PUBLICACIONES MATEMATICAS DEL URUGUAY

VOLUMEN 8

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PUBLICACIONES MATEMÁTICAS DEL URUGUAY
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PUBLICACIONES MATEMÁTICAS DEL URUGUAY

Las Publicaciones Matemáticas del Uruguay constituyen una serie orientada a comunicar nuevos desarrollos en la creación, enseñanza y aplicación de la Matemática. Serán considerados para su publicación en esta serie, trabajos del tipo siguiente:

1. Artículos originales y monográficos, en forma definitiva o preliminar
2. Cursos avanzados que contengan enfoques originales e informes de puesta al día sobre temas de importancia.
3. Avances de investigación
4. Actas de coloquios
5. Resúmenes de resultados o puntos de vista nuevos.

Con estas publicaciones aspiramos a dar cuenta de la labor de los matemáticos que trabajan en el Uruguay, y a estrechar sus vínculos con la comunidad científica internacional. Por ello, solicitamos a todos los que compartan tales objetivos, que nos envíen sus colaboraciones. Proponemos establecer relaciones de canje a todos los responsables de publicaciones científicas que estén en condiciones de concertar acuerdos semejantes.

Consejo Editor.

Publicaciones Matemáticas del Uruguay is a series oriented to new developments in Mathematics. We will consider for publication:

1. Original results in the form of preprints or final versions
2. Lectures on advanced topics and up-to-date reviews on selected topics
3. Proceedings of meetings.

The Editorial Board.

Indice

Gonzalo Pérez Iribarren.

Gonzalo Pérez falleció el 27 de agosto de 1998 en Montevideo. Este texto no es una revisión de su obra como matemático, sino más bien una mezcla de recuerdos personales, entre los que la Matemática está presente como un componente de su personalidad multifacética. Sólo de esa manera creo que se da testimonio veraz de la huella que su vida dejó en nosotros, los que fuimos sus colegas, sus compañeros y amigos.

Quizá no sea posible entender gran parte de sus opciones, si no hacemos un esfuerzo y conseguimos trasladarnos hacia atrás en el tiempo y tratamos de aproximarnos a las condiciones y los valores que flotaban en el ambiente en cada etapa de su vida.

Viene a estudiar a Montevideo en la segunda mitad de los 50, desde Carmelo, donde había crecido. Con una fuerte vocación y talento para la Matemática, comienza estudios en la Facultad de Ingeniería: allí había una pequeña escuela matemática animada por excelentes profesores y el país no ofrecía formaciones científicas profesionales. Docente vocacional desde muy temprano, ingresa por concurso como profesor a la enseñanza media, a los 21 años.

El ambiente juvenil de la época tenía un fuerte tinte filosófico. Estábamos motivados por dilemas universales; la discusión era mucha y en general inflexible e informada. Existía sin duda la política, aunque se trataba de pequeñas minorías, con vocación testimonial y un agudo sentido crítico. En ese Uruguay con la crisis en las entrañas, pero sin embargo satisfecho, Gonzalo participó de los cuestionamientos y de las búsquedas que ya nunca podrían abandonarlo. Recuerdo su grupo de amigos de la Facultad: Germán Casal, José Kreimerman, Carlos Aragone; nombres borrados para casi todos, pero sobre los que él solía proyectar a veces una mirada desde la madurez, reinventando las preguntas.

Después de ese tiempo inicial, impulsado por su fe, abandonó sus estudios y su trabajo de profesor y entró al Seminario Jesuita, primero en Uruguay y luego en Roma.

Sólo puedo atestiguar de ese período en que dejé de verlo - ¿cuánto duró? ¿siete, ocho años? - lo que dejó como profunda huella en su vida: una gran cultura y un rigor incomparable para el abordaje de cada tema, que incluyó naturalmente el dominio de las lenguas latinas vivas, y también del inglés y del alemán, este último a raíz de una estancia larga en Alemania. Y el latín y el griego, que conocía con sutileza literaria y que siempre era capaz de transmitir al lego con su llaneza poco docta, casi de paisano.

Sus años de seminarista coincidieron con una tremenda transformación de la Iglesia Católica. Fueron los años del Papa Roncalli y del Concilio Vaticano II, que anunciaron grandes cambios en la doctrina y también en la aproximación de la Iglesia a la sociedad. Gonzalo vivió ese período en las entrañas del sistema, con la mayor intensidad. Las ideas y emociones de esa época consolidaron su formación espiritual.

Pero no se ordenó sacerdote. Volvió al Uruguay, un país bien distinto al que había dejado unos años antes. Formó una pareja con Beatriz, su compañera y comenzó una vida que le depararía todavía muchas alternativas y riquezas.

De ese período queda un libro de testimonio que escribió sobre la situación social del norte del país. Su retorno a la vida profesional también fue, en cierto modo, un comienzo: es entonces que desarrolla un gran interés por la interacción entre la Matemática y las otras disciplinas científicas, especialmente las ciencias de la vida. Se incorporó al Centro Latinoamericano de Perinatología y desde allí hizo un trabajo que debe considerarse pionero en el Uruguay, que para él fue una constante de su actividad matemática de allí en adelante.

El golpe de Estado de 1973 y luego la intervención de la Universidad llevó a un grupo importante de matemáticos uruguayos - entre los cuales Gonzalo Pérez - a trabajar a la Universidad de Buenos Aires. Pero eso duró poco: a los ocho meses estábamos sin trabajo, perplejos ante las consecuencias inesperadas que las fluctuaciones de la política argentina tenía sobre nuestras vidas.

La mayor parte del grupo se dispersó. Anuncio no declarado de que la dictadura habría de ser larga y determinante para todos y de que los sueños de justicia y de grandes cambios sociales en nuestros países, habían quedado atrás. Con Gonzalo pudimos permanecer un tiempo más en Buenos Aires, haciendo una experiencia intensa en Matemática Aplicada, tanto para empresas públicas como privadas; allí pude aprender cómo la visión de un hombre culto se interna en problemas técnicos complejos, y en qué medida es empobrecedora la partición del conocimiento a la que estamos sometidos.

Gonzalo siempre consideró esa actividad como muy formativa, soñando con que los matemáticos de las generaciones siguientes tuvieran la oportunidad de insertar su saber en las realidades sociales y productivas, como tuvimos la ocasión de hacerlo nosotros entonces, bajo la presión de las circunstancias.

A fines de 1975, Gonzalo Pérez debe emprender nuevamente el camino del exilio, con su familia que ya había empezado a multiplicarse. Poco antes del golpe de Estado de Videla, en marzo de 1976, se instaló en Maracaibo, trabajando como Profesor en la Universidad del Zulia. Completó su formación académica como matemático, que el Uruguay no le había ofrecido: después de su licenciatura en Buenos Aires, hizo su postgrado en Caracas, bajo la dirección de Enrique Cabaña, a quien tuvo la fortuna de reencontrar como orientador científico en el exilio venezolano, como varios de nosotros.

La experiencia "maracucha" del trópico abrumador, inspiraba sus cuentos sorprendidos de sureño trasplantado, en aquel mundo con reglas tan distintas. También le permitió volver a viejos amores: a pintar, como lo había hecho intensamente en su juventud, y ya no dejaría de hacerlo desde entonces. También las amarguras, la muerte de un pequeño hijo, a raíz de una enfermedad fulminante.

Pero no era de allí. En 1983, rehizo el equipaje de su ya numerosa familia viajera y volvió al terruño coloniense, a vivir como se pudiera, pero donde se quería. Sin trabajo regular, se convirtió en profesor particular de una pequeña ciudad del interior, esperando. Durante todo el año 1984 viajaba regularmente a Montevideo, donde formaba parte del grupo de voluntarios que discutía la futura reorganización científica del país y la fundación del PEDECIBA, una actividad cuasi peligrosa para quien tenía sus antecedentes. Al mismo tiempo, recuperó las multiplicidades que lo hacían vivir, recuperó su condición de escritor y fue premiado por su trabajo literario en el "tournant" de la dictadura.

Con la primera fase del restablecimiento de la democracia, se instala en Montevideo, y con la colaboración de Walter Ferrer y de los colegas que van regresando al país en los meses y años siguientes, se propone reconstruir la Matemática uruguaya, devastada por el régimen militar: presos, exilados, jóvenes talentos en el exterior, liquidación de las bibliotecas, cada capitulo requiere dedicación, imaginación, tenacidad. Lo hace con la misma energía que había puesto en cada uno de sus proyectos anteriores; primero en la Facultad de Ingeniería, luego en la de Humanidades y Ciencias y finalmente en la Facultad de Ciencias, cuyo Consejo integró entre 1993 y 1997, como primer titular de la representación de los docentes.

Simultáneamente, vuelve a su tarea académica. Es un profesor ejemplar. Su investigación matemática se centra en los estudios de Estadística No-paramétrica (Estimación funcional, Métodos de Entropía Máxima). Con frecuencia, aunque no siempre, sus estudios de Matemática aparecen motivados por interrogantes que nacen de aplicaciones no triviales a problemas físicos, a los que vuelve a dedicar una parte importante de sus energías; es el caso de sus investigaciones sobre las Cadenas de Markov escondidas, motivadas por sus estudios para modelizar el clima, tema en el que sus aportes fueron objeto de publicación en revistas importantes. Desde otro punto de vista, promovió y fue principal ejecutor de programas de interacción academia-industria, tanto en su actividad personal como en su dirección del Laboratorio de Estadística del Centro de Matemática. Siempre inflexible en cuanto al nivel, siempre riguroso en el estudio y en la crítica.

En una época que no se distingue por el triunfo de la claridad de las posiciones de cada uno, Gonzalo supo exponer sus puntos de vista sin filigrana, reduciendo con frecuencia las verdaderas opciones a imperativos éticos sencillos; su tarea colectiva fue un ejemplo de coherencia entre los dichos y las prácticas. Aplicó su singular inteligencia a mejorar la vida en común, sin pedir nada para sí mismo. Esa es la lección de su vida y los que tuvimos la suerte de tenerla cerca, con el pesar de su ausencia, seguiremos aprendiendo de ella.

Mario Wschebor

Álgebras de Auslander Casi Inclinadas.

Marcelo Lanzilotta *

Resumen

Un álgebra de artin Λ sobre un anillo de artin conmutativo R es casi inclinada si $dgl(\Lambda) \leq 2$, y para todo módulo finitamente generado e indescomponible M se verifica que $dp(M) \leq 1$ o $di(M) \leq 1$. Un álgebra de Δ sobre un anillo de artin conmutativo R es un álgebra de Auslander si $dgl(\Delta) \leq 2$ y $dim.dom(\Delta) \geq 2$. La idea de este trabajo es clasificar todas las álgebras casi inclinadas que son además álgebras de Auslander.

An artin algebra Λ, over a conmutative ring R, is quasitilted when $gldim(\Lambda) \leq 2$, and for each indecomposable finitely generated Λ-module M we have $pd(M) \leq 1$ or $id(M) \leq 1$. An artin algebra Λ over a conmutative artin ring R is an Auslander algebra if $gldim(\Lambda) \leq 2$ and $dom.dim(\Lambda) \geq 2$. On this work we classify the familly of Auslander algebras who also are quasitilted.

1 Preliminares.

La clase de álgebras casi inclinadas, fue introducida en [HRS1] y en [HR], con la idea de encontrar una teoría de inclinación general para categorías abelianas. Específicamente, un álgebra casi inclinada es un álgebra de endomorfismos $\mathrm{End}_{\mathcal{H}}(T)$, donde H es una categoría abeliana y T es un objeto inclinante (o sea $\mathrm{Ext}^1_{\mathcal{H}}(T,T) = 0$; y si $\mathrm{Ext}^1_{\mathcal{H}}(T,X) = 0$, $\mathrm{Hom}_{\mathcal{H}}(T,X) = 0$, entonces $X = 0$).Esta definición es posiblemente de difícil manejo, pero afortunadamente existe la

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siguiente caracterización, demostrada también por Happel, Reiten y Smalø. Un álgebra Λ es **casi inclinada** si y sólo si se cumplen las dos siguientes propiedades: (ci_1) la dimensión global de Λ ($dgl(\Lambda)$) es menor o igual que dos; (ci_2) todo módulo indescomponible sobre Λ tiene dimensión proyectiva o dimensión inyectiva menor o igual a uno.

La idea de este trabajo es clasificar todas las álgebras casi inclinadas que son además álgebras de Auslander.

Definición 1.1

1. *Sea Λ un álgebra de artin. La* **dimensión dominante** *de un módulo $N \in \text{mod}\,\Lambda$ (que notaremos por $dim.dom_\Lambda(N)$), es el mayor entero t o ∞, tal que si consideramos la resolución inyectiva minimal de N:*

$$0 \to N \to I_0 \to I_1 \to \ldots \to I_t \to \ldots$$

tenemos que todos los módulos I_j con $j = 0, \ldots, t-1$ (o $j = 0, \ldots, \infty$), son proyectivos.

2. *Una álgebra de artin Λ, es una* **álgebra de Auslander** *si:*
   (a) $dgl(\Lambda) \leq 2$;
   (b) $dim.dom_\Lambda(\Lambda) = 2$.

3. *Sea A un álgebra de artin. Un módulo M es un* **generador aditivo** *en $\text{mod}\,A$ si todo módulo indescomponible es sumando de M (o sea si $add(M) = \text{mod}\,A$).*

Tomando como referencia [ARS], citamos el siguiente resultado:

Proposición:

Sea Λ un álgebra de artin que satisface $dgl(\Lambda) = 2$ y $dim.dom_\Lambda(\Lambda) = 2$. Sea $Q \in \text{mod}\,\Lambda$ tal que $add(Q) = \{\ B \in \text{mod}\,\Lambda\ /\ B$ es proyectivo e inyectivo $\}$. Entonces:

1. El álgebra $A = \text{End}_\Lambda(Q)^{op}$ es de tipo de representación finito.
2. $\Lambda \cong \text{End}_A(M)^{op}$, donde M es un módulo generador aditivo en $\text{mod}\,A$. ■

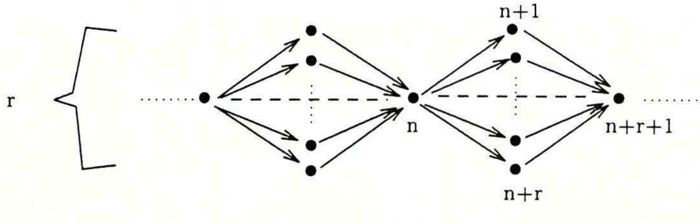
Notaremos $\Lambda = \Lambda(A)$ en las condiciones de la proposición anterior. También necesitamos mencionar el siguiente resultado ([ARS]):

Proposición:
El grafo subyacente al carcaj de Auslander-Reiten Γ_A de un álgebra de artin A con tipo de representación finito coincide con el carcaj ordinario asociado al álgebra de Auslander $\Lambda(A)$.■

2 Clasificación de las álgebras de Auslander casi inclinadas.

Lema 2.1
Sea $\Lambda(A)$ un álgebra de Auslander casi inclinada. Entonces todo módulo indescomponible $M \in ind(A)$, es inyectivo o proyectivo.

Demostración
Probaremos que $DTr^2(M) = 0$, para todo módulo $M \in \text{ind}(A)$. Supongamos que existe $M \in \text{ind}(A)$ tal que $DTr^2(M) \neq 0$. Sea n el vértice del carcaj ordinario de $\Lambda(A)$, asociado al módulo indescomponible $DTr(M)$ en Γ_A. Considero el módulo simple $S_n \in \text{ind}(\Lambda(A))$, y también la situación local en el vértice n del carcaj ordinario de $\Lambda(A)$ (determinada por las sucesiones de Auslander-Reiten que tienen al módulo M como pozo y fuente en cada caso). O sea:



Podemos concluir que $dp(S_n) = 2$, ya que la sucesión

$$0 \to P_{n+r+1} \to \bigoplus_{i=1}^{i=r} P_{n+i} \to P_n \to S_n \to 0$$

es la resolución proyectiva de S_n.
Análogamente podríamos probar que $di(S_n) = 2$. Portanto $\Lambda(A)$ no sería casi inclinada, lo que conduce a una contradicción. Probamos entonces que $DTr^2(M) = 0$, para todo módulo indescomponible M. Esto implica que todo módulo indescomponible es proyectivo o inyectivo, pues si tuviésemos $M \in \text{ind}(A)$ no proyectivo ni inyectivo, entonces $DTr^2(TrD(M)) \neq 0$. ∎

Observación 2.1
Sea Γ_A el carcaj de Auslander-Reiten de un álgebra de artin A. Entonces no existe una flecha saliendo de un módulo inyectivo indescomponible I y llegando a un módulo proyectivo indescomponible P, pues en el caso contrario tendríamos un morfismo irrreductible $I \overset{irr}{\to} P$, lo que sería absurdo porque inevitablemente se partiría. ∎

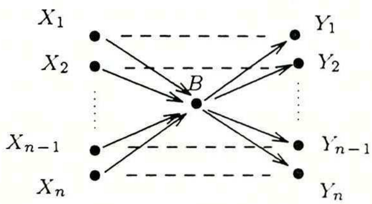
Lema 2.2
Sea A un álgebra de artin de tipo de representación finito, donde todo módulo indescomponible es inyectivo o proyectivo. Supongamos que existe un módulo biyectivo (o sea proyectivo e inyectivo) indescomponible no simple $B \in ind(A)$. Entonces el carcaj de Auslander-Reiten de A es:

$$\Gamma_A : \quad \begin{tikzcd} X \arrow[dr] \arrow[rr, dashed, no head] & & Y \\ & B \arrow[ur] & \end{tikzcd}$$

Demostración
Sea B el módulo biyectivo indescomponible no simple. Localmente tenemos:

$$\begin{tikzcd}[row sep=small]
X_1 \arrow[ddr] & & Y_1 \\
X_2 \arrow[dr] & & Y_2 \\
\vdots & B \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & \vdots \\
X_{n-1} \arrow[ur] & & Y_{m-1} \\
X_n \arrow[uur] & & Y_m
\end{tikzcd}$$

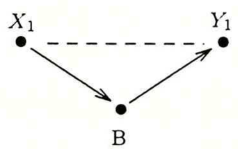
donde los módulos X_i con $i = 1, \ldots, n$, y los Y_j con $j = 1, \ldots, m$, son indescomponibles, $B \to \bigoplus_{j=1}^{j=m} Y_j$ es una fuente, y $\bigoplus_{i=1}^{i=n} X_i \to B$ es un pozo. Por la Observación 2.1, podemos concluir que los módulos X_i con $i = 1, \ldots, n$, no son inyectivos y por lo tanto son proyectivos, y los módulos Y_j con $j = 1, \ldots, m$, no son proyectivos y por lo tanto son inyectivos. Luego existe $DTr(Y_j)$, $\forall\, j = 1, \ldots, m$, y existe $TrD(X_i)$, $\forall\, i = 1, \ldots, n$. Por lo tanto se puede concluir que $n = m$ y reordenando si es necesario podemos asumir que $DTr(Y_j) = X_j$, $\forall\, j = 1, \ldots, m$. Entonces localmente la situación es:



Notaremos $y_j = \ell(Y_j)$ y $x_j = \ell(X_j)$, $\forall\, j = 1, \ldots, m$, y $b = \ell(B)$.

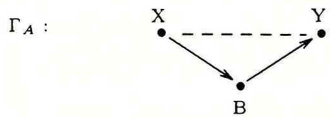
Así tenemos las siguientes informaciones: $\begin{cases} 1. \quad b - x_j \leq y_j, \ \forall\, j = 1, \ldots, m; \\ 2. \quad \displaystyle\sum_{j=1}^{j=m} y_j = b - 1; \\ 3. \quad \displaystyle\sum_{j=1}^{j=m} x_j = b - 1. \end{cases}$

Luego $m.b - (b-1) \leq b - 1$, y por lo tanto $(m-2).b \leq -2$. Las únicas posibilidades son $m = 0$ o $m = 1$.
Si fuese $m = 0$, tendríamos que el módulo B sería simple, lo que contradice nuestra hipótesis. Luego $m = 1$, y por lo tanto localmente tenemos:



donde X_1 es un módulo proyectivo indescomponible, siendo $TrD(X_1) = Y_1$ un módulo inyectivo indescomponible. Luego $x_1 = b - 1 = y_1$.

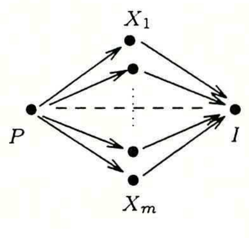
Afirmación: No existen más flechas saliendo de X_1 en Γ_A.
Supongamos que existe $X_1 \to M$ en Γ_A, siendo M un módulo indescomponible no isomorfo a B. Por la hipótesis, M es inyectivo o proyectivo.
Si M es proyectivo, tenemos que $\ell(M) \geq x_1 + 1 = b$, y si M es inyectivo, tenemos que $\ell(M) \geq y_1 + 1 = b$. En ambos casos $\ell(M) + b > y_1 + x_1$, lo que también es absurdo. Así queda probada la afirmación. Consecuentemente podemos concluir también que no existen más flechas llegando en Y_1. Luego $x_1 + y_1 = b$, mas $x_1 = b - 1 = y_1$, por lo tanto $x_1 = y_1 = 1$ y $b = 2$. Entonces X_1 es un módulo proyectivo simple, e Y_1 es un módulo inyectivo simple. El álgebra A es de tipo de representación finito, por lo tanto Γ_A es conexo. Luego concluimos que:



■

Lema 2.3

Sea Γ_A el carcaj de Auslander-Reiten del álgebra A, en la cual tenemos localmente la siguiente situación:



siendo que los módulos X_i, con $i = 1, \ldots, m$, son indescomponibles, P es un módulo proyectivo indescomponible e I es un módulo inyectivo indescomponible. Si A es un álgebra de artin de tipo de representación finito donde todo módulo indescomponible es inyectivo o proyectivo, entonces:

1. *Los módulos X_i, con $i = 1, \ldots, m$, son todos proyectivos o todos inyectivos.*

2. *Se verifica que $m \leq 2$.*

Demostración

1. Supongamos que $m \geq 2$, y que existe $1 \leq k \leq m$ tal que el módulo X_k es proyectivo y existe $1 \leq j \leq m$ tal que X_j es inyectivo. Luego si $x_k = \ell(X_k)$, $x_j = \ell(X_j)$, $p = \ell(P)$, y $i = \ell(I)$, tenemos que $x_k > p$ y $x_j > i$. Entonces $\sum_{h=1}^{h=m} x_h > p + i$, lo que es una contradicción. Mas por la hipótesis, todo módulo indescomponible es proyectivo o inyectivo, por lo tanto los módulos X_i, con $i = 1, \ldots, m$, son todos proyectivos (y no inyectivos si $m \geq 2$), o son todos inyectivos (y no proyectivos si $m \geq 2$).

2. Por lo probado en el punto anterior, los módulos X_h, con $h = 1, \ldots, m$, son todos proyectivos o todos inyectivos. Supongamos que son todos proyectivos. Supongamos también que $m \geq 3$. Entonces por lo visto en el punto anterior, los módulos X_h, con $h = 1, \ldots, m$, no son inyectivos. Luego existe $Y_h = TrD(X_h)$, $\forall h = 1, \ldots, m$. Notamos por $y_h = \ell(Y_h)$, y mantenemos el resto de las notaciones del punto anterior. Entonces tenemos que $y_h \geq i - x_h = (\sum_{j=1}^{j=m} x_j - p) - x_h$, $\forall h = 1, \ldots, m$. Luego obtenemos las siguientes desigualdades:

$$\sum_{h=1}^{h=m} y_h - i \geq \sum_{h=1}^{h=m}\sum_{j=1}^{j=m} x_j - mp - \sum_{h=1}^{h=m} x_h - i = (m-1)\sum_{j=1}^{j=m} x_j - mp - i =$$

$$(m-1)\sum_{j=1}^{j=m} x_j - mp - (\sum_{j=1}^{j=m} x_j - p) \geq (m-2)\sum_{j=1}^{j=m}(p+1) - (m-1)p >$$
$$[m(m-2) - (m-1)]p > 0,$$

si $m \geq 3$. Entonces $\sum_{h=1}^{h=m} y_h - i > 0$, si $m \geq 3$, lo que implicaría que existe el módulo $TrD(I)$, que es una contradicción. Concluimos que $0 \leq m \leq 2$. ∎

Lema 2.4
Sea A un álgebra de artin de tipo de representación finito donde todo módulo

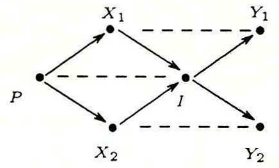
indescomponible es inyectivo o proyectivo, y supongamos que tenemos localmente la siguiente situación en el carcaj de Auslander-Reiten Γ_A:

$$\begin{tikzcd} & X_1 \arrow[dr] & \\ P \arrow[ur] \arrow[dr] \arrow[rr, dashed, no head] & & I \\ & X_2 \arrow[ur] & \end{tikzcd}$$

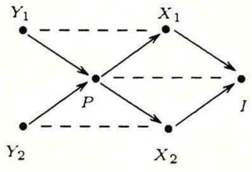
1. *Si los módulos X_1 y X_2 son proyectivos, entonces se verifica que:*
   - *(a) El módulo P es proyectivo simple;*
   - *(b) Existen los módulos $Y_1 = TrD(X_1)$ e $Y_2 = TrD(X_2)$ que son inyectivos (y no proyectivos);*
   - *(c) Los módulos Y_1 e Y_2 son los únicos sucesores del módulo I en Γ_A.*
2. *Si los módulos X_1 y X_2 son inyectivos, entonces se verifica que:*
   - *(a) El módulo I es inyectivo simple;*
   - *(b) Existen los módulos $Y_1 = DTr(X_1)$ e $Y_2 = DTr(X_2)$ que son proyectivos (y no inyectivos);*
   - *(c) Los módulos Y_1 e Y_2 son los únicos predecesores del módulo P en Γ_A.*

Demostración

1. Si existiese una flecha llegando al módulo P en el carcaj Γ_A, debería ser de la forma $I \to P$, lo que es absurdo según la Observación 2.1. Luego P no tiene predecesores en Γ_A, y por lo tanto P es un módulo simple. Por otro lado, según el Lema 2.3, como los módulos X_i, $i = 1, 2$, son proyectivos, entonces no son inyectivos. Luego existen $TrD(X_i) = Y_i$, con $i = 1, 2$. Finalmente no es difícil de ver que no existen más flechas saliendo de I en Γ_A, pues serían del tipo $I \to Q$, donde Q es un módulo proyectivo indescomponible, lo que es absurdo según la Observação 2.1. Luego localmente la situación es:



2. La demostración es análoga, y localmente la situación sería:

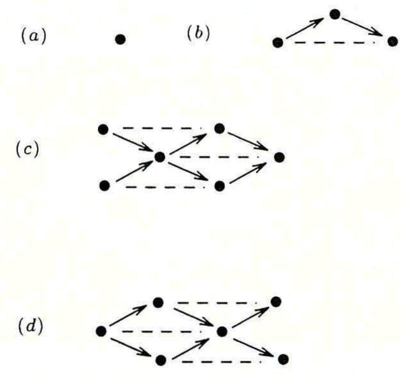


■

Teorema 2.1
Sea Λ un álgebra de artin indescomponible. Las siguientes afirmaciones son equivalentes:

1. *Todo módulo indescomponible es proyectivo o inyectivo.*

2. *El carcaj de Auslander-Reiten Γ_A tiene alguna de las siguientes formas:*



3. *$\Lambda(A)$ es un álgebra de Auslander inclinada.*

4. *$\Lambda(A)$ es un álgebra de Auslander casi inclinada.*

Demostración

(1) $\Longrightarrow$ (2)

Observemos que si el álgebra Λ es indescomponible y de tipo de representación finito, entonces Γ_A es conexa. Esta es la situación aquí ya que todo módulo es proyectivo o inyectivo, y por lo tanto A es de tipo de representación finito. Supongamos que existe un módulo biyectivo $B \in \text{ind}(A)$. Si el módulo B es simple, entonces

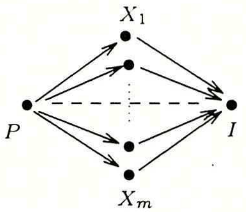
$$\Gamma_A : \overset{B}{\bullet}$$

pues el carcaj es conexo. Si el módulo B no es simple, por el Lema 2.2, tenemos que

$$\Gamma_A : \begin{tikzcd} \overset{X}{\bullet} \arrow[dr] \arrow[rr, dashed, no head] & & \overset{Y}{\bullet} \\ & \underset{B}{\bullet} \arrow[ur] & \end{tikzcd}$$

Sólo resta analizar el caso en que no existe un módulo biyectivo indescomponible.

Sea P un módulo proyectivo no inyectivo. Entonces existe el módulo indescomponible $TrD(P)$, que no es proyectivo, y por lo tanto, por hipótesis es inyectivo. Notaremos $TrD(P) = I$. Así localmente tenemos la siguiente situación:



Entonces por el Lema 2.3, $m \leq 2$ y los módulos X_1 y X_2 son proyectivos o inyectivos. Notaremos por $x_i = \ell(X_i)$, con $1 \leq i \leq m$, $p = \ell(P)$ e $i = \ell(I)$. Supongamos que $m = 2$, y que X_1 y X_2 son proyectivos (y por lo tanto no inyectivos). Entonces localmente la situación es:

$$\begin{tikzcd}
& X_1 \arrow[dr] \arrow[rr, dash, dashed] & & Y_1 \\
P \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & I \arrow[ur] \arrow[dr] & \\
& X_2 \arrow[ur] \arrow[rr, dash, dashed] & & Y_2
\end{tikzcd}$$

donde $Y_1 = TrD(X_1)$ e $Y_2 = TrD(X_2)$ son módulos inyectivos. Luego por el Lema 2.4, el módulo P es simple y los sucesores de I son los módulos Y_1 e Y_2.
Afirmación:

1. P es el único predecesor de X_1 y X_2 en Γ_A.

2. I es el único sucesor inmediato de X_1 y X_2 en Γ_A.

3. I es el único predecesor inmediato de Y_1 e Y_2 en Γ_A.

4. Y_1 e Y_2 no tienen sucesores en Γ_A.

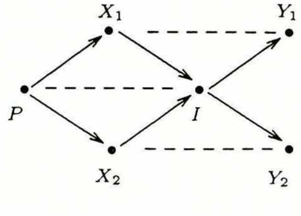
Por el Lema 2.4, los únicos sucesores del módulo I son Y_1 e Y_2. Entonces si $y_1 = \ell(Y_1)$ e $y_2 = \ell(Y_2)$, tenemos que $y_1 + y_2 = i - 1$. Como P es simple, tenemos que $p = 1$, y por lo tanto $i = x_1 + x_2 - 1$. Consideramos las sucesiones de Auslander-Reiten:

$$\xi : \ 0 \to X_1 \to E_1 \to Y_1 \to 0, \text{ donde } I \mid E_1 \text{ y,}$$

$$\zeta : \ 0 \to X_2 \to E_2 \to Y_2 \to 0, \text{ donde } I \mid E_2.$$

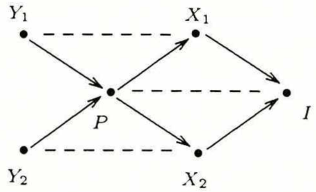
Entonces $y_h \geq i - x_h$, con $h = 1, 2$. Por lo tanto $i - 1 = y_1 + y_2 \geq 2i - x_1 - x_2 = 2i - (i + 1) = i - 1$. Concluimos que $y_1 = i - x_1$ e $y_2 = i - x_2$. Luego $I = E_1$, e $I = E_2$. Así tenemos probado que el único sucesor inmediato de los módulos X_1 y X_2 en Γ_A así como el único predecesor inmediato de los módulos Y_1 e Y_2 es el módulo I. Por hipótesis, todo módulo indescomponible debe ser inyectivo o proyectivo, por lo tanto, sigue de la Observación 2.1 que los módulos Y_1 e Y_2 no tienen sucesores y que el único predecesor de los módulos X_1 y X_2 es P. Entonces tenemos todos los elementos para concluir que el carcaj de Auslander-Reiten del álgebra A es:

Γ_A :

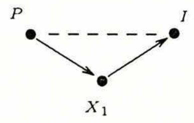


Siendo $m = 2$ resta por analizar el caso en que los módulos X_1 y X_2 son inyectivos. Haciendo un trabajo análogo concluiríamos que el carcaj de Auslander-Reiten de A es:

Γ_A :



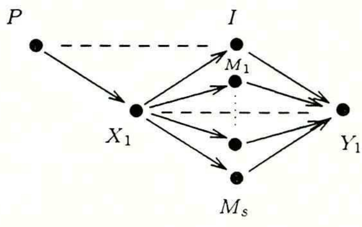
Resta por analizar el caso $m = 1$. Localmente tendríamos:



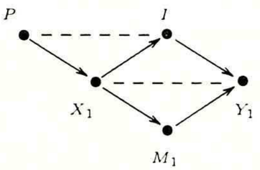
Adoptando las notaciones anteriores, tendríamos que $x_1 = p+i$. El módulo indescomponible X_1 tiene que ser, según la hipótesis, inyectivo (y no proyectivo) o proyectivo (y no inyectivo). Supongamos que X_1 es proyectivo y no inyectivo. Entonces existen $Y_1 = TrD(X_1)$, y un módulo $M = \bigoplus_{h=1}^{h=s} M_h$ con $\ell(M) = y_1 + (x_1 - i) > 0$, tales que

$$0 \to X_1 \to I \oplus M \to Y_1 \to 0$$

es la sucesión de Auslander-Reiten para el módulo X_1. Localmente tenemos la situación:

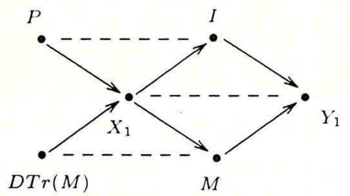


donde Y_1 es un módulo no proyectivo y por lo tanto es inyectivo. Estamos en las hipótesis del Lema 2.3, del cual deducimos que $s = 1$ y que M_1 es inyectivo. Entonces la situación localmente es:



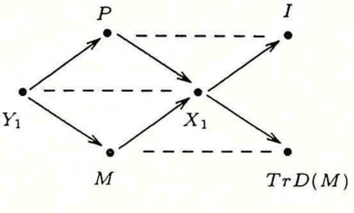
donde el módulo X_1 es proyectivo, y el módulo Y_1 es inyectivo. Por lo tanto podemos notar que estamos en la misma situación local que en el caso $m = 2$ con los módulos del término central de la sucesión de Auslander-Reiten inyectivos. Entonces concluimos de la misma manera que el carcaj de Auslander-Reiten de A tiene la siguiente forma:

Γ_A :

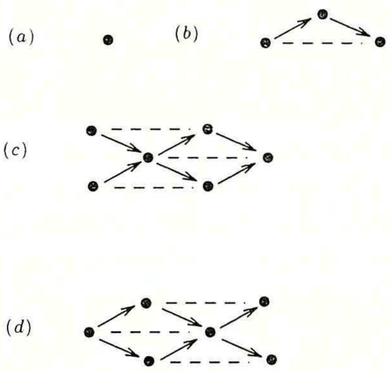


Sólo resta considerar el caso en que el módulo X_1 es inyectivo y no proyectivo. Haciendo un análisis análogo concluiríamos que el carcaj de Auslander-Reiten de A tiene la seguinte forma:

Γ_A :

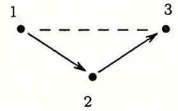


donde Y_1, P y M son módulos proyectivos, e I, X_1, y $TrD(M)$ son módulos inyectivos.
Queda probado así la primera implicación.
(2) $\Longrightarrow$ (3)
Tenemos un álgebra de artin A tal que su carcaj de Auslander-Reiten Γ_A tiene alguna de las siguientes formas:

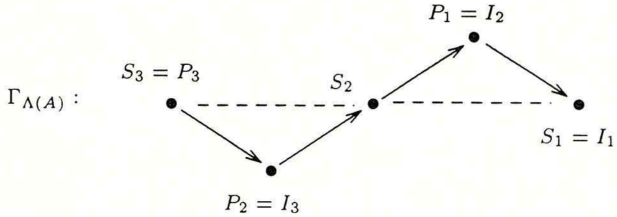


Consideramos el álgebra de Auslander $\Lambda(A)$, cuyo carcaj ordinario Δ coincide con el carcaj subyacente a Γ_A. Evidentemente $\Lambda(A)$ es indescomponible. Vamos a probar que el álgebra $\Lambda(A)$ es inclinada.
En el caso en que $\Delta : \overset{1}{\bullet}$, tendríamos que $\Lambda(A) \cong R\Delta \cong R$, y por lo tanto $\Lambda(A)$ es inclinada.
Consideremos el caso en que:

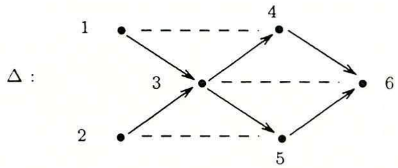
Δ :



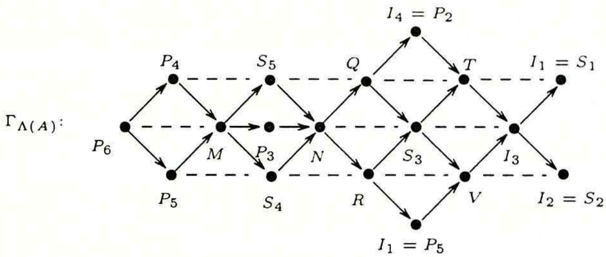
Calculamos el carcaj de Auslander-Reiten de $\Lambda(A)$:



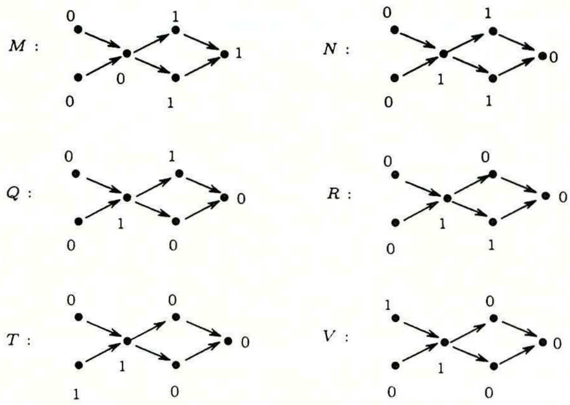
Podemos concluir que $\Lambda(A)$ es inclinada, ya que la clase $\mathcal{S} = \{\ P_2,\ S_2,\ P_1\ \}$ es una sección completa ([Ass]).
Consideramos el caso en que:



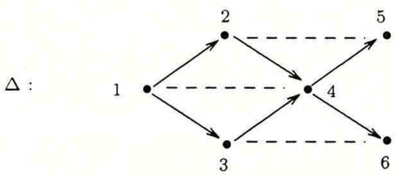
Calculamos el carcaj de Auslander-Reiten de $\Lambda(A)$ y obtenemos:



donde M, N, Q, R, T, V son los siguientes módulos indescomponibles:



no es difícil de ver que $\mathcal{S} = \{ P_3, P_5, R, N, Q, P_2 \}$ es una sección completa. Luego concluimos, que el álgebra $\Lambda(A)$ es un álgebra inclinada. Finalmente restaría el caso en que:



Haciendo un análisis análogo al hecho en el caso anterior, se prueba también aquí que $\Lambda(A)$ es un álgebra inclinada.
(3) $\Rightarrow$ (4)
Es un resultado conocido que toda álgebra inclinada es casi inclinada ([Ass]).
(4) $\Rightarrow$ (1)
El resultado ya fue probado en el Lema 2.1. ■

Bibliografía.

[ARS] M. Auslander, I. Reiten, S. Smalø, *Representation theory of artin algebras*, Cambridge Studies in Advanced Mathematics **36**, Cambridge University Press, (1995).

[Ass] I. Assem, *Tilting theory - an introduction*, Topics in Algebra, Banach Center Publications, Vol. **26** PWN, Warsaw (1990).

[HRS1] D. Happel, I. Reiten, S. Smalø, *Tilting in abelian categories and quasitilted algebras*, Memoirs Amer. Math. Society, **575**, (1996).

[HR] D. Happel, I. Reiten. *An introduction in quaititlted algebras*, preprint, Norwegian University of Science and Technology, Norway.

Stable and unstable sets in a C^o open and dense set of Diffeomorphisms*

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Abstract

On a C^o open dense set of diffeomorphisms, local stable and unstable sets contain connected pieces which have non-trivial homology groups of complementary dimensions.

Resumen

En un C^o abierto y denso de difeomorfismos, los conjuntos estables e inestables locales contienen piezas conexas que tienen grupos de homología no trivial en dimensiones complementarias.

1 Introduction

This work is concerned mainly with a geometrical description of local stable and unstable sets of homeomorphisms on n-manifolds in terms of homology. Our result is that for a wide class of homeomorphisms, local stable and unstable sets satisfy a weak kind of transversality, namely:

Main Result *Given $\varepsilon > 0$, for f in a C^o open and dense set of $\overline{Diff^1(M^n)}$,*

*The final version of this paper will be submitted for publication elsewhere

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there exists $\lambda > 0$ such that if x is in $\Omega(f)$, then the ε-local stable and unstable sets of x contain non-trivial compact connected pieces $\mathbf{C}^f_\varepsilon(x)$ and $\mathbf{D}^f_\varepsilon(x)$ with diam($\mathbf{C}^f_\varepsilon(x)$), diam($\mathbf{D}^f_\varepsilon(x)$)$> \lambda$, which verify $H^{s+1}(B_\lambda(x); \mathbf{C}^f_\varepsilon(x)) \supset \mathbf{Z}_2$ and $H^{u+1}(B_\lambda(x); \mathbf{D}^f_\varepsilon(x)) \supset \mathbf{Z}_2$, where $s + u = n$

Here H^q stands for the Alexander q - cohomology group over $\mathbf{Z}_2$. It might happen, however, that no such a piece contain x, as it is seen in §3. In any event, we have the following

Proposition *For all $m > 1$ there is obtained a C^o open dense set $\mathcal{U}_m$ such that if f, x, λ_f are in the conditions of the statement above, then the orbit of x is λ/m-traced by the orbit of a point which is in $\mathbf{C}^f_\varepsilon(x) \cap \mathbf{D}^f_\varepsilon(x)$(i.e. $d(f^n(x), f^n(y)) < \lambda/m \quad \forall n \in \mathbf{Z}$).*
Moreover $\mathbf{D}^f_\varepsilon(x) \setminus B_{\lambda/m}(x) \not\sim 0$ in $H_{u-1}(B_\lambda(x) \setminus \mathbf{C}^f_\varepsilon(x))$, and also $\mathbf{C}^f_\varepsilon(x) \setminus B_{\lambda/m}(x) \not\sim 0$ in $H_{s-1}(B_\lambda \setminus \mathbf{D}^f_\varepsilon(x))$

In particular, when $s = 1$, the local unstable set of x contains a connected piece $\mathbf{D}^f_\varepsilon(x)$ that separates the ball $B_\lambda(x)$, while the piece $\mathbf{C}^f_\varepsilon(x)$ in the local stable set of x joins two well separated connected component of $\partial B_\lambda(x) \setminus \mathbf{D}^f_\varepsilon(x)$. It should be pointed out that the *ε-local stable set* of $x \in M$ with respect to an homeomorphism $f : M \to M$ is defined here to be the set

$$S^f_\varepsilon(x) = \{y \in M : d(f^n(x), f^n(y)) \le \varepsilon \quad \forall n \ge 0\}$$

The *ε-local unstable set* $U^f_\varepsilon(x)$ is defined analogously, replacing condition $n \ge 0$ by $n \le 0$.

We begin by considering an Axiom A diffeomorphism g satisfying the strong transversality condition (STC). *Topological stability* of this kind of diffeomorphisms yields a continuous surjection $h_f : M \to M$, C^o close to the identity map for f C^o close to g, which verifies $g \circ h_f = h_f \circ f$ ([N2, H]). In particular, points in the non wandering set of such an f will closely follow points in the non wandering set of g. We first restrict our attention to points x in the non wandering set $\Omega(f)$ which follow points in some *basic* set Λ of g. Roughly speaking, what we show is that there are connected components in the local stable and unstable sets of such x which attain the "dimension" of the stable and unstable manifolds of Λ. Further, this is achieved in such a way, that the intersection points of these pieces(which are also seen to exist) trace the orbit of x at a distance substantially smaller than their diameter. In this way there is obtained a C^o neighborhood $\mathcal{N}(g, \Lambda, m)$ of g, where the intersection of the pieces $\mathbf{C}^f_\varepsilon(x)$ and $\mathbf{D}^f_\varepsilon(x)$ traces the orbit of x at a distance bounded by $1/m \inf(diam(\mathbf{C}^f_\varepsilon(x)), diam(\mathbf{D}^f_\varepsilon(x)))$. A finite intersection of these neighborhoods provides an open set $\mathcal{N}(g, m)$ in the C^o topology for which *all* points in $\Omega(f)$ have the desired property, so long as f is a diffeomorphism. Combining this with the fact that Axiom A diffeomorphisms satisfying the STC are C^o dense in $Diff^1(M)$ ([S1]), gives the proof of our result. (Observe that the diameter λ depends on g). We next see that the result can be extended over a C^o open dense set in the closure of $Diff^1(M)$. We remark that this closure equals the set of homeomorphisms when $\dim M \leq 3$; however, this is no longer true for higher dimensions.

We shall proceed in this way: following, basic definitions are given. Section §2 is devoted to proving the main result. The proof is organized in three parts. The first part treats the case of codimension one, which contains the essential ideas of the general case, but in a simplified setting. Since it allows a better geometrical visualization, we thought it might be of interest to include it here. The details to complete the general case are supplied in the third part, while the second is devoted to extending the result over homeomorphisms in the C^o closure of $Diff^1(M)$. The proof makes strong use of a dynamic non sub-additive "metric" which satisfies a condition weaker than, though resembling hyperbolicity. Existence and description of this rather technical tool are postponed until the last section. Finally, section §3 contains examples of local stable and unstable sets exhibiting pathological behavior. We further see that they may be found arbitrarily close to an Anosov diffeomorphism.

I would like to thank J. Lewowicz for giving the seminal idea of this paper, and also R. Ures for useful suggestions. I am specially indebted to J.L. Vieitez for carefully reading earlier versions and making many important remarks.

Let (M, d) be a compact Riemannian n-manifold. For any homeomorphism $f : M \to M$, the set $M \times \mathbf{R}$, where we identify $(x, 1)$ with $(f(x), 0)$ will be written as $\hat{M}^f$. On this set, f induces a flow, which we call the *suspension flow of f*, and is defined by $f^t(x) = (f^{[t]}(x), t - [t])$, where $[t]$ denotes the integral part of t. Sometimes, we will write M_t^f for the projection of $M \times \{t\}$ to $\hat{M}^f$. We shall identify M_0^f with M, and will omit the superscript in $\hat{M}^f$, where there is no likehood of confusion.

Definition 1.1 *Given an homeomorphism $f : M \to M$ we will say that V_f is a* **Lyapunov function for** f^t *if*

(LF1) *$(f^t(x), f^t(y)) \mapsto V_f(f^t(x), f^t(y))$ is a non-negative, continuous and positive definite map (i.e. $V_f(f^t(x), f^t(y)) = 0$ iff $x = y$) defined for points $f^t(x)$, $f^t(y)$ which lie in the same M_t, in a neighborhood of the diagonal $\delta(M_t) = \{(f^t(x), f^t(x)) : \quad f^t(x) \in M_t\}$, with $t \in \mathbf{R}$*

(LF2) *$\dot{V}_f$ and $\ddot{V}_f$ are continuous, where*

$$\dot{V}_f(f^t(x), f^t(y)) = \frac{d}{dt} V_f(f^{t+s}(x), f^{t+s}(y))\bigg|_{s=0} \tag{1.1}$$

$\ddot{V}_f$ is defined analogously substituting V_f by $\dot{V}_f$.

2 Proof of the main result

Throughout this section, we shall fix an Axiom A diffeomorphism g satisfying the STC, and a hyperbolic basic set Λ of g. We further consider a fixed $z \in \Lambda$ once and for all. In §4 we introduce two families of cones $\mathbf{P}_\eta(z,t), \mathbf{N}_\eta(z,t)$ contained in T_zM for all $t \in [0,1)$, which are continuous in $(z,t) \in \Lambda \times (0,1)$ and with the property that

$$\begin{aligned} g'(z)\mathbf{P}_\eta(z,t) &\subset \mathbf{P}_\eta(g(z),t) \\ \left[g^{-1}\right]'(z)\mathbf{N}_\eta(z,t) &\supset \mathbf{N}_\eta(g^{-1}(z),t) \end{aligned}$$

(see Lemma 4.2). There are also obtained Lyapunov functions for diffeomorphisms in a C^o neighborhood of g. More precisely:

Proposition 2.1 *For each positive $\rho < \rho_1$ and $\eta < \eta_o$, there is an assignment $f \mapsto V_f$ defined on a C^o neighborhood $\mathcal{N}(g, \Lambda, \rho)$ of g, such that if $f \in \mathcal{N}(g)$, $x \in h_f^{-1}(z)$, and $|v| < \rho_1$, then $\forall n \in \mathbf{Z}$ and $t \in [0,1)$ we have:*

1. $a_f|v|^2 \le V_f(f^{t+n}(x), f^t(exp_{f^n(x)}v)) \le A_f\,|v|^2$
2. *If a vector* v *is in the cone* $\mathbf{P}_\eta(g^n(z),t)$ *(resp.* $\mathbf{N}_\eta(g^n(z),t)$*) with* $|v| > \rho$ *then* $\dot V_f(f^{t+n}(x), f^t(exp_{f^n(x)}v)) > 0$ *(resp.* < 0*). And conversely, if* v *is such that* $\dot V_f(f^{t+n}(x), f^t(exp_{f^n(x)}v)) > 3\eta\rho_1^2$ *(resp.* $< -3\eta\rho_1^2$*) then* v *is in* $\mathbf{P}_\eta(g^n(z),t)$ *(resp.* $\mathbf{N}_\eta(g^n(z),t)$*)*
3. $\ddot V_f(f^{t+n}(x), f^t(exp_{f^n(x)}v)) > \mu \qquad$ *if* $|v| > \rho$

where μ *is a positive constant which does not depend on* f. *We may further ask that* $\mathcal{N}(g,\Lambda,\rho)$ *be taken so that* $d(h_f, id) < \rho/2$.

Proof of Proposition 2.1 is given in the last section, we remark that all bounds mentioned here are uniform on Λ. Now let $\varepsilon > 0$ be taken to satisfy $\varepsilon < \alpha/2$, where α is the expansivity constant of g on Λ. Consider σ and $\lambda > 0$ so that the function V_g corresponding to g satisfies the conditions $V_g(g^t(z), g^t(y)) < 2\sigma \Rightarrow d(g^t(z), g^t(y)) < \varepsilon/2$ and $d(g^t(z), g^t(y)) < \lambda \Rightarrow V_g(g^t(z), g^t(y)) < \sigma/2$ for all $t \in \mathbf{R}$. Now choose any arbitrarily small $0 < \rho < \lambda$. We shall consider f in the C^o neighborhood $\mathcal{N}(g,\Lambda,\rho)$ obtained in Proposition 2.1 relative to this ρ (η will be fixed later on) and will further fix $x \in h_f^{-1}(z)$ from now on, and so subscripts concerning f will be omitted. Let us denote by

$$D_\sigma^*(t) = \left\{(f^t(y), t) : V(f^t(x), f^t(y)) < \sigma\right\}$$

and define $D_\sigma(t)$ to equal the set $\{f^t(y) : (f^t(y), t) \in D_\sigma^*(t)\}$. Here it is assumed as well that $\mathcal{N}(g,\Lambda,\rho)$ is such that the following inclusions hold for all $t \in \mathbf{R}$

$$B_\rho(f^t(x)) \subset B_\lambda(f^t(x)) \subset D_\sigma(t) \subset B_{\varepsilon/2}(f^t(x))$$

Let us introduce the notation which will be used throughout this section. For $n \in \mathbf{Z}$, $t \in [0,1)$, we write

$$P^*(t+n) = \left\{\left(f^t(exp_{f^n(x)}v), t+n\right) : v \in \mathbf{P}_\eta(g^n(z),t)\right\} \tag{2.1}$$

and also, for all $t \in \mathbf{R}$, we denote $P(t) = \{f^t(y) : (f^t(y), t) \in P^*(t)\}$. The sets $N^*(t)$, $N(t)$, are defined in a similar way, considering the cone $\mathbf{N}_\eta(g^n(z),t)$ instead of $\mathbf{P}_\eta(g^n(z),t)$. It should be pointed out that for all t, $P(t)$ and $N(t)$

are diffeomorphic to cones, and that both $\bigcup_{t\in\mathbf{R}} P^*(t)$ and $\bigcup_{t\in\mathbf{R}} N^*(t)$ are connected sets.
For technical reasons, it will be more fitting to our purposes to work with dynamic balls $D_\sigma(t)$ first. We shall see afterwards that the result holds for euclidean balls $B_\lambda(f^t(x))$.

Remark 2.1 *We should observe that $\partial D_\sigma(t)$ is diffeomorphic to S^{n-1}, because V comes from a positive definite quadratic form (see (4.3)).*

The following lemma will provide a useful property of dynamic balls.

Lemma 2.1 *If we have that $(f^t(y), t)$ is in $D^*_\sigma(t)$ for all $t \in [-T, 0]$ ($[0, T]$) with sufficiently large $T > 0$, then*

1. *there is some $t_o \in [-T, 0)$ ($[0, T]$) s.t. $f^{t_o}(y) \in B_\rho(f^{t_o}(x))$*
2. *If, besides, $V(x, y) = \sigma$, then $\dot{V}(x, y) > 3\eta\rho_1^2 \quad (< -3\eta\rho_1^2)$*

Proof) Assume $f^t(x) \notin B_\rho(f^t(x))$ for all $t \in [-T, 0]$, then Taylor expansion w.r.t. t gives, according to (3) in Proposition 2.1

$$V(f^{-T}(x), f^{-T}(y)) > V(x, y) - T\dot{V}(x, y) + \frac{T^2\mu}{2} > \sigma \qquad \text{if } T \text{ large}$$

Now let

$$\nu = \sup\{V(f^t(x), f^t(y)) : f^t(y) \in B_\rho(f^t(x))\} < \sigma$$

and observe that (1) implies the existence of an instant $t_\nu \in (0, T)$ s.t.

$$V(f^{-t_\nu}(x), f^{-t_\nu}(y)) = \nu \quad \text{and} \quad \dot{V}(f^{-t_\nu}(x), f^{-t_\nu}(y)) > 0$$

then Taylor expansion again, implies $0 < t_\nu < \sqrt{\dfrac{2(\sigma - \nu)}{\mu}}$. But also convexity of V yields

$$\dot{V}(x, y) > \frac{V(x, y) - V(f^{-t_\nu}(x), f^{-t_\nu}(y))}{t_\nu} > \sqrt{\frac{(\sigma - \nu)\mu}{2}} > 3\eta\rho_1^2$$

if η is chosen sufficiently small.

∎

2.1 Codimension one

The first part of this section is devoted to proving the main result under the assumption $\dim \mathbf{E}^u_\Lambda = 1$, $\dim \mathbf{E}^s_\Lambda = n-1$, namely

Theorem A *There exists a compact connected set* $\mathbf{D}^f_\varepsilon(x)$ *contained in* $U^f_\varepsilon(x)$ *that meets both connected components of* $P(0) \cap \partial B_\lambda(x)$

Theorem B *There exists a compact connected set* $\mathbf{C}^f_\varepsilon(x)$ *contained in* $S^f_\varepsilon(x)$ *which separates* $B_\lambda(x)$ *in such a way that both connected components of* $P(0) \cap \partial B_\lambda(x)$ *are in different connected components of* $B_\lambda(x) \setminus \mathbf{C}^f_\varepsilon(x)$. *Besides,* $\mathbf{C}^f_\varepsilon(x) \cap \partial B_\lambda(x)$ *is contained in* $N(0) \cap \partial B_\lambda(x)$.

Theorem C *There is a point* y *in* $\mathbf{C}^f_\varepsilon(x) \cap \mathbf{D}^f_\varepsilon(x)$ *such that*

$$d(f^n(x), f^n(y)) < \rho << \lambda \qquad \forall n \in \mathbf{Z}$$

(i.e. y ρ*-traces the whole orbit of* x*)*

The arguments used in proving the general result do not differ much from these, though specializing to this case yields a much simpler development, which is also easier to visualize geometrically. Hence, we will delay these details until the end of this section, where necessary tools to complete the proof will be outlined. We shall construct $\mathbf{D}^f_\varepsilon(x)$ and $\mathbf{C}^f_\varepsilon(x)$ by finding for arbitrarily large $T > 0$

- an arc $\gamma_T \subset B_\lambda(x)$ joining both components of $P(0) \cap \partial B_\lambda(x)$, and staying in $B_\varepsilon(f^t(x))$ under the image of the flow f^t for an interval of time $[-T, 0]$
- a manifold with boundary V_T separating $B_\lambda(x)$ so that both connected components of $P(0) \cap \partial B_\lambda(x)$ remain in different connected components of $B_\lambda(x) \setminus V_T$, with the property that

$$f^t(V_T) \subset B_\varepsilon(f^t(x)) \qquad \forall t \in [0, T]$$

In our working hypothesis, the set $P^T = \bigcup_{|t| \leq T} P^*(t) \setminus [B_\rho(f^t(x)) \times \{t\}]$ consists of two connected components P_1^T and P_2^T. On the other hand, the set $N^T = \bigcup_{|t|\leq T} N^*(t) \setminus [B_\rho(f^t(x)) \times \{t\}]$ separates the tube $B^T = \bigcup_{|t|\leq T}[B_{\rho_1}(f^t(x)) \setminus B_\rho(f^t(x))] \times \{t\}$ into two connected components each of which contains, respectively P_1^T and P_2^T. These, in turn, can be defined to satisfy $P_i^T \subset P_i^S$ $i = 1, 2$ whenever $T < S$. This is a major advantage in working with suspension flows rather than discrete dynamical systems. We shall write

$$\begin{aligned} P_i^\sigma(t) &= \{f^t(y) : (f^t(y), t) \in P_i^T \cap \partial D_\sigma^*(t) \text{ for some } T > |t|\} \\ N^\sigma t &= \{f^t(y) : (f^t(y), t) \in N^T \cap \partial D_\sigma^*(t) \text{ for some } T > |t|\} \end{aligned}$$

Then we have the following

Proposition A.1 *For arbitrarily large $T > 0$ there is an arc β_T contained in $D_\sigma(0)$, which connects $P_1^\sigma(0)$ with $P_2^\sigma(0)$, and has the property that*

$$f^t(\beta_T) \subset \overline{D_\sigma(t)} \qquad \forall\, t \in [-T, 0]$$

Proof) Let $\alpha_T : [-1, 1] \to \overline{D_\sigma(-T)}$ be an arc with endpoints $\alpha_T(-1) \in P_1^\sigma(-T)$ and $\alpha_T(1) \in P_2^\sigma(-T)$. Then, there is a well defined map $t : [-1, 1] \to \mathbf{R}_0^+ \cup \infty$ defined by

1. $V(f^{t-T}(x), f^t(\alpha_T(s))) < \sigma \qquad \forall t \in [0, t(s))$
2. $V(f^{t-T}(x), f^t(\alpha_T(s))) = \sigma \qquad \forall t = t(s)$

$t(s)$ is the instant at which the orbit $(f^t(\alpha_T(s)), t - T)$ meets $\partial D_\sigma^*(t - T)$ for the first time. We seek an interval $[s_1, s_2] \subset t^{-1}[T, \infty]$ with $f^t(\alpha_T(s_i)) \in P_i^\sigma(0)$ for $i = 1, 2$ and $(s_1, s_2) \subset t^{-1}(T, \infty]$. An arc with the desired form is then obtained by taking $\beta_T = f^t \circ \alpha_T \Big|_{[s_b, s_a]}$ We claim $t(s)$ is continuous on $[-1, 1]$ and differentiable on $t^{-1}(0, \infty)$. Indeed, convexity of $V(f^t(x), f^t(y))$ w.r.t. t on $\partial D_\sigma(t)$ implies that $V(f^t(x), f^t(\alpha_T(s))) > \sigma$ for $t > t(s)$ near $t(s)$. Differentiability follows from the Implicit Function Theorem, since definition of $t(s)$ implies, on account of the previous considerations, that

$$\frac{dV}{dt}(f^{t-T}(x), f^t(\alpha_T(s)))\Big|_{t=t(s)} = \dot{V}(f^{t(s)-T}(x), f^{t(s)}(\alpha_T(s))) > 0 \qquad (2.2)$$

Now, for a residual set of T, we can consider $t^{-1}(T)$ consists of a finite number of points. Take $s_1^o = \inf t^{-1}(T)$ and $s_2^o = \sup t^{-1}(T)$. Let us remark that $f^t(\alpha_T(s_i^o)) \in P_i^\sigma(0)$ for $i = 1, 2$. Indeed, the arc $(f^{t(s)}(\alpha_T(s)), t(s) - T) : s \in [s_2^o, 1]$ is wholly contained in the connected component of $\bigcup_{t\in[-T,0]} [D_\sigma^*(t) \setminus N^T]$ that contains P_2^T, as $t(1) = 0$ and (2.2) holds. But on the other hand, if T is large, we can even assert that $(f^t(\alpha_T(s_2^o)), 0)$ is in $P^*(0)$, as stated in Lemma 2.1. This readily implies the existence of an interval as claimed, for otherwise $\{(f^{t(s)}(\alpha_T(s)), t(s) - T) : s \in [s_1, s_2]\}$ would contain an arc lying in $\bigcup_{t\leq M} \{(f^t(y), t) \in \partial D_\sigma^*(t) : \dot{V}(f^t(x), f^t(y)) > 0\}$ with endpoints in P_1^T and P_2^T; while, as stated above, such an arc should intersect $\bigcup_{t\leq M} N^*(t) \setminus [B_\rho(f^t(x)) \times \{t\}]$, which is a contradiction. Further, this argument also shows that t is onto.

■

As a corollary of the previous arguments, we get the following as well

Proposition A.2 *For arbitrarily large $T > 0$ the arc β_T contains a connected segment γ_T, joining both connected components of $P(0) \cap \partial B_\rho(x)$ and verifying*

$$f^t(\beta_T) \subset B_\varepsilon(f^t(x)) \qquad \forall t \in [-T, 0]$$

Now we will obtain analogues of Prop A1 and A2 for manifolds separating $B_\lambda(x)$

Proposition B.1 *For arbitrarily large $T > 0$ there is a manifold with boundary $W_T \subset \overline{D_\sigma(0)}$ which satisfies*

$$f^t(W_T) \subset \overline{D_\sigma(t)} \qquad \forall t \in [0, T]$$

and separates $\overline{D_\sigma(0)}$ so that $P_1^\sigma(0)$ and $P_2^\sigma(0)$ are in different connected components of $\partial D_\sigma(0) \setminus W_T$

Proof) Let B be the closed unit ball in $\mathbf{R}^{n-1}$, and consider a differentiable embedding $\varphi_T : B \to \overline{D_\sigma(T)}$ such that $\varphi_T(\partial B) \subset N^\sigma(T)$ (cf Remark (2.1)). We may ask that every arc joining $P_1^\sigma(0)$ with $P_2^\sigma(0)$ meet $\varphi_T(B)$. Let us define $t_+ : B \to \mathbf{R}_o^+ \cup \infty$ as in Prop.A.1 , i.e.

1. $V(f^{T-t}(x), f^{-t}(\varphi_T(v))) < \sigma \qquad \forall t \in [0, t_+(v))$
2. $V(f^{T-t}(x), f^{-t}(\varphi_T(v))) = \sigma \qquad \text{if } t = t_+(v)$

In analogous way, we obtain that t_+ is a continuous function onto $\mathbf{R}_o^+ \cup \infty$ and is differentiable on $t_+^{-1}(0, \infty)$. Therefore, $t_+^{-1}[T, \infty]$ is a manifold with boundary for a residual set of T. We write $\mathcal{V}_T$ for the set $f^{-T}(t_+^{-1}[T, \infty])$. Now, if α is an arc with endpoints $\alpha(-1)$ in $P_1^\sigma(0)$, $\alpha(1)$ in $P_2^\sigma(0)$, Proposition A.1 ensures the existence of a segment of $f^t(\alpha)$ joining $P_1^\sigma(T)$ with $P_2^\sigma(T)$. But this segment must meet $\varphi_T(B)$ at a point $\varphi_T(v)$, by our choice of φ_T. Look that this is the same as saying that $t_+(v) \geq T$, and therefore $f^{-T}(\varphi_T(v))$ will be in $\alpha \cap \mathcal{V}_T$, as claimed.
Now, as $\mathcal{V}_T$ consists of a finite number of connected components, we will have that one of them, say W_T, separates $D_\sigma(0)$ leaving $P_1^\sigma(0)$ and $P_2^\sigma(0)$ in different connected components of $D_\sigma(0) \setminus W_T$

∎

We have also shown that

Proposition B.2 *For arbitrarily large $T > 0$ there is a connected manifold with boundary $V_T \subset W_T \cap B_\lambda(x)$ such that*

$$f^t(V_T) \subset B_\varepsilon(f^t(x)) \qquad \forall t \in [0, T]$$

Besides, V_T separates $B_\lambda(x)$ leaving each connected component of $P(0) \cap \partial B_\lambda(x)$ in different connected components of $B_\lambda(x) \setminus V_T$.

To construct the sets we are looking for, we need only show that we can "take limits" of the sets obtained in Propositions A2, B2. In order to do so, the following tool will be applied

Definition 2.1 *For $E \subset \mathbf{R}^n$ the* **δ-parallel body** *of E is defined by*

$$[E]_\delta = \{x \in \mathbf{R}^n : d(x, E) \leq \delta\}$$

If $\mathcal{K}(n)$ denotes the collection of all non-empty compact sets of $\mathbf{R}^n$, *then the* **Hausdorff metric** *on $\mathcal{K}(n)$ is defined by*

$$d_H(E,F) = \inf\{\delta : E \subset [F]_\delta \text{ and } F \subset [E]_\delta\}$$

Observe that the space $(\mathcal{K}(n), d_H)$ is complete, and besides, collections of sets contained in a bounded portion of $\mathbf{R}^n$ are compact subspaces of $(\mathcal{K}(n), d_H)$. This means in particular that any sequence of continua contained in a bounded set $B \subset \mathbf{R}^n$ has subsequence converging to a continuum in $\overline{B}$ (see [F1]). This will allow us to complete the proof of Theorems A and B.

Proof)Observe that the sets γ_n and V_n obtained in Propositions A2 and B2 for $n \in \mathbf{N}$ are all contained in the ball $B_\lambda(x)$. Hence we may take $\mathbf{D}^f_\varepsilon(x)$ and $\mathbf{C}^f_\varepsilon(x)$ as the Hausdorff limit of converging subsequences γ_{n_k}, V_{n_k} .Direct verification shows that $\mathbf{D}^f_\varepsilon(x) \subset U^f_\varepsilon(x)$ and $\mathbf{C}^f_\varepsilon(x) \subset S^f_\varepsilon(x)$. Moreover, $\mathbf{D}^f_\varepsilon(x)$ and $\mathbf{C}^f_\varepsilon(x)$ are connected. Otherwise, $\mathbf{D}^f_\varepsilon(x)$ for instance, could be written as the union of two non empty disjoint compact sets K and L, and letting $d_H(K,L)$ equal δ we would have for large k that $\gamma_{n_k} \subset [K \cup L]_{\delta/4} = [K]_{\delta/4} \cup [L]_{\delta/4}$, which is a contradiction, since each parallel body should contain points of γ_{n_k}, while γ_{n_k} is connected. Clearly $\mathbf{D}^f_\varepsilon(x)$ contains points in both connected components of $P(0) \cap \partial B_\lambda(x)$, and on the other hand, every arc α with endpoints in $P^T_1 \cap \partial B_\lambda(x)$ and in $P^T_2 \cap \partial B_\lambda(x)$ must meet V_{n_k} at a point y_k. Any accumulation point of this sequence is in $\mathbf{C}^f_\varepsilon(x)$. Hence $\alpha \cap \mathbf{C}^f_\varepsilon(x)$ is non void for any such an arc and thus $\mathbf{C}^f_\varepsilon(x)$ separates $B_\lambda(x)$.

∎

It is easy to see that $\mathbf{C}^f_\varepsilon(x)$ must meet $\mathbf{D}^f_\varepsilon(x)$ (at least) at a point y, such a point will satisfy

$$d(f^n(x), f^n(y)) \leq \varepsilon \qquad \forall n \in \mathbf{Z}$$

taking into account that the semiconjugacy h_f between f and g could be made to satisfy $d(h_f, id) < \rho/2$, we have $\quad \forall n \in \mathbf{Z}$

$$\begin{aligned} d(g^n(h_f(x)), g^n(h_f(y))) &= d(h_f(f^n(x)), h_f(f^n(y))) \\ &\leq 2d(h_f, id) + d(f^n(x), f^n(y)) \leq \rho + \varepsilon \end{aligned}$$

where α was the expansivity constant of g on Λ. Thus $h_f(x) = h_f(y)$, what yields $d(f^n(x), f^n(y)) \leq \rho \qquad \forall n \in \mathbf{Z}$.

2.2 Extension to $\overline{Diff^1(M)}$

We are ultimately interested in extending this result to the C^o closure of $Diff^1(M)$. In order to do so, let us consider a homeomorphism f in the neighborhood $\mathcal{N}(g, \Lambda, \rho/2) \subset \mathcal{N}(g, \Lambda, \rho)$ obtained in Proposition 2.1 relative to $\rho/2$, and let $x \in h_f^{-1}(z)$. Suppose we can approximate f by a sequence of diffeomorphisms f_n in $\mathcal{N}_2(g)$. We may choose $x_n \in h_{f_n}^{-1}(z)$ such that $x_n \to x^*$. Observe that $h_f(x^*)$ satisfies $d(f^k(x^*), g^k(h_f(x^*)) < \rho/4$ for all $k \in \mathbf{Z}$, then by taking limits on $d(f_n^k(x_n), g^k(z) < \rho/4$, we get $h_f(x^*) = z$. As a result we will have that $d(f^k(x), f^k(x*)) < \rho/2$. Thus, there is no harm in considering x^* instead of x. We claim that for f we can find pieces $\mathbf{C}_\varepsilon^f(x)$ and $\mathbf{D}_\varepsilon^f(x)$ as obtained in Theorems A and B. Note that, because no differentiability assumption is made in def 2.1, the expressions $P_f(0)$ and $N_f(0)$ make sense. We shall find it convenient to denote by $P_n(0)$, $N_n(0)$ the cones corresponding to (f_n, x_n). Now, as we can assume $\mathbf{C}_{\varepsilon/2}^{f_n}(x_n)$ and $\mathbf{D}_{\varepsilon/2}^{f_n}(x_n)$ are contained in $B_{2\lambda}(x)$, we can take Hausdorff limits of converging subsequences. An easy calculation shows that these limit sets are contained in $S_{\varepsilon/2}^f(x^*)$ and $U_{\varepsilon/2}^f(x^*)$ respectively, and hence they are parts of $S_\varepsilon^f(x)$ and $U_\varepsilon^f(x)$. But also, by our choice of λ, each $\mathbf{D}_{\varepsilon/2}^{f_n}(x_n)$ has points in both connected components of $P_n(0) \cap \partial B_\lambda(x_n)$; and so, by taking limits and recalling that $\mathbf{D}_\varepsilon^f(x)$ is connected, we get the desired property. Similarly, we see that each arc joining both connected components of $P_f(0) \cap \partial B_\lambda(x)$ must intersect $\mathbf{C}_{\varepsilon/2}^{f_n}(x_n)$ if n is large, and it easily follows that it meets $\mathbf{C}_\varepsilon^f(x)$. Moreover, $\mathbf{C}_\varepsilon^f(x) \cap \mathbf{D}_\varepsilon^f(x)$ contains a point $\rho/2$-tracing the orbit of x^*, what implies it ρ-traces the orbit of x.

■

2.3 The general case

We deal now with the general case dim $\mathbf{E}^s_\Lambda = s$, dim $\mathbf{E}^u_\Lambda = u$, where $s + u = n$, towards which we have been heading. We shall give the details only in the translation of the former case to higher dimensions. The rest follows in a very analogous way, and so we shall omit further consideration of it.

Take σ, λ and ρ as at the beginning of this section. In case $s = 0$, then, it trivially follows that $B_\lambda(x) \subset S^f_\varepsilon(x)$, and $\limsup_{n\to\infty} d(f^n(x), f^n(y)) \leq \rho$ for any y in $B_\lambda(x)$. A similar result is obtained if $u = 0$. Thus we will assume in the remainder of this section that $s, u \geq 1$.

An important remark to be made is that the "reduced cones" P^T and N^T now satisfy the following properties:

$$\begin{array}{lll} P^T \not\sim 0 & \text{in} & H_{u-1}(B^T \setminus N^T) \\ N^T \not\sim 0 & \text{in} & H_{s-1}(B^T \setminus P^T) \end{array}$$

Moreover, for each fixed $|t| \leq T$,

$$\begin{array}{lll} P^*(t) \setminus [B_\rho(f^t(x)) \times \{t\}] \not\sim 0 & \text{in} & H_{u-1}(B^T \setminus N^T) \\ N^*(t) \setminus [B_\rho(f^t(x)) \times \{t\}] \not\sim 0 & \text{in} & H_{s-1}(B^T \setminus P^T) \end{array}$$

Proposition A'.1 *For arbitrarily large $T > 0$ there is an s-manifold $W^s_T \subset \overline{D_\sigma(0)}$ with boundary $\partial W^s_T \subset N^\sigma(0)$, such that*

$$f^t(W^s_T) \subset \overline{D_\sigma(t)} \qquad \forall\, t \in [0, T]$$

Moreover $P(0) \setminus B_\lambda(x) \not\sim 0$ in $H_{u-1}(D_\sigma(0) \setminus W^s_T)$

Proof) The proof will proceed much in the spirit of Props A.1 and B.1. Let us choose an embedding φ^s_T of the closed unitary ball $B^s \subset \mathbf{R}^s$ in $\overline{D_\sigma(T)}$, B^s can be imbedded so that $\varphi^s_T(\partial B^s) \subset N^\sigma(0)$ and $P(T) \setminus B_\rho(f^T(x)) \not\sim 0$ in $H_{u-1}(\overline{D_\sigma(T)} \setminus \varphi^s_T(B^s)$. We define $t_+ : B^s \to \mathbf{R}^+_o \cup \infty$ as in Proposition B.1, and take $T > 0$ so that $t_+^{-1}[T, \infty]$ be an s-manifold with boundary, we shall also denote $f^{-T}(t_+^{-1}[T, \infty])$ by $\mathcal{V}^s_T$. We claim that there is a connected

component W^s_T of $\mathcal{V}^s_T$, whose boundary is not null-homologous in $H_{s-1}(\overline{D_\sigma(0)} \setminus P(0))$.
If we define $\mathcal{W}_{\leq T}$ to be the set

$$\mathcal{W}_{\leq T} = \{(f^{t(v)}(v), T - t(v)) : v \in t_+^{-1}[0, T]\}$$

it easily follows that $\mathcal{W}_{\leq T}$ is an s-manifold with boundary which is contained in $\bigcup_{|t| \leq T} \{(f^t(y), t) \in \partial D^*_\sigma(t) : \dot{V}(f^t(x), f^t(y)) < 0\}$, whence $\partial \mathcal{W}_{\leq T} \sim 0$ in $H_{s-1}(B^T \setminus P^T)$, and as a consequence $\varphi^s_T(\partial B^s) \times T \sim \partial \mathcal{V}^s_T \times 0$ in $H_{s-1}(B^T \setminus P^T)$.
Taking $T > 0$ large enough so that Lemma 2.1 apply we will have $\partial \mathcal{V}^s_T \subset N^\sigma(0)$, and the above calculation yields $\partial \mathcal{V}^s_T \not\sim 0$ in $H_{s-1}(N^\sigma(0)) \subset H_{s-1}(\overline{D_\sigma(0)} \setminus P(0))$.This readily implies the existence of a connected component as claimed, for otherwise $\partial \mathcal{V}^s_T = \partial \bigcup \{W^s_T : W^s_T \text{ is a connected component of } \mathcal{V}^s_T\}$ would be null homologous in $H_{s-1}(\overline{D_\sigma(0)} \setminus P(0))$, a contradiction. This is the same as saying that $P(0) \setminus B_\rho(x) \not\sim$ in $H_{u-1}(\overline{D_\sigma(0)} \setminus W^s_T)$
On the other hand, this argument also shows that t_+ is onto. If it were not so, the manifold

$$\mathcal{W} = \{(f^{t(v)}(v), T - t(v)) : v \in B^s\}$$

would be contained in $\bigcup_{|t| \leq M} \{(f^t(y), t) \in \partial D^*_\sigma(t) : \dot{V}(f^t(x), f^t(y)) < 0\}$, for some positive $M > 0$, and the map $v \mapsto (f^{t(v)}(v), T - t(v))$ would give a retract of B^s into a manifold which is isotopic to S^{s-1}.

■

Since the arguments used in the previous proposition are perfectly dual, we get a manifold W^u_T that satisfies an analogue of the previous statement. An isotopy argument thus yields the following:

Proposition A'.2 *For arbitrarily large $T > 0$ there are obtained connected manifolds with boundary V^s_T and V^u_T contained in $B_\lambda(x)$ such that for all $t \in [0, T]$,*

$$f^t(V^s_T) \subset B_\lambda(f^t(x)) \quad \textit{and} \quad f^{-t}(V^u_T) \subset B_\lambda(f^t(x))$$

They further verify that $P(0) \setminus B_\rho(x) \not\sim 0$ in $H_{u-1}(B_\lambda(x) \setminus V_T^s)$ and $N(0) \setminus B_\rho(x) \not\sim 0$ in $H_{s-1}(B_\lambda(x) \setminus V_T^u)$.

The sets $\mathbf{C}_\varepsilon^f(x)$ and $\mathbf{D}_\varepsilon^f(x)$ are constructed as before. We need only show that they have non trivial co-homology groups. Now, the interior of the δ-parallel bodies $\{[\mathbf{C}_\varepsilon^f(x)]_\delta^o\}_{\delta>0}$ and $\{[\mathbf{D}_\varepsilon^f(x)]_\delta^o\}_{\delta>0}$ provide a collection of open neighborhoods directed downward by inclusion, each of which verifies that $P(0) \setminus B_\rho(x)$ is not null-homologous in $H_{u-1}(B_\lambda(x) \setminus [\mathbf{C}_\varepsilon^f(x)]_\delta^o) \approx H^{s+1}(B_\lambda(x); [\mathbf{C}_\varepsilon^f(x)]_\delta^o)$. Therefore we have

$$H^{s+1}(B_\lambda(x); \mathbf{C}_\varepsilon^f(x)) \approx \varinjlim H^{s+1}(B_\lambda(x); [\mathbf{C}_\varepsilon^f(x)]_\delta) \supset \mathbf{Z}_2$$

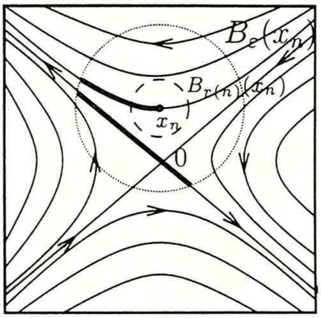
3 Counterexample - A case where no $\mathbf{D}_\varepsilon^f(x)$ contains x

It remains the question of whether or not it can be achieved a piece $\mathbf{D}_\varepsilon^f(x)$ as described in the previous theorem, such that x belong to $\mathbf{D}_\varepsilon^f(x)$. This is, is it always true that the local unstable set of x connects x with both connected components of $P(0) \cap \partial B_\lambda(x)$? The answer is negative, as we show in the following example.

Proposition 3.1 *Let $g : M \to M$ be an orientation preserving Anosov diffeomorphism, such that $\dim E^u = 1$. Then there is a sequence of diffeomorphisms $f_n \overset{C^o}{\to} g$, satisfying the following condition. For every $n \in \mathbf{N}$ there is an $x_n \in \Omega(f_n)$, such that the connected component of the local f_n -unstable set which contains x_n meets $\partial B_\lambda(x_n)$ at exactly one point for sufficiently small λ.*

Proof) We can regard g as an hyperbolic automorphism of $\mathbf{T}^m = \mathbf{R}^m/\mathbf{Z}^m$ ([F2, N1]). Then there is an $A \in \mathcal{M}_m(\mathbf{Z})$, $|detA| = 1$, with no modulus 1 eigenvalues, such that $g \circ \pi = \pi \circ A$ where $\pi : \mathbf{R}^m \to \mathbf{T}^m$ is the canonical projection. Note that in $\mathbf{R}^n$, the transformation $\tilde{g}(x) = Ax$ is the time-one mapping of the flow g^t corresponding to the equation $\quad \dot{x} = \mathrm{C}x \quad$ in $\mathbf{R}^m$, where $e^{\mathrm{C}} = A$.

Let $\{x_n\}_{n\in\mathbf{N}}$ be a sequence in $\mathbf{R}^m$ converging to 0, such that x_n belongs neither to the local g-stable nor to the local g- unstable manifold of 0. Take x_n to be in the cube $Q = \{x \in \mathbf{R}^m : |x^i| < \frac{1}{4} \quad i = 1, \ldots, m\}$, and choose $r(n)$ so that the ball $B_{2r(n)}(x_n)$ be contained in Q and do not intersect the local g-invariant manifolds of 0. Then take a smooth non-negative function $F_n : \mathbf{R}^m \to [0,1]$ to satisfy



$$F_n(x) = \begin{cases} 0 & \text{if } x = x_n \\ 1 & \text{if } x \notin B_{r(n)}(x_n) \\ > 0 & \forall x \neq x_n \end{cases}$$

and consider the equation $\quad \dot{x} = F_n(x)\mathrm{C}x \quad$ in $\mathbf{R}^m$. We shall define $\tilde{f}_n(x)$ to be the time-one map of the flow $f_n^t(x)$ corresponding to this equation. Then, we have that

Lemma 3.1 *$f_n(\pi(x)) := \pi \circ \tilde{f}_n(x)$ for $x \in \overline{2Q}$ defines a diffeomorphism on $\mathbf{T}^m$, if n is sufficiently large, and $f_n \rightrightarrows g$ on $\mathbf{T}^m$*

Now, if we take $N > 0$ such that $x_n \in B_\varepsilon(0)$ $\forall n \geq N$, then the ε-local f_n-unstable set of x_n equals the set $W^u_{2\varepsilon}(0) \cup [o_g^-(x_n) \cap B_\varepsilon(0)]$, where $W^u_{2\varepsilon}(0)$ is the local g-unstable manifold of 0, and $o_g^-(x_n)$ is the backward orbit of x_n with respect to g^t.
It follows that the connected component of $U^n_\varepsilon(x_n)$ which contains x_n is exactly the segment $\{g^t(x_n) : t \in [0, t(\varepsilon)]\}$, where
$t(\varepsilon) = \inf\{t \geq 0 \text{ s.t. } g^t(x_n) \in B_\varepsilon(x_n)\}$; and thus it meets $\partial B_\lambda(x_n)$ only at one point $\forall \lambda \leq \varepsilon$.

■

We are reduced to proving that $\forall \eta > 0$ there is some N such that $\tilde{f}_n(x) = \tilde{g}(x)$ if $n \geq N$ and $x \notin B_\eta(0)$. This will imply on one side that $\tilde{f}_n$

can be projected to $\mathbf{T}^m$ without problems, while on the other hand, will make the distance $d(\tilde{f}_n, \tilde{g})$ equal $\sup_{x \in B_\eta(0)} |\tilde{f}_n(x) - \tilde{g}(x)|$, which is easily seen to tend uniformly to 0.
To see this, let $\eta > 0$ be given, and choose $\delta > 0$, $N > 0$ so that if $n \geq N$then

$$\overline{B_{r(n)}(x_n)} \subset g^t(B_\delta(0)) \subset B_\eta(0) \qquad \forall t \in [-1, 1]$$

For $x \notin B_\eta(0)$ let us write $t_{x,n} = \inf\{t \geq -1 : \quad f_n^t(x) \in \overline{B_{r(n)}(x_n)}\}$. It is immediate that $t_{x,n}$ must be greater than 1; indeed, $F_n(f_n^t(x)) = 1$ as long as $t \in [-1, t_{x,n})$, whence $f_n^t(x) = g^t(x)$ $\forall t \in [-1, t_{x,n})$. Being $t_{x,n} < 1$ would yield $g^t(x) \in B_\delta(0)$ for some $x \notin B\eta(0)$ and $t \in (-1, 1)$, which contradicts our choice of $\delta > 0$. This readily implies that $\tilde{f}_n(x) = \tilde{g}(x)$ for all $x \notin B_\eta(0)$ and $n \geq N$, but also that $\tilde{f}_n(B_\delta(0)) \subset B_\eta(0)$. Now, by considering a new $N_1 > N$ such that $\tilde{f}_n(x) = \tilde{g}(x)$ for all $x \notin B_\delta(0)$ we get

$$d(\tilde{f}_n, \tilde{g}) = \sup_{x \in B_\delta(0)} |\tilde{f}_n(x) - \tilde{g}(x)| \leq \sup_{x \in B_\delta(0)} |\tilde{f}_n(x)| + |\tilde{g}(x)| \leq 2\eta$$

The statement now follows from uniform continuity of π.

■

Examples similar to this can be found to see that the dimension of the local unstable sets may be greater than u. It suffices to proceed as in the previous paragraph, substituting the function F_n appearing in Prop 4.1 by a non-negative smooth function whose singularities consist of a $(k-1)$ dimensional set Γ_k contained in $B_{r(n)}(x)$.

4 Lyapunov functions in a C^o neighborhood of an Axiom A diffeomorphism

This section contains the proof of Proposition 2.1. We shall consider an Axiom A diffeomorphism g satisfying the STC on M, and a hyperbolic basic set Λ of g. We may take a *quadratic form* on Λ (i.e. an assignment $A : T_\Lambda M \to \mathbf{R}$ such that $A_x = A|_{T_x M}$ is a quadratic form on $T_x M$), satisfying certain conditions. First, we will denote, for simplicity, the form

$A_{g(x)}(g'(x)v)$ as $g^{\#}(A)_x(v)$ for (x, v) in $T_\Lambda M$. Then, hyperbolicity of Λ allows us to take the form A, such that it be continuous, positive definite and verifying:

i. $B = g^{\#}(A) - A$ is non-degenerate

ii. $g^{\#}(B) - B$ is positive definite

meaning that these forms are resp. non-degenerate, positive definite on each tangent space (see [L]). We may, and will, extend A over a (no longer invariant) open neighborhood of Λ, $N(\Lambda)$, in such a way that it continue to be positive definite and properties (i), (ii) hold on $\overline{N(\Lambda)}$.

Lemma 4.1 *For a suitable ρ_o there exists a continuous function $U : \{(x, exp_x v) : x \in N(\Lambda),\ |v| < \rho_o\} \to \mathbf{R}_o^+$ such that*

1. $U(x, exp_x v) \geq C|v|^2$

2. $|\ \Delta U(x, exp_x v) - B_x(v)\ | = o(|v|^3)$

3. $\Delta^2 U(x, exp_x v) \geq C|v|^2$,

where $\Delta U(x, y) = U(g(x), g(y)) - U(x, y)$, and $\Delta^2 U$ stands for $\Delta(\Delta U)$ when it makes sense.

Proof) Taking $\rho > 0$ small enough, we can define

$$U(x, exp_x v) := A_x(v) \tag{4.1}$$

and regard $g(exp_x v) - g(x)$ and $g^2(exp_x v) - g^2(x)$ as vectors in $\mathbf{R}^n$. If we denote by $R_{\mathbf{A}}(x, exp_x v)$ the following expression

$$\Delta U(x, exp_x v) - B_x(v) = A_{g(x)}(\ g(exp_x v) - g(x)\ ) - A_{g(x)}(\ g'(x)v\ )$$

and take matrices $\mathbf{A}_x$, $\mathbf{B}_x$ such that $A_x(v) = \langle \mathbf{A}_x v, v\rangle$, $B_x(v) = \langle \mathbf{B}_x v, v\rangle$ for $(x, v) \in T_\Lambda M$, then we have that $|R_{\mathbf{A}}(x, exp_x v)|$ is bounded by

$$\sup_{x\in\Lambda} \|\mathbf{A}_x\|\ \|g(exp_x v) - g(x) - g'(x)v\|\ (\ \|g(exp_x v) - g(x)\| + \|g'(x)v\|\ )$$

which is clearly a term of cubic order.

Similarly, denoting $R_{\mathbf{B}}(x, exp_x v) = B_{g(x)}(g(exp_x v) - g(x)) - B_{g(x)}(g'(x)v)$, we can compute $\Delta^2 U(x, exp_x v) - \left(g^{\#}(B) - B\right)_x(v)$, which equals $R_{\mathbf{A}}(g(x), g(exp_x v)) - R_{\mathbf{A}}(x, exp_x v) + R_{\mathbf{B}}(x, exp_x v)$, thus obtaining the following estimate

$$|\Delta^2 U(x, exp_x v) - \left(g^{\#}(B) - B\right)_x(v)| \leq C_2 |v|^3$$

as $R_{\mathbf{B}}(x, y)$ admits the same kind of bound as $R_{\mathbf{A}}(x, y)$ does (just replace $\mathbf{A}_x$ by $\mathbf{B}_x$). Now, positive definiteness of the form $g^{\#}(B) - B$ together with the last expression, completes the proof.

■

We proceed now to extend some of these features to a C^o neighborhood of g. As it was said in the Introduction, topological stability of g yields a continuous surjection $h_f : M \to M$, C^o close to identity for f C^o close to g, which verifies $g \circ h_f = h_f \circ f$. In particular we may consider f such that

$$h_f^{-1}(\Lambda) \subset N(\Lambda) \qquad \text{if } f \ C^o \text{ close to } g$$

We shall work with suspension flows for technical reasons, some of which are explained before Proposition A.1. In the remainder of this section, we will fix a diffeomorphism f in the conditions just described, $x \in h_f^{-1}(\Lambda)$ and $v \in T_x M$ such that all the following definitions make sense. We will call $y = exp_x v$, and let for $t \in [0, 1)$

$$W_f(f^t(x), f^t(y)) = (1 - \psi(t))\, U(x, y) + \psi(t)\, U(f(x), f(y)) \quad (4.2)$$

where $\psi : [0, 1] \to [0, 1]$ is any smooth increasing function such that $\psi(0) = 0$, $\psi(1) = 1$ and $\psi^{(k)}(0) = 0$, $\psi^{(k)}(1) = 0$, $\forall k \in \mathbf{N}$.
We define

$$\boxed{V_f(f^t(x), f^t(y)) = \int_0^1 \int_{-1}^0 W_f(f^{t+s+u}(x), f^{t+s+u}(x))ds\, du} \quad (4.3)$$

which is positive definite, and we have, in the sense of (1.1) that

$$\dot{V}_f(f^t(x), f^t(y)) = \int_0^1 W_f(f^{u+t}(x), f^{u+t}(y)) - W_f(f^{u+t-1}(x), f^{u+t-1}(y))\, du$$

$$\ddot{V}_f(f^t(x), f^t(y)) = W_f(f^{t+1}(x), f^{t+1}(y)) - 2W_f(f^t(x), f^t(y)) + W_f(f^{t-1}(x), f^{t-1}(y))$$

If we call $\Psi(t) = \int_0^t \psi$, it easily follows that:

$$\dot{V}_f(f^t(x), f^t(y)) = [(1-t) - (\ \Psi(1) - \Psi(t)\ )]\, \Delta_f U(f^{-1}(x), f^{-1}(y)) + \\ + (\ \Psi(1) - \Psi(t) + t - \Psi(t)\ )\, \Delta_f U(x,y) + \Psi(t)\Delta_f U(f(x), f(y))$$

$$\ddot{V}_f(x_t^f, y_t^f) = (\ 1 - \psi(t)\ )\, \Delta_f^2 U(f^{-1}(x), f^{-1}(y)) + \psi(t)\, \Delta_f^2 U(x,y)$$

Here we have written $\Delta_f U(x,y)$ for $[U(f(x), f(y)) - U(x,y)]$ and $\Delta_f^2 U$ for $\Delta_f(\Delta_f U)$.

Observe that if $f = g$, estimates in Lemma 4.1 imply that :

Lemma 4.2 *For each $\eta < \eta_o$ and $z \in \Lambda$ there exist two nonempty and nontrivial families of cones $\{\mathbf{P}_\eta(z,t), \mathbf{N}_\eta(z,t)\}_{t\in[0,1)}$ such that for small $|v| < \rho_1$*

$$\dot{V}_g(g^t(z), g^t(exp_z)) \geq \eta|v|^2 \ \textit{if}\ v \in \mathbf{P}_\eta(z,t) \qquad (\leq -\eta|v|^2 \ \textit{if}\ v \in \mathbf{N}_\eta(z,t))$$

Besides, $(z,t) \mapsto (\mathbf{P}_\eta(z,t), \mathbf{N}_\eta(z,t))$ is continuous on $\Lambda \times (0,1)$; and $\forall t \in [0,1)$

$$g'(z)\mathbf{P}_\eta(z,t) \subset \mathbf{P}_\eta(g(z),t)$$
$$\left[g^{-1}\right]'(z)\mathbf{N}_\eta(z,t) \supset \mathbf{N}_\eta(g^{-1}(z),t)$$

Conversely, if $\dot{V}_g(g^t(z), g^t(exp_z v)) > \frac{5}{2}\eta|v|^2$ $(< -\frac{5}{2}\eta|v|^2)$, with $|v| < \rho_1$ then $v \in \mathbf{P}_\eta(z,t)$ $(\mathbf{N}_\eta(z,t))$

Proof) Call

$$C(z,v,t) = [(1-t) - (\ \Psi(1) - \Psi(t)\ )]\, B_{g^{-1}(z)}([g^{-1}]'(z)v) + \\ (\ \Psi(1) - \Psi(t) + t - \Psi(t)\ )\, B_z(v) + \Psi(t)\, B_{g(z)}(g'(z)v)$$

And take

$$\eta_o = \frac{1}{4} \min_{z\in\overline{N(\Lambda)}} \left\{ \max_{v\in S_z M} B_z(v);\ \max_{v\in S_z M} -B_z(v) \right\}$$

where S_xM is the set of unitary vectors in T_zM. Clearly, η_o is positive, as $B_z(v)$ is a non degenerate quadratic form. Then, it suffices to define

$$\mathbf{P}_\eta(z,t) = \{v \in T_zM : C(z,v,t) \geq 2\eta|v|^2\} \tag{4.4}$$

(In a similar fashion, $\mathbf{N}_\eta(z,t)$ is defined to equal the set $\{v \in T_xM : C(z,v,t) \leq -2\eta|v|^2\}$) It should be noted that

$$\dot{V}_g(g^t(x), g^t(exp_z v)) \quad = \quad o(|v|^3) \; + \; C(z,v,t)$$

then, since $C(z,v,t)$ is a convex combination of $B_{g^{-1}(z)}([g^{-1}]'(z)v)$, $B_z(v)$ and $B_{g(z)}(g'(z)v)$, the claim follows from the fact that $g^\#(B) - B > 0$.

■

Lemma 4.3 *If* $t \in [0,1)$ *and* $z = h_f(x)$ *then*

$$\begin{aligned}
|\dot{V}_g(g^t(z), g^t(exp_z v)) \; - \; \dot{V}_f(f^t(x), f^t(exp_x v))| \; &\leq \; \mathcal{O}(f)|v| \\
|\ddot{V}_g(g^t(z), g^t(exp_z v)) \; - \; \ddot{V}_f(f^t(x), f^t(exp_x v))| \; &\leq \; \mathcal{O}(f)|v|
\end{aligned}$$

where $\mathcal{O}(f) \to 0$ *when* $f \to g$ *in the* C^o *topology.*

Proof) Both expressions can be bounded by a convex combination of $|\Delta U(g^{-1}(z), g^{-1}(exp_z v)) - \Delta_f U(f^{-1}(x), f^{-1}(exp_x v))|$, $|\Delta U(z, exp_z v) - \Delta_f U(x, exp_x v)|$ and $|\Delta U(g(z), g(exp_z v)) - \Delta_f U(f(x), f(exp_x v))|$. Observe that one needs only estimate $|U(g(z), g(exp_z v)) - U(f(x), f(exp_x v))|$ which is easily seen to be bounded by $|\langle(\mathbf{A}_{g(z)} - \mathbf{A}_{f(x)})w_z, w_z\rangle| + |\langle\mathbf{A}_{f(x)}(w_z - w_x), w_z + w_x\rangle|$, using notation in Lemma 4.1, where w_z and w_x stand, respectively, for $g(exp_z v) - g(z)$ and $f(exp_x v) - f(x)$, here regarded as vectors in $\mathbf{R}^n$. Clearly, this last expression admits $Cd(h_f, id)|v|$ as a bound, where C is a positive constant.

■

As a consequence of all the previous discussion, and taking into account that estimates in Lemma 4.1 also imply that $\ddot{V}_g(g^t(x), g^t(exp_z v)) \geq C|v|^2$ Proposition 2.1 follows.

References

[F1] FALCONER, K *The geometry of fractal sets*, Cambridge University Press, Cambridge, 1985

[F2] FRANKS, J *Anosov diffeomorphisms*, Global Analysis, Proc. Symp. Pure Math **14** AMS (1970), 61-93.

[H] HURLEY, M *Combined structural and topological stability are equivalent to Axiom A and the strong transversality condition*, Ergod. Th. & Dynam. Sys., **4** (1984) 81-88

[L] LEWOWICZ, J *Lyapunov functions and topological stability*, J. of Diff. Eq. **38** (1980) 192-209

[N1] NEWHOUSE S *On codimension one Anosov diffeomorphisms*, Am. J. of Math. **92** 3, 761-770.

[N2] NITECKI, Z *On semi-stability of diffeomorphisms* Invent. Math.**14** (1971) 83-122

[S1] SHUB, M *Structurally stable diffeomorphisms are dense*, Bull. AMS **78**(1972) 817-818

[S2] SPANIER, E *Algebraic Topology*, McGraw-Hill, (1966)

A $3D$-manifold with a uniform local product structure is T^3

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Abstract

Let M be a compact connected oriented three dimensional boundaryless manifold and $f : M \to M$ an expansive homeomorphism such that $\Omega(f) = M$. Assume that the stable sets of f form a C^0 foliation of M of codimension 1 and the unstable sets a transverse C^0 foliation of dimension 1. Then M is homeomorphic to T^3.

Resumen

Sea M una variedad sin borde compacta, conexa y orientada de dimensión 3. Sea $f : M \to M$ un homeomorfismo expansivo tal que $\Omega(f) = M$. Supongamos que los conjuntos estables asociados a f forman una foliación continua $\mathcal{F}^s$ de codimensión 1 y que los inestables constituyen una foliación unidimensional transversa a la estable y continua. Entonces M es homeomorfo al toro T^3

1 Introduction

Let M be a compact connected oriented three dimensional manifold and $f : M \to M$ a diffeomorphism. Let $dist : M \times M \to \mathbb{R}$ be a metric defining its topology.

Definition 1.1 *We say that f is expansive if there exists a positive constant α such that if we have $x, y \in M$, $x \neq y$ there is $n \in \mathbb{Z}$ such that $dist(f^n(x), f^n(y)) > \alpha$. The number α is called an expansivity constant for f.*

Let us assume now that there are C^0 foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ preserved by f, such that if x, y are in the same leave W^s of $\mathcal{F}^s$ then

$$\lim_{n \to +\infty} dist(f^n(x), f^n(y)) = 0$$

and if x, y are in the same leave W^u of $\mathcal{F}^u$ then

$$\lim_{n \to -\infty} dist(f^n(x), f^n(y)) = 0$$

Moreover assume that $\Omega(f) = M$. Then M is homeomorphic to T^3.

It should be pointed out that the main purpose of this paper is to show how it can be proved under the hypotheses stated above that the manifold M is T^3. So the reader should be aware that these hypotheses are not minimal. Anyway we wish to say a few words about them. See below for the definitions that are needed.

1. The existence ofthe invarian foliations implies in our case (one of the foliations if of codimension one) that the non-wandering set $\Omega(f)$ is the whole manifold M. The proof of this is rather similar to the proof given by Newhouse in the article "Codimension one Anosov diffeomorphisms" (see [Nw])
2. The assumption in the existence of both foliations is redundant too. Once we have a codimension one stable foliation $\mathcal{F}^s$ it follows the existence of the one dimensional unstable foliation $\mathcal{F}^u$ transverse to the stable one. The proof of this can be proved from the fact that if f is expansive then for all $x \in M$ the connected component $C(x)$ of $W^u_\epsilon(x)$ containing x is a non-trivial continuum (see [Le2]). It is well known that a continuum that has only two non-cut points is an

arc (see [Wi]). But the intersection of $C(x)$ with $W^s_\epsilon(y)$ is at most a point. As for all $y \in M$ $W^s_\epsilon(y)$ is a disk it locally separates M. Therefore if $C(x) \cap W^s_\epsilon(y) = \{z\}$ then $C(x) \backslash \{z\}$ is not connected or z is a non-cut point. From this it is not difficult to prove that $C(x)$ is an arc. Moreover, it can be proved too that $C(x) \backslash \{x\}$ has two connected components.

3. Even the existence of the one dimensional unstable foliation implies the existence of the codimension one stable foliation. But the proof of this fact will appart us far away from our objective.

Let us give the main definitions we will need here after.

Definition 1.2 *For $x \in M$ we define*

$$W^s_\epsilon(x, f) = \{y \in M / dist(f^k(x), f^k(y)) \leq \epsilon \,; k \geq 0\}$$

as the local ϵ-stable set for the point x and the homeomorphism f.

Analogously we define the local ϵ-unstable set for x and f as $W^u_\epsilon(x, f) = W^s_\epsilon(x, f^{-1})$. If there is no ambiguity we shall usually omit any reference to ϵ and f and speak about local stable and unstable sets of the point x denoting them by $W^s_\epsilon(x)$ and $W^u_\epsilon(x)$ respectively. Observe that $W^s_\epsilon(x)$ and $W^u_\epsilon(x)$ verify that $f(W^s_\epsilon(x)) \subset W^s_\epsilon(f(x))$ and $f^{-1}(W^u_\epsilon(x)) \subset W^u_\epsilon(f^{-1}(x))$.

Definition 1.3 *We define*

$$W^s(x, f) = \{y \in M / \lim_{k \to +\infty} dist(f^k(x), f^k(y)) = 0\}$$

as the stable set for the point x and the homeomorphism f.

Analogously we define the unstable set for x, f as $W^u(x, f) = W^s(x, f^{-1})$. We usually will omit the reference to f.

Lemma 1.1 *If α is an expansivity constant for f and $\epsilon \leq \alpha$, then $W^s_\epsilon(x) \subset W^s(x)$, $W^u_\epsilon(x) \subset W^u(x)$.*

As usual $Oxy = \{(x,y,z) \in I\!R^3/z = 0\}$ and $Oz = \{(x,y,z) \in I\!R^3/x = y = 0\}$, a horizontal square is a square located in a plane parallel to Oxy, a vertical line is a segment located in a straight line parallel to Oz.

Definition 1.4 *Given $f : M \to M$ and a point $x \in M$ we say that there is a local product structure in x (abbr.: x has an f-*lps*) if there is a neighborhood $V(x)$, and a homeomorphism $h : V(x) \to [-1,1]^3$, the standard $3D$ cube, such that either h sends the local stable sets of points in $V(x)$ onto horizontal squares and the unstable sets onto vertical lines or, alternatively, sends the local stable sets of points in $V(x)$ onto vertical lines and the unstable sets onto horizontal squares.*

We remark that our definition of local product structure differs from the usual one (see [Sh]). We require the existence of local basic foliations. We say that a set $A \subset M$ has an f-local product structure (abbr. A has an f-lps) if every point of A has an f-lps in M. By definition A is open. The open ball of center x and radius $r > 0$ is the set $B(x,r) = \{y \in M/dist(x,y) < r\}$. As M is a compact $3D$-manifold there is $r_1 > 0$ such that for all $0 < r \leq r_1$ $B(x,r)$ is homeomorphic to B^3 the standard 3-cell of $I\!R^3$. We assume from now on that we take $r \leq r_1$. Throughout this paper we assume that M is a compact connected orientable smooth $3D$-manifold so we will usually avoid to state it.

2 The manifold M is homeomorphic to T^3

Throughout this section we assume that $f : M \to M$ is expansive $\Omega(f) = M$, and that there is a uniform local product structure invariant by f in M. Our goal is to prove that if M is a three dimensional compact connected smooth manifold then it is the three torus T^3. To prove this we will use the following theorem of [St].

Theorem 2.1 *Let M be a closed $3D$-manifold which is irreducible, that is, every tamely embedded 2-sphere in M bounds an embedded ball in*

M. If there is an epimorphism of $\Pi_1(M)$ to $\mathbb{Z}$ whose kernel is finitely generated and not $\mathbb{Z}_2$, then M is a fibre bundle over S^1 with fibre a closed 2-manifold.

Proof. See [St], see also [He-Hi], chapter VIII, theorem 3.2.9. ∎

We will follow closely techniques of [Fr] addapting them to our case in which we cannot make use of a hyperbolic sppliting of the tangent bundle. These techniques will be mixed up with others more closely related to those of [Ro] or [Rs] as in the following lemma.

Lemma 2.2 *Let M be a manifold and $p : \mathbb{R}^3 \to M$ a covering map. Then M is irreducible.*

Proof. Let S be a tamely embedded 2-sphere in M. As S is simply connected, there is a tamely embedded 2-sphere $S' \subset \mathbb{R}^3$ such that $p(S') = S$. For let S' be a connected lift of S and let $x_0 \in S$ and $\bar{x}_0 \in \mathbb{R}^3$ such that $p(\bar{x}_0) = x_0$. Consider a circle $\gamma \subset S$, $x_0 \in \gamma$ and lift γ to $\bar{\gamma}$, $\bar{x}_0 \in \bar{\gamma}$. Let γ be parameterized by $t \in [0,1]$, $\gamma(0) = \gamma(1) = x_0$, and let $\bar{\gamma}(0) = \bar{x}_0$. Then $\bar{\gamma}(1) = \bar{x}_0$. For we may construct a homotopy $H : [0,1] \times [0,1] \to S$ such that $H(t,0) = \gamma(t)$, $H(t,1) = x_0$, $\forall t \in [0,1]$ and lift H to $\bar{H} : [0,1] \times [0,1] \to S'$. Hence S' is simply connected and $p_{|S'}$ is a homeomorphism. It is proved in [Bi], that since S is tame we may approximate it by 2-spheres from both sides. Lifting them we conclude that S' is tame. By Schoenflies Theorem there is an embedded 3 ball $D' \subset \mathbb{R}^3$ such that $\partial D' = S'$. Is we prove that p is injective in D' we are done. Let β be any covering translation. If $\beta \neq id$ then $\beta(D') \cap D' = \emptyset$. For if $x \in \beta(D') \cap D'$ and $y \in D' \backslash \beta(D')$ then $\beta(S') \cap S' \neq \emptyset$ which is not possible because $p_{|S'}$ is a homeomorphism. Thus either $\beta(D') \subset D'$ or $D' \subset \beta(D')$. By Brouwer's Fixed Point Theorem, there is a point x such that $\beta(x) = x$. But this implies that $\beta = id$ contradicting our assumption. Hence p is injective in D' and therefore $p(D') = D$ is a ball bounded by S. ∎

Let $p : \bar{M} \to M$ be the simply connected covering of M, and $\bar{f} : \bar{M} \to \bar{M}$ a lift of $f : M \to M$. Then $\bar{f}$ is a homeomorphism which is also expansive with expansivity constant $\alpha > 0$ if we lift to $\bar{M}$ the metric d of M. Let us call this metric ρ. Moreover we may assume also (see [Du-Hu]) that there is a number $r_0 > 0$ such that

- 1) If $\bar{x}, \bar{y} \in \bar{M}$ and $\rho(\bar{x}, \bar{y}) \leq 2r_0$ then $d(p(\bar{x}), p(\bar{y})) = \rho(\bar{x}, \bar{y})$.
- 2) If $\bar{x} \in \bar{M}$, $y \in M$ and $d(p(\bar{x}), y) \leq 2r_0$ then there is a unique $\bar{y} \in \bar{M}$ such that $\bar{y} \in p^{-1}(y)$ and $\rho(\bar{x}, \bar{y}) = d(p(\bar{x}), y)$.
- 3) All covering transformations are C^0-isometries.
- 4) ρ is a complete metric.

In order to simplify notation we will write $d(\bar{x}, \bar{y})$ instead of $\rho(\bar{x}, \bar{y})$. For any point $x \in M$, $W^s(x) = \cup n \geq 0 f^{-n}(W^s_\epsilon(f^n(x)))$ is an immersed copy of $\mathbb{R}^2$ and $W^u(x) = \cup n \geq 0 f^n(W^u_\epsilon(f^{-n}(x)))$ is an immersed copy of $\mathbb{R}$. The properties cited in 2 enables us, if $\epsilon < r_0$, to lift the local product structure obtaining $\bar{W}^s_\epsilon(\bar{x}) \cong W^s_\epsilon(p(\bar{x}))$, $\bar{W}^u_\epsilon(\bar{x}) \cong W^u_\epsilon(p(\bar{x}))$, and to lift $W^s(x)$ and $W^u(x)$ obtaining $\bar{W}^s(\bar{x})$ and $\bar{W}^u(\bar{x})$ in a similar way. Thus, $\bar{W}^s(\bar{x}) \cong \mathbb{R}^2$ and $\bar{W}^u(\bar{x}) \cong \mathbb{R}$

Proposition 2.3 *For any $\bar{x} \in \bar{M}$, $p_{|\bar{W}^s(\bar{x})}$ is injective.*

Proof. p is a local homeomorphism and if $0 < d(\bar{x}, \bar{y}) < \epsilon$ then $p(\bar{x}) \neq p(\bar{y})$. If $\bar{x}, \bar{y} \in \bar{W}^s(\bar{z})$ and $p(\bar{x}) = p(\bar{y})$, since $f \circ p = p \circ \bar{f}$ then for all $n \geq 0$, $p(\bar{f}^n(\bar{x})) = p(\bar{f}^n(\bar{y}))$. There is $n \geq 0$ such that $d(\bar{f}^n(\bar{x}), \bar{f}^n(\bar{y})) < \epsilon$. Hence $\bar{f}^n(\bar{x}) = \bar{f}^n(\bar{y})$ and therefore $\bar{x} = \bar{y}$. ∎

Similarly $p_{|\bar{W}^u(\bar{x})}$ is injective. Let us sketch a more geomeric proof: if there are $\bar{x}, \bar{y} \in \bar{W}^s(\bar{z})$, $\bar{x} \neq \bar{y}$ such that $p(\bar{x}) = p(\bar{y})$, then we have a non-contractible loop in $W^s(z)$, which is not possible.

Lemma 2.4 *$Per_H(f)$ is dense in M.*

Proof. See [Vi2] Proposition 2.15. ∎

Lemma 2.5 *If $p \in Per_H(f)$ then $clos(W^u(p)) = M$.*

Proof. Let k be the period of p, and consider a point $x \in W^u_\epsilon(p)$ and assume that $x \neq p$ is so close to p that $f^k(x) in W^u_\epsilon(p)$ too. Let us denote by $[x, f^k(x)]$ the arc in $W^u_\epsilon(p)$ bounded by x and $f^k(x)$. Moreover let

$K = \text{clos}(\cup_n f^n([x, f^k(x)])$. Assume $y \in K$ and let N be a product neighbourhood of y. Let q be a periodic point of f in N. Then, by the lps, $W^s_\epsilon(q)$ intersects $\cup_n f^n([x, f^k(x)])$ in a point z. If n_0 is a multiple of the period of q then we have that $\lim_{k\to+\infty} f^{kn_0}(z) = q$ and $f^{kn_0}(z) \in W^u(p)$. Therefore $q \in K$ and $K \supset N$. Hence $K \neq \emptyset$ is open in M so it is equal to M. ∎

Similarly we have $\text{clos}(W^s(p)) = M$ it is enough to consider a fundamental domain in $W^s(p)$ given by an annulus instead of a segment.

Lemma 2.6 *There is a point $x \in M$ such that clos$(\mathcal{O}(x)) = M$. That is, the orbit of x is dense in M.*

Proof. Let V be a neighbourhood in M. Let $p \in V \cap Per_H(f)$ and let $y \in V \cap W^u_\epsilon(p)$. Then, as in 2.5, if k is the period of p, the segment $[p, y] \subset W^u_\epsilon(p)$ contains a fundamental domain of the form $[x, f^k(x)]$ or $[x, f^{2k}(x)]$, the second case would be necessary if f^k reverse the orientation in $W^u(p)$. Thus $f^{2kn}([p, y])$ is dense in M, the proof is the same as that of 2.5, taking into account that we may take as n_0 in the proof of 2.5 a multiple of the period of p. Moreover, if $n > m$ then $f^{2kn}([p, y]) \supset f^{2km}([p, y])$. Hence the orbit of V is dense. We have shown that f is topologically transitive. This is equivalent to the thesis (see [Sh], Chapter VIII). ∎

Remark 2.7 *For all $x \in M$ we have $W^u(x) \cong \mathbb{R}$. Hence $W^u(x)\setminus\{x\}$ is not arcwise connected and has two components which we call $W^{u+}(x)$ and $W^{u-}(x)$*

Lemma 2.8 *For all $x \in M$ we have clos$(W^s(x))$ = clos$(W^{u+}(x))$ = clos$(W^{u-}(x)) = M$.*

Proof. Let $x \in M$ and $W^{u+}_\epsilon(x) = W^{u+}(x) \cap W^u_\epsilon(x)$. Let $x' \neq x$ be a point in $W^{u+}_\epsilon(x)$, and z so close to x' that by the local product structure $W^s_\epsilon(z)$ intersects $W^{u+}_\epsilon(x)$ in a point y. The positive orbit of z is dense in M (as well as the negative orbit), and the distance between $f^n(z)$ and $f^n(y)$ tends to zero as $n \to +\infty$. Hence the orbit of y is dense in M

and consequently $\text{clos}(W^{u+}(x)) = M$. The proof for the other cases is similar. ∎

Let $\Pi_1(M)$ be the fundamental group of M and $\beta \in \Pi_1(M)$ a covering translation.

Corollary 2.9 *For all $\bar{x} \in \bar{M}$ it holds*

$$\bigcup_{\beta \in \Pi_1(M)} \beta(\bar{W}^u(\bar{x}))$$

is dense in $\bar{M}$.

Proof. Let $V \neq \emptyset$ be open in $\bar{M}$. Hence $p(V)$ is open in M and $W^u(p(\bar{x}))$ is dense in $p(V)$. Therefore there are $y \in W^u(p(\bar{x})) \cap p(V)$, $\bar{y}_1 \in \bar{W}^u(\bar{x})$ and $\bar{y}_2 \in V$ such that $p(\bar{y}_1) = p(\bar{y}_2) = y$. We may find $\beta \in \Pi_1(M)$ such that $\beta(\bar{y}_1) = \bar{y}_2$ so $\beta(\bar{W}^u(\bar{x})) \cap V \neq \emptyset$. ∎

Remark 2.10 *We may replace $W^u(\bar{x}))$ by $W^{u+}(\bar{x}))$ or $W^{u-}(\bar{x}))$ in the above proof.*

The following proof is quite similar to that of [Fr], lemma 5.1. Nevertheless, as we don't have a hyperbolic splitting of the tangent bundle there are some differences and we prefer to rewrite it.

Lemma 2.11 *For any pair of points $\bar{x}$, $\bar{y} \in \bar{M}$ there is at most a point in $\bar{W}^u(\bar{x}) \cap \bar{W}^s(\bar{y})$.*

Proof. Assume, arguing by contradiction that there are $\bar{x}, \bar{y} \in \bar{M}$ such that $\bar{x} \neq \bar{y}$ and $\bar{x}, \bar{y} \in \bar{W}^u(\bar{x}) \cap \bar{W}^s(\bar{y})$. In this case we find a closed loop topologically transversal to the foliation $\{\bar{W}^s\}$. Let $\gamma : [0,1] \to \bar{W}^u(\bar{x})$ be a homeomorphism onto the arc in $\bar{W}^u(\bar{x})$ joining $\bar{x} = \gamma(0)$ with $\bar{y} = \gamma(1)$ Since $\bar{W}^s(\bar{x}) = \bar{W}^s(\bar{y})$, there is $n > 0$ such that $\bar{f}^n(\bar{x})$, $\bar{f}^n(\bar{y})$ lie in the same neighbourhood $\bar{N}$ mapped homeomorphically and isometrically (see 2) onto $N \subset M$ by $p : \bar{M} \to M$. Hence there is an $\bar{f}$-lps defined on $\bar{N}$. Let $\beta(t) = \bar{f}^n \circ \gamma(t)$, $0 \leq t \leq 1$. By the local product structure, $\beta(t)$ is topologically transversal to $\{\bar{W}^s\}$. Moreover, $\beta(t)$ intersects $\bar{W}^s(\bar{f}^n(\bar{x}))$ with the same orientation at $\beta(0)$ and $\beta(1)$. For $\bar{M}$ is simply connected and hence orientable. Since there is a lps in $\bar{M}$ invariant by $\bar{f}$, we have that for any $t_0 \in [0,1]$, we have that there is a neighbourhood $E(t_0,\delta)$ such that for all $t \in E(t_0,\delta)$, $\beta(t)$ crosses in the same direction $\bar{W}^s(\beta(t)) \cong I\!R^2$. Therefore, if we have different orientations in $\beta(0)$ and $\beta(1)$ we may decompose $[0,1]$ into two non-void disjoint open sets contradicting that $[0,1]$ is connected. We may change $\beta(t)$ to form.a closed loop $\lambda(t)$ transversal to $\{\bar{W}^s\}$ in the topological sense. By the local product structure of $\bar{N}$ we may assume that we are in the standard cube $[-1,1]^3$ of $I\!R^3$ that $\bar{W}^s(\bar{f}^n(\bar{x}))$ is locally in the $0xy$ plane and that the vertical foliation is the unstable one. It is easy to modify in this cube the local unstable leaves by $\beta(0)$ and $\beta(1)$ without loosing the transversality in order to obtain a loop.

As $\bar{M}$ is simply connected, λ bounds a 2-disk D. That is, there is $\varphi : D^2 \to \bar{M}$ a homeomorphism such that $\varphi_{|\partial D^2} = \lambda$ (identify in the usual way $[0,1]$ with $\partial D^2 = S^1$). Hence for every $t \in \partial D^2$, $\varphi(t)$ is topologically transversal at t to the leaf $\bar{W}^s(\varphi(t))$. By [So], section 2, we may assume that D is in general position with respect to the foliation $\{\bar{W}^s\}$. Thus the intersection of $\{\bar{W}^s\}$ with D has a finite number of singular points of "saddle" and "focus" type. That is, they are locally topologically conjugate to the foliation given by either $\pm(x^2 - y^2)$ or $\pm(x^2 + y^2)$ around $(0,0) \in I\!R^2$. Moreover each singular point lies in a different leaf. Using φ we may lift $\{\bar{W}^s\} \cap D$ to a flow on D^2, such that the curves of the flow are transverse to ∂D^2 and all of them enter in D^2. We now apply the usual Poincare-Bendixson theory to this situation. Let $c : [0,+\infty] \to D^2$ be one of the curves crossing ∂D^2. Let $\omega(c) = \{x \in D^2 / \exists t_n \to +\infty : \lim c(t_n) = x\}$ If $\omega(c) = \{x_0\}$ a single point, then x_0 is one of the saddle points and is only possible to have two such curves for each saddle point. Since by general position there is only a finite number of saddle points, we can find a curve c' for which $\omega(c')$ is more than one point. Hence it is a closed curve, or a crossing eight depending on the fact that $\omega(c')$ contains or not a singular point (it cannot contain an arc joining two singularities because each singularity is in a different leaf, hence $\omega(c')$ contains at most one singularity). In either case $\varphi(\omega(c'))$ is a loop in a single leaf which represents a non trivial holonomy element of that leaf. This contradicts that every leaf of $\{\bar{W}^s\}$ is homeomorphic to $\mathbb{R}^2$, hence simply connected and with trivial holonomy. Thus the thesis follows. ■

Lemma 2.12 *Every leaf $\bar{W}^s(\bar{x})$ of the stable foliation is a proper leaf. Moreover it separates $\bar{M}$ in two connected components.*

Proof. Let us recall that a leaf F of a foliation is said to be proper if the topology induced on F by the topology of the space is the same as the topology induced on F by the topology of the leaves. It is proved in [Ha] that a leaf is proper if and only if there is a transverse submanifold cutting it in a single point.
Take $\bar{W}^u_\epsilon(\bar{x})$. Clearly by the local product structure it is transverse to $\bar{W}^s(\bar{x})$. Moreover, by 2.11 it intersects $\bar{W}^s(\bar{x})$ just in $\bar{x}$. Hence $\bar{W}^s(\bar{x})$ is proper. To prove that $\bar{W}^s(\bar{x})$ separates let us define $V = \bar{M}\backslash\bar{W}^s(\bar{x})$. Then V is a manifold. By the exact homology sequence we have that

$$\to H_1(\bar{M}) \to H_1(\bar{M}, V) \to H_0(V) \to H_0(\bar{M}) \to H_0(\bar{M}, V) \to (0)$$

Since $V \neq \emptyset$ and $\bar{M}$ is arcwise connected it holds that $H_0(\bar{M}, V) = (0)$ which gives us the exact sequence

$$\to H_1(\bar{M}) \to H_1(\bar{M}, V) \to H_0(V) \to H_0(\bar{M}) \to (0)$$

Since $\bar{M}$ is simply connected we have that $H_1(\bar{M}) = (0)$ and $H_0(\bar{M}) = \mathbb{Z}$. Using Alexander-Pontrjaguin's Duality, $H_1(\bar{M}, V) = H^2_c(\bar{W}^s(\bar{x}))$, where H^*_c represents cohomology with compact support. Again by duality $H^2_c(\bar{W}^s(\bar{x})) = H_0(\bar{W}^s(\bar{x})) = \mathbb{Z}$. Therefore $H_0(V) \cong \mathbb{Z} \oplus \mathbb{Z}$ which implies that V has two connected components ■.

Proposition 2.13 *For any $\bar{x}, \bar{y} \in \bar{M}$ we have that $\bar{W}^u(\bar{x}) \cap \bar{W}^s(\bar{y}) \neq \emptyset$.*

Proof. Let $\bar{x} = \bar{x}_0$ and define

$$Q = \{\bar{y} \in \bar{M} / \bar{W}^u(\bar{x}_0) \cap \bar{W}^s(\bar{y}) \neq \emptyset\}$$

We will prove that Q is open and closed in $\bar{M}$. As it is clearly non void, it will follow by the connectedness of $\bar{M}$ that it equals $\bar{M}$. Let $\bar{y} \in Q$ and $\bar{z} \in W^u(\bar{x}_0) \cap \bar{W}^s(\bar{y})$. Let N be such that $\bar{f}^N(\bar{y})$ lies in a product neighbourhood of $\bar{f}^N(\bar{z})$. Then for all y' in a neighbourhood V' of $\bar{f}^N(\bar{y})$ has the property that $\bar{W}^s_\epsilon(\bar{y}')$ intersects $\bar{W}^u_\epsilon(\bar{f}^N(\bar{z}))$. Thus for every point $\bar{y}'$ in the neighbourhood $V = \bar{f}^{-N}(V')$i of $\bar{y}$ we have $\bar{W}^u(\bar{x}_0) \cap \bar{W}^s(\bar{y}') \neq \emptyset$. Hence Q is open. Let us show that Q is closed too. Let $p \in \text{clos}(Q)$. There is a sequence $\{\bar{y}_n\}$ such that $\bar{W}^u(\bar{x}_0) \cap \bar{W}^s(\bar{y}_n) \neq \emptyset$ and $\bar{y}_n \to p$. By the local product structure, $\bar{W}^s_\epsilon(\bar{y}_n)$ cuts $\bar{W}^u_\epsilon(p)$. Hence we may assume that $\bar{y}_n \in \bar{W}^u_\epsilon(p)$. Since $\bar{W}^u_\epsilon(p) \cong I\!R$ we may pick a fixed ordering $<$ for it. Let $\bar{y}_0 \in Q \cap \bar{W}^u_\epsilon(p)$ and suppose, without loss of generality, that $\bar{y}_0 < p$. $\bar{W}^u_\epsilon(p) \cong I\!R$ is embedded in $\bar{M}$ so by Lindeloff's theorem $Q \cap \bar{W}^u_\epsilon(p)$ is a countable union of open intervals. Let $Q \cap \bar{W}^u_\epsilon(p) = \cup_i(a_i, b_i)$ with $a_i < b_i$. To finish the proof it suffices to prove that $a_i, b_i \in Q$ For, clearly $Q \cap \bar{W}^u_\epsilon(p) \neq \emptyset$ and in that case $Q \cap \bar{W}^u_\epsilon(p) = \bar{W}^u_\epsilon(p)$. So assume that for all $\bar{y} : \bar{y} \in [\bar{y}_0, b)$ it holds that $\bar{W}^s(\bar{y}) \cap \bar{W}^u(\bar{x}_0) \neq \emptyset$. We wish to see that the same holds for b. Let us give an orientation to $\bar{W}^u(\bar{x}_0) \cong I\!R$. We define a function ϕ from $[\bar{y}_0, b) \subset \bar{W}^u_\epsilon(p)$ to $\bar{W}^u(\bar{x}_0)$ by $\phi(\bar{y}) = \bar{W}^s(\bar{y}) \cap \bar{W}^u(\bar{x}_0)$. By 2.11 this is well defined. Moreover ϕ is a monotone function. For since $\bar{M}$ is simply connected we have by 2.12 that for all $\bar{y}$ $\bar{W}^s(\bar{y})$ separates $\bar{M}$ in two connected components. So assume that the orientation we have given to $\bar{W}^u(\bar{x}_0)$ is such that $\bar{y} < \bar{y}'$ implies that $\phi(\bar{y}) < \phi(\bar{y}')$ and let $\bar{y}''$ be another point in $[\bar{y}_0, b)$ such that, for instance, $\bar{y}' < \bar{y}''$. Then $\phi(\bar{y}') < \phi(\bar{y}'')$. If this is not the case, then $\phi(\bar{y}'') < \phi(\bar{y}')$ and we may join by $\bar{W}^u(\bar{x}_0)$ the points $\phi(\bar{y})$ and $\phi(\bar{y}'')$ without intersecting $\bar{W}^s(\bar{y}')$ and then using $\bar{W}^s(\bar{y})$ and $\bar{W}^s(\bar{y}'')$ we join $\phi(\bar{y})$ and $\phi(\bar{y}'')$ with $\bar{y}$ and $\bar{y}''$ respectively, again without cutting $\bar{W}^s(\bar{y}')$. So they are in the same connected component with respect to $\bar{W}^s(\bar{y}')$ But it is clear that $\bar{W}^s(\bar{y}')$ locally separates $\bar{y}$ from $\bar{y}''$. Therefore they are in different connected components with respect to $\bar{W}^s(\bar{y}')$. This contradiction implies that $\phi(\bar{y}'') < \phi(\bar{y}')$ is not possible. Thus it holds that $B = \lim_{\bar{y} \to b^-} \phi(\bar{y})$ exists. Now, if B is finite and $\bar{W}^s(b) \cap \bar{W}^u(\bar{x}_0) \neq \emptyset$

then the point of intersection is B, if not, we will have points $\bar{y} \in [\bar{y}_0, b)$ near b such that $\bar{W}^s(\bar{y})$ intersects twice $\bar{W}^u(\bar{x}_0)$ one near B and the other near $\bar{W}^s(b) \cap \bar{W}^u(\bar{x}_0)$, which contradicts 2.11. Let us proof that $B = \infty$ cannot occur.

Lemma 2.14 *The number B is finite.*

Proof. Assume this is not the case. Then the function ϕ has domain $[\bar{y}_0, b) \subset \bar{W}^u_\epsilon(p)$ and image a half-line $\bar{W}^{u+}(\phi(\bar{y}_0))$. This function is a homeomorphism between $[\bar{y}_0, b)$ and $\bar{W}^{u+}(\phi(\bar{y}_0))$ This half-line, by corollary 2.9 and remark 2.10, has the property that

$$\bigcup_{\beta \in \Pi_1(M)} \beta(\bar{W}^{u+}(\phi(\bar{y}_0))$$

is dense in $\bar{M}$. Hence there is $\beta \in \Pi_1(M)$ such that $\beta(\bar{W}^{u+}(\phi(\bar{y}_0))$ has a point $\bar{z}$ in a product neighbourhood N of $[\bar{y}_0, b)$. This product neighbourhood exists because $[\bar{y}_0, b)$ is a compact interval in $\bar{W}^u_\epsilon(p)$. Hence there is a segment of $\bar{W}^u(\bar{z})$ crossing the product neighbourhood N. $\beta^{-1}(N)$ gives a product neighbourhood N' around $\beta^{-1}(z) \in \bar{W}^{u+}(\phi(\bar{y}_0))$. Hence for all points in N' it is defined the intersection between their local stable sets and $\bar{W}^{u+}(\phi(\bar{y}_0))$ Take a point $\bar{y} \in [\bar{y}_0, b]$ and define a function h by

$$\bar{y} \mapsto \beta^{-1}(y) \mapsto w = \bar{W}^s_\epsilon(\beta^{-1}(y) \cap \bar{W}^{u+}(\phi(\bar{y}_0)) \mapsto \phi^{-1}(w) \in [\bar{y}_0, b)$$

We have a map h from the *closed* interval $[\bar{y}_0, b]$ to itself. Hence it has a fixed point $\bar{y}$. But this means that $\bar{W}^s(\bar{y}) = \bar{W}^s(\beta^{-1}(\bar{y}))$. By 2.3 the covering projection $p : \bar{M} \to M$ is injective restricted to $\bar{W}^s(\bar{y})$. But $p(\bar{y}) = p(\beta^{-1}(\bar{y}))$. Hence $\beta = id$, so B is finite. This is absurd. Therefore B is finite. ∎

To finish the proof of 2.13 we must prove that $B \in \bar{W}^s(b)$. Let us take a path $\gamma(t)$, $t \in [0,1]$ in $\bar{W}^s(\bar{y}_0)$ between $\gamma(0) = \bar{y}_0$ and $\gamma(1) = \phi(\bar{y}_0)$. For every t we may give an orientation to $\bar{W}^u(\gamma(t))$ such that if $\bar{y} < \bar{y}'$ then $\bar{W}^s(\bar{y}) \cap \bar{W}^u(\gamma(t)) < \bar{W}^s(\bar{y}') \cap \bar{W}^u(\gamma(t))$. By the local product structure this is well defined in a neighbourhood V of $\bar{y}_0$ in $[\bar{y}_0, b)$. For every $t \in [0,1]$ we may define a function ϕ_t analogous to ϕ which will be monotone too. We claim that for all $\gamma(t)$ and $\bar{y} \in [\bar{y}_0, b]$ we have $\bar{W}^s(\bar{y}) \cap \bar{W}^u(\gamma(t)) \neq \emptyset$ This clearly implies (take $\bar{y} = b$) the thesis. Let t^* be the supremum of those $t \in [0,1]$ such that $\bar{W}^s(\bar{y}) \cap \bar{W}^u(\gamma(t)) \neq \emptyset$ for all $\bar{y} \in [\bar{y}_0, b]$ once again by the local product structure this set is a non-empty open set. If $t^* \neq 1$ then we have that ϕ_{t^*} diverges to ∞ when $\bar{y} \to b$. But in this case we may repeat the arguments of 2.14 hence arriving to a contradiction. This finish the proof. ■

Definition 2.1 *We say that f is splitting if $\bar{M}$ is homeomorphic to $I\!R^2 \times I\!R$ by a homeomorphism which takes each leaf of the unstable foliation $\{\bar{W}^u\}$ to a line and each leaf of the stable foliation $\{\bar{W}^s\}$ to a plane.*

Theorem 2.15 *The expansive homeomorphism f is splitting.*

Proof. If $\bar{W}^u(\bar{z}_0) \in \{\bar{W}^u\}$ and $\bar{W}^s(\bar{z}_0) \in \{\bar{W}^s\}$ then $h : \bar{M} \to \bar{W}^u(\bar{z}_0) \times W^s(\bar{z}_0)$ given by $h(\bar{x}) = (\bar{W}^s(\bar{x}) \cap \bar{W}^u(\bar{z}_0), \bar{W}^u(\bar{x}) \cap \bar{W}^s(\bar{z}_0)$ is the desired homeomorphism. ■

Corollary 2.16 *We have $\bar{M} = I\!R^3$.*

Proposition 2.17 *$\Pi_1(M)$ is free Abelian.*

Proof. Since $\dim(\bar{W}^u(\bar{x})) = 1$ the stable foliation is homeomorphic to $I\!R$; we identify points in the same stable leave. $\Pi_1(M)$ acts sending stable leaves onto stable leaves. Hence it acts as a group of homeomorphisms of $I\!R$. These homeomorphisms are fixed point free because if $\beta \in \Pi_1(M)$ then β restricted to the unstable leaves is injective. By corollary 2.9, for all $\bar{x} \in \bar{M}$, $\cup_{\beta \in \Pi_1(M)} \beta \bar{W}^s(\bar{x})$ is dense in $\bar{M}$. Thus every orbit of $\Pi_1(M)$ is dense in $\{\bar{W}^s\} \cong I\!R$. It follows from this that $\Pi_1(M)$ is minimal (see [He-Hi], pp 243-) and we have that $\Pi_1(M)$ is free abelian (see [He-Hi], part B, chapter VIII, section 3). ■

We have that $\Pi_1(M)$ is free abelian. It is a non trivial group, for by the C^0 version of Haefliger's Theorem, every foliation of codimension 1 of a simply connected manifold, has a non simply connected leaf. (see [So]). Let $x_1, \ldots, x_k$ be a basis of $\Pi_1(M)$ and define an epimorphism of $\Pi_1(M)$

to $\mathbb{Z}$ by sending x_1 to 1, x_i to 0 for $i > 1$ and extending it linearly: $\Pi_1(M) \xrightarrow{e} \mathbb{Z}$. $\ker(e) \neq \mathbb{Z}_2$ and is finitely generated. Hence by 2.1, M is a fibre bundle over S^1 with fibre a closed 2-manifold T. That is, there is $\varphi : M \to S^1$ such that $\varphi^{-1}(x) = T_x \cong T$ and $M \cong T \times \mathbb{R}/\sim$ where $(x,t) \sim (x',t')$ iff there is $n \in \mathbb{Z}$ such that $t' = t + n$ and $x' = H^{-n}(x)$, with H a homeomorphism, $H : T \to T$. Consider the exact homotopy sequence of this fibre bundle: $T \overset{i}{\hookrightarrow} M \xrightarrow{\varphi} S^1$

$$\Pi_n(T) \xrightarrow{i_*} \Pi_n(M) \xrightarrow{\varphi_*} \Pi_n(S^1) \xrightarrow{\Delta} \Pi_{n-1}(T) \overset{i_*}{\Pi}_{n-1}(M)$$

$$\cdots \to \Pi_2(S^1) \xrightarrow{\Delta} \Pi_1(T) \xrightarrow{i_*} \Pi_1(M) \xrightarrow{\varphi_*} \Pi_1(S^1)$$

Since M is covered by $\mathbb{R}^3$ and $\Pi_n(\mathbb{R}^3) = (0)$ for all $n \geq 1$ and $p_* : \Pi_n(\mathbb{R}^3, \bar{x}_0) \cong \Pi_n(M, \bar{x}_0)$, $p(\bar{x}_0) = x_0$, for all $n \geq 2$ we have that $\Pi_n(M) = (0)$, $n \geq 2$ (see [Sp] chapter 7). Moreover, $\mathbb{R}$ is a covering of S^1, so $\Pi_n(S^1) = (0)$, $n \geq 2$. Therefore the sequence is trivial for $n \geq 2$. If $n = 1$ we obtain

$$(0) \to \Pi_1(T) \to \Pi_1(M) \to \Pi_1(S^1) \cong \mathbb{Z}$$

It follows that $\Pi_1(T)$ is mapped injectively in $\Pi_1(M)$. Since $\Pi_1(M)$ is free abelian, $\Pi_1(T)$ is free abelian. Thus $T \cong S^1 \times S^1 = T^2$. As M fibers over S^1, we have that $M \cong T^2 \times \mathbb{R}/\sim$,with $(x, t+1) \sim (H(x), t)$. Hence $\Pi_1(M)$ is the free group on three generators g, h, k with the relations $gh = hg$, $k^{-1}gk = H_*(g)$, with g, h generators of $\Pi_1(T^2)$ and $H_* : \Pi_1(T^2) \to \Pi_1(T^2)$ the homomorphism induced by H. $\Pi_1(M)$ is Abelian, so $H_*(g) = g$ and $H_*(h) = h$. This implies that H is homotopic to the identity of T^2. Hence it is isotopic to the identity. Therefore $M \cong T^2 \times S^1$. We have proved:

Theorem 2.18 *M is homeomorphic to T^3*

References

[Bi] BING R. H., *The Geometric Topology of 3-Manifolds*, AMS Colloquium Publications, **volume 40** 1983.

[Du-Hu] DUVALL P. F. AND HUSCH L. S., *Analysis on Topological Manifolds,* Fundamenta Mathematicae, **77** 1972.

[Fa] FALCONER , *The Geometry of Fractal Sets,* Cambridge University Press, 1985.

[Fr] FRANKS, J., *Anosov diffeomorphisms,* Proceedings of the Symposium in Pure Mathematics, **14** (1970), p. 61-94.

[Ha] HAEFLIGER, A., *Varietés feuilletées,* Annali della Scuola Norm. Sup. Pisa, **16** (1962), p. 367-397.

[He-Hi] HECTOR G. AND HIRSCH U., *Introduction to the Geometry of Foliations,* Aspects of Mathematics, .

[Le1] LEWOWICZ, J., *Persistence in Expansive Systems,* Erg. Th. & Dyn. Sys., **3** (1983), p. 567-578.

[Le2] LEWOWICZ, J., *Expansive homeomorphisms of surfaces,* Bol. Soc. Bras. Mat, **20 (1)** (1989), p. 113-133.

[Ma] MAÑÉ R., *Expansive Homeomorphisms and Topological Dimension,* Trans. Amer. Math. Soc., **252** (1979), p. 313-319.

[Mo] MOISE E. E., *Geometric Topology in Dimensions 2 and 3,* Springer-Verlag, **GTM 47** 1977.

[Nw] NEWHOUSE S., *On Codimension One Anosov Diffeomorphisms,* A.J.Math., **92** (1970), p. 761-770.

[Ro] ROUSSARIE R., *Sur les feuilletages des variétés de dimension trois,* Ann. Inst. Fourier, Grenoble, **21,3** (1971), p. 13-82.

[Rs] ROSENBERG H., *Foliations by planes,* Topology, **7** (1968), p. 131-138.

[Sh] SHUB M., *Global Stability of Dynamical Systems,* Springer-Verlag, 1987.

[So] SOLODOV V.V., *Components of topological foliations,* Math. USSR Sbornik, **47** (1984), p. 329-343.

[Sp] SPANIER E. H., *Algebraic Topology,* Mc. Graw-Hill, 1966.

[St] STALLINGS J. , *On fibering certain 3-manifolds,* Topology of 3-manifolds, **Prentice-Hall** (1962), p. 95-100.

[Vi1] VIEITEZ, J. L., *Expansive Homeomorphisms and Hyperbolic Diffeomorphisms on three manifolds,* Erg. Th. & Dyn. Sys., **16** (1996), p. 591-622.

[Vi2] VIEITEZ, J. L., *Three Dimensional Expansive Diffeomorphisms with Homoclinic Points,* Bol. Soc. Bras. Mat., **27** (1996), p. 55-90.

[Wi] WILDER R. L., *Topology of Manifolds,* AMS Colloquium Publications, **Volume 32** 1979.

Beyond the exceptional covers and their canonical hyper-elliptic potentials

Armando TREIBICH

Contents

The main purpose of this paper is to extend, beyond the exceptional case, our construction of hyperelliptic tangential covers and their canonically associated hyper-elliptic potentials. We recall for the convenience of the reader, in sections §1, §2, §3 and §5, the basic results on the subject, as well as earlier joint work.

The author must state that, except for §8 and §9, all other paragraphs were discussed, worked out and developed together with Jean-Louis Verdier (more than 9 years ago !). I should also stress that the starting point for the full characterization obtained in §9 (i.e. : the "fundamental" equation (27) and the plane model approach of §7) are due to him. Furthermore, §4 and §7 are partly based upon his own hand-written notes. Put another way, this is a joint article which should've been finished years from now.

1 Hyperelliptic tangential covers vs. elliptic KdV finite-gap potentials.

1.1

Let Γ be an integral projective curve of arithmetic genus $g > 0$ and fix a local coordinate, say λ, of Γ at a smooth point $p \in \Gamma$. Choose any effective divisor D of degree g and let m be the greatest integer such that $D - mp$ is still an effective divisor. Then, there exists a unique (so-called *Baker-Akhiezer*) function, denoted by ψ_D, meromorphic over $\mathbf{C}^3 \times (\Gamma \setminus \{p\})$ and satisfying the following conditions (see [8],[9], and also [3],[4]) : for any sufficiently generic $(x, y, t) \in \mathbf{C}^3$, the restriction of ψ_D to $\{(x, y, t)\} \times (\Gamma \setminus \{p\})$ has :

a) its divisor of poles equal to $D - mp$;

b) an essential singularity at p, with the following normalized local development at p :

$$(1)\quad \psi_D(x, y, t; \lambda) = \frac{1}{\lambda^m} \exp(x\lambda^{-1} + y\lambda^{-2} + t\lambda^{-3})(1 + \sum_{j\geq 1}^{\infty} \xi_{j,D}(x, y, t)\lambda^j) .$$

1.2

We shall say in the sequel that (Γ, p) is a *hyperelliptic marked curve* if and only if there exists a projection $f : \Gamma \to \mathbf{P}^1(\mathbf{C})$ such that $f^*(\infty) = 2p$. In the latter case Γ is canonically equipped with a hyperelliptic involution, $\sigma : \Gamma \to \Gamma$ $(\Gamma/_{\sim\sigma} \simeq \mathbf{P}^1(\mathbf{C}))$, which fixes p. Hence p is a Weierstrass point and for any

σ-anti-invariant local coordinate of Γ at p, say λ, we have that $\sigma^*(\lambda) = -\lambda$ and $f(\lambda) = \frac{1}{\lambda^2} + O(\lambda^2)$. Furthermore, for any divisor D as above, the corresponding Baker-Akhiezer function satisfies the following differential equation (Ibid) :

$$(2) \qquad \Big(\frac{\partial^2}{\partial x^2} - u_D\Big)\psi_D(x,y,t;\lambda) = f(\lambda)\psi_D(x,y,t;\lambda) \ ,$$

where $u_D(x,y,t) := -\frac{\partial}{\partial x}\big(\xi_{1,D}(x,y,t)\big)$. In other words, ψ_D is an eigenfunction of the Schrödinger operator $\frac{\partial^2}{\partial x^2} - u_D$, with eigenvalue $f(\lambda)$. Moreover, the function u_D is independent of y and satisfies the Korteweg-de Vries equation. Having fixed a σ-anti-invariant local coordinate of Γ at p, say λ , the initial value function $u(x) = u_D(x,0,0)$ is uniquely determined by the isomorphism class of $\mathcal{O}_\Gamma(D-p)$. We'll say hereafter that $u(x)$ is the KdV finite-gap potential associated to $\big(\Gamma, p, \lambda, \mathcal{O}_\Gamma(D-p)\big)$.

1.3

Let Λ be a lattice of $\mathbf{C}$ and $(X,q) = (\mathbf{C}/\Lambda, 0)$ the corresponding elliptic curve, equipped with the canonical local coordinate z, at its origin $q \in X$. Consider any finite marked projection $\pi : (\Gamma, p) \to (X,q)$, where p is a smooth point of Γ and $\pi(p) = q$. Naturally associated with π we have the Abel (rational) embedding, $A_p : \Gamma \to \operatorname{Jac}\Gamma$, $p' \mapsto \mathcal{O}_\Gamma(p'-p)$, and the homomorphism, $i_\pi : X \to \operatorname{Jac}\Gamma$, $q' \mapsto A_p\big(\pi^*(q'-q)\big)$, of Γ and X into the (generalized) jacobian of Γ .

1.4 Definitions .

1) The marked projection $\pi : (\Gamma, p) \to (X,q)$ is a *tangential cover* if and only if $i_\pi(X)$ and $A_p(\Gamma)$ are tangent at the origin, $A_p(p) = i_\pi(q)$, of $\operatorname{Jac}\Gamma$.

2) $\pi : (\Gamma, p) \to (X,q)$ is a *hyperelliptic tangential cover* if and only if π is a tangential cover and (Γ, p) is a hyperelliptic marked curve.

1.5 Theorem (tangency criterion ; .

A marked morphism $\pi : (\Gamma, p) \to (X,q)$ is a tangential cover if and only if there exists a meromorphic function over Γ, k, called *tangential function*, satisfying the following conditions :

1) k is holomorphic outside $\pi^{-1}(q)$;

2) $k + \pi^*(z^{-1})$ is well defined over a neighborhood of $\pi^{-1}(q) \setminus \{p\}$, and has a simple pole at p.

1.6 Definition .

The moduli space of all torsion-free, rank 1, coherent sheaves of degree $g-1$ over the above curve Γ, denoted by $W(\Gamma)$, exists and will be henceforth called *the compactified jacobian* of Γ . Let ω_Γ denote the canonical sheaf over Γ and, for any $\mathcal{L} \in W(\Gamma)$, let $\mathcal{L}^*$ denote the sheaf $\mathcal{L}^* := \mathcal{H}om(\omega_\Gamma, \mathcal{L})$. Then $\mathcal{L}^*$ is again in $W(\Gamma)$ and $(\mathcal{L}^*)^*$ is isomorphic to $\mathcal{L}$. Consequently the map $\mathcal{L} \mapsto \mathcal{L}^*$ gives rise to an involution of $W(\Gamma)$. We shall say that $\mathcal{L} \in W(\Gamma)$ is a *θ-characteristic* if and only if $\mathcal{L}$ is fixed by the above involution, i.e. if $\mathcal{L} \simeq \mathcal{H}om(\omega_\Gamma, \mathcal{L})$.

1.7 Remarks.

1) The compactified jacobian $W(\Gamma)$ is a projective variety on which the generalized one, $\mathrm{Jac}\,\Gamma$, acts via the tensor product. Whenever Γ is a smooth curve, $W(\Gamma)$ is equal to $\mathrm{Jac}^{g-1}(\Gamma)$. If, as in our case, Γ can be embedded in a smooth algebraic surface, then $W(\Gamma)$ is reduced, irreducible and g-dimensional. Moreover, let Γ° denote the open subset of smooth points of Γ and let Γ°_g denote its g-th symmetric power, then $\mathrm{Jac}^{g-1}(\Gamma)$ is equal to $\{\mathcal{O}_\Gamma(D-p), D \in \Gamma^\circ_g\}$ and is a dense open subset of $W(\Gamma)$ (e.g. [2]) .

2) We can still associate a KdV finite-gap potential to any $\mathcal{L} \in W(\Gamma)$. There exists indeed a unique partial desingularization $j : \bar{\Gamma} \to \Gamma$, of arithmetic genus $\bar{g} \leq g$, and an invertible sheaf $\bar{\mathcal{L}} \in \mathrm{Jac}^{\bar{g}-1}(\bar{\Gamma}) \subset W(\bar{\Gamma})$, such that $j_*(\bar{\mathcal{L}})$ is isomorphic to $\mathcal{L}$ .We shall say that the corresponding potential $u_{\mathcal{L}}(x) := u_{\bar{\mathcal{L}}}(x)$ is associated to $\mathcal{L}$ ([16] §6).

3) Let $\pi : (\Gamma, p) \to (X, q)$ be any marked projection of degree $n \geq 1$, and $\Theta_\Gamma := \{\mathcal{L} \in W(\Gamma), h^\circ(\mathcal{L}) > 0\}$ the natural *theta divisor* of $W(\Gamma)$. For any $\mathcal{L} \in W(\Gamma)$, let $\mathrm{Orb}_{\mathcal{L}} : X \to W(\Gamma)$ denote the corresponding "orbit" morphism, $q' \mapsto \mathcal{L} \otimes i_\pi(q') = \mathcal{L} \otimes \mathcal{O}_\Gamma\big(\pi^*(q'-q)\big)$. Pulling back Θ_Γ by $\mathrm{Orb}_{\mathcal{L}}$ we obtain a divisor of degree n of X, $\sum_{i=1}^n \alpha_i = \mathrm{Orb}^*_{\mathcal{L}}(\Theta_\Gamma)$ (e.g. [16] §A2). We can, therefore, associate to any projection π as above, the morphism

$$I_\pi : W(\Gamma) \to X_n,\ \mathcal{L} \mapsto \mathrm{Orb}^*_{\mathcal{L}}(\Theta_\Gamma)\ ,$$

where X_n denotes the n-th symmetric power of X.

1.8 Definitions.

1) Let $\pi : (\Gamma, p) \to (X, q)$ be a hyperelliptic tangential cover and let $\sigma : \Gamma \to \Gamma$ denote its hyperelliptic involution. Let also $[-1] : (X, q) \to (X, q)$ denote

the inverse homomorphism. Then π is equivariant (i.e. : $\pi \circ \sigma = [-1] \circ \pi$) and there exists a unique σ-anti-invariant tangential function k . In particular, we can choose $\lambda_\pi := \left(k + \pi^*(z^{-1})\right)^{-1}$ as a σ-anti-invariant local coordinate of Γ at p .

2) Recall that the Weierstrass $\wp$-function associated to the lattice Λ is the unique meromorphic even function on X, holomorphic outside q, and having at q the Laurent development : $\wp(z) = \frac{1}{z^2} + O(z^2)$. We'll be mainly interested in $KdV(n, X)$ $(n \geq 1)$, the closure in X_n of the locally closed sub-variety defined by the following system of equations :

$$\sum_{1 \leq j \leq n, j \neq i} \wp'(\alpha_i - \alpha_j) = 0, \quad i = 1, \cdots, n, \quad \alpha_i \neq \alpha_j \quad \left(\sum_{k=1}^{n} \alpha_k \in X_n\right),$$

$\wp'$ being the usual derivative of $\wp$. It turns out that $KdV(n, X)$ parameterizes all KdV solutions of the form $u(x,t) = 2\sum_{i=1}^{n} \wp(x - x_i(t))$, and will be called henceforth, the *n-th KdV locus* (over X) (e.g. [1]).

1.9 Theorem .

Let $\pi : (\Gamma, p) \to (X, q)$ be a hyperelliptic tangential cover of degree n . Then for any $\mathcal{L} \in W(\Gamma)$, the KdV finite-gap potential associated to $(\Gamma, p, \lambda_\pi, \mathcal{L})$ is equal to $u_{\mathcal{L}}(x) = 2\sum_{i=1}^{n} \wp(x - \alpha_i)$, where $\sum_{i=1}^{n} \alpha_i$ is equal to $I_\pi(\mathcal{L})$ and belongs to $KdV(n, X)$. Moreover the morphism $I_\pi : W(\Gamma) \to X_n$ besides factoring through $KdV(n, X) \to X_n$, is also injective and equivariant (i.e. : for any $\mathcal{L} \in W(\Gamma)$, $I_\pi(\mathcal{L}^*) = [-1](I_\pi(\mathcal{L}))$). In particular, the potential $u_{\mathcal{L}}(x)$ is Λ-periodic for any $\mathcal{L} \in W(\Gamma)$, and it's even if and only if $\mathcal{L}$ is a θ-characteristic .

1.10 Remark.

Let $\pi : (\Gamma, p) \to (X, q)$ be as in §1.9, pick any partial desingularization $j : (\bar{\Gamma}, p) \to (\Gamma, p)$, and let j_* denote the closed immersion $j_* : W(\bar{\Gamma}) \to W(\Gamma)$, $\bar{\mathcal{L}} \mapsto j_*(\bar{\mathcal{L}})$. Then, the composed marked morphism, $\bar{\pi} := \pi \circ j : (\bar{\Gamma}, p) \to (X, q)$ is also a hyperelliptic tangential cover, and the corresponding injection, $I_{\bar{\pi}} : W(\bar{\Gamma}) \to KdV_n(X)$, factors as $I_{\bar{\pi}} = I_\pi \circ j_*$ ([11] §6). Consequently $I_{\bar{\pi}}(W(\bar{\Gamma}))$ is a closed subvariety of $I_\pi(W(\Gamma)) (\subset KdV(n, X))$ and the family of Λ-periodic, KdV finite-gap potentials associated to π embodies the one associated to $\bar{\pi}$.

1.11 Definition.

Let $\pi : (\Gamma, p) \to (X, q)$ be as in §1.9. We shall say that π is a *minimal-hyperelliptic tangential cover* if and only if π does not factor through a (non-isomorphic) hyperelliptic tangential cover of the same degree over X.

1.12 Theorem .

Let $R(n, X)$ denote the (finite ; see §2.10) set of isomorphism classes of minimal-hyperelliptic tangential covers of degree n over X, and let $W(n, X)$ denote the (disjoint) reunion of their compactified jacobians. Then, the natural morphism $I_n(X) : W(n, X) \to KdV(n, X)$, made up of all the I_π's, $\pi \in R(n, X)$, is a birational injection. In particular $KdV(n, X)$ splits as the disjoint reunion of the images of the corresponding compactified jacobians :

$$KdV(n, X) = \coprod_{\pi \in R(n,X)} I_\pi(W(\Gamma)) .$$

1.13 Definition.

Let $\pi : (\Gamma, p) \to (X, q)$ be any hyperelliptic tangential cover, $\mathcal{L} \in W(\Gamma)$, and let $u(x) := u_{\mathcal{L}}(x)$ denote the corresponding elliptic KdV finite-gap potential (§1.9).

1) We shall say then that $u(x)$ is the *hyper-elliptic potential* associated to $(\pi; \mathcal{L})$ (or to $(\Gamma, p, \lambda_\pi, \mathcal{L})$) ;

2) We shall say that $u(x)$, considered as a meromorphic function over $X = \mathbf{C}/\Lambda$, $u : X \to \mathbf{P}^1(\mathbf{C})$, is *non-primitive*, if and only if there exists a non-trivial isogeny $\varphi : (X, q) \to (X', q')$ factoring u .

1.14 Remarks.

1) Let $u(x) : X \to \mathbf{P}^1(\mathbf{C})$ be a non-primitive function as in §1.13.2) and $\hat{\varphi} : (X', q') \to (X, q)$ denote the dual isogeny. Then $d := \deg \varphi = \deg \hat{\varphi}$ divides n, $\hat{\varphi}$ factors π and the inverse image morphism, $\varphi^* : X'_{n/d} \to X_n$, $\sum_j \beta_j \mapsto \sum_j \varphi^*(\beta_j)$ factors $I_\pi : W(\Gamma) \to X_n$. More precisely, there exists a hyperelliptic tangential cover $\pi' : (\Gamma, p) \to (X', q')$ such that $n = d \cdot \deg \pi'$, $\pi = \hat{\varphi} \circ \pi'$ and $I_\pi = \varphi^* \circ I_{\pi'}$.

2) Conversely, if $\pi : (\Gamma, p) \to (X, q)$ factors through an isogeny of degree $d > 1$, $\psi : (X', q') \to (X, q)$, as $\pi = \psi \circ \pi'$, then $n = d \cdot \deg \pi'$, $\pi' : (\Gamma, p) \to (X', q')$ is a hyperelliptic tangential cover of degree n/d, and I_π factors as $I_\pi = \varphi^* \circ I_{\pi'}$, where φ is equal to $\hat{\psi}$, the dual isogeny of ψ. Therefore, any hyper-elliptic potential associated to π is non-primitive and factors through $\varphi = \hat{\psi} : X \to X'$. We shall also say that π is *non-primitive*.

1.15

With the above definitions and basic relations in mind, we shall explain hereafter the structure and methods developed to attain the main objectives of our paper.

1) The tangency criterion §1.5 is an intermediate step in the way to the algebraic surface aproach of §2 and §3. The leading parts are played there by the blow-up of a particular ruled surface $\pi_{S^\perp} : S^\perp \to S \to X$ and its quotient by the action of the natural involution $\tau^\perp$, $S^\sim := S^\perp / \sim_{\tau^\perp}$. We start proving that any hyperelliptic tangential cover $\pi : (\Gamma, p) \to (X, q)$ factors uniquely through $\pi_{S^\perp}$. We then say that π has "type" $\mu = (\mu_i)$ if the image of Γ in $S^\perp$ intersects the exceptional divisor $r_i^\perp$ with multiplicity μ_i, for $i = 0, \cdots, 3$ (§2.1, §2.2). The latter amounts to say (roughly speaking) that Γ has $\mu_i + \delta_{i0}$ Weierstrass points over the half period ω_i ($i = 0, \cdots, 3$). For any $\mu \in \mathbb{N}^4$ and $n \geq 1$ we then let $R(n, \mu, X)$ denote the subset of $R(n, X)$ made of all covers $\pi \in R(n, X)$ having "type" μ . The elements of $R(n, \mu, X)$ can be identified with integral divisors of a particular linear system on $S^\perp$. The knowledge of the corresponding linear equivalence class finally allows us to deduce the basic results about $R(n, \mu, X)$. For example, if the congruence conditions $\mu_0 + 1 \equiv \mu_1 \equiv \mu_2 \equiv \mu_3 \equiv n(\mathrm{mod} \cdot 2)$ and $(2n + 1 - \sum_{i=0}^{3} \mu_i^2) \in 4\mathbb{N}$ are not satisfied, the set $R(n, \mu, X)$ is void (§2.3). Otherwise $R(n, \mu, X)$, hence also $R(n, X)$, is finite and any $\pi \in R(n, \mu, X)$ has (arithmetic) genus equal to $\frac{1}{2}(-1 + \sum_{i=0}^{3} \mu_i)$ (§2.10, §2.6).

2) Since the natural involution $\tau^\perp : S^\perp \to S^\perp$ induces the hyperelliptic one on any curve Γ as above, the projection of Γ into $S^\sim := S^\perp / \sim_{\tau^\perp}$ is a rational integral curve of a particular linear system, say $|\lambda(n, \mu)|$, of $S^\sim$ (§3.1). Reciprocally, any rational integral divisor in $|\lambda(n, \mu)|$ gives rise to an element of $R(n, \mu, X)$ (§2.6). The latter result is particularly well suited for proving the existence of nice families of minimal-hyperelliptic tangential covers. As a matter of fact, we start associating an exceptional divisor of $S^\sim$ to any $\mu = (\mu_i) \in \mathbb{N}^4$ satisfying the congruences $\mu_0 +$

$1 \equiv \mu_1 \equiv \mu_2 \equiv \mu_3 (\text{mod}.2)$. The restriction of $\pi_{S^\perp} : S^\perp \to X$ to the inverse image (by $S^\perp \to S^\sim$) of the above exceptional divisor, is then shown to be a minimal-hyperelliptic tangential cover of type μ and degree $n := \frac{1}{2}(-1 + \sum_{i=0}^{3} \mu_i^2)$. We thus obtain the unique element of the moduli space $R(n, \mu, X)$, called hereafter "the exceptional cover of type μ (and degree $n := \frac{1}{2}(-1+\sum_{i=0}^{3} \mu_i^2)$) over X" (see §3.4, §4.1, §4.2, §4.3 and §4.4) .

3) The rest of §4 is devoted to study (for the first time !) the next cases. We start proving that for all $n \geq 1$ and $\mu \in \mathbf{N}^4$ such that $\mu_0 + 1 \equiv \mu_1 \equiv \mu_2 \equiv \mu_3 (\text{mod} \cdot 2)$ and $(2n + 1 - \sum_{i=0}^{3} \mu_i^2) \in 4\mathbf{N}$, the dimension of the linear system $|\lambda(n, \mu)|$ is equal to $\frac{1}{4}(2n + 1 - \sum_{i=0}^{3} \mu_i^2)$ (§4.9) . It then follows that whenever $\sum_{i=0}^{3} \mu_i^2 = 2n - 3$ (the first "non-exceptional case"!) the set $R(n, \mu, X)$ is in a one to one correspondence with the integral and rational fibers of a pencil of elliptic curves on $S^\sim$. Applying the Lefschetz formula (e.g. [7] p.509), relating the Euler-Poincaré characteristic of the fibers of the pencil with that of $S^\sim$, we finally succeed in bounding the cardinal of $R(n, \mu, X)$ (§4.12). The latter result implies the existence of a new family of hyperelliptic tangential covers (other than the exceptional one).

4) On and after the §5 we equip any hyperelliptic tangential cover of degree n and arithmetic genus g, say $\pi : (\Gamma, p) \to (X, q)$, with the following eight "canonical" θ-characteristics (§5.3. 3)) :

$$(3) \qquad \begin{cases} \xi = \mathcal{O}_\Gamma\big((g-1)p + \pi^*(\omega_i - q)\big) \quad \text{and} \\ \\ \xi = \mathcal{O}_\Gamma\big((g-1-n)p + \pi^*(\omega_i)\big) \quad (i = 0, \cdots, 3) , \end{cases}$$

where as usual we let $\{\omega_i, i = 0, \cdots, 3\}$ $(\omega_0 = q)$ denote the set $\frac{1}{2}\Lambda/\Lambda$, of $\frac{1}{2}$-periods of (X, q) . Finding the corresponding even hyper-elliptic potentials (see §1.9) amounts to calculating the intersection of their X-orbits $(\text{Orb}_\xi\, X)$ with the theta divisor Θ_Γ . Since we're dealing with (hyperelliptic) tangential covers, $\text{Orb}_\xi(X)$ is also the orbit of the first KP flow on $W(\Gamma)$ associated to the data (Γ, p, ξ). Accordingly, we can calculate its multiplicity of intersection with Θ_Γ , at any point, by means of the Fay-Segal & Wilson formula of §5 ([5],[11] ; see also [18] §3). The latter problem has been completely solved within the frame of exceptional covers, as explained in the rest of §5. We recall there that for any exceptional cover of type μ , the even hyper-elliptic potentials associated to the above "canonical" θ-characteristics are equal to

$$u_\xi(x) = \sum_{i=0}^{3} a_i(a_i+1)\wp(x-\omega_i) \ , \tag{4}$$

for some $(a_i) \in \mathbf{N}^4$ given in terms of (μ_i) (e.g.:§5.5), such that $2n = \sum_{i=0}^3 a_i(a_i+1)$. Conversely, given any integer vector $(a_i) \in \mathbf{N}^4$, (there exists a unique type μ such that) $u(x) = \sum_{i=0}^3 a_i(a_i+1)\wp(x-\omega_i)$ is the hyper-elliptic potential associated to one of the "canonical" θ-characteristics over an exceptional cover of (type μ), degree $n := \frac{1}{2}\sum_{i=0}^3 a_i(a_i+1)$ and arithmetic genus equal to $\frac{1}{2}\max\{2M, 1+\Sigma-(1+(-1)^{\Sigma})(m+\frac{1}{2})\}$, where $M := \max\{a_i\}$, $\sum := \sum_{i=0}^3 a_i$ and $m := \min\{a_i\}$ (see also [6]).

5) Since our aim is to generalize the above results beyond the exceptional case, we undertake (starting from §6) the analogous study for the next interesting case : $\bigcup_{2n-3=\Sigma\mu_i^2} R(n,\mu,X)$. We tackle the problem of characterizing the "canonically" associated hyper-elliptic potentials, by globalizing our earlier constructions. Let $\mathcal{X}$ denote the moduli space of all elliptic curves endowed with their half-periods, and let $\mathcal{R}(n,\mu)$ denote the algebraic family $\bigcup_{\mathcal{X}} R(n,\mu,X)$. We shall suppose (from §6 till the end) that $\mu^{(2)} = 2n-3$, and associate to any $\pi \in \mathcal{R}(n,\mu)$, one of its canonical θ-characteristics, $\pi \mapsto \xi$ (for ex : $\xi = \mathcal{O}_\Gamma((g-1)p))$. The corresponding hyper-elliptic potential, $u_\xi(x)$, is equal this time to $u_\xi(x) \equiv 2\wp(x-\rho) + 2\wp(x+\rho) + \sum_{i=0}^3 a_i(a_i+1)\wp(x-\omega_i)$, for some $\vec{a} = (a_i) \in \mathbf{N}^4$ such that $n = 2+\frac{1}{2}\sum_{i=0}^3 a_i(a_i+1)$ (§6 (29)) and $\rho \in X\backslash\{\omega_i\}$ satisfying $\sum_{i=0}^3(2a_i+1)^2\wp'(\rho-\omega_i) = 0$.

6) The equation $\sum_{i=0}^3(2a_i+1)^2\wp'(\rho-\omega_i) = 0$ is indeed a necessary condition for such a function to be a hyper-elliptic potential (§6.2). We also realize that the map $\rho \mapsto \big(\wp(\rho) : \wp(\omega_1) : \wp(\omega_2)\big) \in \mathbf{P}^2(\mathbf{C})$ defines an isomorphism between the set of solutions $\{\sum_{i=0}^3(2a_i+1)^2\wp'(\rho-\omega_i) = 0,\ \rho \in X,\ X \in \mathcal{X}\}$ and the affine sextic $\mathcal{C}_{\vec{a}}\backslash\{O_1,O_2,O_3\}$ (§7.3, §7.4). At last we prove that the composition of the latter maps, $\pi \mapsto \xi \mapsto u_\xi(x) \mapsto \big(\wp(\rho) : \wp(\omega_1) : \wp(\omega_2)\big)$, gives rise to a proper injection (in fact an embedding) of $\mathcal{R}(n,\mu)$ into $\mathcal{C}_{\vec{a}} \setminus \{O_1,O_2,O_3\}$ (§7.7, see also §9.6).

7) Conversely, for any $X \in \mathcal{X}$ and any solution of $\sum_{i=0}^3(2a_i+1)^2\wp'(\rho-\omega_i) = 0$ (or equivalently, for any point of $\mathcal{C}_{\vec{a}} \setminus \{O_1,O_2,O_3\}$) it's quite natural to ask whether the function $u(x) = 2\wp(x-\rho) + 2\wp(x+\rho) +$

$\sum_{i=0}^{3} a_i(a_i+1)\wp(x-\omega_i)$ is a hyper-elliptic potential (or not), and to find the corresponding spectral data $(\pi;\xi)$ in the affirmative case. For example, if $\mathcal{C}_{\vec{a}}$ is irreducible $\mathcal{R}(n,\mu)$ must cover the whole affine sextic $\mathcal{C}_{\vec{a}} \setminus \{O_1, O_2, O_3\}$, and any such function $u(x)$ is a hyper-elliptic function associated to a (well determined) canonical θ-characteristic over some $\pi \in R(n,\mu,X)$. More generally, the last statement remains true for any $u(x)$ as above if and only if $\Big(\wp(\rho) : \wp(\omega_1) : \wp(\omega_2)\Big)$ belongs to the image of $\mathcal{R}(n,\mu)$. Consequently, we're naturally lead to find the integer vectors $\vec{a} = (a_i) \in \mathbf{N}^4$ for which $\mathcal{C}_{\vec{a}}$ is reducible and to write down the corresponding decomposition. Surprisingly, the latter problem is (again) reduced in §8 to studying the reducible fibers of a pencil of elliptic curves (i.e. : the one generated by the sextics $\mathcal{C}_{\vec{a}}$ and $\mathcal{C}_0$) and applying the same Lefschetz formula as in §4.

8) As a matter of fact, the complement of the image of $\mathcal{R}(n,\mu)$ in $\mathcal{C}_{\vec{a}} \setminus \{O_1, O_2, O_3\}$ also parameterizes hyper-elliptic potentials, although of a different kind. The latter potentials are carefully constructed through §6.6, the proof of §9.4, §9.6.4) and §9.9. Our "elliptic surface" approach finally enables us to give at the end of §9, a reasonably complete answer to our previous questions : for any $\vec{a} = (a_i) \in \mathbf{N}^4$, any elliptic curve $X \in \mathcal{X}$ and any $\rho \in X \setminus \{\omega_i\}, u(x) = 2\wp(x-\rho) + 2\wp(x+\rho) + \sum_{i=0}^{3} a_i(a_i+1)\wp(x-\omega_i)$ is a hyper-elliptic potential if and only if $\sum_{i=0}^{3}(2a_i+1)^2\wp'(\rho-\omega_i) = 0$. More precisely, let g denote the integer $g := \frac{1}{2}\max\{2M, 1+\Sigma-(1+(-1)^{\Sigma})(m+\frac{1}{2})\}$, where $M := \max\{a_i\}$, $\Sigma := \sum_{i=0}^{3} a_i$ and $m := \min\{a_i\}$, and let $\Pi_{\vec{a}}(x,y,z)$ be the homogeneous polynomial defined in §9.9. Then, if $\sum_{i=0}^{3}(2a_i+1)^2\wp'(\rho-\omega_i) = 0$ but $\Pi_{\vec{a}}\Big(\wp(\rho), \wp(\omega_1), \wp(\omega_2)\Big) \neq 0$, the hyper-elliptic potential $u(x)$ above is associated to a canonical θ-characteristic over a hyperelliptic tangential cover of degree $n = 2 + \frac{1}{2}\sum_{i=0}^{3} a_i(a_i+1)$, genus $= g$ and type $\mu = (\mu_i)$ such that $\sum_{i=0}^{3}\mu_i^2 = 2n-3$. Otherwise (i.e. : if $\sum_{i=0}^{3}(2a_i+1)^2\wp'(\rho-\omega_i) = 0$ and $\Pi_{\vec{a}}\Big(\wp(\rho), \wp(\omega_1), \wp(\omega_2)\Big) = 0$) $u(x)$ is associated to a "non-canonical" θ-characteristic over an exceptional cover of same degree n, genus $= g+1$ and type $\hat{\mu} = (\hat{\mu}_i)$ such that $\sum_{i=0}^{3}\hat{\mu}_i^2 = 2n+1$ (see §9.9 for a more accurate statement).

9) Let me finish this long introductory paragraph with an open problem. For any elliptic $X \in \mathcal{X}$, any $\mu = (\mu_i)$ as in 1.15.4) and any $n \in \mathbf{N}$ such that $\sum_{i=0}^{3}\mu_i^2 = 2n+1$, the set $\mathcal{R}(n,\mu,X)$ contains a unique element :

the "exceptional cover of type μ" over X. It was proven in [6] §3.5 iii) that for a "sufficiently generic" X, the exceptional cover of type μ over X is smooth. In case $\sum_{i=0}^{3} \mu_i^2 = 2n - 3$ (see §2.3) the set $\mathcal{R}(n, \mu, X)$ is finite and for the generic elliptic curve X_{gen}, its cardinal is equal to $\#\mathcal{R}(n, \mu, X_{\text{gen}}) = 6 - 2j$, j being equal to $j := \#\{i, \mu_i = 0\}$ (§4.12). We conjecture that the corresponding covers are smooth.

2 The hyperelliptic tangential covers as divisors of the projective surface $S^{\perp}$.

2.1

Let $\mathcal{E}$ be the unique (up to isomorphism) indecomposable locally free sheaf of rank 2 and degree 0 over the elliptic curve (X, q) . Let $S = \mathbf{P}(\mathcal{E})$ denote the associated projective space bundle, canonically equipped with the projection $\pi_S : S \to X$. It's classically known that S is the unique ruled surface over X having only one section, denoted hereafter by C_0, of self-intersection 0. The inverse homomorphism $[-1] : (X, q) \to (X, q)$ can be lifted to an involution of S, $\tau : S \to S$, having two fixed points over each $\frac{1}{2}$-period, ω_i $(i = 0, \cdots, 3)$, of (X, q) : one in C_0, denoted by s_i, and the other one denoted by r_i . Let $e : S^{\perp} \to S$ denote the blow-up of S at the eight fixed points of τ. We can still lift $\tau : S \to S$ to an involution, $\tau^{\perp} : S^{\perp} \to S^{\perp}$. The corresponding exceptional divisors, denoted by $\{r_i^{\perp}, s_i^{\perp}, i = 0, \cdots, 3\}$, get fixed by $\tau^{\perp}$, and the quotient $S^{\sim} := S^{\perp}/ \sim_{\tau^{\perp}}$ is a smooth rational surface. The degree 2 projection $\phi : S^{\perp} \to S^{\sim}$ is intimately related to the set of all hyperelliptic tangential covers. In fact, as recalled hereafter, any such cover can be uniquely factored through $\pi_{S^{\perp}} := \pi_S \circ e : S^{\perp} \to S$, and can be reconstructed out of suitable rational integral curves in $S^{\sim}$.

2.2 Definition.

Let $\pi : (\Gamma, p) \to (X, q)$ be a hyperelliptic tangential cover and denote by $i^{\perp} : \Gamma \to S^{\perp}$ the unique morphism such that $\pi = \pi_S \circ e \circ i^{\perp}$. We call *type* of π, the intersection multiplicity vector $\mu = (\mu_i) := \left(i^{\perp}(\Gamma) \cdot r_i^{\perp}\right) \in \mathbf{N}^4$, and denote by $\mu^{(1)}$ and by $\mu^{(2)}$, the sums $\mu^{(1)} := \sum_{i=0}^{3} \mu_i$ and $\mu^{(2)} := \sum_{i=0}^{3} \mu_i^2$ respectively.

2.3 Proposition.

Let $\pi : (\Gamma, p) \to (X, q)$ be a hyperelliptic tangential cover of degree n and type μ, and $i^{\perp} : \Gamma \to S^{\perp}$ the canonical factorization of π via $e : S^{\perp} \to S$. Then,

1) the image $\Gamma^{\perp} := i^{\perp}(\Gamma)$ is $\tau^{\perp}$-invariant and linearly equivalent to $e^*(nC_0 + S_0) - s_0^{\perp} - \sum_0^3 \mu_i r_i^{\perp}$, where $S_0 = \pi_S^*(q)$;

2) the restriction of $\tau^{\perp}$ to $\Gamma^{\perp}$ induces the hyperelliptic involution on Γ, and $\phi(\Gamma^{\perp})$ is a rational integral curve of $S^{\sim}$. Furthermore, we have the following (in)equalities between n, $\mu^{(1)}$, $\mu^{(2)}$ and the arithmetic genus, say g, of Γ :

3) $\mu_0 + 1 \equiv \mu_1 \equiv \mu_2 \equiv \mu_3 \equiv n (\mathrm{mod} \cdot 2)$;

4) $\mu^{(2)} \leq 2n + 1$ and $(2n + 1 - \mu^{(2)}) \in 4\mathbf{N}$;

5) $2g + 1 \leq \mu^{(1)}$ and $\frac{1}{2}g(g + 1) \leq n$.

2.4 Definition.

We shall say that an integer vector $\mu \in \mathbf{N}^4$ *fits in (with* $n \in \mathbf{N}$), if and only if $\mu_0 + 1 \equiv \mu_1 \equiv \mu_2 \equiv \mu_3 (\equiv n)(\mathrm{mod} \cdot 2)$.

2.5 Lemma.

For any couple $(n, \mu) \in \mathbf{N} \times \mathbf{N}^4$ such that μ fits in with n, there exists a unique linear equivalence class in $\mathrm{Pic}(S^{\sim})$, denoted by $\lambda(n, \mu)$, such that

$$\phi^*\left(\lambda(n, \mu)\right) \equiv e^*(nC_0 + S_0) - s_0^{\perp} - \sum_{0 \leq i \leq 3} \mu_i r_i^{\perp} \tag{5}$$

2.6 Theorem.

For any (n, μ) as in §2.4, let $C^{\sim}$ be a rational integral curve in $|\lambda(n, \mu)|$, $j : \mathbf{P}^1(\mathbf{C}) \to C^{\sim}$ its desingularization, and consider the fibre product, $\hat{\Gamma} := \mathbf{P}^1(\mathbf{C}) \times_{C^{\sim}} \phi^*(C^{\sim})$, of j and the restriction of $\phi : S^{\perp} \to S^{\sim}$ to $\phi^*(C^{\sim})$. Then, the first projection $\hat{\Gamma} \to \mathbf{P}^1(\mathbf{C})$ defines a hyperelliptic involution on $\hat{\Gamma}$, fixing the smooth point $p := \hat{\Gamma} \cap s_0^{\perp}$, while the second projection $\hat{\Gamma} \to \phi^*(C^{\sim})$, composed with $\pi_{S^{\perp}} : S^{\perp} \to X$ is a minimal-hyperelliptic tangential cover, say $\hat{\pi} : (\hat{\Gamma}, p) \to (X, q)$, of degree n, type μ and arithmetic genus $\hat{g} := \frac{1}{2}(\mu^{(1)} - 1)$.

2.7 Remarks.

1) Let $j : (\Gamma, p) \to (\hat{\Gamma}, p)$ be any marked morphism of degree 1. Then $\pi := \hat{\pi} \circ j : (\Gamma, p) \to (X, q)$ is also a hyperelliptic tangential cover of degree n and type μ, but arithmetic genus $\leq \frac{1}{2}(\mu^{(1)} - 1)$ (with equality if and only if j is an isomorphism).

2) Conversely, for any hyperelliptic tangential cover $\pi : (\Gamma, p) \to (X, q)$, of degree n and type μ, its canonical image in $S^{\sim}$, $\phi \circ i^{\perp}(\Gamma)$, is a rational integral curve in $|\lambda(n, \mu)|$. Furthermore, let $\hat{\pi} : (\hat{\Gamma}, p) \to (X, q)$ be the hyperelliptic tangential cover (of degree n, type μ and arithmetic genus $\hat{g} := \frac{1}{2}(\mu^{(1)} - 1)$) canonically associated to $\phi \circ i^{\perp}(\Gamma)$. Then π is easily seen to factor through $\hat{\pi}$, and $\hat{\pi}$ is minimal-hyperelliptic (§1.11).

2.8 Definition.

For any couple (n, μ) as in §2.4 we denote by $R(n, \mu, X)$ the set of equivalence classes of hyperelliptic tangential covers of degree n, type μ and arithmetic genus $g := \frac{1}{2}(\mu^{(1)} - 1)$.

2.9 Remarks.

1) According to §2.7, any hyperelliptic tangential cover of degree n and type μ is isomorphic to a partial desingularization of a unique element of $R(n, \mu, X)$. In particular $\pi \in R(n, \mu, X)$ if and only if π is a minimal-hyperelliptic tangential cover of degree n and type μ.

2) For any hyperelliptic tangential cover, $\pi : (\Gamma, p) \to (X, q)$, of degree n and type μ, the arithmetic genus of Γ is smaller or equal to $\frac{1}{2}(\mu^{(1)} - 1)$, with equality if and only if $\pi \in R(n, \mu, X)$.

3) We finally learn from §2.6 that there is a natural one-to-one correspondence between the set $R(n, \mu, X)$ and the set of rational integral curves in $S^{\sim}$, linearly equivalent to $\lambda(n, \mu)$.

4) As a matter of fact, for any rational curve $C^{\sim} \in |\lambda(n, \mu)|$, the corresponding normalization morphism $j : \mathbf{P}^1(\mathbf{C}) \to C^{\sim} \subset S^{\sim}$ can not be deformed ([15] §4.10), implying the following result.

2.10 Theorem.

For any elliptic curve (X,q), and any couple $(n,\mu) \in \mathbf{N} \times \mathbf{N}^4$ such that μ fits in with n, the set $R(n,\mu,X)$ defined in §2.8 is finite (but may be empty).

3 On countable many exceptional divisors on the rational surface $S^\sim$.

Choose any fibre ℓ of $\pi_S : S \to X$, over the complement of $(X,q)_2 = \{\omega_i\}$, and let $r_i^\sim$ and $s_i^\sim$ $(i = 0, \cdots, 3)$ be the reduced images of $r_i^\perp$ and $s_i^\perp$ in $S^\sim$. For any divisor D of S let $D^\perp (\subset S^\perp)$ be its strict transform by $e : S^\perp \to S$, and let $D^\sim$ be the reduced image of $D^\perp$ in $S^\sim$, $D^\sim := \phi(D^\perp)$ (§2.1).

3.1 Lemma.

1) The surface $S^\sim$ is rational.

2) The group $\mathrm{Pic}(S^\sim)$ is free of rank 10, and the 10 divisors $\{C_0^\sim, \ell^\sim, S_i^\sim = \pi_S^*(\omega_i)^\sim, r_i^\sim, i = 0, \cdots, 3\}$ make up a $\mathbf{Z}$-basis.

3) For any $i = 0, \cdots, 3$ the curve $\ell^\sim$ is linearly (and numerically) equivalent to $\ell^\sim \equiv 2S_i^\sim + r_i^\sim + s_i^\sim$.

4) For any $n \geq 1$, and $\mu \in \mathbf{N}^4$ which fits in whith n, the linear equivalence class $\lambda(n,\mu)$ of §2.5 is equivalent to

$$(6) \qquad n(C_0^\sim + 2\ell^\sim) - (n-1)S_0^\sim - \frac{1}{2}(n + \mu_0 - 1)r_0^\sim$$
$$- \sum_{1 \leq j \leq 3} \{nS_j^\sim + \frac{1}{2}(n + \mu_j)r_j^\sim\}.$$

5) For any couple (n,μ) as in §3.1.4), and any divisor $D \in |\lambda(n,\mu)|$, there exists an integer $m \geq 0$, an integer vector $(\nu_i) = \nu \in \mathbf{N}^4$ which fits in with m, and an irreducible curve $C^\sim$ such that :

- $C^\sim \in |\lambda(m,\nu)|$;
- $\mu_i \leq \nu_i$ for $0 \leq i \leq 3$;
- $m \leq n$ and $m \equiv n (\mathrm{mod} \cdot 2)$;
- $\mu^{(2)} \leq \nu^{(2)} \leq 2m + 1 \leq 2n + 1$;

- D decomposes as

$$D = C^{\sim} + (n-m)C_0^{\sim} + \tfrac{1}{2}\sum_{0\leq i\leq 3}\{(n-m)s_i^{\sim} + (\nu_i - \mu_i)r_i^{\sim}\}.$$

3.2 Remark.

We easily deduce from §3.1.5) that for $|\lambda(n,\mu)|$ to be non-empty, we must impose the condition $\mu^{(2)} \leq 2n+1$. Better still, we calculate hereafter the dimension of $|\lambda(n,\mu)|$ and give as well a numerical criterion for an integral divisor of $S^{\sim}$ to belong to $|\lambda(n,\mu)|$.

3.3 Lemma.

Let $K, K^{\perp}$ and $K^{\sim}$ be the canonical divisors of S, $S^{\perp}$ and $S^{\sim}$ respectively, and choose any couple $(n,\mu) \in \mathbf{N} \times \mathbf{N}^4$ such that μ fits in with n (see §2.4). Then :

1) the divisors $e^*(K)$ and $\phi^*(K^{\sim})$ are both linearly equivalent to

   $K^{\perp} - \sum_{0\leq i\leq 3}(s_i^{\perp} + r_i^{\perp})$;

2) we have $K^{\sim}\cdot K^{\sim} = 0$, $\lambda(n,\mu)\cdot K^{\sim} = -1$ and $\lambda(n,\mu)^2 = \frac{1}{2}(2n-1-\mu^{(2)})$;

3) any irreducible divisor $D \in |\lambda(n,\mu)|$ satisfies the equality $D\cdot K^{\sim} = -1$ (as well as : $D\cdot \ell^{\sim} = n$, $D\cdot r_i^{\sim} = \mu_i$, for $i = 0,\cdots,3$), intersects $s_0^{\sim}$, and its arithmetic genus is equal to $p_a(D) := \frac{1}{4}(2n+1-\mu^{(2)})$.

   Conversely, for any irreducible divisor $D \subset S^{\sim}$ satisfying the equality $D\cdot K^{\sim} = -1$ and intersecting $s_0^{\sim}$, let $m := D\cdot \ell^{\sim}$ and $\nu_i := D\cdot r_i^{\sim}$; (for any $i = 0,\cdots,3$). Then m is a positive integer, the vector $\nu = (\nu_i) \in \mathbf{N}^4$ fits in with m, and $D \in |\lambda(m,\nu)|$.

3.4 Proposition.

Let $(n,\mu) \in \mathbf{N}\times\mathbf{N}^4$ be any couple such that μ fits in with n. Then, the projective space $|\lambda(n,\mu)|$ is empty if $2n+1 < \mu^{(2)}$, and has dimension $\dim|\lambda(n,\mu)| = \frac{1}{4}(2n+1-\mu^{(2)})$ otherwise. Moreover, if $2n+1 \geq \mu^{(2)}$, the generic element of $|\lambda(n,\mu)|$ is a smooth irreducible curve of genus $\frac{1}{4}(2n+1-\mu^{(2)})$.

3.5 Corollary.

There is a one-to-one correspondence between the set $\bigcup_{(n,\mu)} R(n,\mu,X)$ (of isomorphism classes of minimal-hyperelliptic tangential covers over (X,q)) and the set of rational curves $C^{\sim}$ in $S^{\sim}$, intersecting $s_0^{\sim}$ and $\ell^{\sim}$, and such that $C^{\sim}\cdot K^{\sim} = -1$.

4 The exceptional covers and beyond.

4.1

Let us first recall that by an exceptional curve on a smooth projective surface we mean an integral divisor D, isomorphic to $\mathbf{P}^1(\mathbf{C})$ and of self-intersection $D\cdot D = -1$. Choose any integer vector $\mu \in \mathbf{N}^4$ such that $\mu_0+1 \equiv \mu_1 \equiv \mu_2 \equiv \mu_3(\text{mod}.2)$. Then $2n+1 := \mu^{(2)}$ is an odd integer, μ fits in with n, $\dim|\lambda(n,\mu)| = 0$, and the unique effective divisor in $|\lambda(n,\mu)|$ is an exceptional curve of $S^{\sim}$, denoted hereafter by $C_\mu^{\sim}$. The latter exceptional curves can be numerically characterized and give rise to the so-called, exceptional (hyperelliptic tangential) covers.

4.2 Theorem.

Let $\mu \in \mathbf{N}^4$ be as in §4.1, $n := \frac{1}{2}(\mu^{(2)}-1)$ and let $C_\mu^{\sim}$ denote the unique exceptional curve in $S^{\sim}$, linearly equivalent to $\lambda(n,\mu)$. Then $C_\mu^{\sim}$ intersects $s_0^{\sim}$ and $\ell^{\sim}$, and we have the following equalities : $C_\mu^{\sim}\cdot s_0^{\sim} = 1$, $C_\mu^{\sim}\cdot \ell^{\sim} = n$ and $C_\mu^{\sim}\cdot r_i^{\sim} = \mu_i$; for any $i = 0,\cdots,3$. Conversely, given any exceptional curve $C^{\sim}$ on $S^{\sim}$, if $C^{\sim}$ intersects $s_0^{\sim}$ and $\ell^{\sim}$, then $C^{\sim}\cdot s_0^{\sim} = 1$. Moreover, the integer vector $\nu = (\nu_i) = (C^{\sim}\cdot r_i^{\sim})$ fits in with $m := C^{\sim}\cdot\ell^{\sim}$, $\nu^{(2)} = 2m+1$ and $C^{\sim} = C_\nu^{\sim}$.

4.3 Definition.

Let $\mu \in \mathbf{N}^4$ and $C_\mu^{\sim}$ be as in §4.2, and consider the curve $C_\mu^{\perp} := \phi^*(C_\mu^{\sim}) \subset S^{\perp}$, equipped with the smooth point $p_\mu := C_\mu^{\perp}\cap s_0^{\perp} = \phi^{-1}(C_\mu^{\sim}\cap s_0^{\sim})$. Then, the marked projection $\pi_\mu : (C_\mu^{\perp}, p_\mu) \to (X,q)$, obtained by restricting $\pi_{S^{\perp}} : S^{\perp} \to X$ to $C_\mu^{\perp}$, is a minimal-hyperelliptic tangential cover of type μ, degree $n := \frac{1}{2}(\mu^{(2)}-1)$ and arithmetic genus $= \frac{1}{2}(\mu^{(1)}-1)$ (cf. §2.6). We'll say that π_μ is the *exceptional cover* of type μ over (X,q).

4.4 Theorem.

Let $\pi : (\Gamma, p) \to (X, q)$ be a hyperelliptic tangential cover of degree n and type μ. The following properties are equivalent, and imply that $2n+1 = \mu^{(2)}$ (compare with [15] §6.1) :

i) the canonical morphism $i^{\perp} : \Gamma \to S^{\perp}$ is an embedding ;

ii) π is minimal-hyperelliptic and $\phi\Big(i^{\perp}(\Gamma)\Big)$ is an exceptional curve of $S^{\sim}$;

iii) the arithmetic genus of Γ is equal to $\frac{1}{2}(\mu^{(1)} - 1)$ and $\phi\Big(i^{\perp}(\Gamma)\Big) = C_{\mu}^{\sim}$;

iv) π is isomorphic to the exceptional cover π_{μ}.

4.5 Definition.

The set $\{(n,\mu) \in \mathbf{N} \times \mathbf{N}^4,\ \mu \text{ fits in with } n \text{ and } \mu^{(2)} \leq 2n+1\}$ will be equipped with the following partial ordering : we say that (m,ν) is *smaller* than (n,μ) (and write $(m,\nu) \prec (n,\mu)$) if and only if $n-m \in 2\mathbf{N}$ and $\nu - \mu \in 2\mathbf{N}^4$.

In the latter case we have the following decomposition

$$(7)\ \lambda(n,\mu) \equiv \lambda(m,\nu) + (n-m)C_0^{\sim} + \frac{n-m}{2} \sum_{0 \leq i \leq 3} s_i^{\sim} + \frac{1}{2} \sum_{0 \leq i \leq 3} (\nu_i - \mu_i) r_i^{\sim} .$$

Adding, in fact, the effective divisor $\lambda(n,\mu) - \lambda(m,\nu)$, defines an injection of $|\lambda(m,\nu)|$ as a projective sub-space of $|\lambda(n,\mu)|$. We shall identify $|\lambda(m,\nu)|$ with its image in $|\lambda(n,\mu)|$ (whenever $(m,\nu) \prec (n,\mu)$).

4.6 Lemma.

Let $(n,\mu) \in \mathbf{N} \times \mathbf{N}^4$ be such that μ fits in with n and $\mu^{(2)} \leq 2n+1$, and let $D \subset S^{\sim}$ be any effective divisor linearly equivalent to $\lambda(n,\mu)$. Then D is reducible if and only if there exists $(m,\nu) \prec (n,\mu)$ ($(m,\nu) \neq (n,\mu)$) such that $D \in |\lambda(m,\nu)| \subset |\lambda(n,\mu)|$.

Proof : In fact,D is reducible if and only if the $\tau^{\perp}$-invariant divisor of $S^{\perp}$, $\phi^*(D)$, is reducible. On the other hand $e_*(\phi^*(D))$ is a τ-invariant divisor of S, linearly equivalent to $nC_0 + S_0$. Hence, there exists $m \leq n$, $m \equiv n(\text{mod}.2)$, and an integral τ-invariant divisor $\Gamma \in |mC_0 + S_0|$, such that $e_*(\phi^*(D))$ decomposes as $\Gamma + (n-m)C_0$. We can recover $\phi^*(D)$ out of $\Gamma + (n-m)C_0$ as

$$\phi^*(D) = \Gamma^{\perp} + (n-m)C_0^{\perp}, \quad \text{mod}(\langle s_i^{\perp}, r_i^{\perp},\ i = 0, \cdots, 3\rangle),$$

where we let $\Gamma^{\perp}$ and $C_0^{\perp}$ denote the $\tau^{\perp}$-invariant strict transforms of Γ and C_0 in $S^{\perp}$. In particular, the reduced image of $\Gamma^{\perp}$ in $S^{\sim}$ belongs to $|\lambda(m,\nu)|$, where $\nu = (\nu_i)$ is the type of $\Gamma^{\perp}$, i.e. : $\nu_i := \Gamma^{\perp} \cdot r_i^{\perp}$ for $i = 0, \cdots, 3$, and

$$D = \phi(\Gamma^{\perp}) + (n-m)C_0^{\sim} + \frac{1}{2}(n-m) \sum_{0 \leq i \leq 3} s_i^{\sim} + \frac{1}{2} \sum_{0 \leq i \leq 3} (\nu_i - \mu_i) r_i^{\sim} \in |\lambda(n,\mu)| \, .$$

We've thus obtained a canonical decomposition of D as the sum of an irreducible divisor $\phi(\Gamma^{\perp}) \in |\lambda(m,\nu)|$, plus another effective term $\equiv \lambda(n,\mu) - \lambda(m,\nu)$. It follows that $(m,\nu) \prec (n,\mu)$. Hence D is reducible if and only if $(m,\nu) \neq (n,\mu)$. In the latter case, D belongs to the natural image of $|\lambda(m,\nu)|$ in $|\lambda(n,\mu)|$. ∎

4.7 Proposition.

For any couple (n,μ) such that μ fits in with n and $2n+1 \geq \mu^{(2)}$, the generic element of $|\lambda(n,\mu)|$ is an irreducible divisor.

Proof : There only exists a finite number of couples (m,ν) such that $(m,\nu) \prec (n,\mu)$ and $(m,\nu) \neq (n,\mu)$. For each one of them $\dim|\lambda(m,\nu)| = \frac{1}{4}(2m+1-\nu^{(2)}) < \frac{1}{4}(2n+1-\mu^{(2)}) = \dim|\lambda(n,\mu)|$. In particular, the reunion of the corresponding proper projective subspaces, $\bigcup_{(m,\nu)} |\lambda(m,\nu)|$, which according to §4.6 parametrizes the reducible divisors in $|\lambda(n,\mu)|$, is a closed proper subset. Hence, the generic element in $|\lambda(n,\mu)|$ is irreducible. ∎

4.8 Proposition.

Let us suppose now that $2n+1 > \mu^{(2)}$. Then, the complete linear system $|\lambda(n,\mu)|$ has a unique base point. This base point belongs to $s_0^{\sim}$ and is a smooth point of any element $x \in |\lambda(n,\mu)|$.

Proof Let us first recall that $\lambda(n,\mu) \equiv \lambda(n-2,\mu) + 2C_0^{\sim} + \sum_{0\leq i \leq 3} s_i^{\sim}$ and $\lambda(n,\mu) \cdot C_0^{\sim} = \lambda(n,\mu) \cdot s_j^{\sim} = s_0^{\sim} \cdot s_j^{\sim} = 0$ for $j = 1,2,3$, but $s_0^{\sim} \cdot C_0^{\sim} = 1$. In particular $h^{\circ}\big(\mathcal{O}_{C_0^{\sim}}(\lambda(n,\mu) - s_0^{\sim})\big) = 0$. Hence, the divisor $D^{\sim} = 2C_0^{\sim} + \sum_{1 \leq j \leq 3} s_j^{\sim}$ being connected, we conclude that $h^{\circ}\Big(\mathcal{O}_{D^{\sim}}\big(\lambda(n,\mu) - s_0^{\sim}\big)\Big) = 0$. It follows then, by looking at the cohomology sequence associated to the exact sequence

$$(8)\ 0 \to \mathcal{O}_{S^{\sim}}\Big(\lambda(n-2,\mu)\Big) \to \mathcal{O}_{S^{\sim}}\Big(\lambda(n,\mu) - s_0^{\sim}\Big) \to \mathcal{O}_{D^{\sim}}\Big(\lambda(n,\mu) - s_0^{\sim}\Big) \to 0,$$

that $h^\circ\Big(\mathcal{O}_{S^\sim}(\lambda(n,\mu)-s_0^\sim)\Big) = h^\circ\Big(\mathcal{O}_{S^\sim}(\lambda(n-2,\mu))\Big)(= h^\circ\Big(\mathcal{O}_{S^\sim}(\lambda(n,\mu))\Big)-1$ by §3.4). Look, on the other hand, at the exact sequence

$$(9) \qquad 0 \to \mathcal{O}_{S^\sim}\Big(\lambda(n,\mu)-s_0^\sim\Big) \to \mathcal{O}_{S^\sim}\Big(\lambda(n,\mu)\Big) \to \mathcal{O}_{s_0^\sim}\Big(\lambda(n,\mu)\Big) \to 0\,.$$

Since $H^\circ\Big(\mathcal{O}_{S^\sim}(\lambda(n,\mu)-s_0^\sim)\Big)$ injects as an hyperplane of $H^\circ\Big(\mathcal{O}_{S^\sim}(\lambda(n,\mu))\Big)$, the image of $H^\circ\Big(\mathcal{O}_{S^\sim}(\lambda(n,\mu))\Big)$ into $H^\circ\Big(\mathcal{O}_{s_0^\sim}(\lambda(n,\mu))\Big)$ is a sub-space of dimension 1. Furthermore, since $\lambda(n,\mu)\cdot s_0^\sim = 1$, every irreducible divisor in $|\lambda(n,\mu)|$ intersects $s_0^\sim$ at only one and the same point of $s_0^\sim$. The latter is a smooth point of any element of $|\lambda(n,\mu)|$. We'll finally prove, by induction over n, that $|\lambda(n,\mu)|$ has no other base point. Suppose in fact that $2n+1-\mu^{(2)}=4$, in which case $\lambda(n,\mu)^2 = \frac{1}{2}(2n-1-\mu^{(2)}) = 1$. Hence, any two irreducible elements of $|\lambda(n,\mu)|$ can only intersect at one point, and this point must be the unique base point already found. Suppose now that $2n+1-\mu^{(2)} > 4$ (hence $2(n-2)+1-\mu^{(2)} \geq 4$), and that $|\lambda(n-2,\mu)|$ has a unique base point (in $s_0^\sim$). We know that for any $x \in |\lambda(n-2,\mu)|$, $x + 2C_0^\sim + \sum_{0\leq i\leq 3} s_i^\sim \in |\lambda(n,\mu)|$. It easily follows that any base point of $|\lambda(n,\mu)|$ can only lie in $C_0^\sim$ or $s_i^\sim$ $(i=0,\cdots,3)$. We also know that $\lambda(n,\mu)\cdot C_0^\sim = \lambda(n,\mu)\cdot s_j^\sim = 0$ $(j=1,2,3)$. Hence any base point of $|\lambda(n,\mu)|$ can only lie in $s_0^\sim$ and must, therefore, be equal to the one already found. ∎

4.9 Corollary.

For any $(n,\mu) \in \mathbf{N}\times\mathbf{N}^4$ such that μ fits in with n, the generic element of $|\lambda(n,\mu)|$ is a smooth irreducible curve of genus $\frac{1}{4}(2n+1-\mu^{(2)})$.

Proof : If $2n+1 < \mu^{(2)}$, $|\lambda(n,\mu)|$ is empty and there's nothing to prove.
If $2n+1 = \mu^{(2)}$, the unique element of $|\lambda(n,\mu)|$ is an exceptional curve (see §4.1). Hence it is irreducible and smooth, of genus $0 = \frac{1}{4}(2n+1-\mu^{(2)})$.
If $2n+1 > \mu^{(2)}$ the result follows by coupling the Proposition §4.8 with Bertini's Theorem and §3.3. 3). ∎

4.10

Studying the rational curves $x \in |\lambda(n-2,\mu)|$, for any couple (n,μ) satisfying the inequality $2n+1 > \mu^{(2)}$, seems quite difficult in general. We shall restrain in the sequel to "the first non-exceptional case" : $2n+1-\mu^{(2)} = 4$. In the latter case, $\dim|\lambda(n,\mu)| = 1$, the generic element in $|\lambda(n,\mu)|$ is an elliptic curve, and there exists a unique base point (depending only upon n, and) denoted by $b(n) \in s_0^\sim$.

Recall also (§4.1) that $C_\mu^\sim$, the unique element in $|\lambda(n-2,\mu)|$, is an exceptional curve and $y_\infty := C_\mu^\sim + 2C_0^\sim + \sum_{0\le i\le 3} s_i^\sim$ belongs to $|\lambda(n,\mu)|$. By blowing up the base point $b(n)$ we get the surface $S_n^\sim$, equipped with the monoidal transformation $S_n^\sim \to S^\sim$ and the natural projection $S_n^\sim \to |\lambda(n,\mu)| \simeq \mathbf{P}^1(\mathbf{C})$. In $S_n^\sim$, the strict transform of $C_\mu^\sim$ is still an exceptional curve. By contracting it we finally get a rational elliptic surface $S_{(n,\mu)}^\sim \to |\lambda(n,\mu)| \simeq \mathbf{P}^1(\mathbf{C})$, with Picard group $\mathrm{Pic}(S_{(n,\mu)}^\sim) \cong \mathbf{Z}^{10}$. Hence, the Euler-Poincaré characteristic of $S_{(n,\mu)}^\sim$ is equal to $\chi(S_{(n,\mu)}^\sim) = 12$, and the fibre over y_∞ is equal to $2C_0^\sim + \sum_{0\le i\le 3} s_i^\sim$, where we let $C_0^\sim$ and $s_i^\sim$ $(i = 0,\cdots,3)$ denote also their images in $S_{(n,\mu)}^\sim)$. The latter curves have the following intersection diagram (in $S_{(n,\mu)}^\sim$) :

$$(10) \qquad C_0^\sim \cdot C_0^\sim = s_i^\sim \cdot s_i^\sim = -2\,, C_0^\sim \cdot s_i^\sim = 1\,, s_i^\sim \cdot s_j^\sim = 1$$

4.11 Remarks.

1) For any $y \in |\lambda(n,\mu)|$, let us denote by χ_y its Euler-Poincaré characteristic. We know that the generic fibre of $S_{(n,\mu)}^\sim \to |\lambda(n,\mu)|$ is an elliptic curve. Hence the function $y \mapsto \chi_y$ vanishes outside a finite number of points of $|\lambda(n,\mu)|$, and the (so-called Lefschetz) formula (e.g. : [7] p. 509) tells us that

$$(11) \qquad 12 = \chi(S_{(n,\mu)}^\sim) = \sum_{y\in|\lambda(n,\mu)|} \chi_y \,.$$

2) The $E-P$ characteristic χ_y is $\neq 0$ if and only if y is reducible, or irreducible but singular. Let us denote by Red (resp : Irr) the subset of reducible (resp : irreducible and singular) fibres. The particular fiber $y_\infty = 2C_0^\sim + \sum_{0\le i\le 3} s_i^\sim$ belongs to Red and all its irreducible components are isomorphic to $\mathbf{P}^1(\mathbf{C})$. It follows from their intersection diagram (10) that $\chi_{y_\infty} = 6$. The remaining reducible fibers $(\mathrm{Red}\setminus\{y_\infty\})$ are in a one-to-one correspondence with the indices $i = 0,\cdots,3$ for which $\mu_i = 0$ (§4.6). Let in fact i_0 be such an index, and define the integer vector $(\vec{e}_{i_0}(j)) \in \mathbf{N}^4$ by the formula $\vec{e}_{i_0}(j) = \delta_{i_0 j}$, $j = 0,\cdots,3$. Since $\mu_{i_0} = 0$ we easily check then that $(n,\mu+2\vec{e}_{i_0}) \prec (n,\mu)$. Moreover, the exceptional curve $C_{\mu+2\vec{e}_{i_0}}^\sim$ is the unique element in $\left|\lambda(n,\mu+2\vec{e}_{i_0})\right|$, the reducible curve $y_{i_0} := C_{\mu+2\vec{e}_{i_0}}^\sim + r_{i_0}^\sim$ belongs to $|\lambda(n,\mu)|$, and

$$r_{i_0}^\sim \cdot r_{i_0}^\sim = -r_{i_0}^\sim \cdot C_{\mu+2\vec{e}_{i_0}}^\sim = -2\,.$$

Hence, either $r_{i_0}^\sim$ intersects $C_{\mu+2\vec{e}_{i_0}}^\sim$ in two distinct points and $\chi_{y_{i_0}} = 2$, or $r_{i_0}^\sim$ is tangent to $C_{\mu+2\vec{e}_{i_0}}^\sim$ and $\chi_{y_{i_0}} = 3$.

3) On the other hand Irr has been defined as the subset of all singular irreducible divisors in $|\lambda(n,\mu)|$. The arithmetic genus of any $y \in |\lambda(n,\mu)|$ being equal to $\frac{1}{4}(2n+1-\mu^{(2)}) = 1$, we deduce that Irr consists of all integral rational curves in $|\lambda(n,\mu)|$. As such, any curve $y \in$ Irr must have a unique singular point : either a node and $\chi_y = 1$, or a cusp and $\chi_y = 2$. Let us, as usual, denote by $R(n,\mu,X)$ the set of equivalence classes of minimal-hyperelliptic tangential covers of degree n and type μ over (X,q). Taking into account the equation (11) and the above remarks on χ_y, $y \in \text{Red} \cup \text{Irr}$, we immediately obtain the following bounds for $\#R(n,\mu,X)$, the cardinal of $R(n,\mu,X)$.

4.12 Theorem.

For any $(n,\mu) \in \mathbf{N}\times\mathbf{N}^4$ such that μ fits in with n and $\mu^{(2)} = 2n-3$, $\#R(n,\mu,X)$ is bounded as follows :

1) $\#R(n,\mu,X) = 0$ if $\#\{i, \mu_i = 0\} = 3$;

2) $0 \leq \#R(n,\mu,X) \leq 2$ if $\#\{i, \mu_i = 0\} = 2$;

3) $2 \leq \#R(n,\mu,X) \leq 4$ if $\#\{i, \mu_i = 0\} = 1$;

4) $3 \leq \#R(n,\mu,X) \leq 6$ if $\#\{i, \mu_i = 0\} = 0$;

4.13 Remark.

The inequalities §4.12. 2) could be immediately improved. We give indeed, in the proof of §9.4, a completely down-to-earth construction of two distinct (non-primitive) elements of $R(n,\mu,X)$, whenever $\#\{i, \mu_i = 0\} = 2$. Hence, $\#R(n,\mu,X) = 2$ (if $\#\{i, \mu_i = 0\} = 2$).

5 The canonical θ-characteristics and their potentials : the case $\mu^{(2)} = 2n+1$.

5.1

Let $\pi : (\Gamma,p) \to (X,q)$ be any hyperelliptic tangential cover of degree $n \geq 1$. Recall that for any $\xi \in W(\Gamma)$, the pull-back of the canonical theta divisor Θ_Γ by the orbit morphism $\text{Orb}_\xi : X \to W(\Gamma)$, $q' \mapsto \xi\big(\pi^*(q'-q)\big) = \xi \otimes \mathcal{O}_\Gamma\big(\pi^*(q'-q)\big)$, is a divisor of degree n of X, $\text{Orb}_\xi^*(\Theta_\Gamma) = \sum_{i=1}^n \alpha_i$. Furthermore, $\sum_{i=1}^n \alpha_i$ belongs to $KdV(n,X)$, the n-th KdV locus, and is uniquely determined by the isomorphism class of the data $(\pi;\xi)$ (see [1] or [15] §2.7 §2.9). It also turns out that the hyper-elliptic potential associated to $(\pi;\xi)$ is equal to $u_\xi(x) = 2\sum_{i=1}^n \wp(x-\alpha_i)$ (§1.9 or [16] §7.10). Better still, for any $\alpha = q' - q \in \operatorname{Jac} X (\simeq X)$ the hyper-elliptic potential associated to $\xi\big(\pi^*(q'-q)\big)$ is equal to $2\sum_{i=1}^n \wp\big(x-(\alpha_i-\alpha)\big)$. In other words, $u_{\xi(\pi^*(q'-q))}(x) = u_\xi(x+\alpha)$. Finding $\mathrm{Orb}_\xi^*(\Theta_\Gamma)$ amounts to calculate the multiplicity of intersection between Θ_Γ and $\mathrm{Orb}_\xi(X)$, at any point $\xi' = \mathrm{Orb}_\xi(q')$, $q' \in X$. Our main tool (§5.4) is an easy consequence of the following restricted version of the Fay-Segal & Wilson formula.

5.2 Proposition.

Let $\pi : (\Gamma, p) \to (X, q)$ be any hyperelliptic tangential cover and ξ any point of $W(\Gamma)$. Then, the multiplicity of intersection between Θ_Γ and $\mathrm{Orb}_\xi(X)$ at ξ is equal to

$$\mathrm{mult}_\xi\big(\Theta_\Gamma, \mathrm{Orb}_\xi(X)\big) = \begin{cases} (2h^\circ(\xi)-1)\cdot h^\circ(\xi) & \text{if} \quad h^\circ(\xi) > h^\circ\big(\xi(-p)\big) \\ (2h^\circ(\xi)+1)\cdot h^\circ(\xi) & \text{if} \quad h^\circ(\xi) = h^\circ\big(\xi(-p)\big) \ . \end{cases} \tag{12}$$

5.3 Remarks.

1) It follows from the formula (12), that the multiplicity of intersection is a triangular number which increases with the value of $h^\circ(\xi)$. In particular, if $h^\circ\big(\xi(-mp)\big) > 0$, for some $m \geq 0$, we have the following lower bound :

$$\mathrm{mult}_\xi\left(\Theta_\Gamma, \mathrm{Orb}_\xi(X)\right) \geq \frac{1}{2}(m+1)(m+2) \ . \tag{13}$$

2) For any $i = 0, \cdots, 3$ $\xi_i := \xi\big(\pi^*(\omega_i - \omega_0)\big)$ has the same X-orbit as ξ, its potential is equal to $u_{\xi_i}(x) \equiv u_\xi(x-\omega_i)$, and

$$\sum_{i=0}^{3} \mathrm{mult}_{\xi_i}\left(\Theta_\Gamma, \mathrm{Orb}_\xi(X)\right) \leq \Theta_\Gamma \cdot \mathrm{Orb}_\xi(X) = \deg(\pi) = n \ . \tag{14}$$

3) The divisors $\{2np, 2\pi^*(\omega_i), i = 0, \cdots, 3\}$ have same degree and get fixed by the hyperelliptic involution of Γ. Hence, the're linearly equivalent. Moreover, denoting by g the arithmetic genus of Γ, we deduce that the canonical sheaf is equal to $\omega_\Gamma \simeq \mathcal{O}_\Gamma\big((2g-2)p\big)$ and that the invertible sheaves :

$$
(15)\qquad \begin{aligned} \xi_{A,i} &= \mathcal{O}_\Gamma\big((g-1)p + \pi^*(\omega_i - q)\big),\\ \xi_{B,i} &= \mathcal{O}_\Gamma\big((g-1-n)p + \pi^*(\omega_i)\big), \quad i = 0,\cdots,3 \end{aligned}
$$

are θ-characteristics on Γ. Whenever π is an exceptional cover we can find the corresponding even hyper-elliptic potentials as outlined hereafter (§5.4, §5.5). Due to the fact that for any $j = 1,2,3$, $\xi_{A,j}$ and $\xi_{B,j}$ belong to $\mathrm{Orb}_{\xi_{A,0}}(X)$ and $\mathrm{Orb}_{\xi_{B,0}}(X)$ respectively, it suffices to calculate $u_{\xi_{A,0}}$ and $u_{\xi_{B,0}}$ (see §5.3.2)). The first basic result in this direction is a general lower bound for the corresponding intersection multiplicities, obtained as a direct application of §5.2 and §5.3.1) (but only valid for minimal-hyperelliptic tangential covers).

5.4 Theorem.

Let $\pi : (\Gamma, p) \to (X, q)$ be any minimal-hyperelliptic tangential cover of degree n, type μ and arithmetic genus $g := \frac{1}{2}(\mu^{(1)} - 1)$, and let $\xi_{A,i}$ and $\xi_{B,i}$ be the canonical θ-characteristics defined in §5.3.3), for any half-period $\omega_i \in (X,q)_2$. Let also $m_i^{A,0}$ (resp. : $m_i^{B,0}$) denote the multiplicity of intersection

$$
(16)\qquad \begin{aligned} m_i^{A,0} &:= \mathrm{mult}_{\xi_{A,i}}\big(\Theta_\Gamma, \mathrm{Orb}_{\xi_{A,0}}(X)\big)\\ \Big(\mathrm{resp} : m_i^{B,0} &:= \mathrm{mult}_{\xi_{B,i}}\big(\Theta_\Gamma, \mathrm{Orb}_{\xi_{B,0}}(X)\big)\Big)\,. \end{aligned}
$$

Then $m_0^{A,0} = \frac{1}{2}g(g+1)$, and for $j = 1,2,3$ and $i = 0,\cdots,3$ we have the following lower bounds :

$$
(17)\qquad m_j^{A,0} \geq \frac{1}{2}(g - \mu_0 - \mu_j)(g - \mu_0 - \mu_j + 1) \ \text{ and }
$$

$$
m_i^{B,0} \geq \frac{1}{2}(g - \mu_i)(g - \mu_i + 1)\,.
$$

5.5 Corollary.

Let $\pi_\mu : (\Gamma, p) \to (X, q)$ be the exceptional cover of type μ (degree $n := \frac{1}{2}(\mu^{(2)} - 1)$ and arithmetic genus $g := \frac{1}{2}(\mu^{(1)} - 1)$). Then, the even hyper-elliptic potentials associated to the canonical θ-characteristics $\xi_{A,0} = \mathcal{O}_\Gamma\big((g-1)p\big)$ and

$\xi_{B,0} = \mathcal{O}_\Gamma\big((g-1-n)p + \pi^*(q)\big)$, are equal respectively to :

$$(18)\begin{cases} u_{\xi_{A,0}}(x) & = \\ g(g+1)\wp(x) & + \quad \sum_{1\le j\le 3}(g-\mu_0-\mu_j)(g-\mu_0-\mu_j+1)\wp(x-\omega_j) \\ & \text{and} \\ u_{\xi_{B,0}}(x) & = \quad \sum_{0\le i\le 3}(g-\mu_i)(g-\mu_i+1)\wp(x-\omega_i)\ . \end{cases}$$

Proof : We easily check indeed that

$$(19)\qquad \begin{aligned} \sum_{0\le i\le 3}(g-\mu_i)(g-\mu_i+1) \quad &= \quad \mu^{(2)}-1, \\ \sum_{1\le j\le n, j\ne i}\wp'(\alpha_i-\alpha_j) = 0, \text{and}& \\ g(g+1)+\sum_{1\le j\le 3}(g-\mu_0-\mu_j)(g-\mu_0-\mu_j+1) \quad &= \quad \mu^{(2)}-1\ . \end{aligned}$$

Applying the lower bounds (17) and §5.3.2) we immediately conclude that

$$(20)\qquad \begin{cases} \mu^{(2)}-1 & \le \quad 2\sum_{0\le i\le 3} m_i^{A,0} \le 2n \\ & \text{and} \\ \mu^{(2)}-1 & \le \quad 2\sum_{0\le i\le 3} m_i^{B,0} \le 2n \end{cases}$$

Now, in our (exceptional) case, $\mu^{(2)}-1$ is equal to $2n$. Hence, for any $j=1,2,3$ and $i=0,\cdots,3$ we must have

$$(21)\qquad \begin{cases} 2m_j^{A,0} & = \quad (g-\mu_0-\mu_j)(g-\mu_0-\mu_j+1) \\ & \text{and} \\ 2m_i^{B,0} & = \quad (g-\mu_i)(g-\mu_i+1)\ , \end{cases}$$

and $u_{\xi_{A,0}}(x)$ as well as $u_{\xi_{B,0}}(x)$, are given by (18). ∎

5.6 Remark.

According to Corollary §5.5, whenever π is an exceptional cover the hyper-elliptic potentials associated to the θ-characteristics $\{\xi_{A,i}\ ,\ \xi_{B,i},\ i = 0, \cdots, 3\}$, can only have poles at the half-periods of (X, q). It turns out, as recalled hereafter, that this property characterizes the canonical θ-characteristics on the exceptional covers.

5.7 Theorem.

For any $(a_i) \in \mathbf{N}^4 \setminus \{\vec{0}\}$ there exists a unique exceptional cover $\pi : (\Gamma, p) \to (X, q)$ such that the even function $u(x) = \sum_{0 \le i \le 3} a_i(a_i + 1)\wp(x - \omega_i)$ is the hyper-elliptic potential associated to one of the eight canonical θ-characteristics, $\{\xi_{A,i}, \xi_{B,i}, i = 0, \cdots, 3\}$, on Γ. We can go further into detail, by adding that the degree of π is equal to $n := \frac{1}{2}\sum_{0 \le i \le 3} a_i(a_i + 1)$, and the arithmetic genus of Γ is equal to :

$$g := \frac{1}{2}\max\big(2M, 1 + \Sigma - (1 + (-1)^{\Sigma})(m + \frac{1}{2})\big)\ , \tag{22}$$

where $\Sigma = \sum_{0 \le i \le 3} a_i$, $M = \max\{a_i\}$ and $m = \min\{a_i\}$.

Sketch of the proof : the proof given in [18] §4.4 & §5.3, is quite down-to-earth and runs as follows. For any $(a_i) \in \mathbf{N}^4 \setminus \{\vec{0}\}$ there is a particular $\mu \in \mathbf{N}^4$, given by an explicit formula developed in [18] §4.4 and §5.4. 3), such that μ fits in with $n := \frac{1}{2}\sum_{0 \le i \le 3} a_i(a_i+1)$ and $\mu^{(2)} = 2n+1$. Furthermore, let $\pi : (\Gamma, p) \to (X, q)$ be the exceptional cover of type μ and $\{\xi_{A,i}, \xi_{B,i},\ i = 0, \cdots, 3\}$ the set of θ-characteristics defined in 5.3. 3) above. A direct verification of the formula (18), given in Corollary §5.5, shows that $u(x) = \sum_{0 \le i \le 3} a_i(a_i + 1)\wp(x - \omega_i)$ is the hyper-elliptic potential associated to $\xi_{A,i}$ or $\xi_{B,i}$, for some $i = 0, \cdots, 3$. Finally, the arithmetic genus of Γ is known to be equal to $g := \frac{1}{2}(\mu^{(1)} - 1)$. Replacing (μ_i) in terms of (a_i), by means of the formula of [18] §4.4 & §5.4.3), we easily deduce the expression (22). ∎

6 Finding new hyper-elliptic potentials.

6.1

We've applied in §5, the $F - S\&W$ multiplicity formula (§5.2) and the lower bounds (17), to any exceptional cover, obtaining a complete characterization of the corresponding canonical θ-characteristics (§5.3. 2)) in terms of the associated hyper-elliptic potentials (§5.5, §5.7). Building up over §5.2, (17), §5.5 and §5.7 we shall be able to deal hereafter, with the ("first non-exceptional") case $\mu^{(2)} = 2n - 3$, finding an infinite family of even hyper-elliptic potentials of the form $u(x) = 2\wp(x-\rho) + 2\wp(x+\rho) + \sum_{i=0}^{3} a_i(a_i+1)\wp(x-\omega_i)$, where $(a_i) \in \mathbf{N}^4$ and ρ satisfies the equation $\sum_{i=0}^{3}(2a_i+1)^2\wp'(\rho-\omega_i) = 0$ (see §6.3).

6.2 Proposition.

Let $(a_i) \in \mathbf{N}^4$ and $\rho \in X \setminus \{\omega_i, i = 0, \cdots, 3\}$, be such that the even function $u(x) = 2\wp(x-\rho) + 2\wp(x+\rho) + \sum_{i=0}^{3} a_i(a_i+1)\wp(x-\omega_i)$ is a hyper-elliptic potential. Then ρ must satisfy the following "fundamental" equation

$$(23) \qquad \sum_{i=0}^{3}(2a_i+1)^2\wp'(\rho-\omega_i) = 0\ .$$

Proof : Let us denote by $v(x,t)$ the unique KdV solution having as "initial value" $v(x,0) \equiv u(x)$. We know that there exists a set of $n := 2 + \frac{1}{2}\sum_{i=0}^{3} a_i(a_i+1)$ distinct analytical functions $\{x_k(t),\ 1 \leq k \leq n\}$, such that $v(x,t) = 2\sum_{k=1}^{n}\wp\big(x - x_k(t)\big)$, and the divisor $\sum_{k=1}^{n}\big(x_k(t)\big)$ belongs to $KdV(n,X)$ for any $t \in \mathbf{C}$ (e.g. §1.9). Hence $\sum_{k=1}^{n}\big(x_k(t)\big)$ converges as t goes to 0, to the divisor $(\rho) + (-\rho) + \frac{1}{2}\sum_{i=0}^{3} a_i(a_i+1)(\omega_i)$, while satisfying (in particular) $\sum_{k=2}^{n}\wp'\big(x_2(t) - x_k(t)\big) = 0$ $(t \neq 0)$. Taking the limit $(t \to 0)$ of the latter equality we obtain $\sum_{k=2}^{n}\wp'\big(x_1(0) - x_k(0)\big) = \wp'(2\rho) + \frac{1}{2}\sum_{i=0}^{3} a_i(a_i+1)\wp'(\rho-\omega_i) = 0$. Since $8\wp'(2\rho)$ is classically known to be equal to $\sum_{i=0}^{3}\wp'(\rho-\omega_i)$, we finally deduce that $\sum_{i=0}^{3}(2a_i+1)^2\wp'(\rho-\omega_i) = 0$ as asserted. ∎

6.3 Theorem.

Let $\big[\pi : (\Gamma,p) \to (X,q)\big] \in R(n,\mu,X)$ be any minimal-hyperelliptic tangential cover of type μ (hence, arithmetic genus $g := \frac{1}{2}(\mu^{(1)} - 1)$) and degree n such that $\mu^{(2)} = 2n - 3$. Then, for any canonical θ-characteristic $\xi \in \{\xi_{A,k}, \xi_{B,k}\ , k = 0, \cdots, 3\}$, there exists an integer vector $\vec{a} \in \mathbf{N}^4$ and a point $\rho \in X$ $(2\rho \neq 0)$ satisfying $\sum_{i=0}^{3}(2a_i+1)^2\wp'(\rho-\omega_i) = 0$, such that the hyper-elliptic potential associated to $(\pi;\xi)$ is equal to

$$(24) \qquad u(x) = 2\wp(x-\rho) + 2\wp(x+\rho) + \sum_{i=0}^{3} a_i(a_i+1)\wp(x-\omega_i)\ .$$

Whenever $\xi = \xi_{A,k}$ (resp. $\xi = \xi_{B,k}$), the vector $\vec{a} = (a_i) \in \mathbf{N}^4$ appearing in (24) will be denoted by $A^k(\mu)$ (resp : by $B^k(\mu)$). In that case, $\vec{a} = A^\circ(\mu)$ (resp. : $\vec{a} = B^\circ(\mu)$) is given by the following formula :

$$(25)\begin{cases} a_0 = g = \frac{1}{2}\left(\mu^{(1)} - 1\right) \text{ and } a_j = \frac{1}{2}\left(|\mu^{(1)} - 2\mu_0 - 2\mu_j| - 1\right) \text{ for } j = 1,2,3 \\ \\ (\text{resp : } a_i = \frac{1}{2}\left(|\mu^{(1)} - 2\mu_i| - 1\right) \text{ for } i = 0,\cdots,3)\ . \end{cases}$$

Knowing, on the other hand, that $u_{\xi_{A,k}}(x) \equiv u_{\xi_{A,0}}(x - \omega_k)$ and that $u_{\xi_{B,k}}(x) \equiv u_{\xi_{B,0}}(x - \omega_k)$, we can also deduce from (25) the explicit formulas for $A^k(\mu)$ and $B^k(\mu)$ $(k = 1,2,3)$.

Proof: Let us suppose that ξ is equal, either to $u_{\xi_{A,0}}$, or to $u_{\xi_{B,0}}$, and denote by m_i the multiplicity of intersection at $\xi_i := \xi(\pi^*(\omega_i - \omega_0)),(i = 0,\cdots,3)$ between Θ_Γ and $\mathrm{Orb}_\xi(X)$. Choose $\vec{a}$ equal to $A^\circ(\mu)$ (resp. : to $B^\circ(\mu)$). A straightforward calculation shows that in both cases we have $\mu^{(2)} - 1 = \sum_{i=0}^{3} a_i(a_i + 1)$. On the other hand, it follows from 5.3.2) and the lower bounds (17), that $\sum_{i=0}^{3} m_i \le n$ and $\frac{1}{2}a_i(a_i + 1) \le m_i$, for $i = 0,\cdots,3$. Hence $\mu^{(2)} - 1 = \sum a_i(a_i + 1) \le \sum_{i=0}^{n} 2m_i \le 2n$.

Now, in our particular case $\mu^{(2)} - 1 = 2n - 4$, implying the inequalities $n - 2 = \frac{1}{2}\sum_{i=0}^{3} a_i(a_i + 1) \le \sum_{i=0}^{3} m_i \le n$. We can also argue that the hyper-elliptic potential $u_\xi(x)$ is an even function having (at least) two poles outside the set of $\frac{1}{2}$-periods $\{\omega_i\}$. Otherwise $u_\xi(x)$ should be one of the potentials §5.7 and π an exceptional cover (contradiction !). We finally deduce that $\sum_{i=0}^{3} m_i = n - 2 = \frac{1}{2}\sum_{i=0}^{3} a_i(a_i + 1)$, while $m_i \ge \frac{1}{2}a_i(a_i + 1)$ for $i = 0,\cdots,3$. Hence $m_i = \frac{1}{2}a_i(a_i + 1)$ for $i = 0,\cdots,3$ as announced, and there exists $\rho \in X \setminus \{\omega_i\}$, satisfying the fundamental equation (23), such that $u_\xi(x) = 2\wp(x - \rho) + 2\wp(x + \rho) + \sum_{i=0}^{3} a_i(a_i + 1)\wp(x - \omega_i)$ (§6.2). ∎

6.4 Remarks.

1) To any $\pi \in R(n, \mu, X)$ as above (with $n = \frac{1}{2}(\mu^{(2)} + 3)$) we've associated eight hyper-elliptic potentials as in (24). The corresponding integer vectors $\vec{a} \in \mathbf{N}^4$ satisfy the equality $\sum_{i=0}^{3} a_i(a_i + 1) = 2n - 4 = \mu^{(2)} - 1$, and only depend upon μ and the choice of the θ-characteristic. The eight maps A^k and B^k, $\mu \mapsto \vec{a}$ $(k = 0,\cdots,3)$ defined by each choice, have been thoroughly studied in [18] §4, §5 : they are injective and together they cover the whole $\mathbf{N}^4$.

2) Let us denote hereafter by $\mathbf{M} \subset \mathbf{N}^4$ the subset of all possible types, $\mathbf{M} = \{\mu = (\mu_i) \in \mathbf{N}^4 / \mu_0 + 1 \equiv \mu_1 \equiv \mu_2 \equiv \mu_3 (\mathrm{mod} \cdot 2)\}$, and by $M^{\#k}$ $(k =$

$0,\cdots,3)$ the subset of $\mathbf{M}$ consisting of all types having k zero coordinates : $\mathbf{M}^{\#k} = \{\mu \in \mathbf{M}/\#\{j,\mu_j = 0\} = k\}$. Recall that for any elliptic curve (X,q) and any type $\mu \in \mathbf{M}$, the set $R(n,\mu,X)$ $\left(n = \frac{1}{2}(\mu^{(2)}+3)\right)$ is void if and only if μ belongs to $\mathbf{M}^{\#3} = \{(2k+1,0,0,0), k \in \mathbf{N}\}$. On the other hand, a straight-forward verification shows that $\mu \in \mathbf{M}^{\#3}$ if and only if, for all $k = 0,\cdots,3$, $A^k(\mu)$ and $B^k(\mu)$ belong to $\{(a,a,a,a), a \in \mathbf{N}\}$. We almost immediately deduce the following converse of §6.3 :

6.5 Corollary.

For any vector $\vec{a}$ in the complement of $\{(b,b,b,b), b \in \mathbf{N}\}$, (i.e. : $\vec{a} \in \mathbf{N}^4 \setminus (1,1,1,1)\mathbf{N}$), there exists $\rho \in X \setminus \{\omega_i\}$, satisfying $\sum_{i=0}^{3}(2a_i+1)^2\wp'(\rho-\omega_i) = 0$, such that $u(x) = 2\wp(x-\rho) + 2\wp(x+\rho) + \sum_{i=0}^{3} a_i(a_i+1)\wp(x-\omega_i)$ is one of the hyper-elliptic potentials "canonically" associated (§6.3) to some $\pi \in R(n,\mu,X)$ (where μ is such that $\vec{a} \in \{A^k(\mu), B^k(\mu)\}$, and $n := \frac{1}{2}(\mu^{(2)}+3) = 2 + \frac{1}{2}\sum_{i=0}^{3} a_i(a_i+1)$).

Proof : For any such $\vec{a}$ there exists $\mu \in \mathbf{M} \setminus \mathbf{M}^{\#3}$ and $k \in \{0,\cdots,3\}$ such that, either $A^k(\mu) = \vec{a}$, or $B^k(\mu) = \vec{a}$. Furthermore, the set $R(n,\mu,X)$ $(n = \frac{1}{2}(\mu^{(2)}+3))$ is not empty. Hence, the hyper-elliptic potential associated in §6.3, either to $(\pi;\xi_{A,k})$ or to $(\pi;\xi_{B,k})$ for any $\pi \in R(n,\mu,X)$, is equal to $2\wp(x-\rho) + 2\wp(x+\rho) + \sum_{i=0}^{3} a_i(a_i+1)\wp(x-\omega_i)$, for some ρ in X satisfying $\sum_{i=0}^{3}(2a_i+1)^2\wp'(\rho-\omega_i) = 0$ (§6.3). ∎

6.6 Remark (the case $\vec{a} \in \{(b,b,b,b), b \in \mathbf{N}\}$).

Suppose, on the contrary, that $\vec{a} = (b,b,b,b)$ for some $b \in \mathbf{N}$, and choose any $\rho \in X$ satisfying $\sum_{i=0}^{3}(2a_i+1)^2\wp'(\rho-\omega_i) = (2b+1)^2\sum_{i=0}^{3}\wp'(\rho-\omega_i) = 0$. Then, 2ρ is a non-zero half-period (because $\sum_{i=0}^{3}\wp'(x-\omega_i) \equiv 8\wp'(2x)$), and $u(x) = 2\wp(x-\rho) + 2\wp(x+\rho) + b(b+1)\sum_{i=0}^{3}\wp(x-\omega_i)$ is a non-primitive (§1.13. 2)) hyper-elliptic potential associated to an exceptional cover of type $\mu = (2b+1,2,0,0)$, $(2b+1,0,2,0)$ or $(2b+1,0,0,2)$.
More precisely, suppose that $2\rho = \omega_1$ and denote by $\varphi : X \to X/<\omega_1> =: Y$ the canonical projection, and by $\hat{\varphi} : Y \to X$ the dual homomorphism. In particular $\mathrm{Ker}(\varphi) = \{\omega_0,\omega_1\}$, and composing φ with $\hat{\varphi}$ gives "the multiplication by 2". Hence, $\omega_0' := \varphi(\omega_0) = \varphi(\omega_1)$, $\omega_1' := \varphi(\omega_2) = \varphi(\omega_3)$, $\omega_2' := \varphi(\rho)$ and $\omega_3' := \varphi(\rho+\omega_2)$ are the four half-periods of Y. Analogously, we can easily check that $\hat{\varphi}(\omega_0') = \hat{\varphi}(\omega_1') = \omega_0$ and $\hat{\varphi}(\omega_2') = \hat{\varphi}(\omega_3') = \omega_1$. Consider now the exceptional cover $\pi' : (\Gamma,p) \to (Y,q')$, of (degree $m := b^2+b+1$, arithmetic

genus $g := b+1$ and) type ν equal, either to $\nu = (b+1, b, 1, 1)$ if $b \equiv 1(\text{mod}.2)$, or to $\nu = (b, b+1, 1, 1)$ if $b \equiv 0(\text{mod}.2)$. The hyper-elliptic potential associated to $(\pi'; \xi) := \Big(\pi'; \mathcal{O}_\Gamma\big((g-1-m)p + \pi'^*(\omega_2')\big)\Big)$ is equal to (see §5.5) :

$$v(y) = b(b+1)\wp_Y(y) + b(b+1)\wp_Y(y - \omega_1') + 2\wp_Y(y - \omega_2') ,$$

where we denote by $\wp_Y$ the $\wp$-Weierstrass function of (Y, q'). Composing the above potential $v(y)$ with $\varphi : X \to Y$, we obtain (up to a constant) the hyper-elliptic potential associated to $(\pi; \xi) := (\hat{\varphi} \circ \pi'; \xi)$. (see §1.14). Now, we also know that $\wp_Y\big(\varphi(x)\big) \equiv \wp(x) + \wp(x - \omega_1) + \text{cst}$. Hence $\wp_Y\big(\varphi(x) - \omega_1'\big) = \wp_Y\big(\varphi(x - \omega_2)\big) \equiv \wp(x - \omega_2) + \wp(x - \omega_3) + \text{cst}$, and $\wp_Y\big(\varphi(x) - \omega_2'\big) = \wp_Y\big(\varphi(x - \rho)\big) \equiv \wp(x - \rho) + \wp(x - \rho) + \text{cst}$ (since $-\rho - \omega_1 = \rho$). We finally conclude that, up to a constant,

$$u(x) = 2\wp(x - \rho) + 2\wp(x + \rho) + b(b+1)\sum_{i=0}^{3} \wp(x - \omega_i) = v\big(\varphi(x)\big)$$

is a non-primitive hyper-elliptic potential associated to the data $\Big(\hat{\varphi} \circ \pi'; \mathcal{O}_\Gamma\big((g-m-1)p + \pi'^*(\omega_2')\big)\Big)$. In fact, the type of π' being equal to $\nu = (b+1, b, 1, 1)$ or $(b, b+1, 1, 1)$, Γ has a unique Weierstrass point over ω_2', denoted hereafter by p_2'. Hence, the divisor $\pi'^*(\omega_2')$ is linearly equivalent to $(m-1)p + p_2'$, and $u(x)$ is associated to $\Big(\hat{\varphi} \circ \pi'; \mathcal{O}_\Gamma\big((g-2)p + p_2'\big)\Big)$. An analogous construction works in case $2\rho = \omega_2$ or $2\rho = \omega_3$.

7 A plane model for $\mathcal{R}(n, \mu)$ ($\mu^{(2)} = 2n - 3$).

7.1

The couple of Weierstrass functions $(\wp, \wp')$ defines the classical embedding of the elliptic curve $(X, q) = (\mathbf{C}/\Lambda, 0)$ into the projective plane : $\forall z \neq q$, $z \mapsto (\wp(z) : \wp'(z) : 1) \in \mathbf{P}^2(\mathbf{C})$. The image of X is the closure of the smooth affine cubic $\{y^2 = 4\prod_{i=1}^{3}(x - e_i)\}$, where $e_i = \wp(\omega_i)$, $\sum_{i=1}^{3} e_i = 0$ and $e_i \neq e_j$ if $i \neq j$. A straightforward verification shows the following two results.

7.2 Lemma.

The functions $\wp$ and $\wp'$ (associated to the lattice Λ) satisfy the following functional equalities : for any $\rho \in X$ and $j \in \{j,k,\ell\} = \{1,2,3\}$ we have

$$\text{(26)}\qquad \begin{cases} \big(\wp(\rho) - e_j\big)\big(\wp(\rho - \omega_j) - e_j\big) = (e_k - e_j)(e_\ell - e_j) \\ \text{and} \\ \wp'(\rho - \omega_j)\big(\wp(\rho) - e_j\big)^2 = -\wp'(\rho)(e_k - e_j)(e_\ell - e_j)\ . \end{cases}$$

7.3 Proposition.

For any $\vec{a} = (a_i) \in \mathbf{N}^4$ and $\rho \in X \setminus \{\omega_i\ , i = 0, \cdots, 3\}$ the "fundamental" equation $\sum_{i=0}^{3}(2a_i + 1)^2\wp'(\rho - \omega_i) = 0$ is equivalent to

$$(2a_0+1)^2 \prod_{i=1}^{3}\big(\wp(\rho)-e_i\big)^2 = \sum_{j=1}^{3}(2a_j+1)^2(e_k-e_j)(e_\ell-e_j)\big(\wp(\rho)-e_k\big)^2\big(\wp(\rho)-e_\ell\big)^2\ .$$

(27)

7.4 Definition.

The equation (27) above can be considered as defining a surface of degree 6 in the projective space $\{(e_0 : e_1 : e_2 : e_3)\} = \mathbf{P}^3(\mathbf{C})$ $\big(e_0 := \wp(\rho)\big)$. The intersection of the latter surface with the hyperplane $\mathbf{H} := \{e_1 + e_2 + e_3 = 0\} \subset \mathbf{P}^3(\mathbf{C})$ is a plane sextic, denoted hereafter by $\mathcal{C}_{\vec{a}}$.

7.5 Remarks.

1) We've associated to any couple $(\pi;\xi)$, such that $\pi \in R(n,\mu,X)$, $\mu^{(2)} = 2n-3$ and $\xi \in \{\xi_{A,k}, \xi_{B,k},\ k = 0,\cdots,3\}$, (see §6.3) a hyper-elliptic potential of the form (24), and in particular, an integer vector $\vec{a} = (a_i)$ and a couple of points $\{\rho, -\rho\}$ satisfying $\sum_{i=0}^{3}(2a_i + 1)^2\wp'(\rho - \omega_i) = 0$. In other words, each θ-characteristic, ξ, as above, defines an injection (§1.9) of $R(n,\mu,X)$ into $\mathcal{C}_{\vec{a}}$, where $\vec{a} = A^k(\mu)$ (resp. : $\vec{a} = B^k(\mu)$), if $\xi = \xi_{A,k}$ (resp. : $\xi = \xi_{B,k}$).

2) Changing the lattice Λ by $\Lambda' = \alpha\Lambda$ $(\alpha \in \mathbf{C}^*)$ amounts to change the local coordinate z by αz. It easily follows that the corresponding (Weierstrass) $\wp$-function, as well as it values $\{e_i = \wp(\omega_i),\ i = 1,2,3\}$ (resp. : $\wp'$), get

multiplied by α^2 (resp. : by α^3). The resulting elliptic curve, $X' = \mathbf{C}/\Lambda'$, is isomorphic to the initial one. Hence $R(n, \mu, X')$ is equal to $R(n, \mu, X)$ and gets mapped to the same set of points of $\mathcal{C}_{\vec{a}}$ (§1.12).

7.6 Definition.

Let us denote by $\mathcal{P}$ the rational projection from the point $(1:0:0:0) \in \mathbf{H} \simeq \mathbf{P}^2(\mathbf{C})$ to the projective line $\{e_0 = 0\} \subset \mathbf{H}$, and by $\mathcal{X}$, the moduli space of all elliptic curves equipped with the choice of their half-periods. For any $\vec{a} \in \mathbf{N}^4$, the restriction of $\mathcal{P}$ to $\mathcal{C}_{\vec{a}} \setminus \cup_{j=1}^3 \Delta_j$ (where $\Delta_j = \{e_k - e_\ell = 0\}$), defines a finite morphism of degree 6 onto $\{e_0 = 0\} \setminus \cup_{j=1}^3 \Delta_j$, which is isomorphic to $\mathcal{X}$ (a projective line minus three points).

Let us consider, on the other hand, the one-dimensional algebraic family $\mathcal{R}(n, \mu) := \cup_{X \in \mathcal{X}} R(n, \mu, X)$. We can extend the (eight) injections $R(n, \mu, X) \to \mathcal{C}_{\vec{a}} \setminus \cup_{j=1}^3 \Delta_j$ defined in §7.5.1) (for $n = \frac{1}{2}(\mu^{(2)} + 3)$), to injective morphisms $\mathcal{R}(n, \mu) \to \mathcal{C}_{\vec{a}} \setminus \cup_{j=1}^3 \Delta_j$, denoted respectively by $\mathcal{A}^k$ and $\mathcal{B}^k$ $(k = 0, \cdots, 3)$. We can also check that all $\mathcal{C}_{\vec{a}}$'s intersect the cubic $\cup_{j=1}^3 \Delta_j$ at the three points $O_1 := (1:-2:1:1)$, $O_2 := (1:1:-2:1)$ and $O_3 := (1:1:1:-2)$. In particular $\mathcal{C}_{\vec{a}} \setminus \cup_{j=1}^3 \Delta_j = \mathcal{C}_{\vec{a}} \setminus \{O_1, O_2, O_3\}$.

7.7 Proposition.

For any $k = 0, \cdots, 3$ and type $\mu \in \mathbf{M}$ let us fix $n := \frac{1}{2}(\mu^{(2)} + 3)$ and $\vec{a} := A^k(\mu)$ (resp. : $\vec{a} := B^k(\mu)$). Then the morphism $\mathcal{A}^k : \mathcal{R}(n, \mu) \to \mathcal{C}_{\vec{a}} \setminus \{O_1, O_2, O_3\}$ (resp. $\mathcal{B}^k : \mathcal{R}(n, \mu) \to \mathcal{C}_{\vec{a}} \setminus \{O_1, O_2, O_3\}$) is proper. In particular the image of $\mathcal{R}(n, \mu)$ is a closed subvariety of $\mathcal{C}_{\vec{a}} \setminus \{O_1, O_2, O_3\}$.

Proof : Composing any one of the latter morphisms with the rational projection $\mathcal{P} : \mathbf{H} \to \{e_0 = 0\}$ (§7.6), we recover the canonical quasi-finite map $\mathcal{R}(n, \mu) \to \mathcal{X}$. We shall deduce the properness of $\mathcal{A}^k$ and $\mathcal{B}^k$ $(k = 0, \cdots, 3)$ from the properness of $\mathcal{R}(n, \mu) \to \mathcal{X}$. Let indeed $W(n, \nu, X)$ denote the disjoint reunion of compactified jacobians of all elements of the finite set $R(n, \nu, X)$ $(\nu^{(2)} \leq 2n + 1)$, and let $W(n, X)$ denote the disjoint reunion $W(n, X) := \coprod_{\nu^{(2)} \leq 2n+1} W(n, \nu, X)$. The birational injective morphism constructed in §1.12, and denoted by $I_n(X)$, sends $W(n, X)$ onto $KdV(n, X)$, the closed subvariety of the n-th symmetric power X_n, defined by the following set of equations :

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} \wp'(x_i - x_j) = 0 \ , \quad i = 1, \cdots, n \ . \tag{28}$$

Letting X vary over the whole $\mathcal{X}$ we obtain $\mathcal{W}(n,\nu) := \cup_{\mathcal{X}} W(n,\nu,X)$, the algebraic family of compactified jacobians over $\mathcal{R}(n,\nu)$, as well as the disjoint reunion $\mathcal{W}(n) := \coprod_{\nu^{(2)}\leq 2n+1} \mathcal{W}(n,\nu)$. We shall also consider the universal elliptic curve $\bar{\mathcal{X}} \to \mathcal{X}$, equipped with the corresponding relative Weierstrass functions (also denoted by) $\wp$ and $\wp'$, and let $(\bar{\mathcal{X}}/\mathcal{X})_\setminus$ denote its n-th relative symmetric power. Clearly, the family $\mathcal{K}d\mathcal{V}(n) := \cup_{\mathcal{X}} KdV_n(X)$ is a closed subvariety of $(\bar{\mathcal{X}}/\mathcal{X})_n$, which can be defined in terms of $\wp'$ as in (28). In particular the natural projection $\mathcal{K}d\mathcal{V}(n) \to \mathcal{X}$ is proper and $\mathcal{K}d\mathcal{V}(n)$ splits as the disjoint reunion

$$\mathcal{K}d\mathcal{V}(n) = \coprod_{\nu^{(2)}\leq 2n+1} \mathcal{I}_n(\mathcal{W}(n,\nu)) \ ,$$

where $\mathcal{I}_n : \mathcal{W}(n) \to \mathcal{K}d\mathcal{V}(n)$ globalizes the morphism $I_n(X) : W(n,X) \to KdV(n,X)$ defined in §1.12. Choosing a type $\mu \in \mathbf{M}$ such that $\mu^{(2)} = 2n-3$, and a canonical θ-characteristic as in §7.5, we obtain a natural section $\mathcal{R}(n,\mu) \to \mathcal{W}(n,\mu) \subset \mathcal{W}(n)$, which composed with $\mathcal{I}_n$ gives rise to a closed (proper) injection $\mathcal{R}(n,\mu) \to \mathcal{K}d\mathcal{V}(n)$. Hence, the projection $\mathcal{R}(n,\mu) \to \mathcal{K}d\mathcal{V}(n) \to \mathcal{X}$ is proper. ∎

7.8 Remarks.

1) In order to generalize the theorem §6.5 (i.e. : to find all hyper-elliptic potentials as in (24)), we shall need to deal with the whole set of moduli spaces $\{\mathcal{R}(n,\mu),\ n = \frac{1}{2}(\mu^{(2)}+3)\}$. We shall prove indeed that for any $\vec{a} = (a_i) \in \mathbf{N}^4$, there exists ($(n,\mu)$ and) a morphism over $\mathcal{X}$, $\mathcal{R}(n,\mu) \to \mathcal{C}_{\vec{a}} \setminus \{O_1,O_2,O_3\}$ (as in §7.6), defining an isomorphism between $\mathcal{R}(n,\mu)$ and an irreducible component of $\mathcal{C}_{\vec{a}} \setminus \{O_1,O_2,O_3\}$. In particular, the degree of the natural projection $\mathcal{R}(n,\mu) \to \mathcal{X}$ will be equal to 6 if and only if $\mathcal{C}_{\vec{a}}$ is an irreducible sextic. Recall on the other hand, that the degree of $\mathcal{R}(n,\mu) \to \mathcal{X}$ is bounded by 4, whenever $\#\{i, \mu_i = 0\} \geq 1$. Hence, for any such μ and $\vec{a} \in \{A^k(\mu), B^k(\mu)\ ,\ k = 0,\cdots,3\}$, the sextic $\mathcal{C}_{\vec{a}}$ must be reducible. Finally, by realizing (the strict transform of) any $\mathcal{C}_{\vec{a}}$ $(\vec{a} \in \mathbf{N}^4)$ in a suitable pencil of elliptic curves, we'll obtain a characterization of the set $\{\vec{a} \in \mathbf{N}^4,\ \mathcal{C}_{\vec{a}} \text{ is reducible}\}$ (§8.6).
The following result comes right away (§9.2) : "for any $\vec{a} = (a_i) \in \mathbf{N}^4$, such that $\{2a_i+1,\ i = 0,\cdots,3\}$ is linearly independent over $\mathbf{Z}^* = \{1,-1\}$, and $X \in \mathcal{X}$, the function $u(x) = 2\wp(x-\rho)+2\wp(x+\rho)+\sum_{i=0}^{3} a_i(a_i+1)\wp(x-\omega_i)$ is one of the hyper-elliptic potentials associated to some data $(\pi;\xi)$ as in §6.3, if and only if $\rho \in X \setminus \{\omega_i\}$ satisfies the "fundamental" equation (23)

: $\sum_{i=0}^{3}(2a_i+1)^2\wp'(\rho-\omega_i)=0$.

2) The set of integers $\{2a_i+1,\ i=0,\cdots,3\}$ is indeed linearly dependent over $\mathbf{Z}^*=\{1,-1\}$ if and only if $\mathcal{C}_{\vec{a}}$ is reducible (§8.6). In the latter case (at most) one of the irreducible components of $\mathcal{C}_{\vec{a}}$ corresponds to (some) $\mathcal{R}(n,\mu)$. The other components still give rise to hyper-elliptic potentials, although related to exceptional covers (see §9.6. 4)). In other words, we shall see that for any $(a_i)\in\mathbf{N}^4$, and for any elliptic curve (X,q), $u(x)=2\wp(x-\rho)+2\wp(x+\rho)+\sum_{i=0}^{3}a_i(a_i+1)\wp(x-\omega_i)$is a hyper-elliptic potential if and only if $\sum_{i=0}^{3}(2a_i+1)^2\wp'(\rho-\omega_i)=0\ \ (2\rho\neq 0)$.

3) At last, knowing the irreducible decomposition of any $\mathcal{C}_{\vec{a}}$ we shall also be able to improve the estimates §4.12, by calculating $\#R(n,\mu,X)$ for the generic elliptic curve $X=X_{\text{gen}}$. The property §9.6.2) can in fact be rephrased as saying that $\#R(n,\mu,X_{\text{gen}})=6-2j_0$, where j_0 denotes the number of zero coordinates in (μ_i) (i.e. $j_0=\#\{i,\mu_i=0\}$).

8 An elliptic surface approach for studying the plane model $\mathcal{C}_{\vec{a}}$.

From now on we shall identify the hyperplane $\mathbf{H}=\{e_1+e_2+e_3=0\}\subset\mathbf{P}^3(\mathbf{C})$ with the projective plane $\mathbf{P}^2(\mathbf{C})=\{(e_0:e_1:e_2)\}$, and let O_1,O_2 and O_3 denote (respectively) the points $(1:-2:1),(1:1:-2)$ and $(1:1:1)$ of $\mathbf{H}$ (or the points $(-2,1),(1,-2)$ and $(1,1)$ of the affine plane $\{e_0\neq 0\}\subset\mathbf{H}$).

8.1 Lemma.

For any $\vec{a}\in\mathbf{N}^4$ the sextic $\mathcal{C}_{\vec{a}}\subset\mathbf{H}$ (§7.4) has a singularity of order 3 at the points $\{O_1,O_2,O_3\}$, and its strict transform by $\varphi:\hat{\mathbf{H}}\to\mathbf{H}$, the blow up of $\{O_1,O_2,O_3\}$, is a connected divisor.

Proof: Let $P(\alpha,\beta,\gamma)$ denote the polynomial $(\beta-\alpha)(\gamma-\alpha)(e_0-\beta)^2(e_0-\gamma)^2$. Then, for any $\vec{a}\in\mathbf{N}^4$ the sextic $\mathcal{C}_{\vec{a}}$ is defined by the vanishing of the following homogeneous polynomial :

$$(29)\ F_{\vec{a}}=(2a_0+1)^2\prod_{j=1}^{3}(e_0-e_j)^2-\sum_{i=1}^{3}(2a_i+1)^2P(e_i,e_k,e_\ell)\ (\{i,k,\ell\}=\{1,2,3\}).$$

Replacing e_1,e_2 and e_3 by $x:=\frac{e_1}{e_0}$, $y:=\frac{e_2}{e_0}$ and $-(x+y):=\frac{e_3}{e_0}$ respectively, we obtain a polynomial defining the affine sextic $\mathcal{C}_{\vec{a}}\cap\{e_0\neq 0\}$. A direct calculation shows that $\mathcal{C}_{\vec{a}}$ has singularities of order 3 at $\{O_1, O_2, O_3\}$. Moreover the corresponding tangent cones are given by the following equations ($\alpha_j := 2a_j + 1,\ j = 1, 2, 3$) :

$$\left\{\begin{array}{l}
\text{the tangent cone at } O_3 = (1,1) : \\
\{[x-y][\alpha_2 x - \alpha_1 y + \alpha_1 - \alpha_2][\alpha_2 x + \alpha_1 y - \alpha_1 - \alpha_2] = 0\}; \\
\text{the tangent cone at } O_2 = (1,-2) : \\
\{[2x+y][(\alpha_1 - \alpha_3)x + \alpha_1, y + \alpha_1 + \alpha_3][(\alpha_1 + \alpha_3)x + \alpha_1, y + \alpha_1 - \alpha_3] = 0\}; \\
\text{the tangent cone at } O_1 = (-2,1) : \\
\{[x+2y][\alpha_2 x + (\alpha_2 + \alpha_3)y + \alpha_2 - \alpha_3][\alpha_2 x + (\alpha_2 - \alpha_3)y + \alpha_2 + \alpha_3] = 0\} .
\end{array}\right. \tag{30}$$

Let us suppose now that $\mathcal{C}_{\vec{a}}$ decomposes as $\mathcal{C}_{\vec{a}} = \mathcal{C}' \cup \mathcal{C}''$, and let m_1', m_2' and m_3' denote the order of $\mathcal{C}'$ at O_1, O_2 and O_3 respectively. We know that $6 = \deg \mathcal{C}_{\vec{a}} = \deg \mathcal{C}' + \deg \mathcal{C}''$. Hence, $\mathcal{C}'$ and $\mathcal{C}''$ intersect with multiplicity $5 = 5.1$, $8 = 4.2$ or $9 = 3.3$. On the other hand, their strict transforms by $\varphi : \hat{\mathbf{H}} \to \mathbf{H}$, denoted as usual by $\hat{\mathcal{C}}'$ and $\hat{\mathcal{C}}''$, intersect with multiplicity :

$$\hat{\mathcal{C}}' \cdot \hat{\mathcal{C}}'' = \mathcal{C}' \cdot \mathcal{C}'' - m_1'(3 - m_1') - m_2'(3 - m_2') - m_3'(3 - m_3') ,$$

which is easily seen to be $\neq 0$, because $0 \leq m_i' \leq 3$ and $m_i'(3 - m_i') = 0$ or 2 ($i = 1, 2, 3$). Hence $\hat{\mathcal{C}}' \cap \hat{\mathcal{C}}'' \neq \emptyset$. ∎

8.2 Lemma.

Choose any $(a_1, a_2, a_3) \in \mathbf{N}^3$ and let $\hat{\mathcal{C}}_{(-\frac{1}{2}, a_1, a_2, a_3)}$ and $\hat{\mathcal{C}}_0$ denote, respectively, the strict transforms of the sextics $\mathcal{C}_{(-\frac{1}{2}, a_1, a_2, a_3)}$ and $\mathcal{C}_{(0, -\frac{1}{2}, -\frac{1}{2} - \frac{1}{2})}$ by the blow up $\varphi : \hat{\mathbf{H}} \to \mathbf{H}$ of $\{O_1, O_2, O_3\}$. Let also E_j ($j = 1, 2, 3$) denote the exceptional curve $\varphi^*(O_j)$. Then $\hat{\mathcal{C}}_{(-\frac{1}{2}, a_1, a_2, a_3)}$ is disjoint from $\hat{\mathcal{C}}_0$ and linearly equivalent to $\hat{\mathcal{C}}_0 + E_1 + E_2 + E_3$.

Proof: It follows from the equations (30) above that the tangent cones of $\mathcal{C}_{(-\frac{1}{2}, a_1, a_2, a_3)}$ and $\mathcal{C}_{(0, -\frac{1}{2}, -\frac{1}{2} - \frac{1}{2})}$ at O_1, O_2 and O_3 are transverse and have order 3 and 4 respectively. Hence, their strict transforms do not intersect (since $\hat{\mathcal{C}}_{(-\frac{1}{2}, a_1, a_2, a_3)} \cdot \hat{\mathcal{C}}_0 = \mathcal{C}_{(-\frac{1}{2}, a_1, a_2, a_3)} \cdot \mathcal{C}_{(0, -\frac{1}{2}, -\frac{1}{2} - \frac{1}{2})} - 3(3.4) = 36 - 36 = 0$) and

$$\begin{aligned}\hat{\mathcal{C}}_{(-\frac{1}{2},a_1,a_2,a_3)} &\equiv \varphi^*\left(\mathcal{C}_{(-\frac{1}{2},a_1,a_2,a_3)}\right) - 3\sum_{j=1}^{3} E_j \equiv \varphi^*\left(\mathcal{C}_{(0,-\frac{1}{2},-\frac{1}{2}-\frac{1}{2})}\right) - 3\sum_{j=1}^{3} E_j \\ &\equiv \hat{\mathcal{C}}_0 + \sum_{j=1}^{3} E_j \, .\end{aligned}$$

∎

8.3 Proposition.

For any $(a_1, a_2, a_3) \in \mathbf{N}^3$ the divisors $\hat{\mathcal{C}}_{(-\frac{1}{2},a_1,a_2,a_3)}$ and $\hat{\mathcal{C}}_0 + \sum_{i=1}^{3} E_i$ of $\hat{\mathbf{H}}$ (see §8.2), generate a pencil of generically smooth irreducible curves of genus 1. Furthermore,by blowing up the nine (eventually infinitesimal) base points of the latter pencil, we obtain a rational elliptic fibration $(\hat{\mathbf{H}} \leftarrow)\hat{\hat{\mathbf{H}}} \rightarrow \mathbf{P}^1(\mathbf{C})$, with Euler-Poincaré characteristic $\chi(\hat{\hat{\mathbf{H}}}) = 15$.

Proof: We know indeed that for any $\lambda \in \mathbf{C}$ and $\vec{a} = (\lambda, a_1, a_2, a_3) \in \mathbf{C} \times \mathbf{N}^3$, $\mathcal{C}_{\vec{a}}$ is a plane sextic of arithmetic genus $1 + \frac{1}{2} \cdot 6 \cdot (6-3) = 10$, having singularities of order 3 at O_1, O_2 and O_3. Hence, its strict transform $(\hat{\mathcal{C}}_{\vec{a}})$ in $\hat{\mathbf{H}}$ has arithmetic genus $= 1$, and is also a connected divisor (§8.1). On the other hand, $\mathcal{C}_{(-\frac{1}{2},a_1,a_2,a_3)}$ and $\mathcal{C}_{(0,-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})}$ being transverse at O_1, O_2 and O_3, we easily check that $\hat{\mathcal{C}}_{(-\frac{1}{2},a_1,a_2,a_3)}$ intersects $\hat{\mathcal{C}}_0 + E_1 + E_2 + E_3$, at smooth points of the latter divisor. In particular, Bertini's theorem asserts that the generic element of the corresponding pencil is smooth, and connected (see §8.1), hence also irreducible. The $E-P$ characteristic of $\hat{\mathbf{H}}$ is equal to $\chi(\hat{\mathbf{H}}) = \chi(\mathbf{P}^2(\mathbf{C})) + 3 = 6$. By blowing-up the nine base points of the pencil, we finally end up with a rational elliptic surface, $\hat{\hat{\mathbf{H}}} \rightarrow \mathbf{P}^1(\mathbf{C})$, of $E-P$ characteristic $\chi(\hat{\hat{\mathbf{H}}}) = \chi(\hat{\mathbf{H}}) + 9 = 15$. For example, if the a_j's $(j = 1, 2, 3)$ are all distinct $\hat{\mathcal{C}}_{(-\frac{1}{2},a_1,a_2,a_3)}$ is transverse to $\hat{\mathcal{C}}_0 + E_1 + E_2 + E_3$, and it suffices to blow up the nine points of $\hat{\mathcal{C}}_{(-\frac{1}{2},a_1,a_2,a_3)} \cap (E_1 \cup E_2 \cup E_3)$. ∎

8.4 Remark.

According to the (so-called) "Lefschetz formula" ([7] p.509), the $E-P$ characteristics of all singular fibers of any such pencil (in $\hat{\hat{\mathbf{H}}}$), must add up to $\chi(\hat{\hat{\mathbf{H}}}) = 15$.

The knowledge of a few reduced fibers (plus easy estimates for their $E - P$ characteristics) will finally enable us to distinguish the smooth and irreducible ones (§8.5 and §8.6).

8.5 Lemma.

For any $\vec{a} \in \mathbf{N}^4$ such that $\{2a_i+1,\ i=0,\cdots,3\}$ is linearly dependent over $\mathbf{Z}^* = \{1,-1\}$, the sextic $\mathcal{C}_{\vec{a}} = \{F_{\vec{a}}(e_0,e_1,e_2)=0\}$ (see (29)) contains an irreducible conic passing by O_1, O_2 and O_3, and $\hat{\hat{\mathcal{C}}}_{\vec{a}}$, its strict transform in $\hat{\hat{\mathbf{H}}}$, has an $E-P$ characteristic not smaller than $\chi(\hat{\hat{\mathcal{C}}}_{\vec{a}}) \geq 2$. More precisely, denoting $2a_i+1$ by α_i $(i=0,\cdots,3)$ we can easily check that :

I) if $\alpha_0 = \alpha_1+\alpha_2+\alpha_3$, then $F_{\vec{a}}(e_0,e_1,e_2)$ is a multiple of $R_{\vec{a}}^{+++} := \alpha_2 e_1^2 + (\alpha_1+\alpha_2-\alpha_3)e_1e_2+\alpha_1e_2^2+(\alpha_3-\alpha_1)e_0e_1+(\alpha_3-\alpha_2)e_0e_2-(\alpha_1+\alpha_2+\alpha_3)e_0^2$;

II) if $\alpha_0 = |-\alpha_1+\alpha_2+\alpha_3|$, then $F_{\vec{a}}(e_0,e_1,e_2)$ is a multiple of $R_{\vec{a}}^{-++} := \alpha_2e_1^2 + (\alpha_2-\alpha_3-\alpha_1)e_1e_2-\alpha_1e_2^2+(\alpha_1+\alpha_3)e_1e_0+(\alpha_3-\alpha_2)e_0e_2-(\alpha_2+\alpha_3-\alpha_1)e_0^2$;

III) if $\alpha_0 = |\alpha_1-\alpha_2+\alpha_3|$, then $F_{\vec{a}}(e_0,e_1,e_2)$ is a multiple of : $R_{\vec{a}}^{+-+} := \alpha_2e_1^2 + (\alpha_2+\alpha_3-\alpha_1)e_1e_2-\alpha_1e_2^2+(\alpha_1-\alpha_3)e_1e_0-(\alpha_2+\alpha_3)e_0e_2+(\alpha_1-\alpha_2+\alpha_3)e_0^2$;

IV) if $\alpha_0 = |\alpha_1+\alpha_2-\alpha_3|$, then $F_{\vec{a}}(e_0,e_1,e_2)$ is a multiple of : $R_{\vec{a}}^{++-} := \alpha_2e_1^2 + (\alpha_1+\alpha_2+\alpha_3)e_1e_2+\alpha_1e_2^2-(\alpha_1+\alpha_3)e_1e_0-(\alpha_2+\alpha_3)e_0e_2-(\alpha_1+\alpha_2-\alpha_3)e_0^2$.

Proof: Each conic $\{R=0\} \subset \mathbf{H}$ ($R = R_{\vec{a}}^{+++}, R_{\vec{a}}^{++-}, R_{\vec{a}}^{+-+}$, or $R_{\vec{a}}^{-++}$) is smooth and tangent to $\mathcal{C}_{\vec{a}}$ at O_1, O_2 and O_3. In particular the sum of multiplicities of intersection between $\{R=0\}$ and $\mathcal{C}_{\vec{a}}$ at $\{O_1,O_2,O_3\}$ is not less than 12. But $\{R=0\}\cdot\mathcal{C}_{\vec{a}} = 2\cdot 6 = 12$. Hence $\{R=0\}\cap\mathcal{C}_{\vec{a}}$ is equal to $\{O_1,O_2,O_3\}$, unless $\{R=0\}\subset\mathcal{C}_{\vec{a}}$. On the other hand, we can easily check that in each of the above cases (I,II,III & IV) $R(0,e_1,e_2)$ divides $F_{\vec{a}}(0,e_1,e_2)$. In other words, $\{R=0\}\cap\mathcal{C}_{\vec{a}}\cap\{e_0=0\}\neq\emptyset$. Hence $\mathcal{C}_{\vec{a}}$ must contain the conic $\{R=0\}$. Let $\mathcal{C}_{\vec{a}} = \{R=0\}\cup\{T=0\}$ be the corresponding decomposition. After the first blow-up (of O_1, O_2 and O_3), the strict transforms of $\{R=0\}$ and $\{T=0\}$ still intersect with multiplicity 2. It follows that the $E-P$ characteristic of $\hat{\mathcal{C}}_{\vec{a}}$ is already ≥ 2, and can only increase after the second series of blow-ups. Thus, we've proved that $\chi(\hat{\hat{\mathcal{C}}}_{\vec{a}}) \geq 2$. ∎

8.6 Proposition.

For any $(a_1,a_2,a_3)\in\mathbf{N}^3$, let $\hat{\hat{\mathbf{H}}} \to \mathbf{P}^1(\mathbf{C})$ denote the rational elliptic fibration constructed in §8.3 and $\hat{\hat{\mathcal{C}}}_{(-\frac{1}{2},a_1,a_2,a_3)}$ denote the strict transform of $\mathcal{C}_{(-\frac{1}{2},a_1,a_2,a_3)}$

in $\hat{\hat{\mathbf{H}}}$. Then, the above fibration has $m := 2 + \#\{a_1, a_2, a_3\}$ reduced fibers, with $E-P$ characteristics adding up to 14, and $\hat{\hat{\mathcal{C}}}_{(-\frac{1}{2},a_1,a_2,a_3)}$ as the unique irreducible fiber with a node. All other fibers are smooth and irreducible.

Proof: We know already that the $E-P$ characteristics of all singular fibers of $\hat{\hat{\mathbf{H}}} \to \mathbf{P}^1(\mathbf{C})$ add up to $\chi(\hat{\hat{\mathbf{H}}}) = 15$, and an easy calculation shows that the sextic $\mathcal{C}_{(-\frac{1}{2},a_1,a_2,a_3)}$ (see §8.2) has a node at $(1:0:0) \in \mathbf{H}$. In particular, its strict transform, $\hat{\hat{\mathcal{C}}}_{(-\frac{1}{2},a_1,a_2,a_3)}$ is, either an irreducible rational curve with a node (and $\chi(\hat{\hat{\mathcal{C}}}_{(-\frac{1}{2},a_1,a_2,a_3)}) = 1$), or a reducible one (and $\chi(\hat{\hat{\mathcal{C}}}_{(-\frac{1}{2},a_1,a_2,a_3)}) \geq 2$). Moreover, the reducible fiber corresponding to the sextic $\mathcal{C}_{(0,-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})} = \{\prod_{j=1}^{3}(e_0 - e_j)^2 = 0\}$ $(\sum_{j=1}^{3} e_j = 0)$ is isomorphic to $\hat{\mathcal{C}}_0 + E_1 + E_2 + E_3$ (see §8.2), and its $E-P$ characteristic is equal to 6.
On the other hand, we can check that the $(m - 1 = 1 + \#\{a_1, a_2, a_3\})$ reducible fibers obtained in §8.5 have $E-P$ characteristics adding up to (at least) 8. There are indeed three cases to be considered.

- $\#\{a_1, a_2, a_3\} = 3$. There exist four distinct values of a_0, such that the set $\{2a_i + 1,\ i = 0, \cdots, 3\}$ is linearly dependent over $\mathbf{Z}^* = \{1, -1\}$. Hence four reducible fibers, $\hat{\hat{\mathcal{C}}}_{\vec{a}}$, with $\chi(\hat{\hat{\mathcal{C}}}_{\vec{a}}) \geq 2$ (§8.5).

- $\#\{a_1, a_2, a_3\} = 2$. There exists a couple $(a, b) \in \mathbf{N}^2$ $(a \neq b)$ such that, up to the order, $(a_1, a_2, a_3) = (a, a, b)$, and only three values of a_0 such that $\{2a_i + 1,\ i = 0, \cdots, 3\}$ is linearly dependent over $\mathbf{Z}^*$. Namely : $2a_0 + 1 = 2b + 1$, $4a + 3 + 2b$ or $|4a + 1 - 2b|$. In the first case $(a_0 = b)$ the corresponding reducible sextic, $\mathcal{C}_{\vec{a}} = \{F_{\vec{a}} = 0\}$, is the reunion of three irreducible conics passing by O_1, O_2 and O_3. It follows indeed from §8.5.I) & II) that $F_{\vec{a}}$ factors as $F_{\vec{a}} = F_{(b,a,a,b)} = R_{\vec{a}}^{-++} \cdot R_{\vec{a}}^{+-+} \cdot R$, for some irreducible polynomial $R := R_{(b,a,a,b)}$ (which can be easily written down). Hence $\mathcal{C}_{\vec{a}} = \{R_{\vec{a}}^{-++} = 0\} \cup \{R_{\vec{a}}^{+-+} = 0\} \cup \mathcal{C}on_{\vec{a}}$, where $\mathcal{C}on_{\vec{a}} := \{R_{(b,a,a,b)} = 0\}$. Moreover, the latter conics are transverse at O_1 and O_2, but $\mathcal{C}_{\vec{a}}^{-++} := \{R_{\vec{a}}^{-++} = 0\}$ is tangent to $\mathcal{C}_{\vec{a}}^{+-+} := \{R_{\vec{a}}^{+-+} = 0\}$ and transverse to $\mathcal{C}on_{\vec{a}}$, at O_3 (see (35) for $\alpha_1 = \alpha_2 \neq \alpha_3$). Hence, the strict transform of $\mathcal{C}_{\vec{a}}$ in $\hat{\hat{\mathbf{H}}}$ is equal to $\hat{\hat{\mathcal{C}}}_{\vec{a}} = \hat{\hat{\mathcal{C}}}_{\vec{a}}^{-++} \cup \hat{\hat{\mathcal{C}}}_{\vec{a}}^{+-+} \cup \hat{\hat{\mathcal{C}on}}_{\vec{a}}$, and $\hat{\hat{\mathcal{C}}}_{\vec{a}}^{-++} \cdot \hat{\hat{\mathcal{C}}}_{\vec{a}}^{+-+} = 0$ (but $\hat{\hat{\mathcal{C}}}_{\vec{a}}^{-++} \cdot \hat{\hat{\mathcal{C}on}}_{\vec{a}} = \hat{\hat{\mathcal{C}}}_{\vec{a}}^{+-+} \cdot \hat{\hat{\mathcal{C}on}}_{\vec{a}} = 1$). In particular $\chi(\hat{\hat{\mathcal{C}}}_{\vec{a}}) = 4$. We have, therefore, three reducible fibers (§8.5) with $E-P$ characteristics adding up to (at least) $4 + 2 + 2 = 8$.

- $\#\{a_1, a_2, a_3\} = 1$. Since $a_1 = a_2 = a_3$, the lemma §8.5 gives us only two values of a_0 such that $\{2a_i + 1,\ i = 0, \cdots, 3\}$ is linearly dependent over $\mathbf{Z}^*$. Namely : $a_0 = a_1$ and $a_0 = 3a_1$. For $a_0 = a_1 = a_2 = a_3$, $F_{\vec{a}}$ factors as the product of the three irreducible quadratic polynomials : $\{R_{\vec{a}}^{++-}, R_{\vec{a}}^{+-+}, R_{\vec{a}}^{-++}\}$. In other words

$$\begin{aligned} F_{(a_0,a_0,a_0,a_0)} &= \\ &(2a_0+1)^2\left[e_1^2 + 3e_1e_2 + e_2^2 - 2e_0e_1 - 2e_0e_2 - e_0^2\right] \\ &\left[e_1^2 + e_1e_2 - e_2^2 - 2e_0e_2 + e_0^2\right]\left[e_1^2 - e_1e_2 - e_2^2 + 2e_1e_0 - e_0^2\right] \end{aligned}$$

Hence $\mathcal{C}_{\vec{a}} = \{R_{\vec{a}}^{++-} = 0\} \cup \{R_{\vec{a}}^{+-+} = 0\} \cup \{R_{\vec{a}}^{-++} = 0\}$, and its strict transform in $\hat{\hat{\mathbf{H}}}$, made of three disjoint copies of $\mathbf{P}^1(\mathbf{C})$, has $E-P$ characteristic $= 6$. We have, therefore, two reducible fibers (§8.5) with $E-P$ characteristics adding up to (at least) 6+2=8.

Summing up all our arguments we conclude, since $\chi(\hat{\hat{\mathbf{H}}}) = 15$, that the $(1 + \#\{a_1, a_2, a_3\})$ reducible fibers obtained in §8.5, have the minimal possible $E-P$ characteristics (i.e. : $(2,2,2,2), (4,2,2)$ and $(6,2)$), while $\chi(\hat{\hat{\mathcal{C}}}_0) = 6$. Hence $\chi\left(\hat{\hat{\mathcal{C}}}_{(-\frac{1}{2},a_1,a_2,a_3)}\right) = \chi(\hat{\hat{\mathbf{H}}}) - 8 - 6 = 1$ (i.e.: $\hat{\hat{\mathcal{C}}}_{(-\frac{1}{2},a_1,a_2,a_3)}$ is a rational integral curve having a node at $(1:0:0)$) and all other fibers must be smooth. In other words, for any $\lambda \in \mathbf{C}$ $\hat{\hat{\mathcal{C}}}_{(\lambda,a_1,a_2,a_3)}$ is an elliptic curve if and only if $(2\lambda+1)^2 \notin \{0, (\alpha_1+\alpha_2+\alpha_3)^2, (-\alpha_1+\alpha_2+\alpha_3)^2,$ $(\alpha_1-\alpha_2+\alpha_3)^2, (\alpha_1+\alpha_2-\alpha_3)^2\}$. ∎

9 The full characterization of a new class of even hyper-elliptic potentials.

9.1

The proposition §8.6 and its proof, furnish a detailed study of the subset of reducible sextics in $\{\mathcal{C}_{\vec{a}}, \vec{a} \in \mathbf{N}^4\}$ and provide the final step for clearing up the relationship between $\mathcal{R}(n,\mu)$ ($\mu^{(2)} = 2n-3$) and $\mathcal{C}_{\vec{a}}$, for any $\vec{a} \in \{A^k(\mu), B^k(\mu),\ k = 0, \cdots, 3\}$ (§9.6). We then apply it to deduce the full characterization of all even hyper-elliptic potentials of the following type (§9.7, §9.9) :

$$u(x) = 2\wp(x-\rho) + 2\wp(x+\rho) + \sum_{i=0}^{3} a_i(a_i+1)\wp(x-\omega_i)\ ,$$

where $\vec{a} = (a_i) \in \mathbf{N}^4$ and $2\rho \neq 0$. Recall that the subset of all possible types, $\mathbf{M} \subset \mathbf{N}^4$, decomposes as a disjoint reunion $\mathbf{M} = \coprod_{j=0}^{3} \mathbf{M}^{\#j}$, where μ belongs to $\mathbf{M}^{\#j}$ if and only if $\mu \in \mathbf{M}$ and $\#\{i, \mu_i = 0\} = j$. Furthermore, for any $\vec{a} \in \mathbf{N}^4$ there exists a unique type $\mu \in \mathbf{M}$ such that $\vec{a} \in \{A^k(\mu), B^k(\mu),\ k = 0, \cdots, 3\}$ ([18] §4.4, §4.5, §5.3). Let n denote the positive integer $n := \frac{1}{2}(\mu^{(2)} + 3) = 2 + \frac{1}{2}\sum_{i=0}^{3} a_i(a_i + 1)$, and $F_{\vec{a}}$ denote the homogeneous polynomial (29) such that $\mathcal{C}_{\vec{a}} = \{F_{\vec{a}} = 0\} \cap \mathbf{H}$. An exhaustive investigation of all possibilities finally leads us to studying four distinct cases.

9.2 Theorem (the case $\mu \in \mathbf{M}^{\#0}$).

The following properties are equivalent :

i) $\mu \in \mathbf{M}^{\#0}$ (i.e. : $\mu_i \neq 0$ for all $i = 0, \cdots, 3$) ;

ii) the set of integers $\{2a_i + 1,\ i = 0, \cdots, 3\}$ is linearly independent over $\mathbf{Z}^* = \{1, -1\}$;

iii) the injection $\mathcal{R}(n, \mu) \to \mathcal{C}_{\vec{a}} \setminus \{O_1, O_2, O_3\}$ (see §7.7) is an isomorphism ;

iv) for any elliptic curve X and $\rho \in X \setminus \{\omega_i\}$, the even function

$$u(x) = 2\wp(x - \rho) + 2\wp(x + \rho) + \sum_{i=0}^{3} a_i(a_i + 1)\wp(x - \omega_i)\ ,$$

is one of the hyper-elliptic potentials canonically associated to some $\pi \in \mathcal{R}(n, \mu, X)$ (see §6.3), if and only if $\sum_{i=0}^{3}(2a_i + 1)^2\wp'(\rho - \omega_i) = 0$.

Proof: The equivalence i) $\Leftrightarrow$ ii) can be directly checked from the formulae of A^k and B^k ($k = 0, \cdots, 3$) (e.g. : §6.3 or [18] §4.4, §5.4.3)). It follows from §8.6 that $\mathcal{C}_{\vec{a}} \setminus \{O_1, O_2, O_3\}$ is always smooth, and irreducible if and only if $\{2a_i + 1\}$ is linearly independent over $\mathbf{Z}^*$. Recall, on the other hand, that the image of $\mathcal{R}(n, \mu)$ by the injective morphism $\mathcal{A}^k$ or $\mathcal{B}^k$ is a closed subvariety of $\mathcal{C}_{\vec{a}} \setminus \{O_1, O_2, O_3\}$ (7.7), and that degree$(\mathcal{R}(n, \mu) \to \mathcal{X}) <$ $\cancel{6}$if $\mu \notin \mathbf{M}^{\#0}$ (§4.12). Hence, §9.2.i) and §9.2.ii) imply that the above proper injections $\mathcal{R}(n, \mu) \to \mathcal{C}_{\vec{a}} \setminus \{O_1, O_2, O_3\}$ must also be surjective, and even an isomorphism because $\mathcal{C}_{\vec{a}}$ is smooth outside $\{O_1, O_2, O_3\}$. Finally §9.2.iii) easily implies §9.2.iv), which in turn means that $\deg(\mathcal{R}(n, \mu) \to \mathcal{X}) = 6$. Consequently $\mu \in \mathbf{M}^{\#0}$. ∎

9.3 Theorem (the case $\mu \in \mathbf{M}^{\#1}$).

The following properties are equivalent:

i) $\mu \in \mathbf{M}^{\#1}$ (i.e. : $\#\{i, \mu_i = 0\} = 1$) ;

ii) the set of integers $\{2a_i + 1, i = 0, \cdots, 3\}$ is linearly dependent over $\mathbf{Z}^* = \{1, -1\}$, but $\vec{a} \notin \{(c, c, d, d,), (c, d, c, d), (c, d, d, c) \; ; c, d \in \mathbf{N}\}$;

iii) the homogeneous polynomial $F_{\vec{a}}(e_0, e_1, e_2)$ (see (29)) factors as $F_{\vec{a}} = T \cdot R$, where $R \in \{R_{\vec{a}}^{+++}, R_{\vec{a}}^{-++}, R_{\vec{a}}^{+-+}, R_{\vec{a}}^{++-}\}$ (see §8.5). Moreover, the plane quartic $\mathcal{Q}_{\vec{a}} := \{T = 0\}$ is an irreducible component of $\mathcal{C}_{\vec{a}}$, and the given injection $\mathcal{R}(n, \mu) \to \mathcal{C}_{\vec{a}} \setminus \{O_1, O_2, O_3\}$, defines an isomorphism between $\mathcal{R}(n, \mu)$ and $\mathcal{Q}_{\vec{a}} \setminus \{O_1, O_2, O_3\}$;

iv) $F_{\vec{a}}$ factors as in §9.3.iii), and for any elliptic curve X and any $\rho \in X$ $(2\rho \neq 0)$,

$$u(x) = 2\wp(x - \rho) + 2\wp(x + \rho) + \sum_{i=0}^{3} a_i(a_i + 1)\wp(x - \omega_i) \; ,$$

is one of the hyper-elliptic potentials canonically associated to some $\pi \in R(n, \mu, X)$ (see §6.3) if and only if $T\big(\wp(\rho), e_1, e_2\big) = 0$.

Proof: The equivalence of the properties i) and ii) follows directly from the corresponding formulae (§6.3), and implies that $F_{\vec{a}}$ factors as in §9.3.iii) (see §8.5). For example, if $2a_0 + 1 = \sum_{j=1}^{3}(2a_j + 1)$ then $R_{\vec{a}}^{+++}$ divides $F_{\vec{a}}$, $F_{\vec{a}} = R_{\vec{a}}^{+++}T$, and the quartic $\mathcal{Q}_{\vec{a}} = \{T = 0\}$ is irreducible because $\vec{a} \notin \{(c, c, d, d), (c, d, d, c), (c, d, c, d); c, d \in \mathbf{N}\}$ (see §8.6). In particular, the proper morphism (§7.7) $\mathcal{R}(n, \mu) \to \mathcal{C}_{\vec{a}} \setminus \{O_1, O_2, O_3\}$ must cover, either the affine conic $\{R = 0\} \setminus \{O_1, O_2, O_3\}$, or the affine quartic $\mathcal{Q}_{\vec{a}} \setminus \{O_1, O_2, O_3\}$. We also learn from §4.12 that for any $X \in \mathcal{X}$, $\#\mathcal{R}(n, \mu, X) \geq 2$. Hence $\deg\big(\mathcal{R}(n, \mu) \to \mathcal{X}\big) \geq 2$, with equality if and only if the projection $\mathcal{R}(n, \mu) \to \mathcal{X}$ is non-ramified. On the other hand, it can be easily verified (by means of the Hurwitz formula, coupled with a local development of $R(e_0, e_1, e_2)$ around $\{O_1, O_2, O_3\}$) that the projection $\{R = 0\} \setminus \{O_1, O_2, O_3\} \to \mathcal{X}$, is in fact ramified. Therefore $\deg\big(\mathcal{R}(n, \mu) \to \mathcal{X}\big) > 2$ and $\mathcal{R}(n, \mu)$ must cover the affine quartic $\mathcal{Q}_{\vec{a}} \setminus \{O_1, O_2, O_3\}$. Finally, since $\mathcal{Q}_{\vec{a}}$ is smooth outside $\{O_1, O_2, O_3\}$, the above injection defines an isomorphism between $\mathcal{R}(n, \mu)$ and $\mathcal{Q}_{\vec{a}} \setminus \{O_1, O_2, O_3\}$. The property §9.3.iii) easily implies §9.3.iv), which in turn implies that $F_{\vec{a}}$ factors and $\deg(\mathcal{R}(n, \mu) \to \mathcal{X}) = 4$. Hence $\#\{i, \mu_i = 0\} = 1$ (i.e. : $\mu \in \mathbf{M}^{\#1}$). ∎

9.4 Theorem (the case $\mu \in \mathbf{M}^{\#2}$).

The following properties are equivalent :

i) $\mu \in \mathbf{M}^{\#2}$;

ii) there exist $c, d \in \mathbf{N}$, $c \neq d$, such that $\vec{a}$ is equal, either to (c, c, d, d), to (c, d, c, d) or to (c, d, d, c) ;

iii) $\mathcal{C}_{\vec{a}} \setminus \{O_1, O_2, O_3\}$ splits as a reunion of three irreducible affine conics, and $\mathcal{R}(n, \mu)$ gets mapped isomorphically onto one of them (denoted in §8.6 by $\mathcal{C}on_{\vec{a}}$). Moreover, let $\vec{a}$ be as in §9.4.ii). Then, for any elliptic curve X and $\rho \in X$, $u(x) = 2\wp(x - \rho) + 2\wp(x + \rho) + \sum_{i=0}^{3} a_i(a_i + 1)\wp(x - \omega_i)$ is one of the even hyper-elliptic potentials canonically associated to some $\pi \in R(n, \mu, X)$ (see §6.3), if and only if 2ρ is equal to ω_1 (when $\vec{a} = (c, c, d, d)$), to ω_2 (when $\vec{a} = (c, d, c, d)$), or to ω_3 (when $\vec{a} = (c, d, d, c)$). In particular $u(x)$ is non-primitive.

Proof: The properties §9.4.i) & §9.4.ii) are easily seen to be equivalent, and imply that $R(n, \mu, X) \neq \emptyset$, for any $X \in \mathcal{X}$, as shown below. They also imply that $\deg(\mathcal{R}(n, \mu) \to \mathcal{X}) \leq 2$ (§4.12), and that $\mathcal{C}_{\vec{a}}$ splits as a reunion of three smooth conics (§8.6). Hence, the injection $\mathcal{R}(n, \mu) \to \mathcal{C}_{\vec{a}} \setminus \{O_1, O_2, O_3\}$ defines an isomorphism of $\mathcal{R}(n, \mu)$ with one of the irreducible components of $\mathcal{C}_{\vec{a}} \setminus \{O_1, O_2, O_3\}$. Conversely, the splitting of $\mathcal{C}_{\vec{a}}$ as a reunion of conics, and $\mathcal{R}(n, \mu)$ being $\neq \emptyset$, imply that $\mu \in \mathbf{M}^{\#2} \cup \mathbf{M}^{\#3}$ but $\mu \notin \mathbf{M}^{\#3}$. Hence $\mu \in \mathbf{M}^{\#2}$. Let us finally prove that $R(n, \mu, X)$ is $\neq \emptyset$ for any $X \in \mathcal{X}$, if $\mu \in \mathbf{M}^{\#2}$ (i.e. : if μ is equal, either to $(2\ell + 1, 2k, 0, 0)$, to $(2\ell + 1, 0, 2k, 0)$ or to $(2\ell + 1, 0, 0, 2k)$, for some $k, \ell \in \mathbf{N}$, $k \neq 0$). Let us suppose, for example, that $\mu = (2\ell + 1, 2k, 0, 0)$, hence $n = 2(\ell^2 + \ell + k^2 + 1)$, and consider the projection $r : X \to Y := X/_{\langle \omega_0, \omega_1 \rangle}$. Choose $\rho \in X$ such that $2\rho = \omega_1$. We can easily check that $\nu_0 := r(\omega_0) = r(\omega_1)$, $\nu_1 := r(\omega_2) = r(\omega_3)$, $\nu_2 := r(\rho)$ and $\nu_3 := r(\rho + \omega_2) = \nu_2 + \nu_1$ are the half-periods of Y. Furthermore, the dual morphism $\hat{r} : Y \to X$ has kernel $\ker \hat{r} = \langle \nu_0, \nu_1 \rangle$. Consider the exceptional covers $\pi' : \Gamma' \to Y$ and $\pi'' : \Gamma'' \to Y$, of type $\mu' = (\ell + 1, \ell, k - 1, k + 1)$ and $\mu'' = (\ell + 1, \ell, k + 1, k - 1)$, respectively, if $\ell \equiv k - 1(\text{mod.}2)$; otherwise we choose $\mu' = (\ell, \ell + 1, k - 1, k + 1)$ and $\mu'' = (\ell, \ell + 1, k + 1, k - 1)$. They have same degree, $\ell^2 + \ell + k^2 + 1 = \frac{n}{2}$, and same arithmetic genus $\frac{1}{2}(\mu'^{(1)} - 1) = \ell + k = \frac{1}{2}(\mu''^{(1)} - 1)$. Composing π' and π'' with $\hat{r} : Y \to X$ we get two hyperelliptic tangential covers of degree n over X, type $\mu = (2\ell + 1, 2k, 0, 0)$, and arithmetic genus $\ell + k = \frac{1}{2}(\mu^{(1)} - 1)$. Consequently $R(n, \mu, X)$ contains $\{\hat{r} \circ \pi' , \hat{r} \circ \pi''\}$. Moreover, composing the projection $r : X \to Y$ with the canonical hyper-elliptic potentials associated to

π' and π'', we get those associated to $\hat{r}\circ\pi'$ and $\hat{r}\circ\pi''$, respectively. We can check in particular (exercice !) that the pair of functions :

$$\begin{cases} u(x) = 2f(x-\tau) + c(c+1)f(x) + d(d+1)f(x-\omega_2) \\ (\text{where } 2\tau = \omega_1 \text{ and } f(x) \equiv \wp(x) + \wp(x-\omega_1))\ , \end{cases}$$

are thus obtained.

At last, although not quoted in §9.4, we should pointout that, for any $X \in \mathcal{X}$, $R(n,\mu,X)$ not only contains but is equal to $\{\hat{r}\circ\pi'\ ,\hat{r}\circ\pi''\}$. This is due to the fact that $\#R(n,\mu,X) \le 2$ (§4.12.2)), and that $\hat{r}\circ\pi'$ is not isomorphic to $\hat{r}\circ\pi''$ (see §4.12). ∎

9.5 Theorem (the case $\mu \in \mathbf{M}^{\#3}$).

The following properties are equivalent :

i) $\mu \in \mathbf{M}^{\#3} = \{(2k+1,0,0,0)\ ,k\in\mathbf{N}\}$;

ii) $\vec{a} \in \{(b,b,b,b)\ ,b\in\mathbf{N}\}$;

iii) $\mathcal{C}_{\vec{a}}$ splits as the reunion of three conics and $\mathcal{R}(n,\mu)=\emptyset$.

In the latter case the equation $\sum_{i=0}^{3}(2a_i+1)^2\wp'(\rho-\omega_i)=0$ is equivalent to saying that $4\rho=0$ but $2\rho\neq0$. For any such solution ρ, the even function $u(x)=2\wp(x-\rho)+2\wp(x+\rho)+b(b+1)\sum_{i=0}^{3}\wp(x-\omega_i)$ is a non-primitive hyper-elliptic potential associated to an exceptional cover of degree $n:=2(b^2+b+1)$ and type equal, either to $(2k+1,2,0,0,0)$, to $(2k+1,0,2,0)$ or to $(2k+1,0,0,2)$.

Proof: The equivalence follows (again) from §4.12 and §8.6, while the last statement has been proved in §6.6. ∎

We finally achieve our main purposes by means of the next three corollaries. The first one summarizes the global properties of the moduli spaces $\{\mathcal{R}(n,\mu)\ ,\mu\in\mathbf{M}\ ,\mu^{(2)}=2n-3\}$ (compare, for example, with §9.2, [14] §6.7 and [15] §2.1.) and their plane models $\{\mathcal{C}_{\vec{a}}\ ,\vec{a}\in\mathbf{N}^4\}$. The two final ones give the announced full characterization (compare with [18] §3.2.). They all follow, more or less immediately, from §9.2., §9.3., §9.4. and §9.5.

9.6 Corollary.

Let $\vec{a}\in\mathbf{N}^4$ and $\mu\in\mathbf{M}$ be such that $\vec{a}\in\{A^k(\mu)\ ,B^k(\mu)\ ,k=0,\cdots,3\}$, and denote by n (resp : by j_0) the integer $n:=2+\frac{1}{2}\sum_{i=0}^{3}a_i(a_i+1)=\frac{1}{2}(\mu^{(2)}+3)$ (resp : $j_0:=\#\{i,\mu_i=0\}$). Then :

1) $\mathcal{R}(n,\mu)$ is smooth and irreducible (but void if $j_0 = 3$, i.e. : if $\mu \in \mathbf{M}^{\#3}$) ;

2) the natural projection $\mathcal{R}(n,\mu) \to \mathcal{X}$ has degree $6-2j_0$, and factors through an embedding of $\mathcal{R}(n,\mu)$ into $\mathcal{C}_{\vec{a}} \setminus \{O_1, O_2, O_3\}$;

3) the projection $\mathcal{C}_{\vec{a}} \to \{e_0 = 0\} \simeq \mathbf{P}^1(\mathbf{C})$ (see the proof of §7.7) provides a natural compactification of $\mathcal{R}(n,\mu) \to \mathcal{X}$;

4) the complement of the image of $\mathcal{R}(n,\mu)$ in $\mathcal{C}_{\vec{a}} \setminus \{O_1, O_2, O_3\}$ also parameterizes even hyper-elliptic potentials, although associated to "non-canonical" θ-characteristics over exceptional covers.

Proof: Clearly, it is enough to give an outline of the proof of §9.6.4). For $\mu \in \mathbf{M}^{\#0}$ or $\mu \in \mathbf{M}^{\#3}$ there's nothing to prove (see §9.2 & §9.5). Consider now any $\mu \in \mathbf{M}^{\#1}$, for which we know that (generically over $\mathcal{X}$), $\#R(n,\mu,X) = 4$. Take for example $\mu = (0,\mu_1,\mu_2,\mu_3)$ (hence $\mu_0 = 0$ and $n = \frac{1}{2}(3+\sum_{j=1}^{3}\mu_j^2)$), and let $\hat{\pi} : (\hat{\Gamma},\hat{p}) \to (X,q)$ denote the exceptional cover of type $\hat{\mu} := (2,\mu_1,\mu_2,\mu_3)$, arithmetic genus $\hat{g} := \frac{1}{2}(\hat{\mu}^{(1)} - 1) = \frac{1}{2}(\mu^{(1)} - 1) + 1$ and degree $\frac{1}{2}(\hat{\mu}^{(2)} - 1) = \frac{1}{2}(\mu^{(2)} + 3) = n$. The intersection in $S^{\perp}$ between $r_0^{\perp}$ and the embedding of $\hat{\Gamma}$ in $S^{\perp}$ (§2.2), has degree equal to $\hat{\mu}_0 = 2$. Hence, $r_0^{\perp}$ "intersects" $\hat{\Gamma}$ at two Weierstrass points, denoted hereafter by $\hat{p}_1$ and $\hat{p}_2$. Moreover, a straightforward application of the $F-S$ & W formula, as developed in §5.3 and §6.3, shows that the hyper-elliptic potentials associated to $\mathcal{O}_{\hat{\Gamma}}\big((\hat{g}-2)\hat{p} + \hat{p}_\ell\big)$, as well as $\mathcal{O}_{\hat{\Gamma}}\big((n+\hat{g}-2)\hat{p}+\hat{p}_\ell-\hat{\pi}^*(q)\big)$ $(\ell = 1,2)$, and their translates by $\{\omega_i, i = 0,\cdots,3\}$, also have the above type (§9.1.). In this way we get the two missing hyper-elliptic potentials. An analogous construction works for any $\mu \in \mathbf{M}^{\#2}$ (see §9.9 and its proof). ∎

9.7 Corollary.

For any $\vec{a} \in \mathbf{N}^4$, any elliptic curve $X \in \mathcal{X}$ and any $\rho \in X \setminus \{\omega_i,\ i = 0,\cdots,3\}$, the even function

$$u(x) = 2\wp(x-\rho) + 2\wp(x+\rho) + \sum_{i=0}^{3} a_i(a_i+1)\wp(x-\omega_i)\ , \tag{31}$$

is a hyper-elliptic potential if and only if $\sum_{i=0}^{3}(2a_i+1)^2\wp'(\rho-\omega_i) = 0$.

9.8 Remark.

To get the full picture we should complete §9.7 by "calculating" the spectral data attached to each one of the above (31) hyper-elliptic potentials (or at least the genus of their spectral curves).
Recall at this point that for any $\vec{a} \in \mathbf{N}^4$ there exists a unique type $\mu = \mu(\vec{a}) \in \mathbf{M}$, such that $\vec{a} \in \{A^k(\mu), B^k(\mu)\ , k = 0, \cdots, 3\}$ (e.g. : formulae [18] §4.4, §4.5 & §5.3). We've also learnt (e.g. : §7.7, §9.2.iii), §9.3.iii), §9.4.iii)) that, by choosing the right canonical θ-characteristic (for ex : $\xi_{A,0}$ if $\vec{a} = A^\circ(\mu)$) for any element in the moduli space $\mathcal{R}(n, \mu)$ ($n := \frac{1}{2}(\mu^{(2)} + 3) = 2 + \frac{1}{2}\sum_{i=0}^{3} a_i(a_i + 1)$), we give rise to a proper embedding $\mathcal{R}(n, \mu) \hookrightarrow \mathcal{C}_{\vec{a}} \setminus \{O_1, O_2, O_3\}$. It then follows from §9.2, §9.3, §9.4 & §9.5, that the above mentioned "calculation" amounts to characterizing the natural image of $\mathcal{R}(n, \mu)$ in $\mathcal{C}_{\vec{a}}$. The latter is an irreducible component equal, either to the whole sextic (if $\mu \in \mathbf{M}^{\#0}$), a quartic $\mathcal{Q}_{\vec{a}}$ (if $\mu \in \mathbf{M}^{\#1}$), a conic $\mathcal{C}on_{\vec{a}}$ (if $\mu \in \mathbf{M}^{\#2}$) or void (if $\mu \in \mathbf{M}^{\#3}$). The corresponding homogeneous polynomials (in case $\mu \in \mathbf{M}^{\#1} \cup \mathbf{M}^{\#2}$) have indeed been found, thanks to the MAPLE-user Fatma Jeeawock. However, we think it's worth presenting a simpler characterization, which by-passes the calculation of $\mu(\vec{a})$, as well as the polynomials defining $\mathcal{Q}_{\vec{a}}$ or $\mathcal{C}on_{\vec{a}}$.

9.9 Corollary.

For any $\vec{a} \in \mathbf{N}^4$, let us denote by $\Pi_{\vec{a}}(e_0, e_1, e_2)$ the product

$$\text{(32)} \qquad \Pi_{\vec{a}} = R_{\vec{a}}^{+++} \cdot R_{\vec{a}}^{-++} \cdot R_{\vec{a}}^{+-+} \cdot R_{\vec{a}}^{++-} \qquad \text{(see §8.5)}\ ,$$

by $\mu(\vec{a}) \in \mathbf{M}$ the unique type such that $\vec{a} \in \{A^k(\mu(\vec{a})), B^k(\mu(\vec{a}))\ , k = 0, \cdots, 3\}$, and by n the integer $n := 2 + \frac{1}{2}\sum_{i=0}^{3} a_i(a_i + 1) = \frac{1}{2}(\mu(\vec{a})^{(2)} + 3)$. Then, for any elliptic curve $X \in \mathcal{X}$, the solutions of $\sum_{i=0}^{3}(2a_i + 1)^2\wp'(\rho - \omega_i) = 0$ give rise to two different kinds of hyper-elliptic potentials (§9.7) :

I) If $\sum_{i=0}^{3}(2a_i+1)^2\wp'(\rho-\omega_i) = 0$ but $\Pi_{\vec{a}}(\wp(\rho), e_1, e_2) \neq 0$, the hyper-elliptic potential $u(x)$ (31) is associated to $(\Gamma, p, \lambda_\pi, \xi)$, where $\pi \in R(n, \mu(\vec{a}), X)$, and ξ is one of the canonical θ-characteristics ($\xi = \xi_{A,0}$ if $\vec{a} = A^\circ(\mu(\vec{a}))$, etc.). In particular the arithmetic genus of the spectral curve Γ, is equal to :

$$g := \frac{1}{2}(\mu(\vec{a})^{(1)} - 1) = \frac{1}{2}\max\{2M, 1 + \Sigma - (1 + (-1)^\Sigma)(m + \frac{1}{2})\}\ ,$$

where $M = \max\{a_i\}$, $\Sigma = \sum_{i=0}^{3} a_i$ and $m = \min\{a_i\}$.

II) If $\sum_{i=0}^{3}(2a_i+1)^2\wp'(\rho-\omega_i)=0$ and $\Pi_{\vec{a}}\big(\wp(\rho),e_1,e_2\big)=0$, there exists $k\in\{0,\cdots,3\}$, an exceptional cover $\hat{\pi}:(\hat{\Gamma},\hat{p})\to(X,q)$ of degree n and type $\hat{\mu}$ (where $\hat{\mu}_i=\mu(\vec{a})_i+2\delta_{ik}$, $i=0,\cdots,3$), and a Weierstrass point $\hat{p}'\in\hat{\Gamma}$ over ω_k, such that the hyper-elliptic potential $u(x)$ (31) is associated to $(\hat{\Gamma},\hat{p},\lambda_{\hat{\pi}},\xi)$, where $\xi\otimes\mathcal{O}_{\hat{\Gamma}}(\hat{p}-\hat{p}')$ is one of the canonical θ-characteristics (for ex. : $\xi=\xi_{A,0}\otimes\mathcal{O}_{\hat{\Gamma}}(\hat{p}'-\hat{p})$ if $\vec{a}=A^\circ\big(\mu(\vec{a})\big)$). In particular, the arithmetic genus of the spectral curve $\hat{\Gamma}$, is equal to :

$$\hat{g}:=\frac{1}{2}(\hat{\mu}^{(1)}-1)=g+1=1+\frac{1}{2}\max\{2M,1+\Sigma-\big(1+(-1)^{\Sigma}\big)(m+\frac{1}{2})\}\ .$$

Proof: All we still need to prove is that the vanishing of $\Pi_{\vec{a}}$ distinguishes the image of $\mathcal{R}\big(n,\mu(\vec{a})\big)$ from its complement, in $\mathcal{C}_{\vec{a}}\setminus\{O_1,O_2,O_3\}$. We know already that the intersection $(\mathcal{C}_{\vec{a}}\setminus\{O_1,O_2,O_3\})\cap\{\Pi_{\vec{a}}=0\}$ contains the above complement (see §9.3.iii), §9.4.iii) & §9.5.iii)). On the other hand, we can easily verify that for any $R\in\{R_{\vec{a}}^{+++},R_{\vec{a}}^{-++},R_{\vec{a}}^{+-+},R_{\vec{a}}^{++-}\}$, the conic $\{R=0\}$ intersects the sextic $\mathcal{C}_{\vec{a}}$ at O_j $(j=1,2,3)$ with multiplicity 4. Hence, $\{R=0\}$ doesn't intersect any irreducible component of $\mathcal{C}_{\vec{a}}$ (different from $\{R=0\}$) outside $\{O_1,O_2,O_3\}$. We have thus shown that $(\{R=0\}$, hence) $\{\Pi_{\vec{a}}=0\}$ is disjoint from the image of $\mathcal{R}\big(n,\mu(\vec{a})\big)$ as required.
At last we shall make a few comments about the type $\hat{\mu}$ and explain the choice of the "non-canonical" θ-characteristics, for the case §9.9.II). Recall that the complement of the image of $\mathcal{R}\big(n,\mu(\vec{a})\big)$ is non-void if and only if $\mu(\vec{a})$ belongs to $\cup_{j=1}^{3}\mathbf{M}^{\#j}$. The cases $\mu(\vec{a})\in\mathbf{M}^{\#1}$ and $\mu(\vec{a})\in\mathbf{M}^{\#3}$ have already been studied to full extent (see §9.6.4) and §6.6.). Suppose, finally, that $\mu(\vec{a})\in\mathbf{M}^{\#2}$. To simplify the discussion, we shall assume that $\mu(\vec{a}):=(2\ell+1,2k,0,0)$ $(k\neq0)$, and $A^\circ\big(\mu(\vec{a})\big)=\vec{a}$. Let $\hat{\pi}:(\hat{\Gamma},\hat{p})\to(X,q)$ denote the exceptional cover of type $\hat{\mu}:=(2\ell+1,2k,2,0)$ (resp : $\hat{\mu}:=(2\ell+1,2k,0,2)$), degree $n:=\frac{1}{2}(\hat{\mu}^{(2)}+1)=\frac{1}{2}\big(\mu(\vec{a})^{(2)}+3\big)$ and arithmetic genus $\hat{g}:=\frac{1}{2}\big(\mu(\vec{a})^{(1)}+1\big)$, and let $\hat{p}'$ denote any one of the two Weierstrass points of $\hat{\Gamma}$ over ω_2 (resp : over ω_3). A straightforward application of the lower bounds of (§5.4), as in the proof of §6.3, yields that the hyper-elliptic potential associated to $\xi_{A,0}\otimes\mathcal{O}_{\hat{\Gamma}}(\hat{p}'-\hat{p})=\mathcal{O}_{\hat{\Gamma}}\big((\hat{g}-2)\hat{p}+\hat{p}'\big)$ is equal to $u(x)$ (31), for some $\rho\in X$ $(2\rho\neq0)$ satisfying §9.9.II). Hence, each one of the above choices, $\hat{\mu}=(2\ell+1,2k,2,0)$ and $\hat{\mu}=(2\ell+1,2k,0,2)$, gives rise to a pair of hyper-elliptic potentials corresponding to the fiber over $X\in\mathcal{X}$ of each conic contained in $\mathcal{C}_{\vec{a}}\setminus\mathcal{A}^\circ\Big(\mathcal{R}\big(n,\mu(\vec{a})\big)\Big)$, the complement of the image of $\mathcal{R}\big(n,\mu(\vec{a})\big)$. ∎

References

[1] **AIRAULT H., Mc KEAN H.P. & MOSER J.** "Rational and elliptic solutions of the Korteweg-de Vries equation and a related many body problem." Comm. Pure Appl. Math., vol 30 (1977), 95-148.

[2] **ALTMAN A. & KLEIMAN S.** "Compactifying the Picard scheme." Adv. in Math., vol 35, (1980), 50-112.

[3] **DUBROVIN B.A.** "Theta functions and non-linear equations." Russ. Math. Surveys, vol 36 (2) (1981).

[4] **DUBROVIN B.A, MATVEEV V.B. & NOVIKOV S.P.** "Non linear equations of KdV type, finite zone linear operators, and abelian varieties." English translation : Russian math. Surv., vol. 31 (1976), 59-146.

[5] **FAY J.D.** "On the even-order vanishing of Jacobian theta functions." Duke Math. J., vol 51 (1984), 109-132.

[6] **GESZTESY F. & WEIKARD R.** Math. Zeitsch. vol 219 (1995).

[7] **GRIFFITHS Ph. & HARRIS J.** "Principles of Algebraic Geometry." Ed. : Wiley-Intersc (1978).

[8] **KRICHEVER I.M.** "Algebraic-geometric construction of the Zakharov-Shabat equations and their periodic solutions." Dokl. Akad. Nauk 227 (N°2) (1976), English transl. : Soviet Math. Dokl., vol 17 (N°2) (1976), 394-397.

[9] **KRICHEVER I.M.** "Methods of algebraic geometry in the theory of non-linear equations." English transl. : Russ. Math. Surv. vol 32 (1977).

[10] **KRICHEVER I.M.** "Elliptic solutions of the KP equation and integrable systems of particles." English transl. : Funct. An. vol 14 (N°4) (1980).

[11] **SEGAL G. & WILSON G.** "Loop groups and equations of KdV type." Publ. Math. de l'I.H.E.S., vol 61 (1985), 5-65.

[12] **SMIRNOV A.O.** "Elliptic solutions of the Korteweg-de Vries equation." Mat. Zametki 46 : 6 (1989), 66-73 ; English transl. Math. Notes 45 (1989).

[13] **SMIRNOV A.O.** "Finite gap elliptic solutions of the KdV equation." Acta Appl. Math., vol 36 (1994), 125-166.

[14] **TREIBICH A.** "Tangential polynomials and elliptic solitons." Duke Math. J., vol 59, N°3 (1989), 611-627.

[15] **TREIBICH A. & VERDIER J.L.** with an appendix by OESTERLE J. : "Solitons elliptiques." Prog. in Math. vol 88, The Grothendieck Festschrift, Vol III, Ed. : Birkhauser-Boston, 437-480 (1990).

[16] **TREIBICH A. & VERDIER J.L.** "Variétés de Kritchever des solitons elliptiques de KP." Proceedings of the Indo-French Conference on Geometry, Bombay 1989, Ed. : Hindustan Book Agency, Delhi (1993), 187-232.

[17] **TREIBICH A. & VERDIER J.L.** "Revêtements tangentiels..." C.R. Acad. Sc. Paris, t.311, Série I, 51-54 (1990).

[18] **TREIBICH A. & VERDIER J.L.** "Revêtements exceptionnels et sommes de 4 nombres triangulaires." Duke Math. J., vol 68, N°2 (1992).

[19] **TREIBICH A. & VERDIER J.L.** "Au delà des potentiels et revêtements tangentiels hyperelliptiques exceptionnels." C.R. Acad. Sc. Paris, t. 325, Série I, 1101-1106, (1997).

Index of terms

Index of notations

(2.8) $R(n, \mu, X)$;

(§3) $\ell, \ell^{\sim}, s_i^{\sim}, r_i^{\sim}$ $(i = 0, \cdots, 3)$;
(3.1) $S_i^{\sim}$ $(i = 0, \cdots, 3)$;
(3.3) $K, K^{\perp}, K^{\sim}$;

(4.1) $C_{\mu}^{\sim}$;
(4.3) $C_{\mu}^{\perp}, p_{\mu}$;
(4.11) χ_y ;

(5.3) $\xi_{A,i}, \xi_{B,i}$ $(i = 0, \cdots, 3)$;
(5.4) $m_i^{A,0}$, $m_i^{B,0}$ $(i = 0, \cdots, 3)$;

(6.4) $\mathbf{M}$, $\mathbf{M}^{\# i}$, A^i, $B^i : \mathbf{M} \to \mathbf{N}^4$ $(i = 0, \cdots, 3)$;

(7.4) $\mathbf{H}$, $\mathcal{C}_{\vec{a}}(\vec{a} \in \mathbf{N}^4)$;
(7.6) $\mathcal{X}, \mathcal{R}(n, \mu)$, $\mathcal{A}^i$, $\mathcal{B}^i : \mathcal{R}(n, \mu) \to \mathcal{C}_{\vec{a}}$ $(i = 0, \cdots, 3)$, O_1, O_2, O_3 ;

(7.7) $\mathcal{W}(n, \mu)$, $\mathcal{K}d\mathcal{V}(n)$, $\bar{\mathcal{X}} \to \mathcal{X}$, $(\bar{\mathcal{X}}/\mathcal{X})_n$, $\mathcal{W}(n), \mathcal{I}_n$;

(8.1) $P(\alpha, \beta, \gamma)$, $F_{\vec{a}}(e_0, e_1, e_2, e_3)(\vec{a} \in \mathbf{N}^4)$, $\varphi : \hat{\mathbf{H}} \to \mathbf{H}$;
(8.2) E_j $(j = 1, 2, 3)$;
(8.3) $\hat{\hat{\mathbf{H}}}$;
(8.5) $\hat{\hat{\mathcal{C}}}_{\vec{a}}, R_{\vec{a}}^{+++}, R_{\vec{a}}^{-++}, R_{\vec{a}}^{+-+}, R_{\vec{a}}^{++-}$ $(\vec{a} \in \mathbf{N}^4)$;
(8.6) $Con_{\vec{a}}, \mathcal{C}_{\vec{a}}^{+++}, \mathcal{C}_{\vec{a}}^{-++}, \mathcal{C}_{\vec{a}}^{+-+}, \mathcal{C}_{\vec{a}}^{++-}, \hat{\hat{\mathcal{C}}}_{\vec{a}}^{+++}, \hat{\hat{\mathcal{C}}}_{\vec{a}}^{-++}, \hat{\hat{\mathcal{C}}}_{\vec{a}}^{+-+}, \hat{\hat{\mathcal{C}}}_{\vec{a}}^{++-}$
$(\vec{a} \in \mathbf{N}^4)$;

(9.3) $\mathcal{Q}_{\vec{a}}$;
(9.9) $\prod_{\vec{a}}$

Sobre diagramas de Enriques y constelaciones tóricas

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Abstract

The so called Enriques diagram is the graph naturally associated to a constellation of infinitely near points to an algebraic regular variety, graph also provided with the binary relation associated to proximity on blown-up points of the constellation.

In the present work we characterize combinatorically the Enriques diagrams which are toric (i.e. associated to equivariant constellations with respect to an algebraic torus action) and we get the minimum dimension of a toric constellation which induces a given toric Enriques diagram.

Secondly we characterize combinatorically the linear proximity relation in the toric case and by this method we prove that the characteristic cone of a toric constellation is regular (if) and only if it is inducible by a two dimensional constellation; this gives an inverse statement to a classical result of Zariski.

Resumen

Se llama diagrama de Enriques al grafo asociado naturalmente a una constelación de puntos infinitamente cercanos a una variedad algebraica regular, grafo además provisto de la relación binaria correspondiente a la proximidad de los puntos estallados de la constelación. En este trabajo se caracterizan combinatoriamente los diagramas de Enriques que son tóricos (i.e. asociados a una constelación equivariante bajo la acción de

un toro algebraico) y se determina la dimensión mínima de la constelación tórica que induce un tal diagrama de Enriques.

En segundo lugar se caracteriza combinatoriamente la relación de proximidad lineal en el caso tórico y se prueba usando este método que el cono característico de una constelación tórica es regular (si) y sólo si es inducible por una constelación de dimensión dos, lo que constituye un recíproco de un resultado clásico de Zariski.

1. Introducción

1.1 Sea X una variedad algebraica regular de dimensión $d \geq 2$, definida sobre un cuerpo algebraicamente cerrado K, y sea O un punto cerrado de X. Una *constelación* de puntos infinitamente cercanos a $O \in X$, es un conjunto finito $\mathcal{C} = \{Q_0 := O, Q_1, ..., Q_n\}$ definido recurrentemente: $X_0 := X$ y para $i = 1, ..., n$ Q_i es un punto de X_i que se proyecta sobre Q_0 donde $\sigma_i : X_{i+1} \to X_i$ es el estallido de X_i de centro Q_i. La variedad que resulta de estallar X_n en Q_n se denota $X(\mathcal{C}) := X_{n+1}$. Se llama dimensión de $\mathcal{C}$ a la dimensión de X. La dimensión impuesta a la variedad X es al menos 2, pues los puntos estallados han de tener codimensión al menos 2, para que los objetos definidos tengan sentido y no sean triviales.

Si P y Q son puntos de la constelación $\mathcal{C}$ se dice que P es *infinitamente cercano* a Q (notación: $P \geq Q$), si $P = Q$ o si Q es la imagen de P por la composición de los estallidos. La relación $\geq$ es una relación de orden parcial.

Si los puntos de la constelación están totalmente ordenados por la relación $\geq$, se tiene $Q_n \geq ... \geq Q_0$ y en ese caso se dice que la constelación $\mathcal{C}$ es una *constelación en cadena.*

Asociada a la constelación $\mathcal{C}$, se define un grafo (orientado) $\Gamma_{\mathcal{C}}$ con raíz cuyos vértices están en biyección con los puntos de $\mathcal{C}$, la raíz corresponde al punto O, y las aristas corresponden a los pares $(Q_j,\ Q_i)$ tales que $Q_j \geq Q_i$,

$Q_j \neq Q_i$ y no existe Q_k intermedio $Q_j \geq Q_k \geq Q_i$, $Q_j \neq Q_k \neq Q_i$. Si $\mathcal{C}$ es una constelación en cadena, el grafo $\Gamma_{\mathcal{C}}$ es una cadena. El grafo $\Gamma_{\mathcal{C}}$ es en general un árbol; todos los árboles considerados en lo que sigue son *árboles finitos y con raíz.*

Dada la constelación $\mathcal{C}$ y $Q \in \mathcal{C}$ el conjunto $\mathcal{C}^Q := \{P \mid Q \geq P\}$ es una constelación en cadena; se llama *nivel* de Q al número $l(Q) := \#\mathcal{C}^Q - 1$, es decir $l(Q)$ es el número de aristas en la cadena que corresponde a $\mathcal{C}^Q$ en el grafo $\Gamma_{\mathcal{C}}$.

Para cada punto Q_i de la constelación $\mathcal{C} = (Q_0, ..., Q_n)$ se denota por B_{Q_i} (o B_i) el divisor excepcional del estallido $\sigma_{i+1} : X_{i+1} \to X_i$ de centro Q_i, y E_{Q_i} (o E_i) su transformada estricta en cualquiera de los X_j, con $i+1 \leq j \leq n+1$.

Se denota E_i^* a la transformada total, en cualquiera de las variedades $X_{i+1}, ..., X_{n+1}$, del divisor excepcional B_i del estallido de X_i de centro Q_i.

Definición 1.2 Si P, Q son puntos de la constelación $\mathcal{C}$ se dice que P es *próximo* a Q (notación: $P \to Q$) si $P \in E_Q$.

Se observa que : $P \to Q$ implica $P \geq Q$.

La relación de proximidad se caracteriza a continuación ([C.G.L.1]):

Teorema 1.3 *Sea Γ un árbol provisto de una relación binaria ($\rightsquigarrow$) en el conjunto de sus vértices. Entonces Γ es el grafo asociado a una constelación $\mathcal{C}$ y $\rightsquigarrow$ está inducida por la relación de proximidad en C si y sólo si para cualesquiera vértices p, q, r de Γ se verifican las siguientes condiciones :*

(i) $q \rightsquigarrow p \Rightarrow q \geq p, q \neq p$

(ii) $q \geq p$ *y* $l(q) = l(p) + 1 \Rightarrow q \rightsquigarrow p$

(iii) $r \geq q \geq p$, $q \neq p$ *y* $r \rightsquigarrow p \Rightarrow q \rightsquigarrow p$.

Definición 1.4 Un *diagrama de Enriques* es un par $(\Gamma, \rightsquigarrow)$, donde Γ es un árbol y $\rightsquigarrow$ es una relación binaria en el conjunto de los vértices de Γ, que verifica las propiedades (i),(ii) y (iii) del teorema anterior. Si los vértices de Γ están totalmente ordenados se dice que $(\Gamma, \rightsquigarrow)$ es un *diagrama de Enriques en cadena.*

2. Diagramas de Enriques realizables tóricamente

Para comenzar la sección se recuerdan algunas definiciones y notaciones, por más detalle ver [O].

2.1 Sea $N \cong \mathbb{Z}^d$ un retículo de dimensión $d \geq 2$. Sea Σ un abanico en $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$, K un cuerpo; se denota por X_Σ la variedad tórica sobre K asociada a Σ provista con la acción de un toro algebraico $T \cong (K^*)^d$.

Dada una base cualquiera $\mathcal{B}$ de N, sea $\Delta = \langle \mathcal{B} \rangle$ el cono (regular) d-dimensional en $N_{\mathbb{Q}}$ generado por $\mathcal{B}$. Sea $X_0 := X_{\Sigma_0} (\cong K^d)$ la variedad tórica afín asociada al abanico Σ_0 formado por todas las caras de Δ. Sea Q_0 la T-órbita 0-dimensional en X_0. Una *constelación tórica* d-dimensional de puntos infinitamente cercanos con origen Q_0 es una constelación $\mathcal{C} := \{Q_0, ..., Q_n\}$ tal que cada Q_j es una T-órbita 0-dimensional en la variedad tórica X_j obtenida estallando X_{j-1} en el punto Q_{j-1} para $1 \leq j \leq n$. Se notará $X_0(\mathcal{C}) = X_{\Sigma_0}(\mathcal{C})$ al estallido de X_n en Q_n. Una *constelación tórica en cadena* es una constelación tórica totalmente ordenada por la relación $\geq$.

De acuerdo a la notación introducida X_1 es el estallido de X_0 con centro en Q_0, es decir la variedad tórica X_1 es la asociada al abanico Σ_1 , este es la subdivisión elemental minimal de Σ_0 que contiene como cara al rayo definido por el vector $u = \Sigma_{i=1}^{d} v_i$. Ahora supóngase que $\mathcal{B} = \{v_1, v_2, ..., v_d\}$ es una base ordenada del retículo N. Para cada entero $i, 1 \leq i \leq d$, sea $\mathcal{B}_i$ la base ordenada de N obtenida reemplazando v_i por u en la base $\mathcal{B}$ y sea $\Delta_i := \langle \mathcal{B}_i \rangle$ el cono regular generado por $\mathcal{B}_i$. Es decir $\Delta_1, \Delta_2, ..., \Delta_n$ son los conos maximales de Σ_1. El divisor excepcional B_0 es la clausura en X_1 de la órbita asociada al rayo definido por u. Cada T-órbita 0-dimensional en X_1 corresponde a un cono $\Delta_i, 1 \leq i \leq d$. La elección del punto $Q_1 \geq Q_0$ es equivalente a la elección de un entero $a_1, 1 \leq a_1 \leq d$, para determinar un cono Δ_{a_1} del abanico Σ_1. La subdivisión Σ_2 de Σ_1 correspondiente al estallido de Q_1 es obtenida reemplazando Δ_{a_1} (y sus caras) en Σ_1 por los conos $\Delta_{a_1 i} := \langle \mathcal{B}_{a_1 i} \rangle$ (y sus caras), $1 \leq i \leq d$, donde $\mathcal{B}_{a_1 i}$ es la base ordenada de N obtenida de $\mathcal{B}_{a_1}$ por la sustitución de su i-ésimo vector por $\sum_{v \in \mathcal{B}_{a_1}} v$.

La elección de $Q_2 \geq Q_1$ es equivalente a la elección de un entero $a_2, 1 \leq a_2 \leq d$, el cual determina un cono (regular) $\Delta_{a_1 a_2}$. En general por recurrencia : $\mathcal{B}_{a_1 \ldots a_i a_{i+1}}$ se obtiene a partir de $\mathcal{B}_{a_1 \ldots a_i} = \{w_1, \ldots, w_d\}$ reemplazando su a_{i+1}-ésimo vector $w_{a_{i+1}}$ por $\sum_{j=1}^{d} w_i$ y análogamente el cono $\Delta_{a_1 \ldots a_i a_{i+1}} = \langle \mathcal{B}_{a_1 \ldots a_i a_{i+1}} \rangle$ corresponde al punto Q_{i+1}. Las observaciones precedentes permiten codificar las cadenas tóricas y por ende las constelaciones tóricas, pues para cada $Q \in \mathcal{C}$, la constelación $\mathcal{C}^Q$ es una constelación en cadena. Se codifican así las constelaciones tóricas por medio de árboles con aristas ponderadas, lo cual se establece en la proposición 2.3.

Definición 2.2 Sea Γ un árbol, $A(\Gamma)$ el conjunto de sus aristas. Sea d un entero, $d \geq 2$; se llamará d-ponderación de Γ a una aplicación $\alpha : A(\Gamma) \rightarrow \{1, 2, \ldots, d\}$ tal que dos aristas con origen común no tienen el mismo peso. El par (Γ, α) se llamará *árbol d-ponderado.* Todo árbol d-ponderado es necesariamente d-nario (i.e. tiene a lo sumo d aristas con la misma raíz).

Proposición 2.3 *([C.G.L.2]) Sea $\mathcal{B}$ una base ordenada del retículo $N \cong \mathbb{Z}^d$ y n un entero positivo.*

(a) La correspondencia, que asocia a cada sucesión $(a_1, \ldots, a_n)$ de enteros con $1 \leq a_i \leq d$, $1 \leq i \leq n$, la constelación tórica en cadena $(Q_0, Q_1, \ldots, Q_n)$ donde Q_0 es la T-órbita correspondiente al cono $\Delta = \langle \mathcal{B} \rangle$ y donde Q_i, $1 \leq i \leq n$, es la T-órbita en X_i correspondiente al cono $\Delta_{a_1 \ldots a_i}$ del abanico Σ_i,

es una biyección entre las referidas sucesiones $(a_1, ..., a_n)$ y el conjunto de las cadenas tóricas d-dimensionales con $n+1$ puntos.

(b) La correspondencia de (a) induce una biyección natural entre el conjunto de las constelaciones tóricas de dimensión d de origen $Q_0 \in X_{\Sigma_0}$ y el de los árboles d-ponderados.

Definición 2.4 Sean (Γ, α) un árbol d-ponderado y (Γ', α') un árbol d'-ponderado, se dice que son *isomorfos* si existe una biyección φ entre los vértices de ambos tal que :

1. φ preserva las raíces y las aristas;
2. φ es compatible con las ponderaciones: $\forall\ t_1, t_2 \in A(\Gamma)$ tales que $\alpha(t_1) = \alpha(t_2)$ se verifica $\alpha'(\varphi(t_1)) = \alpha'(\varphi(t_2))$.

Notación Si Γ es un árbol d-ponderado y q y p son vértices de Γ con $p \leq q$, $q = p(a_1, ..., a_i)$ significa que, las aristas de menor a mayor nivel de la cadena de longitud i $(p, ..., q)$, están ponderadas por los enteros $a_1, ..., a_i$ en ese orden.También se escribe $p(a, b^{[i]}) := p(a, b, ..., b)$ donde b aparece t veces; si $t = 0$, $p(a, b^{[0]}) := p(a)$.

Ejemplo Sean $\Gamma = \{q_0, q_1, q_2, q_3, q_4\}$ y $\Gamma' = \{p_0, p_1, p_2, p_3, p_4\}$, dos árboles 3-ponderados de acuerdo a la siguiente definición:

$q_1 = q_0(1),\ q_2 = q_0(2),\ q_3 = q_0(1,1),\ q_4 = q_0(1,3)$

$p_1 = p_0(1),\ p_2 = p_0(2),\ p_3 = p_0(1,2),\ p_4 = p_0(1,3)$

Entonces, Γ y Γ' son árboles isomorfos -hay una biyección entre los vértices de ambos que preserva las raíces y las aristas-, pero no son isomorfos como árboles 3-ponderados. Los abanicos asociados a estos dos árboles 3-ponderados no son isomorfos en el sentido de [O], de acuerdo a la proposición 2.8.

Definición 2.5 Si Σ y Σ' son dos abanicos en $N_{\mathbb{Q}}$, un *mapa de abanicos*

$f : (N, \Sigma') \to (N, \Sigma)$ es un homomorfismo $\mathbb{Z}$-lineal cuya extensión escalar $f : N_{\mathbb{Q}} \to N_{\mathbb{Q}}$ verifica : $\forall\ \sigma' \in \Sigma'$ existe $\sigma \in \Sigma$ tal que $f(\sigma') \subset \sigma$.

Definición 2.6 Dos constelaciones tóricas $\mathcal{C}$ y $\mathcal{C}'$ d-dimensionales, definidas respectivamente a partir de los abanicos Σ_0 y Σ_0' de N, son *isomorfas* si $X_{\Sigma_0}(\mathcal{C})$ y $X_{\Sigma_0'}(\mathcal{C}')$ son equivariantemente isomorfas.

Proposición 2.7 *Sea $\mathcal{C}$ una constelación tórica de origen $Q_0 \in X_{\Sigma_0}$. Entonces las constelaciones tóricas $\mathcal{C}'$ de origen $Q_0 \in X_{\Sigma_0}$ isomorfas a $\mathcal{C}$ están biunívocamente determinadas por los automorfismos de N asociados a las permutaciones de la base $\mathcal{B}$.*

Demostración

($\Rightarrow$) Cada isomorfismo equivariante $\phi\ : X(\mathcal{C}') \to X(\mathcal{C})$ determina un único mapa de abanicos invertible f que satisface $f(\Delta) = \Delta$. Luego f transforma la base $\mathcal{B} = \{v_1, ..., v_d\}$ en $\{w_1, ..., w_d\}$ otra base del retículo N; donde $w_i = \sum_j a_{ij} v_j$ con $a_{ij} \in \mathbb{Z}_{\geq 0}$ y $det(a_{ij}) = \pm 1$. Se concluye por lo tanto que la matriz (a_{ij}) es una una matriz de permutación.

($\Leftarrow$) Sea $\overline{h}$ el automorfismo de N que lleva $\mathcal{B}$ en sí misma. Se va a probar que existen una constelación tórica $\mathcal{C}'$ y un abanico Σ' subdivisión de Σ_0 tales que : $X_{\Sigma'} = X_{\Sigma_0}(\mathcal{C}')$ y $\overline{h} : (N, \Sigma) \to (N, \Sigma')$ es un mapa de abanicos que induce un isomorfismo equivariante $\phi : X_{\Sigma_0}(\mathcal{C}) \to X_{\Sigma_0}(\mathcal{C}')$.

Sea h la permutación del conjunto $\{1, 2, ..., d\}$ que verifica $\overline{h}(v_j) = v_{h(j)}$, donde $\mathcal{B} = \{v_1, ..., v_d\}$. Los puntos de $\mathcal{C}'$ se definen como los correspondientes a los conos $\Delta_{h(a_1)...h(a_i)}$ si $\Delta_{a_1...a_i}$ es un cono que corresponde a un punto de $\mathcal{C}$. Los conos d-dimensionales de Σ' son $\Delta_{h(a_1)...h(a_i)}$ cada vez que $\Delta_{a_1...a_i}$ es un cono d-dimensional de Σ, así se tiene $X_{\Sigma'} = X_{\Sigma_0}(\mathcal{C}')$. Finalmente para probar que $\overline{h}$ es un isomorfismo entre los abanicos Σ y Σ' basta con verificar que $\overline{h}(\Delta_{a_1...a_i}) = \Delta_{h(a_1)...h(a_i)}$, lo cual se demuestra por inducción sobre i.

Para $i = 1$ se verifica $\overline{h}(\Delta_{a_1}) = \Delta_{h(a_1)}$.

La hipótesis de inducción es que si $\mathcal{B}_i = \{w_1, ..., w_d\}$ es el conjunto de los extremales de $\Delta_{a_1...a_i}$ con el orden inducido por el de la base $\mathcal{B}$ de acuerdo a lo establecido en el parágrafo 2.1, entonces $\mathcal{B}'_i = \{z_1, ..., z_d\}$ con $z_{h(j)} := \bar{h}(w_j)$ es el conjunto de los extremales de $\Delta_{h(a_1)...h(a_i)}$ con el orden inducido por $\mathcal{B}$. En particular se tiene que $\bar{h}(\Delta_{a_1...a_i}) = \Delta_{h(a_1)...h(a_i)}$.

Se va a probar el enunciado correspondiente a $\Delta_{a_1...a_i a_{i+1}}$. Sea $\mathcal{B}_{i+1} = \{w_1, ..., \sum_j w_j, ..., w_d\}$ la base obtenida de $\mathcal{B}_i$ sustituyendo $w_{a_{i+1}}$ por $\sum_j w_j$, es decir el conjunto de los extremales de $\Delta_{a_1...a_i a_{i+1}}$ con el orden inducido por $\mathcal{B}$.

Ahora $\Delta_{h(a_1)...h(a_i)h(a_{i+1})} = \langle z_1, ..., \sum_j z_j, ..., z_d \rangle = \langle \bar{h}(w_1), ..., \sum_j \bar{h}(w_j), ..., \bar{h}(w_d) \rangle = \bar{h}(\Delta_{a_1...a_i a_{i+1}})$, puesto que $\sum_j z_j$ en el segundo miembro de la igualdad sustituye a $z_{h(a_{i+1})} = \bar{h}(w_{a_{i+1}})$ es decir en el tercer miembro $\sum_j \bar{h}(w_j)$ sustituye al elemento $\bar{h}(w_{a_{i+1}})$, lo que justifica la última igualdad. □

La correspondencia de la proposición 2.3 es independiente de la elección de la base en el retículo N, en el sentido preciso siguiente:

Proposición 2.8

Existe una biyección entre las clases de isomorfismos de constelaciones tóricas de dimensión d y las clases de isomorfismos de árboles d-ponderados.

Demostración

($\Rightarrow$) Sean $\mathcal{C}$ y $\mathcal{C}'$ dos constelaciones tóricas isomorfas definidas respectivamente a partir de los abanicos Σ_0 y Σ'_0 de N, que provienen respectivamente de las bases $\mathcal{B}$ y $\mathcal{B}'$. Primeramente se observa que es suficiente probar que si $\mathcal{B} = \mathcal{B}'$ con un orden cualquiera $\mathcal{B} = \{v_1, ..., v_d\}$, los árboles d-ponderados asociados a $\mathcal{C}$ y $\mathcal{C}'$ respecto de $\mathcal{B}$ son isomorfos.

Sea h la permutación del conjunto $\{1, 2, ..., d\}$ que corresponde al isomorfismo $\phi : X_0(\mathcal{C}) \to X_0(\mathcal{C}')$ en el sentido considerado en la prueba de la proposición 2.7. En dicha prueba se observó que el mapa de abanicos $\bar{h}$ verifica $\bar{h}(\Delta_{a_1...a_i}) = \Delta_{h(a_1)...h(a_i)}$ cada vez que $\Delta_{a_1...a_i}$ corresponde a un punto de $\mathcal{C}$. Esta correspondencia determina un isomorfismo entre los árboles d-ponderados asociados a $\mathcal{C}$ y $\mathcal{C}'$.

($\Leftarrow$) Sean $\mathcal{C}$ y $\mathcal{C}'$ las constelaciones d-dimensionales asociadas, en las bases ordenadas $\mathcal{B}$ y $\mathcal{B}'$ respectivamente, a los árboles d-ponderados (Γ, α) y (Γ, α') isomorfos. Es suficiente probar que $\mathcal{C}$ y $\mathcal{C}'$son isomorfas en el caso en que $\mathcal{B}$ y $\mathcal{B}'$ son la misma base ordenada.

Sea h la permutación del conjunto $\{1, 2, ..., d\}$ tal que $h(\alpha(t)) = \alpha'(\varphi(t))$, donde φ es la aplicación de la definición 2.4.

Si Σ y Σ' son respectivamente los abanicos correspondientes a $X_0(\mathcal{C})$ y $X_0(\mathcal{C}')$, sea $\bar{h}$ el automorfismo de N tal que $\bar{h}(v_j) = v_{h(j)}$.

Si $(a_1, ..., a_i)$ es la sucesión que pondera una cadena de Γ con origen en la raíz $q_0 \in \Gamma$, el punto de $\mathcal{C}$ que corresponde al vértice $q = q_0(a_1, ..., a_i)$ es el determinado por el cono $\Delta_{a_1 ... a_i}$. Si $q' = \varphi(q) = q_0'(h(a_1), ..., h(a_i))$, el punto de $\mathcal{C}'$ correspondiente a q' es el determinado por el cono $\Delta_{h(a_1)...h(a_i)}$. Además se tiene $\bar{h}(\Delta_{a_1...a_i}) = \Delta_{h(a_1)...h(a_i)}$, lo cual es suficiente para concluir que $\bar{h}$ es un isomorfismo entre los abanicos Σ y Σ'. $\square$

2.9 Dado un árbol Γ, una d-ponderación de Γ induce una partición del conjunto de las aristas $A(\Gamma)$, en la que las clases están formadas por las aristas con igual ponderación.

En consecuencia, a cada clase de isomorfismo de constelaciones tóricas de dimensión d le corresponde una única clase de isomorfismo de árboles provista de una partición del conjunto de las aristas, partición de cardinal menor o igual a d.

Recíprocamente, a una clase de isomorfismo de árboles, provista de una partición del conjunto de las aristas tal que dos aristas con origen común no son equivalentes y dado un entero d mayor o igual al cardinal de la partición, le corresponde una única clase de isomorfismo de constelaciones tóricas de dimensión d.

La dimensión mínima d_α de una constelación que corresponde a un árbol d_α-ponderado isomorfo a un árbol d-ponderado (Γ, α) dado, es el cardinal d_α de la partición de $A(\Gamma)$ inducida por α. En efecto (Γ, α) es isomorfo a un árbol d_α-ponderado y d_α es el menor entero con esta propiedad.

En lo sucesivo se utilizará la siguiente caracterización de la relación de proximidad en una constelación tórica.

Proposición 2.10 *([C.G.L.2]) Sea $\mathcal{C} := (Q_0, ..., Q_n)$ la cadena tórica correspondiente a la sucesión de enteros $(a_1, ..., a_n)$. Sean j y k dos índices tales que $0 \leq j < k \leq n$. Entonces $Q_k \to Q_j$ si y sólo si $a_{j+1} \notin \{a_i \mid j+2 \leq i \leq k\}$.*

Demostración

Si $k = j+1$, puesto que $\mathcal{C}$ es una cadena resulta que $Q_{j+1} \to Q_j$, es decir la condición del enunciado se cumple trivialmente. En general el divisor excepcional B_j en X_{j+1} corresponde al rayo l_j definido por el a_{j+1}-ésimo vector de la base ordenada $\mathcal{B}_{a_1 \dots a_{j+1}}$. El punto Q_k es próximo a Q_j si y sólo si l_j es una cara de $\Delta_{a_1 \dots a_k}$, lo cual significa que el a_{j+1}-ésimo vector de la base $\mathcal{B}_{a_1 \dots a_i}$ no fue sustituído para ningún i tal que $j+1 \leq i \leq k$, i.e. $a_{j+1} \neq a_i$, para i tal que $j+2 \leq i \leq k$. □

Notación Sea una constelación tórica $\mathcal{C}$ de origen Q_0, si Q es un punto de $\mathcal{C}$ de nivel $n \geq 1$ y $(a_1, ..., a_n)$ es la sucesión de enteros que codifica la cadena $\mathcal{C}^Q$, se escribirá $Q = Q_0(a_1, ..., a_n)$. Si $Q = Q_0(a_1, ..., a_n)$ y $P = Q_0(a_1, ..., a_{n-1})$ se denotará $Q = P(a_n)$.

Definición 2.11 Si p es un vértice en un diagrama de Enriques $(\Gamma, \rightsquigarrow)$ se llama *índice de proximidad* de p al número $ind(p) := \#\{q \mid p \rightsquigarrow q\}$. El conjunto de los vértices de Γ *consecutivos* a p es denotado $p^+ := \{q \mid q \geq p,\ l(q) = l(p) + 1\}$.

Un diagrama de Enriques $(\Gamma, \rightsquigarrow)$ se dice *realizable tóricamente* si es isomorfo al diagrama de Enriques asociado a alguna constelación tórica $\mathcal{C}$. En ese caso se dice que $\mathcal{C}$ es una *realización tórica* de $(\Gamma, \rightsquigarrow)$.

Proposición 2.12 *Un diagrama de Enriques en cadena $(\Gamma, \rightsquigarrow)$ es realizable tóricamente si y sólo si el índice de proximidad es una función no decreciente, i.e. $ind(p) \geq ind(q)$ $\forall\, p \geq q$.*

La dimensión mínima $d_{\mathcal{P}}$ que permite realizar tóricamente una cadena es $d_{\mathcal{P}} = max\{2, ind(q)\}$ donde q es el vértice maximal.

Demostración

$(\Rightarrow)$ Sea $\mathcal{C} = (Q_0, ..., Q_n)$ una constelación tórica en cadena codificada por $Q_j = Q_0(a_1, ..., a_j)$, $j = 1, ..., n$ y tal que su diagrama de Enriques es $(\Gamma, \rightsquigarrow)$. Se denota q_j al vértice de Γ correspondiente al punto Q_j de $\mathcal{C}$.

Basta mostrar que $ind(q_{k+1}) \geq ind(q_k)$, para cada índice k, $0 \leq k \leq n-1$.

Si $a_{k+1} \neq a_{j+1}$ para todo j tal que $Q_k \to Q_j$, entonces $Q_{k+1} \to Q_j$ y por lo tanto $ind(q_{k+1}) = ind(q_k) + 1$, pues $Q_{k+1} \to Q_k$.

Si existe j_0 tal que $Q_k \to Q_{j_0}$ con $Q_{k+1} \not\to Q_{j_0}$, entonces $a_{k+1} = a_{j_0+1}$. Si $j \neq j_0$ y $Q_k \to Q_j$, en tonces $Q_{k+1} \to Q_j$:

(a) si $k > j > j_0$, $Q_k \to Q_{j_0}$ $\Leftrightarrow a_{j_0+1} \notin \{a_l \,|j_0 + 2 \leq l \leq k\}$, en particular $(a_{k+1} =) a_{j_0+1} \neq a_{j+1}$, por lo tanto $Q_{k+1} \to Q_j$;

(b) si $k > j_0 > j$, $Q_k \to Q_j$ $\Leftrightarrow a_{j+1} \notin \{a_l \,|j + 2 \leq l \leq k\}$, en particular $(a_{k+1} =) a_{j_0+1} \neq a_{j+1}$, por lo tanto $Q_{k+1} \to Q_j$.

Dicho de otra manera, a lo sumo para un valor de j puede ocurrir $Q_k \to Q_j$ y $Q_{k+1} \not\to Q_j$, pero como por otra parte se agrega la proximidad $Q_{k+1} \to Q_k$, entonces se tiene $ind(q_{k+1}) \geq ind(q_k)$.

$(\Leftarrow)$ Si $(\Gamma, \rightsquigarrow)$ es un diagrama de Enriques en cadena (satisface el enunciado del teorema 1.3), tal que su índice de proximidad es no decreciente, para mostrar que es realizable tóricamente se definirá una $d_{\mathcal{P}}$-ponderación α en Γ tal que las constelaciones asociadas a (Γ, α) sean realizaciones de $(\Gamma, \rightsquigarrow)$.

La prueba se hace por recurrencia sobre el número de vértices de Γ. Sea $\Gamma = (q_0, ..., q_n)$, para $0 \leq k < n$, se probará que si $(\Gamma' = (q_0, ..., q_k), \rightsquigarrow)$ es realizable tóricamente en dimensión $d' = max\{2, ind(q_k)\}$, entonces $(\Gamma'' = (q_0, ..., q_k, q_{k+1}), \rightsquigarrow)$ es realizable en dimensión $d'' = max\{2, ind(q_{k+1})\}$.

Sea α' la d'-ponderación en Γ' que corresponde a la realización tórica de $(\Gamma', \rightsquigarrow)$, para $j = 1, ..., k$ sea a_j el peso de la arista$[q_{j-1}, q_j]$. Para extender α' a una d''-ponderación α'' de Γ'', basta con definir a_{k+1} el peso de la arista $[q_k, q_{k+1}]$, lo cual se hace así :

(a) Si $ind(q_{k+1}) = ind(q_k)$, se toma $a_{k+1} = a_{j_0+1}$, donde $q_{k+1} \not\rightsquigarrow q_{j_0}$ y $q_k \rightsquigarrow q_{j_0}$; el índice j_0 que verifica lo anterior es único pues si hubiera al menos dos tales índices entonces se tendría $ind(q_{k+1}) < ind(q_k)$.

(b) Si $ind(q_{k+1}) > ind(q_k)$, se elige $a_{k+1} \notin \{a_l \mid 1 \leq l \leq k\}$ y $1 \leq a_{k+1} \leq d''$. Es inmediato que las constelaciones tóricas d''-dimensionales definidas por (Γ'', α'') tienen diagrama de Enriques $(\Gamma'', \rightsquigarrow)$.

Se observa finalmente que la dimensión mínima de las constelaciones tóricas en cadena con diagrama de Enriques dado, es $d_{\mathcal{P}}$.

Es suficiente probar que si $\mathcal{C} = (Q_0, ..., Q_n)$ es una constelación tórica en cadena de dimensión d cuyo diagrama de Enriques es $(\Gamma, \rightsquigarrow)$, entonces $d \geq ind(q_n)$. Supongamos $\mathcal{C}$ codificada por $Q_j = Q_0(a_1, ..., a_j)$, sea $s = ind(q_n)$ y sean Q_{j_l}, para $l = 1, ..., s$ los puntos de $\mathcal{C}$ aproximados por Q_n. Necesariamente los enteros $a_{j_1+1}, ..., a_{j_s+1}$ son todos diferentes, por lo tanto $d \geq \#\{a_1, ..., a_n\} \geq s = ind(q_n)$. □

Ejemplo Sea $\Gamma = (q_0, q_1, q_2, q_3)$ una cadena con las proximidades $q_3 \rightsquigarrow q_2 \rightsquigarrow q_1 \rightsquigarrow q_0$ y $q_2 \rightsquigarrow q_0$. Se trata de un diagrama de Enriques, de acuerdo al teorema 1.3. Este diagrama de Enriques en cadena no es realizable tóricamente puesto que el índice de proximidad no es creciente: $ind(q_2) = 2 > ind(q_3) = 1$. Es la cadena más corta que no es realizable tóricamente.

La caracterización general de los diagramas de Enriques realizables tóricamente es la siguiente:

Teorema 2.13 *Un diagrama de Enriques* $(\Gamma, \rightsquigarrow)$ *es realizable tóricamente si y sólo si se verifica:*

(i) la función índice de proximidad es no decreciente: $ind(q) \geq ind(p)$ $\forall\ q, p \in \Gamma,\ p \geq q$

(ii) si r *es próximo a* q*, existe a lo sumo un vértice* s *consecutivo a* r*, que no es próximo a* q*, i.e. si* $r \rightsquigarrow q$ *entonces* $\#\{s \in r^+ \mid s \not\rightsquigarrow q\} \leq 1$*. Si (i) y (ii) se verifican, la dimensión mínima de una constelación tórica cuyo diagrama de Enriques es* $(\Gamma, \rightsquigarrow)$ *es* $d_{\mathcal{P}} = max\{2, max_{q\in\Gamma}\{ind(q) + s(q)\}\}$*, donde* $s(q) = \#\{r \in q^+ \mid ind(r) > ind(q)\}$.

Demostración

$(\Rightarrow)$ Sea Γ un diagrama de Enriques realizable tóricamente, toda cadena $\Gamma^q = \{p\ : q \geq p\}$ es realizable tóricamente, luego la proposición anterior nos asegura que la condición (i) se satisface.

Para verificar la condición (ii) : sea Γ el grafo de Enriques de la constelación $\mathcal{C}$, sea $(a_1, ..., a_j)$ la codificación de la cadena $\mathcal{C}^Q$ que corresponde a la codificación de $\mathcal{C}$. Si $Q = Q_0(a_1, ..., a_j) \rightarrow P = Q_0(a_1, ..., a_i)$ con $j > i$, entonces (si existe) el único $R \in Q^+$ que verifica $R \not\rightarrow P$ es $R = Q_0(a_1, ..., a_j, a_{i+1})$.

$(\Leftarrow)$ La suficiencia se prueba por inducción en el número de vértices del grafo Γ.

Sea x un vértice de Γ de nivel máximo (> 0) y q tal que $x \in q^+$. Se denotan por $r_1 := x,\ r_2, ..., r_t$ a los vértices del conjunto q^+, estos vértices son maximales en Γ. Se define el subgrafo pleno de Γ, $\Gamma' := \Gamma \setminus \{r_1, ..., r_t\}$ con la restricción de la relación $\rightsquigarrow$. Por hipótesis de inducción $(\Gamma', \rightsquigarrow)$ es realizable tóricamente en dimensión $d' = max\{2, max_{q\in\Gamma'}\{ind(q) + s'(q)\}\}$, donde $s'(p) = \#\{r \in \Gamma' \cap p^+ \mid ind(r) > ind(p)\}$. Sea α' la d'-ponderación en Γ' correspondiente a una de las realizaciones tóricas de $(\Gamma', \rightsquigarrow)$. Es suficiente probar que α' se extiende a una $d_{\mathcal{P}}$-ponderación α en Γ, tal que (Γ, α) determina una realización tórica de $(\Gamma, \rightsquigarrow)$.

Sea $\{r_1, ..., r_s\} \subset \{r_1, ..., r_s, ..., r_t\}$ el conjunto de los vértices para los que existe (un único) p_j tal que $q \to p_j$ y $r_j \not\to p_j$ para $j = 1, ..., s \leq t$, i.e. para cada r_j, $1 \leq j \leq s$ se tiene $ind(r_j) = ind(q)$.

Para cada $p \in \Gamma$ tal que $q \to p$ se denota $1 \leq a_p \leq d'$ el peso de la arista de origen p en la cadena Γ'^q. Para cada $r_1, ..., r_s$ se define el peso de la arista $[q, r_i]$ como a_{p_i}; así en virtud de la condición (ii) de la hipótesis se tiene la extensión de α' a una d'-ponderación sobre el subgrafo $\Gamma' \cup \{r_1, ..., r_s\}$ de Γ.

Para ponderar las aristas $[q, r_i]$ con $s < i \leq t$, se necesitan $t - s$ enteros diferentes y no pertenecientes a $\{1 \leq a_p \leq d' \mid q \to p\}$, lo cual muestra que es posible definir la $d_{\mathcal{P}}$-ponderación buscada α.

Por la construcción precedente se obtiene una constelación de dimensión $d_{\mathcal{P}}$.

Finalmente se prueba que $d_{\mathcal{P}}$ es la dimensión mínima posible.

Sea C una constelación tórica cualquiera de dimensión d cuyo diagrama de Enriques es $(\Gamma, \rightsquigarrow)$, para mostrar que $d \geq d_{\mathcal{P}}$ es suficiente probar que $d \geq ind(q) + s(q)$, $\forall q \in \Gamma$.

Si q es un vértice maximal de Γ, entonces $s(q) = 0$, luego $d \geq ind(q)$ de acuerdo a la proposición 2.12.

Si q no es un vértice maximal de Γ, sea $q^+ = \{q_1, ..., q_t\}$, se considera la codificación de C, $Q_j = Q(b_j)$, $j = s + 1, ..., t$, son los puntos de Q^+ tales que $ind(q_j) > ind(q)$ (Q es el punto de C correspondiente al vértice q de Γ). Entonces los b_j son diferentes dos a dos y $b_j \notin A := \{a \mid Q \geq P(a) \text{ y } Q \to P\}$. Ahora $\# A = ind(q)$ y por lo tanto $d \geq ind(q) + s(q)$. $\square$

3. Proximidad lineal y conos característicos.

Se comienza recordando la noción general de constelación ponderada idealística. Luego se hacen algunas consideraciones sobre el caso tórico.

Definición 3.1 Una *constelación ponderada* (cluster) es un par $\mathcal{A} = (\mathcal{C}, \underline{m})$ donde $\mathcal{C} = (Q_0, ..., Q_n)$ es una constelación y $\underline{m} = (m_0, ...m_n)$ es una sucesión de enteros no negativos que pondera los puntos de $\mathcal{C}$.

A una constelación ponderada $\mathcal{A} = (\mathcal{C}, \underline{m})$ se le asocia un divisor con soporte excepcional $D(\mathcal{A}) := \sum m_i E_i^*$ en $X(\mathcal{C})(= X_{n+1})$.

Definición 3.2 La constelación ponderada $\mathcal{A} = (\mathcal{C}, \underline{m})$ con origen O es *idealística* si existe un ideal $I \subset \mathcal{O}_{X,O}$ completo (i.e. integralmente cerrado) y de soporte finito contenido en $\mathcal{C}$ tal que $I\mathcal{O}_{X(\mathcal{C})}$ es el haz de ideales asociado a $-D(\mathcal{A})$ i.e. $I\mathcal{O}_{X(\mathcal{C})} = \mathcal{O}_{X(\mathcal{C})}(-D(\mathcal{A}))$
Fijada la constelación $\mathcal{C}$, la *galaxia* de $\mathcal{C}$ es el conjunto $\mathcal{G}_\mathcal{C} := \{(\mathcal{C}, \underline{m}) \mid (\mathcal{C}, \underline{m})$ es idealística $\}$

Observación Con la operación natural de suma de ponderaciones $\mathcal{G}_\mathcal{C}$ es un semigrupo, por más detalles ver ([C.G.L. 2]).

Las desigualdades de proximidad ([E. C.]) describen la galaxia de una constelación $\mathcal{C}$ cualquiera en dimensión dos:

$$m_Q \geq \sum_{P \to Q} m_P \ \ \forall \ Q \in \mathcal{C}$$

Las condiciones numéricas que caracterizan la galaxia de una constelación tórica dada de dimensión cualquiera se refieren a continuación.

Definición 3.3 Sean P y Q puntos de una constelación tórica $\mathcal{C}$. Se dice que P está *linealmente próximo* a Q (notación: $P \overset{l}{\twoheadrightarrow} Q$) si P pertenece a la transformada estricta de la adherencia de alguna T-órbita 1-dimensional l contenida en el divisor excepcional B_Q. Se denota $\mathcal{C}_Q(l) := \{P \in \mathcal{C} \mid P \overset{l}{\twoheadrightarrow} Q\}$.

La galaxia de una constelación tórica $\mathcal{C}$ está dada por las desigualdades :

$$m_Q \geq \sum_{P \in \mathcal{C}_Q(l)} m_P,$$

donde $Q \in \mathcal{C}$ y l recorre las 1-órbitas contenidas en B_Q ver [C.G.L.2]. Si la dimensión de la constelación tórica es dos, las desigualdades anteriores coinciden con las desigualdades de proximidad.

La siguiente proposición permite describir en términos de la codificación de una constelación tórica la relación de proximidad lineal.

Proposición 3.4 *Sea Σ' una subdivisión del abanico Σ y sea σ un cono de Σ. Entonces la transformada estricta por el morfismo canónico $f : X_{\Sigma'} \to X_\Sigma$ de la clausura de la órbita correspondiente a σ (denotada $\overline{O_\sigma}$) es la unión de las órbitas O_τ donde $\tau \in \Sigma'$ es tal que σ es cara de τ ($\sigma \prec \tau$).*

Demostración

Los abanicos Σ y Σ' definen las variedades tóricas X_Σ y $X_{\Sigma'}$. Es sabido que la fibra excepcional E del morfismo canónico f es la unión de las órbitas $O_{\sigma'}$ tales que $\sigma' \in \Sigma' \setminus \Sigma$. Por lo tanto la transformada estricta de $\overline{O_\sigma}$ es $\overline{\{x \in X_{\Sigma'} \setminus E \mid f(x) \in \overline{O_\sigma}\}}$. Se sabe que en X_Σ se tiene que $\overline{O_\sigma} = \bigcup_{\tau \succ \sigma} O_\tau$

Si $x \in X_{\Sigma'} \setminus E$, entonces $x \in O_\tau$ con $\tau \in \Sigma$ y por lo tanto $f(x) = x$. Es decir $\{x \in X_{\Sigma'} \setminus E \mid f(x) \in \overline{O_\sigma}\} = \bigcup_{\tau \succ \sigma, \tau \in \Sigma} O_\tau$.
Luego en $X_{\Sigma'}$: $\overline{\bigcup_{\tau \succ \sigma, \tau \in \Sigma} O_\tau} = \bigcup_{\tau \succ \sigma, \tau \in \Sigma} \overline{O_\tau} = \bigcup_{\tau \succ \sigma, \tau \in \Sigma} (\bigcup_{\tau' \succ \tau, \tau' \in \Sigma'} O_{\tau'}) = \bigcup_{\tau' \succ \sigma, \tau' \in \Sigma'} O_{\tau'}$. □

Notación Sea una constelación tórica $\mathcal{C}$ codificada respecto de una base ordenada, si $Q \in \mathcal{C}$ se denotará $Q(a, b^{[t]}) := Q(a, b, ...b)$, donde b aparece t veces; si $t = 0$ se define $Q(a, b^{[0]}) = Q(a)$.

Proposición 3.5 *Sea $\mathcal{C}$ una constelación tórica de dimensión d y sea $Q \in \mathcal{C}$ y l una T-órbita 1-dimensional. Dada una codificación de $\mathcal{C}$ se tiene que :* $\mathcal{C}_Q(l) = \{P \mid P = Q(a, b^{[t]})$ *o* $P = Q(b, a^{[t]}),\ t \geq 0\}$, *para a y b fijos y distintos con* $1 \leq a \leq d,\ 1 \leq b \leq d$.

Demostración

Sea Q_0 el origen de la constelación $\mathcal{C}$, sea $Q = Q_0(a_1, ..., a_j)$. B_Q -el divisor excepcional del estallido de centro en Q- corresponde al rayo r_j definido por el a_{j+1}-ésimo vector de la base $\mathcal{B}_{a_1 \ldots a_{j+1}}$ ver el apartado 2.1.

La T-órbita l contenida en B_Q corresponde a una cara σ $d-1$-dimensional (del abanico que define a X_{j+1}) que contiene al a_{j+1}-ésimo vector de la base $\mathcal{B}_{a_1 \ldots a_{j+1}}$. La cara σ es la intersección de dos de los conos maximales -de Σ_{j+1}- introducidos por el estallido de Q, es decir σ queda determinada por dos puntos T-estables $Q(a)$ y $Q(b)$ con $a \neq b$.

Ahora $P \in \mathcal{C}_Q(l)$ si y sólo si P pertenece a la transformada estricta de $\bar{l} = \overline{O_\sigma}$ en alguna de las variedades X_h con $j+1 \leq h \leq n+1$, esto ocurre si y sólo si el cono que corresponde al punto P tiene a σ como cara, según lo observado en la proposición 3.4.

Sea $P = Q(a_{j+1}, ..., a_k)$. Si $k = j+1$ es claro que $a_{j+1} = a$ o $a_{j+1} = b$. Ahora si $k > j+1$ es claro también que $a_{j+1} = a$ o $a_{j+1} = b$, supongamos que $a_{j+1} = a$. El punto P corresponde al cono $\Delta_{a_1 \ldots a_j a a_{j+2} \ldots a_k}$ del abanico Σ_k, $P \in \mathcal{C}_Q(l)$ si y sólo si este cono tiene a σ como cara, esto equivale a que para obtener $\Delta_{a_1 \ldots a_j a a_{j+2} \ldots a_k}$ a partir de $\Delta_{a_1 \ldots a_j a}$ no se ha sustituído ninguno de los vectores que definen a σ, es decir siempre se reemplazó el b-ésimo vector de las sucesivas bases a partir de la base $\mathcal{B}_{a_1 \ldots a_j a}$; esto termina la prueba. □

3.6 Son necesarias algunas consideraciones para establecer la noción de cono característico de una constelación tórica.

Sea $\sigma :\ X(\mathcal{C}) := X_{n+1} \to X_0 := X$ la composición de los estallidos de centro en los puntos de $\mathcal{C}$.

$Z_1(X(\mathcal{C})|X)$ es el grupo de los 1-ciclos en $X(\mathcal{C})$ cuyos soportes son excepcionales respecto del morfismo σ.

El producto de intersección $(D, C) \mapsto D \cdot C$ define una aplicación : $Pic(X(\mathcal{C})) \times Z_1(X(\mathcal{C})|X) \to \mathbb{Z}$.

Sea

$$N^1(X(\mathcal{C})|X) := \frac{Pic(X(\mathcal{C}))}{\equiv}$$

donde $\equiv$ denota la equivalencia numérica :
$D \equiv 0$ si y sólo si $D \cdot C = 0 \quad \forall\ C \in Z_1(X(\mathcal{C})|X)$.

El grupo $N^1(X(\mathcal{C})|X)$ se identifica naturalmente con el grupo de los divisores excepcionales $E = \oplus \mathbb{Z} E_i = \oplus \mathbb{Z} E_i^*$ módulo equivalencia numérica. En el caso tórico el cono generado por la galaxia $\mathcal{G}_{\mathcal{C}}$ en $N^1(X(\mathcal{C})|X) \otimes_{\mathbb{Z}} \mathbb{Q}$ coincide con el *cono característico* del morfismo σ. (i.e. los divisores D tales que $\mathcal{O}_{X(\mathcal{C})}(-D)$ es generado por sus secciones globales en un entorno del divisor excepcional de σ).

El teorema de Zariski sobre la factorización de ideales de polinomios definidos por puntos base infinitamente cercanos [Z] puede ser formulado en un contexto más general: para toda constelación $\mathcal{C}$ de dimensión dos, el cono característico del morfismo σ es regular.

En [C.G.L.2] se demuestra que el cono característico de una constelación tórica en cadena en dimensión cualquiera es regular, las siguientes proposiciones permiten otra demostración de ese resultado.

Proposición 3.7 *Las realizaciones tóricas en dimensión mínima d de un diagrama de Enriques en cadena $(\Gamma, \rightsquigarrow)$, son constelaciones isomorfas.*

Demostración

Todas las d-ponderaciones en Γ que preservan $\rightsquigarrow$, definen árboles d-ponderados isomorfos. Luego las constelaciones tóricas que realizan la cadena $(\Gamma, \rightsquigarrow)$ en dimensión mínima d son isomorfas. $\square$

Proposición 3.8 *En una constelación tórica en cadena la relación de proximidad determina la relación de proximidad lineal.*

Demostración

Sea $C = (Q_0, Q_1, ..., Q_n)$ la cadena, sean $Q_i \leq Q_{i+1} \leq ... \leq Q_j$ con $0 \leq i < j \leq n$.

Para $j = i + 1$ se tiene $Q_j \twoheadrightarrow Q_i$.

Para $j > i + 1$ se tiene $Q_j \twoheadrightarrow Q_i$ sii las únicas relaciones de proximidad en la cadena $(Q_i, ..., Q_j)$ son las mínimas inducidas por $Q_j \to Q_i$ es decir son únicamente $Q_{j-s} \to Q_i$, para $s = 0, 1, ..., j - i - 1$ y $Q_{s+1} \to Q_s$, para $s = i, ..., j - 1$. $\square$

Corolario 3.9 *Las realizaciones tóricas en dimensión mínima de un diagrama de Enriques en cadena dado, poseen galaxias isomorfas.*

Demostración

La galaxia de una constelación tórica está definida por las desigualdades de proximidad lineal, de acuerdo a las dos proposiciones anteriores las relaciones de proximidad lineal no se alteran en las diferentes realizaciones y el isomorfismo es inmediato. $\square$

Corolario 3.10 *Sea C una constelación tórica en cadena. Entonces existe C' una constelación tórica en cadena de dimensión dos y una biyección entre los de puntos de C y los de C' que preserva las relaciones de proximidad lineal.*

Demostración

Si Γ es el grafo de C es posible definir sobre las aristas de Γ una 2-ponderación α tal que la constelación tórica C' que corresponde a (Γ, α) y

$\mathcal{C}$ tienen las mismas relaciones de proximidad lineal vía la identificación de sus grafos. $\square$

Corolario 3.11 *El cono característico de una constelación tórica en cadena es regular.*

Demostración

Sea $\mathcal{C}$ una constelación tórica en cadena. Sea $\mathcal{C}'$ como en el corolario 3.10. Luego los conos característicos de $\mathcal{C}$ y $\mathcal{C}'$ son isomorfos. La prueba se reduce al resultado conocido -que se deduce de [Z]- de que el cono característico de una constelación cualquiera en dimensión dos es regular. $\square$

Se caracteriza a continuación la proximidad lineal en una constelación tórica $\mathcal{C}$ en términos de familias de subárboles de su grafo asociado $\Gamma_{\mathcal{C}}$.

3.12 Si Γ es un árbol, se llama *bi-cadena* de Γ a todo subárbol de Γ formado por dos cadenas de Γ que tienen la misma raíz y ninguna arista en común.

Si Γ es el árbol asociado a la constelación tórica $\mathcal{C}$, q el vértice correspondiente al punto $Q \in \mathcal{C}$ y l una 1-órbita contenida en B_Q, se denota $\Gamma_q(l)$ al subgrafo pleno de Γ de raíz q, cuyos otros vértices corresponden a los puntos $R \in \mathcal{C}$ tales que $R \overset{l}{\twoheadrightarrow} Q$. Cada subgrafo Γ es una cadena o una bi-cadena de Γ. Se denota $\Gamma(q)$ a la familia de los subgrafos $\Gamma_q(l)$ maximales, donde l recorre las 1-órbitas de B_Q. Un vértice $q \in \Gamma$ es simple (resp. ramificado) si $\# q^+ = 1$ (resp. $\# q^+ > 1$).

Recurriendo a la caracterización de la relación de proximidad lineal se demuestra la siguiente:

Proposición 3.13 *Sea $\mathcal{C}$ una constelación tórica y Γ su árbol asociado.*

1. *(a) Para cada $q \in \Gamma$, la familia $\Gamma(q)$ es no vacía y sus elementos son cadenas o bi-cadenas de raíz q.*

(b) *Si* γ, $\gamma' \in \Gamma(q)$ *y* $\gamma \subset \gamma'$, *entonces* $\gamma = \gamma'$

2. (a) *Dos elementos distintos de* $\cup_p \Gamma(p)$ *tienen a lo sumo una arista común.*

   (b) *Dos aristas que tienen raíz en un vértice* q *de ramificación (resp. la arista de raíz en un vértice simple* q*) pertenecen (resp. pertenece) a un (y sólo un) elemento de* $\Gamma(q)$.

3. (a) *Para cada* $q \in \Gamma$ *y* $r \in q^+$ *existe a lo sumo un vértice* $s \in r^+$ *tal que la cadena* (q, r, s) *no está contenida en ningún elemento de* $\Gamma(q)$.

   (b) *Si* $(p, ..., q, r)$ *es una cadena contenida en algún* $\gamma \in \Gamma(p)$ *y* $s \in r^+$ *verifica 3.(a), entonces la cadena* $(p, ..., q, r, s)$ *está contenida en* γ.

Definición 3.14 Se llama *PL-diagrama de Enriques* de una constelación tórica C a su grafo asociado Γ provisto de la estructura de proximidad lineal formada por la familia de subgrafos plenos $\{\Gamma(p) \mid p \in \Gamma\}$.

Teorema 3.15 *La pareja* $(\Gamma, \{\Gamma(p) \mid p \in \Gamma\})$ *formada por un árbol* Γ *y una familia de subgrafos plenos* $\Gamma(q)$, *es el PL-diagrama de Enriques de una constelación tórica* C *si y sólo si se verifican las propiedades 1,2 y 3 de la proposición anterior.*

La dimensión mínima de las constelaciones cuyo PL-diagrama de Enriques es el dado es $d_{PL} = max\{2, max\{\#p^+ + n_p \mid p \in \Gamma\}\}$ *donde* $n_p = max_{q \in p^+} \#\{\gamma \in \Gamma(p) | \ \gamma$ *es una cadena de más de dos vértices y* q *es vértice de* $\gamma\}$.

Demostración

Se definirá una d_{PL}-ponderación α en Γ, tal que la constelación tórica C asociada a (Γ, α) satisfaga el teorema.

La demostración será realizada en dos etapas :

1. Se exhibe la recurrencia que permite ponderar las aristas de Γ respetando los siguientes criterios:

   (a) Si $\gamma = (p_0, p_1, ..., p_n) \in \Gamma(p_0)$, la ponderación en γ será $p_i = p_0(a, b^{[i]})$ para $i = 0, ..., n-1$ con $a \neq b$.

   (b) Si $\gamma \in \Gamma(p_0)$ es la bi-cadena reunión de las cadenas $(p_0, p_1, ..., p_n)$ y $(p_0, q_1, ..., q_m)$ la ponderación en cada una de ellas será $p_i = p_0(a, b^{[i]})$, $i = 0, ..., n-1$; $q_j = p_0(b, a^{[i]})$, $j = 0, ..., m-1$ con $a \neq b$.

2. Luego se demuestra que la constelación $\mathcal{C}$ además satisface que: cada conjunto maximal $\mathcal{C}_P(l)$ -mediante la correspondencia entre puntos de $\mathcal{C}$ y vértices de Γ- está en biyección con un elemento de $\Gamma(p)$; para P fijo la familia $\{\mathcal{C}_P(l)\}_{l \subset B_P}$ está en biyección con la familia $\Gamma(p)$.

Parte 1. Se hace por recurrencia sobre los conjuntos de vértices de $C_i = \bigcup_{l(p) \leq i} \Gamma(p)$, $i = 0, 1, 2, ...$

Para $i = 0$. Van a ser ponderadas las aristas de $C_0 = \Gamma(o)$ donde o es la raíz de Γ.

Primero se ponderan arbitrariamente las aristas de raíz o con los enteros $1, 2, ..., \#o^+$. Ahora por la condición 2.(b) cada par de aristas de raíz o identifica exactamente una bi-cadena de $\Gamma(o)$ cuyas aristas se ponderan de acuerdo a los criterios (a) y (b).

Será de utilidad a lo largo de la demostración la siguiente notación, si $p \in \Gamma$ y $q \in p^+$, $\Gamma(p, q) := \{\gamma \in \Gamma(p) | \gamma$ es una cadena de más de dos vértices y q es vértice de $\gamma\}$.El nivel de una arista es el nivel de su vértice de origen más uno.

Llamemos $o_1, o_2, ..., o_t$ a los vértices de o^+ tales que $\Gamma(o, o_i) \neq \emptyset$, para ellos se denota $\Gamma(o, o_i) = \{\gamma_j^i | j = 1, ..., s_i\}$.

Con i fijo para ponderar $\gamma_1^i, ..., \gamma_{s_i}^i$ se necesitan forzosamente s_i naturales diferentes de los usados para ponderar las aristas de origen o, así se ponderan estas cadenas respetando los criterios (a) y (b). Si han sido ponderadas $\gamma_1^1, ..., \gamma_{s_1}^1, \gamma_1^2, ..., \gamma_{s_2}^2, ..., \gamma_1^k, ..., \gamma_{s_k}^k$ para ponderar $\gamma_1^{k+1}, ..., \gamma_{s_{k+1}}^{k+1}$ es suficiente disponer de s_{k+1} naturales diferentes de los usados para ponderar las aristas de origen o; es decir las ponderaciones de $\Gamma(o, o_i)$ y $\Gamma(o, o_j)$ son independientes si $i \neq j$.

De manera que han sido ponderadas las aristas de $\Gamma(o)$ usando $\#o^+ + n_0$ números naturales, donde $n_0 = max\ s_i = max_{o_i \in o^+} \#\Gamma(o, o_i)$.

Conviene observar que las aristas de nivel 2 están todas codificadas, salvo a lo sumo -condición 3.(a)- para cada i una arista de origen $o_i \in o^+$, a la que se la pondera con el mismo entero que a la arista $[o, o_i]$.

Paso inductivo. Suponiendo que han sido ponderadas las aristas de $C_{h-1} = \bigcup_{l(p) \leq h-1} \Gamma(p)$ con $h \geq 1$ hay que demostrar que es posible ponderar las de $C_h = \bigcup_{l(p) \leq h} \Gamma(p)$ respetando los criterios (a) y (b).

- Al estar ponderadas las aristas de C_{h-1}, de hecho están ponderadas todas las aristas de nivel $(h+1)$, salvo eventualmente, si $p \in \Gamma$ es tal que $l(p) = h-1$ y $\{p_1, ..., p_t\} = p^+$, a lo sumo una arista de origen p_i que se pondera con el mismo entero que pondera la arista $[p, p_i]$.

- Para determinar la ponderación de las aristas de $C_h = \bigcup_{l(p) \leq h} \Gamma(p)$ basta con tener ponderadas las aristas de nivel $h+2$.

Se observará como ponderar $\Gamma(p_1)$. Ya tienen ponderación las aristas de origen p_1 y se sabe que esto induce la ponderación de las bi-cadenas de $\Gamma(p_1)$.

Si $p_1^+ = \{q_1, ..., q_s\}$ faltan ponderar las cadenas $\Gamma(p_1, q_i)$ para $i = 1, .., s$.

Para ponderar cada cadena de $\Gamma(p_1, q_1)$ es necesario un entero positivo diferente de los que ponderan las aristas $[p_1, q_1], ..., [p_1, q_s]$. Para las otras cadenas de $\Gamma(p_1, q_i)$ se hace lo mismo observando que, si han sido ponderadas $\Gamma(p_1, q_1), ..., \Gamma(p_1, q_j)$ la ponderación de las cadenas de $\Gamma(p_1, q_{j+1})$ es independiente de las anteriores.

Para ponderar cada $\Gamma(p_i)$ se procede igual que con $\Gamma(p_1)$: ignorando las familias $\Gamma(p_j)$ ponderadas anteriormente.

La consistencia de esta ponderación está asegurada por la condición 2.(a), vale decir, cada nuevo peso introducido mediante la recurrencia es compatible con los anteriores.

Parte 2. Primeramente se observa que $C_p(l)$ maximal significa $\#C_p(l) > 2$ salvo que $\#p^+ = 1$ con $p^+ = q$ y q maximal en Γ. En esta situación se tiene $\Gamma(p) = \{(p,q)\}$ y la parte 2. de la demostración es inmediata.

Se llamará $\Gamma_p\{a,b\}$ a la cadena o bi-cadena maximal en Γ entre las que tienen origen p y cuyos otros vértices son de la forma $p(a, b^{[i]})$ o $p(b, a^{[j]})$, en la ponderación α definida.

Sea $\gamma \in \Gamma(p)$ una bi-cadena -la prueba es análoga si γ es una cadena-; sean a y b los pesos de las aristas de origen p que pertenecen a γ, entonces $\gamma \subset \Gamma_p\{a,b\}$.

Sea $q = p(a,b,...,b)$ vértice de $\Gamma_p\{a,b\}$, entonces $p(a,b)$ es vértice de γ y por 3.(b) resulta que q también es vértice de γ con lo que $\Gamma_p\{a,b\} \subset \gamma$, es decir $\gamma = \Gamma_p\{a,b\}$.

Sea $\Gamma_p\{a,b\}$ maximal no trivial, es decir $\Gamma_p\{a,b\}$ es una cadena de longitud mayor que 1 o una bi-cadena. Supóngase que $\Gamma_p\{a,b\}$ es una cadena de vértice maximal $q = p(a, b^{[s]})$. La cadena $(p, p(a), p(a,b))$ está contenida en algún elemento de $\Gamma(p)$ por 3.(a) y por la manera en que se ha ponderado, luego por 3.(b) resulta que $(p,...,q) \subset \gamma \in \Gamma(p)$, resta probar que $(p,...,q) = \gamma$. Es claro que γ es cadena sino $\Gamma_p\{a,b\}$ no sería cadena. Finalmente por los criterios de ponderación si $(p,...,q,r) \subset \gamma$ resulta $r = q(b)$ es decir r es vértice de $\Gamma_p\{a,b\}$.

Finalmente se probará la minimalidad de d_{PL}. Basta demostrar, que si $\mathcal{C}$ es una constelación tórica de dimensión d, cuyo PL-diagrama de Enriques es el dado, entonces $d \geq \#p^+ + n_p \ \forall p \in \Gamma$.

Si $\Gamma(p,q) = \emptyset \ \forall q \in p^+$ se tiene $n_p = 0$ y entonces es cierto que $d \geq \#p^+$.

Si $\Gamma(p,q) \neq \emptyset$ para algún $q \in p^+$, sea $n_p = \#\Gamma(p,q_0) = max_{q \in p^+} \#\Gamma(p,q)$. $\mathcal{C}$ define una d-ponderación en Γ, los pesos de las aristas de $\Gamma(p,q_0)$ con origen en q_0 son diferentes a los de las aristas de origen p, luego $d \geq \#p^+ + n_p$. □

Corolario 3.16 *Si una constelación tórica tiene cono característico regular entonces su PL-diagrama de Enriques es el de una constelación de dimensión dos.*

Demostración

Sea $\mathcal{C}$ la constelación cuyo árbol asociado es Γ.

El cono característico de $\mathcal{C}$ es regular si y sólo si $\forall p \in \Gamma$, $\#\Gamma(p) = 1$ -ver [C.G.L.2]-. Esto implica que el grafo es binario.

Sea $p \in \Gamma$:

- si $\#p^+ = 2$, entonces $n_p = 0$;
- si $\#p^+ = 1$, se tiene que $n_p = 0, 1$, porque $\#\Gamma(p) = 1$.

Entonces de acuerdo al teorema 3.15 la dimensión minimal d_{PL} es dos. □

Ejemplo Se definen dos constelaciones tóricas $\mathcal{C}$ y $\mathcal{C}'$ de puntos infinitamente cercanos a $O \in X \cong K^3$, que no son isomorfas y sin embargo tienen diagramas de Enriques y PL-diagramas de Enriques isomorfos.

$\mathcal{C}$ y $\mathcal{C}'$ son bi-cadenas de nueve puntos.

Las sucesiones que codifican $\mathcal{C}$ son: $(1, 1, 2, 3)$ y $(2, 2, 1, 3)$.

Las sucesiones que codifican $\mathcal{C}'$ son: $(1, 1, 2, 3)$ y $(2, 2, 3, 1)$.

El siguiente enunciado da una condición suficiente para que la constelación $\mathcal{C}$ del teorema 3.15 quede unívocamente determinada a menos de isomorfismos; en particular en esas condiciones el PL-diagrama de Enriques determina la relación de proximidad entre los puntos de $\mathcal{C}$.

Corolario 3.17 *Sea* $(\Gamma, \{\Gamma(p) \mid p \in \Gamma\})$ *un PL-diagrama de Enriques tal que para todo* $p \in \Gamma$ *se verifica que* $\Gamma(p)$ *está formado únicamente por bi-cadenas o* $\Gamma(p) = \{p\}$, *entonces existe una única constelación* $\mathcal{C}$ *-a menos de isomorfismos- de dimensión* d_{PL}, *cuyo PL-diagrama de Enriques es el dado.*

Demostración

La prueba surge analizando la recurrencia usada para ponderar las aristas de Γ en la demostración del teorema 3.15. Sea o la raíz de Γ que corresponde al punto O origen de $\mathcal{C}$. Se ponderan arbitrariamente con los enteros $\{1, 2, ..., t := \#o^+\}$ las aristas de origen o, si las hay. Dado que $\Gamma(o)$ sólo contiene bi-cadenas -a menos que Γ tenga un sólo vértice-, queda determinada la ponderación de todas las aristas de nivel 2. La hipótesis asegura que resulta determinada la ponderación de Γ. Obsérvese que $\Gamma(p) = \{p\}$ significa que p es maximal en Γ.

Finalmente se observa que en estas condiciones $d_{PL} = \#o^+$. $\square$

Referencias

[C.G.L.1] Campillo A., González-Sprinberg G., Lejeune-Jalabert M. : Enriques diagrams, resolutions and toric clusters, C. R. Acad. Sci. Paris, t. 320, Serie I, p.329-334 (1995).

[C.G.L.2] Campillo A., González-Sprinberg G., Lejeune-Jalabert M. : Clusters of infinitely near points, Math. Annalen 306(1), 169-194 (1996).

[E. C.] Enriques F., Chisini O.: Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche, 1915 Libro IV.

[K] Kleiman S.: Toward a numerical theory of ampleness, Ann. of Math. 84 (1966), 293-344.

[L] Lipman J.: On complete ideals in regular local rings, Algebraic geometry and conmutative algebra in honor of M. Nagata (1987), 293-306.

[O] Oda T.: Convex bodies and algebraic geometry. An introduction to the theory of toric varieties, Springer-Verlag, 1988.

[Z] Zariski O.: Polynomials ideals defined by infinitely near base points, Amer. J. Math. 60 (1938) 151-204.

A χ^2 goodness of fit test based on Transformed Empirical Processes

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to Prof. Gonzalo Pérez Iribarren, in memoriam

Abstract

A goodness of fit χ^2 test is presented. In this test, the classical empirical process is replaced by a transformed empirical process. The test is implemented in two particular cases and its efficacy is compared with that of the classical χ^2 test, obtaining an improvement in both cases.

Keywords and phrases: Goodness of fit, Transformed Empirical Processes

1 Introduction

The aim of the present paper is to construct a consistent goodness of fit test designed to detect a particular sequence of alternatives. This is, a test that has all the fairly good consistency properties of the Pearson's χ^2 test, plus an improved performance in power in the case of a determined sequence of alternatives, contiguous to the null hypothesis. The reasons to use this type of test (χ^2 based on Transformed Empirical Processes) instead of others, are the same reasons that one has to prefer the Pearson's χ^2 among the other tests based on the classical empirical process (consistency in the case of simpler

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tests, simplicity in the case of other tests like Kolmogorov-Smirnov, Cramer-Von Mises, etc.). Without a substantial increase in the amount of calculation required, one obtains an improvement in the performance of the test mentioned above.

2 Transformed Empirical Processes

A.Cabaña and E.M.Cabaña propose in [Cabaña & Cabaña, 1997], a method for transforming the empirical process of a sample of i.i.d. random variables, into a process that converges in law to a V-Wiener process, being V a conveniently chosen measure. Besides the interest that this has in itself, it presents an important advantage to statistics, because it deals with objets that converge in law to processes with independent increments. Just as classical statistics is based on the classical empirical process, we could consider basing ourselves on the Transformed Empirical Process. Apart from this, the transformation proposed in [Cabaña & Cabaña, 1997] is not unique, and this allows us to choose the most convenient one, according to the problem stated. The Transformed Empirical Process (in short: TEP) of a sample $X_1, X_2, \ldots, X_n$ of i.i.d. random variables (with distribution function F), associated to the distribution function F_0, the score function a (with $\|a\|^2 = \int a^2(x)dF_0(x) = 1$) and the isometry $\mathcal{T}$ in $L_2(\mathbb{R}, dF_0)$, with range orthogonal to the constant function 1, was introduced in [Cabaña & Cabaña, 1997], and is defined as:

$$w_n^{a,\mathcal{T}}(A) = \int \mathcal{T}(a\mathbb{I}_A)db_n = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \mathcal{T}(a\mathbb{I}_A)(X_i) \tag{1}$$

where $\mathbb{I}_A$ is the indicator function of A, $b_n(x) = \sqrt{n}(F_n(x) - F_0(x))$ is the empirical process and F_n is the empirical distribution function of the sample $X_1, X_2, \ldots, X_n$.

If $F = F_0$, when n tends to infinity, the empirical process b_n converges in law to a F_0-brownian bridge, while the Transformed Empirical Process converges, under certain hypothesis, to a V-Wiener Process (see Cabaña & Cabaña, 1997). This is a Gaussian process w^V such that:

$$E(w^V(A)) = 0 \ , \ \forall A \in \mathcal{B} \tag{2}$$

and

$$E(w^V(A)w^V(B)) = V(A \cap B) \ , \ \forall A, B \in \mathcal{B} \tag{3}$$

where $\mathcal{B}$ is the Borel $\sigma-$algebra of $I\!R$ and V is a measure with density a^2 with respect to F_0, that is $A \ : \ V(A) = \int_A a^2(x)dF_0(x)$.

Let us suppose that we want to test the null hypothesis $\mathcal{H}_0 : F = F_0$, against the sequence of alternatives $\mathcal{H}_n : F = F^{(\delta/\sqrt{n})}$ contiguous to $\mathcal{H}_0$, (see Le Cam & Yang, 1990), with density $f^{(\delta/\sqrt{n})}$ with respect to F_0 . Let us suppose that there exists a function $k \in L^2(I\!R, dF_0)$ such that:

$$\lim_{n\to+\infty} \left\| \frac{\sqrt{n}}{\delta}\left(\sqrt{f^{(\delta/\sqrt{n})}} - 1\right) - \frac{k}{2} \right\|_{L^2} = 0 \ , \ \delta \neq 0 \tag{4}$$

This equation implies:

$$\int k(x)dF_0(x) = 0 \tag{5}$$

and k is the limit in $L^1(I\!R, dF_0)$ of $\frac{\sqrt{n}}{\delta}(f^{(\delta/\sqrt{n})} - 1)$, when $n \to \infty$.

Under $\mathcal{H}_n$, $b_n(A)$ converges in law to a F_0-brownian bridge b^{F_0} plus the deterministic bias $\delta \int_A k(x)dF_0(x)$ (see Oosterhoff & Van Zwet, 1979). On the other hand $w_n^{(a,\mathcal{T})}$ converges, under $\mathcal{H}_n$ when the conditions stated above hold, to a V-Wiener w^V plus the deterministic bias $\delta \int k(x)\mathcal{T}(a\mathbb{I}_A)(x)dF_0(x)$.

The present paper proposes to consider a χ^2 goodness of fit test, in which the classical empirical process is replaced by a Transformed Empirical Process, chosen to maximize the power, for a given sequence of contiguous alternatives. Let us consider a partition of the space (in our case $I\!R$) in adjacent intervals $A_1, A_2, \ldots, A_m$. Following the usual recommendations of practical statisticians, we will consider intervals with the same probability V, that is

$$V(A_1) = V(A_2) = \cdots = V(A_m),$$

where the measure V will be conveniently chosen, according to the hypothesis of the test.
Under $\mathcal{H}_0$, the random vector

$$\left(w_n^{a,\mathcal{T}}(A_1), w_n^{a,\mathcal{T}}(A_2), \ldots, w_n^{a,\mathcal{T}}(A_m)\right)$$

converges in law to a multivariate normal with mean $(0, \ldots, 0)$ and covariance matrix

$$\begin{pmatrix} \frac{1}{m} & 0 & \cdots & 0 \\ 0 & \frac{1}{m} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{m} \end{pmatrix}$$

Hence $m \sum_{i=1}^{m} \left(w_n^{a,\mathcal{T}}(A_i)\right)^2$ has, in the limit, distribution χ^2 with m degrees of freedom, while under the contiguous sequence of alternatives $\mathcal{H}_n : F = F^{(\delta/\sqrt{n})}$ the asymptotic distribution is a noncentred χ^2 with m degrees of freedom, and bias given by :

$$\begin{aligned} m\,\delta^2 \sum_{i=1}^{m} \left(\int k\mathcal{T}(a\mathbb{I}_{A_i})dF_0\right)^2 &= m\,\delta^2 \sum_{i=1}^{m} \left(\int_{A_i} a\mathcal{T}^{-1}k dF_0\right)^2 \\ m\,\delta^2 \sum_{i=1}^{m} \left(\int_{A_i} \frac{\mathcal{T}^{-1}k}{a} dV\right)^2 &= m\,\delta^2 \sum_{i=1}^{m} \left(\int_{A_i} h dV\right)^2 \end{aligned} \tag{6}$$

where $h = \frac{\mathcal{T}^{-1}k}{a}$. This suggests a test, based on the properties of the Transformed Empirical Process, with maximum bias, in order to improve the power. By virtue of the Cauchy-Schwarz's inequality :

$$\sum_{i=1}^{m} \left(\int_{A_i} h dV\right)^2 \leq \sum_{i=1}^{m} \int_{A_i} h^2 dV \int_{A_i} dV = \frac{1}{m} \sum_{i=1}^{m} \int_{A_i} h^2 dV = \frac{\|k\|^2}{m} \tag{7}$$

because $\int h^2 dV = \int (\mathcal{T}^{-1}k)^2 dF_0 = \|k\|^2$. On the other hand, if we take $a = \frac{\mathcal{T}^{-1}(k)}{\|k\|}$, we have $h = \|k\|$ and then

$$\sum_{i=1}^{m}\left(\int_{A_i} hdV\right)^2 = \|k\|^2 \sum_{i=1}^{m}\frac{1}{m^2} = \frac{\|k\|^2}{m} \tag{8}$$

The conclusion is that the function $a = \frac{T^{-1}(k)}{\|k\|}$ is optimum, in the sense that it maximizes the asymptotic bias.
In the following section we will perform the proposed χ^2 test, in two particular cases for the normal distribution : shifts in the mean and changes in dispersion.

3 Implementation of the test

In the two following examples, we will use the isometry $\mathcal{T}_{\mathrm{L}}$ defined in $L^2(I\!R, \Phi)$ by :

$$\mathcal{T}_{\mathrm{L}}\, g(x) = g(x) - \int_{-\infty}^{x} \frac{g(t)\varphi(t)}{1-\Phi(t)}\, dt \ , \ \forall g \in L^2(I\!R, \Phi) \tag{9}$$

where Φ is the standard normal distribution function and φ its density. The inverse of $\mathcal{T}_{\mathrm{L}}$ is

$$\left(\mathcal{T}_{\mathrm{L}}^{-1} h\right)(x) = h(x) + \frac{1}{1-\Phi(x)} \int_{-\infty}^{x} h(t)\varphi(t)\, dt \tag{10}$$

To see that $\mathcal{T}_{\mathrm{L}}$ is an isometry on $L^2(I\!R, \Phi)$, the reader should be refered to [Cabaña & Cabaña, 1997].

Let us present the two examples

3.1 Case 1. Shifts in the mean

Let us suppose that $F_0(x) = \Phi(x)$ and $F_n(x) = \Phi\left(x - \delta/\sqrt{n}\right)$, where Φ is the standard normal distribution function The function k is :

$$k(x) = \lim_{n\to+\infty} \frac{\sqrt{n}}{\delta}\left(f^{(\delta/\sqrt{n})}(x) - 1\right) = \lim_{n\to+\infty}\frac{\sqrt{n}}{\delta}\left(\frac{\varphi(x-\delta/\sqrt{n})}{\varphi(x)} - 1\right) = x.$$

It is possible to show that $k(x) = x$ satisfies (4). We choose the score function

$$a(x) = \mathcal{T}_{\mathrm{L}}^{-1} x = x + \frac{1}{1-\Phi(x)} \int_{-\infty}^{x} t\varphi(t)\, dt = x - \frac{\varphi(x)}{1-\Phi(x)} \tag{11}$$

and $A_1, A_2, \ldots, A_m$ with the same probability V, that is: $V(A_1) = \cdots = V(A_m)$, where

$$V((-\infty, y]) = \int_{-\infty}^{y} a^2(x)\varphi(x)\, dx = \int_{-\infty}^{y} \left(x - \frac{\varphi(x)}{1-\Phi(x)} \right)^2 \varphi(x)\, dx =$$

$$\int_{-\infty}^{y} x^2 \varphi(x)\, dx - \int_{-\infty}^{y} \frac{2x\varphi^2(x)}{1-\Phi(x)}\, dx + \int_{-\infty}^{y} \frac{\varphi^3(x)}{(1-\Phi(x))^2}\, dx =$$

$$= \Phi(y) - y\varphi(y) + \frac{\varphi^2(y)}{1-\Phi(y)} \tag{12}$$

Let us take $m = 10$ and define the intervals $A_j = (t_{j-1}, t_j]$, for $j = 1, \ldots, 10$ with the conventions $t_0 = -\infty$ and $t_{10} = +\infty$, so that $V(A_j) = \frac{1}{10}, for j = 1, \ldots, 10$. Let us evaluate the Transformed Empirical Process on a generic interval

$$\sqrt{n}\, w_n^{a,\mathcal{T}}(A_j) = \sqrt{n} \int \mathcal{T}(a\mathbb{I}_{A_j}) db_n = \sum_{i=1}^{n} \mathcal{T}(a\mathbb{I}_{A_j})(X_i) =$$

$$\sum_{i=1}^{n} \left(a(X_i)\mathbb{I}_{(t_{j-1},t_j]}(X_i) - \int_{-\infty}^{X_i} \frac{a(x)\mathbb{I}_{(t_{j-1},t_j]}(x)\varphi(x)}{1-\Phi(x)} dx \right) =$$

$$\sum_{i=1}^{n} \left(a(X_i)\mathbb{I}_{(t_{j-1},t_j]}(X_i) - \int_{X_i \wedge t_{j-1}}^{X_i \wedge t_j} \frac{a(x)\varphi(x)}{1-\Phi(x)} dx \right) =$$

$$\sum_{i=1}^{n} \left(a(X_i)\mathbb{I}_{(t_{j-1},t_j]}(X_i) - \left[\frac{-\varphi(x)}{1-\Phi(x)} \right]_{x = X_i \wedge t_{j-1}}^{x = X_i \wedge t_j} \right) =$$

$$\sum_{i=1}^{n}\left(X_i\,\mathbb{I}_{(t_{j-1},t_j]}(X_i)+\frac{\varphi(t_j)}{1-\Phi(t_j)}\mathbb{I}_{\{X_i>t_j\}}-\frac{\varphi(t_{j-1})}{1-\Phi(t_{j-1})}\mathbb{I}_{\{X_i>t_{j-1}\}}\right). \qquad (13)$$

The critical region for our test, at level α is:

$$\left\{\ \sum_{j=1}^{10}\left(w_n^{a,\mathcal{T}}(A_j)\right)^2>\frac{1}{10}\,\chi^2_{10}(\alpha)\ \right\} \qquad (14)$$

where $\chi^2_{10}(\alpha)$ is the $1-\alpha$ percentile for the χ^2 distribution with 10 degrees of freedom.

The following powers were obtained after simulations of both tests : the classical χ^2 and the modified χ^2, were performed to 5000 samples of size 100 (both at level $\alpha = 0.05$), for different values of the initial displacement δ.

δ	classical χ^2	χ^2 with TEPs
0	0,0466	0,0514
0,5	0,0560	0,0684
1,0	0,0812	0,1210
1,5	0,1222	0,1984
2.0	0,2236	0,3148
2.5	0,3568	0,4574
3,0	0,4942	0,6028
3,5	0,6418	0,7468
4,0	0,8040	0,8642
4,5	0,8976	0,9336
5,0	0,9528	0,9746

table 1

Table 1 shows the comparison between the rate of rejections of the two tests, with different values of the initial displacement, in case 1.

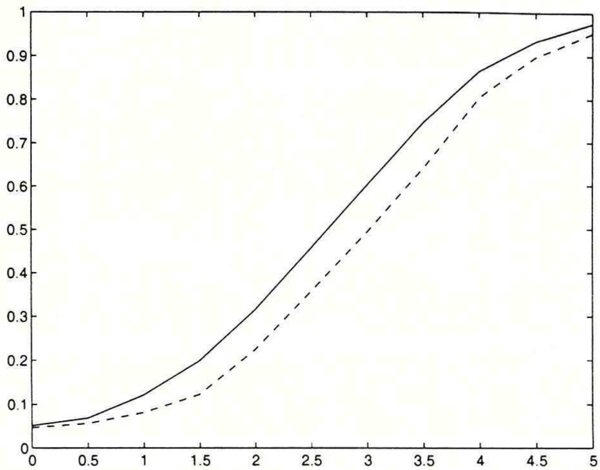

figure 1

`Case 1.` Variation in the power of the two tests with respect to the initial displacement δ, the dashed line corresponding to the classical χ^2 test, and the other to the χ^2 test based on the Transformed Empirical Process

3.2 Case 2. Changes in Dispersion

Let us suppose that $F_0(x) = \Phi(x)$ and $F_n(x) = \Phi\left((1 - \frac{\delta}{\sqrt{n}})x\right)$, where Φ is the standard normal distribution function. The function k is :

$$k(x) = \lim_{n\to+\infty} \frac{\sqrt{n}}{\delta}\left(f^{(\delta/\sqrt{n})}(x) - 1\right) =$$

$$= \lim_{n\to+\infty} \frac{\sqrt{n}}{\delta}\left(\frac{\varphi\left((1-\frac{\delta}{\sqrt{n}})x\right)}{\varphi(x)} - 1\right) = x^2 - 1 \tag{15}$$

It is possible to show that $k(x) = x^2 - 1$ satisfies (4).
Let us choose the score function

$$a(x) = \frac{1}{\|k\|}\mathcal{T}_{\mathrm{L}}^{-1}(k) = \frac{1}{\sqrt{2}}\left(x^2 - 1 + \frac{1}{1-\Phi(x)}\int_{-\infty}^{x}(t^2-1)\varphi(t)\,dt\right) =$$

$$= \frac{1}{\sqrt{2}}\left(x^2 - 1 - \frac{x\varphi(x)}{1-\Phi(x)}\right) \tag{16}$$

and compute the measure V of a generic interval

$$V((-\infty, y]) = \int_{-\infty}^{y} a^2(x)\varphi(x)\,dx = \frac{1}{2}\int_{-\infty}^{y}\left(x^2 - 1 - \frac{x\varphi(x)}{1-\Phi(x)}\right)^2 \varphi(x)\,dx =$$

$$\frac{1}{2}\int_{-\infty}^{y}\left(x^4\varphi(x) + \varphi(x) + \frac{x^2\varphi^3(x)}{(1-\Phi(x))^2} - 2x^2\varphi(x) - \frac{2x^3\varphi^2(x)}{1-\Phi(x)} + \frac{2x\varphi^2(x)}{1-\Phi(x)}\right)dx =$$

$$= 2\Phi(y) - y\varphi(y) - y^3\varphi(y) + \frac{y^2\varphi^2(y)}{1-\Phi(y)}. \tag{17}$$

With $m = 10$ and $A_j = (t_{j-1}, t_j]$, $j = 1, \dots, 10$ with $t_0 = -\infty$ and $t_{10} = +\infty$, so that $V(A_j) = \frac{1}{10}$, $j = 1, \dots, 10$. We evaluate the Transformed Empirical Process on a generic interval

$$w_n^{a,\mathcal{T}}(A_j) = \int \mathcal{T}(a\mathbb{I}_{A_j})db_n = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathcal{T}(a\mathbb{I}_{A_j})(X_i) =$$

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(a(X_i)\mathbb{I}_{(t_{j-1},t_j]}(X_i) - \int_{-\infty}^{X_i}\frac{a(x)\mathbb{I}_{(t_{j-1},t_j]}(x)\varphi(x)}{1-\Phi(x)}dx\right) =$$

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(a(X_i)\mathbb{I}_{(t_{j-1},t_j]}(X_i) - \int_{X_i\wedge t_{j-1}}^{X_i\wedge t_j}\frac{a(x)\varphi(x)}{1-\Phi(x)}dx\right) =$$

$$\frac{1}{\sqrt{2n}}\sum_{i=1}^{n}\left((X_i^2-1)\,\mathbb{I}_{(t_{j-1},t_j]}(X_i)-\left[\frac{-x\varphi(x)}{1-\Phi(x)}\right]_{x=X_i\wedge t_{j-1}}^{x=X_i\wedge t_j}\right)=$$

$$\frac{1}{\sqrt{2n}}\sum_{i=1}^{n}\left((X_i^2-1)\,\mathbb{I}_{(t_{j-1},t_j]}(X_i)+\frac{t_j\varphi(t_j)}{1-\Phi(t_j)}\mathbb{I}_{\{X_i>t_j\}}-\frac{t_{j-1}\varphi(t_{j-1})}{1-\Phi(t_{j-1})}\mathbb{I}_{\{X_i>t_{j-1}\}}\right) \tag{18}$$

The following results were obtained after a simulation in which both tests, the classical χ^2 and the modified χ^2, were performed to 5000 samples of size 100 (both at the level $\alpha = 0.05$), for different values of δ.

δ	classical χ^2	χ^2 with TEP	δ	classical χ^2	χ^2 with TEP
0	0.0466	0.0454	1.6	0.2368	0.5768
0.2	0.0530	0.0610	1.8	0.3276	0.6742
0.4	0.0536	0.0946	2.0	0.4018	0.7716
0.6	0.0724	0.1400	2.2	0.5032	0.8482
0.8	0.0850	0.1970	2.4	0.5916	0.9092
1.0	0.1112	0.2698	2.6	0.6864	0.9508
1.2	0.1416	0.3588	2.8	0.7664	0.9748
1.4	0.1864	0.4680	3.0	0.8434	0.9910

table 2

Comparison between the rate rejections of the two tests, with different values δ, in case 2

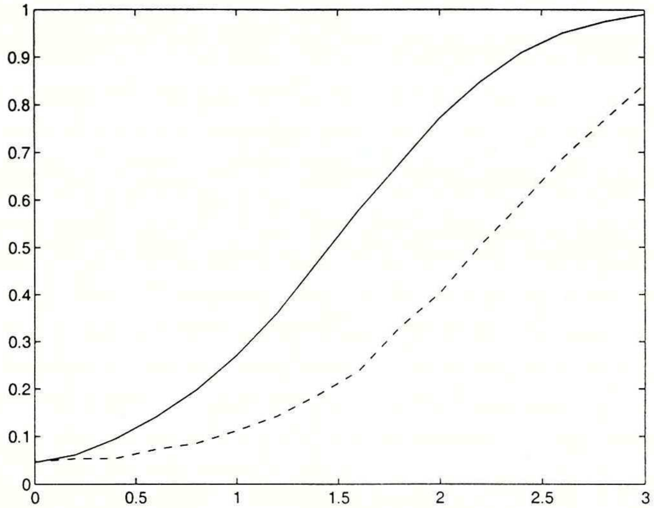

figure 2

`Case 2.` Variation in the power of the two tests with respect to δ. The dashed line corresponding to the classical χ^2 test, and the other one to the χ^2 test based on the Transformed Empirical Process .

3.3 Conclusions

3.3.1 Comparison for simulated samples:

An improvement with respect to the classical χ^2 test, has been obtained in both cases. In the case of changes in dispersion, the difference between the two tests is wider than in the other case. This is to a certain extent reasonable, because the efficacy of the classical χ^2 is, in the first case, very good while in the second case, it is not so good. The reason to include the preceding comparisons is that, in general, asymptotic results don't represent the behaviour of the test with small samples.

3.3.2 Asymptotic comparison:

Let $T_n^{(1)}$ be the statistic of the modified χ^2 test and $T_n^{(2)}$ the statistic of the classical χ^2 test. The limiting distributions of these statistics under the sequence of alternatives $\mathcal{H}_n$ are χ^2 with m and $m-1$ degrees of freedom respectively, and biases $\eta_1^2 = \delta^2\|k\|^2$ and $\eta_2^2 = m\,\delta^2\sum_{j=1}^{m}\left(\int_{R_j} kdF_0\right)^2$ respectively. Where R_j, $j = 1,\ldots,m$ are the regions of the classical χ^2 test. It is not difficult to show that both tests, the classical χ^2 and the modified χ^2, with asymptotic level α, satisfy the conditions A, B and C stated in theorem 3 of [Rothe, 1981]. According to an asymptotic (with respect to the the level) version of this theorem, the ARPE (Asymptotic Relative Pitman Efficiency) of the modified χ^2 test with respect to the classical χ^2, at the level α is :

$$e_{12}(\alpha,\beta) = \frac{H_{2,\alpha}^{-1}(\beta)}{H_{1,\alpha}^{-1}(\beta)} \tag{19}$$

where

$$H_{i,\alpha}(t) = 1 - F_{\chi^2(d_i,tc_i)}(F^{-1}_{\chi^2(d_i,0)}(1-\alpha)),\ i = 1,2.$$

with $c_1^2 = \|k\|^2$, $c_2^2 = m\sum_{j=1}^{m}\left(\int_{R_j} kdF_0\right)^2$, $d_1 = m$ and $d_2 = m-1$.

In words : $H_{i,\alpha}^{-1}(\beta) = \dfrac{\eta_i(\alpha,\beta)}{c_i}$, where $\eta_i(\alpha,\beta)$ is the (uniquely determined) noncentrality parameter so that the β percentile of a noncentred χ^2 distribution with d_i degrees of freedom and the $1-\alpha$ percentile of a centred χ^2 distribution with d_i degrees of freedom, coincide (i.e. $F^{-1}_{\chi^2(d_i,\eta_i)}(\beta) = F^{-1}_{\chi^2(d_i,0)}(1-\alpha)$).
Then, for an asymptotic comparison, it suffices to compare the distributions $\chi^2(m,\eta_1)$ and $\chi^2(m-1,\eta_2)$.
Finally we compare the asymptotic distribution of the test statistics $T_n^{(1)}$ and $T_n^{(2)}$, under the sequence of alternatives $\mathcal{H}_n$, for the two preceding examples.

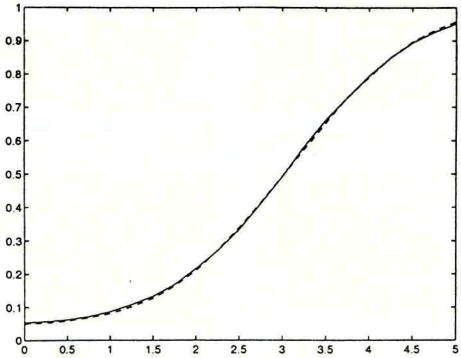

figure 3

`Case 1: Shifts in the mean.` Variation in the asymptotic power of the two tests with respect to δ, the dashed line corresponding to the classical χ^2 test, and the other to the χ^2 test based on the Transformed Empirical Process .

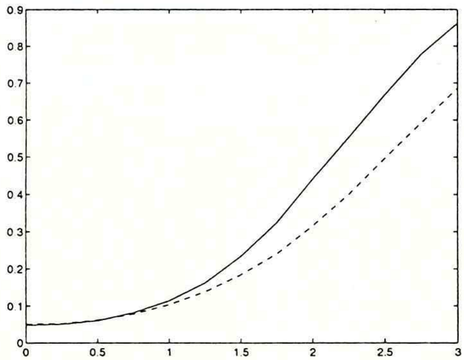

figure 4

`Case 2: Changes in dispersion.` Variation in the asymptotic power of the two tests with respect to δ, the dashed line corresponding to the classical χ^2 test, and the other to the χ^2 test based on the Transformed Empirical Process .

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References

Cabaña A. & Cabaña E.M.: (1997) *Transformed Empirical Processes and Modified Kolmogorov Smirnov Tests for Multivariate Distributions,* Ann. Statist. Vol 25, No. 6, pp. 2388-2409.

Le Cam L. & Yang G.L.: (1990) *Asymptotics in Statistics,* Springer-Verlag, New York

Oosterhoff J. & Van Zwet W.R.: (1979) *A Note on Contiguity and the Hellinger Distance,* Contributions to Statistics, 157-166, Reidel, Dordrecht

Rothe G.: (1981) *Some Properties of the Asymptotic Relative Pitman Efficiency,* Ann. Statist. Vol 9, N. 3, 663-669

Optimal Stopping and Maximal Inequalities for Poisson Processes

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Abstract

Closed form solutions for some optimal stopping problems for stochastic processes driven by a Poisson processes $N = (N_t)_{t\geq 0}$ are given.

First, cost functions and optimal stopping rules are described for the problems

$$s(x) = \sup_{\tau} E(\max[x, \sup_{0\leq t\leq\tau} (N_t - at)] - c\tau),$$

$$v(x) = \sup_{\tau} E(\max[x, \sup_{0\leq t\leq\tau} (bt - N_t)] - c\tau),$$

with a, b, c positive constants and τ a stopping time.

Based on the obtained results, maximal inequalities in the spirit of [1] are obtained.

To complete the picture, solutions to the problems

$$c(x) = \sup_{\tau} E(x + N_\tau - a\tau)^+, \qquad p(x) = \sup_{\tau} E(x + b\tau - N_\tau)^+$$

are given.

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1 Introduction and main results

1.1. Let be given a Poisson process $N = (N_t)_{t\geq 0}$ with intensity $\lambda > 0$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. Denote by $\overline{\mathcal{M}}$ the class of all stopping times (that can take the value $+\infty$), by $\mathcal{M}$ the class of finite valued stopping times, and by $\mathcal{M}_0$ the class of stopping times with finite expectation. All stopping times are considered with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$.

In this paper we face the problem of giving the cost functions and optimal stopping times in the following optimal stopping problems:

$$s(x) = \sup_{\tau} E(\max[x, \sup_{0\leq t\leq \tau} (N_t - at)] - c\tau), \tag{1}$$

$$v(x) = \sup_{\tau} E(\max[x, \sup_{0\leq t\leq \tau} (bt - N_t)] - c\tau), \tag{2}$$

where a, b and c are positive constants, (c is the "price" of one unity of observation), and $x \in \mathbf{R}$.

Problems (1) and (2) are related to the pricing of "Russian Options" introduced by L. Shepp and A.N. Shiryaev in [2]. In our case, the price process for the risky asset is driven by a Poisson process instead of a geometric Brownian motion.

The solutions to these problems are then used to obtain maximal inequalities of the form

$$\sup_{\tau\in\mathcal{M}_0} E[\sup_{0\leq t\leq \tau} (N_t - at))] \leq \phi(E\tau) \tag{3}$$

$$\sup_{\tau\in\mathcal{M}_0} E[\sup_{0\leq t\leq \tau} (bt - N_t))] \leq \psi(E\tau) \tag{4}$$

where $\phi = \phi(x)$, $\psi = \psi(x)$, $x > 0$ are the minimal possible functions, that satisfy (3) and (4).

In relation with this, we refer to the paper of L. Dubbins, L.A. Shepp and A.N. Shiryaev [1] devoted to the study of optimal stopping rules and maximal inequalities for Bessel Processes, the work of L. Dubbins and G. Schwartz [2], and to some recent results for linear diffusions of S.E. Graversen and G. Peškir [4].

Finally, to complete the picture, we give the cost function and optimal stopping time in the following problems

$$c(x) = \sup_{\tau \in \overline{\mathcal{M}}} E(x + N_\tau - a\tau)^+ \tag{5}$$

$$p(x) = \sup_{\tau \in \overline{\mathcal{M}}} E(x + b\tau - N_\tau)^+ \tag{6}$$

where $x \in \mathbf{R}$, z^+ denotes $\max(z, 0)$.

1.2. In order to formulate our results, let us introduce the function $u = u(x) = u(x; d), x \geq 0$ with d a positive constant, defined by

$$u(x; d) = \sum_{k=0}^{+\infty} [e^{\frac{x-k}{d}} P_k(\frac{x-k}{d}) - (k+1)] \mathbf{I}_{[k,k+1)}(x) \tag{7}$$

where $P_k = P_k(x)$ is a polynomial of order k:

$$P_k(x) = \sum_{l=0}^{k} d_{k-l} \frac{(-1)^l x^l}{l!}, \tag{8}$$

and the coefficients $d_k, k \geq 0$ are defined by the following recurrence relation:

$$d_0 = 1, \quad d_{k+1} = 1 + e^{1/d} \sum_{l=0}^{k} d_{k-l} \frac{(-1)^l}{d^l l!} = 1 + e^{1/d} P_k(1/d). \tag{9}$$

Some properties of the function $u = u(x)$ are revisted in section 2.

Denote now by τ_s^* the optimal stopping time in the problem (1), that is, the stopping time for which the supremum is realized. The price function $s = s(x)$ and the optimal stopping time τ_s^* are given in the following

Theorem 1

(i) *If $a + c \leq \lambda$, then $s(x) = +\infty$, for all $x \geq 0$.*

(ii) *If $a > \lambda$ and $c = 0$, then*

$$s(x) = E(\max[x, \sup_{0 \leq t \leq +\infty} (N_t - at)]) = \frac{\lambda}{2(a-\lambda)} + \frac{a-\lambda}{\lambda} u(x; \frac{a}{\lambda}) \tag{10}$$

(iii) *If $a > 0$ and $c \geq \lambda$, then $s(x) = x$ and $\tau_s^* = 0$.*
(iv) *If $c < \lambda < a + c$, then*

$$s(x) = \begin{cases} x, & \text{if } x \geq x^* \\ x_s^* + \frac{a+c-\lambda}{\lambda}[u(x, \frac{a}{\lambda}) - u(x_s^*; \frac{a}{\lambda})], & \text{if } 0 \leq x < x^* \end{cases} \tag{11}$$

and

$$\tau_s^* = \inf\{t \geq 0 \colon X_t \geq x_s^*\}, \tag{12}$$

where $X = (X_t)_{t \geq 0}$ is a stochastic process defined by

$$X_t = \max[x, \sup_{0 \leq r \leq t} (N_r - ar)] - (N_t - at), \tag{13}$$

and the positive constant x_s^* is the solution of the equation

$$\frac{a + c - \lambda}{\lambda} u'(x; \frac{a}{\lambda}) = 1. \tag{14}$$

We also have, when $x < x_s^*$

$$E(\tau_s^*) = \frac{1}{\lambda}[u(x_s^*, \frac{a}{\lambda}) - u(x; \frac{a}{\lambda})]. \tag{15}$$

Denote now by τ_v^* the optimal stopping time for the problem (2).The price function $v = v(x)$ and the optimal stopping time are given in the following

Theorem 2

(i) *If $c + \lambda \leq b$, then $v(x) = +\infty$, for all $x \geq 0$.*
(ii) *If $\lambda > b$ and $c = 0$, then*

$$v(x) = E(\max[x, \sup_{0 \leq t < +\infty} (bt - N_t)]) = x + \frac{1}{\alpha^*} e^{-\alpha^* x}, \tag{16}$$

where the constant α^* is the unique positive root of the equation

$$\frac{b}{\lambda}\alpha + e^{-\alpha} - 1 = 0. \tag{17}$$

(iii) *If $c \geq b$, then $v(x) = x$ and $\tau_v^* = 0$.*

(iv) *If $c < b < c+\lambda$, then*

$$v(x) = \begin{cases} x, & \text{if } x \geq x_v^*, \\ x + \frac{c}{\lambda} u(x_v^* - x, \frac{b}{\lambda}), & \text{if } 0 \leq x < x_v^*, \end{cases} \tag{18}$$

and

$$\tau_v^* = \inf\{t \geq 0 \colon Y_t \geq x_v^*\},$$

where $Y = (Y_t)_{t \geq 0}$ is a stochastic process defined by

$$Y_t = \max[x, \sup_{0 \leq r \leq t} (br - N_r)] - (bt - N_t), \tag{19}$$

and the positive constant x_v^* is the solution of the equation

$$\frac{c}{\lambda} u'(x; \frac{b}{\lambda}) = 1. \tag{20}$$

We also have, when $0 \leq x < x_v^*$

$$E(\tau_v^*) = \frac{1}{c+\lambda-b}[x_v^* - x - \frac{c}{\lambda} u(x_v^* - x, \frac{b}{\lambda})]. \tag{21}$$

1.3. Let $\mathcal{M}_T = \{\tau \in \mathcal{M}_0 \colon E(\tau) \leq T\}$ denote the set of all the stopping times τ with expected value less or equal than $T > 0$. Functions ϕ and ψ in (3) and (4) can be defined in the following way

$$\phi(T) = \sup_{\tau \in \mathcal{M}_T} E(\sup_{0 \leq t \leq \tau} (N_t - at)), \tag{22}$$

$$\psi(T) = \sup_{\tau \in \mathcal{M}_T} E(\sup_{0 \leq t \leq \tau} (bt - N_t)). \tag{23}$$

In what follows, the stopping times such that the supremum in (22) and (23) is realized will be called $\mathcal{M}_T$-optimal.

Theorem 3 *Let $0 < T < +\infty$.*

(i) *Denote by $x_s^*(T)$ the root of the equation*

$$u(x; \frac{a}{\lambda}) = \lambda T. \tag{24}$$

Then

$$\phi(T) = x_s^*(T) + (\lambda - a)T \tag{25}$$

and the stopping time

$$\tau_s^*(T) = \inf\{t \geq 0 \colon X_t \geq x_s^*(T)\},$$

is $\mathcal{M}_T$-optimal for the problem (22), where in the process $X = (X_t)_{t \geq 0}$ defined in (13) we take $x = 0$.

Furthermore, when $\lambda = a$ we have

$$\frac{\sqrt{4T+1} - 1}{2} \leq \phi(T) \leq \sqrt{T} \tag{26}$$

and in consequence

$$\lim_{T \to +\infty} \frac{\phi(T)}{\sqrt{T}} = 1. \tag{27}$$

(ii) *Denote by $x_v^*(T)$ the positive root of the equation*

$$\frac{xu'(x; \frac{b}{\lambda}) - u(x; \frac{b}{\lambda})}{1 + (1 - \frac{b}{\lambda})u'(x; \frac{b}{\lambda})} = \lambda T, \tag{28}$$

Then

$$\psi(T) = x_v^*(T) + (b - \lambda)T, \tag{29}$$

and the stopping time

$$\tau_v^*(T) = \inf\{t \geq 0 \colon Y_t \geq x_v^*(T)\},$$

is $\mathcal{M}_T$-optimal for the problem (23), where in the process $Y = (Y_t)_{t \geq 0}$ defined in (19) we take $x = 0$.

Furthermore, when $\lambda = b$ we have

$$\lim_{T \to +\infty} \frac{\psi(T)}{\sqrt{T}} = 1. \tag{30}$$

1.4. Denote by τ_c^* the optimal stopping time in the problem (5). The price function and the optimal stopping time for this problem are given in the following

Theorem 4

(i) *If $a \le \lambda$, then $c(x) = +\infty$ for all $x \in \mathbf{R}$.*

(ii) *If $\lambda < a$, then*

$$c(x) = \begin{cases} x, & \text{if } x \ge x_c^*, \\ x + (\frac{a}{\lambda} - 1)u(x_c^* - x, \frac{a}{\lambda}), & \text{if } x < x_c^*, \end{cases} \tag{31}$$

with $x_c^* = \frac{\lambda}{2(a-\lambda)}$. The optimal stopping time is

$$\tau_c^* = \inf\{t \ge 0 \colon x + N_t - at \ge x_c^*\}. \tag{32}$$

Denote finally by τ_p^* the optimal stopping time in the problem (6). The cost function and the optimal stopping time for this problem are given in the following

Theorem 5

(i) *If $b \ge \lambda$, then $p(x) = +\infty$ for all $x \in \mathbf{R}$.*

(ii) *Assume $b < \lambda$. Denote by α^* the positive root of the equation (17). Then, the cost function is*

$$p(x) = \begin{cases} x, & \text{if } x \ge x_p^*, \\ x_p^* e^{\alpha^*(x - x_p^*)}, & \text{if } x < x_p^*, \end{cases} \tag{33}$$

with $x_p^* = \frac{1}{\alpha^*}$, and the optimal stopping time is

$$\tau_p^* = \inf\{t \ge 0 \colon x + bt - N_t \ge x_p^*\}. \tag{34}$$

2 Some auxiliar results

2.1. In this section we formulate some technical results concerning the function $u = u(x), x \ge 0$ defined in (7). Let us introduce the operator $\mathcal{K} = \mathcal{K}(d)$ defined on the set of continuously differentable functions by the following relation

$$\mathcal{K}w(x) = dw'(x) + w(x - x \wedge 1) - w(x), \qquad x \ge 0. \tag{35}$$

As will be shown in section 3, the operator $\lambda\mathcal{K}$ is the infinitesimal operator associated to the process $X = (X_t)_{t \ge 0}$ defined in (13) with $d = \frac{a}{\lambda}$.

Lemma 1 *The function $u = u(x)$ defined in (7) is continously differentable, and satisfies the following relation*

$$\mathcal{K}u(x) = du'(x) - u(x) + u(x - x \wedge 1) = 1, \qquad x \geq 0. \tag{36}$$

Proof. ¿From the definition of the function $u = u(x)$ (see (7), (8) and (9)) follows, that this function is smooth on the intervals $(k, k+1)$ for any nonnegative integer k.

If $x \in [0, 1)$, we have $u(x) = e^{x/d} - 1$, giving that $u(0) = 0$ and

$$\mathcal{K}u(x) = du'(x) - u(x) = 1.$$

If $k \geq 1$ and $x \in (k, k+1)$, equation (36) is

$$du'(x) - u(x) = 1 - u(x-1).$$

In view of (7) for $x \in (k, k+1)$

$$u(x) = e^{\frac{x-k}{d}} P_k(\frac{x-k}{d}) - (k+1),$$

and from (8) follows that $P_k'(x) = -P_{k-1}(x)$. Then

$$du'(x) - u(x) = e^{\frac{x-k}{d}} P_k'(\frac{x-k}{d}) + k + 1$$

$$= 1 - e^{\frac{x-k}{d}} P_{k-1}(\frac{x-k}{d}) + k = 1 - u(x-1) \tag{37}$$

In this way (36) is proved for $x \in (k, k+1)$. It is clear from (35), that in order to verify (36) when $x = 1, 2, \ldots$ it is enough to see that the functions $u = u(x)$ and $u' = u'(x)$ are continuous in these points.

Let us examine the contінity of u. In view of (7)

$$u(k) = P_k(0) - (k+1) = d_k - (k+1)$$

and by (9)

$$\lim_{x \uparrow k} u(x) = e^{1/d} P_{k-1}(1/d) - k = d_k - (k+1),$$

and the continuity of u follows. For the function u' in view of (37)

$$\lim_{x \downarrow k} u'(x) = \frac{1}{d} \lim_{x \downarrow k} [1 + u(x-1) - u(x)]$$

$$= \frac{1}{d}[1 + u(k-1) - u(k)],$$

and

$$\lim_{x \uparrow k} u'(x) = \frac{1}{d} \lim_{x \uparrow k} [1 + u(x - x \wedge 1) - u(x)]$$

$$= \frac{1}{d}[1 + u(k-1) - u(k)],$$

concluding the proof. □

2.2. In the following Lemma we study the behaviour of $u = u(x)$ and its derivative.

Lemma 2 *The functions $u = u(x)$ and $u' = u'(x)$ satisfy*

a) $u(0) = 0$, $u'(0) = \frac{1}{d}$.

b) u and u' are strictly increasing.

c) $\lim_{x\to+\infty} u(x) = +\infty$. $\lim_{x\to+\infty} u'(x) = \begin{cases} \frac{1}{d-1}, & d > 1 \\ +\infty, & 0 < d \leq 1. \end{cases}$

d) If $d = 1$, for all $x \geq 0$ we have

$$x^2 \leq u(x) \leq x^2 + x.$$

e) If $d = 1$, then

$$\lim_{x\to+\infty} \frac{xu'(x) - u(x)}{x^2} = 1.$$

Proof. a) As $u(x) = e^{x/d} - 1$ when $x \in [0, 1)$ we obtain

$$u(0) = 0 \quad \text{and} \quad u'(0) = \frac{1}{d}.$$

b) In order to prove that u and u' are strictly increasing, as they are both absolutely continuous, we will see that

$$u''(x) \text{ is positive and continuous for all } x \neq 1, \quad x \geq 0, \tag{38}$$

As $u(x) = e^{x/d} - 1$ if $x \in (0,1)$, by differentiation $u''(x) = \frac{1}{d^2}e^{x/d}$ is continuous and positive if $x \in [0,1)$. Take now $x > 1$. From (36), for $x > 1$ we obtain

$$du''(x) = \int_{x-1}^{x} u''(y)dy. \tag{39}$$

As u'' is bounded on compact intervals, the continuity if $x \neq 1$ follows. In view of (39) follows that

$$u''(x) > 0, \qquad x > 1.$$

In fact, let $x_0 = \inf\{x \geq 0 \colon u''(x) = 0\}$. We have $u''(x_0) = 0$ and $u''(x) > 0$ when $x < x_0$ contradicting (39).

So $u = u(x)$ and $u' = u'(x)$ are both strictly increasing functions.

c) Let us now compute the limits at infinite. If $x > 1$, then from (36) follows that

$$du'(x) = u(x) - u(x-1) + 1 = 1 + \int_{x-1}^{x} u'(y)dy.$$

¿From this, taking into account the monotonicity of the function $u' = u'(x)$ we obtain

$$1 + u'(x-1) < du'(x) < 1 + u'(x)$$

and taking limits we obtain the second limit in c). Finally, this ensures the convergence of $u(x) \to +\infty$ when $x \to +\infty$, proving c).

d) Let $d = 1$. Denote $\Delta(x) = u(x) - x^2$. We want to establish

$$\Delta(x) \geq 0 \quad \text{for all} \quad x \geq 0. \tag{40}$$

For this function

$$\mathcal{K}\Delta(x) = \mathcal{K}u - \mathcal{K}(x^2) = 0.$$

This means

$$\Delta'(x) = \Delta(x) - \Delta(x-1) = \int_{x-1}^{x} \Delta'(t)dt.$$

If $x \in (0,1)$, we know $\Delta(x) = e^x - 1 - x^2$, and in consequence $\Delta'(x) > 0$ on this interval. So, with the same argument as in the proof of (38) we obtain that $\Delta'(x) > 0$ for all $x > 0$. As $\Delta(0) = 0$, (40) is proved.

Denote now

$$\delta(x) = x^2 + x - u(x).$$

In a similar way, as was done for Δ, we obtain that

$$\delta'(x) = 2x + 1 - e^x > 0 \quad \text{for} \quad 0 < x < 1,$$

and for $x > 1$

$$\delta'(x) = \int_{x-1}^{x} \delta'(t)dt.$$

concluding that $\delta(x) \geq 0$ for all $x \geq 0$ and the proof of d).

e) From d) we obtain

$$\lim_{x\to+\infty} \frac{u(x)}{x^2} = 1 \tag{41}$$

Now, denoting $w(x) = xu'(x) - u(x)$, we have

$$w'(x) = xu''(x).$$

So, by L'Hôpital rule

$$\lim_{x\to+\infty} \frac{w(x)}{x^2} = \lim_{x\to+\infty} \frac{xu''(x)}{2x} = \lim_{x\to+\infty} \frac{u''(x)}{2}.$$

But, in view of (41), and L'Hôpital rule again

$$\lim_{x\to+\infty} \frac{u''(x)}{2} = \lim_{x\to+\infty} \frac{u'(x)}{2x} = \lim_{x\to+\infty} \frac{u(x)}{x^2} = 1.$$

proving d). □

3. We will also need the following

Lemma 3 *Let $d > 1$. Then the function $w = w(x)$ defined by*

$$w(x) = x - (d-1)u(x), \qquad x \geq 0,$$

is strictly increasing and

$$\lim_{x\to+\infty} w(x) = \frac{1}{2(d-1)}. \tag{42}$$

Proof. By part c) of Lemma 2

$$w'(x) = 1 - (d-1)u'(x) > 0 \quad \text{for all} \quad x \geq 0$$

so the funtion w is stricltly incresing. In consequence the limit in (42) exists.

Now, by (36)

$$\mathcal{K}w(x) = dw'(x) - w(x) = 1 - x, \qquad x < 1,$$

and

$$\mathcal{K}w(x) = dw'(x) - w(x) + w(x-1) = 0, \qquad x \geq 1. \tag{43}$$

Integrating (43) over $[1, x]$, follows

$$\begin{aligned} d[w(x) - w(1)] &= \int_1^x [w(y) - w(y-1)]dy \\ &= \int_{x-1}^x w(y)dy + \int_0^1 w(y)dy \end{aligned}$$

Taking into account (36)

$$dw(x) = \int_{x-1}^x w(y)dy + \frac{1}{2}.$$

Now, the monotonicity of $w(x)$ gives

$$w(x-1) + \frac{1}{2} < dw(x) < w(x) + \frac{1}{2},$$

and taking limits as $x \to +\infty$ we conclude the proof. □

3 Proofs of the Theorems

3.1. The process $X = (X_t)_{t\geq 0}$, defined in (13), is an homogeneous Markov process with stochastic differential

$$dX_t = adt - (X_{t-} \wedge 1)dN_t.$$

If $w = w(x),\ x \geq 0$, is a continously differentiable function, then, Itô's formula for pure jump processes gives (see [6] ch. 3 §6)

$$w(X_t) = w(x) + \lambda \int_0^t \mathcal{K}w(X_{r-})dr + M_t, \quad t \geq 0, \tag{44}$$

with $\mathcal{K} = \mathcal{K}(\frac{a}{\lambda})$ in (35) and the process $M = (M_t)_{t\geq 0}$ given by

$$M_t = \int_0^t [w(X_{r-} - X_{r-} \wedge 1) - w(X_{r-})]d(N_r - \lambda r)$$

is a local martingale. From (44) and Lemma 1 follows, that the process $(u(X_t) - \lambda t)_{t\geq 0}$ is a local martingale, with the function $u = u(x; \frac{a}{\lambda})$ defined in (7) and $d = \frac{a}{\lambda}$.

Proof of Theorem 1.

(i) Assume $a + c \leq \lambda$. As $E(N_t - (a+c)t) = (\lambda - a - c)t \geq 0$ and the process $\{N_t - (a+c)t\}$ has stationary independent increments, it follows that $P(\sup_t(N_t - (a+c)t) = +\infty) = 1$ (see [7]). Thi s means, that for the stopping time

$$\tau_H = \inf\{t \geq 0 \colon N_t - (a+c)t \geq H\}.$$

we have $P(\tau_H < +\infty) = 1$. Then

$$s(x) \geq s(0) \geq E(\sup_{0\leq r\leq \tau_H}(N_r - ar) - c\tau_H)$$

$$\geq E((N_{\tau_H} - (a+c)\tau_H) \geq H.$$

As H is arbitrary, the proof of (i) is concluded.

(ii) Let $c = 0$ and $a > \lambda$. For $t > 0$ denote

$$s_t(x) = E(\max[x, \sup_{0\leq r\leq t}(N_r - ar)]).$$

By monotnonous convergence

$$s_t(x) \uparrow s(x), \qquad t \to +\infty.$$

Taking into account, that $E(N_t) = \lambda t,\ t \geq 0$, we obtain that

$$s_t(x) = E(X_t - (a - \lambda)t), \qquad X_0 = x,$$

with the process $X = (X_t)_{t\geq 0}$ defined in (13).

Denote $w(x) = x - \frac{a-\lambda}{\lambda}u(x)$. By Lemma 3, we have

$$\lim_{x\to+\infty} w(x) = \frac{\lambda}{2(a-\lambda)}.$$

Furthermore, as $X_t \geq at - N_t$, and $a > \lambda$, then, $X_t \to +\infty$ with probability one as $t \to +\infty$. As the process $\{\frac{a-\lambda}{\lambda}u(X_t) - (a-\lambda)t\}$ is a martingale, we obtain

$$s(x) - \frac{a-\lambda}{\lambda}u(x) = \lim_{t\to+\infty} E(s_t(x)) - E(\frac{a-\lambda}{\lambda}u(X_t) - (a-\lambda)t)$$

$$= \lim_{t\to+\infty} E(X_t - \frac{a-\lambda}{\lambda}u(X_t)) = \lim_{t\to+\infty} E(w(X_t)) = \frac{\lambda}{2(a-\lambda)}$$

by bounded convergence, concluding the proof of (ii).

(iii) First we observe, that when $c > 0$ and $a + c > \lambda$ the supremum in (1) can be taken over $\mathcal{M}_0$. To see this, take an arbitrary stopping time $\tau \in \mathcal{M}$ with $E(\tau) = +\infty$. Consider δ such that $\lambda - a < \delta < c$.

$$E(\max[x, \sup_{0\leq t\leq\tau}(N_t - at)] - c\tau) \leq x + E[\sup_{0\leq t<+\tau}(N_t - at) - c\tau]$$

$$\leq x + E[\sup_{0\leq t<+\tau}(N_t - (a+\delta)t) - (c-\delta)\tau] = -\infty$$

because, in accordance with (10)

$$E(\sup_{0\leq t<+\infty}(N_t - (a+\delta)t)) = \frac{\lambda}{a+\delta-\lambda} < +\infty.$$

Now, if $c \geq \lambda$, $a > 0$ and $E(\tau) < +\infty$, then $E(N_\tau) = \lambda E(\tau)$ and

$$E(\max[x, \sup_{0\leq t\leq\tau}(N_t - at)] - c\tau)) \leq x + E(\sup_{0\leq t\leq\tau}(N_t - at) - c\tau)$$

$$\leq x + E(N_\tau - c\tau) \leq x - (c-\lambda)E(\tau) \leq x.$$

On the other hand, taking $\tau_s^* = 0$

$$E(\max[x, \sup_{0\leq t\leq\tau_s^*}(N_t - at)] - c\tau_s^*) = x,$$

and follows that $\tau_s^* = 0$ is the optimal stopping time, and $s(x) = x,\ x \geq 0$.

(iv) Let $c < \lambda < a + c$. In view of Lemma 2 (14) has an unique solution x_s^* such that $0 < x_s^* < +\infty$.

Let us first prove, that for τ_s^* defined by (12), identity (15) holds.

As $u(x) \leq u(x_s^*)$ if $x \leq x_s^*$, then

$$-\lambda t \leq u(X_{\tau_s^* \wedge t}) - \lambda(\tau_s^* \wedge t) \leq u(x_s^*)$$

and, as a consequence, the local martingale $\{u(X_{\tau_s^* \wedge t}) - \lambda(\tau_s^* \wedge t)\}_{t \geq 0}$ is in fact a martingale. Therefore

$$\lambda E(\tau_s^* \wedge t) = E(u(X_{\tau_s^* \wedge t})) - u(x) \tag{45}$$

¿From this, we deduce that $E(\tau_s^*) < +\infty$, so $P(\tau_s^* < +\infty) = 1$, concluding that

$$u(X_{\tau_s^* \wedge t}) \to u(x_s^*) \quad \text{as} \quad t \to +\infty.$$

Now, taking limits in (45) as $t \to +\infty$ we obtain (15).

We know by (iii) that we can take the supremum over $\mathcal{M}_0$. If $\tau \in \mathcal{M}_0$, $E(\tau) < +\infty$, then $E(N_\tau) = \lambda E(\tau)$. In consequence

$$E(\max[x, \sup_{0 \leq t \leq \tau} (N_t - at)] - c\tau) = E(X_\tau - (a + c - \lambda)\tau).$$

Denote by $\tilde{s} = \tilde{s}(x)$ by

$$s(x) = \begin{cases} x, & \text{if } x \geq x^* \\ x_s^* + \frac{a+c-\lambda}{\lambda}[u(x, \frac{a}{\lambda}) - u(x_s^*; \frac{a}{\lambda})], & \text{if } 0 \leq x < x^* \end{cases}$$

We want to prove $\tilde{s} = s$ with s in (1). The following relation takes place

(A)$s(x) = \max(x, x_s^*) - (a + c - \lambda)E(\tau_s^*) = E(X_{\tau_s^*} - (a + c - \lambda)\tau_s^*)$,

that is direct if $x \geq x_s^*$ and follows from (15) when $x < x_s^*$. In consequence, in order to complete the proof of the Theorem, it remains to see that for any stopping time $\tau \in \mathcal{M}_0$

(B)$\tilde{s}(x) \geq E(X_\tau - (a + c - \lambda)\tau)$.

In view of Lemma 2 the function $u = u(x)$ is convex, so $s(x) \geq x$. To prove (B) it is enough to see that the process $Z = (Z_t)_{t \geq 0}$ defined by

$$Z_t = \tilde{s}(X_{\tau \wedge t}) - (a + c - \lambda)(\tau \wedge t)$$

is a supermartingale. ¿From (44) follows that the process Z is a local supermartingale, if

$$\lambda \mathcal{K}\tilde{s}(x) - (a + c - \lambda) \leq 0, \qquad x \geq 0, \tag{46}$$

with $\mathcal{K} = \mathcal{K}(\frac{a}{\lambda})$ defined in (35).

For $x \leq x_s^*$ (46) takes places in view of Lemma 1 (in fact we have an identity). If $x > x_s^*$, then $\tilde{s}(x) = x$ and

$$\lambda \mathcal{K}\tilde{s}(x) - (a + c - \lambda) = a + \lambda \tilde{s}(x - x \wedge 1) - \lambda x - (a + c - \lambda)$$

$$\leq a + \lambda \tilde{s}(x_s^* - x_s^* \wedge 1) - \lambda x_s^* - (a + c - \lambda) = \lambda \mathcal{K}\tilde{s}(x_s^*) - (a + c - \lambda) = 0,$$

where we used the convexity of the function $\tilde{s}$. In this way the process $Z = (Z_t)_{t \geq 0}$ is a local supermartingale, and as

$$Z_t \geq -(a + c - \lambda)t, \qquad t \geq 0,$$

and $E(\tau) < +\infty$, we deduce that Z is a supermartingale and the proof is concluded. □

3.2. The process $Y = (Y_t)_{t \geq 0}$ defined in (19) is an homogeneous Markov process with stochastic differential

$$dY_t = dN_t - \mathbf{I}_{\{Y_{t-} > 0\}} b dt. \tag{47}$$

If $w = w(x),\ x \geq 0$, is a continously differentiable function, Itô's formula gives (see [6])

$$w(Y_t) = w(x) + \lambda \int_0^t \mathcal{L}w(Y_{r-}) dr + M_t, \tag{48}$$

where

$$\mathcal{L}w(x) = -\frac{b}{\lambda} \mathbf{I}_{\{x > 0\}} w'(x) + w(x+1) - w(x), \tag{49}$$

and the process $M = (M_t)_{t \geq 0}$, given by

$$M_t = \int_0^t [w(Y_{r-} + 1) - w(Y_{r-})] d(N_r - \lambda r)$$

is a local martingale.

Proof of Theorem 2.

(i) Let $\lambda + c \leq b$. Take $H > 0$ and define

$$\tau_H = \inf\{t \geq 0 \colon (b-c)t - N_t \geq H\}.$$

As on the proof of (i) in Theorem 1 we conclude that $P(\tau_H < +\infty) = 1$. We have

$$\begin{aligned} v(x) &\geq E(\max[x, \sup_{0 \leq t \leq \tau_H} (bt - N_t)] - c\tau_H) \\ &\geq E((b-c)\tau_H - N_{\tau_H}) \geq H. \end{aligned}$$

As H is arbitrary, the proof of (i) is concluded.

(ii) Let $c = 0$ and $b < \lambda$. For $t > 0$ denote

$$v_t(y) = E(\max[x, \sup_{0 \leq r \leq t} (br - N_r)]).$$

By monotonous convergence

$$v_t(y) \uparrow v(y) \quad \text{as} \quad t \to +\infty.$$

Taking into account the definition of Y in (19) and the fact that $E(N_t) = \lambda t$, we obtain that

$$v_t(x) = E[Y_t - (\lambda - b)t], \qquad Y_0 = x.$$

Consider the function $\tilde{v} = \tilde{v}(x)$ defined by

$$\tilde{v}(x) = x + \frac{1}{\alpha^*} e^{-\alpha^* x},$$

with α^* the positive root of the equation (17) that, as $b < \lambda$ has an unique positive solution. A direct computation shows that the function $\tilde{v} = \tilde{v}(x)$ satisfies the equation

$$\lambda \mathcal{L} \tilde{v}(x) - (\lambda - b)t = 0.$$

In view of (46) the process $K = (K_t)_{t \geq 0}$ defined by

$$K_t = \tilde{v}(Y_t) - (\lambda - b)t$$

is a local martingale.

¿From the definition of $\tilde{v} = \tilde{v}(x)$ follows that

$$x \leq \tilde{v}(x) \leq x + \frac{1}{\alpha^*}, \qquad x \geq 0, \tag{50}$$

and

$$\lim_{x \to +\infty} |\tilde{v}(x) - x| = 0.$$

Notice now, that

$$\max(0, N_t - bt) \leq Y_t \leq x + N_t \tag{51}$$

so the local martingale $K = (K_t)_{t \geq 0}$ is uniformly integrable and in consequence a martingale. Finally, from (51) and $\lambda > b$ follows that

$$Y_t \to +\infty, \quad t \to +\infty \quad P\text{-a.s.} \tag{52}$$

¿From this fact, and (50), we obtain

$$\lim_{t \to +\infty} |\tilde{v}(Y_t) - Y_t| = 0.$$

So, we have

$$\begin{aligned} v(x) &= \lim_{t \to +\infty} v_t(x) = \lim_{t \to +\infty} E(Y_t - (b - \lambda)t) \\ &= \lim_{t \to +\infty} E(\tilde{v}(Y_t) - (b - \lambda)t) = \tilde{v}(x), \end{aligned}$$

and the proof of (ii) is concluded.

(iii) Let $c \geq b$. Then, for any $\tau \in \mathcal{M}$

$$E(\max[x, \sup_{0 \leq t \leq \tau} (bt - N_t)] - c\tau) \leq y + (b - c)E(\tau) \leq y,$$

and, if we take $\tau_s^* = 0$

$$E(\max[x, \sup_{0 \leq t \leq \tau_s^*} (bt - N_t) - c\tau_s^*)] = x,$$

proving (iii).

(iv) Assume $c < b < c + \lambda$. Let us see, that in (2) it is enough to take the supremum over $\mathcal{M}_0$. Take δ positive, such that $b - \lambda < \delta < c$. Consider an arbitrary stopping time τ with $E(\tau) = +\infty$. As by (16)

$$E(\sup_{0 \le t < +\infty} ((b-\delta)t - N_t) < \infty,$$

we obtain

$$E(\max[x, \sup_{0 \le t \le \tau} (bt - N_t)] - c\tau)) \le x +$$

$$+ E(\sup_{0 \le t < +\infty} ((b-\delta)t - N_t) - (c - \delta)E(\tau) = -\infty,$$

See now, that in view of Lemma 2 equation (20) has a unique positive solution x_v^*. Define now the function $\tilde{v} = \tilde{v}(x)$ by

$$\tilde{v}(x) = \begin{cases} x, & x \ge x_v^* \\ x + \frac{c}{\lambda} u(x_v^* - x; \frac{b}{\lambda}), & 0 \le x < x_v^*. \end{cases}$$

In order to prove (iv) we verify $\tilde{v} = v$, with v in (2). For this, it is enough to verify the following two assertions:

(A) If $\tau_v^* = \inf\{t \ge 0 : Y_t \ge x_v^*\}$, then

$$\tilde{v}(y) = E(\max(y, \sup_{0 \le t \le \tau_v^*} (bt - N_t)) - c\tau_v^*).$$

(B) For all $\tau \in \mathcal{M}_0$

$$\tilde{v}(x) \ge E(\max[x, \sup_{0 \le t \le \tau} (bt - N_t)] - c\tau),$$

We begin by (B). If $\tau \in \mathcal{M}_0$ then $E(N_\tau) = \lambda E(\tau)$ and in consequence

$$E(\max[x, \sup_{0 \le t \le \tau} (bt - N_t)] - c\tau) = E(Y_\tau - (c + \lambda - b)\tau)$$

$$\le E(\tilde{v}(Y_\tau) - (c + \lambda - b)\tau),$$

because $\tilde{v}(x) \ge x$, as the function u is convex (Lemma 2).

Then, in order to conclude the proof of (B) it is enough to see that the process $V = (V_t)_{t\geq 0}$ given by

$$V_t = \tilde{v}(Y_{t\wedge\tau}) - (c+\lambda-b)(t\wedge\tau)$$

is a supermartingale. As $V_t \geq -(c+\lambda-b)t$, by (46) it is enuogh to verify

$$\lambda\mathcal{L}\tilde{v}(x) - (c+\lambda-b) \leq 0 \quad \text{for} \quad x\geq 0. \tag{53}$$

with $\mathcal{L}$ defined in (48). Extend the definition of the function u in a continuous way, as $u(x) = 0$ if $x < 0$. Now, using Lemma 1 we obtain, for $0 \leq x < x_v^*$

$$\lambda\mathcal{L}\tilde{v}(x) = -b\mathbf{I}_{\{y>0\}}[1 - \frac{c}{\lambda}u'(x_v^* - x)] + \lambda + c[u(x_v^* - x - 1) - u(x_v^* - x)]$$

$$= -b + \lambda + c\mathcal{K}u(y)|_{y=x_v^*-x} = c+\lambda-b \leq 0.$$

When $x \geq x_v^*$, then $\tilde{v}(x) = x$ and

$$\lambda\mathcal{L}\tilde{v}(x) - (c+\lambda-b) = -c < 0,$$

and in this way the inequality (51) holds and (B) is proved.

Let us see (A). Define $V^* = (V_t^*)_{t\geq 0}$ by

$$V_t^* = \tilde{v}(Y_{t\wedge\tau_v^*}) - (c+\lambda-b)(t\wedge\tau_v^*).$$

In view of (46) and (52) V^* is a local martingale. Furthermore, this process is bounded on each interval of the form $[0,T]$, $T < +\infty$, and in consequence is a martingale. So

$$\tilde{v}(x) = E(V_t^*) = E(\tilde{v}(Y_{t\wedge\tau_v^*})) - (c+\lambda-b)E(t\wedge\tau_v^*). \tag{54}$$

As $0 \leq \tilde{v}(Y_{t\wedge\tau_v^*}) \leq \max(x, x_v^*)$, we can take limits as $t\to+\infty$ in (54) obtaining

$$\tilde{v}(x) = E(\tilde{v}(Y_{\tau_v^*} - (c+\lambda-b)\tau_v^*) = E(Y_{\tau_v^*} - (c+\lambda-b)\tau_v^*)$$

$$= E(\max[x, \sup_{0\leq t\tau_v^*}(bt - N_t) - c\tau_v^*).$$

concluding (A). From (54), we also deduce (21)

$$E(\tau_v^*) = \frac{1}{c+\lambda-b}(\tilde{v}(x_v^*) - \tilde{v}(x)) = \frac{1}{c+\lambda-b}(x_v^* - x - \frac{c}{\lambda}u(x_v^* - x)).$$

concluding the proof of Theorem 2. □

3.3. *Proof of Theorem 3.*

(i) For $T > 0$ equation (24) has an unique positive root $x_s^*(T)$ by Lemma 2. The constant $c(T)$ defined by

$$\frac{a+c-\lambda}{\lambda}u'(x_s^*(T); \frac{a}{\lambda}) = 1,$$

satisfies

$$\lambda - a < c(T) < \lambda.$$

So, by (iv) in Theorem 1 the stopping time

$$\tau_s^*(T) = \inf\{t \geq 0 \colon X_t \geq x_s^*(T)\},$$

with $X = (X_t)_{t\geq 0}$ defined by (11) is optimal for the problem (1). By the election of $c = c(T)$, and (15)

$$E(\tau_s^*(T)) = T.$$

Also, in view of (1), (11) and (24)

$$E(\sup_{0\leq t\leq \tau_s^*(T)} (N_t - at)) = s(0) + c(T)T = x_s^*(T) - (\lambda - a)T,$$

and for any $\tau \in \mathcal{M}_T$

$$E(\sup_{0\leq t\leq \tau} (N_t - at)) \leq s(0) + c(T)E(\tau) \leq x_s^*(T) - (\lambda - a)T.$$

proving (25). Finally, (26) and (27) are direct consequences of c) in Lemma 2, concluding (i).

(ii) For a positive T, define $x_v^*(T)$ as the root of the equation

$$\frac{xu'(x) - u(x)}{1 + (1 - \frac{b}{\lambda})u'(x)} = \lambda T, \tag{55}$$

where $u = u(x) = u(x; \frac{b}{\lambda})$. In order to see that the equation (55) has an unique root, we note, that for $b > \lambda$ the function

$$f(x) = \frac{xu'(x) - u(x)}{1 + (1 - \frac{b}{\lambda})u'(x)}$$

is increasing (because is the quotient of an increasing function over a decreasing one), $f(0) = 0$ and $f(x) \to +\infty$ if $u'(x) \to (\frac{b}{\lambda} - 1)^{-1}$.

If $b \leq \lambda$, L'Hôpital rule gives $\lim_{x\to+\infty} f(x) = +\infty$, and, as $f(0) = 0$, we confirm the existence of only one root, because

$$f'(x) = \frac{u''(x)[x + u(x)(1 - \frac{b}{\lambda})]}{[1 + (1 - \frac{b}{\lambda})u'(x)]^2} \geq 0.$$

Let us see that the constant $c(T)$ defined by the relation

$$\frac{c(T)}{\lambda} u'(x_v^*(T); \frac{b}{\lambda}) = 1 \tag{56}$$

satisfies $b - \lambda < c(T) < b$. As, by Lemma 2 $u'(x_v^*(T)) > u'(0) = \frac{\lambda}{b}$,

$$c = \frac{\lambda}{u'(x_v^*(T))} < b.$$

On the other side, if $b \leq \lambda$ the second inequality is inmediate. If $\lambda < b$, then by b) in Lemma 2,

$$u'(x_v^*(T)) < \lim_{x\to+\infty} u'(x) = \frac{\lambda}{b - \lambda}$$

and the second inequality follows.

Then, we are in case (iv) of Theorem 2. The stopping time

$$\tau_v^*(T) = \inf\{t \geq 0 : Y_t \geq x_v^*(T)\}$$

is optimal for the problem (2) with $c = c(T)$, $x = 0$. Then,

$$E(\tau_v^*(T)) = \frac{1}{c + \lambda - b}(x_v^*(T) - \frac{c}{\lambda}u(x_v^*)) = T.$$

Furthermore, (2), (18) and (56) gives

$$E(\sup_{0\le t\le \tau_v^*(T)}(bt-N_t)) = v(0)+c(T)E(\tau_v^*(T)) = \frac{c(T)}{\lambda}u(x_v^*(T))+c(T)T$$

$$= \frac{u(x_v^*)(T)+\lambda T}{u'(x_v^*(T))} = x_v^*(T)-(\lambda-b)T,$$

concluding the proof of (31).

To see (30), take $b=\lambda$, and denote $w(x)=xu'(x)-u(x)$. We have $x_v^*(T)=w^{-1}(T)=\psi(T)$. We know,

$$\lim_{x\to+\infty}\frac{w(x)}{x^2}=1,$$

this means

$$\frac{T}{(w^{-1}(T))^2}\to 1,$$

and taking square roots, we obtain (30) concluding the proof of Theorem 3. □

3.4. *Proof of Theorem 4.*

(i) Assume $\lambda > a$. For $H>0$, denote

$$\tau_H = \{t\ge 0 : x+N_t-at\ge H\}.$$

As in the proof of (i) in Theorem 1, we obtain $P(\tau_H<+\infty=1)$. Then

$$c(x)=\sup_{\tau\ge 0}E(x+N_\tau-a\tau)^+\ge E(x+N_{\tau_H}-a\tau_H)^+\ge H.$$

As H is arbitrary, the proof of (i) is concluded.

(ii) Take $a>\lambda$. Denote $U_t=x+N_t-at$. The process $U=(U_t)_{t\ge 0}$ has an infinitesimal operator of the form

$$\mathcal{U}f(x)=\lambda(f(x+1)-f(x))-af'(x).$$

Itô's formula in this case reads, with c in (31)

$$c(U_t)=c(x)+\int_0^t\mathcal{U}c(U_{r-})dr+M_t,\qquad t\ge 0, \tag{57}$$

with the process $M = (M_t)_{t\geq 0}$ given by

$$M_t = \int_0^t [c(U_{r-}+1) - c(U_{r-}))]d(N_r - \lambda r)$$

a local martingale. Let us now verify that for the function $c = c(x)$ defined in (31)

$$\mathcal{U}c(x) = 0 \quad \text{if} \quad x < x_c^*. \tag{58}$$

$$\mathcal{U}c(x) < 0 \quad \text{if} \quad x \geq x_c^*. \tag{59}$$

In fact, (59) is inmediate. In order to see (58), taking into account (36)

$$\mathcal{U}c(x) = \lambda(c(x+1) - c(x)) - ac'(x)$$

$$= \lambda - a + (a-\lambda)(u(x_c^*-1) - u(x_c^*-x) + \frac{a}{\lambda}u'(x_c^*-x)) = 0.$$

Now, from (57) we obtain that the stopped local martingale $M^* = \{M_{t\wedge\tau_c^*}\}_{t\geq 0}$ with τ_c^* in (32) is uniformly bounded:

$$-c(x) \leq M_{t\wedge\tau_c^*} \leq c(x_c^*) + 1,$$

so, taking expected values and limits, we obtain $E(M_{\tau_c^*}) = 0$ that means, by (31). As on the set $\{\tau_c^* = +\infty\}$ we have

$$c(x + N_{\tau_c^*} - a\tau_c^*) = (x + N_{\tau_c^*} - a\tau_c^*)^+ = 0.$$

we deduce

(A) $c(x) = E(c(x + N_{\tau_c^*} - a\tau_c^*)) = E(x + N_{\tau_c^*} - a\tau_c^*)^+$.

To complete the proof, we will see

(B) For any $\tau \in \bar{\mathcal{M}}$

$$c(x) \geq E(x + N_\tau - a\tau)^+.$$

For the process M defined by (57), as $-\mathcal{U}c(x) \geq 0$ we have

$$M_t \geq -c(x).$$

This fact, and Fatou's Lemma gives, that the process M is a supermartingale. Now, using this fact, and $c(x) \geq x^+$ we obtain

$$c(x) \geq E(c(x + N_\tau - a\tau)) \geq E(x + N_\tau - a\tau)^+,$$

concluding the proof of Theorem 4. □

3.5. *Proof of Theorem 5.*

(i) Is analogous to the proof of (i) in Theorem 4.

(ii) Take $b < \lambda$. Denote $D_t = x + bt - N_t$. The process $D = (D_t)_{t\geq 0}$ has in infinitesimal operator of the form

$$\mathcal{D}f(x) = bf'(x) - \lambda(f(x-1) - f(x)).$$

It is direct to see that

$$\begin{cases} \mathcal{D}p(x) = 0, & \text{if} \quad x < x_p^*. \\ \mathcal{D}p(x) = b - \lambda, & \text{if} \quad x > x_p^*. \end{cases} \tag{60}$$

Itô's formula in this case is

$$p(D_t) = p(x) + \int_0^t \mathcal{D}(D_{r-})dr + M_t, \quad t \geq 0, \tag{61}$$

with the process $M = (M_t)_{t\geq 0}$ given by

$$M_t = \int_0^t [p(D_{r-} - 1) - p(D_{r-}))]d(N_r - \lambda r)$$

a local martingale. Now, from (60) we obtain that the stopped local martingale $M^* = \{M_{t\wedge\tau_p^*}\}_{t\geq 0}$ with τ_p^* defined in (34) is uniformly bounded:

$$-p(x) \leq M_{t\wedge\tau_p^*} \leq p(x_p^*) = x_p^*,$$

so, taking expected values and limits, we obtain $E(M_{\tau_p^*}) = 0$. As on the set $\{\tau_p^* = +\infty\}$ we have $= p(D_{\tau_p^*}) = (D_{\tau_c^*})^+ = 0$, (61) gives

(A) $p(x) = E(p(x + b\tau_p^* - N_{\tau_p^*}) = E(x + b\tau_p^* - N_{\tau_p^*})^+$.

To complete the proof, we will see

(B) For any $\tau \in \bar{\mathcal{M}}$

$$p(x) \geq E(x + b\tau - N_\tau)^+.$$

For the process M defined by (61), as $-\mathcal{D}p(x) \geq 0$ we have

$$M_t \geq -p(x).$$

This fact, and Fatou's Lemma gives, that the process M is a supermartingale. Now, using this fact, and $p(x) \geq x^+$ we obtain

$$p(x) \geq E(p(x + b\tau - N_\tau)) \geq E(x + b\tau - N_\tau)^+,$$

concluding the proof of the Theorem □

References

[1] Shepp L.A., Shiryaev A.N. The Russian Option: Reduced regret. The Annals of Applied Probability. Vol 3 (1993) 631–640.

[2] Dubins L., Shepp L.A., Shiryaev A.N. On optimal stopping and maximal inequalities for Bessel processes. Theory of Probability and its Applications. Vol 38 (1993) 226–261.

[3] Dubins L. Schwarz G. A sharp inequality for sub-martingales and stopping times. Asterisque. Vol 157-158 (1988) 129–145.

[4] Graversen S.E., Peškir G. Optimal Stopping and Maximal Inequalities for Linear Diffusions. Journal of Theoretical Probability. Vol. 1 No.1 (1998) 259–277.

[5] Shiryaev A.N. Optimal Stopping Rules. Springer, Berlin. 1978.

[6] Liptser R. Sh., Shiryaev A.N. The Theory of Martingales Kluwer, Dodrecht. 1989.

[7] Koroliuk V.S., Portenko N.I., Skorohod A.V., Turbin A.F. Manual of the Theory of Probability and Mathematical Statistics. Nauka, Moscow. 1985, (In russian).

Procesos Empíricos Transformados y Pruebas de Kolmogorov-Smirnov con un parámetro estimado.

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Resumen

En este trabajo se define una nueva clase de procesos empíricos transformados, que modifica la que ha sido introducida en literatura por Alejandra y Enrique M. Cabaña (1997, ver [3]), con el propósito de desarrollar pruebas de bondad de ajuste del tipo Kolmogorov-Smirnov (K-S) con un parámetro estimado.

Las pruebas construidas, entre ellas una prueba de exponencialidad, son particularmente potentes frente a seleccionadas sucesiones de alternativas contiguas.

En cada uno de los problemas estudiados se presenta un análisis numérico para la comparación entre la prueba de K-S clásica y la modificada.

1 Introducción

Se estudia el problema estadístico siguiente: poner a prueba la hipótesis de que las observaciones $X_1, \ldots, X_n$ independientes provienen de una cierta distribución univariada continua - también en $\underline{\theta}$ - $F(x, \underline{\theta})$ donde $\underline{\theta} = \binom{\underline{\theta}_1}{\xi}$ es un vector de p parámetros, con $p-1$ parámetros conocidos ($\underline{\theta}_1 = \underline{\theta}_{1,0}$) y uno desconocido ξ, cuyo estimador se indica con $\hat{\xi}_n$.

Indiquemos con θ el verdadero valor desconocido del parámetro ξ y con $\underline{\hat{\theta}}_n = \binom{\underline{\theta}_{1,0}}{\hat{\xi}_n}$ un estimador adecuado de $\underline{\theta}$.

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Después de hacer la transformación básica

$$u = F(x, \hat{\underline{\theta}}_n) \tag{1}$$

toda la información muestral es proporcionada por la medida aleatoria

$$\hat{b}_n^U([0,u]) = \sqrt{n}\,[\,\hat{G}_n(u) - u\,], \quad 0 \leq u \leq 1 \tag{2}$$

conocida como *proceso empírico uniforme estimado* (o parámetrico), donde $\hat{G}_n(u) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{F(X_i;\hat{\underline{\theta}}_n) \leq u\}}$ es la proporción de valores $X_1, \ldots, X_n$ tales que $F(X_i; \hat{\underline{\theta}}_n) \leq u, \ 0 \leq u \leq 1$.

A $\hat{G}_n(u)$ se la suele denominar *función de distribución empírica uniforme estimada* asociada a las variables aleatorias $X_1, \ldots, X_n$ que tienen ley F en presencia de parámetros desconocidos.

Es bien conocido que muchos de los estadísticos de pruebas no parámetricos que se utilizan para contrastar la hipótesis nula compuesta (parámetrica)

$$H_0 : F \in \mathcal{F}_0 = \{F(x; \underline{\theta}),\ \underline{\theta} \in \Theta^p\} \tag{3}$$

con $H_1 : F \notin \mathcal{F}_0$ se pueden definir como funcionales de $\hat{b}_n^U$. Por ejemplo, el estadístico (clásico) de Kolmogorov-Smirnov está dado por

$$\sup_{0 \leq u \leq 1} \mid \hat{b}_n^U(u) \mid . \tag{4}$$

En este artículo se construirán pruebas asintóticas de K-S modificadas en el sentido que se empleará en (4) una clase de procesos estocásticos apropiadamente elegidos (definida en la sección siguiente) en lugar del proceso empírico uniforme estimado usual.

A continuación se mencionan algunos resultados importantes sobre el comportamiento asintótico de la sucesión de elementos aleatorios $\{\hat{b}_n^U(u)\}$ que, junto a otros importantes resultados de estadística asintótica que se introducirán en la próxima sección, permitirán conocer el comportamiento asintótico de nuestra clase de procesos.

En 1955 Darling ([5]) demostró que bajo H_0 la ley conjunta del vector $(\hat{b}_n^U(u_1), \ldots, \hat{b}_n^U(u_k))$ converge a la de un vector Gaussiano $(\hat{b}^U(u_1), \ldots, \hat{b}^U(u_k))$ para cada conjunto finito $(u_1,\, u_2,\, \ldots, u_k)$.

En el caso de un estimador $\hat{\xi}_n$ de ξ que satisface las bien conocidas *condiciones de Cramér* ([4], pag.477-489) Darling obtuvo que el proceso Gaussiano centrado limite de la sucesión $\{\hat{b}_n^U(u)\}$ admite la representación siguiente:

$$\hat{b}^U(u) = b(u) + \psi(u) \int_0^1 \psi''(s)\, b(s)\, ds \tag{5}$$

donde b es un *puente browniano típico,* $\psi(u) = c\, g(u)$, c una cierta constante de normalización y

$$g(u) = \left.\frac{\partial F(x,\theta)}{\partial \theta}\right|_{x=F^{-1}(u,\theta)},\ 0 \leq u \leq 1\,.$$

En general, $g(u)$ depende de θ.

Durbin ([7], 1973) ha estudiado la *convergencia débil* de la sucesión $\{\hat{b}_n^U(u)\}$ bajo la sucesión de alternativas siguiente:

$$H_{1,n}(\delta)\ : \underline{\theta} = \underline{\theta}_n = \begin{pmatrix} \underline{\theta}_{1,0} + \frac{\delta}{\sqrt{n}} \\ \theta \end{pmatrix} \tag{6}$$

donde θ es un valor fijo y verdadero de ξ y δ un valor asignado que sirve para distinguir entre las sucesiones de alternativas.

Las condiciones introducidas por Durbin (para los detalles consultar [7], pag.280-281) que aseguran la convergencia débil de la sucesión $\hat{b}_n^U$ bajo la sucesión de alternativas (6) son de dos tipos: sobre la sucesión $\hat{\xi}_n$ de estimadores de ξ y sobre la ley F.

En particular se supone que:

$$\sqrt{n}(\hat{\xi}_n - \theta) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} l(X_i\,, \underline{\theta}_n) + \underline{A}\ \delta + \epsilon_{1,n} \tag{7}$$

donde l es una función medible, A una matriz dada de orden $(p-1) \times 1$, y $\epsilon_{1,n} \to_p 0$.

Sea $L = \mathbf{E}\{l(X\,, \underline{\theta}_0)\, l(X\,, \underline{\theta}_0) \mid \underline{\theta} = \underline{\theta}_0\}$. Vale el teorema siguiente:

Teorema 1.1 (Durbin, Convergencia débil de $\{\hat{b}_n^U\}$)

Bajo la sucesión de alternativas (6), $\hat{b}_n^U$ *converge débilmente en el espacio* $D[0,1]$ *al proceso Gaussiano* $\hat{b}^U = \{\hat{b}^U(u)\,,\ 0 \leq u \leq 1\}$ *con esperanza*

$$\mathbf{E}\hat{b}^U(u) = \delta(g_1(u) - \underline{A}^t g_2(u))\,, \tag{8}$$

y función de covariancia

$$\mathbf{Cov}\{\hat{b}^U(u)\,,\hat{b}^U(v)\} = (u \wedge v) - u\,v - h(u)\,g_2(v) - h(v)\,g_2(u) + g_2(u)\,L\,g_2(v) \tag{9}$$

donde

$$\begin{pmatrix} g_1(u) \\ g_2(u) \end{pmatrix} = g(u) = g(u, \underline{\theta}_0)\,,\ con\ g(u,\underline{\theta}) = \left.\frac{\partial F(x,\underline{\theta})}{\partial \underline{\theta}}\right|_{x=F^{-1}(u,\underline{\theta})} \tag{10}$$

y

$$h(u) = h(u, \underline{\theta}_0)\,,\ con\ h(u,\underline{\theta}) = \int_{-\infty}^{F^{-1}(u,\underline{\theta})} l(x\,,\underline{\theta})\,dF(x,\,\underline{\theta})\,. \tag{11}$$

Se omite la demostración del teorema - basada sobre las técnicas de convergencia débil excelentemente expuestas en el libro de Billingsley ([1]) - que está muy detallada en ([7]).

Se observa que el teorema de Durbin muestra claramente cómo hallar las funciones h,g_1,g_2, una vez que se haya elegido el estimador del parámetro incógnito ξ que aparece en la ley F y consecuentemente se haya encontrado la función l.

El proceso asintótico $\hat{b}^U$ no es de distribución libre; sin embargo, cuando F pertenece a una familia de posición y/o escala $\hat{b}^U$ no depende del parámetro incógnito ξ.

Es importante destacar que para cada problema de bondad de ajuste en presencia de un solo parámetro estimado la función de covariancia del proceso $\hat{b}^U$ tiene la forma

$$\mathbf{Cov}\{\hat{b}^U(u)\,,\hat{b}^U(v)\} = (u \wedge v) - u\,v - B(u)\,B(v)\,. \tag{12}$$

Como consecuencia se obtiene que el proceso Gaussiano $\hat{b}^U$ con función de covariancia (12) admite bajo H_0 la representación siguiente en términos de un *movimiento browniano típico* w:

$$\hat{b}^U(u) = w(u) - u\,w(1) - B(u)\int_0^1 B'(s)dw(s)\,,\ 0 \leq u \leq 1\,. \tag{13}$$

Entonces, una vez halladas las funciones h, g_1, g_2, es inmediato conseguir una representación cómoda de $\hat{b}^U$ en términos de w del tipo (13).

Se observa que cuando $\int_0^1 B'(s)ds = 0$ se obtiene, integrando por partes,

$$\int_0^1 B'(s)dw(s) = \int_0^1 B'(s)db(s) = -\int_0^1 B''(s)\,b(s)\,ds$$

y, por lo tanto, (13) coincide con la representación (5) de Darling.

2 Una nueva clase de procesos empíricos transformados

Se introduce una clase de procesos empíricos transformados que permite construir estadísticos de prueba del tipo Kolmogorov-Smirnov consistentes contra cualquier alternativa y especialmente potentes frente a las sucesiones de alternativas contiguas del tipo (6) que el experimentador considera de interés para la investigación.

De esta manera se adapta al caso de hipótesis compuestas en presencia de un parámetro estimado la teoría desarrollada por Alejandra y Enrique M. Cabaña para hipótesis simples ([2],[3]).

Definición 2.1 (Proceso Empírico Transformado estimado)

Sea $\mathcal{T}$ una isometría de $L^2([0,1], dt)$ con recorrido ortogonal a la función $\mathbf{1}$ y $a \in L^2([0,1], dt)$ una función que llamaremos función de graduación. Se considera:

$$\hat{w}_n^{(a,\mathcal{T})}([0,u]) = \hat{w}_n^{(a,\mathcal{T})}(u) = \int_0^1 \mathcal{T}(a\,\mathbb{1}_{([0,u])})(s)\,d\hat{b}_n^U(s),\ 0 \le u \le 1 \tag{14}$$

donde $\hat{b}_n^U$ es el proceso empírico uniforme estimado definido en (2) .

La medida aleatoria (14) está expresada como integral estocástica respecto de la medida inducida por las trayectorias del proceso $\hat{b}_n^U$. Claramente:

$$\hat{w}_n^{(a,\mathcal{T})}([0,u]) = \frac{1}{\sqrt{n}}\sum_{i=1.}^{n} \hat{w}^{(a,\,\mathcal{T})}_{F(X_i;\,\underline{\hat{\theta}}_n)}([0,u]) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \mathcal{T}(a\,\mathbb{1}_{([0,u])})(F(X_i;\,\underline{\hat{\theta}}_n)). \tag{15}$$

Respecto de los procesos empíricos usuales aparecen dos nuevos parámetros: una función a y una isometría $\mathcal{T}$.

Una aplicación del Teorema (1.1) permite concluir que la distribución asintótica de (14) es la de

$$\hat{w}^{(a,\mathcal{T})}(u) = \int_0^1 \mathcal{T}(a\,1\!\!1_{([0,u])})(s)\,d\hat{b}^U(s),\; 0 \leq u \leq 1\,. \tag{16}$$

Dada la ley límite de (14) bajo H_0, una aplicación del teorema de Oosterhoff y van Zwet (ver [9], [2] pag.23-30) que caracteriza las condiciones bajo las cuales hay contigüidad entre las hipótesis nula y alternativa y luego el Tercer Lema de Le Cam ([8], [2] pag.14-15) permiten establecer la ley límite de (14) bajo $H_{1\,n}$.

La función a contiene toda la información asintótica que proviene de la contigüidad existente entre las hipótesis nula y alternativa y será determinada de manera tal que la distancia asintótica entre el proceso límite bajo H_0 y bajo $H_{1\,n}$ sea la más grande posible.

Para la determinación de a se utilizará un criterio heurístico que, en general, depende del tipo de estadístico de prueba elegido.

La isometría $\mathcal{T}$ será elegida de manera conveniente desde el punto de vista analítico.

En las próximas tres secciones se presentarán tres pruebas de ajuste distintas, de tipo K-S, y para cada una de ellas se obtendrán las funciónes de graduación óptimas y las representaciones explícitas del proceso empírico (14) y del límite débil (16) de la sucesión (14) correspondientes a la isometría de Laguerre para una ley uniforme en $[0, 1]$ (ver sección 3, (33)). Un análisis númerico para la comparación de las potencias de nuestros estadísticos K-S modificados y de los estadistícos K-S clásicos concluirá cada aplicación.

3 Caso de observaciones que provienen de una población normal con media desconocida y variancia conocida

En esta sección se muestra cómo construir una prueba K-S modificada para la hipótesis compuesta (caso 1):

$$H_0 : F \in \mathcal{F}_0 = \left\{ \Phi \left(x; \begin{pmatrix} \mu_0 \\ 1 \end{pmatrix} \right), \ \mu_0 \in I\!R \right\}$$

particularmente potente cuando se considera la siguiente sucesión de alternativas contiguas (un cambio de variancia):

$$H_{1\,n}(\delta) : F \in \mathcal{F}_n = \left\{ \Phi \left(x; \begin{pmatrix} \mu_0 \\ 1 + \frac{\delta\, c}{\sqrt{n}} \end{pmatrix} \right), \ \mu_0 \in I\!R \right\} . \tag{17}$$

Un primer paso hacia la construcción de nuestro proceso empírico transformado es aplicar el corolario del teorema de Oosterhoff y Van Zwet (para su enunciado y demostración en el caso de hipótesis simples ver [2], pag.28-30).

Dado que aparece el parámetro incógnito μ_0 se deben cumplir *para cada valor de* μ_0 las condiciones siguientes:

- $\Phi\,(x;\ (\mu_0,\ 1)')$ y $\Phi\left(x;\ (\mu_0,\ 1 + \frac{\delta\, c}{\sqrt{n}})'\right)$ tienen, respectivamente, densidades $f_0(x)$ y $f_n(x)$ respecto de la medida de Lebesgue en $I\!R$;
- las funciones $k_n(x) = \frac{2\sqrt{n}}{\delta} \left(\sqrt{\frac{f_n(x)}{f_0(x)}} - 1 \right)$ son uniformemente acotadas por una función K tal que $\int K^2\, d\Phi\,(x;\ (\mu_0,\ 1)') < \infty$;
- existe una función k que cumple $\int k^2\, d\Phi\,(x;\ (\mu_0,\ 1)') = 1$ y tal que $\lim_{n\to\infty} \int (K_n - k)^2\, d\Phi\,(x;\ (\mu_0,\ 1)') = 0$

Cuando estas hipótesis son satisfechas hay contigüidad entre las sucesiones de medidas correspondientes a las hipótesis nulas y alternativas para cada valor del parámetro incógnito μ_0.

Se sobrentienderá este tipo de observación en las secciones siguientes.

Después de verificar las hipótesis del corolario se necesita identificar el límite de la sucesión k_n de funciones de $L^2\,[I\!R,\ d\Phi\,(x;\ (\mu_0,\ 1)')]$.

Claramente

$$\sqrt{\frac{f_n(x)}{f_0(x)}} = \left(\frac{1}{1+(\delta\, c)/(\sqrt{n})} \right)^{\frac{1}{4}} exp \left\{ \frac{1}{4} \, (1 - \frac{1}{1+(\delta\, c)/(\sqrt{n})})\, (x - \mu_0)^2 \right\} ,$$

y se obtiene

$$\lim_{n\to\infty} k_n(x) = \tfrac{c}{2}\left[(x-\mu_0)^2 - 1\right] = k(x,\mu_0)\,.$$

Se evalua la constante de normalización c de manera tal que

$$\| \; k(x,\mu_0) \; \|^2_{L^2\left[\mathbb{R},d\Phi(x;\,(\mu_0,\,1)')\right]} = 1\,. \tag{18}$$

Se obtiene $c = \sqrt{2}$ y por lo tanto la función k que satisface el requerimiento (18) es

$$k(x,\mu_0) = \frac{1}{\sqrt{2}}\left[(x-\mu_0)^2 - 1\right]\,. \tag{19}$$

Para que k no dependa de μ_0 se aplica la transformación

$$u = \Phi\left(x; \begin{pmatrix} \mu_0 \\ 1 \end{pmatrix}\right) = \Phi(x-\mu_0)\,,$$

y está claro que

$$x = \Phi^{-1}(u) + \mu_0 = \Phi^{-1}\left(u; \begin{pmatrix} \mu_0 \\ 1 \end{pmatrix}\right)\,.$$

Finalmente

$$k(x,\mu_0) = k(\Phi^{-1}(u)+\mu_0,\mu_0) = \frac{1}{\sqrt{2}}\left[\left(\Phi^{-1}(u)\right)^2 - 1\right] = \kappa(u)\,,\ 0 \le u \le 1 \tag{20}$$

y la función κ es la que resume toda la información proveniente de la contigüidad.

Ahora se desea aplicar el teorema de Durbin enunciado en la sección anterior. De esta forma se obtendrá una representación cómoda del proceso asintótico uniforme estimado $\hat{b}^U$ en términos del movimiento Browniano típico w.

Se utiliza $\hat{\mu}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ como estimador del parámetro incógnito μ_0, entonces $\sqrt{n}(\hat{\mu}_n - \mu_0) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(X_i - \mu_0)$, o sea $l(X_i,\mu_0) = X_i - \mu_0$ y resulta además $L = 1,\ A = 0,\ \epsilon_{1\,n} = 0$.

Se concluye que bajo $H_{1\,n}$ la sucesión de elementos aleatorios

$$\left\{ \hat{b}^U_n(u) = \sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n} 1\!\!1_{\{\Phi(X_i;\,(\hat{\mu}_n,\,1)')\le u\}} - u\right]\ ,0 \le u \le 1 \right\} \tag{21}$$

converge débilmente en $D[0,1]$ al elemento aleatorio $\hat{b}^U$, donde

$$\hat{b}^U = \{\hat{b}^U(u),\ 0 \le u \le 1\}$$

es un proceso Gaussiano para el que se saben hallar la esperanza y la función de covariancia que dependen de las funciones g_1, g_2, h.

Se observa que las variables aleatorias $\left\{\Phi\left(X_i; \begin{pmatrix} \hat{\mu}_n \\ 1 \end{pmatrix}\right), 1 \le i \le n\right\}$ - no independientes - están idénticamente distribuidas (no uniformemente en [0,1]) según una ley que no depende del parámetro desconocido μ_0 cuando se utiliza un estimador que cumple las condiciones usuales para un estimador de posición (ver [6], donde se determina la distribución exacta de estas variables).

Como consecuencia la distribución de $\hat{b}_n^U$ es independiente de μ_0.

Cálculo de g_1: $g_1(u) = \left[\frac{\partial}{\partial \sigma^2}\left[\Phi\left(x;\ (\mu_0,\ \sigma^2)'\right)\right]_{\sigma^2=1}\right]\Bigg|_{x=\Phi^{-1}(u)+\mu_0}$

$= \frac{1}{2}\int_{-\infty}^{x} \phi(w-\mu_0)\left[(w-\mu_0)^2 - 1\right] dw\Bigg|_{x=\Phi^{-1}(u)+\mu_0} = -\frac{1}{2}\,\phi(\Phi^{-1}(u))\,\Phi^{-1}(u).$

Por lo tanto el valor esperado del proceso $\hat{b}^U$ bajo H_{1n} está dado por:

$$\mathbf{E}\hat{b}^U(u) = -\frac{\delta\, c}{2}\,\phi(\Phi^{-1}(u))\,\Phi^{-1}(u)\,,\ 0 \le u \le 1\,. \tag{22}$$

Para hallar la covariancia del proceso $\hat{b}^U$ se calculan:

- $h(u) = \int_{-\infty}^{x}(w-\mu_0)\,\phi(w-\mu_0)\,dw\Bigg|_{x=\Phi^{-1}(u)+\mu_0}$

$= \int_{-\infty}^{\Phi^{-1}(u)} w\phi(w)\,dw = -\phi(\Phi^{-1}(u))\,;$

- $g_2(u) = \left[\frac{\partial}{\partial \mu_0}\Phi\left(x;\ (\mu_0,\ 1)'\right)\right]_{x=\Phi^{-1}(u)+\mu_0}$

$= \int_{-\infty}^{x}(w-\mu_0)\,\phi(w-\mu_0)\,dw\Bigg|_{x=\Phi^{-1}(u)+\mu_0} = -\phi(\Phi^{-1}(u))\,.$

Resumiendo, en el caso 1:

$$h(u) = -\phi(\Phi^{-1}(u))\,, \tag{23}$$

$$g_1(u) = -\frac{1}{2}\,\phi(\Phi^{-1}(u))\,\Phi^{-1}(u)\,, \tag{24}$$

$$g_2(u) = -\phi(\Phi^{-1}(u))\,. \tag{25}$$

La función de covariancia del proceso $\hat{b}^U$ bajo $H_{1\,n}$ (igual a la covariancia bajo H_0 por el Tercer Lema de Le Cam) es:

$$\mathbf{Cov}\{\hat{b}^U(u), \hat{b}^U(v)\} = (u \wedge v) - u\,v - \phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))\,. \tag{26}$$

Entonces la representación de $\hat{b}^U$ bajo $H_{1\,n}$ en términos de w cuando $0 \leq u \leq 1$ es:

$$\hat{b}^U(u) = w(u) - uw(1) + \phi(\Phi^{-1}(u)) \int_0^1 \Phi^{-1}(s)\,dw(s) - \frac{\delta\,c}{2}\,\phi(\Phi^{-1}(u))\,\Phi^{-1}(u)\,. \tag{27}$$

Escribimos ahora el proceso asintótico transformado estimado bajo $H_{1\,n}$ que en la sección anterior fué definido como la integral estocástica respecto del proceso asintótico uniforme estimado (27):

$$\begin{aligned}
\hat{w}^{(a,\mathcal{T})}(u) &= \int_0^1 \mathcal{T}(a\,\mathbb{1}_{([0,u])})(s)\,d\hat{b}^U(s) \\
&= \int_0^1 \left(\mathcal{T}(a\,\mathbb{1}_{([0,u])})(s) + \Phi^{-1}(s)\int_0^u a(r)\mathcal{T}^{-1}(-\Phi^{-1}(r))\,dr\right) dw(s) \\
&\quad -\frac{\delta}{\sqrt{2}}\int_0^1 \mathcal{T}(a\,\mathbb{1}_{([0,u])})(s)\,[1-((\Phi^{-1}(s))^2]\,.ds
\end{aligned} \tag{28}$$

Para hallar la función de graduación a óptima para este problema se utilizará un criterio heurístico cuya aplicación requiere el cálculo previo del valor esperado del proceso $\hat{w}^{(a,\mathcal{T})}$ en (28) bajo $H_{1\,n}$ y de su variancia bajo H_0. Bajo $H_{1\,n}$ se obtiene:

$$\mathbf{E}\hat{w}^{(a,\mathcal{T})}(u) = \delta\,\langle\,\mathcal{T}(a\,\mathbb{1}_{([0,u])})\,,\,\kappa\,\rangle_{L^2([0,1],ds)} = \delta\,\langle\,a\,\mathbb{1}_{([0,u])}\,,\,\mathcal{T}^{-1}\kappa\,\rangle\,, \tag{29}$$

con κ definida en (20), y bajo H_0:

$$\mathbf{Var}\hat{w}^{(a,\mathcal{T})}(u) = \|\,a\,\mathbb{1}_{([0,u])}\,\|^2 - \langle\,a\,\mathbb{1}_{([0,u])}\,,\,\mathcal{T}^{-1}\Phi^{-1}\rangle^2\,. \tag{30}$$

Se propone emplear el criterio heurístico siguiente:

$$max \frac{\langle a\, \mathbb{1}_{([0,u])}\,,\, \mathcal{T}^{-1}\kappa \rangle^2}{\| a\, \mathbb{1}_{([0,u])} \|^2 - \langle a\, \mathbb{1}_{([0,u])}\,,\, \mathcal{T}^{-1}\Phi^{-1} \rangle^2} \tag{31}$$

donde el máximo se considera cuando $a \in L^2([0,1], dt)$ y $u \in [0,1]$.

Se observa que en el denominador de (31) aparece la función Φ^{-1} - ortogonal a la función κ -, que se puede ver como el primer polinomio de Hermite $H_1(x) = x$ en la variable Φ^{-1}. Unas consideraciones geométricas permiten establecer que la función de graduación óptima $\hat{a}$ que resuelve el problema (31) es:

$$\hat{a} = \alpha_1\, \mathcal{T}^{-1}\kappa + \alpha_2\, \mathcal{T}^{-1}\Phi^{-1}\,. \tag{32}$$

Se elige de manera conveniente la solución particular $\hat{a} = \mathcal{T}^{-1}\kappa$ que coincide con la solución hallada en ([2],[3]) en un marco de pruebas de hipótesis simples. De esta forma queda determinado el parámetro funcional del proceso en (28).

Ahora se elige una isometría apropiada en $L^2([0,1], ds)$. Una conveniencia analítica sugiere emplear (al igual que en [2], [3]) la isometría de Laguerre correspondiente a la distribución de una ley uniforme en $[0,1]$:

$$\left(\mathcal{T}_{L,Uni[0,1]}\, g\right)(u) = g(u) - \int_0^u \frac{g(t)}{1-t} dt\,, \tag{33}$$

cuya inversa es

$$\left(\mathcal{T}^{-1}_{L,Uni[0,1]}\, h\right)(u) = h(u) + \frac{1}{1-u}\int_0^u h(t) dt\,. \tag{34}$$

Por lo tanto

$$\begin{aligned}
\hat{a}(u) = (\mathcal{T}_{\mathrm{L}}^{-1}\,\kappa)(u) &= \frac{1}{\sqrt{2}}\left[\left(\Phi^{-1}(u)\right)^2 - 1\right] \\
&+ \frac{1}{1-u}\int_0^u \frac{1}{\sqrt{2}}\left[\left(\Phi^{-1}(s)\right)^2 - 1\right] ds \\
&= \frac{1}{\sqrt{2}}\left[\left(\Phi^{-1}(u)\right)^2 - 1 - \frac{\Phi^{-1}(u)\,\phi(\Phi^{-1}(u))}{1-u}\right]\,.
\end{aligned} \tag{35}$$

Para escribir el proceso asintótico transformado bajo H_0 asociado a la $\hat{a}$ recién evaluada y a la isometría $\mathcal{T}_{\mathrm{L}}$ se debe calcular:

$$\begin{aligned}
\mathcal{T}_{\mathrm{L}}(\hat{a}\,1\!\!1_{([0,u])})(s) &= (\hat{a}\,1\!\!1_{([0,u])})(s) - \int_0^s \frac{(a\,1\!\!1_{([0,u])})(v)}{1-v}\,dv \\
&= \hat{a}(s)\,1\!\!1_{\{s\le u\}} - \int_0^{s\wedge u} \frac{\hat{a}(v)}{1-v}\,dv \\
&= \hat{a}(s)\,1\!\!1_{\{s\le u\}} + \frac{1}{\sqrt{2}}\,\frac{\Phi^{-1}(s\wedge u)\,\phi(\Phi^{-1}(s\wedge u))}{1-s\wedge u} \\
&= \frac{1}{\sqrt{2}}\left[\left(\Phi^{-1}(s)\right)^2 - 1\right] 1\!\!1_{\{s\le u\}} + \frac{1}{\sqrt{2}}\frac{\Phi^{-1}(u)\,\phi(\Phi^{-1}(u))}{1-u} 1\!\!1_{\{s>u\}}\,. \qquad (36)
\end{aligned}$$

Finalmente, reemplazando la fórmula (36) en (28) se obtiene:

$$\hat{w}^{(\hat{a},\mathcal{T}_{\mathrm{L}})}(u) = \int_0^1 \mathcal{T}_{\mathrm{L}}(\hat{a}\,1\!\!1_{([0,u])})(s)\,\left(dw(s) - \left(\int_0^1 \Phi^{-1}(r)\,dw(r)\right)\Phi^{-1}(s)\,ds\right)$$

$$\begin{aligned}
&= \int_0^1 \frac{1}{\sqrt{2}}\left[\left(\Phi^{-1}(s)\right)^2 - 1\right] 1\!\!1_{\{s\le u\}}\,dw(s) \\
&\quad + \int_0^1 \frac{1}{\sqrt{2}}\frac{\Phi^{-1}(u)\,\phi(\Phi^{-1}(u))}{1-u} 1\!\!1_{\{s>u\}}\,dw(s) \\
&\quad - \left(\int_0^1 \Phi^{-1}(r)\,dw(r)\right)\int_0^1 \frac{1}{\sqrt{2}}\left[\left(\Phi^{-1}(s)\right)^2 - 1\right] 1\!\!1_{\{s\le u\}}\Phi^{-1}(s)ds \\
&\quad - \left(\int_0^1 \Phi^{-1}(r)\,dw(r)\right)\int_0^1 \frac{1}{\sqrt{2}}\frac{\Phi^{-1}(u)\,\phi(\Phi^{-1}(u))}{1-u} 1\!\!1_{\{s>u\}}\Phi^{-1}(s)ds\,. \qquad (37)
\end{aligned}$$

Esta fórmula es muy cómoda para la obtención de las trayectorias de $\hat{w}^{(\hat{a},\mathcal{T}_{\mathrm{L}})}$ con un método de Monte-Carlo porque en definitiva se trata de engendrar las trayectorias de w. De esta manera se pueden generar realizaciones de la variable aleatoria

$$\sup_{0\le u\le 1} \mid \hat{w}_n^{(\hat{a},\mathcal{T}_{\mathrm{L}})}([0,u]) \mid \qquad (38)$$

de la cual parece muy difícil hallar la distribución de probabilidad exacta.

Dado un nivel de significación asintótico α se puede determinar la región crítica asintótica aproximada de la prueba de K-S modificada, o sea el valor $c(\alpha)$ tal que

$$\mathbf{P}\{\sup_{0\leq u\leq 1} \mid \hat{w}^{(\hat{a},\mathcal{T}_{\mathrm{L}})} \mid \geq c(\alpha)\} = \alpha\,.$$

La comparación de las dos pruebas puede plantearse sobre sus potencias aproximadas para distintos valores de δ y para un tamaño muestral n dado.

El proceso empírico transformado estimado asociado a las variables aleatorias $X_1, \ldots, X_n$, a la función de graduación óptima $\hat{a}$ en (35) y a la isometría $\mathcal{T}_{\mathrm{L}}$ puede escribirse como:

$$\begin{aligned}
\hat{w}_n^{(\hat{a},\mathcal{T}_{\mathrm{L}})}([0,u]) &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \hat{w}_{\Phi(X_i-\hat{\mu}_n)}^{(\hat{a},\mathcal{T}_{\mathrm{L}})}([0,u]) \\
&= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathcal{T}_{\mathrm{L}}(\hat{a}\, \mathbb{1}_{([0,u])})(\Phi(X_i - \hat{\mu}_n)) \\
&= \text{(en virtud de la ecuación (36))} \\
&\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ \frac{1}{\sqrt{2}} \left[\left(\Phi^{-1}(\Phi(X_i - \hat{\mu}_n)) \right)^2 - 1 \right] \mathbb{1}_{\{\Phi(X_i-\hat{\mu}_n)\leq u\}} \right. \\
&\left. + \frac{1}{\sqrt{2}} \frac{\Phi^{-1}(u)\, \phi(\Phi^{-1}(u))}{1-u} \mathbb{1}_{\{\Phi(X_i-\hat{\mu}_n)>u\}} \right\} \\
&= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ \frac{1}{\sqrt{2}} \left[(X_i - \hat{\mu}_n)^2 - 1 \right] \mathbb{1}_{\{\Phi(X_i-\hat{\mu}_n)\leq u\}} \right. \\
&\left. + \frac{1}{\sqrt{2}} \frac{\Phi^{-1}(u)\, \phi(\Phi^{-1}(u))}{1-u} \mathbb{1}_{\{\Phi(X_i-\hat{\mu}_n)>u\}} \right\}, \quad 0 \leq u \leq 1\,.
\end{aligned} \tag{39}$$

Por supuesto es necesario generar realizaciones de las variables aleatorias X_i bajo $H_{1\,n}$ con distintos valores del parámetro δ para disponer de las trayectorias del proceso empírico transformado estimado bajo distintas sucesiones de alternativas contiguas.

La región crítica asintótica de la prueba de K-S modificada ha sido determinada con 300.000 trayectorias del proceso $\hat{w}^{(\hat{a},\mathcal{T}_{\mathrm{L}})}$. El valor $c(\alpha)$ así obtenido después ha sido utilizado cuando se han efectuado 5.000 simulaciones del proceso empírico transformado con $n = 100$ y diferentes valores de δ de 0 a 5 con paso 0.25.

Este mismo procedimiento ha sido realizado con el proceso asintótico uniforme estimado bajo H_0 (ver fórmula (27) con $\delta = 0$) y con el proceso empírico uniforme estimado cuya expresión es la siguiente:

$$\hat{b}_n^U(u) = \sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{\Phi(X_i-\hat{\mu}_n)\leq u\}} - u\right], 0 \leq u \leq 1. \tag{40}$$

Se detallan a continuación los resultados numéricos obtenidos: en la prueba de K-S clásica $c(0.05) = 0.9437$, mientras que en la prueba de K-S modificada $c(0.05) = 2.1851$.

El estudio de la potencia de las dos pruebas mencionadas (con $n = 100$) ha sido comparado con la prueba unilateral óptima (que no es consistente frente a cualquier alternativa) del cociente de verosimilitudes. Se observa el buen comportamiento de la prueba modificada (bilateral) en comparación a las otras dos pruebas.

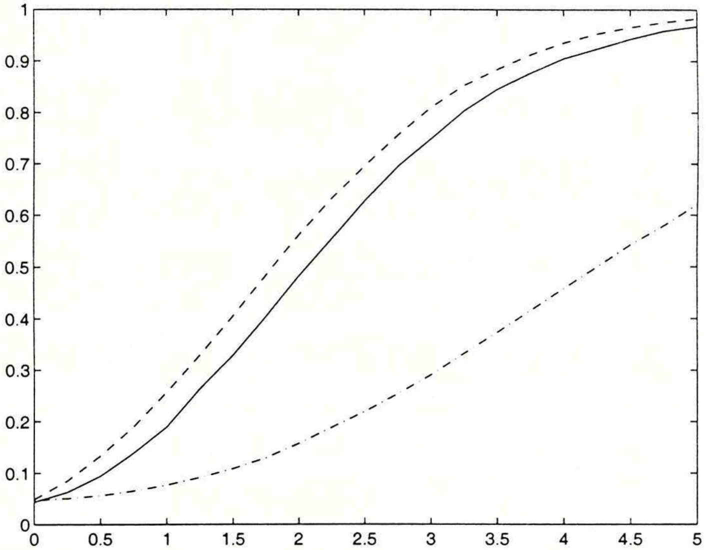

Figura 1

Curvas de potencia aproximada para la prueba K-S clásica (- .), la prueba K-S modificada (-) y la prueba del cociente de verosimilitudes para distintos valores de δ.

4 Caso de observaciones que provienen de una población normal con media conocida y variancia desconocida

Se construye un estadístico de prueba del tipo Kolmogorov-Smirnov para la hipótesis compuesta (caso 2):

$$H_0 : F \in \mathcal{F}_0 = \left\{ \Phi \left(x; \begin{pmatrix} 0 \\ \sigma_0^2 \end{pmatrix} \right), \ \sigma_0 \in I\!R^+ \right\}$$

que pueda detectar de manera especialmente eficiente un apartamiento de la hipótesis nula en la dirección de un cambio de posición:

$$H_{1\,n}(\delta) : F \in \mathcal{F}_n = \left\{ \Phi \left(x; \begin{pmatrix} \frac{\delta\, c}{\sqrt{n}}\, \sigma_0 \\ \sigma_0^2 \end{pmatrix} \right), \ \sigma_0 \in I\!R^+ \right\} . \qquad (41)$$

El caso 2 no contiene particulares novedades respecto del caso 1; sin embargo se destaca que la sucesión de alternativas se debe construir de modo de respetar la propiedad de invariancia frente a cambios de posición y/o dispersión.

Por ejemplo, en este caso no es correcto expresar la sucesión de alternativas contiguas como:

$$H_{1\,n}(\delta) : F \in \mathcal{F}_n = \left\{ \Phi \left(x; \begin{pmatrix} \frac{\delta\, c}{\sqrt{n}} \\ \sigma_0^2 \end{pmatrix} \right), \ \sigma_0 \in I\!R^+ \right\} ,$$

pues entonces, si también se llega a la misma función $k(x, \sigma_0)$ que depende de las alternativas (41), ocurre que la constante de normalización c depende del parámetro incógnito, un hecho no deseado.

Como en el caso 1, hay contigüidad entre las medidas de probabilidades que definen las hipotesis H_0 y $H_{1\,n}$ aplicando el resultado de Oosterhoff y van Zwet. En este caso

$$\frac{f_n(x)}{f_0(x)} = \left[exp \left\{ -\frac{1}{4}\, \frac{\delta^2\, c^2}{n} + \frac{1}{2\,\sigma_0}\, \frac{\delta\, c}{\sqrt{n}} \right\} \right]^2$$

y

$$\lim_{n\to\infty} k_n(x) = \lim_{n\to\infty} \frac{2\sqrt{n}}{\delta}\left(\sqrt{\frac{f_n(x)}{f_0(x)}}-1\right) = \frac{c}{\sigma_0}\, x = k(x,\sigma_0)\,. \tag{42}$$

De la condición

$$\| \, k(x,\sigma_0) \, \|^2_{L^2[I\!R,\, d\Phi(x;(0,\,\sigma_0^2)')]} = 1\,. \tag{43}$$

se desprende que $c = 1$ y la función k es $k(x,\sigma_0) = \frac{x}{\sigma_0}$.

Aplicando la transformación "canónica" para este problema

$$u = \Phi\left(x; \begin{pmatrix} 0 \\ \sigma_0^2 \end{pmatrix}\right) = \Phi\left(\frac{x}{\sigma_0}\right), \tag{44}$$

se obtiene

$$k(x,\sigma_0) = k(\sigma_0\,\Phi^{-1}(u),\sigma_0) = \Phi^{-1}(u) = \kappa(u) \quad 0 \le u \le 1\,. \tag{45}$$

Como estimador de σ_0^2 se utiliza $\hat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^n X_i^2$. Entonces $\sqrt{n}(\hat{\sigma}_n^2 - \sigma_0^2) = \frac{1}{\sqrt{n}}\sum_{i=1}^n (X_i^2 - \sigma_0^2)$ y $l(X_i,\sigma_0) = X_i^2 - \sigma_0^2$, $L = 2\,\sigma_0^4$,$A = 0$,$\epsilon_{1\,n} = 0$.

El teorema de Durbin asegura que la sucesión de elementos aleatorios

$$\left\{ \hat{b}_n^U(u) = \sqrt{n}\left[\frac{1}{n}\sum_{i=1}^n 1\!\!1_{\{\Phi(X_i;\,(0,\,\hat{\sigma}_n^2)')\le u\}} - u\right], 0 \le u \le 1 \right\} \tag{46}$$

converge débilmente en D[0,1] al proceso Gaussiano $\hat{b}^U = \{\hat{b}^U(u),\ 0 \le u \le 1\}$ con esperanza

$$\mathbf{E}\hat{b}(u) = \delta\, g_1(u)$$

y función de covariancia

$$\mathbf{Cov}\{\hat{b}^U(u), \hat{b}^U(v)\} = (u \wedge v) - u\,v - h(u)g_2(v) - h(v)g_2(u) + g_2(u)\,2\sigma_0^4\,g_2(v)\,.$$

Es necesario hallar las funciones h, g_1, g_2 y se encuentra que en el caso 2:

$$h(u) = -\Phi^{-1}(u)\,\phi(\Phi^{-1}(u))\,\sigma_0^2\,, \tag{47}$$

$$g_1(u) = -\phi(\Phi^{-1}(u))\,, \tag{48}$$

$$g_2(u) = -\frac{1}{2\,\sigma_0^2}\,\Phi^{-1}(u)\phi(\Phi^{-1}(u))\,. \tag{49}$$

Bajo $H_{1\,n}$:

$$\mathbf{E}\hat{b}^U(u) = -\delta\,\phi(\Phi^{-1}(u))\,, \tag{50}$$

$$\mathbf{Cov}\{\hat{b}^U(u),\hat{b}^U(v)\} = (u\wedge v) - uv - \frac{1}{2}\Phi^{-1}(u)\Phi^{-1}(v)\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v)). \tag{51}$$

El procedimiento descrito en la sección 1, con $B(u) = \frac{1}{\sqrt{2}}\,\phi(\Phi^{-1}(u))\,\Phi^{-1}(u)$ y $B'(u) = \frac{1}{\sqrt{2}}\,\left[1 - (\Phi^{-1}(s))^2\right]$ permite hallar la representación de $\hat{b}^U$:

$$\hat{b}^U(u) = w(u) - uw(1) + \tfrac{1}{2}\phi(\Phi^{-1}(u))\,\Phi^{-1}(u)\int_0^1 \left[(\Phi^{-1}(s))^2 - 1\right] ds$$
$$-\delta\,\phi(\Phi^{-1}(u))\,,\ 0 \le u \le 1\,. \tag{52}$$

Es fácil verificar que (52) tiene covariancia (51).

El proceso transformado asintótico bajo $H_{1\,n}$ es la integral estocástica respecto del proceso $\hat{b}^U$ en (52):

$$\hat{w}^{(a,\mathcal{T})}(u) = \int_0^1 \mathcal{T}(a\,\mathbb{1}_{([0,u])})(s)\,d\hat{b}^U(s) = \int_0^1 \Big(\mathcal{T}(a\,\mathbb{1}_{([0,u])})(s)$$
$$-\frac{1}{2}\left[\left(\Phi^{-1}(s)\right)^2 - 1\right] \times \int_0^1 \mathcal{T}(a\,\mathbb{1}_{([0,u])})(r)\,\left(\left(\Phi^{-1}(r)\right)^2 - 1\right)\,dr\Big)\,dw(s)$$
$$+\delta\int_0^1 \mathcal{T}(a\,\mathbb{1}_{([0,u])})(s)\,\Phi^{-1}(s)\,ds\,. \tag{53}$$

Bajo $H_{1\,n}$:

$$\mathbf{E}\hat{w}^{(a,\mathcal{T})}(u) = \delta\,\langle\,\mathcal{T}(a\,\mathbb{1}_{([0,u])})\,,\,\kappa\,\rangle_{L^2([0,1],ds)} = \delta\,\langle\,a\,\mathbb{1}_{([0,u])}\,,\,\mathcal{T}^{-1}\kappa\,\rangle\,, \tag{54}$$

con κ definido en (45), y bajo H_0

$$\mathbf{Var}\hat{w}^{(a,\mathcal{T})}(u) = \|\,a\,\mathbb{1}_{([0,u])}\,\|^2 - \frac{1}{2}\langle\,a\,\mathbb{1}_{([0,u])}\,,\,\mathcal{T}^{-1}[(\Phi^{-1})^2 - 1]\rangle^2\,. \tag{55}$$

Se consigue determinar la función de graduación óptima con el mismo criterio heurístico del caso 1; maximizar cuando $a \in L^2([0,1], dt)$ y $u \in [0,1]$ el cociente:

$$\max \frac{\langle\, a\, \mathbb{1}_{([0,u])}\,,\, \mathcal{T}^{-1}\kappa\,\rangle^2}{\|\, a\, \mathbb{1}_{([0,u])}\,\|^2 - \langle\, a\, \mathbb{1}_{([0,u])}\,,\, \mathcal{T}^{-1}(\frac{1}{\sqrt{2}}[(\Phi^{-1})^2 - 1])\,\rangle^2}\,. \tag{56}$$

A diferencia del caso 1, ahora aparece en el denominador el segundo polinomio de Hermite normalizado en la variable Φ^{-1}: $\frac{1}{\sqrt{2}}\,[(\Phi^{-1})^2 - 1]$. Consideraciones geométricas idénticas a las del caso 1 permiten concluir que la función de graduación óptima para el problema de hipótesis (41) es

$$\hat{a} = \alpha_1\, \mathcal{T}^{-1}\kappa + \alpha_2\, \mathcal{T}^{-1}(\frac{1}{\sqrt{2}}[(\Phi^{-1})^2 - 1])\,. \tag{57}$$

y es cómodo para los cálculos sucesivos elegir la solución $\hat{a} = \mathcal{T}^{-1}\kappa$. Utilizando la isometría de Laguerre se obtiene:

$$\hat{a}(u) = (\mathcal{T}_{\mathrm{L}}^{-1}\,\kappa)(u) = \Phi^{-1}(u) + \frac{1}{1-u}\int_0^u \Phi^{-1}(s)\,ds = \Phi^{-1}(u) - \frac{\phi(\Phi^{-1}(u))}{1-u}\,, \tag{58}$$

y que

$$\begin{aligned} \mathcal{T}_{\mathrm{L}}(\hat{a}\,\mathbb{1}_{([0,u])})(s) &= (\hat{a}\,\mathbb{1}_{([0,u])})(s) - \int_0^s \frac{(a\,\mathbb{1}_{([0,u])})(v)}{1-v}\,dv \\ &= \Phi^{-1}(s)\mathbb{1}_{\{s\leq u\}} + \frac{\phi(\Phi^{-1}(u))}{1-u}\mathbb{1}_{\{s>u\}}\,. \end{aligned} \tag{59}$$

Sustituyendo (59) en (53) con $\delta = 0$ se halla la expresión del proceso asintótico transformado estimado bajo H_0 asociado a la función de graduación (58) y a la isometría $\mathcal{T}_{\mathrm{L}}$:

$$\begin{aligned} &\hat{w}^{(\hat{a},\mathcal{T}_{\mathrm{L}})}(u) \\ &= \textstyle\int_0^1 \mathcal{T}_{\mathrm{L}}(\hat{a}\mathbb{1}_{([0,u])})(s)d(w(s) + \frac{1}{2}\phi(\Phi^{-1}(s))\Phi^{-1}(s)\int_0^1[(\Phi^{-1}(r))^2 - 1]dw(r)) \\ &= \textstyle\int_0^1 \left[\Phi^{-1}(s)\mathbb{1}_{\{s\leq u\}} + \frac{\phi(\Phi^{-1}(u))}{1-u}\mathbb{1}_{\{s>u\}}\right] d\Big(w(s) + \end{aligned}$$

$$+\tfrac{1}{2}\phi(\Phi^{-1}(s))\,\Phi^{-1}(s)\left(\int_0^1\left[(\Phi^{-1}(r))^2-1\right]dw(r)\right)\Bigg)=$$

$$=\int_0^1\Phi^{-1}(s)\mathbb{1}_{\{s\le u\}}\,dw(s)+\tfrac{\phi(\Phi^{-1}(u))}{1-u}\int_0^1\mathbb{1}_{\{s>u\}}dw(s)+$$

$$+\left(\int_0^1\left[(\Phi^{-1}(r))^2-1\right]dw(r)\right)\int_0^1\Phi^{-1}(s)\tfrac{1}{2}[1-(\Phi^{-1}(s))^2-1]\mathbb{1}_{\{s\le u\}}ds+$$

$$+\left(\int_0^1\left[\left(\Phi^{-1}(r)\right)^2-1\right]dw(r)\right)\frac{\phi(\Phi^{-1}(u))}{1-u}\int_0^1\frac{1}{2}[1-\left(\Phi^{-1}(s)\right)^2-1]\mathbb{1}_{\{s>u\}}ds \tag{60}$$

La ecuación (60) permite individualizar la región crítica asintótica de la prueba de K-S modificada por un nivel de significación asintótico α.

Presentamos los resultados numéricos correspondientes a la simulación de 50.000 trayectorias tanto del proceso $\hat{w}^{(\hat{a},\mathcal{T}_{\rm L})}$ como del proceso $\hat{b}^U$ bajo H_0: para la prueba clásica de K-S $c(0.05)=1.3042$ mientras que para la prueba K-S modificada $c(0.05)=2.1312$.

Dados los dos valores que individualizan las regiones críticas asintóticas de las dos pruebas se pueden hallar las correspondientes curvas de potencia aproximada por un tamaño muestral dado. Se precisa en este caso simular las trayectorias de:

- el proceso empírico transformado estimado (asociado a la función de graduación óptima $\hat{a}$ en (58) y a la isometría $\mathcal{T}_{\rm L}$) para distintos valores del parámetro δ (y por lo tanto se necesitan generar realizaciones de las variables aleatorias X_i bajo $H_{1,n}$) y con $u\in[0,1]$:

$$\begin{aligned}\hat{w}_n^{(\hat{a},\mathcal{T}_{\rm L})}([0,u]) &= \frac{1}{\sqrt{n}}\sum_{i=1}^n \hat{w}^{(\hat{a},\mathcal{T}_{\rm L})}_{\Phi(\frac{X_i}{\hat{\sigma}_n})}([0,u]) =\\ &= \frac{1}{\sqrt{n}}\sum_{i=1}^n \mathcal{T}_{\rm L}(\hat{a}\,\mathbb{1}_{([0,u])})\,\Phi(\frac{X_i}{\hat{\sigma}_n})\\ &= \text{(en virtud de la ecuación (59))}\\ &\frac{1}{\sqrt{n}}\sum_{i=1}^n\left\{\Phi^{-1}(\Phi(\frac{X_i}{\hat{\sigma}_n}))\mathbb{1}_{\{\Phi(\frac{X_i}{\hat{\sigma}_n})\le u\}}+\frac{\phi(\Phi^{-1}(u))}{1-u}\mathbb{1}_{\{\Phi(\frac{X_i}{\hat{\sigma}_n})>u\}}\right\}\\ &= \frac{1}{\sqrt{n}}\sum_{i=1}^n\left\{\frac{X_i}{\hat{\sigma}_n}\mathbb{1}_{\{\Phi(\frac{X_i}{\hat{\sigma}_n})\le u\}}+\frac{\phi(\Phi^{-1}(u))}{1-u}\mathbb{1}_{\{\Phi(\frac{X_i}{\hat{\sigma}_n})>u\}}\right\};\end{aligned} \tag{61}$$

- el proceso empírico uniforme estimado:

$$\hat{b}_n^U(u) = \sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\left\{\Phi\left(\frac{X_i}{\hat{\sigma}_n}\right)\leq u\right\}} - u\right], 0 \leq u \leq 1\,. \tag{62}$$

La figura siguiente presenta los resultados obtenidos con $n = 100$ y 2.500 replicaciones de los procesos empíricos indicados arriba; se observa un crecimiento de la potencia al pasar de la prueba K-S clásica a la modificada.

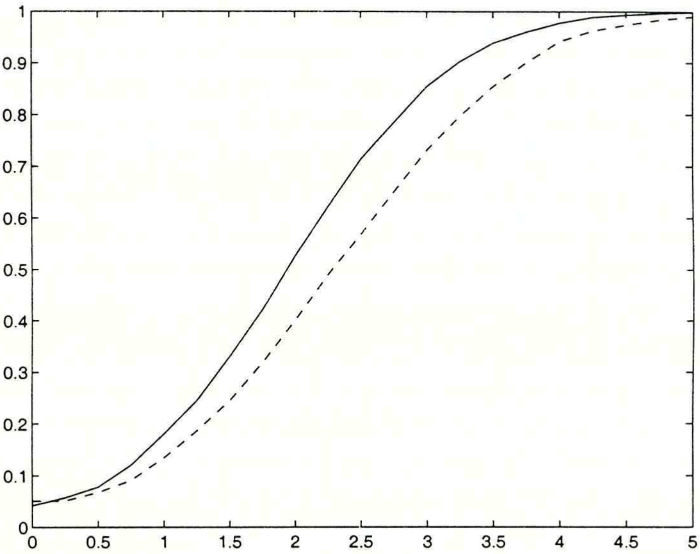

Figura 2

Curvas de potencia aproximada para la prueba K-S clásica (- -) y la prueba K-S modificada (-) para distintos valores de δ..

5 Una prueba de exponencialidad

Se contrasta la hipótesis de que las observaciones provienen de una distribución exponencial con media incógnita:

$$H_0 : F \in \mathcal{F}_0 = \left\{1 - \exp\left(-\tfrac{x}{\theta_0}\right),\ \theta_0 \in I\!R^+\right\}$$

contra cualquier alternativa y se construye un estadístico de prueba particularmente sensible frente alternativas contiguas del tipo Weibull:

$$H_{1\,n}(\delta) : F \in \mathcal{F}_n = \left\{1 - \exp\left[-\left(\frac{x}{\theta_0}\right)^{1+\frac{\delta c}{\sqrt{n}}}\right],\ \theta_0 \in I\!R^+\right\}. \qquad (63)$$

En este caso

$$f_n(x) = \tfrac{1+\lambda}{\theta_0}\left(\tfrac{x}{\theta_0}\right)^{\lambda} exp\left[-\left(\tfrac{x}{\theta_0}\right)^{1+\lambda}\right],\ f_0(x) = \tfrac{1}{\theta_0}\, exp\left(-\tfrac{x}{\theta_0}\right),$$

con $1 + \lambda = 1 + \frac{\delta c}{\sqrt{n}}$.

Entonces

$$\lim_{n\to\infty} \frac{\sqrt{n}}{\delta}\left(\frac{f_n}{f_0} - 1\right) = c\left(1 + \log\frac{x}{\theta_0} - \frac{x}{\theta_0}\log\frac{x}{\theta_0}\right) = k(x, \theta_0). \qquad (64)$$

Se obtiene que la constante de normalización c tal que

$$\| \, k(x, \theta_0) \, \|^2_{L^2(I\!R^+, \frac{1}{\theta_0} e^{-\frac{x}{\theta_0}} dx)} = 1.$$

es $c = \frac{1}{\sqrt{[(1-\gamma)^2 + \frac{\pi^2}{6}]}}$, donde γ es la constante de Euler.

Utilizando la transformación canónica $u = 1 - exp(-\frac{x}{\theta_0})$ resulta

$$\kappa(u) = \frac{1}{\sqrt{[(1-\gamma)^2 + \frac{\pi^2}{6}]}}\{1 + \log(-\log(1-u))\,[1 + \log(1-u)]\}. \qquad (65)$$

Ahora necesitamos determinar las funciones que intervienen en el Teorema (1.1) para individualizar la estructura del proceso Gaussiano estimado $\hat{b}^U$, límite débil de la sucesión de procesos empíricos uniformes paramétricos:

$$\left\{\hat{b}^U_n(u) = \sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n} 1\!\!1_{\{1-\exp(-\frac{X_i}{\hat{\theta}_{0\,n}})\le u\}} - u\right],\ 0 \le u \le 1\right\}. \qquad (66)$$

Tenemos que

$$h(u) = (1-u)\,\theta_0\,\log(1-u)\,, \tag{67}$$

$$g_1(u) = (u-1)\,\log(1-u)\,\log(-\log(1-u))\,, \tag{68}$$

$$g_2(u) = \frac{1-u}{\theta_0}\,\log(1-u)\,. \tag{69}$$

Entonces bajo $H_{1\,n}$:

$$\mathbf{E}\hat{b}^U(u) = \delta\,c\,g_1(u) = \frac{-\delta(1-u)}{\sqrt{[(1-\gamma)^2+\frac{\pi^2}{6}]}}\,\log(1-u)\log(-\log(1-u))\,, \tag{70}$$

$$\mathbf{Cov}\{\hat{b}^U(u),\hat{b}^U(v)\} = (u\wedge v) - u\,v - (1-u)\,\log(1-u)\,(1-v)\,\log(1-v)\,, \tag{71}$$

y la representación del proceso asintótico $\hat{b}^U$ en términos de w es:

$$\begin{aligned}\hat{b}^U(u) &= w(u) - u\,w(1) + (1-u)\log(1-u)\int_0^1 (1+\log(1-s))\,dw(s)\\ &+ \frac{-\delta\,(1-u)}{\sqrt{[(1-\gamma)^2+\frac{\pi^2}{6}]}}\,\log(1-u)\log(-\log(1-u))\,.\end{aligned} \tag{72}$$

El proceso transformado asintótico bajo $H_{1\,n}$ está dado por:

$$\begin{aligned}\hat{w}^{(a,\mathcal{T})}(u) = \int_0^1 &\mathcal{T}(a\,\mathbb{1}_{([0,u])})(s)\,d\{w(s) - sw(1)\\ &+(1-s)\,\log(1-s)\,\int_0^1 (1+\log(1-r))\,dw(r)\\ &+\frac{-\delta\,(1-s)}{\sqrt{[(1-\gamma)^2+\frac{\pi^2}{6}]}}\,\log(1-s)\log(-\log(1-s))\}\,.\end{aligned} \tag{73}$$

y

$$\mathbf{E}\hat{w}^{(a,\mathcal{T})}(u) = \delta\,\langle\,\mathcal{T}(a\,\mathbb{1}_{([0,u])})\,,\,\kappa\,\rangle_{L^2([0,1],ds)}\,. \tag{74}$$

Mientras que bajo H_0:

$$\mathbf{Var}\hat{w}^{(a,\mathcal{T})}(u) = \| \, a \, \mathbb{1}_{([0,u])} \, \|^2 - \langle \, \mathcal{T}(a \, \mathbb{1}_{([0,u])}) \, , \, \eta \, \rangle^2_{L^2([0,1],ds)}, \tag{75}$$

con $\eta(s) = 1 + \log(1-s)$.

Para hallar la función de graduación óptima $\hat{a}$ hay que resolver el problema (maximizando cuando $a \in L^2([0,1], dt)$ y $u \in [0,1]$):

$$\max \frac{\langle \, \mathcal{T}(a \, \mathbb{1}_{([0,u])}) \, , \, \kappa \, \rangle^2}{\| \, a \, \mathbb{1}_{([0,u])} \, \|^2 - \langle \, \mathcal{T}(a \, \mathbb{1}_{([0,u])}) \, , \, \eta \, \rangle^2} \tag{76}$$

cuya solución es $\hat{a} = \alpha_1 \mathcal{T}^{-1}\kappa + \alpha_2 \mathcal{T}^{-1}\eta$ y se elige la solución particular $\hat{a} = \mathcal{T}^{-1}\kappa$. Empleando la isometría de Laguerre se concluye que:

$$\begin{aligned}
\hat{a}(u) &= (\mathcal{T}_{\mathrm{L}}^{-1} \, \kappa)(u) = \kappa(u) + \frac{1}{1-u} \int_0^u \kappa(s) \, ds \\
&= \frac{1}{\sqrt{[(1-\gamma)^2 + \frac{\pi^2}{6}]}} \left\{ \frac{d}{du} g_1(u) + \frac{1}{1-u} \, g_1(u) \right\},
\end{aligned} \tag{77}$$

con $g_1(u)$ definida en (68), y que:

$$\begin{aligned}
\mathcal{T}_{\mathrm{L}}(\hat{a} \, \mathbb{1}_{([0,u])})(s) &= (\hat{a} \, \mathbb{1}_{([0,u])})(s) - \int_0^s \frac{(a \, \mathbb{1}_{([0,u])})(v)}{1-v} \, dv \\
&= \frac{1}{\sqrt{[(1-\gamma)^2 + \frac{\pi^2}{6}]}} \left\{ \frac{d}{ds} \, g_1(s) + \frac{1}{1-s} \, g_1(s) \right\} \mathbb{1}_{\{s \le u\}} \\
&\quad - \frac{1}{\sqrt{[(1-\gamma)^2 + \frac{\pi^2}{6}]}} \int_0^{s \wedge u} \frac{1}{1-v} \left[\frac{d}{dv} \, g_1(v) + \frac{1}{1-v} \, g_1(v) \right] dv \\
&= \kappa(s) \mathbb{1}_{\{s \le u\}} - \frac{1}{\sqrt{[(1-\gamma)^2 + \frac{\pi^2}{6}]}} \frac{1}{1-u} \, g_1(u) \, \mathbb{1}_{\{s > u\}} \, .
\end{aligned} \tag{78}$$

El proceso asintótico transformado bajo H_0 (en términos de w) asociado a la función de graduación óptima $\hat{a}$ y a la isometría de Laguerre $\mathcal{T}_{\mathrm{L}}$ es:

$$\hat{w}^{(\hat{a},\mathcal{T}_{\mathrm{L}})}(u) = \int_0^1 \mathcal{T}_{\mathrm{L}}(\hat{a} \, \mathbb{1}_{([0,u])})(s) \, d\{w(s) - sw(1)$$

$$+(1-s)\log(1-s)\left(\int_0^1(1+\log(1-r))\,dw(r)\right)$$

$$=\int_0^1\left[\kappa(s)\mathbb{1}_{\{s\le u\}}-\frac{1}{\sqrt{[(1-\gamma)^2+\frac{\pi^2}{6}]}}\frac{1}{1-u}g_1(u)\,\mathbb{1}_{\{s>u\}}\right]\,d\{w(s)$$

$$+(1-s)\log(1-s)\left(\int_0^1(1+\log(1-r))\,dw(r)\right)\}$$

$$=\int_0^1\kappa(s)\mathbb{1}_{\{s\le u\}}\,dw(s)-\frac{1}{\sqrt{[(1-\gamma)^2+\frac{\pi^2}{6}]}}\frac{1}{1-u}g_1(u)\int_0^1\mathbb{1}_{\{s>u\}}\,dw(s)$$

$$-\left(\int_0^1(1+\log(1-r))\,dw(r)\right)\int_0^1\kappa(s)\mathbb{1}_{\{s\le u\}}(1+\log(1-s))\,ds$$

$$+\left(\int_0^1(1+\log(1-r))\,dw(r)\right)\frac{1}{\sqrt{[(1-\gamma)^2+\frac{\pi^2}{6}]}}$$

$$\times\int_0^1\frac{1}{1-u}g_1(u)\,\mathbb{1}_{\{s>u\}}\,(1+\log(1-s))\,ds\,. \tag{79}$$

La fórmula (79) permite engendrar bajo H_0 realizaciones de la variable aleatoria $\sup_{0\le u\le 1}\mid \hat{w}^{(\hat{a},\mathcal{T}_{\mathrm{L}})}(u)\mid$.

La fórmula (78) evaluada en $1-\exp(-\frac{X_i}{\hat{\theta}_{0\,n}})$ permite escribir el proceso empírico transformado estimado (asociado a $\hat{a}$ en (77) y a $\mathcal{T}_{\mathrm{L}}$):

$$\hat{w}_n^{(\hat{a},\mathcal{T}_{\mathrm{L}})}([0,u])=\tfrac{1}{\sqrt{n}}\textstyle\sum_{i=1}^n\hat{w}^{(\hat{a},\mathcal{T}_{\mathrm{L}})}_{1-\exp(-\frac{X_i}{\hat{\theta}_{0\,n}})}([0,u])=$$

$$=\tfrac{1}{\sqrt{n}}\textstyle\sum_{i=1}^n\mathcal{T}_{\mathrm{L}}(\hat{a}\,\mathbb{1}_{([0,u])})(1-\exp(-\frac{X_i}{\hat{\theta}_{0\,n}}))$$

$$=\tfrac{1}{\sqrt{n}}\textstyle\sum_{i=1}^n\Big\{\kappa(1-\exp(-\frac{X_i}{\hat{\theta}_{0\,n}}))\,\mathbb{1}_{\{1-\exp(-\frac{X_i}{\hat{\theta}_{0\,n}})\le u\}}$$

$$-\frac{1}{\sqrt{[(1-\gamma)^2+\frac{\pi^2}{6}]}}\,\tfrac{1}{1-u}\,g_1(u)\,\mathbb{1}_{\{1-\exp(-\frac{X_i}{\hat{\theta}_{0\,n}})>u\}}\Big\}$$

=(en virtud de (65))

$$
\begin{aligned}
&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Bigg\{\frac{1}{\sqrt{[(1-\gamma)^2+\frac{\pi^2}{6}]}}\Bigg\{1+\log(-\log(1-1-\exp(-\frac{X_i}{\hat{\theta}_{0\,n}})))\\
&\quad\times[1+\log(1-1-\exp(-\frac{X_i}{\hat{\theta}_{0\,n}}))]\Bigg\}\,\mathbb{1}_{\{1-\exp(-\frac{X_i}{\hat{\theta}_{0\,n}})\le u\}}\\
&\quad-\frac{1}{\sqrt{[(1-\gamma)^2+\frac{\pi^2}{6}]}}\,\frac{1}{1-u}\,g_1(u)\,\mathbb{1}_{\{1-\exp(-\frac{X_i}{\hat{\theta}_{0\,n}})>u\}}\Bigg\}\\
&=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Bigg\{\frac{1}{\sqrt{[(1-\gamma)^2+\frac{\pi^2}{6}]}}\Bigg\{1+\log(\frac{X_i}{\hat{\theta}_{0\,n}})\,[1-\frac{X_i}{\hat{\theta}_{0\,n}}]\Bigg\}\,\mathbb{1}_{\{1-\exp(-\frac{X_i}{\hat{\theta}_{0\,n}})\le u\}}\\
&\quad-\frac{1}{\sqrt{[(1-\gamma)^2+\frac{\pi^2}{6}]}}\,\frac{1}{1-u}\,g_1(u)\,\mathbb{1}_{\{1-\exp(-\frac{X_i}{\hat{\theta}_{0\,n}})>u\}}\Bigg\}. \qquad (80)
\end{aligned}
$$

Dado un nivel de significación asintótico $\alpha = 0.05$, se han simulado bajo H_0 10.000 trayectorias para los procesos clásico (72) y transformado (79) obteniéndose $c(0.05) = 1.0720$ para la prueba K-S clásica y $c(0.05) = 2.0540$ para la prueba K-S modificada.

Dado un tamaño muestral $n = 100$ han sido realizadas - bajo $H_{1\,n}$ y para distintos valores de δ de 0 a 5 con paso 0.25 - 2500 simulaciones tanto del proceso empírico uniforme estimado (66) como del proceso empírico transformado (80).

La comparación entre las potencias aproximadas de las dos pruebas puede apreciarse en la figura siguiente:

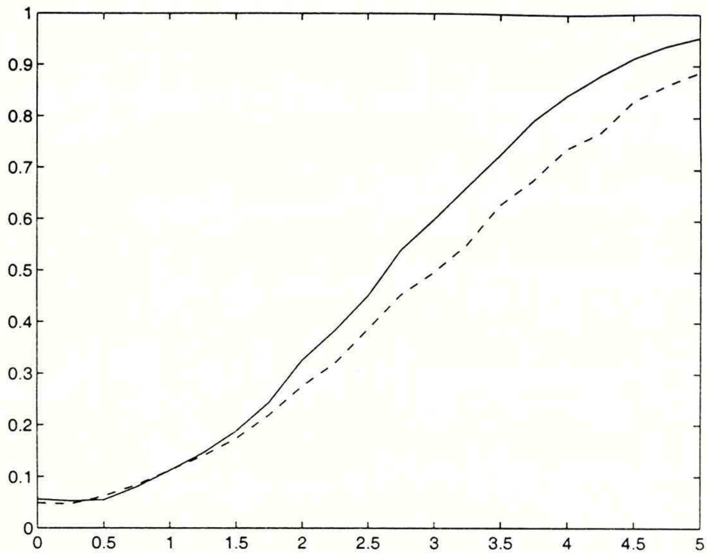

Figura 3
Curvas de potencia aproximada para la prueba K-S clásica (- -) y la prueba K-S modificada (-) para distintos valores de δ.

6 Conclusiones

Se dispone de una técnica muy clara para hallar el estadístico de K-S modificado apropiado para cualquier prueba de hipótesis en presencia de un párametro estimado y con alternativas contiguas.

En cada una de las aplicaciones desarrolladas la implementación de la prueba de K-S modificada ha permitido comprobar que la potencia de la prueba modificada es siempre mayor que la de la prueba K-S clásica.

Se observa que es posible construir estadísticos de K-S modificados para pruebas con más de un parámetro estimado. Por ejemplo una prueba de normalidad, donde la familia de alternativas, al igual que en la prueba de exponencialidad, será especificada con una familia de distribuciones de probabilidad distinta de la que aparece en la hipótesis nula.

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Deseo expresar mi profunda gratitud al Prof. Ing. Enrique M. Cabaña bajo cuya dirección he realizado el presente trabajo. Su dedicación y generosidad me han permitido lograr los resultados aquí desarrollados.

Bibliografía.

[1] Billingsley, P. (1968). *Convergence of Probability Measures.* Wiley, New York.

[2] Cabaña, E. M. (1997). *Contigüidad, Pruebas de Ajuste y Procesos Empíricos Transformados.* Décima Escuela Venezolana de Matemáticas, Caracas.

[3] Cabaña, A., Cabaña, E. M. (1997). *Transformed Empirical Processes and Modified Kolmogorov-Smirnov Tests for multivariate distributions.* Ann. Statist., Vol.25, 2388-2409.

[4] Cramér, H. (1946). *Mathematical Methods of Statistics.* Princeton University Press.

[5] Darling, D. A. (1955). *The Cramér-Smirnov test in the parametric case.* Ann. Math. Statist., Vol.26, 1-20.

[6] David, F. N., Johnson, N. L. (1948). *The Probability Integral Transformation when Parameters are Estimated from the Sample.* Biometrika, Vol.XXXV, 182-190.

[7] Durbin, J. (1973). *Weak convergence of the sample distribution function when parameters are estimated.* Ann. Statist., Vol.1, 279-290.

[8] Le Cam, L., Yang, G.L. (1990). *Asymptotics in Statistics. Some basic concepts.* Springer-Verlag, New York.

[9] Oosterhoff, J., van Zwet, W.R. (1979). *A note on contiguity and Hellinger distance.* Contributions to Statistics, 157-166, Reidel, Dordrecht.

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[9] PARANJAPE, S. R. and PARK, C. Distribution of the supremum of the two-parameter Yeh-Wiener process on the boundary, *J. Appl. Probability* **10**, 1973, 875-880.

[10] PYKE, R. *Multidimensional Empirical Processes: Some Comments in Statistical Inference and Related Topics.* (M. L. Puri, ed.) Academic Press, New York, 1975, 45-48.

[11] WIDDER, D. V. *The Heat Equation.* Academic Press, New York, 1975.

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