $\mathfrak{D}_T$ be the closure of the range of D_T. Let V be the isometry acting in $H := E \oplus \mathfrak{D}_T$, with domain D = E, given by $Vh \equiv (Th, D_T h)$. Then $U \in \mathcal{L}(G)$ as in (1) is a minimal unitary dilation of T. Set $M = = \vee\{U^n E: n \geq 0\}$ and $W = U_{|M}$; then $W \in \mathcal{L}(M)$ is a minimal isometric dilation of T. Uniqueness is proved as in Naimark's dilation theorem. Thus, we may speak of "the" minimal (isometric or unitary) dilation of T and assume always that $M \subset G$.

An alternative proof can be based directly in Naimark's theorem as follows. Let $T(.): \mathbb{Z} \to \mathcal{L}(E)$ be given by $T(n) = T^n$ if $n \geq 0$ and $T(n) = T^{*-n}$ if $n \leq 0$. If $h: \mathbb{Z} \to E$ is such that supp $h \subset [0,k]$, it may be considered as a vector in E^{k+1}; let $R = [r_{ij}]_{0 \leq i,j \leq k} \in \mathcal{L}(E^{n+1})$ be given by $r_{i+1,i} = T$ if $0 \leq i < k$ and $r_{ij} = 0$ if $i - j \neq 1$; then $(I - R)^{-1} = I + R + \ldots + R^k$. If $v = (I - R)^{-1} h$, then $\sum\{<T(m-n)h(m),h(n)>: m,n \in \mathbb{Z}\} = <[(I - R^*)^{-1} + (I - R)^{-1} - I]h,h> = \|v\|^2 - \|Rv\|^2$; consequently, T(.) is of positive type and Nagy's theorem follows.

(3) Exercise Let $T \in \mathcal{L}(E)$, $W \in \mathcal{L}(M)$ and $U \in \mathcal{L}(G)$ be as in (2). Then $P_E W = TP_E$, $W^*_{|E} = T^*$ and $W(M \ominus E) \subset (M \ominus E)$. Set $M' = \vee\{U^n E: n \leq 0\}$ and $W' = U^*_{|M'}$; then $W' \in \mathcal{L}(M')$ is the minimal isometric dilation of T^* and $G = M \oplus (M' \ominus E)$.

The commutant lifting theorem

The famous abstract extension of Sarason's interpolation theorem due to Nagy and Foias can be stated as follows.

(4) THEOREM For $j = 1,2$ let $T_j \in \mathcal{L}(E_j)$ be a contraction in a Hilbert space with minimal isometric dilation $W_j \in \mathcal{L}(M_j)$ and minimal unitary dilation $U_j \in \mathcal{L}(G_j)$. Let $X \in \mathcal{L}(E_1,E_2)$ be such that $XT_1 = T_2X$. Then:

i) $\exists\, \gamma \in \mathcal{L}(M_1,M_2)$ such that $\gamma W_1 = W_2\gamma$, $P_{E_2}\gamma = XP_{E_1}$ and $\|\gamma\| = \|X\|$;

ii) $\exists\, \tau \in \mathcal{L}(G_1,G_2)$ such that $\tau U_1 = U_2\tau$, $\tau M_1 \subset M_2$, $P_{E_2}\tau_{|M_1} = XP_{E_1}$ and $\|\tau\| = \|X\|$.

Proof

We apply theorem (II.1) with $B_1 = M_1$, $B_2 = \vee\{U_2{}^n E_2: n \leq 0\}$ and $A \in \mathcal{L}(B_1,B_2)$ such that $A = XP_{E_1}$. Thus, $AU_{1|B_1} = XP_{E_1}W_1 = XT_1P_{E_1} = T_2XP_{E_1} = P_{E_2}W_2A = P_{B_2}U_2A$. So there exists $\tau \in \mathcal{L}(G_1,G_2)$ such that $\tau U_1 = U_2\tau$, $A = P_{B_2}\tau_{|B_1}$ and $\|A\| = \|\tau\|$. Consequently $\|\tau\| = \|X\|$ and $XP_{E_1} = P_{E_2}\tau_{|B_1}$. Since $P_{B_2}\tau_{|E_1} = X$ and $G_2 = M_2 \oplus (B_2 \ominus E_2)$, $\tau E_1 \subset M_2$ so also $\tau M_1 \subset M_2$.

Note that $P_{B_2}\tau_{|M_1} = XP_{E_1}$ implies that $\tau(M_1 \ominus E_1)$ is orthogonal to B_2, so $\tau^*\vee\{U_2{}^n E_2: n \leq 0\} \subset \vee\{U_1{}^n E_1: n \leq 0\}$.

Now let $\gamma \in \mathcal{L}(M_1,M_2)$ be such that $\gamma = \tau_{|M_1}$. The result follows.

- A complete study of the Nagy and Nagy-Foias theorems is given in

[NF] and [FF]. The approach to generalized interpolation presented in this paper extends the proof of the commutant lifting theorem given in [A].

V. ON PARROTT'S EXTENSION OF THE COMMUTANT LIFTING THEOREM

In this section $\{T_1,T_2,X\}$ denotes a given commutant, so, for $j = 1,2$, $T_j \in \mathcal{L}(E_j)$ is a contraction with minimal unitary dilation $U_j \in \mathcal{L}(F_j)$ minimal isometric dilation $W_j \in \mathcal{L}(M_j)$, and $X \in \mathcal{L}(E_1,E_2)$ is such that $XT_1 = T_2X$. When $T \in \mathcal{L}(E)$ is a contraction, the function $T(.): \mathbb{Z} \to \mathcal{L}(E)$ is defined by $T(m) = T^m$ if $m \geq 0$ and $T(m) = T^{*-m}$ if $m \leq 0$.

(1) PROPOSITION If $\|X\| \leq 1$ the following conditions are equivalent: (a) $\exists\ \hat{X} \in \mathcal{L}(F_1,F_2)$ that extends X and is such that $\hat{X}U_1 = U_2\hat{X}$ and $\|\hat{X}\| \leq 1$; (b) $I - T_1^*T_1 - X^*X + (T_2X)^*T_2X \geq 0$; (c) $\sum\{\langle T_2(m-n)Xh(m),Xh(n)\rangle: m,n\in\mathbb{Z}\} \leq \sum\{\langle T_1(m-n)h(m),h(n)\rangle: m,n\in\mathbb{Z}\}$ holds for every finitely supported function $h: \mathbb{Z} \to E$.

Sketch of the proof

We must have $\hat{X}(U_1{}^n e) = U_2{}^n Xe$, $\forall\ n \in \mathbb{Z}$, $e \in E$, so (a) holds iff $\|\sum_m U_2{}^m Xh(m)\|_{F_2}{}^2 \leq \|\sum_m U_1{}^m h(m)\|_{F_1}{}^2$ is always true, i.e., iff (c) holds.

If v is a positive integer such that supp $h \subset [-v,v]$ we may assume that $h \in E_1{}^{2v+1}$; let $\tau = [\tau_{jk}]_{-v\leq j,k\leq v}$ and $R = [R_{jk}]_{-v\leq j,k\leq v} \in \mathcal{L}(E_1{}^{2v+1})$ be given by $\tau_{jk} = T_1(k-j)$, $R_{j,j+1} = T_1$ and $R_{jk} = 0$ if $k-j \neq 1$. Then $(I-R)^{-1} \in \mathcal{L}(E_1{}^{2v+1})$; for $u = (I-R)^{-1}h \in E_1{}^{2v+1}$ we have

$\sum\langle T_1(m-n)h(m),h(n)\rangle = \|u\|^2 - \|Ru\|^2 = \sum\|u(m)\|^2 - \sum\|T_1u(m)\|^2$ and

$\sum\langle T_2(m-n)Xh(m),Xh(n)\rangle = \sum\|Xu(m)\|^2 - \sum\|T_2Xu(m)\|^2$; thus, (c) and (b) are equivalent.

Condition (b) above holds in the following particular cases: i) X is an isometry; ii) T_2 is an isometry; iii) $\|T_1 e\|^2 + \|Xe\|^2 \leq \|x\|^2$, $\forall\, e \in E_1$; iv) $X^*T_2 = T_1X^*$. In fact, the last implies that $X^*XT_1 = T_1X^*X$, so $D_XT_1 = T_1D_X$ and (b) follows from $\|T_1D_Xe\|^2 \leq \|D_Xe\|^2$ for every $e \in E_1$.

Consequently, if $T \in \mathcal{L}(E)$ is a contraction with minimal unitary dilation $U \in \mathcal{L}(F)$, every $A \in \mathcal{L}(E)$ that bicommutes with T (i.e., such that A commutes with T and T^*) has a (unique) extension $\hat{A} \in \mathcal{L}(F)$ that commutes with U and is such that $\|\hat{A}\| = \|A\|$.

Let L(X) be the set of liftings of X, i.e.,

$$L(X) = \{\tau \in \mathcal{L}(F_1,F_2): \tau U_1 = U_2\tau,\ P_{E_2}\tau_{|E_1} = X,\ \|\tau\| = \|X\|\}.$$

Let $\tau \in L(X)$; then $P_{E_2}\tau_{|M_1} = XP_{E_1|M_1}$; consequently. τ^*E_2 is orthogonal to $M_1 \ominus E_1$; duality considerations show that $\tau E_1 \subset M_2$, so $\tau M_1 \subset M_2$.

From now on we assume that $\|X\| = 1$. Thus, conditions (1.b) and (1.c) are equivalent to the existence of an extension $\hat{X}$ of X that belongs to L(X).

(2) <u>Exercise</u> $X^*T_2 = T_1X^*$ iff $\exists\, \hat{X} \in L(X)$ such that $P_{E_2}\hat{X} = XP_{E_1}$.

The Nagy-Foias commutant lifting theorem states that L(X) is never empty. An extension due to Parrott of that result is closely connected with the following.

(3) <u>THEOREM</u> Let the commutant $\{T_1, T_2, X\}$ be given. Set $(A_1,A_2) \in \mathcal{A}$ if $A_j \in \mathcal{L}(E_j)$ bicommutes with T_j, $j = 1,2$, and $XA_1 = A_2X$, $XA_1^* = A_2^*X$; let $\hat{A}_j \in \mathcal{L}(F_j)$ be the extension of A_j that commutes with U_j and is such that $\|\hat{A}_j\| = \|A_j\|$, $j = 1,2$. There exists $\tau \in L(X)$ such that $\tau\hat{A}_1 = \hat{A}_2\tau$, $\forall\, (A_1,A_2) \in \mathcal{A}$.

Our proof is based on the following

(4) <u>LEMMA</u> Let V be an isometry with domain D and range R, both closed subspaces of the Hilbert space H, and Δ the set of operators $\delta \in \mathcal{L}(H)$ such that D and R are invariant under δ and δ^*, and $V\delta_{|D} = \delta V$. Then the minimal unitary dilation $U \in \mathcal{L}(F)$ of $VP_D \in \mathcal{L}(H)$ is a unitary extension of V to $F \supset H$ such that every $\delta \in \Delta$ has a (unique) extension $\hat{\delta} \in \mathcal{L}(F)$ that

commutes with U; moreover, $\|\hat{\delta}\| = \|\delta\|$ and $\hat{\delta}^*$ extends δ^*.

Let N and M be the orthogonal complements of D and R in H, respectively; then $F = (\oplus_{n<0} U^n M) \oplus H \oplus (\oplus_{n>0} U^n N)$ and $\hat{\delta} \in \mathcal{L}(F)$ is defined by $\hat{\delta}(U^n v) = U^n(\delta v)$ for $v \in M$, $n < 0$ and for $v \in N$, $n > 0$. The assertion follows.

(5) <u>Exercise</u> Prove theorem (3) in the following way.

a) Set $M_1 = \vee\{U_1{}^n E_1 : n \geq 0\}$ and $M'_2 = \vee\{U_2{}^n E_2 : n \leq 0\}$. Let H be a Hilbert space such that $H = M_1 \vee M'_2$ and $P^H{}_{M'_2|M_1} = X' := XP^{M_1}{}_{E_1}$.

Prove that every $(A_1, A_2) \in \mathcal{A}$ defines an operator $A \in \mathcal{L}(H)$ by $A(g'_2 + g_1) = \hat{A}_2 g'_2 + \hat{A}_1 g_1$, $\forall\, g'_2 \in M'_2$ and $g_1 \in M_1$.

b) Set $D = U_2{}^* M'_2 \vee M_1$; let V be the isometry given by $V(U_2{}^* g'_2 + g_1) \equiv g'_2 + U_1 g_1$ and $U \in \mathcal{L}(F)$ as in (4). It may be assumed that $F = F_1 \vee F_2$ and that $U_{|F_j} = U_j$. Since $A \in \Delta$ it extends to $\hat{A} \in \mathcal{L}(F)$ such that $\hat{A}U = U\hat{A}$; thus $\hat{A}_{|F_j} = \hat{A}_j$.

c) Set $\tau = P^F{}_{F_2|F_1}$; then $\tau \in L(X)$ and $\tau\hat{A}_1 = \hat{A}_2\tau$.

From theorem (3) we obtain the following result of Parrott.

(6) <u>THEOREM</u> Let $T \in \mathcal{L}(E)$ be a contraction, $W \in \mathcal{L}(M)$ its minimal isometric dilation and $X \in \mathcal{L}(E)$ such that $TX = XT$. Let $\mathfrak{U}$ be the algebra of all the operators in $\mathcal{L}(E)$ that bicommute with T and X; let $A' \in \mathcal{L}(M)$ be the unique extension of $A \in \mathfrak{U}$ that commutes with W. There exists $X' \in \mathcal{L}(M)$ such that: $X'A' = A'X'$, $\forall\, A \in \mathfrak{U}$; $P_E A' W^m X'^n{}_{|E} = AT^m X^n$ for every $m,n \geq 0$ and $A \in \mathfrak{U}$; $\|X'\| = \|X\|$.

<u>Proof</u>. Assume $\|X\| = 1$ and let $\tau \in L(X)$ be given by theorem (3), with $T_1 = T_2 = T$, $M_1 = M$ and $U_{1|M} = W$. If $A \in \mathfrak{U}$, $(A,A) \in \mathcal{A}$ and $A' = \hat{A}_{|M}$. Since M is invariant by τ, the operator $X' := \tau_{|M} \in \mathcal{L}(M)$ commutes with $W = U_{|M}$ and A', and is such that $P_E X' = XP_E$. The result follows.

• Parrott's extension of the commutant lifting theorem was established in [P].

VI. THE COTLAR-SADOSKY LIFTING THEOREM

Let $\mathbf{P}$ be the space of trigonometric polynomials, i.e., the linear span of $\{e_n : n \in \mathbb{Z}\}$; set $\mathbf{P}_+ = \{\sum a_n e_n \in \mathbf{P}: a_n = 0 \text{ if } n < 0\}$ and $\mathbf{P}_- = \{\sum a_n e_n \in \mathbf{P}: a_n = 0 \text{ if } n \geq 0\}$. If $V = [v_{jk}]_{j,k=1,2}$ is a matrix with entries in $M(\mathbb{T})$ and $F = (f_1, f_2) \in C(\mathbb{T}) \times C(\mathbb{T})$, we set $(VF,F) = \sum\{\int_{\mathbb{T}} f_j \bar{f}_k dv_{jk}: j,k = 1,2\}$. It is said that V is a positive matrix measure if $[v_{jk}(\Delta)]_{j,k=1,2}$ defines a positive operator in $\mathbb{C}^2$, for every Borel set $\Delta \subset \mathbb{T}$.

(1) Exercise $V = [v_{jk}]_{j,k=1,2}$ is a positive matrix measure iff $(VF,F) \geq 0$ for every $F \in C(\mathbb{T}) \times C(\mathbb{T})$.

The Cotlar-Sadosky lifting theorem can be stated as follows.

(2) THEOREM Let the matrix measure $V = [v_{jk}]_{j,k=1,2}$ be such that $(VF,F) \geq 0$ for every $F = (f_1, f_2) \in \mathbf{P}_+ \times \mathbf{P}_-$. There exists a function $h \in H^1$ such that, setting $w_{jj} = v_{jj}$, $j = 1,2$, $w_{12} = v_{12} + h\, dm$, and $w_{21} = (w_{12})^-$, then $W := [w_{jk}]_{j,k=1,2}$ is a positive matrix measure and $(VF,F) = (WF,F)$ holds for every $F = (f_1, f_2) \in \mathbf{P}_+ \times \mathbf{P}_-$.

Proof

In order to apply theorem (II.1) note that v_{jj} is a positive measure and let U_j be the shift in $G_j := L^2(\mathbb{T}, v_{jj})$, $j = 1,2$. Call B_1 (B_2) the closure of $\mathbf{P}_+$ ($\mathbf{P}_-$) in G_1 (G_2) and let $A \in \mathbf{L}(B_1, B_2)$ be defined by $\langle Af_1, f_2 \rangle = \int_{\mathbb{T}} f_1 \bar{f}_2 dv_{12}$, $\forall (f_1, f_2) \in \mathbf{P}_+ \times \mathbf{P}_-$. Since $(VF,F) \geq 0$ for every $F = (f_1, f_2) \in \mathbf{P}_+ \times \mathbf{P}_-$, $\|A\| \leq 1$. There exists $\tilde{A} \in \mathbf{L}(G_1, G_2)$ such that $\tilde{A}U_1 = U_2\tilde{A}$, $A = P_{B_2}\tilde{A}_{|B_1}$ and $\|A\| = \|\tilde{A}\|$. Thus, $\tilde{A}$ is given by the multiplication by the function $g = \tilde{A}e_0$ and $\langle \tilde{A}f_1, f_2 \rangle = \int_{\mathbb{T}} f_1 \bar{f}_2 g\, dv_{22}$, $\forall (f_1, f_2) \in C(\mathbb{T}) \times C(\mathbb{T})$; since $\int_{\mathbb{T}} e_n dv_{12} = \int_{\mathbb{T}} e_n g\, dv_{22}$ for every

$n > 0$, the F. and M. Riesz theorem shows that $\exists\, h \in H^1$ such that $w_{12} := v_{12} + h\, dm = g\, dv_{22}$; since $\|\tilde{A}\| \leq 1$, $(WF,F) \geq 0$ for every $F = (f_1,f_2) \in C(\mathbb{T}) \times C(\mathbb{T})$. The proof is over.

<u>Remarks on Fourier series and the Helson-Szegö theorem</u>

The Fourier series of $f \in L^1$ is given by $S_k(f) = \sum\{\hat{f}(n)e_n\colon |n|\leq k\}$; the functional in L^1 given by $f \to \hat{f}(n)$ is continuous for every $n \in \mathbb{Z}$. If $f \in L^2 \equiv L^2(\mathbb{T},m)$ then $\lim_{k\to\infty} \|S_k(f) - f\| = 0$. Helson and Szegö characterized - as stated in theorem (5) below - the positive measures $\mu \in M(\mathbb{T})$ such that the same happens in $L^2(\mathbb{T},\mu)$, a characterization that also answers a question concerning the "prediction theory" of stochastic processes.

Let P_+, P_-, P_k ($k \in \mathbb{Z}$) and $P_{k,m}$ ($k,m \geq 0$) be the operators in $\mathbf{P}$ defined by $P_+[\sum\hat{f}(n)e_n] \equiv \sum\{\hat{f}(n)e_n\colon n\geq 0\}$, $P_-[\sum\hat{f}(n)e_n] \equiv \sum\{\hat{f}(n)e_n\colon n<0\}$, $P_k[\sum\hat{f}(n)e_n] \equiv \hat{f}(k)$, and $P_{k,m}[\sum\hat{f}(n)e_n] \equiv \sum\{\hat{f}(n)e_n\colon -k\leq n\leq m\}$. We keep the same names for continuous eextensions of these operators to spaces where $\mathbf{P}$ is dense. Thus, $P_k(f) = \hat{f}(k)$, $\forall\, k \in \mathbb{Z}$ and $f \in L^1(\mathbb{T},m)$.

(3) <u>PROPOSITION</u> Let $\mu \in M(\mathbb{T})$ be positive. If P_0 is bounded in $L^2(\mathbb{T},\mu)$ then so is P_k for every $k \in \mathbb{Z}$, $L^2(\mathbb{T},\mu) \subset L^1(\mathbb{T},m)$ and $P_k(f) = \hat{f}(k)$, $\forall\, k \in \mathbb{Z}$ and $f \in L^2(\mathbb{T},\mu)$. The following conditions are equivalent: (i) P_+ is bounded in $L^2(\mathbb{T},\mu)$; (ii) the operators $P_{k,m}$, $k,m \geq 0$, are uniformly bounded in $L^2(\mathbb{T},\mu)$; (iii) P_k is bounded for every $k \in \mathbb{Z}$ and $\lim_{k\to\infty} \|S_k(f) - f\| = 0$ for every $f \in L^2(\mathbb{T},\mu)$; (iv) the operators $P_{m,m}$, $m \geq 0$, are uniformly bounded in $L^2(\mathbb{T},\mu)$.

<u>Sketch of the proof</u>

$P_k = P_0 S^{-k}$, $\forall\, k \in \mathbb{Z}$; if P_0 is bounded $e_0 \notin \vee\{e_n\colon n \neq 0\}$ in $L^2(\mathbb{T},\mu)$ so there exists $g \in L^2(\mathbb{T},\mu)$ such that $g\, d\mu = dm$ and

$\int |f| \, dm \le \|f\|_{L^2(\mathbb{T},\mu)} \|g\|_{L^2(\mathbb{T},\mu)}, \ \forall f \in L^2(\mathbb{T},\mu)$.

(i) $\Rightarrow$ (ii): $P_{k,m} = S^{k+1}P_-S^{-k-m-1}P_+S^k$.

(ii) $\Rightarrow$ (iii): $\lim_{k\to\infty} \|S_k(f) - f\| = 0$ holds for every f in $\mathbb{P}$ which is dense in $L^2(\mathbb{T},\mu)$ and $P_{k,k}(f) = S_k(f), \ \forall f \in L^2(\mathbb{T},\mu)$.

(iii) $\Rightarrow$ (iv): follows from the uniform boundedness principle.

(iv) $\Rightarrow$ (i): $P_+f = \lim_{k\to\infty} S^kP_{k,k}S^{-k}f, \ \forall f \in \mathbb{P}$.

Let the operator H be defined in $\mathbb{P}$ by $H = i[P_- - (P_+ - P_o)]$; then $f + iHf = 2P_+f - P_of$ is analytic, so Hf is the harmonic conjugate of f that verifies $Hf(0) = 0$. Since $I + H^2 = P_o$, H is bounded in $L^2(\mathbb{T},\mu)$ iff the same holds for P_+. it can be seen that, for example when $f \in L^2(\mathbb{T},m)$, Hf is given by a "singular integral", $Hf(e^{ix}) = \lim_{\varepsilon\to 0^+} \int_{\varepsilon\le|y|\le\pi} f[e^{i(x-y)}] \operatorname{cotg}(y/2) \, dy$, which is called the Hilbert transform of f.

(4) <u>Exercise</u> $H \in \mathcal{L}[L^2(\mathbb{T},\mu)]$ iff $\exists$ $M > 1$ such that, setting $V = [v_{jk}]_{j,k=1,2}$ with $v_{11} = v_{22} = (M-1)\mu$ and $v_{12} = v_{21} = (M+1)\mu$, $(VF,F) \ge 0$ holds for every $F = (f_1,f_2) \in \mathbb{P}_+ \times \mathbb{P}_-$.

(5) <u>THEOREM</u> Let $\mu \in M(\mathbb{T})$ be positive. The following conditions are equivalent: (i) $H \in \mathcal{L}[L^2(\mathbb{T},\mu)]$; (ii) P_k is bounded for every $k \in \mathbb{Z}$ and $\lim_{k\to\infty} \|S_k(f) - f\| = 0$ for every $f \in L^2(\mathbb{T},\mu)$; (iii) $d\mu = e^{u+Hv} \, dm$, with $u,v \in L^\infty(\mathbb{T},m)$ and $\|v\|_\infty < \pi/2$.

<u>Sketch of the proof</u>

Assume (i); by (4) and the Cotlar-Sadosky theorem $\exists$ $h \in H^1$ such that, setting $w_{jj} = (M-1)\mu$, $j = 1,2$, $w_{12} = (M+1)\mu + h \, dm$, and $w_{21} = (w_{12})^-$, then $W := [w_{jk}]_{j,k=1,2}$ is a positive matrix measure. It follows that $\mu(A) = 0$ for any set A such that $m(A) = 0$, so $d\mu = f \, dm$. For $r = (M-1)/(M+1)$

[#] $\quad f^2 (r^2 - 1) - 2(\operatorname{Re} h)f - |h|^2 \ge 0$

holds a.e. Set $h_1 = \operatorname{Re} h$ and $h_2 = \operatorname{Im} h$; since $f \ge 0$, $h_1 \le 0$ and

$|h_1/h|^2 \geq 1 - r^2$, so $|h_2/h| \leq r$. Then the theory of Hardy spaces H^p shows that h is an "outer function", so $h = c\, e^{Hv - i(v+\pi)}$, with c a positive constant and $v+\pi = \arg h$; thus, $\|v\|_\infty < \pi/2$. Now, [#] shows that $(f/|h|)$ is bounded from below and from above by positive constants, so $\exists\, u \in L^\infty$ such that $f = e^{u+Hv}$.

- The Cotlar-Sadosky lifting theorem was established in [CS].

VI. ON THE BAND EXTENSION PROBLEM

The Naimark type approach can be developed in order to handle some of the problems that Gohberg, Kaashoek and Woerdman have solved by the "band method".

We are given the integers N and p such that $0 \leq p < N-1$, the Hilbert spaces H_j, $1 \leq j \leq N$, and the operators $A_{ij} \in \mathcal{L}(H_j, H_i)$, $1 \leq i,j \leq N$, $|i-j| \leq p$. The "band" $A^{(p)} := \{[A_{ij}]: |i-j| \leq p\}$ is positive if the operators $[A_{kj}]_{i \leq k,j \leq i+p} \in \mathcal{L}[\oplus(H_j: i \leq j \leq i+p)]$ are positive for $1 \leq i \leq N-p$; $A^{(p)}$ is positive definite (p.d.) if $[A_{kj}]_{i \leq k,j \leq i+p}$ is positive definite for $1 \leq i \leq N-p$. Recall that an operator in a Hilbert space is positive definite if it is positive and boundedly invertible. A positive operator $F = [F_{kj}]_{1 \leq k,j \leq N} \in \mathcal{L}[\oplus(H_j: 1 \leq j \leq N)]$ such that $F_{ij} = A_{ij}$ whenever $|i-j| \leq p$ is called a positive extension of the given band. We shall prove that

(1) <u>THEOREM</u> Every positive band $A^{(p)} = \{[A_{ij}]: |i-j| \leq p\}$ has positive extensions. If $A^{(p)}$ is positive definite, it has positive definite extensions and there exists one of them $\overline{A}$ such that $[\overline{A}^{-1}]_{rs} = 0$ if $|s-r| > p$.

<u>Proof</u>

If $r \wedge s$ denotes the minimum of r and s, set $\mathbb{C} = \{(i,j) \in \mathbb{Z}^2: 1 \leq i \leq N,\ i \leq j \leq (i+p) \wedge N\}$, $H_{ij} = H_j$ for every $(i,j) \in \mathbb{C}$ and $\tilde{H} = \oplus\{H_{ij}: (i,j) \in \mathbb{C}\}$. Every $f \in \tilde{H}$ is naturally given by $[f_{ij}]_{(i,j) \in \mathbb{C}}$, $f_{ij} \in H_j$; its support is the set $\operatorname{supp} f := \{(i,j) \in \mathbb{C}: f_{ij} \neq 0\}$. Let the sesquilinear hermitian positive semidefinite form $[.,.]$ in $\tilde{H}$ be given by $[f,f'] \equiv$

$\Sigma\{\langle A_{jk}f_{ik},f'_{ij}\rangle_{H_j}$: (i,j), (i,k) $\in \mathbb{C}\}$. The vector space $\tilde{H}$ and [.,.] generate a Hilbert space G and an operator $\varepsilon \in \mathbf{L}(\tilde{H},G)$ onto a dense subspace of G such that $\langle \varepsilon f,\varepsilon f'\rangle_G \equiv [f,f']$.

Set $D' = \{f \in \tilde{H}: f_{ii} = 0,\ 1\leq i\leq N\}$ and, for $f \in D'$, let $g = \tau f \in \tilde{H}$ be given by $g_{ij} = f_{i-1,j}$ if (i,j), (i-1,j) $\in \mathbb{C}$ and $g_{ij} = 0$ if (i,j) $\in \mathbb{C}$ but (i-1,j) $\notin \mathbb{C}$. Setting $W\varepsilon f = \varepsilon\tau f$ for every $f \in D'$ an isometry W is defined in the closure D of $\varepsilon D'$ in G with range $R = WD \subset G$.

For $1\leq t\leq N$ let $\lambda_t \in \mathbf{L}(H_t,G)$ be given, for any $v \in H_t$, by $\lambda_t v = \varepsilon v'$, where $v' \in \tilde{H}$ is such that supp $v' = \{(t,t)\}$ and $v'_{tt} = v$. It follows that

$G = \oplus_{1\leq i\leq N} \vee\{W^{i-j}\lambda_j H_j: i\leq j\leq (i+p)\wedge N\}$,

$D = \oplus_{1\leq i\leq N-1} \vee\{W^{i-j}\lambda_j H_j: i<j\leq (i+p)\wedge N\}$ and

$R = \oplus_{2\leq i\leq N} \vee\{W^{i-j}\lambda_j H_j: i\leq j\leq (i+p-1)\wedge N\}$.

If (i,j) $\in \mathbb{C}$, $u \in H_j$ and $v \in H_i$, supp $\tau^{i-j}u' = \{(i,j)\}$ and supp $v' = \{(i,i)\}$, so $\langle W^{i-j}\lambda_j u,\lambda_i v\rangle_G = \langle W^{i-j}\varepsilon u',\varepsilon v'\rangle_G = \langle \varepsilon\tau^{i-j}u',\varepsilon v'\rangle_G = [\tau^{i-j}u',v'] = \langle A_{ij}u,v\rangle_{H_i}$. Thus:

$$\lambda_i^* W^{i-j}\lambda_j = A_{ij} \ , \ \forall\ (i,j)\ (i,j) \in \mathbb{C}\ .$$

If $U \in \mathbf{L}(X)$ is a unitary operator such that $X \supset G$ and $U_{|D} = W$, a positive extension A of the band $A^{(p)}$ is given by $A_{ij} = \lambda_{U,i}{}^* U^{i-j}\lambda_{U,j}$, $1\leq i,j\leq N$, with $\lambda_{U,j} = i_G\ \lambda_j$, $1\leq j\leq N$, and i_G the injection of G in X. (In fact, it can be shown that every positive extension of the positive band $A^{(p)}$ is obtained in this way).

Now let $\overline{U} \in \mathbf{L}(X)$ be the minimal unitary dilation of the contraction $WP_D \in \mathbf{L}(G)$. Then, with obvious notation $\overline{A} := [\overline{\lambda}_i{}^*\overline{U}^{i-j}\overline{\lambda}_j]_{1\leq i,j\leq N}$ is a positive extension of $A^{(p)}$.

For $1<i\leq N-p$ set $J_i = \vee\{W^{i-j}\lambda_j H_j: i\leq j\leq i+p\} \ominus \vee\{W^{i-j}\lambda_j H_j: i\leq j<i+p\}$. Then, if $1\leq i<N-p$ and $p < k \leq N-i$,

(a) $\vee\{\overline{U}^{i-j}\overline{\lambda}_j H_j: i\leq j\leq i+k\} =$

$\vee\{W^{i-j}\lambda_j H_j: i\leq j\leq i+p\} \oplus \overline{U}^{-1} J_{i+1} \oplus \dots \oplus \overline{U}^{-(k-p)} J_{i+k-p}$.

In fact, since $G \perp \overline{U}^{-n}[G \ominus R]$, $\forall\ n \geq 1$, and the subspaces J_i are contained in $[G \ominus R]$, the orthogonality relations in (a) hold; thus,

$\vee\{\overline{U}^{i-j}\overline{\lambda}_j H_j\colon i \leq j \leq i+k\} =$

$(\vee\{\overline{U}^{i-j}\overline{\lambda}_j H_j\colon i \leq j \leq i+k-1\}) \vee (\vee\{\overline{U}^{i-j}\overline{\lambda}_j H_j\colon i+k-p \leq j \leq i+k\}) =$

$(\vee\{\overline{U}^{i-j}\overline{\lambda}_j H_j\colon i \leq j \leq i+k-1\}) \vee \overline{U}^{-(k-p)}(\vee\{W^{i+k-p-j}\lambda_j H_j\colon i+k-p \leq j \leq i+k\}) =$

$(\vee\{\overline{U}^{i-j}\overline{\lambda}_j H_j\colon i \leq j \leq i+k-1\}) \vee \overline{U}^{-(k-p)}(\vee\{W^{i+k-p-j}\lambda_j H_j\colon i+k-p \leq j < i+k\} \oplus J_{i+k-p})$

$= (\vee\{\overline{U}^{i-j}\overline{\lambda}_j H_j\colon i \leq j \leq i+k-1\}) \vee \overline{U}^{-(k-p)} J_{i+k-p}$.

Now assume that $A^{(p)}$ is p.d. Let [+] denote the algebraic direct sum. There exists a positive constant c such that $c\|f\|_{\tilde{H}} \leq \|\varepsilon f\|_G$ if $f \in \tilde{H}$, so the operator $\varepsilon \in \mathcal{L}(\tilde{H},G)$ is boundedly invertible. Then the operators λ_j are one to one, $1 \leq j \leq N$, and $\vee\{W^{i-j}\lambda_j H_j\colon i \leq j \leq i+p\} =$ $[+]\{W^{i-j}\lambda_j H_j\colon i \leq j \leq i+p\}$, $1 \leq i \leq N-p$. Thus

(b) $\vee\{\overline{U}^{-j}\overline{\lambda}_j H_j\colon i \leq j \leq i+s\} = [+]\{\overline{U}^{-j}\overline{\lambda}_j H_j\colon i \leq j \leq i+s\}$, $1 \leq i \leq N-s$,

holds for $s = p$; we shall prove it by induction for $s > p$. By (a) we may assume that $\vee\{\overline{U}^{-j}\overline{\lambda}_j H_j\colon i \leq j \leq i+s+1\} = [+]\{\overline{U}^{-j}\overline{\lambda}_j H_j\colon i \leq j \leq i+s\} \oplus \overline{G}_{i+s+1}$, $i+s+1 \leq N$, with $\overline{G}_{t+p} = [+]\{\overline{U}^{-j}\overline{\lambda}_j H_j\colon t \leq j \leq t+p\} \ominus [+]\{\overline{U}^{-j}\overline{\lambda}_j H_j\colon t \leq j < t+p\}$, $1 < t \leq N-p$. Then, any $f \in \vee\{\overline{U}^{-j}\overline{\lambda}_j H_j\colon i \leq j \leq i+s+1\}$ can be written as $f =$ $\sum\{\overline{U}^{-j}\overline{\lambda}_j h_j\colon i \leq j \leq i+s+1\}$ with $h_j \in H_j$; if $f = 0$, then $\overline{U}^{-i-s-1}\overline{\lambda}_{i+s+1} h_{i+s+1} \in [+]\{\overline{U}^{-j}\overline{\lambda}_j H_j\colon i+s+1-p \leq j < i+s+1\}$, so $h_i = 0$ and $0 =$ $f = \sum\{\overline{U}^{-j}\overline{\lambda}_j h_j\colon i+1 \leq j \leq i+s+1\}$, $i+s+1 \leq N$; the induction hypothesis shows that $h_j = 0$, $i \leq j \leq i+s+1$. So (b) holds for any $s < N$, and so does

(c) $\vee\{\overline{U}^{-j}\overline{\lambda}_j H_j\colon i \leq j \leq i+s+1\} = [+]\{\overline{U}^{-j}\overline{\lambda}_j H_j\colon i \leq j \leq i+s\} \oplus \overline{G}_{i+s+1}$, $i+s+1 \leq N$.

Set $\overline{E} = \vee\{\overline{U}^{-j}\overline{\lambda}_j H_j;\ 1 \leq j \leq N\}$; then $\overline{E} = [+]\{\vee\{\overline{U}^{-j}\overline{\lambda}_j H_j\colon 1 \leq j \leq N\}$. Set $H =$ $\oplus(H_j\colon 1 \leq j \leq N)$ and let $B \in \mathcal{L}(H,\overline{E})$ be given by $B = \sum\{\overline{U}^{-j}\overline{\lambda}_j P_{H_j}\colon 1 \leq j \leq N\}$; then $\overline{A} = B^{*}B$ and B is a bijection, so $\overline{A}$ is p.d.

Set $A' = \overline{A}^{-1}$ and $B' = B^{-1}$ so $A' = B'B'^{*}$. Assume $r+p < s$; we must

show that $A'_{rs} = 0$, i.e., that $B'^*H_s \perp B'^*H_r$. Now, $B'^*H_k =$
$\overline{E} \ominus ([+]\{\overline{U}^{-j}\overline{\lambda}_j H_j : 1 \leq j \leq N, j \neq k\})$; (c) shows that $\overline{E} =$
$[+]\{\overline{U}^{-j}\overline{\lambda}_j H_j : 1 \leq j \leq 1+t\} \oplus \overline{G}_{2+t} \oplus \ldots \oplus \overline{G}_N$. If $k-p > r$,
$\overline{G}_k \subset [+]\{\overline{U}^{-j}\overline{\lambda}_j H_j : r < j \leq N\} \subset \overline{E} \ominus B'^*H_r$, so $B'^*H_r \subset \overline{E} \ominus \{\overline{G}_{r+p+1} \oplus \ldots \oplus \overline{G}_N\}$
$= [+]\{\overline{U}^{-j}\overline{\lambda}_j H_j : 1 \leq j \leq r+p\} \subset \overline{E} \ominus B'^*H_s$. The proof is over.

- Concerning the band method see [GKW].

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On some extensions of the commutant lifting theorem: corrections

Rodrigo Arocena

In our paper [A] some mistakes concerning theorem (VIII.1) were included. We hope to correct them in the following, where we keep notations and numerations.

In fact, the mistakes appeared in the derivation of formulas (VI.2) and (VI.3), which should be corrected as follows.

From formula (I.2) we obtain $E_1 \vee (W''_1 M_1) = E_1 \oplus \Lambda(0,1) \oplus W''_1(M_1 \ominus E_1)$,

with $\Lambda(0,1) = \{(W''_1 - T''_1)b : b \in E_1\}^-$.

Assume that $M_1 = E_1 \vee (W''_1 M_1)$; it follows that

$(M_1 \ominus E_1) = \Lambda(0,1) \oplus W''_1(M_1 \ominus E_1)$,

so $W'_1(M_1 \ominus E_1) = W'_1 W''_1(M_1 \ominus E_1) \oplus \{W'_1(W''_1 - T''_1)b : b \in E_1\}^-$, and consequently

$$(R' \ominus R) = \left[\{(I - XP^{M_1}_{E_1})W'_1 a : a \in E_1\}^- \oplus \{W'_1(W''_1 - T''_1)b : b \in E_1\}^-\right]$$

$$\ominus \{(I - XP^{M_1}_{E_1})W'_1 W''_1 c : c \in E_1\}^-.$$

Now, the correspondence given, for any $a \in E_1$ and $b \in E_1$, by

$(I - XP^{M_1}_{E_1})W'_1 a \oplus W'_1(W''_1 - T''_1)b \rightarrow D_{XT'_1} a \oplus D_{T''_1} b$ is an isometry of

$\left[\{(I - XP^{M_1}_{E_1})W'_1 a : a \in E_1\}^- \oplus \{W'_1(W''_1 - T''_1)b : b \in E_1\}^-\right]$ onto $\mathfrak{D}_{XT'_1} \oplus \mathfrak{D}_{T''_1}$ that, for every $c \in E_1$, takes $(I - XP^{M_1}_{E_1})W'_1 W''_1 c = (I - XP^{M_1}_{E_1})W'_1 T''_1 c \oplus W'_1(W''_1 - T''_1)c$ to $D_{XT'_1} T''_1 c \oplus D_{T''_1} c$. Thus:

(VI.2) If $M_1 = E_1 \vee (W''_1 M_1)$,

$$(R' \ominus R) \approx \left[\mathfrak{D}_{XT'_1} \oplus \mathfrak{D}_{T''_1}\right] \ominus \{D_{XT'_1} T''_1 c \oplus D_{T''_1} c : c \in E_1\}^-.$$

From formula (I.3) we obtain $E_2 \vee (\widetilde{W}''_2 \widetilde{M}_2) = \Lambda_*(0,1) \oplus \widetilde{W}''_2 \widetilde{M}_2$,

with $\Lambda_*(0,1) = \{(I - \widetilde{W}''_2 T''_2)a : a \in E_2\}^-$.

Assume that $\widetilde{M}_2 = E_2 \vee (\widetilde{W}''_2 \widetilde{M}_2)$; it follows that

$\widetilde{M}_2 = \{(I - \widetilde{W}''_2 T''_2)a : a \in E_2\}^- \oplus \widetilde{W}''_2 \widetilde{M}_2$, so

$\widetilde{W}'_2 \widetilde{M}_2 = \widetilde{W}'_2 \widetilde{W}''_2 \widetilde{M}_2 \oplus \{\widetilde{W}'_2(I - \widetilde{W}''_2 T''_2)a : a \in E_2\}^-$, and consequently

$$(D' \ominus D) = \left[\{\widetilde{W}'_2(I - \widetilde{W}''_2 T''_2)a : a \in E_2\}^- \oplus \{(I - \widetilde{W}'_2 X T'_1)b : b \in E_1\}^-\right]$$

$\ominus \{(I-\tilde{W}'_2\tilde{W}''_2XT'_1T''_1)c: c\in E_1\}^-$.

The correspondence $\tilde{W}'_2(I-\tilde{W}''_2T''_2)a \oplus (I-\tilde{W}'_2XT'_1)b \to D_{T''_2}a \oplus D_{T'_2X}b$ shows that:

(VI.3) If $\tilde{M}_2 = E_2 \vee (\tilde{W}''_2\tilde{M}_2)$,

$$(D'\ominus D) \approx [\mathbf{D}_{T''_2} \oplus \mathbf{D}_{T'_2X}] \ominus \{D_{T''_2}T'_2Xc \oplus D_{T'_2X}c: c \in E_1\}^-.$$

Now, the same proof as in [A] shows that:

(VIII.1) <u>THEOREM</u> For $j = 1,2$ let T'_j and T''_j be commuting contractions in a Hilbert space E_j, U'_j and U''_j commuting unitary operators in a Hilbert space F_j such that $\{U'^m_j U''^n_j: m,n \in \mathbb{Z}\}$ is a unitary dilation of the semigroup $\{T'^m_j T''^n_j: m,n \geq 0\}$. Let $X \in \mathcal{L}(E_1,E_2)$ be such that $X T'_1 = T'_2 X$ and $X T''_1 = T''_2 X$. Assume that one of the following conditions hold:

i) $M_1 = E_1 \vee [\vee\{U'^m_1 U''^{n+1}_1 E_1: m,n\geq 0\}]$ and $(XT'_1)T''_1$ is a regular factorization of $XT'_1T''_1 = T'_2T''_2X$;

ii) $\tilde{M}_2 = E_2 \vee [\vee\{U'^m_2 U''^{n-1}_2 E_2: m,n\leq 0\}]$ and $T''_2(T'_2X)$ is a regular factorization of $XT'_1T''_1 = T'_2T''_2X$.

Then there exists $\tau \in \mathcal{L}(F_1,F_2)$ such that $\tau U'_1 = U'_2\tau$, $\tau U''_1 = U''_2\tau$, $P^{F_2}_{E_2}\tau_{|E_1} = X$ and $\|\tau\| = \|X\|$.

Note that (i) holds if T''_1 is a unitary operator and that (ii) holds when T''_2 is a unitary operator.

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