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INDICE

Modified Anderson-Darling test with selective power improvement.

Alejandra Cabaña and Enrique M. Cabaña

ABSTRACT

Transformed empirical processes (TEPs) have been used by the authors in a previous paper to construct consistent and selectively efficient goodness-of-fit tests of the Kolmogorov - Smirnov type.
A straightforward application of the same ideas to the construction of tests of the Cramér - von Mises type with the same properties leads to cumbersome computations.
This short note exhibits some of the inconveniences encountered, and introduces a new family of quadratic statistics of the Cramér - von Mises type, in order to circumvent the difficulties.
AMS Classification numbers: 62G10, 62G20, 62G30, 60E20.

RESUMEN

Los procesos empíricos transformados (TEPs) han sido utilizados por los autores para la construcción de pruebas de ajuste coherentes y selectivamente eficientes del tipo de Kolmogorov - Smirnov.
Para aplicar las mismas ideas, de manera directa, a la construcción de pruebas de ajuste del tipo de Cramér - von Mises con las mismas propiedades de coherencia y potencia selectiva, se requiere realizar cálculos muy complicados.
En esta breve nota se describen algunas de las dificultades que se encuentran al intentar tal generalización, y se introduce una nueva familia de estadísticos del tipo de Cramér - von Mises que permite evitar esas dificultades.
Números de clasificación AMS: 62G10, 62G20, 62G30, 60E20.

1 Introduction

The design of tests suited to detect a specific kind of alternative is a common procedure in nonparametric statistics. Linear rank tests (see [7]) are typical examples. In applying them, the statistician chooses the scores in such a way that the power of the test is optimized for a specific family of alternatives. As for other fixed alternatives, the resulting tests may be inefficient or even unable to detect them.

Other well known nonparametric tests have the property of rejecting any fixed alternative for sufficiently large sample sizes. Kolmogorov - Smirnov and Cramér - von Mises tests, for instance, have this consistency property.

In a previous paper (see [3]), the authors have proposed a way of obtaining goodness-of-fit tests, having these two just mentioned desirable properties. They are both consistent against every fixed alternative and specially efficient against a given sequence of contiguous ones. The critical regions are of the Kolmogorov-Smirnov type, and *Transformed Empirical Processes* (TEPs) play there the role of the empirical process in the classical K-S tests. The families of tests in [3] depend on a functional parameter, that has to be adequately chosen in order to achieve the optimal efficiency against the given sequence of alternatives, maintaining the consistency. The resulting tests can be applied when the statistician requires consistency, and, in addition, is specially interested in avoiding an *error of type II* when the alternatives are of some specific kind.

Tests of the Cramér - von Mises type, based on quadratic functionals of the empirical process, can be modified as well, by replacing the empirical process by a TEP, to produce consistent and selectively efficient tests. After describing the TEPs and their asymptotic distributions (§2), we show in §3 and §4 that, when the modified Cramér - von Mises statistics are defined in the apparently simplest way, the optimum score functions and the asymptotic distributions of the test statistics may be quite difficult to obtain.

In order to overcome these difficulties, we introduce in §5 a particular family of statistics for which the optimum score functions are easily obtained. In fact, they are the same that optimize the behaviour of the modified Kolmogorov - Smirnov tests for the same TEPs, and under the same family of contiguous alternatives.

In addition, these test statistics are asymptotically distribution free both under the null hypothesis and under the alternatives. The asymptotic distribution of the test statistic under the null hypothesis, and under the privileged alternatives, depends only on the weight function, and the *size* but not the *shape* of the alternatives. Furthermore, we show elsewhere ([4]) that the shape of the weight function has a little effect on the resulting power.

As a consequence, tables of critical levels and asymptotic powers of the tests can be constructed by simulation. Such tables are provided in the last section (§6).

Some theoretical comments on the distribution of the test statistic, particularly for the case of a constant weight function, are also contained in [4]. A particular application to the derivation of normality tests is developed in [5];

though the general form of the test statistics may look rather complicated, for this latter case, the test statistic is simply a quadratic form evaluated on a vector of sums of polynomials in the sample points.

2 The Transformed Empirical Processes and their asymptotic distributions.

Let $\{X_1, X_2, \ldots, X_n\}$ denote a sample of independent real random variables with distribution function F, and let us consider a sequence of probability distributions $F^{(n)}$ *contiguous* to a given probability distribution F_0 (see [8],[9]), and such that

i. $F^{(n)}$ has density f_n with respect to F_0,

ii. the functions k_n defined by $\sqrt{f_n} = 1 + \frac{\delta k_n}{2\sqrt{n}}$ are absolutely and uniformly bounded by K such that $\int K^2 dF_0 << \infty$,

iii. there exists k such that $\int k^2 dF_0 = 1$, and $\lim_{n\to\infty} \int (k_n - k)^2 dF_0 = 0$.

Families of Transformed Empirical Processes depending on a functional parameter (*score function*) have been introduced by one of the authors in [2], with the purpose of designing goodness-of-fit tests of the Kolmogorov-Smirnov type for the null hypothesis $\mathcal{H}_0$: "$F = F_0$". Those tests share the following two properties: (a) they are consistent against any fixed alternative "$F \neq F_0$" and (b) they are specially sensitive against the particular sequence of alternatives $\mathcal{H}_n$: "$F = F^{(n)}$".

In this article we focus our attention in introducing new tests that also share properties (a) and (b), based on a quadratic statistic of the Anderson-Darling type.

Following [3], we define the Transformed Empirical Process (TEP) of the sample $\{X_1, X_2, \ldots, X_n\}$, associated to the distribution function F_0, the isometry $\mathcal{T}$ on $L_2 = L_2(\mathbf{R}, dF_0)$ with range equal to the orthogonal complement of the constant function 1, and the *score function* a with $\|a\|^2 = \int a^2(x) dF_0(x) = 1$ as

$$w_n^{(a,\mathcal{T})}(x) = \int \mathcal{T}(a 1_x) db_n, \tag{1}$$

where 1_x is the indicator function of $(-\infty, x]$, $b_n(x) = \sqrt{n}(F_n(x) - F_0(x))$ is the empirical process and $F_n(x) = \frac{1}{n}\sum_{i=1}^{n} 1_{\{X_i \leq x\}}$ is the sample distribution function.

Let $\mathcal{J} = \{(-\infty, x] : x \in \mathbf{R}\}$ and let $w^{(V)}$ denote a Wiener process on $\mathbf{R}$ with covariance function

$$V(x) = \mathbf{E}\left(w^{(V)}(x)\right)^2 = \int_{-\infty}^{x} a^2(t) dF_0(t), \ \mathbf{E} w^{(V)}(x) w^{(V)}(y) = V(x \wedge y). \tag{2}$$

It is shown in [3] that, under suitable conditions, the TEP $w_n^{(a,\mathcal{T})}(x), x \in \mathbf{R}$ converges in law to

$$w^{(V)}(x) + \delta \int k\mathcal{T}(a1_x)dF_0, x \in \mathbf{R} \tag{3}$$

in the space of right continuous functions with left limits, as n tends to infinity. In particular, $w_n^{(a,\mathcal{T})}$ converges in law to $w^{(V)}$ under $\mathcal{H}_0$.

We refer to [3] for general conditions ensuring the convergence to (3).

3 The modified Cramér - von Mises statistics.

Classical tests of Cramér - von Mises type are based on the quadratic statistics

$$S_n = \int (b_n(x))^2 \psi(F_0(x))dF_0(x). \tag{4}$$

In this expression, ψ is a *weight function* to be adequately chosen. The *Cramér - von Mises test* corresponds to the selection $\psi(x) = 1$, and other selections of the weight function lead to other tests with similar properties. In particular, the *Anderson-Darling test* is based on the quadratic statistic $\mathrm{AD}_n = \int (b_n(x))^2 \frac{dF_0(x)}{F_0(x)(1-F_0(x))}$ with $\psi(F_0(x)) = (\mathbf{E}(b_n(x))^2)^{-1}$.

It is well known that b_n is asymptotically distributed as a Brownian bridge $b^{(F_0)}$ associated to the probability F_0 (that is, $b^{(F_0)}$ is a centered Gaussian process with covariances $\mathbf{E}b^{(F_0)}(x)b^{(F_0)}(y) = F_0(x \wedge y) - F_0(x)F_0(y))$ and therefore S_n is asymptotically distributed as $\int (b(s))^2 \psi(s)ds$ where b denotes a standard Brownian bridge.

When the empirical process b_n is replaced by the TEP $w_n^{(a,\mathcal{T})}$, the analogue of (4) is

$$S_n^{(a,\mathcal{T})} = \int (w_n^{(a,\mathcal{T})}(x))^2 \psi(V(x))dV(x). \tag{5}$$

The variance $V(x) = \mathbf{Var} w_n^{(a,\mathcal{T})}(x)$ plays in (5) the same role as F_0 in (4) with the same purpose of simplifying the description of the asymptotic distributions. Now it is easily verified that $S_n^{(a,\mathcal{T})}$ is asymptotically distributed as $\int w^2(s)\psi(s)ds$, where w is a standard Wiener process, under $\mathcal{H}_0$.

When $\mathcal{H}_n$ applies instead, the asymptotic distribution of $S_n^{(a,\mathcal{T})}$ is that of

$$\int \left(w^{(V)}(x) + \delta \int k\mathcal{T}(a1_{(-\infty,x]})dF_0 \right)^2 \psi(V(x))dV(x)$$

$$= \int \left(w(V(x)) + \delta \int_{-\infty}^{x} a^{-1}(\mathcal{T}^{-1}k)dV \right)^2 \psi(V(x))dV(x)$$

$$= \int \left(w(s) + \delta \int_0^s h(r)dr \right)^2 \psi(s)ds,$$

with

$$h(V(x)) = (\mathcal{T}^{-1}k)(x)/a(x). \tag{6}$$

The function h must satisfy

$$\int_0^1 h^2(s)ds = \int_{-\infty}^{\infty} h^2(V(x))dV(x) = \int_{-\infty}^{\infty} (\mathcal{T}^{-1}k)^2 dF_0 = \int_{-\infty}^{\infty} k^2 dF_0 = 1.$$

A reasonable heuristic criterion to improve the efficiency of tests with critical regions $S_n^{(a,\mathcal{T})} > c$ (where c is a suitable constant) is to maximize the asymptotic bias

$$B = \mathbf{E}(S_n^{(a,\mathcal{T})}|\mathcal{H}_n) - \mathbf{E}(S_n^{(a,\mathcal{T})}|\mathcal{H}_0) = \delta^2 \int_0^1 \left(\int_0^s h(r)dr \right)^2 \psi(s)ds.$$

4 The optimization problem.

4.1 The general setting.

The criterion sketched in the previous section poses the problem of finding h in $L^2(0,1)$ with $\|h\| = 1$, such that $\int_0^1 \left(\int_0^s h(r)dr\right)^2 \psi(s)ds$ is maximum, for the given nonnegative weight function ψ. The quantity to be maximized can be written as

$$\int_0^1 h(r_1)dr_1 \int_0^1 h(r_2)dr_2 \int_{r_1 \vee r_2}^1 \psi(s)ds = \int_0^1 \int_0^1 K(r_1, r_2)h(r_1)h(r_2)dr_1 dr_2,$$

with kernel $K(r_1, r_2) = \int_{r_1 \vee r_2}^1 \psi(s)ds$. The maximum is the largest eigenvalue of the Fredholm operator

$$f \mapsto \int_0^1 K(\cdot, r)f(r)dr, \tag{7}$$

and it is attained at the corresponding normalized eigenfunction h.

In order to obtain the eigenfunctions h and eigenvalues λ of (7) we must solve

$$\lambda h(x) = \int_0^1 \int_{x \vee r}^1 \psi(s)ds h(r)dr = \int_x^1 \psi(s)ds \int_0^s h(r)dr.$$

The solutions satisfy the differential equation $\lambda h'(x) = -\psi(x) \int_0^x h(r)dr$ with boundary condition $h(1) = 0$. Therefore, the primitive $H(t) = \int_0^t h(s)ds$ a primitive of h satisfies the conditions

$$\lambda H''(t) = -\psi(t)H(t),\ H(0) = 0,\ H'(1) = 0. \tag{8}$$

The equations (8) characterize the function H, together with the condition $\|h\| = 1$ that can be written as

$$1 = \int_0^1 (H'(t))^2 dt = -\int_0^1 H(t)H''(t)dt = \lambda^{-1} \int_0^1 H^2(t)\psi(t)dt. \qquad (9)$$

Once solved (8), (9) in H, $h = H'$ is known and the differential equation

$$V'(x)h^2(V(x)) = (\mathcal{T}^{-1}k(x))^2$$

with the initial condition $V(0) = 0$ has to be solved in order to obtain the score function $a(x) = \sqrt{V'(x)}$. Notice that this is equivalent to the integral equation

$$\int_0^{V(x)} h^2(y)dy = \int_0^x (\mathcal{T}^{-1}k(t))^2 dt. \qquad (10)$$

The sign of a is determined from (6) as $\text{sgn}a = \text{sgn}(h \circ V)\text{sgn}(\mathcal{T}^{-1}k)$.

4.2 The solutions in two particular cases.

The difficulties of the preceding approach are better evaluated by means of some examples: let us first obtain the optimum score function for the case $\psi(x) = 1$.

In this case, (8) implies that $H(t)$ is proportional to $\sin\sqrt{\lambda^{-1}}t$, $\sqrt{\lambda^{-1}} = \nu\pi - \frac{\pi}{2}$. The eigenfunctions are $h(t) = \sqrt{2}\cos(\nu\pi - \frac{\pi}{2})t$, so that the maximum of the eigenvalues $(\nu\pi - \frac{\pi}{2})^{-2}$ is obtained for $\nu = 1$ and the corresponding eigenfunction is $h(t) = \sqrt{2}\cos\frac{\pi}{2}t$

Now (10) reads $V(x) + \frac{1}{\pi}\sin\pi V(x) = \int_0^x (\mathcal{T}^{-1}k(y))^2 dy$ and can be solved in $V(x)$ because the function $z \mapsto z + \frac{1}{\pi}\sin\pi z$ is strictly increasing, but even in this simple case, such procedure does not give us a closed formula for V neither for the score function $a = \sqrt{V'}$.

Our second example is the analogue of the Anderson - Darling test, obtained with $\psi(x) = 1/x$. This particular selection of the weight function imitates the criterion applied for the definition of the classical Anderson - Darling statistic, that is, to choose the weight so as the expectation of the integrand is constant. In the present case, $\psi(V(x)) = \left(\mathbf{E}\left(w_n^{(a,\mathcal{T})}(x)\right)^2\right)^{-1} = 1/V(x)$ is obtained. The corresponding statistic is $\text{AD}_n^{(a,\mathcal{T})} = \int (w_n^{(a,\mathcal{T})}(x))^2 dV(x)/V(x)$.

The differential equation in (8) and the initial condition $H(0) = 0$ lead to the series expansion $h(t) = H'(t) = c\sum_{n=0}^{\infty}(-1)^n(\lambda^{-1}t)^n/(n!^2) = cJ_0(2\sqrt{\lambda^{-1}t})$, where J_0 denotes as usual the Bessel function of the first kind of order 0 and

c is a constant to be determined. The additional condition $H'(1) = 0$ implies that $\zeta = 2\sqrt{\lambda^{-1}}$ must be one of the roots of J_0.

The normalization $\|h\| = 1$ gives $c = \zeta(2\int_0^\zeta zJ_0^2(z)dz)^{-1/2}$. Up to this point, h is determined up to the selection of the root ζ. In order to obtain the maximum λ, we choose ζ equal to the minimum root ζ_1 of J_0.

After integrating in the left hand side of (10), we get

$$\frac{V(x)(J_{-1}(\sqrt{V(x)}\zeta)J_1(\sqrt{V(x)}\zeta) - J_0^2(\sqrt{V(x)}\zeta))}{J_{-1}(\zeta)J_1(\zeta) - J_0^2(\zeta)} = \int_0^x (\mathcal{T}^{-1}k(y))^2 dy,$$

and it remains to replace ζ by ζ_1, solve in V and compute $a = \sqrt{V'}$.

We believe unnecessary to go further in order to show that the calculations involved in using these modified quadratic statistics make them rather unmanageable.

5 A new family of quadratic statistics.

Consider first the family of statistics

$$T_{n,x}^{(a)} = \int \left(\int c_{(x,y)}(z)dw_n^{(a)}(z)\right)^2 \frac{dV(y)}{\int c_{(x,y)}(z)dV(z)} \tag{11}$$

with

$$c_{(x,y)}(s) = \begin{cases} 1 & \text{if } x << s << y, y << x << s \text{ or } s << y << x, \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

In particular, $T_{n,-\infty}^{(a)}$ equals (5).

Then we integrate $T_{n,x}^{(a)}$ with respect to $dV(x)$ thus defining the new statistic

$$T_n^{(a)} = \iint \left(\int c_{(x,y)}(z)dw_n^{(a)}(z)\right)^2 \frac{dV(x)dV(y)}{\int c_{(x,y)}(z)dV(z)}. \tag{13}$$

Although (13) looks intricate because of the multiple integration, we show below that $T_n^{(a)}$ does not have the disadvantages sketched in regard with the examples in §4.2. On the contrary, the optimum score function and the asymptotic distributions under $\mathcal{H}_0$ and $\mathcal{H}_n$ are extremely simple.

5.1 Asymptotic behaviour of $T_n^{(a)}$ under fixed alternatives. Consistency of the tests.

From $w_n^{(a)} = \frac{1}{\sqrt{n}}\sum_{i=1}^n w_{\{X_i\}}^{(a)}$, with $w_{\{X_i\}}^{(a)}$ equal to the TEP corresponding to the sample $\{X_i\}$ of size one, the expectation and variance of $\int c_{(x,y)}(z)dw_n^{(a)}(z)$

are respectively

$$\mathbf{E}\int c_{(x,y)}(z)dw_n^{(a)}(z) = \sqrt{n}\,\mathbf{E}\int c_{(x,y)}(z)dw_{\{X_1\}}^{(a)}(z) \tag{14}$$

and

$$\mathbf{Var}\int c_{(x,y)}(z)dw_n^{(a)}(z) = \mathbf{Var}\int c_{(x,y)}(z)dw_{\{X_1\}}^{(a)}(z).$$

It is easily shown by applying the same arguments used in [3] that when $F \neq F_0$, then there exist x_0 and y_0 such that

$$\mathbf{E}\int c_{(x,y)}(z)dw_{\{X_1\}}^{(a)}(z) \neq 0 \text{ for } X \sim F \tag{15}$$

for $x = x_0$ and $y = y_0$. Then, the continuity of the left-hand member of (15) as a function of x, y implies that there are neighbourhoods I of x_0 and J of y_0 such that $|\mathbf{E}\int c_{(x,y)}(z)dw_{\{X_1\}}^{(a)}(z)|$ is greater than a certain constant $k > 0$ for $x \in I$ and $y \in J$.

PROPOSITION 1 *When the score function a is $F_0 -$ a.s. different from zero, and $X \sim F \neq F_0$, then $\mathbf{P}\{\lim T_n^{(a)} = +\infty\} = 1$.*

Proof. From the previous context and the assumption on a it follows that $V(I)$ and $V(J)$ do not vanish. Hence (14) tends to infinity as $n \to \infty$ so that the required conclusion is readily obtained. □

As a corollary, the test with critical region $T_n^{(a)} >$ constant is consistent for any $F \neq F_0$.

5.2 Asymptotic behaviour of $T_n^{(a)}$ under the sequence of contiguous alternatives.

The process (3) that has the limit distribution of $\{w_n^{(a)}(A) : A \in \mathcal{J}\}$ under the sequence of alternatives $\mathcal{H}_n$: "$F = F^{(\delta/\sqrt{n})}$", can also be written as $\{w^{(V)}(A) + \delta\int_A a(\mathcal{T}^{-1}k)dF_0$, since $\int k\mathcal{T}(1_A a)dF_0 = \int_A a\mathcal{T}^{-1}kdF_0$ because $\mathcal{T}$ is an isometry.

Therefore $T_n^{(a)}$ is asymptotically distributed under $\mathcal{H}_n$ as

$$\iint\left(\int c_{(x,y)}(z)(dw^{(V)}(z) + \delta a(\mathcal{T}^{-1}k)dF_0)\right)^2 \frac{dV(x)dV(y)}{\int c_{(x,y)}(z)dV(z)}$$

$$= \iint\left(\int c_{(x,y)}(z)(dw(V(z)) + \delta h(V(z))dV(z))\right)^2 \frac{dV(x)dV(y)}{\int c_{(x,y)}(z)dV(z)}, \tag{16}$$

where $h(V(z))a(z) = (\mathcal{T}^{-1}k)(z)$. Since $c_{(x,y)}(z) = c_{(V(x),V(y))}(V(z))$, then, with new variables $r = V(x)$, $s = V(y)$, $t = V(z)$, (16) reduces to

$$\iint \left(\int c_{(r,s)}(t)(dw(t) + \delta h(t)dt) \right)^2 \frac{dr\ ds}{\int c_{(r,s)}(t)dt}$$

$$= \int_0^1 \int_0^1 C(t,u)(dw(t) + \delta h(t)dt)(dw(u) + \delta h(u)du), \tag{17}$$

with

$$C(t,u) = \int_0^1 \int_0^1 c_{(r,s)}(t)c_{(r,s)}(u)\frac{dr\ ds}{\lambda(r,s)}, \quad \lambda(r,s) = \int c_{(r,s)}(t)dt. \tag{18}$$

The distribution of (17) depends only on the selected score function, through the function h. In particular, the asymptotic bias under the alternatives is

$$\Delta(a) = \delta^2 \int_0^1 \int_0^1 C(t,u)h(t)h(u)dt\ du. \tag{19}$$

The limit behaviour described by (17) suggests to reject the null hypothesis $\mathcal{H}_0$ when $T_n^{(a)}$ is greater than an adequate constant, and, in order to improve the sensitivity of the test with respect to the given sequence of contiguous alternatives, we propose to choose the score function a that maximizes the asymptotic bias $\Delta(a)$.

PROPOSITION 2 *The asymptotic bias $\Delta(a)$ given by (19) is maximum when the score function a is chosen equal to $\hat{a} = \mathcal{T}^{-1}k$, and its maximum value is $\delta^2/2$.*

Remark. The optimum score $\hat{a} = \mathcal{T}^{-1}k$ is the same that optimizes the power of tests of the Kolmogorov-Smirnov type (see [2, 3]).

Proof. Let us compute, for $t << u$,

$$C(t,u) = \int_0^t dr \int_u^1 \frac{ds}{s-r} + \int_0^t dr \int_0^r \frac{ds}{1+s-r}$$

$$+ \int_t^u dr \int_t^r \frac{ds}{1+s-r} + \int_u^1 dr \int_u^r \frac{ds}{1+s-r} = \gamma(|u-t|), \tag{20}$$

with $\gamma(y) = 1 + |y|\log(|y|) + (1-|y|)\log(1-|y|)$. The expression (20) also holds for $u << t$, since it depends symmetrically on t and u.

The function γ is symmetric with respect to 0 and 1/2, and this implies that $\int_0^1 C(t,u)du$ does not depend on t and equals $\int_0^1 \gamma(y)dy = 1/2$ so that, when h is the constant 1, and hence $a = \hat{a}$, we have $\Delta(\hat{a}) = 1/2$. On the other hand, by Cauchy-Schwartz Inequality,

$$\Delta(a) \leq \int_0^1 \int_0^1 h^2(t)C(t,u)dt\, du = \frac{1}{2}\int_0^1 h^2(t)dt = \frac{1}{2}$$

since the restriction $\|k\| = 1$ implies that h must satisfy

$$\int_0^1 h^2(s)ds = \int (h(V(x)))^2 dV(x) = \int (\mathcal{T}^{-1}k(x))^2 dF_0 = \|\mathcal{T}^{-1}k\|^2 = 1.$$

This ends the proof of our proposition.

6 Performing the test.

6.1 Computing the test statistic.

Let us abbreviate $T_n = T_n^{(\hat{a})}$. The same changes of variables made in §3 lead us to write our optimum test variable as

$$T_n = \int_0^1 \int_0^1 C(t,u)dw_n^{(\hat{a})}(V^{-1}(t))dw_n^{(\hat{a})}(V^{-1}(u)),$$

$$= \frac{1}{n}\sum_{i,j=1}^{n} \int_0^1 \int_0^1 C(t,u)dw_{X_i}^{(\hat{a})}(V^{-1}(t))dw_{X_j}^{(\hat{a})}(V^{-1}(u)).$$

It is easily verified that for any measurable g, $\int g(x)dw_X^{(a)}(x) = \mathcal{T}(ag)(X)$, and therefore, with the notations

$$\mathcal{T}_x g(x,y)|_{x=X} = \mathcal{T}(g(\bullet,y))(X),\ \mathcal{T}_y g(x,y)|_{y=Y} = \mathcal{T}(g(x,\bullet))(Y), \tag{21}$$

we are lead to the expression $T_n = \frac{1}{n}\sum_{i,j=1}^{n} S(X_i, X_j)$ with

$$S(X,Y) = \mathcal{T}_x\mathcal{T}_y\hat{a}(x)\hat{a}(y)C(V(x),V(y))|_{x=X,y=Y} \tag{22}$$

that points out that T_n is a second-order U-statistic.

6.2 Critical regions and power.

The critical region $T_n > \kappa(\alpha)$ with κ defined by

$$\mathbf{P}\left\{\int_0^1 \int_0^1 C(t,u)dw(t)dw(u) > \kappa(\alpha)\right\} = \alpha$$

provides a test for $\mathcal{H}_0$ consistent under any fixed alternative, with asymptotic level equal to α. Its asymptotic power is

$$\pi(\delta) = \mathbf{P}\left\{\int_0^1 \int_0^1 C(t,u)(dw(t) + \delta dt)(dw(u) + \delta du) > \kappa(\alpha)\right\}. \quad (23)$$

The values of $\kappa(\alpha)$ and $\pi(\delta)$ indicated in tables 1 and 2, were obtained by simulations based on 8000 replications. The table indicates two other series of power values, with the purpose of comparison: the computed asymptotic power π^* of the modified Kolmogorov-Smirnov test introduced in [2] for optimum score function, and the asymptotic power $\hat{\pi}(\delta) = 1 - \Phi(\Phi^{-1}(1-\alpha/2) - \delta) + \Phi(\Phi^{-1}(\alpha/2) - \delta)$ of the two-sided test with critical region $|\Lambda|$ >constant, where Λ denotes the typified logarithm of the likelihood ratio. It will be noticed that the performance of all three tests is very much the same.

α	1%	2.5%	5%	10%
$\kappa(\alpha)$	3.78	2.97	2.40	1.84

Table 1: Numerical approximation of the critical values $\kappa(\alpha)$ for sizes α = 1, 2.5, 5 and 10%.

δ	0	0.2	0.4	0.6	0.8	1.0	1.2	1.4
$\pi(\delta)(\%)$	5.0	5.3	6.6	8.9	12.3	16.8	22.3	28.7
$\pi^*(\delta)(\%)$	5.0	5.4	6.7	8.9	12.0	16.2	21.3	27.3
$\hat{\pi}(\delta)(\%)$	5.0	5.5	6.9	9.2	12.6	17.0	22.4	28.8
δ	1.6	1.8	2.0	2.2	2.4	2.6	2.8	3.0
$\pi(\delta)(\%)$	35.7	43.2	52.0	59.9	67.2	74.3	80.2	85.0
$\pi^*(\delta)(\%)$	34.1	41.4	49.1	56.9	64.4	71.4	77.7	83.1
$\hat{\pi}(\delta)(\%)$	36.0	43.6	51.6	59.5	67.0	73.9	80.0	85.1

Table 2: Asymptotic powers $\pi(\delta)$ of the proposed test, $\pi^*(\delta)$ of the Modified K-S Test with optimum score function, and $\hat{\pi}(\delta)$ of the two-sided test based on the logarithm of the likelihood ratio, for a level of significance of 5%, as a function of δ.

6.3 Example: Goodness-of-fit to standard normal.

We finally indicate for completeness (see [3] for details) how to compute the statistic $S(X,Y)$ defined in (22) for the particular isometry

$$\mathcal{T}g = g - \int_{-\infty}^{\bullet} \frac{g(t)d\Phi(t)}{1-\Phi(t)},\ \mathcal{T}^{-1}h = h + \frac{1}{1-\Phi(\bullet)}\int_{-\infty}^{\bullet} h(t)d\Phi(t)$$

in two simple cases:

6.3.1 Case 1: test sensitive to changes in position.

We assume $F_0(x) = \Phi(x) = \int_{-\infty}^{x}\varphi(t)dt$, $\varphi(x) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2}$, $f^{(\tau)}(x) = \frac{\varphi(x-\tau)}{\varphi(x)} = \mathrm{e}^{-\frac{\tau^2-2x\tau}{2}}$, hence $k(x) = \lim_{\tau\to 0^+}\frac{2}{\tau}(\mathrm{e}^{-\frac{\tau^2-2x\tau}{4}} - 1) = x$, $\|k\|^2 = \int_{-\infty}^{\infty} x^2\varphi(x)dx = 1$, $\hat{a}(x) = \mathcal{T}^{-1}(x) = x + \frac{1}{1-\Phi(x)}\int_{-\infty}^{x} t\varphi(t)dt = x - \frac{\varphi(x)}{1-\Phi(x)}$, and therefore $V(x) = \int_{-\infty}^{x}\left(t - \frac{\varphi(t)}{1-\Phi(t)}\right)^2\varphi(t)dt = \int_{-\infty}^{x} t^2\varphi(t)dt + \frac{\varphi^2(x)}{1-\Phi(x)}$. Once we have the analytical expressions of C, $\hat{a}$ and V, S can be computed by means of a simple algorithm, involving numerical integration.

6.3.2 Case 2: test sensitive to changes in dispersion.

Let $F^{(\tau)}(x) = \Phi((1-\frac{\tau}{\sqrt{2}})x)$, so that $f^{(\tau)}(x) = (1-\frac{\tau}{\sqrt{2}})\varphi((1-\frac{\tau}{\sqrt{2}})x)/\varphi(x)$ $= (1-\frac{\tau}{\sqrt{2}})\mathrm{e}^{\frac{1}{2}x^2(1-(1-\frac{\tau}{\sqrt{2}})^2)}$, $k(x) = \lim_{\tau\to 0^+}\frac{2}{\tau}\left(\sqrt{1-\frac{\tau}{\sqrt{2}}}\mathrm{e}^{\frac{1}{4}x^2(\sqrt{2}\tau-\tau^2/2)} - 1\right) = \frac{1}{\sqrt{2}}(x^2-1)$, and $\|k\|^2 = \frac{1}{2}\int_{-\infty}^{\infty}(x^2-1)^2\varphi(x)dx = 1$.

The score function is $\hat{a}(x) = \frac{\mathcal{T}^{-1}(x^2-1)}{\sqrt{2}} = \frac{x^2-1}{\sqrt{2}} + \frac{1}{1-\Phi(x)}\int_{-\infty}^{x}\frac{t^2-1}{\sqrt{2}}\varphi(t)dt$ $= \frac{1}{\sqrt{2}}\left[x^2 - 1 - \frac{x\varphi(x)}{1-\Phi(x)}\right]$, and consequently, $V(x) = \frac{1}{2}\int_{-\infty}^{x}\left[t^2 - 1 - \frac{t\varphi(t)}{1-\Phi(t)}\right]^2 dt$ $= \int_{-\infty}^{x}(t^2-1)^2\varphi(t)dt - \frac{x^2\varphi^2(x)}{1-\Phi(x)}$. As in the previous case, numerical integration can be used to compute each evaluation of S.

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ALEJANDRA CABAÑA
acabana@ivic.ve
Instituto Venezolano de Investigaciones Científicas (IVIC),
Departamento de Matemática
y
Universidad Simón Bolívar,
Departamento de Matemática.
Caracas,
Venezuela.

ENRIQUE M. CABAÑA
ecabana@chaja.ccee.edu.uy
Facultades de Ciencias y de Ciencias Económicas y de Administración,
Universidad de la República.
Montevideo,
Uruguay.

Prueba de bondad de ajuste para distribuciones isótropas en el plano basada en procesos empíricos transformados

Juan Kalemkerian

ABSTRACT

In this work we present a goodness of fit test for isotropic distributions in the plane under contiguous alternatives. The same problem has been studied in [4], but here we employ Transformed Empirical Processes as defined in [3]. Besides, we obtain a lower bound for the asymptotic relative efficiency of our test with respect to the Likelihood Ratio Test, both for the same isotropically-invariant family of alternatives. This improves the ARE calculated in [4] where only a simple alternative is considered for the LRT.

RESUMEN

En el presente trabajo se plantea un problema de bondad de ajuste para distribuciones isótropas en el plano bajo alternativas contiguas. El mismo ya ha sido abordado en [4], pero aquí se le introduce la terminología de los Procesos Empíricos Transformados que fueron definidos en [3]. Además se da una cota inferior para el cálculo de la eficiencia relativa asintótica del test propuesto, con respecto al del cociente de verosimilitudes para la misma alternativa múltiple invariante bajo rotaciones. Esto mejora el cálculo similar realizado en [4] donde la potencia del nuevo test con alternativa múltiple se compara con la del test del cociente de verosimilitudes con alternativa simple.

1 Planteamiento del problema, definición de PET y de la región crítica.

Dado un vector (X_1, X_2) con distribución F_0 se dice que su distribución es isótropa, cuando toda rotación de (X_1, X_2) tiene distribución F_0. Es decir que si (X_1, X_2) tiene distribución F_0 y llamamos $R_\alpha(r\cos\phi\ ,\ r\text{sen}\phi) = (r\cos(\phi + \alpha)\ ,\ r\text{sen}(\phi+\alpha))$ la rotación de ángulo α, entonces el vector $R_\alpha(X_1, X_2)$ también tiene distribución F_0 para cualquier α.
Para probar la hipótesis $H_0 : Z = (X_1, X_2) \sim F_0$ isótropa, plantearemos una región crítica S también isótropa, en el sentido de que cumpla con la siguiente propiedad: $(Z_1, Z_2, ..., Z_n) \in S$ implica $(R_\alpha(Z_1), R_\alpha(Z_2), ..., R_\alpha(Z_n)) \in S\ \forall\alpha \in (0, 2\pi)$, ya que queremos que la misma sea invariante bajo rotaciones de la alternativa.

Un cálculo directo nos da que si F_0 es isótropa y $X_1 = R\cos\Phi$, $X_2 = R\mathrm{sen}\Phi$ tienen distribución conjunta F_0, entonces R y Φ resultan independientes y además $\Phi \sim U[0, 2\pi]$.
De aquí en más, a las coordenadas polares del vector (X_1, X_2) las llamaremos Y y $2\pi Z$ de manera que si la distribución de (X_1, X_2) es F_0 entonces Z e Y serán independientes y además $Z \sim U[0, 1]$.
En adelante, trabajaremos con las coordenadas polares de (X_1, X_2), por lo que sin posibilidad de confusión, a partir de ahora le llamaremos F_0 a la distribución de (Y, Z).

Sean $F_0 = \tilde{F}_0 \times U[0, 1]$ y $F^{(\tau)}$ distribuciones en $\mathbb{R} \times [0, 1]$ tales que si $f^{(\tau)} = \frac{dF^{(\tau)}}{dF_0}$ entonces $\sqrt{f^{(\tau)}} = 1 + \frac{\tau}{2}k_\tau$ donde k_τ cumple las siguientes condiciones:

1. $\exists\, K \in L^1(\mathbb{R} \times [0, 1]\ ,\ dF_O)\ /\ k_\tau^2 \leq K \quad \forall \tau$

2. $\exists\, k \in L^2(\mathbb{R} \times [0, 1]\ ,\ dF_O)\ /\ k_\tau \xrightarrow{L^2} k$ si $\tau \to 0$

Dadas $T : L^2(\mathbb{R} \times [0, 1]\ ,\ dF_O) \longrightarrow L^2(\mathbb{R} \times [0, 1]\ ,\ dF_O)$ isometría con recorrido ortogonal a 1, $a \in L^2(\mathbb{R} \times [0, 1]\ ,\ dF_O)$, $||a|| = 1$ y la familia

$$\mathcal{A} = \{A \subset \mathbb{R} \times [0, 1]\ ,\ A = (-\infty, y] \times [z_1, z_2]\ ,\ y \in \mathbb{R}\ ,\ 0 \leq z_1 < z_2 \leq 1\}$$

definimos el proceso $\{W_{(Y,Z)}^{(a,T)}(A)\}_{A \in \mathcal{A}}$ mediante $W_{(Y,Z)}^{(a,T)}(A) = T(a1_A)(Y, Z)$.

Dada una muestra $(Y_1, Z_1), \ldots, (Y_n, Z_n)$ de (Y, Z), le llamaremos Proceso Empírico Transformado (de aquí en más lo abreviamos como PET) a

$$W_n^{(a,T)}(A) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} W_{(Y_i,Z_i)}^{(a,T)}(A).$$

Dado $u > 0$ llamaremos $[u]$ a la parte fraccionaria de u, es decir, al valor $u - m$ siendo $m \in \mathbf{N}$ tal que $m \leq u < m + 1$. Nos planteamos probar

$$H_0 : (Y, Z) \sim F_0 = \tilde{F}_0 \times U[0, 1]$$

contra las alternativas

$$H_n : \exists\, v \in (0, 1) \text{ tal que } (Y, [Z + v]) \sim F^{(\delta/\sqrt{n})}$$

(En las alternativas se pide que alguna rotación de (X_1, X_2) tenga la distribución $F^{(\delta/\sqrt{n})}$.)
Proponemos un test basado en PETs cuya región crítica sea la siguiente:

$$S = \{\sup_{A \in \mathcal{A}} |W_n^{(a,T)}(A) - W_n^{(a,T)}(A^c)|\ >\ c\}$$

Observamos que, como queremos que nuestra región crítica sea invariante por rotaciones (que en el contexto de nuestra hipótesis nula, son traslaciones en la variable Z), debemos asegurarnos que si $((Y_1, Z_1); ...; (Y_n, Z_n)) \in S$ entonces $((Y_1, [Z_1 + v]); ...; (Y_n, [Z_n + v])) \in S \ \forall v$.

2 Preliminares, contigüidad, resultados asintóticos.

En esta sección describiremos rápidamente la definición de contigüidad y los resultados que necesitaremos. Las demostraciones de los mismos se pueden encontrar en [5] [9] y [11].

DEFINICIÓN *Se llama experimento binario en el espacio probabilizable* $(\Omega, \mathcal{A})$ *a un par de distribuciones de probabilidad* P_0 *y* P_1 *en* $(\Omega, \mathcal{A})$.

DEFINICIÓN *Consideremos* $(\Omega_n, \mathcal{A}_n, P_{0,n}, P_{1,n})_{n \in \mathbf{N}}$, *sucesión de experimentos binarios, diremos que la sucesión de probabilidades* $(P_{1,n})_{n \in \mathbf{N}}$ *es contigua a la sucesión* $(P_{0,n})_{n \in \mathbf{N}}$ *cuando cualquier sucesión de estadísticos* $T_n : \Omega_n \longrightarrow \mathbb{R}$ *que converja en* $P_{0,n}$ *probabilidad a 0, converge también en* $P_{1,n}$ *probabilidad a 0.*
Es decir que si $P_{0,n}(|T_n| > \varepsilon) \to 0 \ \forall \varepsilon > 0$ *entonces* $P_{1,n}(|T_n| > \varepsilon) \to 0 \ \forall \varepsilon > 0$.

DEFINICIÓN *Sean dos probabilidades* P_0 *y* P_1 *y* μ *una medida respecto de la cual* P_0 *y* P_1 *son absolutamente continuas (por ejemplo,* $\mu = P_0 + P_1$ *). Llamaremos distancia de Hellinger entre* P_0 *y* P_1 *a*

$$h(P_0, P_1) = \sqrt{\frac{1}{2} \int \left(\sqrt{\frac{dP_1}{d\mu}} - \sqrt{\frac{dP_0}{d\mu}} \right)^2 d\mu}.$$

Se verifica en forma inmediata que esta definición no depende de la medida μ con tal que P_0 y P_1 sean absolutamente continuas respecto de μ.

El Tercer Lema de Le Cam.

TEOREMA *Si* $(P_{1,n})$ *es una sucesión de probabilidades contigua a* $(P_{0,n})$, *llamamos* $\Lambda_n = \sum_{i=1}^{n} \log(\frac{dP_{1,n}}{dP_{0,n}})$ *al logaritmo del cociente de verosimilitudes, y* T_n *es una sucesión de estadísticos tales que* $(T_n(X_n), \Lambda_n(X_n)) \xrightarrow{\mathcal{L}} F$ *cuando* $X_n \sim P_{0,n}$, *entonces, cuando* $X_n \sim P_{1,n}$

$$(T_n(X_n), \Lambda_n(X_n)) \xrightarrow{\mathcal{L}} G \text{ donde } dG(t, z) = e^z dF(t, z)$$

La demostración se puede encontrar en [5] , [7] y [9]. También en [5] se deducen los siguientes corolarios.

COROLARIO 1 *Si* $P_{1,n}$ *es contigua respecto de* $P_{0,n}$ *y si bajo* $(P_{0,n})$, Λ_n *tiene distribución asintótica* $N(\mu, \sigma^2)$ *no degenerada (* $\sigma^2 \neq 0$*) entonces, bajo* $(P_{1,n})$, Λ_n *tiene distribución asintótica* $N(-\mu, \sigma^2)$ *y además, se cumple necesariamente que* $\mu = -\frac{\sigma^2}{2}$.

COROLARIO 2 *Si cuando* $X_n \sim P_{0,n}$ *la pareja* $(T_n(X_n), \Lambda_n(X_n))$ *es asintóticamente*

$$Normal\left(\left(\begin{array}{c}\tau \\ \mu\end{array}\right);\left(\begin{array}{cc}\Sigma & \gamma^{tr} \\ \gamma & \sigma^2\end{array}\right)\right); \qquad (1)$$

entonces , cuando $X_n \sim P_{1,n}$*, la distribución asintótica de* $(T_n(X_n), \Lambda_n(X_n))$ *es*

$$Normal\left(\left(\begin{array}{c}\tau+\gamma \\ -\mu\end{array}\right);\left(\begin{array}{cc}\Sigma & \gamma^{tr} \\ \gamma & \sigma^2\end{array}\right)\right)$$

y, además, $\mu = -\frac{1}{2}\sigma^2$.

Una condición necesaria y suficiente para la contigüidad.

Consideremos un arreglo triangular $\{X_{n,j} \ : \ 1 \leq j \leq J_n \ , \ n = 1, 2, 3, \ldots\}$ de variables independientes, y, para $i = 0$ ó 1 llamemos $P_{i,n}$ a la distribución de $(X_{n,1}, \ldots, X_{n,J_n})$ en $\mathbb{R}^{J_n}$ cuando $X_{n,j} \sim P_{i,n,j}$. Es decir, $P_{i,n} = \otimes_{j=1}^{J_n} P_{i,n,j}$. Para simplificar la notación supondremos en lo que sigue $J_n = n$.

TEOREMA *La sucesión* $(P_{1,n} = \otimes_{j=1}^{n} P_{1,n,j})_{n\in\mathbf{N}}$ *es contigua a*

$$(P_{0,n} = \otimes_{j=1}^{n} P_{0,n,j})_{n\in\mathbf{N}}$$

si y sólo si

$$(i) \ \lim_{n\to+\infty} \sup \sum_{j=1}^{n} h^2(P_{0,n,j}, P_{1,n,j}) < +\infty$$

y

$$(ii) \ \lim_{n\to+\infty} \sum_{j=1}^{n} P_{1,n,j}\left(\frac{dP_{1,n,j}}{dP_{0,n,j}} \geq c_n\right) = 0 \ \forall \ c_n \to +\infty.$$

La demostración de este teorema es de un trabajo de Oosterhoff y Van Zwet y se puede ver en [11] y también en [5].

TEOREMA *Si consideramos $P_{0,n,j} = F_0$ $\forall n, j$, para cada n las medidas $P_{1,n,j}$ tienen densidad $f^{(\delta/\sqrt{n})}$ respecto de F_0, y el desarrollo*

$$\sqrt{f^{(\delta/\sqrt{n})}} = 1 + \frac{\delta k_n}{2\sqrt{n}}$$

cumple las condiciones (1) y (2) de la introducción, es decir,
(1) *existe una función K tal que $\int K dF_0 < +\infty$ y $k_n^2 \leq K$ $\forall n$*
(2) *existe $k \in L^2(\Omega, dF_0)$ tal que $k_n \xrightarrow{L^2} k$*
entonces,

$$P_{1,n} = \otimes_{j=1}^{n} P_{1,n,j} \text{ es contigua a } P_{0,n} = \otimes_{j=1}^{n} P_{0,n,j}.$$

La demostración se puede encontrar en [5] y se reduce a verificar las hipótesis del teorema anterior. Como aplicación estadística, se demuestra en [5] que bajo ciertas condiciones impuestas sobre a y T, se cumple que bajo $P_{0,n}$, y para todo $A = (A_1, ..., A_k)$ la distribución conjunta asintótica del par $(W_n^{(a,T)}(A), \Lambda_n)$ es normal, lo que veremos en la siguiente sección.

3 Construcción de la isometría T.

Aquí daremos ejemplos de isometrías T y veremos que bajo ciertas hipótesis sobre T y a, los PETs convergerán débilmente, bajo H_0, a un V-proceso de Wiener donde $V(A) = \int_A a^2 dF_0$ y bajo H_n convergerán a un proceso tal que para cada A, la distribución es la misma que la de H_0 más un sesgo que depende de a y que tiene la siguiente forma:

$$\delta \int T(a 1_A - a 1_{A^c}) k(y, [z + v]) dF_0(y, z).$$

Sea $\tilde{T}$ isometría en $L^2(\mathbb{R}, d\tilde{F}_0)$ con recorrido ortogonal a 1.
Definimos $T : L^2(\mathbb{R} \times [0, 1]\ ,\ dF_0) \longrightarrow L^2(\mathbb{R} \times [0, 1]\ ,\ dF_0)$ tal que
$T(g(.,.))(y, z) = g(y, z) - \tilde{g}(y) + \tilde{T}(\tilde{g}(.))(y)$ donde $\tilde{g}(y) = \mathbf{E}(g(Y, Z)/Y = y)$.
Observamos que $\mathbf{E}(\tilde{g}(Y)) = \mathbf{E}(g(Y, Z))$ por lo que $g \in 1^{\perp} \iff \tilde{g} \in 1^{\perp}$.

Se verifica en [5] que T es una isometría con rango ortogonal a 1 y que si $h \in L^2(\mathbb{R} \times [0,1]\ ,\ dF_0)$ es tal que $\tilde{h} \in \text{Rec}(\ \tilde{T}\ )$, entonces

$$T^{-1}(h)(y,z) = h(y,z) - \tilde{h}(y) + \tilde{T}^{-1}(\tilde{h})(y)$$

Si tomamos $A \in \mathcal{A} \Rightarrow A = (-\infty, y'] \times [z_1, z_2]$, si llamamos $B = (-\infty, y']$ y $C = [z_1, z_2]$, y elegimos $a(y,z) = a(y)$, tenemos que

$$W^{(a,T)}_{(y,z)}(A) = T(a\mathbf{1}_A(.,.))(y,z)$$

$$= a(y)\mathbf{1}_B(y)\mathbf{1}_C(z) - a(y)\mathbf{1}_B(y)\mathbf{E}(\mathbf{1}_C(Z)) + (z_2 - z_1)\tilde{T}(a(.)\mathbf{1}_B(.))(y)$$

$$= a(y)\mathbf{1}_B(y)\mathbf{1}_C(z) - a(y)\mathbf{1}_B(y)(z_2 - z_1) + (z_2 - z_1)\tilde{T}(a(.)\mathbf{1}_B(.))(y)$$

Observamos entonces que eligiendo la función a independiente de la variable en z, y con esta definición de T a partir de una $\tilde{T}$ cualquiera en $L^2(\mathbb{R}, d\tilde{F}_0)$, la región crítica propuesta $S = \{\sup_{A \in \mathcal{A}} |W_n^{(a,T)}(A) - W_n^{(a,T)}(A^c)| > c\}$ es invariante por traslaciones de la variable z. Bajo ciertas hipótesis, que veremos en el punto siguiente, tendremos que bajo H_0, el proceso $\{W_n^{(a,T)}(A)\}$ converge débilmente al proceso de Wiener centrado $\{W^V(A)\}$ asociado a la medida $V(A) = \int_A a^2(y) dF_0(y,z)$, mientras que para cada v bajo las alternativas H_n, el proceso $\{W_n(A)\}$ converge débilmente al proceso

$$\left\{W^V(A) + \delta \int k(y,[z+v])T(a\mathbf{1}_A)(y,z)dF_0(y,z)\right\}$$

o sea al mismo proceso de Wiener que el de la hipótesis nula más un sesgo. Para mejorar la eficiencia de nuestro test, en cada caso particular se buscará aquella función a que maximice este sesgo para algún conjunto A elegido convenientemente.

Verifiquemos que bajo las hipótesis (1) y (2), el sesgo es

$$\delta \int k(y,[z+v])T(a\mathbf{1}_A)(y,z)dF_0(y,z).$$

En efecto, bajo H_n tenemos que $(Y,Z) \sim F^{(\delta/\sqrt{n})}(y,[z+v])$, entonces

$$E(W_n^{(a,T)}(A)) = \sqrt{n}E(W^{(a,T)}_{(Y,Z)}(A))$$

$$= \sqrt{n} \int T(a\mathbf{1}_A)(y,z)dF^{(\delta/\sqrt{n})}(y,[z+v])$$

$$= \sqrt{n} \int T(a\mathbf{1}_A)(y,z) f^{(\delta/\sqrt{n})}(y,[z+v]) dF_0(y,[z+v])$$

$$= \sqrt{n} \int T(a\mathbf{1}_A)(y,z) f^{(\delta/\sqrt{n})}(y,[z+v]) dF_0(y,z).$$

La última igualdad se debe a que F_0 es invariante bajo traslaciones en la variable en z.
Concluimos entonces que

$$E(W_n^{(a,T)}(A)) = \sqrt{n} \int T(a1_A)(y,z) \left[1 + \frac{\delta}{2\sqrt{n}} k_n(y,[z+v])\right]^2 dF_0(y,z)$$

$$= \sqrt{n} \int T(a\mathbf{1}_A)(y,z) \left[1 + \frac{\delta}{\sqrt{n}} k_n(y,[z+v]) + \frac{\delta^2}{4n} k_n^2(y,[z+v])\right] dF_0(y,z)$$

Separando en 3 integrales, el primer sumando es 0 porque $\mathrm{Rec}(T) = 1^{\perp}$ y el último tiende a 0 con n por lo que el sesgo asintótico queda

$$\delta \int T(a\mathbf{1}_A)(y,z) k(y,[z+v]) dF_0(y,z)$$

Comportamiento asintótico del PET.

Para los PETs es posible aplicar un Teorema Central del Límite y luego el Tercer Lema de Le Cam para concluir que la distribución asintótica de $W_n^{(a,T)}(A)$ es la de $W^V(A) + \delta \int_A aT^{-1}k dF_0$ siendo $V(A) = \int_A a^2 dF_0$.
La convergencia en distribución de $\{W_n^{(a,T)}(A)\}$ a $\{W^V(A)\}$ sobre la familia $\mathcal{A}$ está asegurada por un Teorema Central del Límite de Ossiander [12] cuando se cumplen las siguientes hipótesis:

1. **TCL1** la familia de integrandos $\mathcal{G} = \{\mathcal{T}(\dashv\mathbf{1}_{\mathcal{A}}) \ / \ \mathcal{A} \in \mathcal{A}\}$ está uniformemente acotada por una función $G \in L^2(I\!\!R \times [0,1], dF_0)$ y

2. **TCL2**

$$\int_0^1 \sqrt{log N_{[\,]}^{(2)}(u,\mathcal{G},F_0) du} < +\infty$$

siendo $N_{[\,]}^{(2)}(\varepsilon,\mathcal{G},F_0)$ igual al mínimo ν para el cual existen conjuntos $\mathcal{L} = \{l_1,...,l_\nu\}$ y $\mathcal{U} = \{u_1,...,u_\nu\}$ de funciones de cuadrado integrable tales que, para cada $f \in \mathcal{G}$ se encuentran $l \in \mathcal{L}$ y $u \in \mathcal{U}$ de modo que $l < f < u$ y $||u - l|| \leq \varepsilon$.

Verifiquemos que se cumplen estas hipótesis si consideramos como $\tilde{T}$ a la transformación de Laguerre: $\tilde{T}(h)(y) = h(y) - \int_{-\infty}^{y} \frac{h(u)}{1-\tilde{F}_0(u)} d\tilde{F}_0(u)$
En dicho caso, nos queda

$$T(a\mathbf{1}_{(-\infty,y']}\mathbf{1}_{[z_1,z_2]})(y,z)$$

$$= a(y)\mathbf{1}_{(-\infty,y']}(y)\mathbf{1}_{[z_1,z_2]}(z) - (z_2-z_1)\int_{-\infty}^{y\wedge y'} \frac{a(t)}{1-\tilde{F}_0(t)} d\tilde{F}_0(t)$$

TEOREMA *Si para algún* $\theta > 0$ *la función* $\frac{a}{(1-\tilde{F}_0)^\theta} \in L^2(\mathbb{R}, d\tilde{F}_0)$ *entonces* T *cumple* **TCL1** .

Demostración.
En efecto, $|T(a\mathbf{1}_{(-\infty,y']}\mathbf{1}_{[z_1,z_2]})(y,z)| \leq |a| + \int_{-\infty}^{y} \frac{|a(t)|d\tilde{F}_0}{1-\tilde{F}_0}$.
Verifiquemos que $\int_{-\infty}^{\cdot} \frac{a}{1-\tilde{F}_0} d\tilde{F}_0 \in L^2$.
Suponemos sin pérdida de generalidad que $\theta < 1/2$.

$$\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{y} \frac{|a(s)|}{1-\tilde{F}_0(s)} d\tilde{F}_0(s)\right)^2 d\tilde{F}_0(y)$$

$$\leq \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{y} \frac{a^2(s)d\tilde{F}_0(s)}{(1-\tilde{F}_0(s))^{2\theta}} \int_{-\infty}^{y} \frac{d\tilde{F}_0(s)}{(1-\tilde{F}_0(s))^{2-2\theta}}\right) d\tilde{F}_0(y)$$

$$\leq \left\|\frac{a}{(1-\tilde{F}_0)^\theta}\right\|^2 \frac{1}{1-2\theta}\int_{-\infty}^{+\infty}\left(\frac{1}{(1-\tilde{F}_0(y))^{1-2\theta}} - 1\right) d\tilde{F}_0(y)$$

$$\leq \left\|\frac{a}{(1-\tilde{F}_0)^\theta}\right\|^2 \frac{1}{1-2\theta}\int_{-\infty}^{+\infty} \frac{d\tilde{F}_0(y)}{(1-\tilde{F}_0(y))^{1-2\theta}}$$

$$= \frac{1}{2\theta(1-2\theta)}\left\|\frac{a}{(1-\tilde{F}_0)^\theta}\right\|^2$$

TEOREMA *Bajo las mismas hipótesis del teorema anterior,* T *cumple* **TCL2**.

Demostración.
Dado $\varepsilon > 0$, construimos $-\infty = y_0 < y_1, ..., < y_m = +\infty$ partición de $\mathbb{R}$ tal que:

$$\int_{y_{i-1}}^{y_i} \frac{a^2}{(1-\tilde{F}_0)^{2\theta}} d\tilde{F}_0 \leq \frac{\theta(1-2\theta)\varepsilon^2}{32}$$

Una partición que satisfaga estos requisitos se puede lograr con

$$m \geq 1 + \frac{32 \left\| \frac{a}{(1-\tilde{F}_0)^\theta} \right\|^2}{\varepsilon^2 \theta (1-2\theta)}$$

intervalos.
Observamos entonces que

$$\int_{-\infty}^{+\infty} \left(\int_{y_{i-1}\wedge y}^{y_i \wedge y} \frac{|a|}{1-\tilde{F}_0} d\tilde{F}_0 \right)^2 d\tilde{F}_0(y) = \int_{y_{i-1}}^{+\infty} \left(\int_{y_{i-1}}^{y_i \wedge y} \frac{|a|}{1-\tilde{F}_0} d\tilde{F}_0 \right)^2 d\tilde{F}_0(y)$$

$$\leq \int_{-\infty}^{+\infty} \left(\int_{y_{i-1}}^{y_i \wedge y} \frac{a^2}{(1-\tilde{F}_0)^{2\theta}} d\tilde{F}_0 \int_{y_{i-1}}^{y} \frac{1}{(1-\tilde{F}_0)^{2-2\theta}} d\tilde{F}_0 \right) d\tilde{F}_0(y)$$

$$\leq \int_{y_{i-1}}^{y_i} \frac{a^2}{(1-\tilde{F}_0)^{2\theta}} d\tilde{F}_0 \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{y} \frac{1}{(1-\tilde{F}_0)^{2-2\theta}} d\tilde{F}_0 \right) d\tilde{F}_0(y)$$

$$= \int_{y_{i-1}}^{y_i} \frac{a^2}{(1-\tilde{F}_0)^{2\theta}} d\tilde{F}_0 \frac{1}{1-2\theta} \int_{-\infty}^{+\infty} \left(\frac{1}{(1-\tilde{F}_0(y))^{1-2\theta}} - 1 \right) d\tilde{F}_0(y)$$

$$\leq \frac{1}{2\theta(1-2\theta)} \int_{y_{i-1}}^{y_i} \frac{a^2}{(1-\tilde{F}_0)^{2\theta}} d\tilde{F}_0 \leq \frac{\varepsilon^2}{64}$$

Por otra parte, construimos $0 = z^{(0)} < z^{(1)} < ... < z^{(n)} = 1$ partición de $[0,1]$ tal que

$$z^{(j)} - z^{(j-1)} < \frac{\varepsilon}{8} \wedge \frac{\varepsilon\sqrt{2\theta(1-2\theta)}}{16 \left\| \frac{a}{(1-\tilde{F}_0)^\theta} \right\|} \quad \forall j.$$

Una partición que satisfaga este requisito se puede lograr tomando $n \geq (1 + 8/\varepsilon) \vee \left(1 + \frac{16 \left\| \frac{a}{(1-\tilde{F}_0)^\theta} \right\|}{\varepsilon\sqrt{2\theta(1-2\theta)}} \right)$ intervalos.

Sean $y' \in [y_{i-1}, y_i]$, $z_1 \in [z^{(j-1)}, z^{(j)}]$, $z_2 \in [z^{(k-1)}, z^{(k)}]$, $j \leq k$
Si $j = k$ o $j = k-1$, tenemos que

$$(a \wedge 0)\mathbf{1}_{[z^{(j-1)}, z^{(k)}]} \leq a\mathbf{1}_{(-\infty, y']}\mathbf{1}_{[z_1, z_2]} \leq (a \vee 0)\mathbf{1}_{[z^{(j-1)}, z^{(k)}]}$$

$$\left\| (a \vee 0)\mathbf{1}_{[z^{(j-1)}, z^{(k)}]} - (a \vee 0)\mathbf{1}_{[z^{(j-1)}, z^{(k)}]} \right\| \leq \sqrt{2}(z^{(k)} - z^{(j-1)})||a|| < \sqrt{2}\frac{\varepsilon}{4} < \frac{\varepsilon}{2}$$

Por otro lado,

$$(z^{(k)}-z^{(j-1)})\int_{-\infty}^{\cdot}\frac{-|a|d\tilde{F}_0}{1-\tilde{F}_0}\leq(z_1-z_2)\int_{-\infty}^{y'\wedge\cdot}\frac{ad\tilde{F}_0}{1-\tilde{F}_0}\leq(z^{(k)}-z^{(j-1)})\int_{-\infty}^{\cdot}\frac{|a|d\tilde{F}_0}{1-\tilde{F}_0}$$

$$\left\|(z^{(k)}-z^{(j-1)})\int_{-\infty}^{\cdot}\frac{|a|d\tilde{F}_0}{1-\tilde{F}_0}-(z^{(k)}-z^{(j-1)})\int_{-\infty}^{\cdot}\frac{-|a|d\tilde{F}_0}{1-\tilde{F}_0}\right\|=$$

$$2(z^{(k)}-z^{(j-1)})\left\|\int_{-\infty}^{\cdot}\frac{|a|d\tilde{F}_0}{1-\tilde{F}_0}\right\|\leq 2(z^{(k)}-z^{(j-1)})\left\|\frac{a}{(1-\tilde{F}_0)^{\theta}}\right\|\frac{1}{\sqrt{2\theta(1-2\theta)}}<\frac{\varepsilon}{2}$$

Por lo tanto, si $z_1\in[z^{(j-1)},z^{(j)}]$, $z_2\in[z^{(k-1)},z^{(k)}]$, con $j=k$ o $j=k-1$, tendremos que $T(a\mathbf{1}_{(-\infty,y']}\mathbf{1}_{[z_1,z_2]})$ está acotado superiormente por:

$$u_j=(a\vee 0)\mathbf{1}_{[z^{(j-1)},z^{(k)}]}+(z^{(k)}-z^{(j-1)})\int_{-\infty}^{\cdot}\frac{|a|d\tilde{F}_0}{1-\tilde{F}_0}$$

e inferiormente por

$$l_j=(a\wedge 0)\mathbf{1}_{[z^{(j-1)},z^{(k)}]}-(z^{(k)}-z^{(j-1)})\int_{-\infty}^{\cdot}\frac{|a|d\tilde{F}_0}{1-\tilde{F}_0}$$

Estas cotas satisfacen

$$||u_j-l_j||<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

Si $j\leq k-2$, definimos

$$l_{ijk}^{(1)}=a\mathbf{1}_{(-\infty,y_{i-1})}\mathbf{1}_{[z^{(j)},z^{(k-1)}]}-|a|\mathbf{1}_{[y_{i-1},y_i]}\mathbf{1}_{[z^{(j-1)},z^{(j)}]\cup[z^{(k-1)},z^{(k)}]}$$

$$u_{ijk}^{(1)}=a\mathbf{1}_{(-\infty,y_{i-1})}\mathbf{1}_{[z^{(j)},z^{(k-1)}]}+|a|\mathbf{1}_{[y_{i-1},y_i]}\mathbf{1}_{[z^{(j-1)},z^{(j)}]\cup[z^{(k-1)},z^{(k)}]}$$

tenemos que

$$l_{ijk}^{(1)}\leq a\mathbf{1}_{(-\infty,y']}\mathbf{1}_{[z_1,z_2]}\leq u_{ijk}^{(1)}$$

donde

$$\|u_{ijk}^{(1)}-l_{ijk}^{(1)}\|=2\left\||a|\mathbf{1}_{[y_{i-1},y_i]}\mathbf{1}_{[z^{(j-1)},z^{(j)}]\cup[z^{(k-1)},z^{(k)}]}\right\|$$
$$\leq 2\left(z^{(j)}-z^{(j-1)}+z^{(k)}-z^{(k-1)}\right)<\frac{\varepsilon}{2}$$

Por otro lado, $-(z_2-z_1)\int_{-\infty}^{y'\wedge\cdot}\frac{a}{1-\tilde{F}_0}d\tilde{F}_0$ está acotada superiormente por

$$u_{ijk}^{(2)}=-(z^{(k-1)}-z^{(j)})\int_{-\infty}^{y_{i-1}\wedge\cdot}\frac{a}{1-\tilde{F}_0}d\tilde{F}_0+$$

$$\left(z^{(j)} - z^{(j-1)} + z^{(k)} - z^{(k-1)}\right) \int_{-\infty}^{\cdot} \frac{|a|}{1-\tilde{F}_0} d\tilde{F}_0 + \int_{y_{i-1}\wedge .}^{y_i \wedge .} \frac{|a|}{1-\tilde{F}_0} d\tilde{F}_0$$

e inferiormente por

$$l_{ijk}^{(2)} = -\left(z^{(k-1)} - z^{(j)}\right) \int_{-\infty}^{y_{i-1}\wedge .} \frac{a}{1-\tilde{F}_0} d\tilde{F}_0$$

$$-(z^{(j)} - z^{(j-1)} + z^{(k)} - z^{(k-1)}) \int_{-\infty}^{\cdot} \frac{|a|}{1-\tilde{F}_0} d\tilde{F}_0 - \int_{y_{i-1}\wedge .}^{y_i \wedge .} \frac{|a|}{1-\tilde{F}_0} d\tilde{F}_0$$

Además, estas cotas satisfacen

$$\|u_{ijk}^{(2)} - l_{ijk}^{(2)}\| \leq 2(z^{(j)} - z^{(j-1)} + z^{(k)} - z^{(k-1)}) \left\| \int_{-\infty}^{\cdot} \frac{|a| d\tilde{F}_0}{1-\tilde{F}_0} \right\| + 2 \left\| \int_{y_{i-1}\wedge .}^{y_i \wedge .} \frac{|a| d\tilde{F}_0}{1-\tilde{F}_0} \right\|$$

$$\leq 2(z^{(j)} - z^{(j-1)} + z^{(k)} - z^{(k-1)}) \left\| \frac{|a|}{(1-\tilde{F}_0)^{\theta}} \right\| \frac{1}{\sqrt{2\theta(1-2\theta)}} + 2 \left\| \int_{y_{i-1}\wedge .}^{y_i \wedge .} \frac{|a| d\tilde{F}_0}{1-\tilde{F}_0} \right\| < \frac{\varepsilon}{2}$$

En conclusión, definiendo $u_{ijk} = u_{ijk}^{(1)} + u_{ijk}^{(2)}$ y $l_{ijk} = l_{ijk}^{(1)} + l_{ijk}^{(2)}$, tenemos que

$$l_{ijk} \leq T\left(a \mathbf{1}_{(-\infty, y']} \mathbf{1}_{[z_1, z_2]}\right) \leq u_{ijk}$$

donde

$$\|u_{ijk} - l_{ijk}\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

En total, hemos construido $\frac{c_1}{\varepsilon} + \frac{c_2}{\varepsilon^3}$ funciones u y otras tantas funciones l tales que $\forall y' \in I\!R$, $\forall z_1 < z_2 \in [0,1]$, $T\left(a \mathbf{1}_{(-\infty, y']} \mathbf{1}_{[z_1, z_2]}\right)$ está acotada superiormente por alguna u e inferiormente por alguna l.
Entonces $N_{[\,]}^{(2)} \leq \frac{c_1}{\varepsilon} + \frac{c_2}{\varepsilon^3}$ por lo que vale **TCL2** .

4 Dos ejemplos concretos de alternativas: cambios de posición y de dispersión.

En esta sección plantearemos dos casos concretos de alternativas: cambios de posición y cambios de dispersión. Hallaremos las respectivas funciones k, y para mejorar la eficiencia del test, adoptaremos como criterio hallar el sesgo máximo sobre las funciones a y los conjuntos A. Esto se hará en cada caso particular y a la función a hallada le llamaremos óptima.

Bajo H_0 : (X_1, X_2) tiene distribución isótropa, si llamamos $y = \sqrt{x_1^2 + x_2^2}$, $z = \frac{\arg(x_1, x_2)}{2\pi}$, tendremos que si f_0 es la densidad bajo H_0, ésta dependerá

sólo de y esto es, $f_0(x_1, x_2) = \overline{f}_0(y)$.
Supongamos que las $F^{(\tau)}$ que aparecen en las H_n son tales que $\exists\ T_\tau : R^2 \longrightarrow R^2$ tal que $F^{(\tau)}$ es la distribución de $T_\tau^{-1}(X_1, X_2) = T_\tau^{-1}(X)$ donde X distribuye como en H_0, y que $\overline{f_0}$ admite derivada primera continua.

Se puede ver en [3] que la función k puede ser calculada en la siguiente forma:

$$k(x) = k(x_1, x_2) = \frac{\overline{f}_0'(y)}{\overline{f}_0(y)} \frac{\langle x, D_x \rangle}{y} + J(x)$$

siendo $J(x) = \frac{\partial}{\partial \tau} |\det J_\tau(x)||_{\tau=0}$ donde $J_\tau(x)$ es la matriz jacobiana de la transformación T_τ, $D_x = \lim_{\tau \to 0} \frac{1}{\tau}(T_\tau(x) - x)$.

Lo que allí se hace es calcular k como $\lim_{\tau \to 0} \frac{1}{\tau} \frac{f^{(\tau)}(x) - f_0(x)}{f_0(x)}$.

Caso I. Cambios de posición.

Supongamos que las alternativas son de la forma $F^{(\tau)}(x_1, x_2) = F_0(x_1 - \tau, x_2)$.

Cálculo de k.
Cuando $T_\tau(x) = x + \tau e_1$ se tiene que $D_x = e_1$,
$J_\tau(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ lo que implica $J(x) = 0$
Por lo tanto

$$k(x) = \frac{\overline{f}_0'(y)}{\overline{f_0}(y)} \frac{x_1}{y} \Rightarrow k(y, z) = \frac{\overline{f}_0'(y)}{\overline{f}_0(y)} \cos(2\pi z)$$

En el caso particular de que en la hipótesis nula tengamos la normal bivariada, tendremos que $F_0(x) = \Phi(x_1)\Phi(x_2) = \frac{1}{2\pi} e^{-\frac{1}{2}(x_1^2 + x_2^2)} = \overline{f}_0(y)$, entonces $\overline{f}_0(y) = \frac{1}{2\pi} e^{-\frac{1}{2}y^2}$, $\overline{f}_0'(y) = -\frac{y}{2\pi} e^{-\frac{1}{2}y^2}$ y por lo tanto

$$k(y, z) = -y \cos(2\pi z)$$

Optimización del sesgo.
Hallemos el par (a, A) que maximice el sesgo.
Llamemos $h(y) = \frac{\overline{f}_0'(y)}{\overline{f}_0(y)}$. Observamos que

$$\tilde{k}(y) = \mathbf{E}[h(Y) \cos(2\pi Z) | Y = y] = 0$$

por lo tanto $T^{-1}(k) = k$. Entonces el sesgo es

$$\delta \int h(y)\cos(2\pi z) T(2a\mathbf{1}_A - a)(y,z) dF_0(y,z)$$

$$= \delta \int T^{-1}(h(y)\cos(2\pi z))(y,z)(2a(y)\mathbf{1}_A(y,z) - a(y)) dF_0(y,z)$$

$$= \delta \int h(y)\cos(2\pi z)(2a(y)\mathbf{1}_A(y,z) - a(y)) dF_0(y,z)$$

$$= 2\delta \int_A h(y)a(y)\cos(2\pi z) dF_0(y,z).$$

Entonces, si llamamos $A = B \times [z_1, z_2]$, el sesgo será

$$\frac{2\delta}{2\pi}(\text{sen}(2\pi z_2) - \text{sen}(2\pi z_1)) \int_B a(y)h(y) d\tilde{F}_0(y)$$

$$\leq \frac{2\delta}{\pi}\langle a, h1_B\rangle \leq \frac{2\delta}{\pi}||h1_B|| \leq \delta\frac{2}{\pi}||h|| \ \forall A, \ \forall a \ / \ ||a|| = 1.$$

Por lo tanto el sesgo máximo se consigue si consideramos $A = \mathbb{R} \times [1/4, 3/4]$. y $\tilde{a} = \frac{h}{||h||}$ En el caso particular de la normal bivariada en la hipótesis nula en que $k(y,z) = -y\cos(2\pi z), h(y) = -y \Rightarrow ||h||^2 = \mathbf{E}(Y^2) = 2$ por lo que el sesgo máximo es $\delta\frac{2\sqrt{2}}{\pi}$, y la función óptima es $\tilde{a}(y) = \frac{-y}{\sqrt{2}}$.

Caso II. Cambios de dispersión.

Supongamos alternativas de la forma $F^{(\tau)}(x_1, x_2) = F_0(\frac{1}{1+\tau}x_1, \frac{1}{1+\tau}x_2)$.

Cálculo de k.
$T_\tau(x) = (1+\tau)x$, entonces $D_x = x$, $\det J_\tau(x) = \det \begin{pmatrix} 1+\tau & 0 \\ 0 & 1+\tau \end{pmatrix} = (1+\tau)^2$, lo que implica $J(x) = 2$. Por lo tanto $k(y,z) = y\frac{\overline{f}_0'(y)}{\overline{f}_0(y)} + 2$.
En el caso particular de que en la hipótesis nula sea $F_0(x) = \Phi(x_1)\Phi(x_2)$ se tiene que $\overline{f}_0(y) = \frac{1}{2\pi}e^{-\frac{1}{2}y^2}$, se obtiene $k(y,z) = 2 - y^2$.

Optimización del sesgo.
En el caso de cambios de dispersión, tenemos $k(y,z)$ independiente de z, por lo que el sesgo es $\delta \int T(a\mathbf{1}_A - a\mathbf{1}_{A^c})k dF_0$.
Buscamos maximizar $\int T(\mathbf{1}_A - \mathbf{1}_{A^c})k dF_0 = \langle T(a\mathbf{1}_A - a\mathbf{1}_{A^c}), k\rangle =$

$\langle a\mathbf{1}_A - a\mathbf{1}_{A^c}, T^{-1}(k)\rangle = \langle a, (\mathbf{1}_A - \mathbf{1}_{A^c})T^{-1}(k)\rangle \leq ||a||||(\mathbf{1}_A - \mathbf{1}_{A^c})T^{-1}(k)|| = ||T^{-1}(k)|| = ||k||$.
Por lo tanto, para optimizar el sesgo, se debe elegir

$$a = \frac{(\mathbf{1}_A - \mathbf{1}_{A^c})T^{-1}(k)}{||(\mathbf{1}_A - \mathbf{1}_{A^c})T^{-1}(k)||} = \frac{(\mathbf{1}_A - \mathbf{1}_{A^c})T^{-1}(k)}{||k||}$$

Pero además se debe elegir a independiente de la variable z y de A, por lo que eligiendo

$$A = I\!R \times [0,1]\ ,\ \tilde{a} = \frac{T^{-1}(k)}{||k||}$$

obtenemos el sesgo máximo. Observemos que a no depende de A ni de z debido a que k depende sólo de y y, como consecuencia, $T^{-1}(k)$ también. En el caso particular de la normal bivariada tendremos que el sesgo máximo es $\delta||k|| = \delta\sqrt{\mathbf{E}(2-Y^2)^2} = 2\delta$ y se obtiene en $\tilde{a}(y) = \frac{1}{2}T^{-1}(2 - .^2)(y)$.

5 Planteo del TRV y comportamiento asintótico.

Aquí plantearemos la región crítica según el criterio de la razón de verosimilitud para alternativas compuestas (de aquí en más TRV) y deduciremos su comportamiento asintótico a partir de los resultados que surgen del Tercer Lema de Le Cam. Tenemos la prueba $H_0 : (Y, Z) \sim F_0 = \tilde{F}_0 \times U[0,1]$ contra las alternativas $H_n : \exists\ v \in (0,1)$ tal que $(Y, [Z+v]) \sim F^{(\delta/\sqrt{n})}$ o sea que $H_n : (Y, Z) \sim F^{(\delta/\sqrt{n})}(y, [z+v])$ para algún v.
Por lo tanto según TRV para hipótesis compuestas, tomamos como región crítica

$$\left\{ \sup_{0\leq v\leq 1} \prod_{i=1}^{n} \frac{dF^{(\delta/\sqrt{n})}}{dF_0}(y_i, [z_i+v]) \geq c \right\}$$

que, tomando logaritmos, queda en la forma:

$$\left\{ \sup_{0\leq v\leq 1} \sum_{i=1}^{n} \log\left(f^{(\delta/\sqrt{n})}(y_i, [z_i+v])\right) \geq c \right\}$$

Denotamos

$$\Lambda_{n,v} := \sum_{i=1}^{n} \log\left(f^{(\delta/\sqrt{n})}(y_i, [z_i+v])\right)$$

y llamamos σ^2 a la varianza asintótica de $\Lambda_{n,v}$ (como veremos en el siguiente cálculo, σ^2 no depende de v), bajo H_0. Aplicando el teorema de Oosterhoff y Van Zwet, la distribución asintótica de $\Lambda_{n,v}$, bajo H_0 es $N(-\frac{\sigma^2}{2}, \sigma^2)$ y bajo H_n es $N(\frac{\sigma^2}{2}, \sigma^2)$. Se verifica fácilmente que si a las funciones k_n les pedimos

que:
i) $k_n \in L^4$
ii) $\mathbf{E}(k_n(Y,Z) - k_n(Y,[Z+v]))^2 \leq \text{cte} \times v^2 \ \forall v, n$
entonces $\{\Lambda_{n,v}\}$, bajo H_0, converge débilmente a un proceso gaussiano cuyas medias y covarianzas pasaremos a calcular en lo que sigue.

Estas nuevas hipótesis se verifican sin dificultad en los casos particulares que hemos considerado.

Cálculo de las covarianzas asintóticas.
Si llamamos $H_{n,v}(Y,Z) = \log\left(f^{(\delta/\sqrt{n})}(Y,[Z+v])\right)$ entonces, $\Lambda_{n,v} = \sum_{i=1}^{n} H_{n,v}(Y_i,Z_i)$ y además, $H_{n,v}(Y,Z) = 2\log\left(1+\frac{\delta}{2\sqrt{n}}k_n(Y,[Z+v])\right)$ por lo que

$$\begin{aligned}\text{COV}(\Lambda_{n,v},\Lambda_{n,0}) &= \sum_{i=1}^{n}\text{COV}(H_{n,v}(Y_i,Z_i),H_{n,0}(Y_i,Z_i)) \\ &= n\text{COV}(H_{n,v}(Y,Z),H_{n,0}(Y,Z)) \\ &= n\left(\mathbf{E}H_{n,v}(Y,Z)H_{n,0}(Y,Z) - \mathbf{E}H_{n,v}(Y,Z)\mathbf{E}H_{n,0}(Y,Z)\right)\end{aligned}$$

Observemos que si en la integral hacemos el cambio variable $u = [z+v] \Rightarrow \mathbf{E}(H_{n,v}(Y,Z)) = \mathbf{E}(H_{n,0}(Y,Z))$ (con un cambio de variable análogo se obtiene que el proceso asintótico es estacionario) y observemos también que como $\log(1+x) = x - \frac{1}{(1+c)^2}\frac{x^2}{2}$ con $|c| \leq |x|$ obtenemos

$$\begin{aligned}n\mathbf{E}(H_{n,v}(Y,Z))\mathbf{E}(H_{n,0}(Y,Z)) &= n\mathbf{E}^2(H_{n,0}(Y,Z)) \\ &= 4n\mathbf{E}^2\left(\frac{\delta}{2\sqrt{n}}k_n - \frac{1}{(1+c_n)^2}\frac{\delta^2}{8n}k_n^2\right)\end{aligned}$$

con $|c_n| \leq |\frac{\delta}{2\sqrt{n}}k_n|$. Usando que $(a+b)^2 \leq 2(a^2+b^2)$ y que $k_n^2 \leq K$ tenemos que

$$\mathbf{E}^2\left(k_n - \frac{1}{(1+c_n)^2}\frac{\delta k_n^2}{4\sqrt{n}}\right) \leq 2\mathbf{E}^2(k_n) + \frac{\delta^2}{8n}\mathbf{E}^2\left(\frac{k_n^2}{(1+c_n)^2}\right) \longrightarrow \mathbf{E}^2(k) = 0$$

Por otro lado, si llamamos $k_{n,v}$ a $k_n(Y,[Z+v])$, tenemos que

$$\begin{aligned}n\mathbf{E}(H_{n,v}(Y,Z)H_{n,0}(Y,Z)) &= 4n\mathbf{E}\left(\log\left(1+\frac{\delta}{2\sqrt{n}}k_n\right)\log\left(1+\frac{\delta}{2\sqrt{n}}k_{n,v}\right)\right) \\ &= 4\delta^2\mathbf{E}\left(\frac{k_n}{2} - \frac{1}{(1+c_n)^2}\frac{\delta k_n^2}{8\sqrt{n}}\right)\left(\frac{k_{n,v}}{2} - \frac{1}{(1+c_{n,v})^2}\frac{\delta k_{n,v}^2}{8\sqrt{n}}\right)\end{aligned}$$

$$= \delta^2 \mathbf{E}\left(k_n - \frac{1}{(1+c_n)^2}\frac{\delta k_n^2}{4\sqrt{n}}\right)\left(k_{n,v} - \frac{1}{(1+c_{n,v})^2}\frac{\delta k_{n,v}^2}{4\sqrt{n}}\right)$$

que converge a $\delta^2 \mathbf{E}(k(Y,Z)k(Y,[Z+v]))$.
Entonces la varianza asintótica es $\delta^2||k||^2$, por lo tanto, el proceso asintótico (llamémosle Λ_v) es gaussiano estacionario con esperanzas $\frac{-\delta^2}{2}||k||^2$ y función de covarianza

$$\mathbf{E}(\Lambda_0\Lambda_v) = \delta^2 E(k(Y,[Z+v])k(Y,Z)).$$

6 Una cota inferior para la eficiencia relativa asintótica.

En esta sección daremos una cota inferior para el límite con α y β (probabilidades de error tipo I y II respectivamente) tendiendo a cero de la eficiencia relativa asintótica de PET respecto a TRV. Más precisamente probaremos que

$$\liminf_{\alpha,\beta \longrightarrow 0} ARE(PET, TRV) \geq \frac{d^2}{||k||^2}$$

siendo $d = \sup_{a,A} \int T(a1_A - a1_{A^c})(y,z)k(y,[z+v])dF_0(y,z)$ (máximo sesgo posible), por lo que la prueba será óptima si y sólo si $d = ||k||$.
De aquí se deducirá que la prueba basada en PET es óptima en el caso de cambios de dispersión y calcularemos la cota inferior en el caso de cambios de posición.
Le llamaremos n_{PET} al tamaño de muestra necesario para obtener un nivel α y una potencia $1-\beta$ para la prueba PET y análogamente, definimos n_{TRV} para la prueba de la razón de verosimilitud. En ambas pruebas, como no sabemos la distribución exacta usaremos la región crítica asintótica.
Usaremos para dichos cálculos, el siguiente resultado debido a Marcus y Shepp cuya demostración se puede ver en [10].
Si $\{X_t\}_{t\in T}$ es un proceso gaussiano separable centrado con varianzas acotadas (T cualquiera), sea $\sigma^2 = \sup\{\sigma_t^2, t \in T\}$ entonces, para cualquier $\varepsilon > 0$

$$\exists\ L > 0 \text{ tal que } P(\sup_{t\in T} X_t > c) \leq LP((1+\varepsilon)\sigma Z > c) = L\Phi\left(\frac{-c}{(1+\varepsilon)\sigma}\right)$$

donde L es una constante que depende únicamente de ε y $Z \sim N(0,1)$.
Como nuestro V- proceso de Wiener puede ser llevado a uno estándar en $[0,1]\times[0,1]$, que es separable podemos entonces aplicarle el resultado de Marcus y Shepp. En efecto, sea $G(y) = \int_{-\infty}^{y} a^2 d\tilde{F}_0$ y $\mathcal{B} = \{(0,a]\times[b,c]\ 0<a<1\ ,\ 0<b<c<1\}$, entonces si $B \in \mathcal{B}$ definiendo $W(B) = W^V((-\infty, G^{-1}(a)]\times[b,c])$ se obtiene un proceso de Wiener estándar en $(0,1)\times(0,1)$ ya que

$\mathbf{E}(W^2(B)) = \int_{-\infty}^{G^{-1}(a)} a^2(y)d\tilde{F}_0(y)(c-b) = G(G^{-1}(a))(c-b) = a(c-b)$ y con un cálculo análogo, se obtiene que si $B_1 = (0, a_1] \times [b_1, c_1]$,
$B_2 = (0, a_2] \times [b_2, c_2] \Rightarrow \mathbf{E}(W(B_1)W(B_2)) = (a_1 \wedge a_2)\text{long}([b_1, c_1] \cup [b_2, c_2])$

Estimación de $\mathbf{n_{PET}}$.

$$\alpha = P_{H_0}(\sup_{A \in \mathcal{A}} |W_n(A) - W_n(A^c)| > c) \to P(\sup_{A \in \mathcal{A}} |W(A) - W(A^c)| > c)$$

$$\leq L\Phi\left(\frac{-c}{1+\varepsilon}\right) \text{ lo que implica que } c \leq -(1+\varepsilon)\Phi^{-1}\left(\frac{\alpha}{L}\right).$$

Para la potencia,

$$1-\beta = P_{H_n}(\sup_{A \in \mathcal{A}} \{|W_n(A)-W_n(A^c)|\} > c) \to P(\sup_{A \in \mathcal{A}} |W(A)-W(A^c)+\delta d_A| > c)$$

$$\geq P(\sup_{A \in \mathcal{A}} W(A)-W(A^c)+\delta d_A > c) \geq P(W(\tilde{A})-W(\tilde{A}^c)+\delta d > c) = \Phi(\delta d - c)$$

donde en esta última desigualdad se tomó $\tilde{A}$ como el A que maximiza el sesgo y d es el sesgo máximo. Entonces,
$\frac{\Phi^{-1}(1-\beta)+c}{d} \geq \delta$ por lo que si $\delta = s\sqrt{n}$,

$$\sqrt{n_{PET}} \leq \frac{\Phi^{-1}(1-\beta) - (1+\varepsilon)\Phi^{-1}(\frac{\alpha}{L})}{sd}$$

Estimación de $\mathbf{n_{TRV}}$.
Bajo H_0 , el proceso

$$\{\Lambda_{n,v}\}_{\{0 \leq v \leq 1\}} = \left\{ \sum_{i=1}^{n} log\left(f^{(\delta/\sqrt{n})}(Y_i, [Z_i + v])\right) \right\}_{0 \leq v \leq 1}$$

es asintóticamente gaussiano con medias $-\frac{\delta^2}{2}||k||^2$ y función de covarianza $\rho(v) = \delta^2 E[k(Y, Z)k(Y, [Z+v])]$.
Consideremos el proceso estandarizado

$$\Lambda'_{n,v} = \frac{\Lambda_{n,v} + \frac{\delta^2}{2}||k||^2}{\delta||k||}$$

Bajo H_0, $\{\Lambda_{n,v}\}$ es asintóticamente gaussiano estacionario con medias $-\frac{\delta^2}{2}||k||^2$ y función de covarianza $\rho(v)$, por lo que $\{\Lambda'_{n,v}\}$ será asintóticamente gaussiano centrado estacionario con función de covarianza $\frac{\rho(v)}{\delta^2||k||^2}$. Llamémosle $\{\Lambda_v'\}$ a este proceso asintótico.

$$\alpha = P_{H_0}(\sup_v \Lambda_{n,v}' > c) \to P(\sup_v \Lambda_v' > c) \geq P(\Lambda_0' > c) = \Phi(-c)$$

$$\Longrightarrow -\Phi^{-1}(\alpha) \leq c$$

Usando ahora el Corolario 2 del Tercer Lema de Le Cam, tendremos que bajo H_n el proceso $\{\Lambda_{n,v}\}$ es asintóticamente gaussiano estacionario con medias $-\frac{\delta^2}{2}||k||^2 + \rho(v)$ y función de covarianza $\rho(v)$ por lo que $\{\Lambda_{n,v}'\}$ será asintóticamente gaussiano estacionario con medias $\frac{\rho(v)}{\delta||k||}$ y función de covarianza $\frac{\rho(v)}{\delta^2||k||^2}$.

Hagamos una regresión lineal de Λ_v' en Λ_0', es decir, escribimos $\Lambda_v' = a\Lambda_0' + Z_v$ con $\{Z_v\}$ gaussiano, y hallamos a para que el proceso $\{Z_v\}$ sea independiente de Λ_0'.

Entonces, tomando la covarianza con Λ_0', nos queda $a = \frac{\rho(v)}{\delta^2||k||^2}$.

Dado $\varepsilon > 0$, elijamos $c^* > 0$ tal que $P(\sup_v Z_v \leq c^*) \geq 1 - \varepsilon$. Dicho c^* depende únicamente de ε ya que el proceso $\{Z_v\}$ es gaussiano centrado con varianzas $1 - \frac{\rho^2(v)}{\delta^4||k||^4}$ que son acotadas.

Observemos que como haremos tender α a cero, si α es suficientemente chico, c será mayor que c^*. Por lo tanto,

$$\beta = P_{H_n}(\sup_v \Lambda_{n,v}' \leq c) \to P(\sup_v \Lambda_v' \leq c)$$

$$\geq P(|\Lambda_0'| \leq c - c^*)P(\sup_v Z_v \leq c^*) \geq (1-\varepsilon)P(|\Lambda_0'| \leq c - c^*)$$

Pero como $\frac{P(|\Lambda_0'| \leq c - c^*)}{P(\Lambda_0' \leq c)}$ tiende a uno cuando α y β tienden a cero, si α es suficientemente chico, este cociente quedará mayor o igual que $1 - \varepsilon$.

Por lo tanto,

$$\beta \geq (1-\varepsilon)^2 P(\Lambda_0' \leq c) = (1-\varepsilon)^2 \Phi(c - \delta||k||)$$

$$-\Phi^{-1}\left(\frac{\beta}{(1-\varepsilon)^2}\right) + c \leq \delta||k|| \, ,$$

y como $c \geq -\Phi^{-1}(\alpha)$, tomando $\delta = s\sqrt{n}$ se obtiene

$$\sqrt{n_{TRV}} \geq \frac{-\Phi^{-1}(\alpha) - \Phi^{-1}\left(\frac{\beta}{(1-\varepsilon)^2}\right)}{s||k||}.$$

Para el cálculo de la acotación para el límite de $ARE(PET, TRV)$ serán necesarios los siguientes dos lemas cuyas demostraciones son de simple cálculo elemental y se pueden ver en [5].

LEMA 1 *Si k y h son constantes, entonces* $\lim_{\alpha\to 0} \frac{\Phi^{-1}(k\alpha)}{\Phi^{-1}(h\alpha)} = 1$.

LEMA 2 *Si* $\lim_{\alpha\to 0} \frac{a(\alpha)}{b(\alpha)} = h \geq 1$ *y* $\lim_{\alpha\to 0} \frac{c(\alpha)}{d(\alpha)} = 1$, *$b$ y d positivas, entonces*

$$\limsup_{\alpha\to 0} \frac{a(\alpha)+c(\alpha)}{b(\alpha)+d(\alpha)} \leq h.$$

Cálculo de la cota inferior para la eficiencia relativa asintótica de PET respecto de TRV.

Tenemos que

$$\sqrt{n_{TRV}} \geq \frac{\Phi^{-1}\left(1-\frac{\beta}{(1-\varepsilon)^2}\right)+\Phi^{-1}(1-\alpha)}{s||k||};$$

$$\sqrt{n_{PET}} \leq \frac{\Phi^{-1}(1-\beta)-(1+\varepsilon)\Phi^{-1}(\frac{\alpha}{L})}{sd} = \frac{\Phi^{-1}(1-\beta)+(1+\varepsilon)\Phi^{-1}(1-\frac{\alpha}{L})}{sd}$$

Entonces

$$\limsup_{\alpha,\beta\to 0} \frac{\sqrt{n_{PET}}}{\sqrt{n_{TRV}}} \leq \lim_{\alpha,\beta\to 0} \frac{\Phi^{-1}(1-\beta)+(1+\varepsilon)\Phi^{-1}(1-\frac{\alpha}{L})}{\Phi^{-1}(1-\frac{\beta}{(1-\varepsilon)^2})+\Phi^{-1}(1-\alpha)}\frac{||k||}{d} \leq \frac{(1+\varepsilon)||k||}{d}$$

y como ε es arbitrario, se tiene que

$$\limsup_{\alpha,\beta\to 0} \frac{\sqrt{n_{PET}}}{\sqrt{n_{TRV}}} \leq \frac{||k||}{d}$$

y por lo tanto

$$\liminf_{\alpha,\beta\to 0} ARE(PET, TRV) \geq \frac{d^2}{||k||^2}.$$

Vemos de esta manera que la eficiencia de PET respecto de TRV será tanto más cercana a uno cuanto más se aproxime el sesgo máximo d al valor $||k||$. En el caso de cambios de dispersión, tenemos $d = ||k||$ por lo que

$$\liminf_{\alpha,\beta\to 0} ARE(PET, TRV) \geq 1$$

y la prueba en este caso es óptima.
En el caso de cambios de posición tenemos $k(y,z) = h(y)\cos 2\pi z$ siendo $h(y) = \frac{\overline{f}'_0(y)}{\overline{f}_0(y)}$ por lo que $||k||^2 = ||h||^2 \int_0^1 \cos^2(2\pi z)dz = \frac{1}{2}||h||^2$. Además tenemos que $d = \frac{2}{\pi}||h||$ por lo que

$$\liminf_{\alpha,\beta\to 0} ARE(PET,TRV) \geq \frac{8}{\pi^2} \cong 0,81.$$

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JUAN KALEMKERIAN
`jkalem@cmat.edu.uy`
Centro de Matemática. Facultad de Ciencias.
Montevideo,
Uruguay.

Russian Options for a Diffusion with Negative Jumps

Ernesto Mordecki and Walter Moreira

ABSTRACT

Closed solutions to the problem of pricing a Russian option when the underlying process is a diffusion with negative jumps are obtained. More precisely, the underlying process is assumed to have the form of a Wiener process with drift and negative mixed–exponentially distributed jumps driven by a Poisson process. This result generalizes those of Shepp and Shiryaev (1993) for the Wiener process and Gerber, Michaud and Shiu (1995) for pure–jumps process.

RESUMEN

Se dan fórmulas explícitas para el precio de una opción Rusa cuando el proceso que modela los precios es una difusión con saltos negativos. Más precisamente, el proceso subyacente tiene la forma de un proceso de Wiener con deriva mas un proceso de Poisson compuesto con saltos distribuidos según mezclas de exponenciales. Este resultado generaliza el de Shepp y Shiryaev (1993) para el proceso de Wiener y el de Gerber, Michaud y Shiu (1995) para procesos de saltos puros.

1 Introduction and main results

1.1 Consider a model of a financial market with two assets, a savings account $B = (B_t)_{t\geq 0}$, and a stock $S = (S_t)_{t\geq 0}$. The evolution of B is deterministic, with

$$B_t = B_0 e^{rt}; \quad B_0 > 0, \quad r > 0,$$

and the stock is random, and evolves according to the formula

$$S_t = S_0 e^{X_t}; \quad S_0 > 0, \tag{1}$$

where $X = (X_t)_{t\geq 0}$ is a stochastic process defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$, a stochastic basis that satisfy the usual conditions. Consider also the supremum process, denoted by $(S_t^*)_{t\geq 0}$, and given by

$$S_t^* = \sup_{0\leq r\leq t} S_r.$$

In this model L. Shepp and A. N. Shiryaev [SS93] introduced an American option type on the maximum value of the stock, baptized as *Russian option*. Related to this, we mention the European options on the maximum introduced by A. Conze and Viswanathan [CV91], called *look–back options*. In [SS93] and [SS94] closed solutions were obtained for the problem of pricing Russian options in the perpetual case, in the framework of the Black–Scholes–Merton (1973) model (see [BS73]), this is to say, when X is a Wiener process with drift. Afterwards, Gerber, Michaud and Shiu, in [GMS95] gave closed solutions to prices of perpetual Russian options when the underlying process was a *risk process*, more precisely, a compound Poisson process with mixed exponentially distributed negative jumps and deterministic drift.

1.2 The purpose of the present paper is to unify these results, that is, to give closed solutions to the following optimal stopping problem.

- The process X in the stock (1) has the form

$$X_t = \left(a - \frac{\sigma^2}{2}\right)t + \sigma W_t - \sum_{i=1}^{N_t} Y_i, \tag{2}$$

where $W = (W_t)_{t\geq 0}$ is a standard Wiener process, $\sigma > 0$, $N = (N_t)_{t\geq 0}$ is a Poisson process with intensity c, and $Y = (Y_k)_{k\in\mathbb{N}}$ is a sequence of non–negative independent random variables with common distribution

$$F(y) = 1 - \sum_{i=1}^{n} A_i e^{-\alpha_i y}, \quad y \geq 0, \tag{3}$$

where $A_i > 0$ for $i = 1, 2, \dots, n$; $\sum_{i=1}^{n} A_i = 1$; and $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n$. The processes W, N and Y are independent.

- The payoff $(f_t)_{t\geq 0}$ of the perpetual American option takes the form

$$f_t = e^{-\lambda t}\max\left[S_t^*, S_0\psi_0\right]$$

with $\lambda \geq 0$ a discount factor and $\psi_0 \geq 1$.

To price this contract we can assume that

$$r = a + c\int_0^{+\infty} (e^{-y} - 1)\, dF(y),$$

and this implies that P is a martingale measure. Anyhow, we consider a more general situation, introducing a dividend rate ρ, given by

$$\rho = r - a - c\int_0^{+\infty} (e^{-y} - 1)\, dF(y) \tag{4}$$

under the restriction $\rho \geq 0$. With this assumptions the process $(e^{(\rho-r)t}S_t)_{t\geq 0}$ is a martingale under P.

Rational pricing of Russian options in complete markets led to the consideration of an optimal stopping problem. We solve the following question: find a function $C(\psi_0)$ and a stopping time τ^* such that

$$C(\psi_0) = \sup_{\tau\in\mathcal{M}} \mathsf{E}\, e^{-(\lambda+r)\tau} \max\bigl[S^*_\tau, S_0\psi_0\bigr] = \mathsf{E}\, e^{-(\lambda+r)\tau^*} \max\bigl[S^*_{\tau^*}, S_0\psi_0\bigr] \tag{5}$$

where $\mathcal{M}$ is the class of all P–finite stopping times.

1.3 Dual Martingale measure. In the case considered, according to (2), X is a Lévy process. If $q \in \mathbb{R}$, Lévy–Khinchine's formula states

$$\mathsf{E}\, e^{iqX_t} = \exp\Biggl\{t\Bigl[\Bigl(a - \frac{\sigma^2}{2}\Bigr)iq - \frac{\sigma^2}{2}q^2 + c\int_{\mathbb{R}} (e^{iqx} - 1)\, dF(x)\Bigr]\Biggr\}. \tag{6}$$

Taking into account (3), if $z \in \mathbb{C}$ with $\mathsf{Re}(z) > -\alpha_1$, the characteristic exponent $\Psi = \Psi(z)$ defined through

$$\mathsf{E}\, e^{zX_t} = e^{t\Psi(z)},$$

completely determines the law of X, and takes the form

$$\begin{aligned}\Psi(z) &= \Bigl(a - \frac{\sigma^2}{2}\Bigr)z + \frac{\sigma^2}{2}z^2 + c\int_0^{+\infty} (e^{-zy} - 1)\, dF(y)\\ &= \Bigl(a - \frac{\sigma^2}{2}\Bigr)z + \frac{\sigma^2}{2}z^2 - c\sum_{i=1}^{n} A_i \frac{z}{z + \alpha_i}.\end{aligned} \tag{7}$$

Our path–dependent problem is transformed into an optimal stopping problem of a Markov process through a *change of numeraire*, that corresponds to a change of measure, leading to the introduction of the *dual martingale measure*. This procedure was introduced in [SKKM94, SS94, KM94]. In Proposition 1 we construct the measure $\widetilde{\mathsf{P}}$ and show, that under this new probability measure, X is a Lévy process with characteristic exponent

$$\widetilde{\Psi}(z) = \widetilde{a}z + \frac{\sigma^2}{2}z^2 - \widetilde{c}\sum_{i=1}^{n} \widetilde{A}_i \frac{z}{z + \widetilde{\alpha}_i}. \tag{8}$$

The *dual* parameters are given by Girsanov's Theorem,

$$\widetilde{a} = a + \sigma^2/2, \qquad \widetilde{c}\widetilde{F}(dy) = e^{-y}c\,F(dy). \tag{9}$$

This gives that under $\widetilde{\mathsf{P}}$ the process X changes its distribution only trough its parameters, according to

$$\widetilde{c} = c\sum_{i=1}^{n}\frac{A_i\alpha_i}{1+\alpha_i}, \quad \widetilde{\alpha}_i = \alpha_i + 1, \quad \widetilde{A}_i = \frac{A_i\alpha_i}{1+\alpha_i}\Big/\sum_{i=1}^{n}\frac{A_i\alpha_i}{1+\alpha_i}, \tag{10}$$

for $i = 1, \dots, n$. We denote also by $\widetilde{\Psi}$ the analytical continuation of the characteristic exponent of X under $\widetilde{\mathsf{P}}$.

1.4 Main Result. We are in position to formulate our main result.

Theorem 1 *Consider the market model in 1.1. Assume that ρ in (4) satisfies $\rho \geq 0$. Then, the solution to the optimal stopping problem (5) for $\psi_0 \geq 1$ has cost function*

$$C(\psi_0) = S_0\begin{cases}\widetilde{\psi}\left[C_0\left(\dfrac{\psi_0}{\widetilde{\psi}}\right)^{\beta_0} + \cdots + C_{n+1}\left(\dfrac{\psi_0}{\widetilde{\psi}}\right)^{\beta_{n+1}}\right] & \text{if } 1 \leq \psi_0 < \widetilde{\psi} \\ \psi_0 & \text{if } \widetilde{\psi} \leq \psi_0,\end{cases} \tag{11}$$

where $\beta_0, \dots, \beta_{n+1}$ are the real roots of the equation

$$\widetilde{\Psi}(-\beta) = \lambda + \rho, \tag{12}$$

with $\widetilde{\Psi}$ defined in (8), and satisfy

$$\beta_0 < 0 < 1 < \beta_1 < \alpha_1 + 1 < \cdots < \beta_n < \alpha_n + 1 < \beta_{n+1}. \tag{13}$$

Coefficients $C_0, \dots, C_{n+1}$ are given by

$$C_i = \prod_{k=1}^{n}\left(\frac{\alpha_k + 1 - \beta_i}{\alpha_k}\right)\prod_{\substack{k=0\\k\neq i}}^{n+1}\left(\frac{\beta_k - 1}{\beta_k - \beta_i}\right),$$

and $\widetilde{\psi} > 1$ is the only root of the equation in ψ

$$\beta_0 C_0 \psi^{-\beta_0} + \cdots + \beta_{n+1}C_{n+1}\psi^{-\beta_{n+1}} = 0. \tag{14}$$

The optimal stopping time is

$$\tau^* = \inf\left\{t \geq 0\colon \frac{\max\left[S_t^*, S_0\psi_0\right]}{S_t} \geq \widetilde{\psi}\right\} \tag{15}$$

and it is P–a.s. finite.

2 Proof

The first step of the proof consist in a change of numeraire that led us to the solution of a different optimal stopping problem, having the advantage that the underlying process is not path–dependent. The second part is the solution of the deterministic free boundary problem for an integro–differential operator, related to the generator of this auxiliary process.

Let us introduce a probability measure $\widetilde{\mathsf{P}}$ on $(\Omega, \mathcal{F})$ by its restrictions to $\mathcal{F}_t$, as

$$\frac{d\widetilde{\mathsf{P}}_t}{d\mathsf{P}_t} = e^{\rho t} \frac{B_0 S_t}{S_0 B_t}, \tag{16}$$

and stochastic processes $(M_t)_{t \geq 0}$ and $(\psi_t)_{t \geq 0}$ by

$$M_t = \max[S_t^*, S_0 \psi_0], \qquad \psi_t = \frac{M_t}{S_t}. \tag{17}$$

PROPOSITION 1 *(a) There exists a probability measure* $\widetilde{\mathsf{P}}$ *such that* $\widetilde{\mathsf{P}}|_{\mathcal{F}_t} = \widetilde{\mathsf{P}}_t$ *with* $\widetilde{\mathsf{P}}_t$ *defined in* (16).

(b) Under $\widetilde{\mathsf{P}}$, *the process* X *is a Lévy process with characteristic exponent*

$$\widetilde{\Psi}(iu) = i\widetilde{a}u - \frac{\sigma^2}{2} u^2 + \widetilde{c} \int_0^{+\infty} (e^{-iux} - 1)\, d\widetilde{F}(x)$$

for real u, *with*

$$\widetilde{a} = a + \sigma^2/2, \qquad \widetilde{c}\,\widetilde{F}(dy) = e^{-y} c\, F(dy).$$

(c) If $\widetilde{\mathsf{E}}$ *denotes expectation with respect to* $\widetilde{\mathsf{P}}$, *for an arbitrary bounded stopping time* τ *we have*

$$\mathsf{E}\, e^{-(\lambda+r)\tau} M_\tau = S_0\, \widetilde{\mathsf{E}}\, e^{-(\lambda+\rho)\tau} \psi_\tau. \tag{18}$$

In view of (c) in the previous Proposition, we must solve an optimal stopping problem under $\widetilde{\mathsf{P}}$ for the process $(\psi_t)_{t \geq 0}$. Consider then the infinitesimal generator of ψ, given by

$$L^{\psi} f(z) = -azf'(z) + \frac{\sigma^2}{2} z^2 f''(z) + \widetilde{c} \int_0^{+\infty} [f(ze^x) - f(z)]\, d\widetilde{F}(x).$$

In case f is only once differentiable and convex, by f'' we mean the second derivative from the left. The way to find the solution to this associated optimal stopping problem under $\widetilde{\mathsf{P}}$ is solving the free–boundary problem, consisting in finding a constant $\widetilde{\psi} > 1$ and a real function $V = V(\psi)$ with $\psi \geq 1$ such that

$$\begin{cases} L^{\psi}V(z) - (\lambda+\rho)V(z) = 0 \quad \text{if } 1 \leq z \leq \widetilde{\psi}, \\ V(\widetilde{\psi}) = \widetilde{\psi}, \\ V'(1+) = 0, \\ V'(\widetilde{\psi}-) = 1. \end{cases} \tag{19}$$

The next proposition presents some technical results, while Propositions 3 and 4 contain the key information to solve this problem.

PROPOSITION 2 *(a) The equation in β given by*

$$\widetilde{\Psi}(-\beta) = \lambda + \rho \tag{20}$$

has $n+2$ roots $\beta_0, \beta_1, \dots, \beta_{n+1}$, that satisfy

$$\beta_0 < 0 < 1 < \beta_1 < \alpha_1 < \cdots < \beta_n < \alpha_n + 1 < \beta_{n+1}. \tag{21}$$

(b) Coefficients C_i in Theorem 1 satisfy the following system of linear equations

$$\sum_{i=0}^{n+1} C_i \frac{1}{\widetilde{\alpha}_k - \beta_i} = \frac{1}{\widetilde{\alpha}_k - 1}, \quad \textit{for } k = 1, \dots, n; \tag{22}$$

$$\sum_{i=0}^{n+1} \beta_i C_i = 1; \tag{23}$$

$$\sum_{i=0}^{n+1} C_i = 1; \tag{24}$$

with $\widetilde{\alpha}_k = \alpha_k + 1$. Furthermore, $C_i > 0$ for $i = 0, 1, \dots, n+1$.

(c) The function

$$f(x) = \beta_0 C_0 x^{-\beta_0} + \cdots + \beta_{n+1} C_{n+1} x^{-\beta_{n+1}}, \quad x > 0, \tag{25}$$

has only one root $\widetilde{\psi} > 1$.

The following proposition gives the solution to the free boundary problem.

PROPOSITION 3 *Consider a function V defined by*

$$V(\psi_0) = \begin{cases} \widetilde{\psi}\Big[C_0\Big(\frac{\psi_0}{\widetilde{\psi}}\Big)^{\beta_0} + \cdots + C_{n+1}\Big(\frac{\psi_0}{\widetilde{\psi}}\Big)^{\beta_{n+1}}\Big] & \text{if } 1 \le \psi_0 < \widetilde{\psi} \\ \psi_0 & \text{if } \widetilde{\psi} \le \psi_0. \end{cases}$$

Then, the following holds:

(a) *The function V is convex, continuously differentiable for all $\psi_0 \geq 1$, twice differentiable for all $\psi_0 \neq \widetilde{\psi}$ and $V(\psi_0) \geq \psi_0$ for all $\psi_0 \geq 1$.*

(b) *For all $z \geq 1$*

$$L^{\psi}V(z) - (\lambda+\rho)V(z) \leq 0.$$

(c) *Furthermore, if $1 \leq z \leq \widetilde{\psi}$, then*

$$L^{\psi}V(z) - (\lambda+\rho)V(z) = 0.$$

PROPOSITION 4 *For the function V and the process $\psi = (\psi_t)_{t\geq 0}$ as above,*

$$e^{-(\lambda+\rho)t}V(\psi_t) - V(\psi_0) = \int_0^t e^{-(\lambda+\rho)s}\big[L^{\psi}V(\psi_{s-}) - (\lambda+\rho)V(\psi_{s-})\big]\,ds + Q_s \quad (26)$$

for all $t \geq 0$, where $(Q_t)_{t\geq 0}$ is a local martingale under $\widetilde{\mathsf{P}}$.

PROOF (*of the Theorem*): We verify the following two assertions for the function $C(\psi_0)$ in (11). Observe that $C(\psi_0) = S_0 V(\psi_0)$.

(a) $\mathsf{E}\, e^{-(\lambda+r)\tau} M_\tau \leq C(\psi_0)$, for any $\tau \in \mathcal{M}$;

(b) $\mathsf{E}\, e^{-(\lambda+r)\tau^*} M_{\tau^*} = C(\psi_0)$, for τ^* defined in (15).

Let us verify (a). Take $\tau \in \mathcal{M}$ and $(\tau_n)_{n\geq 1}$ a bounded localizing sequence for the $\widetilde{\mathsf{P}}$–local martingale $(Q_t)_{t\geq 0}$. Then, by Proposition 4 and (b) in Proposition 3, we have

$$e^{-(\lambda+\rho)\tau_n\wedge\tau}V(\psi_{\tau_n\wedge\tau}) - V(\psi_0) \leq Q_{\tau_n\wedge\tau}. \quad (27)$$

As $(Q_{\tau_n\wedge t})_{t\geq 0}$ is a $\widetilde{\mathsf{P}}$–martingale and $Q_0 = 0$, $\widetilde{\mathsf{P}}$–expectations in (27) give $\widetilde{\mathsf{E}}\, e^{-(\lambda+\rho)\tau_n\wedge\tau}V(\psi_{\tau_n\wedge\tau}) \leq V(\psi_0)$. So

$$\begin{aligned}\mathsf{E}\, e^{-(\lambda+r)\tau_n\wedge\tau}M_{\tau_n\wedge\tau} &= S_0\,\widetilde{\mathsf{E}}\, e^{-(\lambda+\rho)\tau_n\wedge\tau}\psi_{\tau_n\wedge\tau} \\ &\leq S_0\,\widetilde{\mathsf{E}}\, e^{-(\lambda+\rho)\tau_n\wedge\tau}V(\psi_{\tau_n\wedge\tau}) \leq S_0 V(\psi_0). \qquad (28)\end{aligned}$$

Now, as $P(\tau < \infty) = 1$, we let $\tau_n \to +\infty$ and the part (a) follows by Fatou's lemma. In order to prove (b), we verify that $(Q_{\tau^*\wedge t})_{t\geq 0}$ is an uniform integrable $\widetilde{\mathsf{P}}$–martingale. By Proposition 4 and (c) in Proposition 3, as $\psi_{(\tau^*\wedge t)-} \leq \widetilde{\psi}$, we have

$$e^{-(\lambda+\rho)\tau^*\wedge t}V(\psi_{\tau^*\wedge t}) - V(\psi_0) = Q_{\tau^*\wedge t}. \qquad (29)$$

Therefore

$$\begin{aligned}-V(\psi_0) \leq Q_{\tau^*\wedge t} &\leq e^{-(\lambda+\rho)\tau^*\wedge t}V(\psi_{\tau^*\wedge t}) \\ &= e^{-(\lambda+\rho)t}V(\psi_t)\mathbb{I}_{\{t<\tau^*\}} + e^{-(\lambda+\rho)\tau^*}V(\psi_{\tau^*})\mathbb{I}_{\{\tau^*\leq t\}} \\ &\leq V(\widetilde{\psi}) + e^{-(\lambda+\rho)\tau^*}\psi_{\tau^*}.\end{aligned}$$

To conclude the uniform integrability of $(Q_{\tau^*\wedge t})_{t\geq 0}$ it is enough to see that $e^{-(\lambda+\rho)\tau^*}\psi_{\tau^*}$ has finite $\widetilde{\mathsf{P}}$ expectation. First observe that $\widetilde{\mathsf{P}}(\tau^* < \infty) = 1$. This follows based on the property of homogeneous independent increments of X, as done in [SS94], see also [Mor00]. By Fatou's Lemma and (28),

$$\begin{aligned}\widetilde{\mathsf{E}}\, e^{-(\lambda+\rho)\tau^*}\psi_{\tau^*} &= \widetilde{\mathsf{E}}\Big[\lim_{t\to+\infty} e^{-(\lambda+\rho)\tau^*\wedge t}\psi_{\tau^*\wedge t}\Big] \\ &\leq \liminf_{t\to+\infty}\widetilde{\mathsf{E}}\, e^{-(\lambda+\rho)\tau^*\wedge t}\psi_{\tau^*\wedge t} \leq V(\psi_0)\end{aligned}$$

as τ^* is $\widetilde{\mathsf{P}}$–finite. Now, we have $\widetilde{\mathsf{E}}(Q_{\tau^*}) = 0$ and thus, by (29),

$$\widetilde{\mathsf{E}}\, e^{-(\lambda+\rho)\tau^*\wedge t}\psi_{\tau^*\wedge t} \longrightarrow \widetilde{\mathsf{E}}\, e^{-(\lambda+\rho)\tau^*}\psi_{\tau^*} = \widetilde{\mathsf{E}}\, e^{-(\lambda+\rho)\tau^*}V(\psi_{\tau^*}) = V(\psi_0).$$

On the other hand

$$\begin{aligned}\mathsf{E}\, e^{-(\lambda+r)\tau^*\wedge t}M_{\tau^*\wedge t} &= \mathsf{E}\, e^{-(\lambda+r)t}M_t\mathbb{I}_{\{t<\tau^*\}} + \mathsf{E}\, e^{-(\lambda+r)\tau^*}M_{\tau^*}\mathbb{I}_{\{\tau^*\leq t\}} \\ &= \widetilde{\mathsf{E}}\, e^{-(\lambda+\rho)t}\psi_t\mathbb{I}_{\{t<\tau^*\}} + \mathsf{E}\, e^{-(\lambda+r)\tau^*}M_{\tau^*}\mathbb{I}_{\{\tau^*\leq t\}} \\ &\to \mathsf{E}\, e^{-(\lambda+r)\tau^*}M_{\tau^*}.\end{aligned}$$

as $t \to +\infty$, since $\widetilde{\mathsf{E}}\, e^{-(\lambda+\rho)t}\psi_t\mathbb{I}_{\{t<\tau^*\}}$ is bounded by $\widetilde{\psi}\,\widetilde{\mathsf{P}}(t < \tau^*)$ and τ^* is $\widetilde{\mathsf{P}}$–finite. Then, part (b) follows from part (c) of proposition 1. This concludes the proof of the Theorem. □

3 Appendix: Proof of Propositions

PROOF (*of Proposition 1*): For the part (a), since $Z_t = e^{\rho t} B_0 S_t / S_0 B_t$ is a martingale, the construction of $\widetilde{\mathsf{P}}$ follows as in §1.3 in [SKKM94].

For the part (b) we compute the characteristic exponent of X under $\widetilde{\mathsf{P}}$. For $u \in \mathbb{R}$ we have

$$\widetilde{\mathsf{E}}\, e^{iuX_t} = \mathsf{E}\Big(e^{iuX_t} e^{\rho t} \frac{B_0 S_t}{S_0 B_t}\Big) = \mathsf{E} \exp\big[(iu+1)X_t + \rho t - rt\big]$$
$$= \exp\big[t(\Psi(iu+1) + \rho - r)\big],$$

with Ψ as in (7). Now, taking into account (4):

$$\Psi(iu+1) + \rho - r = \Big(a - \frac{\sigma^2}{2}\Big)(iu+1) + \frac{\sigma^2}{2}(iu+1)^2$$
$$+ c \int_0^{+\infty} (e^{-(iu+1)x} - 1)\, dF(x)$$
$$= \Big(a + \frac{\sigma^2}{2}\Big) iu - \frac{\sigma^2}{2} u^2 + \widetilde{c} \int_0^{+\infty} (e^{-iux} - 1)\, d\widetilde{F}(x),$$

proving (b).

Now we prove (c). Measures $\widetilde{\mathsf{P}}$ and P are locally mutually absolutely continuous, with density process $Z = (Z_t)_{t \ge 0}$ given by $Z_t = e^{\rho t} B_0 S_t / S_0 B_t$. When τ is bounded, by III.3.4 in [JS87],

$$\mathsf{E}\, e^{-(\lambda + r)\tau} M_\tau = \mathsf{E}\Big(e^{\rho\tau} \frac{B_0 S_\tau}{S_0 B_\tau} \times \frac{S_0 e^{-(\lambda+\rho)\tau} M_{\tau \wedge t}}{S_\tau}\Big)$$
$$= S_0\, \widetilde{\mathsf{E}}\, e^{-(\lambda+\rho)\tau} \psi_\tau.$$

concluding the proof. □

PROOF (*of Proposition 2*): Let us prove (a). Taking into account (8), (9) and (10),

$$\widetilde{\Psi}(-\beta) = -\beta(a + \sigma^2) + \frac{\sigma^2}{2}\beta(\beta+1) + c \sum_{i=1}^n \frac{A_i \alpha_i}{1 + \alpha_i - \beta} - c \sum_{i=1}^n \frac{A_i \alpha_i}{1 + \alpha_i}.$$

So (20) reads

$$-\frac{\sigma^2}{2}\beta^2 + \Big(\frac{\sigma^2}{2} + a\Big)\beta + c \sum_{i=1}^n \frac{A_i \alpha_i}{1+\alpha_i} + \lambda + \rho = c \sum_{i=1}^n \frac{A_i \alpha_i}{1 + \alpha_i - \beta}. \qquad (30)$$

The roots are then given by the intersection of the graphs of a sum of n hyperbolae with a concave parabola. Evaluation at $\beta = 0$ gives that the parabola is bigger that the sum at this points, and the roots satisfy (21). In order to see $1 < \beta_1$ we evaluate both terms in (30) at $\beta = 1$ to see that at this point the parabola is bigger than the sum. For details see [Mor00].

To prove (b) we introduce two auxiliary polynomials

$$P(x) = \prod_{j=1}^{n}(1 + x/\alpha_j), \qquad Q(x) = \prod_{j=0}^{n+1}(1 + x/(\beta_j - 1)),$$

and consider the simple fractional expansion,

$$\frac{P(x)}{Q(x)} = \sum_{j=0}^{n+1} D_j \frac{1}{\beta_j - 1 + x}. \tag{31}$$

In order to determine the coefficients, as we have simple roots,

$$\begin{aligned} D_i &= \frac{P(1-\beta_i)}{Q'(1-\beta_i)} \\ &= \prod_{j=1}^{n}\Big(\frac{\alpha_j + 1 - \beta_i}{\alpha_j}\Big)\Big[\frac{1}{\beta_i - 1}\prod_{\substack{j=0\\ j\neq i}}^{n+1}\Big(\frac{\beta_j - \beta_i}{\beta_j - 1}\Big)\Big]^{-1} = (\beta_i - 1)C_i. \end{aligned}$$

So, (31) becomes

$$\frac{P(x)}{Q(x)} = \sum_{j=0}^{n+1} C_i \frac{\beta_i - 1}{\beta_i - 1 + x}.$$

Now, taking $x = -\alpha_k$ for $k = 1, \dots, n$ and $x = 0$ in (31) we obtain (22) and (24) respectively. To see (23) we multiply both sides of (31) by x and take limits as $x \to \infty$, obtaining

$$\sum_{j=0}^{n+1} C_i(\beta_i - 1) = 0,$$

that in view of (24) concludes the proof. The properties $C_i > 0$ follows from (13).

For the part (c), as $C_i > 0$ for $i = 0, \dots, n+1$, by differentiation in (25) we get that f is decreasing, and $\lim_{x\to\infty} f(x) = -\infty$. We then see $f(1) > 0$. But

$$f(1) = \beta_0 C_0 + \cdots + \beta_{n+1} C_{n+1} = 1$$

in view of (23), proving the existence of a root bigger that one. □

PROOF (*of Proposition 3*): For the first part, clearly V is differentiable for all orders if $\psi \neq \widetilde{\psi}$. Equation (24) shows that $V(\widetilde{\psi}) = \widetilde{\psi}$ meaning that V is continuous, and equation (23) gives $V'(\widetilde{\psi}-) = 1$, showing that V is continuously differentiable, i.e. satisfies the *smooth pasting condition* (see [Shi78]). In what respects convexity, we examine the second derivative on $\psi_0 \in [1, \widetilde{\psi})$,

$$V''(\psi_0) = \widetilde{\psi} \sum_{i=0}^{n+1} \beta_i(\beta_i - 1) C_i \left(\frac{\psi_0}{\widetilde{\psi}}\right)^{\beta_i - 2} \geq 0,$$

because $C_i > 0$ and $\beta_i(\beta_i - 1) > 0$ in view of (21).

For the parts (b) and (c), take first $z > \widetilde{\psi}$. In this case $V(z) = z$ and $V(ze^x) = ze^x$ for $x \geq 0$. So $V''(z) = 0$ and

$$\begin{aligned} L^{\psi} V(z) - (\lambda + \rho) V(z) &= -az + \widetilde{c} \int_0^{+\infty} ze^x \, d\widetilde{F}(x) - z(\widetilde{c} + \lambda + \rho) \\ &= z\Big(-a + \widetilde{c} \sum_{i=1}^{n} \frac{\widetilde{A}_i \widetilde{\alpha}_i}{\widetilde{\alpha}_i + 1} - \widetilde{c} - \lambda - \rho\Big) \\ &= -z(r + \lambda) \leq 0 \end{aligned}$$

for all $z > \widetilde{\psi}$, where $\widetilde{c}$ and $\widetilde{A}$ are given in (10) and ρ in (4). Take now $\widetilde{\psi} \geq z$, so

$$\begin{aligned} L^{\psi} V(z) - (\lambda + \rho) V(z) &= -az\widetilde{\psi} + \sum_{i=0}^{n+1} \beta_i C_i \left(\frac{1}{\widetilde{\psi}}\right) \left(\frac{z}{\widetilde{\psi}}\right)^{\beta_i - 1} \\ &+ \frac{\sigma^2}{2} z^2 \widetilde{\psi}^2 \sum_{i=0}^{n+1} \beta_i(\beta_i - 1) C_i \left(\frac{1}{\widetilde{\psi}^2}\right) \left(\frac{z}{\widetilde{\psi}}\right)^{\beta_i - 2} \\ &+ \widetilde{c}\widetilde{\psi} \int_0^{\log(\widetilde{\psi}/z)} \sum_{i=0}^{n+1} C_i \left(\frac{z}{\widetilde{\psi}}\right)^{\beta_i} e^{\beta_i x} \, d\widetilde{F}(x) \\ &+ \widetilde{c} \int_{\log(\widetilde{\psi}/z)}^{+\infty} ze^x \, d\widetilde{F}(x) - (\widetilde{c} + \lambda + \rho)\widetilde{\psi} \sum_{i=0}^{n+1} C_i \left(\frac{z}{\widetilde{\psi}}\right)^{\beta_i}. \end{aligned}$$

that, after computing the integrals becomes

$$\begin{aligned}
&L^{\psi}V(z) - (\lambda+\rho)V(z) \\
&\quad = \sum_{i=0}^{n+1}\Big(\frac{z}{\widetilde{\psi}}\Big)^{\beta_i} C_i\widetilde{\psi}\Big\{-a\beta_i + \frac{\sigma^2}{2}\beta_i(\beta_i-1) + \widetilde{c}\sum_{k=1}^{n}\frac{\widetilde{A}_k\widetilde{\alpha}_k}{\widetilde{\alpha}_k-\beta_i} - (\widetilde{c}+\lambda+\rho)\Big\} \\
&\quad\quad - \widetilde{c}\sum_{k=1}^{n}\widetilde{A}_k\widetilde{\alpha}_k\Big(\frac{z}{\widetilde{\psi}}\Big)^{\widetilde{\alpha}_k}\widetilde{\psi}\sum_{i=0}^{n+1}\Big\{\frac{C_i}{\widetilde{\alpha}_k-\beta_i} - \frac{1}{\widetilde{\alpha}_k-1}\Big\}.
\end{aligned}$$

By (30) the first sum is zero, and by (22) the second also vanishes. In conclusion, $L^{\psi}V(z) - (\lambda+\rho)V(z) = 0$ if $z \le \widetilde{\psi}$. □

Proof (*of Proposition 4*): First we use Itô's Formula to establish the following

$$V(\psi_t) - V(\psi_0) = \int_0^t L^{\psi}V(\psi_{s^-})\,ds + q_t \tag{32}$$

for all $t \ge 0$, where $(q_t)_{t\ge0}$ is a local martingale. Let us apply Itô's–Meyer formula to the convex function V and process $(\psi_t)_{t\ge0}$ as in Theorem 51 in [Pro90]:

$$\begin{aligned}
V(\psi_t) - V(\psi_0) = \int_0^t V'(\psi_{s^-})\,d\psi_s + \frac{1}{2}\int_{-\infty}^{+\infty} L_t^a(\psi)\,\mu(da) \\
+\sum_{s\le t}\big[V(\psi_s) - V(\psi_{s^-}) - V'(\psi_{s^-})\Delta\psi_s\big],
\end{aligned} \tag{33}$$

where $L_t^a(\psi)$ is the local time of $(\psi_t)_{t\ge0}$ at level a and μ is the second derivative of V in the sense of distributions. Due to the form of V we have $\mu(da) = V''(a)\,da$ with V'' the second derivative from the left. As V'' is bounded

$$\int_{-\infty}^{+\infty} L_t^a(\psi)\,\mu(da) = \int_{-\infty}^{+\infty} L_t^a(\psi)V''(a)\,da = \int_0^t V''(\psi_{s^-})\,d\langle\psi^c,\psi^c\rangle_s \tag{34}$$

by Corollary 1 to Theorem 51 in [Pro90]. In reference to $(\psi_t)_{t\ge0}$, as $(M_t)_{t\ge0}$ is continuous, then

$$d\psi_t = M_t dS_t{}^{-1} + S_t{}^{-1}dM_t. \tag{35}$$

Also, as $S_t{}^{-1} = S_0^{-1}e^{-X_t}$, we have

$$dS_t^{-1} = S_t^{-1}\Big[-a\,dt - \sigma\,dW_t + d\Big(\sum_{s\le t}(e^{-\Delta X_s}-1)\Big)\Big]. \tag{36}$$

So, (35) and (36) give

$$d\psi_t = \psi_{t^-}\left[-a\,dt - \sigma\,dW_t + d\Big(\sum_{s\le t}(e^{-\Delta X_s}-1)\Big)\right] + S_t^{-1}\,dM_t. \tag{37}$$

As $(M_t)_{t\geq 0}$ does not decrease, the last term in (37) is continuous with bounded variation, and

$$d\langle \psi^c, \psi^c\rangle_t = \sigma^2 \psi_{t^-}^2\,dt.$$

Let us now compensate the jump part in (33). From (37), we know $\Delta\psi_t = \psi_{t^-}(e^{-\Delta X_t}-1)$, so

$$\begin{aligned}\sum_{s\le t}\big[V(\psi_s) - V(\psi_{s^-})\big] &= \sum_{s\le t}\big[V(\psi_{s^-}+\Delta\psi_s) - V(\psi_{s^-})\big]\\ &= \sum_{s\le t}\big[V(\psi_{s^-}e^{-\Delta X_s}) - V(\psi_{s^-})\big]\\ &= \int_{\mathbb{R}\times[0,t]}\big[V(\psi_{s^-}e^{-x}) - V(\psi_{s^-})\big]\,\mu^X(dx,ds)\end{aligned}$$

where μ^X is the jump measure of the process X given by

$$\mu(\omega, dx, dt) = \sum_s \mathbb{I}_{\{\Delta X_s(\omega)\neq 0\}}\delta_{(\Delta X_s(\omega),s)}(dx,dt).$$

As μ^X is an extended Poisson measure, according to II.1.21 in [JS87], its compensator under $\widetilde{\mathsf{P}}$ is given by $\widetilde{c}\,\widetilde{F}(-dx)\times dt$. From this

$$\sum_{s\le t}\big[V(\psi_s) - V(\psi_{s^-})\big] = \int_{\mathbb{R}\times[0,t]}\big[V(\psi_{s^-}e^{x}) - V(\psi_{s^-})\big]\widetilde{c}\,ds\,d\widetilde{F}(x) + q_t^1$$

with $(q_t^1)_{t\geq 0}$ a local martingale. All this computations and (33) give

$$\begin{aligned}V(\psi_t) - V(\psi_0) = \int_0^t \Big\{&-a\psi_{s^-}V'(\psi_{s^-}) + \frac{\sigma^2}{2}\psi_{s^-}^2 V''(\psi_{s^-})\\ &+\widetilde{c}\int_0^{+\infty}\big[V(\psi_{s^-}e^{x}) - V(\psi_{s^-})\big]\,d\widetilde{F}(x)\Big\}\,ds\\ &+ q_t^1 + \int_0^t \psi_{s^-}V'(\psi_{s^-})(-\sigma)\,dW_s + \int_0^t \frac{V'(\psi_{s^-})}{S_s}\,dM_s.\end{aligned} \tag{38}$$

As $\Delta X_t \leq 0$ the support of the measure dM_s is concentrated on the set where $M_t = S_t$, that is to say, when $\psi_s = 1$. But Proposition 2 gives $V'(1) = 0$. So,

$$\int_0^t \frac{V'(\psi_{s-})}{S_s}\,dM_s = \int_0^t \frac{V'(1)}{S_s}\mathbb{I}_{\{\psi_s=1\}}\,dM_s = 0$$

since $V'(\psi_{s-})\mathbb{I}_{\{\psi_s=1\}} = V'(1)\mathbb{I}_{\{\psi_s=1\}}$. If we define $q = (q_t)_{t\geq 0}$ by

$$q_t = q_t^1 + \int_0^t \psi_{s-} V'(\psi_{s-}).(-\sigma)\,dW_s$$

then (38) coincides with (32). The last step is the application of Itô's Formula to the process given by $\left(e^{-(\lambda+\rho)t}V(\psi_t)\right)_{t\geq 0}$. We obtain (26), where $(Q_t)_{t\geq 0}$ is a stochastic integral with respect to $(q_t)_{t\geq 0}$. □

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Ernesto Mordecki
mordecki@cmat.edu.uy
Centro de Matemática. Facultad de Ciencias.
Universidad de la República.
Montevideo.
Uruguay.

Walter Moreira
walterm@cmat.edu.uy
Centro de Matemática. Facultad de Ciencias.
Universidad de la República.
Montevideo.
Uruguay.

Statistical applications of the approximation of the occupation measure of a diffusion.

Gonzalo Perera and Mario Wschebor

ABSTRACT

In this paper we announce without proofs an hypothesis testing method to fit the diffusion coefficient σ of a $d-$dimensional stochastic differential equation $dX_t = \sigma(X_t)\, dW_t + b(X_t)\, dt$ on the basis of the observation of regularizations of a single trajectory of the solution. Our method is based on a Central Limit Theorem for the occupation measure of a Brownian semimartingale that extends a previous result of the authors (1998).

RESUMEN

En este trabajo anunciamos un método de prueba de hipótesis para hacer inferencia sobre el coeficiente de difusión σ en una ecuación diferencial estocástica $dX_t = \sigma(X_t)dW_t + b(X_t)dt$ en dimensión d. No se incluyen demostraciones. El método está basado en la observación de regularizaciones de la solución y en un Teorema Central del Límite para la medida de ocupación de una semimartingala Browniana, que extiende un resultado anterior de los autores (1998).

Mathematics Subject Classification (1991): 62G10, 62M02, 60F05, 60J60, 60J55 *Key words:* Hypothesis testing, occupation measures, limit theorems, diffusion coefficient
Short Title: Statistical applications of occupation measures[1].

Let

$$X_t = x_0 + \int_0^t a_s dW_s + \int_0^t b_s ds,\ t \geq 0 \tag{1}$$

be an Itô semimartingale with values in $\mathcal{R}^d$, d a positive integer.

We use the following notations:

$\mathcal{W} = \{W_s : s \geq 0\} = \{(W_s^1, ..., W_s^d)^T : s \geq 0\}$ is a Brownian motion in $\mathcal{R}^d$, $(.)^T$ denotes transposition, $x_0 \in \mathcal{R}^d$.

$\mathcal{F} = \{\mathcal{F}_s : s \geq 0\}$ is the filtration generated by $\mathcal{W}$.

$$a_s = \left(a_s^{j,k}\right)_{j,k=1,...,d},\ b_s = \left(b_s^j\right)_{j=1,...,d},\ s \geq 0.$$

[1]This paper contains the announcement of new results; an extended version with detailed proofs will be submitted for publication elsewhere.

are stochastic processes with continuous paths adapted to $\mathcal{F}$, the first one taking values in the real matrices of $d \times d$ elements and the second one in $\mathcal{R}^d$. We denote $a = \{a_s : s \geq 0\}$, $b = \{b_s : s \geq 0\}$, $\mathcal{X} = \{X_t : t \geq 0\}$.

The purpose of this paper is to study inference methods on the noise part in (1) from the observation of a regularization of the actual path X_t during a time interval $0 \leq t \leq \tau$. This is done in the Example C of 2 for diffusions that verify some additional requirements.

A well-known difficulty for this problem is that if one considers different values of a the induced measures on the space of trajectories become mutually singular, so that there is no straightforward method based on likelihood methods(see for example Kutoyants (1984, 1994) or Prakasa Rao (1999) for general references on this point).

Several estimation methods for diffusion coefficients and related problems have been studied in the literature. See for instance Brugière (1991), Dacunha-Castelle & Florens-Zmirou (1986), Florens-Zmirou (1993), Jacod (2000), Génon-Catalot (1990), Génon-Catalot & Jacod (1993, 1994), Génon-Catalot, Jeantheau & Laredo (1998), Hoffmann (1997 a), 1997 b), 1999 a), 1999 b), 1999 c), 2000), Yoshida (1992).

Most of these results are based on discrete observations of the process $\mathcal{X}$ and deal with approximation of the local time of $\mathcal{X}$. For discrete sampling Azaïs (1989) and Jacod (1998) contain approximation methods for local times, including, in the second reference, the speed of convergence.

In this paper we present a new method for hypothesis testing on the diffusion coefficient, that is based upon the observation of functionals defined on regularizations of the path of the underlying process $\mathcal{X}$ instead of the discrete sampling framework. Our approach is based on certain integral functionals related to the occupation measure. Theorem 1 below is the key result in order to obtain the asymptotic distribution of our estimates. In Perera & Wschebor (1998) a similar statement to that of Theorem 1 has been proved in dimension 1. The proofs of this new extended version and a comparison with previous estimation methods will be included in an extended version of the present paper. A first result in the spirit of Theorem 1, valid when $\mathcal{X}$ is Brownian Motion, was given by Berzin-Joseph & León (1994) (see also Berzin-Joseph & León (1997)). A similar result for stationary Gaussian processes is in Berzin-Joseph, León & Ortega (1998).

On the other hand, and more important from the standpoint of applications, the statement of Theorem 2 is more adequate for statistical purposes. We have included some comments and given some general examples in which the computations can be expressed in a reasonable simple form.

We assume the following additional hypotheses on the processes $\{a_s : s \geq 0\}$ and $\{b_s : s \geq 0\}$:

For each $j, k = 1, ..., d, \; s \geq 0, \; \varepsilon > 0$:

$$\frac{a_{s+\varepsilon}^{j,k} - a_s^{j,k}}{\sqrt{\varepsilon}} = \left(\overrightarrow{a}_s^{j,k}\right)^T Z_{s,\varepsilon}^{j,k} + r_{s,\varepsilon}^{j,k} \tag{2}$$

where:

$\overrightarrow{a}_s^{j,k}, \; Z_{s,\varepsilon}^{j,k}$ are random vectors with values in $\mathcal{R}^d$.

We also put for $j, k = 1, ..., d$, $Z_{s,\varepsilon}^{i,j,k}$ (respectively $\overrightarrow{a}_s^{i,j,k}$) for the i-th coordinate of $Z_{s,\varepsilon}^{j,k}$ (respectively $\overrightarrow{a}_s^{j,k}$) $i = 1, ..., d$, and

$$Z_{s,\varepsilon} = \left(Z_{s,\varepsilon}^{i,j,k}\right)_{i,j,k=1,....,d}, \; \overrightarrow{a}_s = \left(\overrightarrow{a}_s^{i,j,k}\right)_{i,j,k=1,....,d}.$$

$r_{s,\varepsilon}^{j,k}$ is a random variable with values in $\mathcal{R}^1$.

$\overrightarrow{a}_s^{j,k}$ is $\mathcal{F}_s$-mesurable and $Z_{s,\varepsilon}^{j,k}, \; r_{s,\varepsilon}^{j,k}$ are $\mathcal{F}_{s+\varepsilon}$-mesurable, and verify for almost every pair $s, t, s \neq t$:

$$\left(Z_{s,\varepsilon}, Z_{t,\varepsilon}, W_{\cdot}^{\varepsilon,s}, W_{\cdot}^{\varepsilon,t}\right) \Longrightarrow \left(Z_s, Z_t, \widetilde{W}_{\cdot}^s, \widetilde{W}_{\cdot}^t\right) = V(s,t) \quad as \;\; \varepsilon \to 0 \tag{3}$$

where $\Longrightarrow$ denotes weak convergence of probability measures in the space

$$\mathcal{R}^{d^3} \times \mathcal{R}^{d^3} \times [C([0, +\infty) \to \mathcal{R})] \times [C([0, +\infty) \to \mathcal{R})]$$

and:

- For each $\varepsilon > 0, \; t \geq 0, \; W_{\cdot}^{\varepsilon,t}$ is a new Brownian motion with values in $\mathcal{R}^d$ defined by
$$W_u^{\varepsilon,t} = \frac{W_{t+\varepsilon u} - W_t}{\sqrt{\varepsilon}}, \;\; u \geq 0$$
- $\left\{\widetilde{W}_{\cdot}^t : t \geq 0\right\}$ is a collection of independent Brownian motions with values in $\mathcal{R}^d$.
- The distribution of $V(s,t)$ is symmetric, that is, $V(s,t)$ and $-V(s,t)$ have the same law.
- $V(s,t)$ is independent from $\mathcal{F}_\infty$.
- $\{s, t\} \cap \{s', t'\} = \emptyset \quad \Longrightarrow \quad V(s,t)$ and $V(s', t')$ are independent.

Finally, we assume the following boundedness properties.

First:

$$\sup_{s\in[0,T]}\sup_{j,k=1\dots,d} E\left\{\left\|r^{j,k}_{s,\varepsilon}\right\|^p\right\} \to 0 \quad as \quad \varepsilon \to 0 \quad \text{for every } T, p > 0.$$

and second, for every $T, \varepsilon_0 > 0$ and every $p > 0$ the $L^p(\Omega)$-norms of the coordinates of:

$$a_s, b_s, \overrightarrow{a}_s, Z_{s,\varepsilon}$$

are uniformly bounded as $0 \le s \le T,\ 0 < \varepsilon \le \varepsilon_0$.

Let us first check that in a basic set of examples the preceeding hypotheses hold true.

Example 1: Diffusions.

With the above notations, let

$$a_s = \sigma(s, X_s),\ b_s = b(s, X_s)$$

where

$$\sigma(s,x) = (\sigma^{j,k}(s,x))_{j,k=1,\dots,d}; \quad b(s,x) = (b^j(s,x))_{j=1,\dots,d}, \quad s \ge 0,\ x \in \mathcal{R}^d$$

satisfy the usual hypotheses to assure the existence and uniqueness of strong solutions of the system of stochastic differential equations

$$\begin{aligned} dX_t &= \sigma(t, X_t)\ dW_t + b(t, X_t)dt \\ X_0 &= x_0 \end{aligned}$$

such as Lipschitz local behaviour and degree one polynomial bound at ∞ (see Ikeda & Watanabe (1989)).

Furthermore we assume that σ is of class C_b^2 (that is C^2 with bounded derivatives).

Let us verify that (2) and the subsequent conditions hold true. Check first that for every $p > 0$, as $\varepsilon \downarrow 0$:

$$\frac{a^{j,k}_{s+\varepsilon} - a^{j,k}_s}{\sqrt{\varepsilon}} = \frac{\sigma^{j,k}(s+\varepsilon, X_{s+\varepsilon}) - \sigma^{j,k}(s, X_s)}{\sqrt{\varepsilon}}$$

$$= \frac{(D_x\sigma^{j,k})(s, X_s)(X_{s+\varepsilon} - X_s)}{\sqrt{\varepsilon}} + o_{L^p}(1)$$

where the notation $A^{j,k}(s,\varepsilon) = o_{L^p}(1)$ means that

$$\sup_{j,k=1,\dots,d} E\left\{ \sup_{0\le s\le T} \left|A^{j,k}(s,\varepsilon)\right|^p \right\} \to 0 \quad as\ \varepsilon \downarrow 0.$$

for each $T > 0$.

It follows that:

$$\frac{a^{j,k}_{s+\varepsilon} - a^{j,k}_s}{\sqrt{\varepsilon}} = (D_x\sigma^{j,k})(s,X_s)\frac{\int_s^{s+\varepsilon}\sigma(t,X_t)\,dW_t + \int_s^{s+\varepsilon} b(t,X_t)\,dt}{\sqrt{\varepsilon}} + o_{L^p}(1)$$

$$= (D_x\sigma^{j,k})(s,X_s)\frac{1}{\sqrt{\varepsilon}}\left\{\int_0^1 \sigma(s+\varepsilon v, X_{s+\varepsilon v})\,d_v(W_{s+\varepsilon v} - W_s) + \varepsilon\int_0^1 b(s+\varepsilon v, X_{s+\varepsilon v})\,dv\right\} + o_{L^p}(1)$$

$$\begin{aligned} = {} & (D_x\sigma^{j,k})(s,X_s).\sigma(s,X_s).\frac{W_{s+\varepsilon} - W_s}{\sqrt{\varepsilon}} \\ & +(D_x\sigma^{j,k})(s,X_s)\int_0^1 [\sigma(s+\varepsilon v, X_{s+\varepsilon v}) - \sigma(s,X_s)]\,d_v W_v^{\varepsilon,s} + o_{L^p}(1) \\ & = (D_x\sigma^{j,k})(s,X_s).\sigma(s,X_s).\frac{W_{s+\varepsilon} - W_s}{\sqrt{\varepsilon}} + o_{L^p}(1) \end{aligned}$$

which means that we have (2) with:

$$\left(\overrightarrow{a}^{j,k}_s\right)^T = (D_x\sigma^{j,k})(s,X_s).\sigma(s,X_s)$$

$$Z^{j,k}_{s,\varepsilon} = \frac{W_{s+\varepsilon} - W_s}{\sqrt{\varepsilon}} = W_1^{\varepsilon,s}$$

(3) is easily verified.

Example 2. Smoother integrands.

Suppose that for every $T > 0$, $\{a_s : s \ge 0\}$ satisfies a condition of type:

$$\sup_{j,k=1,\dots,d;0\le s\le T} \left|a^{j,k}_{s+\varepsilon} - a^{j,k}_s\right| \le C_T.\varepsilon^{\alpha(T)} \quad (0 < \varepsilon < 1)$$

where C_T stands for a random variable depending on T, $C_T \in L^p(\Omega)$ for all $p > 0$ and $\alpha(T) > \frac{1}{2}$. Then, the above conditions are obviously satisfied with

$$\overrightarrow{a}_s = 0,\ Z_{s,\varepsilon} = 0.$$

Example 3. Non-Markovian martingales.

A class of examples that is neither contained in **1.** nor in **2.** is the following. Take $d = 1$ and

$$a_s = f(W_s),\ b_s = 0$$

where the function $f : \mathcal{R} \to \mathcal{R}$ is a C^3-function such that

$$\underline{f} = \inf\{f(x) : x \in \mathcal{R}\} > 0,$$

f'' and f''' are bounded, $f''(0) \neq 0$ and

$$\sup\{|f''(x)| : x \in \mathcal{R}\} < 2.\underline{f}.$$

For example, $f(x) = 1 + C.g(x)$, $g(.)$ of class C^3, non-negative with compact support, $g''(0) \neq 0$ and the constant C small enough so that $0 < C.\sup_{x\in\mathcal{R}} |g''(x)| < 2$. In this case

$$\frac{a_{s+\varepsilon} - a_s}{\sqrt{\varepsilon}} = \frac{f'(W_s)\,(W_{s+\varepsilon} - W_s)}{\sqrt{\varepsilon}} + \frac{\frac{1}{2}f''(W_s + \theta(W_{s+\varepsilon} - W_s))(W_{s+\varepsilon} - W_s)^2}{\sqrt{\varepsilon}}$$

with $(0 < \theta < 1)$, so that the conditions hold with $\overrightarrow{a}_s = f'(W_s)$, $Z_{s,\varepsilon} = W_1^{\varepsilon,s}$.

However, the process $\mathcal{X}$ is non-Markovian independently of the choice of the filtration, so that it is not a diffusion (c.f. Nualart & Wschebor, 1991, p. 106).

Let us now describe the general procedure that we will be considering.

Instead of observing the path of the underlying stochastic process $\{X_t : 0 \leq t \leq \tau\}$ during a time interval, which generally speaking is not feasible from physical point of view, we will observe a regularization

$$X_\varepsilon(t) = \int_{-\infty}^{+\infty} \psi_\varepsilon(t - s)X_s\ ds$$

where $\varepsilon > 0$, for each $x \in \mathcal{R}$, $\psi(x) = \left(\psi^{j,k}(x)\right)_{j,k=1,\dots,d}$ is a deterministic matrix kernel, each function $\psi^{j,k}(x)$ being C^∞real-valued, support contained in the interval $[-1, 1]$,

$$\int_{-\infty}^{+\infty} \psi(x)\ dx = \left(\int_{-\infty}^{+\infty} \psi^{j,k}(x)\ dx\right)_{j,k=1,\dots,d} = I$$

(I denotes the identity matrix $d \times d$) and

$$\psi_\varepsilon(t) = \frac{1}{\varepsilon}\psi(\frac{t}{\varepsilon})$$

In fact we do not observe the complete smoothed path but only a functional defined on it having the general form:

$$\theta_{\varepsilon,T} = \theta_{\varepsilon,T}(f,g) = \int_0^T f(X_\varepsilon(t))g(\|\sqrt{\varepsilon}X'_\varepsilon(t)\|)\, dt \tag{4}$$

$'$ denotes differentiation with respect to t and $f : \mathcal{R}^d \to \mathcal{R}$, $g : \mathcal{R}^+ \to \mathcal{R}$ satisfy:

(i) f is of class C_b^2.

(ii) g is of class C^2. $|g'(r)| \leq C_g(1 + r^m)$ for some $m \geq 1$, some constant C_g and all $r \in \mathcal{R}^+$.

Our aim is to give a Central Limit Theorem for $\theta_{\varepsilon,T}$ as a first step to statistical inference on a.

Two interesting functionals are the following:

1) (Normalized curve length) Let $g(r) = r$ and $f(.)$ a C_b^2- approximation of 1_B the indicator function of a subset in $\mathcal{R}^d$ with a sectionally smooth boundary.

In this case, the functional $\theta_{\varepsilon,T}$ is an approximation of $\sqrt{\varepsilon}.l_\varepsilon(\tau; B)$, $l_\varepsilon(\tau; B)$ denoting the length of the part of the curve $t \rightsquigarrow X_\varepsilon(t)$ $(0 \leq t \leq \tau)$ that is contained in the "observation window" B, a subset of the state space. In the relevant situations $l_\varepsilon(\tau; B) \to +\infty$ as $\varepsilon \downarrow 0$ and $\varepsilon^{\frac{1}{2}}$ is the appropriate renormalization of the length to have a non-trivial limiting behaviour.

2) (Normalized kinetic energy) Let $g(r) = r^2$ and $f(.)$ is as in the previous example, in which case $\theta_{\varepsilon,T}$ is an approximation of $\varepsilon.E_\varepsilon(\tau; B)$, $E_\varepsilon(\tau; B)$ denoting the kinetic energy of the same part of the smoothed path.

THEOREM 1 *With the hypotheses above*

$$\left(W_T, \frac{1}{\sqrt{\varepsilon}}\left[\theta_{\varepsilon,T}(f,g) - \int_0^T E\{f(X_t)g(\|\Sigma_t^{\frac{1}{2}}\xi\|)/\mathcal{F}_\infty\}dt\right]\right) \Longrightarrow \left(W_T, W^*_{\bar{\sigma}^2(T)}\right)$$

as $\varepsilon \downarrow 0$ *where*

- $\Rightarrow$ *denotes weak convergence of probability measures in the space* $C([0, +\infty), \mathcal{R}^2)$,
- W^* *denotes a Wiener process independent of* $\mathcal{F}_\infty$,
- *for* $u \in \mathcal{R}$, $\Sigma_u = \int_{-1}^1 \psi(v)a_u a_u^T \psi^T(v)dv$,
- ξ *is a standard Gaussian random variable with values in* $\mathcal{R}^d$, *independent of* $\mathcal{F}_\infty$.

- $\overline{\sigma}^2(\tau) = \int_0^\tau du \iint_{-1}^1 s(u,v,v')dvdv'$ *where* $s(u,v,v') =$

$$E\Big\{f^2(X_u)g'(\|\eta_{u,v}\|)g'(\|\eta'_{u,v'}\|) \times \left(sg(\eta_{u,v})\right)^T \psi(-v)a_u a_u^T \psi^T(-v')sg(\eta'_{u,v'})/\mathcal{F}_\infty\Big\}$$

where

$$sg(y) = \frac{y}{\|y\|}(y \in \mathcal{R}^d,\ y \neq 0),$$

and where the conditional distribution of the pair of $\mathcal{R}^d$*-valued random variables* $(\eta_{u,v}, \eta'_{u,v'})$ *given the* σ*-algebra* $\mathcal{F}_\infty$ *is centered Gaussian and*

$$E\left\{\eta_{u,v}\eta_{u,v}^T/\mathcal{F}_\infty\right\} = E\left\{\eta'_{u,v'}\eta'^T_{u,v'}/\mathcal{F}_\infty\right\} = \Sigma_u$$

$$E\left\{\eta_{u,v}\eta'^T_{u,v'}/\mathcal{F}_\infty\right\} = \int_{-1}^{v\wedge v'} \psi(-w)a_u a_u^T \psi^T(-w + |v' - v|)dw$$

Remark 1 It is not possible to use for statistical purposes the above Theorem as it has been stated, since its application requires the knowledge of the path $\{X_t\ (0 \leq t \leq \tau)\}$ which can not be observed. The next Theorem points to solve this problem in a framework that is more limited than the previous one, that is, for diffusions with coefficients that are only space functions and regularisations that are isotropic.

Remark 2 Note that the drift b does not appear in the statement of Theorem 1, either in the centering term or in the asymptotic probability distribution. The same happens in the statistical version below (Theorem 2). This is of course useful to make inference on a.

For the next Theorem we will add a certain number of restrictions to the preceeding framework. We only consider the case of diffusions with coefficients that do not depend on time, that is

$$a_s = \sigma(X_s), \quad b_s = b(X_s)$$

where σ and b satisfy the hypotheses stated in the above Example 1.

We will also assume that

$$\psi(x) = \psi^*(x).I$$

where ψ^* is real-valued (isotropic regularization).

Then, we may replace $\sigma(X_t)$ by $\sigma(X_\varepsilon(t))$ in the centering term which becomes:

$$C_\varepsilon(\tau) = \int_0^\tau f(X_\varepsilon(t))\, E\left\{g\left[\|\psi^*\|_2 \left\|\sigma^T(X_\varepsilon(t)).\xi\right\|)\right] / \mathcal{F}_\infty\right\} dt$$

($\|\psi^*\|_2$ denotes the L^2-norm ot the function ψ^*) and the asymptotic law of $W^*_{\overline{\sigma}^2_\varepsilon(\tau)}$ is asymptocally the same as the one of $W^*_{\overline{\sigma}^2(\tau)}$, where

$$\overline{\sigma}^2_\varepsilon(\tau) = \int_0^\tau f^2(X_\varepsilon(u))\, du \iint_{-1}^1 \psi^*(-v)\psi^*(-v')\, dvdv'$$

$$E\left\{g'(\|\eta_{u,v}\|)g'(\|\eta'_{u,v'}\|).\left(sg(\eta_{u,v})\right)^T.\sigma(X_\varepsilon(t)).\sigma^T(X_\varepsilon(t)).\left(sg(\eta'_{u,v'})\right)/\mathcal{F}_\infty\right\}$$

Σ_u is replaced by

$$\|\psi^*\|^2\, \sigma(X_\varepsilon(t)).\sigma^T(X_\varepsilon(t))$$

and $E\left\{\left(sg(\eta_{u,v})\right)\left(sg(\eta'_{u,v'})\right)^T\right\}$ by

$$K(v,v').\sigma(X_\varepsilon(t)).\sigma^T(X_\varepsilon(t))$$

with

$$K(v,v') = \int_{-1}^{v\wedge v'} \psi^*(-w)\psi^*(-w+|v-v'|)\, dw \tag{5}$$

Note that the replacement in the centering term of X_t by $X_\varepsilon(t)$ is by no means evident, since it is necessary to divide by $\sqrt{\varepsilon}$.

THEOREM 2 *Under the above conditions,*

$$\left(W_\tau, \frac{1}{\sqrt{\varepsilon}}\left[\left[\int_0^\tau f(X_\varepsilon(t))g(\varepsilon^{\frac{1}{2}}X'_\varepsilon(t))dt - C_\varepsilon(\tau)\right]\right]\right) \tag{6}$$

$$\Longrightarrow \left(W_\tau, W^*_{\overline{\sigma}^2(\tau)}\right) \quad \text{as } \varepsilon \downarrow 0$$

and

$$\overline{\sigma}^2_\varepsilon(\tau) \approx \overline{\sigma}^2(\tau)$$

where weak convergence, ξ, W^* and $\overline{\sigma}^2(\tau)$ are as in (1).

Example A.

In Theorem 1 let us put $d = 1$ and $F(x, y) = f(x).|y|$.
We have:

$$\Sigma_u = |a_u|^2 \, \|\psi\|_2^2 ,$$

$$\theta_{\varepsilon,\tau}(F) = \int_0^\tau f(X_\varepsilon(t)).\varepsilon^{\frac{1}{2}} \left|X'_\varepsilon(t)\right| \, dt = \varepsilon^{\frac{1}{2}} \int_{-\infty}^{+\infty} f(u) N_u^{X_\varepsilon}([0,\tau]) \, du$$

where for $g : I \to \mathcal{R}$, $N_u^g(I)$ denotes the number of roots of the equation $g(t) = u$ in the interval I.

In this case, the centering term becomes:

$$\int_0^\tau f(X_t)\,|a_t|\,\|\psi\|_2 \sqrt{\frac{2}{\pi}} \, dt = \sqrt{\frac{2}{\pi}} \, \|\psi\|_2 \int_{-\infty}^{+\infty} f(u) \widetilde{L}_u^X([0,\tau]) \, du$$

Here $\widetilde{L}_u^X([0,\tau])$ stands for the local time of the process $\mathcal{X}$ for the mesure having density $|a_t|$ with respect to Lebesgue measure λ. That is, if

$$\mu_\tau(B) = \int_0^\tau 1_{\{X_t \in B\}} \, |a_t| \, dt$$

then

$$\widetilde{L}_u^X([0,\tau]) = \frac{d\mu_\tau}{d\lambda}(u)$$

The asymptotic variance is given by

$$\overline{\sigma}^2(\tau) = \int_0^\tau f^2(X_u)\,|a_u|^2 \, du \iint_{-1}^{1} \psi(-v)\psi(-v') \left[2.P(\eta_v \eta_{v'} > 0) - 1\right] \, dv dv'$$

where the distribution of the random variable $\begin{pmatrix} \eta_v \\ \eta_{v'} \end{pmatrix}$ in $\mathcal{R}^2$ is centered Gaussian with covariance matrix

$$\begin{pmatrix} \|\psi\|_2^2 & \int_{-1}^{v\wedge v'} \psi(-w)\psi(-w + |v - v'|) \, dw \\ \int_{-1}^{v\wedge v'} \psi(-w)\psi(-w + |v - v'|) \, dw & \|\psi\|_2^2 \end{pmatrix}$$

Summing up, the statement of (1) takes the form:

$$\frac{1}{\sqrt{\varepsilon}} \left[\varepsilon^{\frac{1}{2}} \int_{-\infty}^{+\infty} f(u) N_u^{X_\varepsilon}([0,\tau]) \, du - \sqrt{\frac{2}{\pi}} \, \|\psi\|_2 \int_{-\infty}^{+\infty} f(u) \widetilde{L}_u^X([0,\tau]) \, du \right] \Longrightarrow W^*_{\overline{\sigma}^2(\tau)} \quad (7)$$

when $\varepsilon \downarrow 0$, the convergence taking place in the space $C\left(\left([0,+\infty) \to \mathcal{R}\right)\right.$.

It is known that if $\mathcal{X}$ is a continuous semimartingale as $\varepsilon \downarrow 0$, the expression in brackets tends to zero almost surely (this is essentially the result in Azaïs & Wschebor, 1997). The convergence (7) is a result on the fluctuations contained in Perera & Wschebor (1998). In case $\mathcal{X}$ is Brownian motion, this has been first proved by León & Berzin (1994) using Wiener Chaos expansions.

Example B.

Suppose that we are in the conditions of Theorem 2, with $d = 1$ and $g(r) = r$. We also assume that $\inf_{x\in\mathcal{R}} \sigma(x) > 0$.

Suppose that we want to test the null hypothesis

$$H_0 : \sigma(x) = \sigma_0(x) \quad \text{for all } x \in \mathcal{R}$$

against the alternative

$$H_\varepsilon : \sigma(x) = \sigma_\varepsilon(x) = \sigma_0(x) + \sqrt{\varepsilon}\sigma_1(x) + \gamma(x,\varepsilon) \quad \text{for all } x \in \mathcal{R}$$

where $\gamma(x,\varepsilon) = o(\sqrt{\varepsilon})$ and $D_x\gamma(x,\varepsilon) = o(\sqrt{\varepsilon})$ *as* $\varepsilon \downarrow 0$, uniformly on $x \in \mathcal{R}$. Here $\sigma_0(.)$, $\sigma_1(.)$ and $\gamma(.,\varepsilon)$ are given functions of class C_b^2 with at most degree one polynomial growth at ∞.

The application of Theorems 1.1 and 1.2 is not straightforward under the present conditions, since the process X_t depends now on epsilon. However, one can check that the same proofs remain valid, sustituting X_t by the process X_t^ε, which is the solution of

$$dX_t^\varepsilon = \sigma_\varepsilon(X_t^\varepsilon)\, dW_t + b(X_t^\varepsilon)dt,\ X_0^\varepsilon = x_0$$

and setting $X_\varepsilon(t) = (\psi_\varepsilon * X^\varepsilon)(t)$. Under the hypothesis H_ε, one has:

$$\frac{1}{\sqrt{\varepsilon}}\left[\sqrt{\varepsilon}\int_{-\infty}^{+\infty} f(u)N_u^{X_\varepsilon^\varepsilon}([0,\tau])\, du - \sqrt{\frac{2}{\pi}}\,\|\psi\|_2\int_0^\tau f(X_\varepsilon(t))\, \sigma_0(X_\varepsilon(t))dt\right]$$

$$\approx \sqrt{\frac{2}{\pi}}\,\|\psi\|_2\int_0^\tau f(X_\varepsilon(t))\, \sigma_1(X_\varepsilon(t))dt + W^*_{\overline{\sigma}_\varepsilon^2(\tau)} \tag{8}$$

One should interpret (8) in the following way:

As $\varepsilon \downarrow 0$ the law of the left hand member converges in $C([0,+\infty),\mathcal{R})$ to the law of the random process

$$\sqrt{\frac{2}{\pi}}\,\|\psi\|_2\int_0^\tau f(X(t))\, \sigma_1(X(t))dt + W^*_{\overline{\sigma}_0^2(\tau)} \tag{9}$$

and *furthermore* the right-hand member in (8) converges weakly to the process (9) as $\varepsilon \downarrow 0$. Note that this is well adapted to statistical purposes since both the centering term and the asymptotic distribution in (8) can be computed from the hypotheses and from functionals defined on the smoothed path $\{X_\varepsilon(t) : 0 \leq t \leq \tau\}$.

Example C.

Suppose again that we are in the conditions of Theorem 2 that is, the process $\mathcal{X}$ is the solution of a stochastic differential equation, the remaining conditions hold true and $g(r) = r$. We will extend the previous example to dimension $d > 1$.

Suppose that we want to test the null hypothesis

$$H_0 : \Gamma(x) = \sigma(x).\sigma^T(x) = \Gamma_0(x) \quad \text{for all } x \in \mathcal{R}^d$$

against the alternative

$$H_\varepsilon : \Gamma(x) = \Gamma_0(x) + \sqrt{\varepsilon}\Gamma_1(x) + \Gamma_2(x, \varepsilon) \quad \text{for all } x \in \mathcal{R}^d \tag{10}$$

where $\|\Gamma_2(x, \varepsilon)\|_{d\times d} = o(\sqrt{\varepsilon})$ and $\|D_x\Gamma_2(x, \varepsilon)\| = o(\sqrt{\varepsilon})$ *as* $\varepsilon \to 0$, uniformly on $x \in \mathcal{R}^d$. $\Gamma_0(.)$ and $\Gamma_1(.)$ are positive semidefinite $d \times d$ matrices having elements that are functions of class C_b^2 and at most degree two polynomial growth at ∞ and $\|.\|_{d\times d}$ is any norm on $d \times d$ matrices. Furthermore, assume that $\Gamma_0(.)$ satisfies a strong ellipticity condition of the type $y^T\Gamma_0(x)y \geq \delta \|y\|^2$ for some $\delta > 0$ and any $x, y \in \mathcal{R}^d$.

The result takes the form that under H_ε:

$$\frac{1}{\sqrt{\varepsilon}}\left[\int_0^\tau f(X_\varepsilon(t))\left[\left\|\sqrt{\varepsilon}X'_\varepsilon(t)\right\| - \|\psi^*\|_2 .J_0\left(\Gamma_0(X_\varepsilon(t))\right)\right] dt\right] \approx$$

$$\approx \frac{1}{2}\|\psi^*\|_2 \int_0^\tau f(X_\varepsilon(t)).J_1\left(\Gamma_0(X_\varepsilon(t)), \Gamma_1(X_\varepsilon(t))\right) dt + W^*_{\overline{\sigma}_\varepsilon^2(\tau)}$$

$$\overline{\sigma}_\varepsilon^2(\tau) \approx \int_0^\tau f^2(X_\varepsilon(u))\, du \iint_{-1}^1 \psi^*(-v)\psi^*(-v')\; J_2(\Gamma_0(X_\varepsilon(u)), \widetilde{K}(v, v'))dvdv'$$

where for A positive definite and B, C positive semidefinite $d \times d$ matrices, ξ a standard normal random vector in $\mathcal{R}^d$ and k a real number $|k| \leq 1$, we put:

$$J_0(A) = E\left\{\left(\xi^T A\xi\right)^{\frac{1}{2}}\right\}, \quad J_1(A, B) = E\left\{\frac{\xi^T B\xi}{\left(\xi^T A\xi\right)^{\frac{1}{2}}}\right\}, \; J_2(C, k) = E\left\{\frac{\eta^T C\eta'}{\|\eta\|\|\eta'\|}\right\}$$

where η, η' are normal centered random vectors in $\mathcal{R}^d$, $E\{\eta.\eta'\} = k.C$, $E\{\eta.\eta^T\} = E\{\eta'.\eta'^T\} = C$ and $\widetilde{K}(v, v') = \frac{K(v,v')}{\|\psi^*\|_2^2}$. (see (5)).

In some cases the functions J_0, J_1, J_2 can be computed explicitely by more or less complicated formulae. An important such case is $\Gamma_0 = I$, when the noise in the stochastic differential equation is purely Brownian under the null hypothesis, and we want to test this hypothesis against contiguous ones of the form (10). One obtains:

$$J_0(I) = \begin{cases} \frac{(2p)!}{(2^p p!)^2}\sqrt{8\pi p} & \text{if} \quad d = 2p \\ \frac{(2^p p!)^2}{(2p)!}\frac{1}{\sqrt{2\pi}} & \text{if} \quad d = 2p+1 \end{cases},$$

$$J_1(I, B) = \frac{J_0(I)}{d} tr(B),$$

$$J_2(I, k) = \frac{1}{\sqrt{\pi} 2^{\frac{d}{2}-1}\Gamma(\frac{d-1}{2})} \int_{-\infty}^{+\infty} x\mathcal{I}_d(x) e^{-\frac{1}{2}\left(x - \frac{k}{\sqrt{1-k^2}}\right)^2} dx,$$

$$\mathcal{I}_d(x) = \int_0^{+\infty} \frac{\rho^{d-2}}{\sqrt{x^2+\rho^2}} e^{-\frac{1}{2}\rho^2} d\rho.$$

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GONZALO PERERA
gperera@fing.edu.uy
Instituto de Matemática y Estadística "Rafael Laguardia".
Facultad de Ingeniería.
y
Centro de Matemática. Facultad de Ciencias.
Universidad de la República.
Montevideo.
Uruguay.

MARIO WSCHEBOR
wscheb@fcien.edu.uy
Centro de Matemática. Facultad de Ciencias.
Universidad de la República.
Montevideo.
Uruguay.

Sobre la entropía topológica de un flujo Hamiltoniano óptico

César J. Niche

RESUMEN

En este trabajo probamos dos fórmulas para la entropía topológica, en un nivel de energía $\Sigma = H^{-1}(e)$, de un flujo Hamiltoniano $\mathcal{F}$-óptico, inducido por un Hamiltoniano $H \in C^{\infty}(M)$, donde (M, ω) es una variedad simpléctica y $\mathcal{F}$ una distribución Lagrangiana. En la primera de ellas, la entropía topológica es

$$h_{top}(\varphi_t|_{\Sigma}) = \lim_{\epsilon \to 0} \lim_{t \to \infty} \frac{1}{t} \log \int_{N_\epsilon} |det\,(d\varphi_t)_x|_{\alpha(x)}|\, dx$$

donde $N_\epsilon = H^{-1}(e - \epsilon, e + \epsilon)$, φ_t es el flujo inducido por H y $\alpha(x) \subset T_x M$ es el subespacio Lagrangiano dado por $\mathcal{F}$.

En la segunda, asumiendo que la restricción del flujo a Σ admite una distribución continua invariante de hiperplanos transversales a $X_H(x)$, la entropía topológica es, para una modificación adecuada de la distribución Lagrangiana $\mathcal{F}$, dada por $\tilde{\alpha}(x) = (\alpha(x) \cap T_x\Sigma) + \langle X_H(x) \rangle$,

$$h_{top}(\varphi_t|_{\Sigma}) = \lim_{t \to \infty} \frac{1}{t} \log \int_{\Sigma} |det\,(d\varphi_t)_x|_{\tilde{\alpha}(x)}|\, dx.$$

ABSTRACT

In this article, we prove two formulas for the topological entropy, on an energy level $\Sigma = H^{-1}(e)$, of an $\mathcal{F}$-optical Hamiltonian flow induced by $H \in C^{\infty}(M)$, where (M, ω) is a symplectic manifold, endowed with a Lagrangian distribution $\mathcal{F}$.

In the first formula, the topological entropy is

$$h_{top}(\varphi_t|_{\Sigma}) = \lim_{\epsilon \to 0} \lim_{t \to \infty} \frac{1}{t} \log \int_{N_\epsilon} |det\,(d\varphi_t)_x|_{\alpha(x)}|\, dx$$

where $N_\epsilon = H^{-1}(e - \epsilon, e + \epsilon)$, φ_t is the flow induced by H and $\alpha(x) \subset T_x M$ is the Lagrangian subspace given by the distribution $\mathcal{F}$. In the second one, assuming that the restriction of the flow to Σ admits a continuous invariant distribution of hyperplanes in $T_x\Sigma$ transversal to $X_H(x)$, then the topological entropy is, for a suitable modification of the Lagrangian distribution, given by $\tilde{\alpha}(x) = (\alpha(x) \cap T_x\Sigma) + \langle X_H(x) \rangle$,

$$h_{top}(\varphi_t|_{\Sigma}) = \lim_{t \to \infty} \frac{1}{t} \log \int_{\Sigma} |det\,(d\varphi_t)_x|_{\tilde{\alpha}(x)}|\, dx.$$

1 Introducción

Para una variedad Riemanniana compacta (M, g), donde la métrica g es C^∞, Mañé [8] probó las siguientes fórmulas para la entropía topológica del flujo geodésico $\varphi_t : SM \to SM$, donde SM es el fibrado unitario tangente.

TEOREMA 1 (TEOREMA 1.1, MAÑÉ [8]) *Para una variedad Riemanniana compacta y sin borde (M, g), donde g es C^∞, se cumple que*

$$h_{top} = \lim_{T\to\infty} \frac{1}{T} \log \int_{M\times M} n_T(x, y)\, dx\, dy = \lim_{T\to\infty} \frac{1}{T} \log \int_{SM} ex(d\varphi_T)_\theta\, d\theta$$

TEOREMA 2 (TEOREMA 1.4, MAÑÉ [8]) *Con las mismas hipótesis,*

$$h_{top} = \lim_{T\to\infty} \frac{1}{T} \log \int_{SM} |det\,(d\varphi_T)_\theta|_{V(\theta)}|\, d\theta = \lim_{T\to\infty} \frac{1}{T} \log \int_M vol\, \varphi_T(S_x M)\, dx$$

donde notamos como $n_T(x, y)$ al número de geodésicas de longitud menor o igual que T que conectan a x con y , mientras que la expansión es

$$ex\,(d\varphi_T)_\theta = \max_{S\subset T_\theta SM} |det\,(d\varphi_T)_\theta|_S|$$

y $V(\theta) = Ker(d\pi)_\theta$, con $\pi : SM \to M$ la proyección usual.
A pesar que los resultados fueron deducidos en un contexto Riemanniano, surge de manera natural una estructura simpléctica en TM, que enriquece la descripción de la dinámica del flujo geodésico. Para tratar de motivar nuestro trabajo, orientado a probar, en un contexto diferente, una igualdad similar a la primera del Teorema 2, describimos brevemente este enfoque simpléctico del flujo geodésico. Esta descripción está basada en Paternain [12]. Otras referencias son Arnold y Givental [1] y Mañé [8]. Consideremos una variedad Riemanniana (M, g) con métrica C^∞. En ella, podemos definir de manera canónica, independiente de g, el subespacio vertical $V(\theta) \subset T_\theta TM$, donde $\theta = (x, v) \in TM$, como $V(\theta) = Ker(d\pi)_\theta$, con $\pi : TM \to M$ la proyección usual. Para definir un subespacio complementario, utilizamos el mapa de conexión $K : TTM \to TM$, el cual se define, para $\zeta \in T_\theta TM$ y una curva adaptada $z : (-\epsilon, \epsilon) \to TM$ como

$$K_\theta(\zeta) = (\nabla_{\dot\alpha} Z)(0)$$

con $z(t) = (\alpha(t), Z(t))$, para $\alpha = \pi \circ z$. Al subespacio horizontal, que si depende de la métrica a través de la conexión ∇, lo definimos como $H(\theta) = Ker\, K_\theta$.

Podemos ver entonces que $T_\theta TM = H(\theta) \oplus V(\theta)$, por lo que $\zeta = (\zeta_h, \zeta_v) = ((d\pi)_\theta(\zeta), K_\theta(\zeta))$. De manera natural, definimos en TM la métrica de Sasaki

$$\langle\langle \zeta, \eta \rangle\rangle = \langle (d\pi)_\theta(\zeta), (d\pi)_\theta(\eta)\rangle + \langle K_\theta(\zeta), K_\theta(\eta)\rangle$$

lo cual hace que los subespacios vertical y horizontal sean ortogonales. A partir de esta afirmación, podemos introducir una estructura casi compleja J en TM, a través de

$$J_\theta : T_\theta TM \to T_\theta TM, \qquad J_\theta(\zeta_h, \zeta_v) = (-\zeta_v, \zeta_h).$$

Esto permite definir una forma simpléctica en TM como

$$\Omega_\theta(\zeta, \eta) = \langle\langle J_\theta(\zeta), \eta \rangle\rangle. \tag{1}$$

Con esta 2-forma, los subespacios $V(\theta)$ y $H(\theta)$ son Lagrangianos. En esta descripción de la geometría simpléctica de TM, podemos ver al flujo geodésico como un flujo Hamiltoniano.

PROPOSICIÓN 1 *Con la forma simpléctica (1) y para el campo geodésico $G(\theta)$, tenemos que*

$$(dH)_\theta(\zeta) = \Omega_\theta(G(\theta), \zeta), \qquad \zeta \in T_\theta TM$$

para el Hamiltoniano $H : TM \to \mathbb{R}$, dado por $H(x, v) = \frac{1}{2}\langle v, v\rangle_x$.

Parametrizando las geodésicas por longitud de arco, podemos describir la dinámica en el fibrado compacto SM. Si consideramos a $S(\theta) \subset T_\theta SM$ como el complemento ortogonal al campo geodésico $G(\theta)$ en la métrica de Sasaki, podemos verificar que $\Omega|_{S(\theta)\times S(\theta)}$ no degenera y que $S(\theta) \cap V(\theta)$ es un subespacio Lagrangiano de $S(\theta)$. Todo esto nos permite demostrar los Teoremas 1 y 2, cuyas pruebas están basadas en varios lemas auxiliares, en la propiedad twist del fibrado vertical y la desigualdad de Przytycki. Recordamos ahora estos dos últimos resultados.

PROPOSICIÓN 2 *Sea E un subespacio Lagrangiano de $T_\theta TM$, con la forma simpléctica definida en la ecuación (1). Entonces, el conjunto*

$$\{t \in R : (d\varphi_t)_\theta(E) \cap V(\varphi_t(\theta))\} \neq \{0\}$$

es discreto, donde $\varphi_t : TM \to TM$ es el flujo geodésico de (M, g). Esta es la propiedad twist del fibrado vertical $V(\theta)$.

PROPOSICIÓN 3 (PRZYTYCKI [13]) *Para un flujo $\psi_t : X \to X$, de clase C^2, donde X es una variedad compacta, se cumple que*

$$h_{top}(\psi) \leq \liminf_{t\to\infty} \frac{1}{t} \log \int_X ex\,(d\psi_t)_x\,dx.$$

Dado el enfoque simpléctico que hemos visto en los párrafos previos, cabe preguntarse si resultados como los de los Teoremas 1 y 2, no pueden ser obtenidos para flujos Hamiltonianos en general, que cumplan cierto tipo de propiedad twist para una distribución Lagrangiana dada, la desigualdad de Przytycki o algún resultado similar a los lemas auxiliares de la prueba del resultado de Mañé o modificaciones de estos.

El concepto de Hamiltoniano óptico, introducido por Bialy y Polterovich [2], establece la noción adecuada de propiedad twist en nuestro contexto: un Hamiltoniano óptico mueve la distribución Lagrangiana en el tiempo, "torciédola" siempre en la misma dirección respecto a si misma y de manera que, en caso que $(d\varphi_t)_x(\alpha(x))$ y $\alpha(\varphi_t(x))$ se corten no trivialmente, una pequeña perturbación en t hace que la intersección sea trivial.
Luego, un teorema de Kozlovski [7] para difeomorfismos, válido también para flujos, da una versión más útil de la desigualdad de Przytycki para el contexto en el cual trabajamos.
Para enunciar este resultado, presentamos la siguiente notación. Sea $A : \mathbb{R}^m \to \mathbb{R}^m$ un mapa lineal, el cual induce un mapa $A^{\wedge k} : \mathbb{R}^{m\wedge k} \to \mathbb{R}^{m\wedge k}$ en el álgebra de las k-formas en $\mathbb{R}^m$. La norma $||A^{\wedge k}||$ tiene un fuerte significado geométrico. Si $Vol_k(v_1, \ldots, v_k)$ es el k-volumen de un paralelepípedo generado por $v_1, \ldots, v_k \in \mathbb{R}^m$ entonces

$$||A^{\wedge k}|| = \sup_{v_i \in \mathbb{R}^m} \frac{Vol_k(Av_1, \ldots, Av_k)}{Vol_k(v_1, \ldots, v_k)}.$$

Luego, $A^{\wedge k}$ induce un mapa $A^{\wedge}$ en el álgebra exterior completa y

$$||A^{\wedge}|| = \max_{1\leq i\leq m} ||A^{\wedge i}||.$$

Es claro que en nuestro contexto, $||A|| = ex\,A$.

TEOREMA 3 (KOZLOVSKI [7]) *Para una variedad compacta X y un difeomorfismo $f : X \to X$, con $f \in C^\infty(X)$, se cumple que*

$$h_{top}(f) = \lim_{n\to\infty} \frac{1}{n} \log \int_X ex\,(df^n)_x\,dx.$$

Entonces, la fórmula de Kozlovski muestra que la entropía topológica es la tasa de crecimiento exponencial del promedio, sobre la variedad X, de la máxima expansión en volumen, tomada sobre todos los elementos de volumen de cualquier dimensión en T_xX.

Al ser el nivel de energía Σ una variedad de dimensión impar, no puede ser simpléctica. Por lo tanto, nuestros resultados deben ser probados para variedades simplécticas que contengan a Σ y en las que "se pase al límite" o en fibrados vectoriales simplécticos, en cuyo caso debemos hacer "descender" al fibrado la distribución Lagrangiana y la opticidad de H respecto a ella.

En estas condiciones, podemos probar los siguientes resultados.

TEOREMA 4 *Sea una variedad simpléctica (M, ω), con una distribución Lagrangiana $\mathcal{F}$ y un Hamiltoniano $\mathcal{F}$-óptico $H : M \to \mathbb{R}$, con $H \in C^\infty(M)$. Entonces para un nivel de energía compacto $\Sigma = H^{-1}(e)$, donde e es un valor regular, se cumple que*

$$h_{top}(\varphi_t|_\Sigma) = \lim_{\epsilon\to 0}\lim_{t\to\infty}\frac{1}{t}\log\int_{N_\epsilon}|det\,(d\varphi_t)_x|_{\alpha(x)}|\,dx$$

donde φ_t es el flujo asociado al Hamiltoniano H, $\alpha(x)$ es el subespacio Lagrangiano de T_xM dado por la distribución $\mathcal{F}$ y N_ϵ es la variedad $N_\epsilon = H^{-1}(e-\epsilon, e+\epsilon)$.

TEOREMA 5 *Sea una variedad simpléctica (M, ω), con una distribución Lagrangiana $\mathcal{F}$ y un Hamiltoniano $\mathcal{F}$-óptico $H : M \to \mathbb{R}$, con $H \in C^\infty(M)$. Si $\Sigma = H^{-1}(e)$, donde e es un valor regular y $\varphi_t|_\Sigma$ admite una distribución invariante y continua T de hiperplanos de $T_x\Sigma$ transversales al campo Hamiltoniano, entonces se cumple que*

$$h_{top}(\varphi_t|_\Sigma) = \lim_{t\to\infty}\frac{1}{t}\log\int_{\Sigma}|det\,(d\varphi_t)_x|_{\tilde\alpha(x)}|\,dx$$

donde φ_t es el flujo asociado al Hamiltoniano H y $\tilde\alpha(x)$ es el subespacio Lagrangiano de T_xM, dado por $\tilde\alpha(x) = (\alpha(x)\cap T_x\Sigma) + \langle X_H(x)\rangle$.

Estos dos resultados son de interés dada la generalidad de la condición de opticidad. Por ejemplo, $H \in C^\infty(T^*M, \omega)$, donde ω es la forma simpléctica canónica, es $\mathcal{F}$-óptico respecto a la distribución de espacios tangentes a las fibras $\mathcal{F} = \{dq = 0\}$ sii H es convexo en cada fibra, esto es, sii $H_{pp} > 0$. En particular, los Hamiltonianos usuales de la Mecánica Clásica, son ópticos respecto a esta distribución. Una aplicación de estos resultados a flujos magnéticos, puede verse en Burns y Paternain [5].

Este trabajo está organizado de la siguiente manera. En la sección §2 presentamos las definiciones y resultados referentes a Hamiltonianos ópticos que serán utilizados en nuestro trabajo. En §3 introducimos los elementos necesarios para medir la expansión del flujo Hamiltoniano y probamos una versión generalizada de la Proposición 4.18 de Paternain [12], clave en la prueba de nuestros resultados. En §4 probamos el Teorema 4 y finalmente en §5, luego de hacer descender la distribución Lagrangiana y la opticidad de H a un fibrado vectorial simpléctico adecuado, probamos el Teorema 5.

2 Hamiltonianos ópticos

En esta sección introducimos las definiciones y resultados necesarios para comprender la idea de Hamiltoniano óptico, la cual verifica cierta forma de propiedad twist, necesaria para nuestro resultado. Nos basamos fundamentalmente en Bialy y Polterovich [2], [3].
Dado un espacio vectorial simpléctico (E^{2n}, ω), sea $\Lambda(E)$ la familia de sus subespacios Lagrangianos. Este conjunto Λ tiene estructura de variedad diferenciable difeomorfa a $U(n)/O(n)$ y por lo tanto su dimensión es $n(n+1)/2$. Para un $\lambda \in \Lambda$, el tangente $T_\lambda\Lambda$ se identifica naturalmente con $S^2(\lambda)$, el espacio de las formas bilineales simétricas sobre λ, de la siguiente manera: dada una curva $\lambda(t) \subset \Lambda$ con $\lambda(0) = \lambda$, esta puede ser descrita como

$$\lambda(t) = S(t)\lambda, \qquad S(0) = Id, \; S(t) \in Sp(E, \omega).$$

Entonces, el vector $\dot{\lambda}(0) \in T_\lambda\Lambda$ tiene asociada la forma bilineal simétrica

$$(\zeta, \eta) \mapsto \omega(\zeta, \dot{S}(0)\eta), \qquad \zeta, \eta \in \lambda.$$

Fijemos ahora un espacio Lagrangiano $\alpha \in \Lambda$. Si notamos como Λ_α al conjunto de los Lagrangianos que no son transversales a α, podemos verificar que

$$\Lambda_\alpha = \bigcup_{1 \le k \le n} \Lambda_\alpha^{(k)} \tag{2}$$

es una variedad estratificada, donde $\Lambda_\alpha^{(k)}$ son los Lagrangianos cuya intersección con α es de dimensión k. De forma análoga al párrafo anterior, si $\lambda \in \Lambda_\alpha^{(k)}$ podemos identificar naturalmente a $T_\lambda\Lambda/T_\lambda\Lambda_\alpha^{(k)}$ con $S^2(\alpha \cap \lambda)$. Decimos entonces que, para $\lambda \in \Lambda_\alpha^{(k)}$, un vector de $T_\lambda\Lambda/T_\lambda\Lambda_\alpha^{(k)}$ es α-positivo si la forma bilineal correspondiente en $\alpha \cap \lambda$ es definida positiva. Un vector tangente en $T_\lambda\Lambda$ es α-positivo si su imagen por la proyección a $T_\lambda\Lambda/T_\lambda\Lambda_\alpha^{(k)}$ es α-positiva.

Las ideas anteriores pueden extenderse naturalmente a un contexto más amplio. Dada una variedad M, sea $\pi : E \to M$ un fibrado vectorial simpléctico, esto es, un fibrado vectorial real $\pi : E \to M$, tal que cada fibra E_q posee una forma simpléctica bilineal ω_q, que varía diferenciablemente con q. Sea $\Theta : \mathcal{A} \to M$ el fibrado de Lagrange-Grassmann asociado, en el cual cada fibra es difeomorfa a $\Lambda(T_xM)$. Si $\mathcal{F}$ es una sección de Θ, definimos como $\mathcal{A}_\mathcal{F}$ al subfibrado de $\mathcal{A}$, tal que sus subespacios Lagrangianos no cortan transversalmente a $\mathcal{F}$. Claramente $\mathcal{A}_\mathcal{F}$ es una subvariedad estratificada de $\mathcal{A}$ de codimensión 1, donde cada fibra de $\pi_\mathcal{F} : \mathcal{A}_\mathcal{F} \to M$ es difeomorfa al Λ_α de (2). Decimos entonces que un vector de $T_\lambda\mathcal{A}$, con $\lambda \in \mathcal{A}_\mathcal{F}$, es $\mathcal{F}$-positivo, si su imagen bajo la proyección $pr : T_\lambda\mathcal{A} \to T_\lambda\Lambda/T_\lambda\Lambda_\alpha^{(k)}$ es α-positiva. Una curva diferenciable en $\mathcal{A}$ es $\mathcal{F}$-positiva, para una distribución Lagrangiana $\mathcal{F}$, si sus vectores tangentes son $\mathcal{F}$-positivos en las intersecciones con $\mathcal{A}_\mathcal{F}$.
El siguiente resultado es clave en la prueba de nuestros lemas y proposiciones técnicos.

Proposición 4 (Prop. 1.6, Bialy y Polterovich [2]) *Todo vector tangente $\mathcal{F}$-positivo es transversal a la variedad estratificada $\mathcal{A}_\mathcal{F}$.*

Ahora pasamos al contexto natural de nuestro trabajo, el de una variedad simpléctica (M, ω), con una función $H : M \to \mathbb{R}$. Dada una distribución Lagrangiana $\mathcal{F}$, notamos como $\alpha(x)$ al subespacio correspondiente en x a $\mathcal{F}$. Si φ_t es el flujo asociado al Hamiltoniano H, consideramos la curva $(\varphi_t)_*(\alpha(\varphi_{-t}(x))$ en $\mathcal{A}$. Decimos que una función Hamiltoniana $H : M \to \mathbb{R}$ es $\mathcal{F}$-óptica si para cada $x \in M$, el vector

$$\frac{d}{dt}\,(\varphi_t)_*\,(\alpha(\varphi_{-t}(x)))\,|_{t=0}$$

es $\alpha(x)$-positivo. Un flujo Hamiltoniano es óptico, si el Hamiltoniano que lo genera es óptico.

El ejemplo más sencillo de Hamiltoniano óptico es el siguiente. Sean una variedad Riemanniana (M, g) y su fibrado cotangente (T^*M, ω), donde ω es la forma simpléctica canónica $\omega = -dq \wedge dp$, con (q, p) coordenadas locales de T^*M; $H : T^*M \to \mathbb{R}$, está dado por $H(q, p) = \frac{1}{2}||p||^2$. Entonces, H es $\mathcal{F}$-óptico respecto a la distribución de espacios tangentes a las fibras T_xM, dada

por $\mathcal{F} = \{dq = 0\}$. De manera más general, una función H en T^*M es $\mathcal{F}$-óptica sii su restricción a cada fibra es estrictamente convexa, esto es $H_{pp} > 0$. Esto incluye, por ejemplo, a todos los Hamiltonianos usuales de la Mecánica Clásica.
Moser [10], da un vínculo entre la propiedad twist de un difeomorfismo de un anillo en el plano y la propiedad óptica de un Hamiltoniano: un difeomorfismo twist C^∞ de un anillo cerrado $A \subset \mathbb{R}^2$, que preserva el área y fija los bordes de A, es el tiempo uno del flujo generado por un Hamiltoniano $H = H(t, x, y) \in C^\infty(\mathbb{R}, A)$, que es 1−periódico en t y en x, y cumple que $H_{yy} > 0$.

3 Resultados previos

En esta sección establecemos algunos resultados y herramientas básicas para probar los resultados enunciados.En primer lugar, introducimos una familia contractible de métricas Riemannianas adaptadas a la forma simpléctica ω, con las cuales podemos calcular determinantes de mapas lineales y tomar formas de volumen. Luego, probamos una generalización a un contexto algo más amplio, de la Proposición 4.18 de Paternain [12]. Esta es clave para poder obtener una igualdad de la tasa de crecimiento exponencial de los promedios, sobre el nivel de energía, de la expansión y del determinante del diferencial del flujo, restricto a la distribución Lagrangiana dada.

3.1 Métricas Riemannianas compatibles

Para poder "medir" ángulos y volúmenes en (M, ω), es útil introducir una métrica Riemanniana compatible. Los siguientes resultados, prueban la existencia de una familia de métricas compatibles con la forma ω. Nuestra referencia es McDuff y Salamon [9].
Sea (V, ω) un espacio vectorial simpléctico. Una estructura casi compleja J es un mapa $J : V \to V$, tal que $J^2 = -Id$. Esta estructura se dice compatible con ω si

$$\begin{aligned} \omega(v, w) = \omega(Jv, Jw), \quad \forall v, w \in V \\ \omega(v, Jv) > 0, \quad \forall v \in V. \end{aligned} \tag{3}$$

De esta manera, J induce un producto interno en V, a través de

$$g_J(v, w) = \omega(v, Jw).$$

Un fibrado vectorial simpléctico (E, ω) sobre una variedad M, es un fibrado vectorial real $\pi : E \to M$, tal que cada fibra E_q posee una forma simpléctica bilineal ω_q, que varía diferenciablemente con q. Una estructura casi compleja en el fibrado $\pi : E \to M$ es un mapa $J : E \to E$, tal que $J^2 = -Id_E$. Esta estructura compleja se dice compatible con ω si cada J_q en E_q es compatible con ω_q, $q \in M$. Entonces, para cada J, la forma bilineal

$$g_J : E \times E \to \mathbb{R}, \qquad g_J(v, w) = \omega(v, Jw)$$

es simétrica y definida positiva.
El siguiente resultado garantiza la existencia de una estructura casi compleja compatible en cualquier fibrado vectorial simpléctico.

PROPOSICIÓN 5 *Sea $\pi : E \to M$ un fibrado vectorial, con fibras de dimensión $2n$. Entonces*

1. *Para cada forma bilineal simpléctica ω en E, existe una estructura casi compleja J, compatible con ω. El espacio $\mathcal{J}(E, \omega)$ de tales estructuras es contractible.*

2. *Sea J una estructura casi compleja en E. Entonces existe una forma simpléctica bilineal ω que es compatible con J. El espacio de tales formas es contractible.*

Como consecuencia de esta Proposición, toda variedad simpléctica tiene asociada una familia contractible de métricas Riemannianas. De ahora en más, la medida de volumen en M, será el volumen Riemanniano inducido por una de estas métricas.
Recordamos que para un mapa lineal $L : E \to F$, con E y F espacios de Hilbert de dimensión finita, podemos definir el determinante de L, a menos de un signo, de la siguiente manera. Dada una base ortonormal $\{v_1, v_2, \ldots, v_n\}$ de E, consideramos la matriz con entradas $a_{ij} = \langle L(v_i), L(v_j)\rangle$ y definimos $|det\, L| = \sqrt{det\, A}$. Podemos definir entonces, para E_1, E_2 subespacios de E con $dim\, E_i = \frac{1}{2}\, dim\, E$, el ángulo entre ellos como

$$ang(E_1, E_2) = |det(P\,|_{E_1})|$$

donde $P : E \to E_2^{\perp}$ es la proyección usual. Claramente, el ángulo entre dos subespacios es una función continua y $ang(E_1, E_2) = 0$ sii $E_1 \cap E_2 \neq \emptyset$.

3.2 Cociclo lineal simpléctico

Introducimos aquí la idea de cociclo lineal simpléctico. Para la justificación de ideas similares, ver el Apéndice de Katok y Mendoza, en Katok y Hasselblatt [6].

Sea $\pi : S \to X$ un fibrado vectorial simpléctico C^∞, con X una variedad compacta y $\phi_t : X \to X$ un flujo de clase C^∞, que preserva una medida dx. Decimos que una familia de mapas diferenciables $\phi_t^*(x) : S(x) \to S(\phi_t(x))$ es un cociclo lineal simpléctico, si cumple, $\forall x \in X, t \in \mathbb{R}$

1. $\phi_{t+s}^* = \phi_t^*(\phi_s(x))\, \phi_s^*(x)$;
2. $\phi_t \circ \pi = \pi \circ \phi_t^*$;
3. $\forall x \in X, t \in \mathbb{R}$, el mapa $\phi_t^*(x) : S(x) \to S(\phi_t(x))$ es un isomorfismo lineal simpléctico.

Sea $\mathcal{F}$ una distribución Lagrangiana diferenciable en S, con $\alpha(x)$ el subespacio correspondiente en $x \in X$. Decimos que el cociclo lineal simpléctico $\phi_t^*(x)$ es $\mathcal{F}$-óptico, si lo es en el sentido de §2.

El objetivo de esta subsección es el de probar el siguiente resultado, generalización a este contexto de la Proposición 4.18 de Paternain [12].

Sea $\pi : S \to X$ un fibrado vectorial simpléctico, con X compacta, ϕ_t un flujo de clase C^∞ que preserva una medida dx y ϕ_t^* un cociclo lineal simpléctico. Notamos como $\Lambda(S)$ al fibrado de Lagrange-Grassmann asociado a $\pi : S \to X$.

Proposición 6 *Si el cociclo ϕ_t^* es $\mathcal{F}$-óptico, entonces existe una constante $C > 0$ tal que para todo $t \in \mathbb{R}$, se cumple que*

$$\int_X |det\, \phi_t^*(x)|_{\alpha(x)}|\, dx \geq C \int_X ex\, \phi_t^*(x)\, dx.$$

Para probar esta proposición, necesitamos varios lemas auxiliares.

Lema 1 *Existen $\delta > 0$, un entero $m \geq 1$ y una función semicontinua superior*

$$\tau : \Lambda(S) \times \mathbb{R} \to \{0, 1/m, 2/m, \ldots, 1\}$$

tal que para todo $(x, E) \in \Lambda(S), t \in \mathbb{R}$, se cumple que

$$ang(\phi_{t+\tau}^*(x)(E), \alpha(\phi_{t+\tau}(x))) > \delta.$$

Demostración
Si probamos que existe una constante $\delta > 0$ y un entero $m \geq 1$ tales que, para todo $(x, E, t) \in \Lambda(S) \times \mathbb{R}$, el conjunto

$$Q(x, E, t) = \{i \in \mathbb{Z}, 0 \leq i \leq m : ang(\phi^*_{t+i/m}(x)(E), \alpha(\phi_{t+i/m}(x)) > \delta\}$$

es no vacío, podríamos definir

$$\tau(x, E, t) = min\{i/m, i \in Q(x, E, t)\}$$

la cual sería la función buscada.
Supongamos que la afirmación inicial no se cumple. Entonces, para cualquier $m \geq 1$, existe una sucesión (x_m, E_m, t_m) tal que

$$ang(\phi^*_{s+t_m}(x_m)(E), \alpha(\phi_{s+t_m}(x))) \leq \frac{1}{2^m} \tag{4}$$

donde $s \in A_m$, con

$$A_m = \{j/2^m, j \in \mathbb{Z}, 0 \leq j \leq 2^m\}.$$

La compacidad de $\Lambda(S)$ implica la existencia de una subsucesión convergente a un punto (x, E). Como consecuencia de (4) y de la continuidad del ángulo, existe un m_k, tal que $\forall s \in A_{m_k}$, se cumple que

$$ang(\phi^*_s(x)(E), \alpha(\phi_s(x))) = 0$$

por lo que para todo $s \in [0, 1]$, tenemos que

$$\phi^*_s(x)(E) \cap \alpha(\phi_s(x)) \neq \{0\}$$

lo cual contradice la Proposición 4. □

LEMA 2 *Existen $\gamma > 0$, en entero $n \geq 1$ y una función semicontinua superior*

$$\rho : \Lambda(S) \to \{0, 1/n, 2/n, \dots, 1\}$$

tal que para todo $(x, E) \in \Lambda(S)$, se cumple que

$$ang(E, \phi^*_\rho(x_-)\,(\alpha(x_-))) > \gamma$$

donde $x_- = \phi_{-\rho}(x)$.

Demostración
La prueba es análoga a la del Lema anterior. □

Al igual que antes, la expansión de $\phi_t^*(x)$ es

$$ex\, \phi_t^*(x) = \max_{E \subset S(x)} |det\, \phi_t^*(x)|_E|.$$

LEMA 3 *Para cada $x \in X, t \in \mathbb{R}$, existe un subespacio Lagrangiano $R_t(x) \subset S(x)$, que depende mediblemente de t y x, tal que*

1. $|det\, \phi_t^*(x)|_{R_t(x)}| = ex\, \phi_t^*(x)$;

2. *si E es un subespacio de $S(x)$, con $dim\, E = \frac{1}{2}\, dim\, S(x)$, entonces*

$$|det\, \phi_t^*(x)|_E| \geq ang(E, R_t^{\perp}(x))\, ex\, \phi_t^*(x).$$

Demostración
Consideramos la descomposición polar para el cociclo lineal simpléctico

$$\phi_t^*(x) = O_t(x)\, L_t(x)$$

donde $L_t(x) : S(x) \to S(x)$ es una transformación lineal simétrica y definida positiva y $O_t(x) : S(x) \to S(\phi_t(x))$ es una isometría lineal. Al ser $\phi_t^*(x)$ un mapa simpléctico, lo es su traspuesto y en consecuencia también lo es

$$L_t(x) = (\phi_t^*(x)\, \phi_t^*(x)^t)^{1/2}.$$

Por ello, los valores propios de $L_t(x)$ son reales y cumplen que $\lambda_{i+n} = \lambda_i^{-1}$, para $i = 1, \ldots, n$. Dado el mapa $J(x)$ de la Proposición 5, si v_i es un vector propio asociado a λ_i, entonces $J(x)v_i$ es un vector propio con valor propio λ_i^{-1}, ya que $L_t \circ J = J \circ L_t^{-1}$. Podemos construir una base ortonormal de $S(x)$ de la forma

$$\{v_1, \ldots, v_n, J(x)v_1, \ldots, J(x)v_n\}.$$

Claramente el subespacio $R_t(x)$ generado por los n primeros vectores, asociados a los valores propios $\lambda_i \geq 1$, generan un subespacio Lagrangiano que cumple (a). Para probar la parte b), notemos que $L_t(x)$ deja invariante a $R_t(x)$ y $R_t^{\perp}(x)$, lo cual es claro, ya que ambos subespacios son generados por vectores propios. Supongamos ahora que $E \cap R_t^{\perp}(x) = \{0\}$ (en caso contrario, el ángulo es 0 y no hay nada que probar). Al ser $L_t(x) \circ P = P \circ L_t(x)$, con P la proyección ortogonal $P : S(x) \to R_t(x)$, tenemos que

$$|det\, L_t(x)|_{R_t(x)}||det\, P|_E| = |det\, P|_{L_t(x)(E)}||det\, L_t(x)|_E| \leq |det\, L_t(x)|_E|$$

de donde tenemos que

$$ex\, \phi_t^*(x)\, ang(E, R_t^\perp(x)) \leq |det\, \phi_t^*(x)|_E|$$

Resta probar solamente la medibilidad de $R_t(x)$ respecto a t y x. Sea $\mathbb{F}$ el fibrado vectorial sobre X, dado por los pares (x, h), con $x \in X$ y $h : S(x) \to S(x)$ un mapa lineal simétrico. Dados enteros positivos p y $l_i, 1 \leq i \leq p$, definimos como $\mathbb{F}(l_1, \ldots, l_p)$ al conjunto de los (x, h) tales que tienen p valores propios λ_i, con multiplicidades l_i. Este conjunto es un Boreliano, al igual que $\mathbb{P}(l_1, \ldots, l_p) \subset X \times \mathbb{R}$, dado por $\mathbb{P}(l_1, \ldots, l_p) = \{(x, t), L_t(x) \in \mathbb{F}(l_1, \ldots, l_p)\}$. Como $R_t(x)$ es continua en cada $\mathbb{P}$ y estos son una cantidad finita, entonces la medibilidad se deduce de inmediato. □

LEMA 4 *Existen $\delta > 0$, un entero $m \geq 1$ y funciones medibles $\tau_i : X \to \{0, 1/m, 2/m, \ldots, 1\}$ tales que si $\tau = \tau_1 + \tau_2$ y $x_1 = \phi_{-\tau_1}(x)$, $x_2 = \phi_{\tau_2}(x)$ y $\alpha^i = \alpha(x_i)$, entonces para todo x y t se cumple que*

1. $ang(\phi_{\tau_1}^*(x_1)(\alpha^1), R_t^\perp(x)) > \delta$;
2. $ang(\phi_{t+\tau}^*(x_1)(\alpha^1), \alpha^2) > \delta$

Demostración

Si encontramos δ_1, δ_2, enteros m_1, m_2 y funciones medibles $\tau_i : X \times \mathbb{R} \to \{0, 1/m_i, 2/m_i, \ldots, 1\}$ tales que se cumple $a)$ para $\delta = \delta_1$ y $b)$ para $\delta = \delta_2$, entonces podemos obtener el resultado para $m = m_1 m_2$ y $\delta = min\{\delta_1, \delta_2\}$. Sean entonces γ, n, ρ los del Lema 2. Si tomamos

$$\delta_1 = \gamma, \quad m_1 = n, \quad \tau_1(x, t) = \rho(x, R_t^\perp)$$

podemos afirmar que τ_1 es medible, ya que $R_t^\perp$ es medible y ρ es semicontinua superior. Aplicando el Lema 2 a $E = R_t^\perp$, obtenemos que

$$ang(\phi_{\tau_1}^*(x_1)(\alpha^1), R_t^\perp) > \delta_1.$$

lo cual prueba 1). Ahora, para los δ, n, τ del Lema 1, tomamos

$$\delta_2 = \delta, m_2 = m, \tau_2(x, t) = \tau(x, \phi_{\tau_1}^*(x_1)(\alpha^1), t).$$

Al ser τ_1 composición de funciones medibles, es medible y el Lema 1 aplicado a $E = \phi^*_{t+\tau_1+\tau_2}(x_1)(\alpha^1)$, implica que

$$ang(\phi^*_{t+\tau_1+\tau_2}(x_1)(\alpha^1), \alpha^2) > \delta_2$$

lo cual prueba la parte 2). $\square$

Lema 5 *Sean τ_1, x_1, α^1 como en el Lema 4. Entonces, existe una constante $K > 0$, tal que para todo x, t se cumple que*

$$|det\, \phi^*_t(x_1)|_{\alpha^1}| \geq K\, ex\, \phi^*_t(x).$$

Demostración
Si $E = \phi^*_t(x_1)(\alpha^1)$ tomamos $a > 0$ tal que

$$|det\, \phi^*_s(y)|_L| \geq a$$

para todo $s \in [0, 1]$, $(y, L) \in \Lambda(S)$. Entonces, utilizando la parte 1 del Lema 4

$$|det\, \phi^*_t(x_1)|_{\alpha^1}| = |det\, \phi^*_{t-\tau_1}(x)|_E\, ||det\, \phi^*_{\tau_1}(x_1)|_{\alpha^1}| \geq a\, \delta\, ex\, \phi^*_{t-\tau_1}(x).$$

Tomando a' tal que $ex\, \phi^*_s(x) \leq a'$, para $x \in X$, $s \in [0, 1]$, el resultado se cumple para $K = a\delta a'$. $\square$

Finalmente, probamos el resultado clave.

Demostración de la Proposición 6
Tomamos t fijo. A partir del cambio de variables $F : X \to X$, dado por $F(x) = x_1 = \phi_{-\tau_1(x,t)}(x)$, definimos

$$A(i) = \{x \in X : \tau_1(x, t) = i/m\}.$$

En cada $A(i)$, F es inyectiva, ya que si $F(x_1) = F(x_2)$, entonces

$$\phi_{-i/m}(x_1) = \phi_{-i/m}(x_2)$$

y llevándolos "hacia adelante", tenemos que $x_1 = x_2$. Si $\mu = dx$, entonces para cada Boreliano $S \subset A(i)$, se cumple que $\mu(S) = \mu(F(S))$ y si Φ es integrable en X, entonces

$$\int_{F(A(i))} \Phi\, dx = \int_{A(i)} (\Phi \circ F)\, dx.$$

Si $\Phi \geq 0$, entonces

$$\begin{aligned}\int_X (\Phi \circ F)\, dx &= \int_{\bigcup A(i)} (\Phi \circ F)\, dx = \sum_i \int_{A(i)} (\Phi \circ F)\, dx \\ &= \sum_i \int_{F(A(i))} \Phi\, dx \leq \sum_i \int_X \Phi\, dx = (m+1) \int_X \Phi\, dx.\end{aligned}$$

Haciendo $\Phi = |det\, \phi_t^*(x)|_{\alpha(x)}|$ y usando el Lema 5 tenemos que

$$\begin{aligned}\int_X |det\, \phi_t^*(x)|_{\alpha(x)}|\, dx &\geq \frac{1}{m+1} \int_X |det\, \phi_t^*(x_1)|_{\alpha^1}|\, dx \\ &\geq \frac{K}{m+1} \int_X ex\, \phi_t^*(x)\, dx.\end{aligned}$$

lo cual prueba el Lema para $C = \frac{K}{m+1}$. □

4 Prueba del teorema 4

En esta sección, probamos el Teorema 4.

Demostración del Teorema 4
Sea $N'_\epsilon = H^{-1}[e - \epsilon, e + \epsilon]$. Para poder utilizar la igualdad del Teorema 3, construímos el doble $D(N'_\epsilon)$, pegando dos copias de N'_ϵ por su borde, con lo cual obtenemos una variedad compacta, sin borde, de dimensión $2n$. Entonces

$$h_{top}(\varphi_t|_{D(N'_\epsilon)}) = \lim_{t \to \infty} \frac{1}{t} \log \int_{D(N'_\epsilon)} ex\, (d\varphi_t)_x\, dx \tag{5}$$

donde utilizamos el flujo inducido de manera natural en el doble. Al ser $D(N'_\epsilon) = N'_\epsilon \cup N'_\epsilon$, utilizamos la siguiente proposición, para el lado izquierdo de la igualdad (5).

PROPOSICIÓN 7 *Sea $\varphi_t : Y \to Y$ un flujo continuo en una variedad compacta Y. Entonces, para Y_1, Y_2 cerrados invariantes incluidos en Y, tales que $Y = Y_1 \cup Y_2$, se cumple que*

$$h_{top}(\varphi, Y_1 \cup Y_2) = \max_{Y_i, i=1,2} h_{top}(\varphi, Y_i).$$

Por otra parte, al ser

$$\int_{D(N'_\epsilon)} ex\, (d\varphi_t)_x\, dx = 2 \int_{N'_\epsilon} ex\, (d\varphi_t)_x\, dx \tag{6}$$

a través de la Proposición 7 y tomando tasa de crecimiento exponencial en (6), transformamos (5) en

$$h_{top}(\varphi_t|_{N_\epsilon}) = \lim_{t\to\infty} \frac{1}{t} \log \int_{N_\epsilon} ex\,(d\varphi_t)_x\,dx \tag{7}$$

donde $N_\epsilon = N'_\epsilon \;/\; \partial N'_\epsilon$. De ahora en más, utilizaremos el flujo y su diferencial restrictos a N_ϵ, a menos que se indique lo contrario. Es claro que $(d\varphi_t)_x$ es un cociclo lineal simpléctico $\mathcal{F}$-óptico para la distribución $\mathcal{F}$ dada, respecto al flujo φ_t, para el fibrado $\pi : TM|_{N_\epsilon} \to N_\epsilon$. A través de la Proposición 6 y de la desigualdad trivial

$$ex(d\varphi_t)_x \geq |det\,(d\varphi_t)_x|_{\alpha(x)}|$$

transformamos a (7) en

$$h_{top}(\varphi_t|_{N_\epsilon}) = \lim_{t\to\infty} \frac{1}{t} \log \int_{N_\epsilon} |det\,(d\varphi_t)_x|_{\alpha(x)}|\,dx. \tag{8}$$

Sea la función $h(\epsilon) = h_{top}(\varphi_t|_{N_\epsilon})$. Al ser h creciente en ϵ y $h(0) = h_{top}(\varphi_t|_\Sigma)$, tenemos que

$$\liminf_{\epsilon\to 0} h(\epsilon) \geq h_{top}(\varphi_t|_\Sigma). \tag{9}$$

Utilizamos ahora el siguiente resultado de Bowen [4].

Proposición 8 (Corolario 18, Bowen [4]) *Sean X e Y espacios métricos compactos y $\varphi_t : X \to X$ un flujo. Si $\pi : X \to Y$ es un mapa continuo tal que $\pi \circ \varphi_t = \pi$, entonces*

$$h_{top}(\varphi) = \sup_{y\in Y} h_{top}(\varphi|_{\pi^{-1}(y)}).$$

Para $\varphi_t|_{N_\epsilon}$ y $H|_{N_\epsilon}$, tenemos que

$$h(\epsilon) = \sup_{\epsilon_0\in[e-\epsilon,e+\epsilon]} h_{top}(\varphi_t|_{H^{-1}(\epsilon_0)}).$$

Luego, dado $r > 0$, existe un $\epsilon_0(\epsilon, r)$, tal que

$$h(\epsilon) \leq h_{top}(\varphi_t|_{H^{-1}(\epsilon_0)}) + r, \quad \epsilon_0 \in [e-\epsilon, e+\epsilon].$$

Cuando $\epsilon \to 0$, entonces $\epsilon_0 \to e$, por lo que, tomando límite superior en ambos lados y aplicando la semicontinuidad superior de la entropía topológica para flujos C^∞ (Newhouse [11]), al ser r arbitrario, se cumple que

$$\limsup_{\epsilon \to 0} h(t) \leq h_{top}(\varphi_t|_\Sigma). \tag{10}$$

A partir de (8), (9) y (10) llegamos a

$$h_{top}(\varphi_t|_\Sigma) = \lim_{\epsilon \to 0} h_{top}(\varphi_t|_{N_\epsilon}) = \lim_{\epsilon \to 0} \lim_{t \to \infty} \frac{1}{t} \log \int_{N_\epsilon} |det\,(d\varphi_t)_x|_{\alpha(x)}|\,dx$$

lo que prueba el resultado. □

5 Prueba del teorema 5

En esta sección probamos el Teorema 5. Previo a la prueba, presentamos algunos lemas básicos y verificamos que en el fibrado $\pi : S \to \Sigma$, con $S(x) = T_x\Sigma/\langle X_H(x)\rangle$, la distribución Lagrangiana $\mathcal{F}$ induce una distribución $\tilde{\mathcal{F}}_S$, respecto a la cual el Hamiltoniano H induce un cociclo lineal simpléctico que es $\tilde{\mathcal{F}}_S$-óptico.

LEMA 6 *Sea ω_Σ la restricción al nivel de energía Σ de la forma simpléctica ω. Entonces, ω_Σ degenera en $X_H(x)$.*

Demostración
Para $v \in T_x\Sigma$, tenemos que

$$\omega_\Sigma(X_H(x), v) = \omega(X_H(x), v) = dH_x(v) = \frac{d}{dt} H(c(t))|_{t=0}$$

donde $c : (-\epsilon, \epsilon) \to M$ es una curva adaptada a v, esto es $c(0) = x$ y $\dot{c}(0) = v$. Pero Σ es un nivel de energía de H, por lo que $H(c(t)) \equiv cte$, lo cual prueba el Lema. □

LEMA 7 *El fibrado $\pi : S \to \Sigma$ es un fibrado vectorial simpléctico, con la forma $\omega_S = p_*\omega$, donde $p_x : T_x\Sigma \to S(x)$ es la proyección usual y ω es la forma simpléctica de M.*

Demostración
Para $v \in T_x\Sigma$, notamos $p_x(v) = [v]$. Entonces, si $v_1, v_2 \in T_x\Sigma$, definimos

$$\omega_S([v_1], [v_2]) = \omega_\Sigma(v_1, v_2).$$

El Lema 6, garantiza que ω_S no degenera y que está bien definida. La propiedad de antisimetría surge a partir de la definición. □

LEMA 8 *Si $\alpha(x) \subset T_x\Sigma$, entonces $X_H(x) \in \alpha(x)$.*

Demostración
Supongamos que no se cumple la proposición. Si $\zeta \in \alpha(x)$, entonces es cierto que $\omega(X_H(x), \zeta) = 0$ y en consecuencia, $F = \alpha(x) + \langle X_H(x) \rangle$ anula a ω. El hecho que F siga siendo una distribución Lagrangiana, pero de dimensión $n + 1$, lleva a un absurdo. □

PROPOSICIÓN 9 *Sea el subespacio $\tilde{\alpha}(x) \subset T_xM$, dado por*

$$\tilde{\alpha}(x) = \alpha(x) \cap T_x\Sigma + \langle X_H(x) \rangle;$$

Entonces $\tilde{\alpha}(x)$ es un subespacio Lagrangiano de T_xM, que induce una distribución diferenciable $\tilde{\mathcal{F}}$. En particular, $\tilde{\alpha}(x) \subset T_x\Sigma$

Demostración
Por el Lema 8, $dim\,\tilde{\alpha}(x) = n$. Por la construcción de $\tilde{\alpha}(x)$, tenemos que $\tilde{\alpha}(x) \in T_x\Sigma$. Para un vector $v' \in \tilde{\alpha}(x)$, podemos escribir

$$v' = v + a\, X_H(x), \qquad v \in \alpha(x).$$

Entonces, para $v_1', v_2' \in \tilde{\alpha}(x)$, tenemos que

$$\begin{aligned} \omega(v_1', v_2') &= \omega(v_1, v_2) + a_1\, \omega(X_H(x), v_2) \\ &+ a_1\, a_2\, \omega(X_H(x), X_H(x)) + a_2\, \omega(v_1, X_H(x)) = 0 \end{aligned}$$

ya que el primer término se anula al pertenecer los v a $\alpha(x)$, el tercero por ser ω una forma simpléctica y los dos restantes por el Lema 6. Entonces, $\tilde{\alpha}(x)$ es Lagrangiano. □

PROPOSICIÓN 10 *Sea $\tilde{\alpha}(x)$ el subespacio Lagrangiano de la sección $\tilde{\mathcal{F}}$ y $p_x : T_x\Sigma \to S(x)$ la proyección canónica. Entonces $\tilde{\alpha}_S(x) = p_x(\tilde{\alpha}(x))$ es un subespacio Lagrangiano de la fibra $(S(x), \omega_S)$, el cual induce una distribución diferenciable $\tilde{\mathcal{F}}_S$.*

Demostración
Si $[v_1], [v_2] \in \tilde{\alpha}_S(x)$, tomamos $v_1, v_2 \in \tilde{\alpha}(x)$ tales que $[v_1] = p_x(v_1)$, $[v_2] = p_x(v_2)$. Como consecuencia de la definición de ω_S en el Lema 7, ω_S se anula en $\tilde{\alpha}_S(x) \times \tilde{\alpha}_S(x)$. Luego, como $X_H(x) \in \tilde{\alpha}(x)$, $dim\,\tilde{\alpha}_S(x) = dim\,\tilde{\alpha}(x) - 1 = n - 1 = \frac{1}{2}\, dim\, S(x)$, de donde se concluye que $\tilde{\alpha}_S(x)$ es un subespacio Lagrangiano de $S(x)$. □

Notamos, de ahora en más, como $\widetilde{d\varphi_t}(x)$ al diferencial del flujo bajado a $S(x)$.

PROPOSICIÓN 11 *Sea $\tilde{\mathcal{F}}_{\mathcal{S}}$ la distribución Lagrangiana inducida en $\pi : S \to \Sigma$ por la Proposición 10. Si $H \in C^\infty(M)$ es un Hamiltoniano $\mathcal{F}$-óptico, entonces induce un cociclo lineal simpléctico $\widetilde{d\varphi_t}(x)$ que es $\tilde{\mathcal{F}}_{\mathcal{S}}$-óptico.*

Demostración
De la definición de opticidad, surge que dada una curva

$$\lambda(t) = S(t)\alpha(\varphi_t(x)), \qquad S(0) = Id,\ S(t) \in Sp\,(T_{\varphi_t(x)}M, \omega)$$

en el fibrado $\Theta : \mathcal{A} \to \mathcal{M}$, el vector $\dot{\lambda}(0) \in T_{\alpha(x)}\mathcal{A}$, definido como

$$\dot{\lambda}(0) = \frac{d}{dt}\,(\varphi_t)_*\,(\alpha(\varphi_{-t}(x))\,|_{t=0}$$

tiene asociada en $S^2(\alpha(x))$ la forma bilineal simétrica definida positiva

$$(\zeta, \eta) \mapsto \omega(\zeta, \dot{S}(0)\eta). \tag{11}$$

Por construcción, cada subespacio $\tilde{\alpha}(x)$ contiene al campo $X_H(x)$ y está contenido en $T_x\Sigma$, por lo que la forma (11) es semidefinida positiva, ya que $\omega|_\Sigma$ degenera en el campo. Al pasar al cociente $S(x)$, la forma inducida ω_S de la Proposición 7 es simpléctica, por lo que

$$([\zeta], [\eta]) \mapsto \omega([\zeta], [\dot{S}(0)\eta]), \qquad [\zeta], [\eta] \in \tilde{\alpha}_S(x)$$

es nuevamente una forma bilineal simétrica y definida positiva, lo que prueba el resultado. □

Notamos, como $ex_{T_x\Sigma}$ a la expansión en $T_x\Sigma$ y como $ex_{S(x)}$ a la expansión en $S(x)$.

Los próximos dos lemas, utilizados en la prueba del Teorema 5, son los que vinculan las expansiones y determinantes en $T_x\Sigma$ y $S(x)$. Probamos solamente el segundo de ellos, ya que la prueba del primero es elemental.

LEMA 9 *Sea $\pi : E \to X$ un fibrado vectorial sobre una variedad compacta X. Si g_1 y g_2 son dos métricas Riemannianas continuas en E, entonces existe una constante $K > 0$, tal que para cualquier mapa lineal $L : E(x) \to E(y)$, se cumple que*

$$ex_{g_1}L \leq K\, ex_{g_2}L.$$

LEMA 10 *Sean (M, ω) una variedad simpléctica, con una distribución Lagrangiana $\mathcal{F}$, φ_t un flujo Hamiltoniano $\mathcal{F}$-óptico, tal que admite una distribución continua, invariante, transversal a $X_H(x)$ en $T_x\Sigma$. Entonces, existen constantes $K_1, K_2 > 0$ tales que*

$$ex_{T_x\Sigma}\, d\varphi_t(x) \leq K_1\, ex_{S(x)}\, \widetilde{d\varphi_t}(x)$$
$$|det\, \widetilde{d\varphi_t}(x)|_{\tilde{\alpha}_S(x)}| \leq K_2\, |det\, d\varphi_t(x)|_{\tilde{\alpha}(x)}|.$$

Demostración
Al ser $T(x)$ un subespacio de $T_x\Sigma$ de dimensión $2n-2$ y transversal a $X_H(x)$, la forma simpléctica ω restricta a T no degenera, por lo que por la Proposición 5, existe una estructura casi compleja J que induce un producto interno $\langle ., . \rangle_T$ en T. A partir de el, construimos en $T_x\Sigma$ el siguiente producto interno:

$$\langle v, w\rangle_{T_x\Sigma} = \begin{cases} \langle v, w\rangle_T & si \quad v, w \in T \\ 0 & si \quad v \in \langle X_H(x)\rangle, w \in T(x) \\ 1 & si \quad v = w = X_H(x) \end{cases}$$

De esta manera, $T_x\Sigma$ es suma directa ortogonal de $T(x)$ y de $X_H(x)$ y la invariancia y transversalidad de T bajo $\varphi_t(x)$, permiten reproducir esta descomposición en cada fibra de $T\Sigma$. En una base $\{X_H(x), t_1, \ldots, t_{2n-2}\}$ de $T_x\Sigma$, donde $\{t_i\}_{1\leq i\leq 2n-2}$ es una base de $T(x)$, la matriz del diferencial del flujo es

$$(d\varphi_t)_x|_\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & (d\varphi_t)_x|_{T(x)} \end{pmatrix}.$$

Como la expansión puede calcularse como el máximo de los valores absolutos de los determinantes sobre todos los menores adjuntos de cualquier dimensión de la matriz, es claro que

$$ex_{T_x\Sigma}(d\varphi_t)_x = ex_{T(x)}(d\varphi_t)_x.$$

Introducimos entonces un producto interno en $S(x)$, tal que la proyección $P|_{T(x)}$ es una isometría. Claramente, si g_1, g_2 son las métricas dadas por los productos internos en $T(x)$ y $S(x)$, se cumple que

$$ex^{g_1}_{T(x)}(d\varphi_t)_x = ex^{g_2}_{S(x)}(\widetilde{d\varphi_t})_x.$$

Aplicando el Lema 9, obtenemos la primer desigualdad buscada. La segunda de ellas, se obtiene por un método análogo. □

Demostración del Teorema 5
Por la Proposición 11 $\widetilde{(d\varphi_t)}_x$ es un cociclo lineal simpléctico $\tilde{\mathcal{F}}_{\mathcal{S}}$-óptico, para la distribución dada por la Proposición 10, respecto al flujo $\varphi_t|_\Sigma$, en el fibrado vectorial simpléctico $\pi : S \to \Sigma$. A través de la desigualdad trivial

$$ex\widetilde{(d\varphi_t)}_x \geq |det\,\widetilde{(d\varphi_t)}_x|_{\tilde{\alpha}_S(x)}|$$

y de la Proposición 6, tenemos que

$$\liminf_{t\to\infty}\frac{1}{t}\log\int_\Sigma ex_{S(x)}\,\widetilde{(d\varphi_t)}_x\,dx = \liminf_{t\to\infty}\frac{1}{t}\log\int_\Sigma |det\,\widetilde{(d\varphi_t)}_x|_{\tilde{\alpha}_S(x)}|\,dx. \quad (12)$$

Utilizando ahora la igualdad de Kozlovski en $T_x\Sigma$, la primer desigualdad del Lema 10, (12) y la segunda desigualdad del Lema 10, obtenemos que

$$\begin{aligned} h_{top}(\varphi_t|_\Sigma) &= \liminf_{t\to\infty}\frac{1}{t}\log\int_\Sigma ex_{T_x\Sigma}(d\varphi_t)_x\,dx \\ &\leq \liminf_{t\to\infty}\frac{1}{t}\log\int_\Sigma ex_{S(x)}\,\widetilde{(d\varphi_t)}_x\,dx \\ &= \liminf_{t\to\infty}\frac{1}{t}\log\int_\Sigma |det\,\widetilde{(d\varphi_t)}_x|_{\tilde{\alpha}_S(x)}|\,dx \\ &\leq \liminf_{t\to\infty}\frac{1}{t}\log\int_\Sigma |det\,(d\varphi_t)_x|_{\tilde{\alpha}(x)}|\,dx. \end{aligned} \quad (13)$$

Por otra parte, nuevamente Kozlovski y la desigualdad trivial muestran que

$$\begin{aligned} h_{top}(\varphi_t|_\Sigma) &= \lim_{t\to\infty}\frac{1}{t}\log\int_\Sigma ex_{T_x\Sigma}(d\varphi_t)_x\,dx \\ &\geq \limsup_{t\to\infty}\frac{1}{t}\log\int_\Sigma |det\,(d\varphi_t)_x|_{\tilde{\alpha}(x)}|\,dx \end{aligned} \quad (14)$$

por lo que (13) y (14) conducen a que

$$h_{top}(\varphi_t|_\Sigma) = \lim_{t\to\infty}\frac{1}{t}\log\int_\Sigma |det\,(d\varphi_t)_x|_{\tilde{\alpha}(x)}|\,dx$$

lo cual prueba el resultado. □

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César J. Niche
cniche@math.ucsc.edu
Instituto de Matemática y Estadística "Rafael Laguardia".
Facultad de Ingeniería.
Universidad de la República.
Montevideo.
Uruguay.
Current postal address:
Department of Mathematics
University of California at Santa Cruz
USA

Very flat reductive monoids

Alvaro Rittatore

ABSTRACT

This is a preliminary version, to be published elsewhere. Given a semisimple algebraic group G_0 over an algebraically closed field k of arbitrary characteristic, we construct a reductive monoid, the *universal semigroup*, which gives a flat deformation of G_0 to an affine semigroup, the *asymptotic semigroup*, whose algebra of regular functions is obtained by changing the product in $k[G_0]$. This construction, of geometric nature, generalises the construction of the universal and asymptotic semigroups done in characteristic zero by Vinberg.

RESUMEN

Sea k un cuerpo algebraicamente cerrado de característica arbritaria, y sea G_0 un grupo algebraico sobre k semisimple. En este trabajo construímos un monoide reductivo, el *semigrupo universal*, que es a su vez una deformación playa de G_0 en un semigrupo afín, el *semigrupo asintótico*, cuya álgebra de funciones regulares es obtenida mediante un cambio en el producto de $k[G_0]$. Esta construcción, de naturaleza geométrica, generaliza la hecha en característica cero por Vinberg para los semigrupos universal y asintótico. Este trabajo constituye una versión preliminar.

1 Introduction

Let k be an algebraically closed field of arbitrary characteristic, and let M be an algebraic normal irreducible monoid over k of unit group G. Then G acts by right and left multiplication over M, in such a way that the orbit of 1 is an open dense subset isomorphic to $(G \times G)/\Delta(G)$, $\Delta(G)$ the diagonal. Suppose that G is reductive, with commutator G_0; then the quotient $\pi : M \to A_M = M/_{G_0 \times G_0}$ exists and is a commutative monoid, with unit group the torus G/G_0. We call this quotient the *abelianisation* of M. We say that M is *very flat* if the abelianisation is a flat morphism with reduced (as schemes) and irreducible fibres. In this paper we study the geometry of very flat monoids. In particular, given a semisimple group G_0, we find a minimal element of the family $\mathcal{FM}(G_0)$ of very flat monoids with unit group such that its commutator is G_0. This monoid, the *universal semigroup*, is such that its abelianisation is isomorphic to the affine space $\mathbb{A}^n$, n the semisimple rank of G_0, and such that any very flat monoid $M \in \mathcal{FM}(G_0)$ is a fibred product over $\mathbb{A}^n$ of A_M and the universal semigroup. Moreover, let S be the universal semigroup associated to G_0; then the abelianisation $\pi : S \to A_S \cong \mathbb{A}^n$ gives a flat deformation of G_0 to the algebraic semigroup $\pi^{-1}(0)$. We call $\pi^{-1}(0)$ the *asymptotic semigroup* associated to G_0. The algebra of regular functions of the asymptotic semigroup is obtained from $k[G_0]$ by changing the product (and leaving the coproduct as is).

These constructions are valid in arbitrary characteristic, and thus generalise those done by Vinberg for characteristic zero in [15] and [16].

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2 Preliminaries

In this section we recall the basic facts about the classification of *reductive embeddings* – i.e. embeddings of a reductive group – to be used in this work.

DEFINITION 1 Let G be a reductive group. An homogeneous space G/H is *spherical* if there exists a Borel subgroup B, of G such that BH is open in G. If X is a normal irreducible G-variety with an open orbit isomorphic to G/H, then X is called a *G/H-spherical variety.* We say that X is *simple* if the action of G over X has only one closed orbit.

If G is a reductive group, then $G \cong (G \times G)/\Delta(G)$, is a spherical homogeneous space. A *reductive embedding* is a G-spherical variety.

A reductive monoid is an irreducible algebraic monoid with unit group a reductive group. We have the following:

THEOREM 1 ([12]) *The normal reductive monoids are exactly the affine embeddings of reductive groups. The normal commutative reductive monoids are exactly the affine embeddings of tori.* □

NOTATION From now on reductive monoids are supposed to be normal (and irreducible), unless stated otherwise. All reductive groups are supposed connected.

In [15] Vinberg classified all reductive monoids in characteristic zero in terms of the decomposition of their algebra of regular functions for the action of the unit group. This classification is dual of their classification as spherical varieties. We refer the reader to [12] for a complete description of the classification of reductive embeddings, and to [7] for the general case of spherical varieties. We summarize this classification in order to fix notations. Let G/H be a spherical homogeneous space. We denote $k(G/H)^{(B)}$ the set of B-eigenvectors of $k(G/H)$, the field of rational functions of G/H, and $\Lambda_{G/H}$ the set of weights of $k(G/H)^{(B)}$. We consider the space $\mathrm{Hom}_{\mathbb{Z}}(\Lambda_{G/H}, \mathbb{Z}) \otimes \mathbb{Q} = \mathcal{Q}(G/H)$. The restriction to $k(G/H)^{(B)}$ induces an injection of the set of G-invariant valuations of the field $k(G/H)$, into $\mathcal{Q}(G/H)$. Its image is a rational polyhedral cone, the *valuation cone,* denoted $\mathcal{V}(G/H)$. On the other hand, if we denote $\mathcal{D}(G/H)$ the set of the irreducible B-stable divisors of G/H, the *colors*, then there exists a map (not necessarily injective) $\rho : \mathcal{D}(G/H) \to \mathcal{Q}(G/H)$.

To each simple spherical variety X with open orbit G/H corresponds an unique *colored cone,* namely a pair $(\mathcal{C}(X), \mathcal{F}(X))$ constructed as follows: let Y be the unique closed G-orbit of X, and put $\mathcal{F}(X) = \{D \in \mathcal{D}(G/H) \mid \overline{D} \supset Y\}$. Let $\mathcal{B}(X)$ be the subset of $\mathcal{V}(G/H)$ consisting of the valuations associated to the irreducible G-stable divisors of X. Then $\mathcal{C}(X)$ is the strictly convex polyhedral cone generated by $\mathcal{B}(X)$ and $\rho(\mathcal{F}(X))$. This cone verifies that $\mathcal{C}(X)^\circ$, the relative interior of $\mathcal{C}(X)$, intersects the valuation cone.

Conversely, a pair $(\mathcal{C}, \mathcal{F})$ verifying the properties described above determines an unique G/H-spherical variety such that $(\mathcal{C}(X), \mathcal{F}(X)) = (\mathcal{C}, \mathcal{F})$. Non-simple spherical varieties can be classified by means of *colored fans*, which are collections of colored cones with some compatibility restrictions.

Finally, we recall that a dominant G-equivariant morphism $\varphi : X \to X'$ between spherical varieties with open orbits G/H and G/H' respectively, induces a morphism $\varphi_* : \mathcal{Q}(G/H) \to \mathcal{Q}(G/H')$ such that $\varphi_*(\mathcal{E}_X) \subset \mathcal{E}_{X'}$, where $\mathcal{E}_X$ denotes the colored fan associated to X.

Let us fix some notations. Let G be a reductive group, and let T be a maximal torus of G, B a Borel subgroup containing T, and B^- its opposite Borel subgroup. We denote $\Xi(T)$ the set of weights and $\Xi_+(T)$ the semigroup of dominant weights with respect to B. We denote W the Weyl group associated to T, and $C = C(G)$ the Weyl chamber associated to (B, T). We denote $\alpha_1, \dots, \alpha_l$ and $\omega_1, \dots, \omega_l$ the simple roots and fundamental weights associated to (B, T) respectively.

Finally, we denote $\Xi_*(T)$ the set of one parameter subgroups (1-PS) of T. We identify $\Xi_*(T)$ with $\Xi(T)$ by means of a W-invariant form $\langle \cdot, \cdot \rangle$, in such a way that $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$.

The subvarieties $\overline{Bs_{\alpha_i}B^-}$, $s_{\alpha_i} \in W$ the reflection associated to the simple root α_i, $i = 1, \dots, l$, are exactly the irreducible $(B \times B^-)$-stable divisors of

G (the colors).

The combinatorial data associated to the spherical homogeneous space $G = (G \times G)/\Delta(G)$ is the space $\mathcal{Q}(G) = \Xi(T) \otimes \mathbb{Q}$, with valuation cone $\mathcal{V}(G) = -C$ and colors $\alpha_i^\vee$, $i = 1, \ldots, l$ ([12]).

Let S be a reductive monoid with unit group G. We denote $\mathcal{V}(S) = \mathcal{C}(S) \cap \mathcal{V}(G)$.

3 Abelianisation of reductive monoids

We keep the notations of the preceding section.

Let G be a reductive group and let Z be the *connected* center of G. We denote as before G_0 the commutator of G. Let S be a reductive monoid of unit group $G = (G_0 \times Z)/Z_0$, where $Z_0 = Z \cap G_0$. It seems natural to try to describe the geometry of S in terms of the commutative monoid $\overline{T} \subset S$. However, for the very flat monoids, Vinberg ([15],[16]) has shown that it suffices to consider $\overline{Z}$ to give a geometrical picture of S, in characteristic zero. In the rest of this work we show how to extend Vinberg results to arbitrary characteristic, giving a geometric construction of the *enveloping* and *asymptotic semigroups*.

DEFINITION 2 Let S be a (not necessarily normal) reductive monoid of unit group G. The *abelianisation* of S is the geometric quotient $\pi : S \to A_S = S /\!/ (G_0 \times G_0)$ for the action by left and right multiplication, *i.e.* A_S is the affine algebraic variety such that $k[A_S] = k[S]^{G_0 \times G_0}$.

If S is normal, A_S is so. Moreover, $k[A_S] \subset k[S]$ is $(G \times G)$-stable, and $\pi : S \to A_S$ is a $(G \times G)$-morphism.

Instead of using $\overline{Z}$, we will describe the geometry of very flat monoids by means of its abelianisation (which in fact is closely related to $\overline{Z}$):

THEOREM 2 ([15, thm. 3],[11]) *Let S be a reductive monoid (not necessarily normal) with unit group G, and let $\pi : S \twoheadrightarrow A_S$ be the abelianisation of S. Then the following properties are verified:*

i) $\pi^{-1}(\pi(1)) = G_0$.

ii) *A_S is a commutative monoid, with unit group $G/G_0 \cong Z/Z_0$.*

iii) $\pi^{-1}(G/G_0) = G$.

iv) *$\pi(\overline{Z}) = A_S$, and $A_S \cong \overline{Z}/Z_0$.*

v) *If moreover S is normal then the induced map*

$$\pi_* : \mathcal{Q}(G) \cong \mathcal{Q}(G_0) \times \mathcal{Q}(Z) \to \mathcal{Q}(G/G_0) \cong \mathcal{Q}(Z/Z_0)$$

is given by the projection over the second coordinate $(\lambda, \mu) \mapsto \mu$, *and the following properties of* π_* *are verified:*

- $\pi_*(\mathcal{C}(S)) = \mathcal{C}(A_S)$. *In particular,* ${\pi_*}^{-1}(\mathcal{C}(A_S)) = \mathcal{C}(S) + \mathcal{Q}(G_0)$, *and* $\pi_*(\mathcal{C}(S) \cap \mathcal{Q}(Z)) = \mathcal{C}(A_S)$.
- *If we denote* $\pi_Z : \overline{Z} \to A_S$ *the restriction of* π *to* $\overline{Z}$, *then* $\pi_{Z*} : \mathcal{Q}(Z) \to \mathcal{Q}(Z/Z_0) \cong \mathcal{Q}(Z)$ *is the identity. In particular,*

$$\mathcal{C}(A_S) = \mathcal{C}(S) \cap \mathcal{Q}(Z) \ .$$

The algebra $k[A_S]$ *decomposes for the action of* $(Z/Z_0) \times (Z/Z_0)$ *as follows:*

$$k[A_S] = \bigoplus_{\mu \in \mathcal{L}(A_S)} ke^{\mu} \ ,$$

where $\mathcal{L}(A_S) = \{\mu \in \Xi(Z/Z_0) = \Xi(Z)^{Z_0} \ : \ \mu \in (\mathcal{C}(S) \cap \mathcal{Q}(Z))^{\vee}\}$ *(the dual is taken in* $\mathcal{Q}(Z)$ *), and* ke^{μ} *is the simple representation of* $(Z/Z_0) \times (Z/Z_0)$ *of weight* $(\mu, -\mu)$.

PROOF. Assertions i) – iv) are proved in [15, §3] in the characteristic zero case. These proofs remain valid in arbitrary characteristic with slight modifications (see for example [11]). Assertion v) is proved straightforward from i) – iv) by "dualizing". For a complete proof, we refer the reader to [11]. □

Given a very flat monoid, we want to see it as the fibred product of its abelianisation with a "minimal" monoid (over the abelianisation of this one). For this, we must first understand the relationship between monoids and their abelianisation at the level of morphisms:

Let $\varphi : S' \to S$ be a morphism of algebraic monoids. Then φ induces a morphism (of algebraic monoids) between the abelianisations $\varphi_{ab} : A_{S'} \to A_S$ in such a way that the diagram :

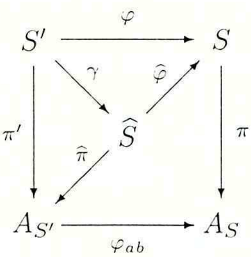
$$\begin{array}{ccc} S' & \xrightarrow{\varphi} & S \\ {\scriptstyle \pi'} \downarrow & & \downarrow {\scriptstyle \pi} \\ A_{S'} & \xrightarrow[\varphi_{ab}]{} & A_S \end{array}$$

is commutative.

Consider the fibred product

$$\widehat{S} = A_{S'} \times_{A_S} S = \{(a, s) \in A_{S'} \times S \ : \ \varphi_{ab}(a) = \pi(s)\}\,.$$

It is clear that $\widehat{S}$ is a submonoid of $A_{S'} \times S$. The canonical projections $\widehat{\pi} : \widehat{S} \to A_{S'}$ and $\widehat{\varphi} : \widehat{S} \to S$ are morphisms of algebraic monoids. By the universal property of the fibred product, there exists an unique morphism (of algebraic monoids) $\gamma : S' \to \widehat{S}$, such that the following diagram is commutative:



The morphism φ is called *excellent* if γ is an isomorphism. In this case, φ sends the commutator of $G(S')$ isomorphically over the commutator of $G(S)$.

Let us study the fibred product $\widehat{S}$ in more detail.

PROPOSITION 1 *Let S be a (not necessarily normal) reductive monoid of unit group G and let $\pi : S \to A_S$ be the abelianisation of S. Let A be a commutative monoid of unit group a torus R, such that there exists a morphism of monoids $\varphi : A \to A_S$. Then:*

i) $\widehat{S} = A \times_{A_S} S = \{(a, s) \in A \times S \mid \varphi(a) = \pi(s)\}$ *is a monoid of unit group* $\widehat{G} = G(\widehat{S}) = R \times_{Z/Z_0} G$. *In particular, the commutator of* $\widehat{G}$ *is* $\widehat{G_0} = \{1\} \times G_0 \cong G_0$.

ii) The connected center $\widehat{Z}$ *of* $\widehat{G}$ *is isomorphic to* $R \times_{Z/Z_0} Z$, *and*

$$\widehat{Z_0} = \widehat{Z} \cap \widehat{G_0} = \{1\} \times Z_0 \cong Z_0.$$

In particular, $\widehat{G} = (G_0 \times \widehat{Z})/\widehat{Z_0} \cong (G_0 \times (R \times_{Z/Z_0} Z))/\widehat{Z_0}$.

iii) The abelianisation of $\widehat{S}$ *is* A.

Moreover, consider the cartesian diagram:

$$\begin{array}{ccc} \widehat{S} & \xrightarrow{\widehat{\varphi}} & S \\ {\scriptstyle \widehat{\pi}}\downarrow & & \downarrow{\scriptstyle \pi} \\ A & \xrightarrow[\varphi]{} & A_S \end{array}$$

Then $\varphi = \widehat{\varphi}_{ab}$ *.*

iv) *Let* $x \in A$ *be such that the fibre* $\pi^{-1}(\varphi(x))$ *is reducible as scheme (resp. irreducible, resp. normal). Then* $\widehat{\pi}^{-1}(x)$ *is reducible as scheme (resp. irreducible, resp. normal).*

v) *If* $\pi : S \to A_S$ *is flat, then* $\widehat{\pi} : \widehat{S} \to A$ *is flat.*

PROOF. It is clear that $\widehat{S}$ is a submonoid of $A \times S$. An element $(a, s) \in \widehat{S}$ is invertible if and only if $a \in R$ and $s \in G$, so $\widehat{G} = R \times_{A_S} G$. As $\varphi(R) \subset Z/Z_0$ and $\pi(Z) = Z/Z_0$, we deduce that $\widehat{G} = R \times_{Z/Z_0} G = R \times_{Z/Z_0} ((Z \times G_0)/Z_0)$.

On the other hand, the commutator of two elements $(r_1, g_1), (r_2, g_2) \in \widehat{G}$ is

$$(r_1 r_2 r_1^{-1} r_2^{-1}, g_1 g_2 g_1^{-1} g_2^{-1}) = (1, g_1 g_2 g_1^{-1} g_2^{-1}) ,$$

so the commutator of $\widehat{G}$ is $\widehat{G}_0 = \{1\} \times G_0 \cong G_0$.

Analogously, we prove that $\widehat{Z} = R \times_{Z/Z_0} Z$. It follows that $\widehat{Z_0} = \{1\} \times Z_0$.

It is easy to see that $(A \times_{A_S} S) /\!/ (G_0 \times G_0) \cong A \times_{A_S} A_S = A$, and $\varphi = \widehat{\varphi}_{ab}$.

In order to prove iv), let $x \in A$ and consider $\mathcal{M} \subset k[A]$, the maximal ideal associated to x. Let $\mathcal{M}' \subset k[A_S]$ be the preimage of $\mathcal{M}$ by φ^*. Then $\mathcal{M}'$ is the maximal ideal of $\varphi(x)$, and

$$k[\widehat{S}]/\mathcal{M}k[\widehat{S}] \cong (k[A]/\mathcal{M}) \otimes_{k[A_S]/\mathcal{M}'} (k[S]/\mathcal{M}'k[S]) = k[S]/\mathcal{M}'k[S] .$$

It follows that the fibre $\widehat{\pi}^{-1}(x)$ is reducible as scheme (resp. irreducible, resp. normal) if and only if $\pi^{-1}(\varphi(x))$ is.

Finally, we recall that flatness is stable under base extension. □

Proposition 2 *We keep the notations of Proposition 1. Suppose that S and A are normal varieties. Then*
i) The normalisation $\widehat{S}_{norm}$ of $\widehat{S}$ – which is a reductive monoid of unit group $\widehat{G}$ ([12]) – verifies:

$$\mathcal{V}(\widehat{S}_{norm}) = \mathbb{Q}^+ \left\{ \begin{array}{r} (\lambda, (\mu_1, \mu_2)) \in (\Xi_*(T_0) \times \Xi_*(\widehat{Z}))^{\widehat{Z_0}} \ : \\ (\lambda, \mu_2) \in \mathcal{V}(S)\,, \ \mu_1 \in \mathcal{C}(A)\,, \varphi(\mu_1(t)) = \pi(\mu_2(t)) \end{array} \right\},$$

where λ is an 1-PS of G_0 and (μ_1, μ_2) is an 1-PS of $R \times Z$.

In particular, $A_{\widehat{S}_{norm}} \cong A$.

ii) If moreover $\pi : S \to A_S$ is flat with normal fibres, then $\widehat{S}$ is normal.

Proof. The 1-PS of $\widehat{G}$ are the triples $(\lambda, \mu_1, \mu_2) \in \Xi_*(G_0) \times \Xi_*(R) \times \Xi_*(Z)$ such that $\pi(\mu_2(t)) = \varphi(\mu_1(t))$. It suffices to show that the 1-PS $(\lambda, \mu_1, \mu_2) \in \widehat{G}$ belonging to $-C(\widehat{G})$ such that the limit $\lim_{t\to 0}(\lambda, \mu_1, \mu_2)(t)$ exists in $\widehat{S}$, are exactly those such that $(\lambda, \mu_2) \in \mathcal{V}(S)$, and $\mu_1 \in \mathcal{C}(A)$ ([12, §4.2]). Recall that $C(\widehat{G})$ is the Weyl chamber of $\widehat{G}$.

Let $\nu = (\lambda, \mu_1, \mu_2) \in \Xi_*(\widehat{G})$ be a 1-PS. Then $\nu \in -C(\widehat{G})$, if and only if $\lambda \in -C(G_0)$. Moreover, the limit $\lim_{t\to 0} \nu(t) = \lim_{t\to 0}(\lambda(t), \mu_1(t), \mu_2(t))$ exists in $\widehat{S}$ if and only if the limits

$$\lim_{t\to 0} \widehat{\varphi}(\lambda(t), \mu_1(t), \mu_2(t)) = \lim_{t\to 0} \lambda(t)\mu_2(t)$$

and

$$\lim_{t\to 0} \widehat{\pi}(\lambda(t), \mu_1(t), \mu_2(t)) = \lim_{t\to 0} \mu_1(t)$$

exist in S and A respectively. It follows that the limit $\lim_{t\to 0} \nu(t)$ exists in $\widehat{S}$ if and only if $(\lambda, \mu_2) \in \mathcal{V}(S)$, and $\mu_1 \in \mathcal{C}(A)$. As an 1-PS of $\widehat{G}$ has a limit in $\widehat{S}$ if and only if it has a limit in $\widehat{S}_{norm}$, assertion i) is verified.

If $\pi : S \to A_S$ is a flat morphism, then $\widehat{\pi} : \widehat{S} \to A$ is flat by base extension. By Proposition 1-iv), if the fibres of π are normal, then the fibres of $\widehat{\pi}$ are normal. It follows from [8, §21.E] that $\widehat{S}$ is normal. □

Remark We can identify $\mathcal{Q}(\widehat{G})$ with $\ker((\varphi_*, 0) - (0, \pi_*)) \subset \mathcal{Q}(R) \times \mathcal{Q}(G)$. Under this identification, $\mathcal{V}(\widehat{S}_{norm}) = (\mathcal{V}(A) \times \mathcal{V}(S)) \cap \mathcal{Q}(\widehat{G})$.

4 Enveloping and asymptotic semigroups of a semisimple group

Let G_0 be a semisimple group. We keep the notations of the preceding paragraphs, and denote $\mathcal{FM}(G_0)$ the class of very flat monoids whose unit group has commutator isomorphic to G_0. This family of monoids has been studied by Vinberg ([15]) in characteristic 0. He has shown that it has a minimal element, the *enveloping semigroup*, that allows to describe the monoids of the family as fibred products. Moreover, the abelianisation of the enveloping semigroup gives us a flat deformation of G_0 over a semigroup, the *asymptotic semigroup*, associated to a certain graded algebra ([16]). In this section, we give a geometric construction of the enveloping and asymptotic semigroups, valid in arbitrary characteristic, and study their geometry.

DEFINITION 3 Let G_0 be a semisimple group. We denote $\mathrm{Env}(G_0)$ the monoid of unit group $G = (G_0 \times T_0)/Z_0$ associated to the colored cone:

$$(\{\mathbb{Q}^+\{(-\omega_1, \omega_1), \dots, (-\omega_l, \omega_l)\} + \mathbb{Q}^+\mathcal{D}(G), \mathcal{D}(G)) \ .$$

We call this monoid the *enveloping semigroup* associated to G_0.

REMARK By Theorem 2, the abelianisation of the enveloping semigroup is a (T_0/Z_0)-toric variety with associated cone $\mathbb{Q}^+(\alpha_1, \dots, \alpha_l)$, thus $A_{\mathrm{Env}(G_0)} \cong \mathbb{A}^l$.

An easy calculation shows that in characteristic zero, the preceding definition coincides with the one given in [15].

NOTATION From now on, we consider a semisimple simply connected group G_0, and its adjoint group $G_{ad} = G_0/Z_0$, where Z_0 is the center of G_0, unless otherwise stated. This is done without loss of generality, by the fact that reductive monoids "behave well" under quotient by finite central subgroups ([11], [12]).

PROPOSITION-DEFINITION ([3],[14]) Let G be a reductive group and Z be its center. There exists an unique simple projective embedding, without colors, of $G_{ad} = G/Z \cong (G_{ad} \times G_{ad})/\Delta(G_{ad})$. We call this embedding the *wonderful embedding of* G_{ad} (or G).

Let us summarize the properties of the wonderful embedding to be used in our work:

The wonderful embedding X is a smooth variety.

If $C = C(G)$ denotes the Weyl chamber of G associated to a Borel subgroup B, the colored cone associated to the wonderful embedding is:

$$(\mathcal{C}(X), \mathcal{F}(X)) = (\mathcal{V}(G_{ad}), \emptyset) = (-C(G_{ad}), \emptyset) .$$

We denote X_i the irreducible component (divisor) of $X \setminus G_{ad}$ that has associated valuation $-\omega_i$. The closed irreducible $(G \times G)$-stable subsets of X are exactly the sets $X_I = \cap_{i \in I} X_i$, $I \subset \{1, \dots, l\}$, with the convention $X_\emptyset = X$.

The unique closed orbit of X is $Y = X_{\{1,\dots,l\}} \cong (G/B) \times (G/B^-)$.

Moreover, there exists a morphism $\gamma : \mathbb{A}^l \to X$, where $\mathbb{A}^l$ is the affine space of dimension l, and an open neighborhood $\mathcal{U} \subset X$ of $1 \in G_{ad}$ such that the map

$$U \times U^- \times \mathbb{A}^l \ni (u, u', a) \mapsto (u, u') \cdot \gamma(a) \in \mathcal{U}$$

is an isomorphism. The morphism γ verifies that for all $(t, s) \in T_0 \times T_0$, $\gamma^{-1}((t, s) \cdot 1) = (t^{\alpha_1} s^{-\alpha_1}, \dots, t^{\alpha_1} s^{-\alpha_1})$ (recall that α_i, $i = 1, \dots, l$, are the simple roots of G_0).

Finally, a G-embedding has no colors if and only if it dominates equivariantly X ([7]).

NOTATION Let $\lambda \in \Xi_+(T)$ be a dominant weight of G. We denote $L(\lambda) \to G/B$ the line bundle associated to the weight λ ([6]). The external tensor product $L(\lambda) \boxtimes L(-\lambda) \to (G/B) \times (G/B^-)$ is denoted $\mathcal{L}(\lambda)$.

We recall some properties about line bundles over the wonderful embedding of G_0 needed for our work.

Let X be the wonderful embedding of G_0, and Y its unique closed $(G_0 \times G_0)$-orbit. The Picard group of X is isomorphic to the subgroup of the group of weights of $G_0 \times G_0$ generated by $(\omega_i, -\omega_i)$, $i = 1, \dots l$ ([14]), in such a way that if L_λ is the line bundle associated to the weight $(\lambda, -\lambda)$, then the restriction of L_λ to the closed orbit $G/B \times G/B^-$ coincides with $\mathcal{L}(\lambda)$. Moreover, L_λ is $(G_0 \times G_0)$-linearized (see [9] for a definition of linearisation), in such a way that over the $(B_0^- \times B_0)$-fixed point of Y, $(B_0^- \times B_0)$ acts by multiplication by $(-\lambda, \lambda)$.

We denote L_i the line bundle L_{ω_i}, which is associated to the divisor $\overline{D_i} = \overline{B_0 s_\alpha B_0^-} \subset X$. If $\lambda = \sum n_i \omega_i$, then $L_\lambda \cong L_1^{\otimes n_1} \otimes \dots \otimes L_l^{\otimes n_l}$. On the other hand, the line bundles L_{α_i} are associated to the divisors X_i ([14]). Moreover, $H^0(X, L_{\alpha_i}) \neq 0$.

Finally, we recall that the bundles L_i are generated by their global sections and that $L_{\sum_{i=1}^{l}\omega_i} = L_1 \otimes \cdots \otimes L_l$ is an ample line bundle ([14]). We consider the action of $G_0 \times G_0$ over the sections of this $(G_0 \times G_0)$-linearized bundle and its induced action on the bundle $\varrho : E = \bigoplus_{i=1}^{l} L_i^\vee \to X$.

In [1], Brion and Polo have described some properties of the algebra

$$R(X) = \bigoplus_{n_1,\dots,n_l \in \mathbb{Z}} H^0(X, L_1^{\otimes n_1} \otimes \cdots \otimes L_l^{\otimes n_l}) ,$$

in arbitrary characteristic. This algebra turns out to be the algebra of regular functions of the enveloping semigroup:

THEOREM 3 *Let G_0 be a semisimple simply connected group, and X its wonderful embedding. Then the algebra of regular functions of the enveloping semigroup of G_0 verifies:*

$$k[\mathrm{Env}(G_0)] \cong R(X) = \bigoplus_{n_1,\dots,n_l \in \mathbb{Z}} H^0(X, L_1^{\otimes n_1} \otimes \cdots \otimes L_l^{\otimes n_l}) . \tag{1}$$

Moreover, $k[\mathrm{Env}(G_0)]$ is generated by its subspaces $H^0(X, L_i)$, $i = 1, \dots, l$, and by $\sigma_1, \dots, \sigma_l$, where $\sigma_i \in H^0(X, L_{\alpha_i})$ is a global $(B \times B^-)$-equivariant section of L_{α_i}.

The sections $\sigma_1, \dots, \sigma_l$ form a regular sequence in $k[\mathrm{Env}(G_0)]$, and the quotient $k[\mathrm{Env}(G_0)]/(\sigma_1, \dots, \sigma_l)$ is isomorphic to $\bigoplus_{n_i \in \mathbb{Z}} H^0(Y, L_1^{n_1} \otimes \cdots \otimes L_l^{n_l})$, where Y is the closed orbit of X.

PROOF. Equation (1) will be a consequence of the construction developed below (cf. Corollary 1). The stated properties of $k[\mathrm{Env}(G_0)] = R(X)$ are proved in [1, Cor. 9 and 10]. □

4.1 A geometrical construction of the enveloping semigroup

In this paragraph, we give a construction of $\mathrm{Env}(G_0)$ using bundles over the wonderful embedding of G_0.

If G_0 is not simply connected, it is known that there exists a simply connected group $\widetilde{G_0}$ such that $G_0 \cong \widetilde{G_0}/\Gamma$, where Γ is a finite central subgroup. In this case, by general properties of reductive embeddings, (see [15] and [12]) we have:

$$\mathrm{Env}(G_0) = \mathrm{Env}(\widetilde{G_0}/\Gamma) = \mathrm{Env}(\widetilde{G_0})/\Gamma ,$$

so we can suppose without loss of generality that G_0 is simply connected. We denote Z_0 its center, and $G_{ad} = G_0/Z_0$ its adjoint group.

The following Proposition, easy to prove, generalizes some results of Fujita and Demazure ([5], [2]). For a proof, we refer the reader to [11].

PROPOSITION 3 *We keep the notations of the preceding paragraphs. Consider the bundle* $\varrho : E = L_1^\vee \oplus \cdots \oplus L_l^\vee \to X$. *Then the algebra of regular functions of* $E_I = \varrho^{-1}(X_I)$ *is of finite type and normal, and satisfies:*

$$H^0(E_I, \mathcal{O}_{E_I}) \cong \bigoplus_{\lambda\in\Xi_+(T_0)} H^0(X_I, L_\lambda) \cong \bigoplus_{n_1,\dots,n_l\geq 0} H^0(X_I, L_1^{\otimes n_1} \otimes \cdots \otimes L_l^{\otimes n_l}) ,$$

In particular,

$$S_0 = \operatorname{Spec} H^0(E, \mathcal{O}_E) = \operatorname{Spec} \bigoplus_{\lambda\in\Xi_+(T_0)} H^0(X, L_\lambda) \tag{2}$$

is a normal affine variety, and the canonical morphism $\varphi : E \to S_0$ *is a* $(G_0 \times G_0)$*-equivariant proper birational morphism, with connected fibres (obvious actions).*

Let E_i *be the* zero section *of* $L_i^\vee$, *i.e. the image of the canonical closed immersion* $\bigoplus_{j\neq i} L_j^\vee \hookrightarrow E$. *Consider* $E' = E \setminus (\cup_{1\leq i\leq l} E_i)$, *and set* $E_I' = \varrho^{-1}(X_I) \cap E'$. *Then*

$$H^0(E_I', \mathcal{O}_{E_I'}) \cong \bigoplus_{\lambda\in\Xi(T_0)} H^0(X_I, L_\lambda)$$

is an algebra of finite type and normal.

Moreover, the canonical morphism

$$\varphi' : E' \to \operatorname{Spec} \bigoplus_{\lambda\in\Xi_+(T_0)} H^0(X, L_\lambda)$$

is birational. □

We will show that the variety S_0 defined in equation (2) is a reductive monoid with unit group $(G_0 \times T_0)/Z_0$. Moreover, $\varphi : E \to S_0$ is the *decoloration* of S_0 (cf. Definition 4 below). This fact will allow us to determine the colored cone associated to S_0 .

In general, S_0 is not very flat: the enveloping semigroup associated to G_0 will be constructed as the normalization of the fibred product between S_0 and $\mathbb{A}^l$, over the abelianisation of S_0 (cf. Theorem 5 below).

THEOREM 4 *We keep the preceding notations and let* $G = (G_0 \times T_0)/Z_0$. *Then* E *is a simple* G*-embedding, without colors, and*

$$G \cong \operatorname{Spec} \bigoplus_{\lambda \in \Xi(T_0)} H^0(G_{ad}, L_\lambda|_{G_{ad}}) =$$

$$\operatorname{Spec} \bigoplus_{n_1, \dots, n_l \in \mathbb{Z}} H^0(G_{ad}, L_1^{\otimes n_1} \otimes \cdots \otimes L_l^{\otimes n_l}|_{G_{ad}}) \hookrightarrow E' \subset E .$$

Moreover, there exists a bijection between the $(G \times G)$*-stable irreducible closed subsets of* E *(and thus the orbits) and the couples* (I, J) *of subsets of* $\{1, \dots l\}$, *given by* $(I, J) \leftrightarrow E_{I,J} = \bigoplus_{i \in I} L_i^\vee|_{X_J}$.

In particular, the colored cone associated to E *in* $\mathcal{Q}(G)$ *is:*

$$\left(\mathbb{Q}^+\{(-\omega_1, \omega_1), \dots (-\omega_l, \omega_l), (0, \alpha_1), \dots , (0, \alpha_l)\}, \emptyset\right) .$$

PROOF. First we observe that the total space E is a smooth variety. The torus T_0 acts by multiplication by ω_i on the fibres of $L_i^\vee$, in such a way that we have an action of T_0 on E . On the other hand, the action of $G_0 \times G_0$ over the sections of the bundles $L_i^\vee$ induces an action of $G_0 \times G_0$ on E .

Thus, we have an action $G_0 \times G_0 \times T_0$ on E . We identify T_0 with the homogeneous space $(T_0 \times T_0)/\Delta(T_0)$ by $t \mapsto (1, t)$, and we get an action of $(G_0 \times T_0) \times (G_0 \times T_0)$ on E . We claim that this action is the one needed in order to prove the theorem.

Let us explicit the action of $G_0 \times G_0 \times T_0$ on E

Consider an affine open subset $V \subset X$ such that there exist trivialisations over V for the bundles $L_i^\vee$. Consider a section s_i of $L_i^\vee$ that does not vanish on V , $i = 1, \dots l$. We have an isomorphism $V \times \mathbb{A}^l \cong \varrho^{-1}(V)$, given by $(x, t_1, \dots , t_l) \mapsto \sum_i t_i s_i(x)$. If $(a, b, z) \in G_0 \times G_0 \times T_0$, then $(a, b) \cdot V$ is an open subset of X , isomorphic to V *as a variety.* the action of (a, b, z) over $\sum_i t_i s_i(x) \in V$ is given by:

$$(a, b, z) \cdot \sum_{i=1}^{l} t_i s_i(x) = \sum_{i=1}^{l} t_i \omega_i(z) \left((a, b) \cdot s_i\right) \left((a, b) \cdot x\right) . \qquad (3)$$

In order to prove that the action of $G_0 \times G_0 \times T_0$ on E induces an action of $G \times G$ such that E is a G-embedding, we must first show that if $z \in Z_0$, then $(1, z^{-1}, z)$ acts trivially on E (*i.e.* the action found factorizes trough the quotient $(G_0 \times T_0)/Z_0 \times (G_0 \times T_0)/Z_0$). Next, we must show that there exists an element such that its isotropy group is $\Delta(G)$. The orbit passing by this element is necessarily open by dimension considerations.

If we substitute (a,b,z) in equation (3) by $(1,z^{-1},z)$, $z\in Z_0$, we obtain:

$$(1,z^{-1},z)\cdot\sum_{i=1}^{l} t_i s_i(x)=\sum_{i=1}^{l} t_i\omega_i(z)\omega_i(z^{-1})s_i(x)=\sum_{i=1}^{l} t_i s_i(x)\ ,$$

so the action of $G_0\times G_0\times T_0$ of E induces one of $G\times G$.

Next, we calculate the $(G\times G)$-orbits of E or, equivalently, we calculate the $(G_0\times G_0\times T_0)$-orbits. As T_0 acts on the fibres, it is clear that every orbit is contained in the preimage by ϱ of one $(G_0\times G_0)$-orbit of X.

On the other hand, for every couple $I,J\subset\{1,\dots,l\}$, there exists a $(G_0\times G_0\times T_0)$-equivariant closed immersion:

$$E_{I,J}=\bigoplus_{i\in I} L_i^\vee|_{X_J}\hookrightarrow\bigoplus_{i=1}^{n} L_i^\vee=E\ .$$

It is clear that the $E_{I,J}$ are the only irreducible $(G\times G)$-stable closed subsets of E. Every $E_{I,J}$ contains an unique open orbit $\mathcal{O}_{I,J}$, obtained by discarding the zero sections over $\mathcal{O}_J$.

In particular, $\operatorname{Spec}\bigoplus_{\lambda\in\Xi(T_0)}\Gamma(G_{ad},L_\lambda|_{G_{ad}})\cong\mathcal{O}_{\{1,\dots,l\},\emptyset}\subset\varrho^{-1}(G_{ad})$ is an open orbit, and if $Y\subset X$ is the unique closed orbit of the wonderful embedding, then the unique closed orbit of E is $\mathcal{O}_{\emptyset,\{1,\dots,l\}}=\theta(Y)$, the zero section. It follows that E is a simple $(G\times G)$-spherical variety.

Let us show that $\mathcal{O}_{\{1,\dots,l\},\emptyset}$ is isomorphic to G as $(G\times G)$-variety. It follows that E is a simple G-embedding. Consider an open neighborhood $\mathcal{U}\subset X$ around $1\in G_{ad}=G_0/Z_0$, isomorphic to $U\times U^-\times\mathbb{A}^l$. The bundles $L_i^\vee$ can be trivialized over $\mathcal{U}$ ([14]), so $\varrho^{-1}(\mathcal{U})\cong U\times U^-\times\mathbb{A}^l\times\mathbb{A}^l$. Consider sections s_i of the the bundles $L_i^\vee$, $i=1,\dots,l$, over a neigbourhood of 1, such that s_i trivializes the bundle $L_i^\vee$. It suffices to show that $\Delta(G)$ is the isotropy group of $\sum_{i=1}^{l}s_i(1)$, that is:

$$(G_0\times G_0\times T_0)_{\sum_{i=1}^{l}s_i(1)}=$$
$$(\Delta(G_0)\times\{1\})\left(\{(1,z^{-1},z)\in G_0\times G_0\times T_0\ :\ z\in Z_0\}\right)\ .$$

¿From equation (3), we deduce that an element (a,b,z) belongs to $(G_0\times G_0\times T_0)_{\sum_{i=1}^{l}s_i(1)}$ if and only if $ab^{-1}\in Z_0$ and $\omega_i(z)((a,b)\cdot s_i)(1)=s_i(1)$, for all $i=1,\dots,l$. In this case, there exists $z_0\in Z_0$ such that $b=az_0$, so

$s_i(1) = \omega_i(z)((a, az_0)\cdot s_i)(1) = \omega_i(z)\omega_i(z_0)s_i(1)$. It follows that $\omega_i(zz_0) = 1$ for all $i = 1,\dots,l$; i.e. $z = z_0^{-1}$, so E is a G-embedding.

Finally, we calculate the colored cone $(\mathcal{C}(E), \mathcal{F}(E))$ associated to E.

In order to find the set of colors $\mathcal{F}(E)$, observe that $\varrho(\overline{Bs_{(\alpha,0)}B^-}) = \overline{B_0 s_\alpha B_0^-} \subset X$, which does not intersect Y. It follows that that E is without colors.

Next, we calculate $\mathcal{C}(E) = \mathcal{V}(E)$. For this, it suffices to find the 1-PS of G which belong to $-C(G) = -C(G_0)\times \mathcal{Q}(T_0)$, such that their limit as t goes to zero exists in E ([12, §4.2]).

An 1-PS $(\lambda,\mu)\in \Xi_*(T_0\times T_0)^{Z_0}$ is given by $(\lambda(t),\mu(t))\cdot\sum s_i(1) = \sum \omega_i(\mu(t)-\lambda(t))s_i(\lambda(t)^{-1})$. Considering the isomorphism $\varrho(\mathcal{U})\cong U\times U^-\times \mathbb{A}^l\times\mathbb{A}^l$ we have

$$\begin{array}{l} \mathcal{U}\supset BB^- \leftrightarrow U\times U^-\times (k^*)^l\times (k^*)^l \subset U\times U^-\times \mathbb{A}^l\times\mathbb{A}^l \\ (\lambda(t),\mu(t)) \leftrightarrow (1,1,\lambda(t)^{-\alpha_1},\dots,\lambda(t)^{-\alpha_l},\omega_1(\mu(t)-\lambda(t)),\dots,\omega_l(\mu(t)-\lambda(t))) \end{array}$$

Then, $(\lambda(t),\mu(t)) \leftrightarrow (1,1,t^{-\langle\lambda,\alpha_1\rangle},\dots,t^{-\langle\lambda,\alpha_l\rangle},t^{\langle\mu-\lambda,\omega_1\rangle},\dots,t^{\langle\mu-\lambda,\omega_l\rangle})$. We deduce that the limit $\lim_{t\to 0}(\lambda(t),\mu(t))$ exists if and only if $\lambda\in -C(G_0)$, et $\mu-\lambda\in\mathbb{Z}^+\{\alpha_1\dots,\alpha_l\}$.

It follows that

$$\mathcal{C}(E) = \mathcal{V}(E) = \mathbb{Q}^+\{(-\omega_1,\omega_1),\dots(-\omega_l,\omega_l),(0,\alpha_1),\dots,(0,\alpha_l)\}\ .$$

□

We recall an useful result about the sections of bundles over the wonderful embedding.

PROPOSITION 4 ([3, prop. 8.2],[14]) *We keep the notations of the preceding paragraphs. Consider $I\subset\{1,\dots,l\}$ and $\lambda=\sum_{i=1}^l n_i\omega_i\in\Xi(T_0)$. We define*

$$\mathcal{L}_{I,\lambda} = \left\{\gamma\in\Xi_+(T_0)\ :\ \gamma = \lambda - \sum_{j\in I^c} t_j\alpha_j\ ,\ t_j\in\mathbb{Z}^+\right\}\ .$$

There exists a morphism $V_\gamma\times V_\gamma^*\hookrightarrow H^0(X_I,L_\lambda)$ if and only if $\gamma\in\mathcal{L}_{I,\lambda}$.

□

COROLLARY 1 *Equation (1) is verified:*

$$k[\mathrm{Env}(G_0)]\cong R(X) = \bigoplus_{n_1,\dots,n_l\in\mathbb{Z}} H^0(X, L_1^{\otimes n_1}\otimes\cdots\otimes L_l^{\otimes n_l})\ .$$

PROOF. Indeed, observe as before that the action of G_0 over the sections $H^0(X, L_1^{\otimes n_1} \otimes \cdots \otimes L_l^{\otimes n_l})$ induces an action of $G \times G$ over $R(X)$, such that the weights of the vectors in $R(X)^{U \times U^-}$ are exactly those appearing in $\mathcal{C}(\mathrm{Env}(G_0))^\vee \cap C(G)$, where $G = G(\mathrm{Env}(G_0))$. Moreover, by Proposition 3 the open orbit in $\mathrm{Spec}\, R(X)$ of the induced action is isomorphic to G. It follows that $\mathrm{Spec}\, R(X) \cong \mathrm{Env}(G_0)$ ([7]). □

DEFINITION 4 *Let X be a normal G-embedding, with associated colored fan $\mathcal{E}$. Consider the normal G-embedding $\widetilde{X}$ associated to the fan $\mathcal{E}'$ obtained from $\mathcal{E}$ by intersecting with $\mathcal{V}(G)$. Then the morphism $\varphi : \widetilde{X} \to X$ is proper and birational, and is minimal for these properties; we call φ (or $\widetilde{X}$) the* decoloration *of X ([13]).*

It follows immediately from the classification of reductive monoids ([12, Prop. 12]) that two reductive monoids are isomorphic if and only if their decolorations are isomorphic.

THEOREM 5 *The variety S_0 defined by equation (2) is a reductive monoid with zero, with unit group $G = (G_0 \times T_0)/Z_0$. The colored cone in $\mathcal{Q}(G)$ associated to S_0 is*

$$
\begin{aligned}
&(\mathcal{C}(S_0), \mathcal{F}(S_0)) = \\
&\qquad \left(\mathbb{Q}^+ \left\{(-\omega_i, \omega_i), (0, \alpha_j^\vee), (\alpha_k^\vee, 0) \ ; \ i, j, k = 1 \ldots, l\right\}, \mathcal{D}(G)\right) .
\end{aligned} \tag{4}
$$

The morphism $\varphi : E \to S_0$ is the decoloration of S_0. In particular, φ is a proper birational morphism, so it is surjective.

Moreover, if $\pi : S \to \mathbb{A}^l$ and $\pi_0 : S_0 \to A_0$ are the abelianisations of $\mathrm{Env}(G_0)$ *and S_0 respectively, then there exists a morphism $\gamma : \mathbb{A}_l \to A_0$ such that* $\mathrm{Env}(G_0) = (\mathbb{A}^l \times_{A_0} S_0)_{norm}$.

PROOF. We have seen that S_0 is an affine normal variety. The action of $G \times G$ on E induces an action on $H^0(E, \mathcal{O}_E) = k[S_0]$. We consider the induced action of $G \times G$ over S_0. By Proposition 3, $\varphi : E \to S_0$ is a birational proper $(G \times G)$-morphism. It follows that $\varphi_0(G)$ is an open $(G \times G)$-orbit of S_0, so S_0 is an affine G-embedding, i.e. a monoid of unit group G. Moreover, the maximal ideal $\mathcal{M} = \bigoplus_{\lambda \in \Xi_+(T_0) \setminus \{0\}} H^0(X, L_\lambda)$ is $(G \times G)$-stable, so the unique closed orbit of S_0 is the closed point corresponding to $\mathcal{M}$. In particular, S_0 has a zero.

The colored cone associated to S_0 is the dual of the cone generated by the weights of the regular functions belonging to $k[S_0]^{U \times U^-}$ (recall that if a monoid has a zero, then all the colors are present). By Proposition 4 this semigroup is:

$$\mathcal{L}_{S_0} = \left\{(\lambda, \mu) \in \Xi_+(T_0) \times \Xi_+(T_0) \ : \ \mu - \lambda \in \mathbb{Z}^+\{\alpha_1 \dots, \alpha_l\}\right\} \ .$$

Dualizing, we obtain equation (4).

Finally, the morphism $\varphi : E \to S_0$ is the decoloration of S_0 because E is without colors and that $\mathcal{C}(E) = \mathcal{V}(S_0)$.

Let $G = (G_0 \times T_0)/Z_0$; in order to avoid confusion, we denote $Z \cong \{1\} \times T_0$ the connected center of G.

The associated cones of $\mathbb{A}^l$ and A_0 as a (Z/Z_0)-toric varieties are $\mathcal{C}(\mathbb{A}^l) = \mathbb{Q}^+\{\omega_1, \dots, \omega_l\} \subset \mathcal{Q}(Z/Z_0) = \mathcal{Q}(Z)$, and $\mathbb{Q}^+\{\alpha_1^\vee, \dots, \alpha_l^\vee\}$ respectively. It follows that the identity id : $\mathcal{Q}(Z/Z_0) \to \mathcal{Q}(Z/Z_0)$ induces a morphism $\gamma : \mathbb{A}^l \to A_0$.

Consider the fibred product $S = \mathbb{A}^l \times_{A_0} S_0$.

By Proposition 2, S_{norm} is a reductive monoid with unit group $G(S) = G \times_{Z/Z_0} (Z/Z_0) = G$, such that the associated colored cone verifies:

$$\mathcal{V}(S_{norm}) = \mathbb{Q}^+ \left\{ \begin{array}{l} (\lambda, (\mu_1, \mu_2)) \in (\Xi(T_0) \times \Xi(\widehat{Z}))^{\widehat{Z_0}} \ : \\ \qquad (\lambda, \mu_2) \in \mathcal{V}(S), \ \mu_1 \in \mathcal{C}(\mathbb{A}^l), \gamma(\mu_1(t)) = \pi_0(\mu_2(t)) \end{array} \right\}$$

or equivalently:

$$\mathcal{V}(S_{norm}) = \mathbb{Q}^+ \left\{ \begin{array}{l} (\lambda, \mu) \in (\Xi(T_0) \times \Xi(Z))^{Z_0} \ : \\ \qquad \mu - \lambda \in \mathbb{Z}^+\{\alpha_1, \dots, \alpha_l\}, \mu \in \mathbb{Z}^+\{\omega_1, \dots, \omega_l\} \end{array} \right\},$$

which is equal to $\mathcal{V}(\mathrm{Env}(G_0))$, so $S \cong \mathrm{Env}(G_0)$ ([12, Prop. 12]). □

The $(G \times G)$-orbits of $\mathrm{Env}(G_0)$, where $G = (G_0 \times T_0)/Z_0$, are in bijection with the colored faces of $\mathcal{C}(\mathrm{Env}(G_0))$. So it is possible to give a combinatorial characterization of the orbits:

THEOREM 6 ([15], [11]) *There exists a bijection between the couples (I, J) of subsets of $\{1, \dots, l\}$ such that every connected component of J (in the Dynkin diagram with indices $1, \dots, l$) intersects I, and the $(G \times G)$-orbits of $\mathrm{Env}(G_0)$. If $\mathcal{O}_{I,J}$ is the orbit associated to (I, J) by the bijection constructed, then*

$$\dim \mathcal{O}_{I,J} = \dim(G \times G)/P_{J^c} \ ,$$

where P_{J^c} is the parabolic subgroup generated by the parabolics $P_{\alpha_i} \times Q_{\alpha_i}$, $i \in J^c$, P_α being the minimal parabolic subgroup associated to the simple root

α, and Q_α its opposite. □

We study now the geometry of the abelianisation of the enveloping semigroup. We keep the notations of the preceding paragraphs. Consider $J \subset \{1, \dots, l\}$ and let I be its complement. Consider the closed subvariety X_J of the wonderful embedding X.

If $\lambda \in \Xi(T)$ and global sections $\sigma_i \in H^0(X, L_{\alpha_i})$, $i = 1, \dots, l$, as in Theorem 3, we consider the injection

$$\prod_{i\in I} \sigma_i^{t_i} : H^0\left(X_J, L_{\lambda - \sum_{i\in I} t_i\alpha_i}\right) \hookrightarrow H^0(X_J, L_\lambda)$$
$$\sigma \mapsto \sigma \prod_{i\in I} \sigma_i^{t_i}$$

We denote its image $F_{\sum_{i\in I} t_i\alpha_i} H^0(X_J, L_\lambda)$, and consider the filtration of $H^0(X_J, L_\lambda)$ by $F_n H^0(X_J, L_\lambda) = H^0(X_J, \mathcal{I}_Y^n \otimes L_\lambda)$. We recall that $Y = X_{\{1,\dots,l\}}$ is the unique closed orbit, and that the sheaf of ideals $\mathcal{I}_Y$ of Y is generated in $\mathcal{O}_{X_J}$ by the regular sequence $\{\sigma_i\ ,\ i \in I\}$ ([1]).

PROPOSITION 5 ([1]) *The graduation $F_n H^0(X_J, L_\lambda)$ is finite, and such that*

$$F_n H^0(X_J, L_\lambda) = \sum_{\sum_{i\in I} t_i = n} F_{\sum_{i\in I} t_i\alpha_i} H^0(X_J, L_\lambda) .$$

We denote $n_0(J, \lambda) \in \mathbb{Z}^+$ the integer such that $F_n H^0(X_J, L_\lambda) = 0$ if $n > n_0$, and such that $F_{n_0} H^0(X_J, L_\lambda) \neq 0$.

The nth layer of the associated graded module verifies:

$$\operatorname{gr}_n H^0(X_J, L_\lambda) = F_n H^0(X_J, L_\lambda)/F_{n+1} H^0(X_J, L_\lambda) \cong \bigoplus_{\substack{\gamma = \lambda - \sum_{i\in I} t_i\alpha_i \\ \sum_{i\in I} t_i = n,\ \gamma\in\Xi_+(T_0)}} H^0(Y, L_\gamma) .$$

Moreover, consider the restriction $r : H^0(X, L_\lambda) \to H^0(X_J, L_\lambda)$. Then

$$r\left(F_n H^0(X, L_\lambda)\right) \subset F_n H^0(X_J, L_\lambda) .$$

PROOF. We prove the last assertion. For the other ones, it suffices to translate the proof of [1, Thm. 7] for the case $J = \emptyset$, to this more general case.

It suffices to prove that $r(F_{\sum t_i\alpha_i}H^0(X,L_\lambda)) \subset F_nH^0(X_J,L_\lambda)$ whenever $\sum t_i = n$. We recall that $F_{\sum t_i\alpha_i}H^0(X,L_\lambda) = \prod \sigma_i^{t_i}H^0(X,L_{\lambda-\sum t_i\alpha_i})$. Taking restriction to X_J we get:

$$\begin{aligned} r\left(F_{\sum t_i\alpha_i}H^0(X,L_\lambda)\right) = & \prod \sigma_i^{t_i}|_{X_J} r\left(H^0(X,L_{\lambda-\sum t_i\alpha_i})\right) \subset \\ & \prod \sigma_i^{t_i}|_{X_J} H^0(X_J,L_{\lambda-\sum t_i\alpha_i}) . \end{aligned}$$

Thus, if there exists an $i \in J$ such that $t_i \neq 0$, then $r\left(F_{\sum t_i\alpha_i}H^0(X,L_\lambda)\right) = 0$, so

$$\begin{aligned} r\left(F_nH^0(X,L_\lambda)\right) \subset & \sum_{\sum_{i\in I} t_i=n} \prod_{i\in I} \sigma_i^{t_i} H^0(X_J,L_{\lambda-\sum t_i\alpha_i}) = \\ & \sum_{\sum_{i\in I} t_i=n} F_{\sum_{i\in I} t_i\alpha_i} H^0(X_J,L_\lambda) = F_nH^0(X_J,L_\lambda) . \end{aligned}$$

□

In [1] it is proved that if $\lambda \in \Xi_+(T_0)$, then the restriction $r : H^0(X,L_\lambda) \to H^0(X_J,L_\lambda)$ is surjective. The following Lemma shows that in fact this is true for all $\lambda \in \Xi(T_0)$.

LEMMA 1 *Let $\lambda \in \Xi(T_0)$. Then the restriction $r : H^0(X,L_\lambda) \to H^0(X_J,L_\lambda)$ is surjective.*

PROOF. It suffices to prove that $r_n = r|_{F_nH^0(X,L_\lambda)} : F_nH^0(X,L_\lambda) \to F_nH^0(X_J,L_\lambda)$ is surjective for all n. We prove the assertion by recursion.

First, we observe that

$$\begin{aligned} 0 \neq F_{n_0(J,\lambda)}H^0(X_J,L_\lambda) \cong & \operatorname{gr}_{n_0(J,\lambda)} H^0(X_J,L_\lambda) = \\ & \bigoplus_{\substack{\gamma=\lambda-\sum_{i\in I} t_i\alpha_i \\ \sum_{i\in I} t_i=n_0(J,\lambda),\ \gamma\in\Xi_+(T_0)}} H^0(Y,L_\gamma) . \end{aligned}$$

In particular, $\operatorname{gr}_{n_0(J,\lambda)} H^0(X,L_\lambda) = \bigoplus_{\substack{\gamma=\lambda-\sum t_i\alpha_i \\ \sum t_i=n_0(J,\lambda),\ \gamma\in\Xi_+(T_0)}} H^0(Y,L_\gamma) \neq 0$.

It follows that $F_{n_0(J,\lambda)}H^0(X,L_\lambda) \neq 0$ and r_{n_0} is surjective. In particular, $n_0(J,\lambda) \leq n_0(\emptyset,\lambda) = n_0$.

Suppose now that r_{n+1} is surjective. Take $s \in F_nH^0(X_J,L_\lambda)$, and consider $\pi_n(s)$, where $\pi_n : F_nH^0(X_J,L_\lambda) \to \operatorname{gr}_n H^0(X_J,L_\lambda)$ is the canonical projection. There exists an element $\widetilde{s} \in F_nH^0(X,\lambda)$ such that $\widetilde{r}_n(\pi(\widetilde{s})) = \pi(s)$, where $\widetilde{r}_n : \operatorname{gr}_n H^0(X, L_\lambda) \to \operatorname{gr}_n H^0(X_J, L_\lambda)$ denotes the (surjective) morphism induced by r. The recursion hypothesis implies that $r(\widetilde{s}+F_{n+1}H^0(X, L_\lambda)) = s + F_{n+1}H^0(X_J, L_\lambda)$, and the proposition is proved. □

Consider the enveloping semigroup and its abelianisation $\pi : \operatorname{Env}(G) \to \mathbb{A}^l$. The associated cone of $\mathbb{A}^l$ is generated by the fundamental weights $\omega_1, \ldots, \omega_l$. If $I \subset \{1, \ldots, l\}$, we denote $\mathcal{O}_I$ the orbit associated to the face generated by $\{\omega_i \mid i \in I\}$. Then the ideal associated to $\overline{\mathcal{O}_I}$ is $\oplus_{\gamma \notin \mathcal{M}_I} ke^\gamma$, where $\mathcal{M}_I$ denotes the semigroup generated by the simple roots α_i, $i \in I$.

THEOREM 7 *We keep the preceding notations. The preimage of* $\overline{\mathcal{O}_I}$ *is isomorphic to* $\operatorname{Spec} \oplus_{\lambda\in\Xi(T_0)} H^0(X_J, L_\lambda)$. *In particular,* $\pi^{-1}(\overline{\mathcal{O}_I})$ *is an irreducible normal affine variety. Moreover, if* $x \in \mathbb{A}^l$ *then* $\pi^{-1}(x)$ *is a irreducible normal affine variety.*

PROOF. Consider the restriction

$$r : k[\operatorname{Env}(G_0)] = \bigoplus_{\lambda\in\Xi(T_0)} H^0(X, L_\lambda) \twoheadrightarrow \bigoplus_{\lambda\in\Xi(T_0)} H^0(X_J, L_\lambda) .$$

It suffices to show that $\ker r = \mathcal{I}_{\overline{\mathcal{O}_I}} k[\operatorname{Env}(G)]$. Observe that $\mathcal{I}_{\overline{\mathcal{O}_I}} k[\operatorname{Env}(G_0)]$ is the ideal generated by $\{\sigma_i \ : \ i \in J\}$, σ_i as in Theorem 3.

Consider the morphism between the graded modules induced by r:

$$\widetilde{r} : \bigoplus_{n=1}^{n_0} \operatorname{gr}_n H^0(X, L_\lambda) \to \bigoplus_{n=1}^{n_0} \operatorname{gr}_n H^0(X_J, L_\lambda) ,$$

where $\operatorname{gr}_n H^0(X_J, L_\lambda) = 0$ if $n_0(J, \lambda) < n \leq n_0$. In order to calculate $\ker(\widetilde{r})$, we consider the restriction to the nth summand:

$$\widetilde{r}_n : \bigoplus_{\substack{\gamma=\lambda-\sum t_i\alpha_i \\ \sum t_i=n,\ \gamma\in\Xi_+(T_0)}} H^0(Y, L_\gamma) \to \bigoplus_{\substack{\gamma=\lambda-\sum_{i\in I} t_i\alpha_i \\ \sum_{i\in I} t_i=n,\ \gamma\in\Xi_+(T_0)}} H^0(Y, L_\gamma)$$

It is clear that $\ker \widetilde{r}_n = \bigoplus_{\sum_{\sum t_i=n} t_i\alpha_i \in \Xi_+(T_0)\setminus\mathcal{M}_I} H^0(Y, L_{\lambda-\sum t_i\alpha_i})$. A standard argument shows that $\ker(r)$ is then generated by $\{\sigma_i \ : \ i \in J\}$, and the first assertion of the theorem is proved.

By Proposition 3, $\pi^{-1}(\overline{\mathcal{O}_I})$ is a irreducible normal variety. As $\mathcal{O}_I$ is an homogeneous space the restriction $\pi|_{\pi^{-1}(\mathcal{O}_I)}$ is a fibration; it follows that if $x \in \mathcal{O}_I$, then $\pi^{-1}(x)$ is a normal variety. In order to prove that $\pi^{-1}(x)$

is irreducible, it suffices to prove that the isotropy group of a point in $\mathcal{O}_I$ (for the action of $G(\mathrm{Env}(G_0)) \times G(\mathrm{Env}(G_0))$) is connected. We recall that $G = G(\mathrm{Env}(G_0)) \cong (G_0 \times Z)/Z_0$, where $Z \cong T_0$ is the connected center of G, and $Z_0 = G_0 \cap Z$.

Consider an idempotent $e_I \in \mathcal{O}_I$, and let $Z_I \subset Z/Z_0$ be its isotropy group for the action of Z/Z_0 on $\mathbb{A}^l$. An element $(g, g') = (\overline{(a, z)}, \overline{(b, w)}) \in (G \times Z)/Z_0 \times (G \times Z)/Z_0$ acts in $\mathbb{A}^l$ by

$$(\overline{(a, z)}, \overline{(b, w)}) \cdot x = zw^{-1} \cdot x = \pi(zw^{-1})x \quad , \quad x \in \mathbb{A}^l ,$$

where $\pi : Z \to Z/Z_0$ denotes the quotient map. It follows that $(g, g') \in (G \times G)_{e_I}$ if and only if $zw^{-1} \in \pi^{-1}(Z_I)$, that is

$$(G \times G)_{e_I} = \big((G_0 \times \pi^{-1}(Z_I))/Z_0 \times (G_0 \times \pi^{-1}(Z_I))/Z_0\big)\, \Delta Z \subset G \times G .$$

If we prove that $H = (G_0 \times \pi^{-1}(Z_I))/Z_0$ is connected, we are done. Consider Z_I', the connected component of the identity of $\pi^{-1}(Z_I)$. We claim that the map $\varphi : G_0 \times Z_I' \to H$, $\varphi(g, z) = \overline{(g, z)}$ is surjective, which implies the assertion. Indeed, let $h \in H$ and take $(g, z) \in G_0 \times \pi^{-1}(Z_I) \subset G_0 \times Z$ a representant of H. Then there exists $z_0 \in Z_0$ such that $zz_0 \in Z_I'$, and $\varphi(gz_0^{-1}, zz_0) = \overline{(gz_0^{-1}, zz_0)} = \overline{(g, z)} = h$. □

4.2 Asymptotic semigroup associated to a semisimple group

We keep the notations of the preceding sections. In this paragraph we study the fibre $\pi^{-1}(0)$ of $\pi : \mathrm{Env}(G_0) \to \mathbb{A}^l$, which will give a generalisation to arbitrary characteristic of the *asymptotic semigroup* defined by Vinberg in [16].

DEFINITION 5 As before, let G_0 be a semisimple simply connected group, X its wonderful embedding, and let Y be the unique closed orbit of X. We define the *asymptotic semigroup associated to* G_0 as:

$$\mathrm{As}(G_0) = \mathrm{Spec}\left(\bigoplus_{\lambda \in \Xi_+(T_0)} H^0(Y, L_\lambda) \right) .$$

If $\Gamma \subset G_0$ is a finite central subgroup of G_0, we put:

$$\mathrm{As}(G_0/\Gamma) = \mathrm{As}(G_0)/\Gamma .$$

In order to describe the combinatorial data associated to the asymptotic semigroup of a semisimple group, it suffices to describe the data associated to its simply connected cover ([11],[12]), so from now on, we suppose G_0 simply connected.

THEOREM 8 *If* $\pi : \mathrm{Env}(G_0) \to \mathbb{A}^l$ *is the abelianisation of the enveloping semigroup of* G_0 *, and* $\pi_0 : S_0 \to A_0$ *the abelianisation of* S_0 *, then*

$$\mathrm{As}(G_0) \cong \pi^{-1}(0) \cong \pi_0^{-1}(0) .$$

In particular, $\mathrm{As}(G_0)$ *is an algebraic semigroup with zero.*

Let $\varphi : E \to S_0$ *be the decoloration of* S_0 *, and let* $\varrho : E \to X$ *the bundle over the wonderful embedding of* G_0 *constructed in the preceding paragraph. Consider the closed* $(G \times G)$*-stable subset of* E *:*

$$E_Y = \varrho^{-1}(Y) = E_{\{1,\dots,l\},\{1,\dots,l\}} \cong L_1^\vee|_Y \oplus \cdots \oplus L_l^\vee|_Y ,$$

where $Y \cong G_0/B_0 \times G_0/B_0^-$ *is the unique closed* $(G_0 \times G_0)$*-orbit of* X *. Then* E_Y *is a simple spherical* $(G_0 \times G_0)$*-variety with open orbit isomorphic to* $(G_0 \times G_0)/((U^- \times U)\Delta(T_0))$ *, and the restriction of* φ *to* E_Y *is a birational proper* $(G_0 \times G_0)$*-morphism from* E_Y *over* $\mathrm{As}(G_0)$ *:*

$$\varphi|_{\varrho^{-1}(Y)} = \varphi_Y : E_Y \to \pi_0^{-1}(0) = \mathrm{As}(G_0) .$$

In particular, $\mathrm{As}(G_0)$ *is a simple* $(G_0 \times G_0)$*-spherical variety, with open orbit isomorphic to* $(G_0 \times G_0)/((U^- \times U)\Delta(T_0))$ *, and thus of dimension* $\dim \mathrm{As}(G_0) = \dim G_0$ *.*

PROOF. Consider the $(G_0 \times G_0)$-action on $\mathrm{As}(G_0)$ induced by the action over the sections $H^0(Y, L_\lambda|_Y)$, $\lambda \in \Xi_+(T_0)$. By Theorem 7, $\mathrm{As}(G_0)$ is isomorphic to $\pi^{-1}(0)$ as a $(G_0 \times G_0)$-variety.

On the other hand, the same arguments used in Theorem 7 allow to prove that $\pi_0^{-1}(0) \cong \mathrm{As}(G_0)$.

By construction, $\varphi_Y : E_Y \to \mathrm{As}(G_0)$ is a birational proper $(G_0 \times G_0)$-morphism. Let us study the geometry of the $(G \times G)$-spherical variety E_Y .

Recall that the closed $(G \times G)$-stable subsets of E_Y are the subsets $E_{I,Y} = E_{I,\{1,\dots,l\}} = \oplus_{i \in I} L_i^\vee|_Y$, $I \subset \{1, \dots l\}$. The line bundles $L_i^\vee$ are $(G_0 \times G_0)$-linearized, so there exit global sections s_i , $i = 1, \dots , l$, such that they do not vanish over the affine open set $(B_0 \times B_0^-) \cdot y_0$, where $y_0 \in Y$ is the unique point of Y fixed by $B_0^- \times B_0$. It follows that

$$\varrho^{-1}((B_0 \times B_0^-) \cdot y_0) \cong ((B_0 \times B_0^-) \cdot y_0) \times \mathbb{A}^l .$$

More generally, if $I \subset \{1, \dots, l\}$ then

$$\varrho^{-1}((B_0 \times B_0^-) \cdot y_0) \cap E_{I,Y} = \oplus_{i \in I} L_i^\vee|_Y \cong ((B_0 \times B_0^-) \cdot y_0) \times \mathbb{A}^{\#I} .$$

In particular, $((B_0 \times B_0^-) \cdot \sum_{i \in I} s_i)(y_0)$ is a open $(B_0 \times B_0^-)$-orbit of $E_{I,Y}$.

If $x_I = \sum_{i \in I} s_i(y_0)$ then $(G_0 \times G_0) \cdot x_I$ is an open $(G_0 \times G_0)$-orbit of $E_{I,Y}$. So the $(G_0 \times G_0)$-orbits and the $(G \times G)$-orbits of E_Y coincide; in particular, E_Y is a simple $(G_0 \times G_0)$-spherical variety, with $\theta(Y)$ as unique closed orbit, where θ is the zero section.

Let us calculate the isotropy group of $x_0 = x_{\{1, \dots, l\}}$.

If $(a, b) \in G_0 \times G_0$, then $(a, b) \cdot x_0 = \sum((a, b) \cdot s_i)((a, b) \cdot y_0)$, so $(a, b) \cdot x_0 = x_0$ if and only if $(a, b) \in (G_0 \times G_0)_{y_0}$, and $(a, b) \cdot s_i(y_0) = s_i(y_0)$ for all $i = 1, \dots l$. It follows that we have $a \in B_0^-$, $b \in B_0$, and $\omega_i(a^{-1}b) = 1$, so $\omega_i(a) = \omega_i(b)$ for all $i = 1, \dots, l$. Then $(a, b) \in (G_0 \times G_0)_{x_0}$ if and only if $(a, b) \in (U^- \times U)\Delta(T_0)$.

On the other hand, as φ_Y is a birational $(G_0 \times G_0)$-morphism, $\mathrm{As}(G_0)$ is a $(G_0 \times G_0)$-spherical variety, with open orbit isomorphic to $(G_0 \times G_0)/((U^- \times U)\Delta(T_0))$. Moreover, the $(G_0 \times G_0)$-action extends onto a $(G \times G)$-action in such way that the orbit of both actions coincide: just take the action induced by the $(G \times G)$-action on E_Y.

Finally, the dimension of E_Y is equal to the dimension of Y plus l, so $\dim E_Y = \dim G_0$ (recall that l is equal to the number of simple roots). □

COROLLARY 2 *The fibres of* π *and* π_0 *are equidimensional, with dimension equal to* $\dim G_0$. □

The asymptotic semigroup being a $(G_0 \times G_0)$-spherical variety, it seems natural to describe the combinatorial data associated to this variety. We begin by the calculation of the combinatorial data associated to the spherical homogeneous space $(G_0 \times G_0)/((U^- \times U)\Delta(T_0))$ (the open orbit of $\mathrm{As}(G_0)$).

PROPOSITION 6 *The combinatorial data associated to the the spherical homogeneous space* $(G_0 \times G_0)/H$, $H = (U^- \times U)\Delta(T_0)$, *is the following:*

i) The space $\mathcal{Q}((G_0 \times G_0)/H)$ *is isomorphic to* $\Xi(T_0) \otimes \mathbb{Q}$.

ii) The valuation cone $\mathcal{V}((G_0 \times G_0)/H)$ *is all the space* $\mathcal{Q}((G_0 \times G_0)/H)$.

iii) The set of colors $\mathcal{D}$ *has* $2l$ *elements, namely :*

$D_\alpha = \overline{((B_0 s_\alpha B_0^-) \times G_0)H}$ *and* $D'_\alpha = \overline{(G_0 \times (B_0^- s_{-\alpha} B_0))H}$, *where* α *is a simple root of* G_0 *associated to* (B_0, T_0).

Moreover,

$$\rho_{(G_0\times G_0)/H}(\nu_{D_\alpha}) = \rho_{(G_0\times G_0)/H}(\nu_{D'_\alpha}) = \alpha^\vee .$$

PROOF. A weight $(\omega,\lambda)\in\Xi(G_0\times G_0)$ is always $(U^-\times U)$-invariant, and it is $\Delta(T)$-invariant if and only if $\lambda=-\omega$, so $\mathcal{Q}((G_0\times G_0)/H)\cong\Xi(T_0)\otimes\mathbb{Q}$.

On the other hand, $U^-\times U$ is a unipotent maximal subgroup of $G_0\times G_0$, so $\mathcal{V}((G_0\times G_0)/H)=\mathcal{Q}((G_0\times G_0)/H)$ ([10], [7, cor. 7.2]).

In order to calculate the colors, observe that if $y_0=(1,1)\in(G_0\times G_0)/H$, then

$$(((G_0\times G_0)/H)\setminus(B_0\times B_0^-))\cdot y_0=\bigcup_{i=1}^{l} D_{\alpha_i}\cup D'_{\alpha_i} .$$

It follows that the colors are the divisors D_α and D'_α, α a simple root.

Consider the projection $p: G_0\times G_0\to(G_0\times G_0)/H$. It has reduced (as schemes) and irreducible fibres, so by [7, §5] in order to calculate the valuations associated to the colors D_α and D'_α it suffices to calculate the valuations of $k(G_0\times G_0)=k(G_0)\otimes k(G_0)$ associated to the colors of $G_0\times G_0$. Those colors are the divisors $\overline{B_0 s_{\alpha_i} B_0^-}\times G_0$ and $G_0\times\overline{B_0^- s_{-\alpha_i} B_0}$, $i=1,\ldots l$. The valuation of $k(G_0)$ associated to $\overline{B_0 s_\alpha B_0^-}$ is $\alpha^\vee$, so $\rho_{G_0\times G_0}(\nu_{D_\alpha})=(\alpha^\vee,0)$, and $\rho_{G_0\times G_0}(\nu_{D'_\alpha})=(0,-\alpha^\vee)$. If $p: G_0\times G_0\to(G_0\times G_0)/H)$ is the projection, then $p_*(\rho_{G_0\times G_0}(\nu_{D_\alpha}))=p_*(\alpha^\vee,0)=\alpha^\vee$, so $\rho_{(G_0\times G_0)/H)}(\nu_{D_\alpha})=\alpha^\vee$. Analogously, $\rho_{(G_0\times G_0)/H}(\nu_{D'_\alpha})=\alpha^\vee$. □

To finish, we describe the colored cone associated to $\mathrm{As}(G_0)$:

PROPOSITION 7 *Let G_0 be a semisimple simply connected group, and $H=(U^-\times U)\Delta(T_0)$ as before. Then*

$$\varphi_Y=\varphi|_{E_Y}: E_Y=E_{\{1,\ldots,l\},\{1,\ldots,l\}}\to\mathrm{As}(G_0)$$

is the decoloration of $\mathrm{As}(G_0)$.

Moreover, the colored cone of $\mathcal{Q}((G_0\times G_0)/H)$ associated to $\mathrm{As}(G_0)$ *is :*

$$(\mathcal{C}(\mathrm{As}(G_0))),\mathcal{F}(\mathrm{As}(G_0)))=\left(\mathbb{Q}^+\{\alpha_1^\vee,\ldots,\alpha_l^\vee\},\{D_{\alpha_i},D'_{\alpha_i}\mid i=1,\ldots,l\}\right) .$$

PROOF. The asymptotic semigroup is an affine spherical variety. In particular, it is simple. The $(G_0\times G_0)$-action on $\mathrm{As}(G_0)$ is induced by the left and right multiplication on S_0, so the unique closed orbit is the zero of S_0. It follows that the set of colors $\mathcal{D}(\mathrm{As}(G_0))$ is all $\mathcal{D}((G_0 \times G_0)/(U^- \times U)\Delta(T_0))$.

On the other hand, E_Y is without colors. Indeed, the closure in E_Y of the colors are obtained by taking the preimages by ϱ of the $(B_0 \times B_0^-)$-stable divisors of Y. But E_Y is a simple spherical variety of unique closed orbit $\theta(Y)$, the zero section of the bundle $E_Y \to Y$, and it is clear that this orbit is not contained in the closure of any color.

As φ_Y is a proper morphism, $\mathcal{C}(E_Y) = \mathcal{V}(E_Y) = \mathcal{V}(\mathrm{As}(G_0))$, so $\varphi_Y : E_Y \to \mathrm{As}(G_0)$ is the decoloration of the asymptotic semigroup.

Now, if we show that $\mathrm{As}(G_0)$ contains no $(G_0 \times G_0)$-stable divisors, the proposition is proved. Such a divisor is the image by φ of a $(G_0 \times G_0)$-stable divisor $D \subset E_Y$.

Recall that the $(G_0 \times G_0)$-stable divisor $E_{\{i\},Y}$, $i = 1, \dots, l$, of E_Y is obtained by taking the zero section of the bundle $L_i^\vee$:

$$E_{\{i\},Y} = E_{\{1,\dots,\hat{i},\dots,l\},\{1,\dots,l\}} = L_1^\vee|_Y \oplus \cdots \oplus \widehat{L_i^\vee|_Y} \oplus \cdots \oplus L_l^\vee|_Y ,$$

where $\widehat{L_i^\vee}$ means that $L_i^\vee$ is not taken into account.

By construction,

$$\varphi_Y(E_{\{i\},Y}) \cong \mathrm{Spec}\left(\bigoplus_{n_j \geq 0} H^0(Y, L_{\sum_{j\neq i} n_j\omega_j})\right)$$

Under this isomorphism, the inclusion $\varphi_Y(E_{\{i\},Y}) \subset \mathrm{As}(G_0)$ is induced by:

$$H^0(Y, \bigotimes_i L_i^{\otimes n_i}) \longrightarrow \begin{cases} H^0(Y, \bigotimes_{j\neq i} L_j^{\otimes n_j}) & \text{if } n_i = 0 \\ 0 & \text{otherwise}. \end{cases}$$

The kernel of this morphism is the ideal associated to $\varphi_Y(E_{\{i\},Y})$, and verifies:

$$\mathcal{P}_{\varphi_Y(E_{\{i\},Y})} = \bigoplus_{\substack{n_j \geq 0,\ j\neq i \\ n_i > 0}} H^0(Y, L_1^{\otimes n_1} \otimes \cdots \otimes L_l^{\otimes n_l}) .$$

In particular, this ideal contains $H^0(Y, L_i)$.

Recall that the bundle $L_i|_Y$ is isomorphic to $\mathcal{L}(\omega_i) = L(\omega_i) \boxtimes L(-\omega_i)$. Consider a non-zero section s of $L(\omega_i) \to G_0/B_0$, (i.e. s does not vanish over an open subset of G_0/B_0). Let s' be a non-zero section of $L(-\omega_i) \to G_0/B_0^-$. then $\varphi_Y(D_i')$ is contained in the intersection of the zeroes of s et s'. By construction, these varieties intersect in a variety of codimension greater or equal to 2, so $\operatorname{codim} \varphi_Y(D_i) \geq 2$, and the proposition is proved. □

4.3 Minimality properties of the enveloping semigroup

In this paragraph we prove that the enveloping semigroup of a semisimple group G_0 is a minimal element of the family $\mathcal{FM}(G_0)$, in the sense of [15, Thm. 5]. In particular, $\operatorname{Env}(G_0)$ is a very flat monoid, and thus the abelianisation $\pi : \operatorname{Env}(G_0) \to \mathbb{A}^l$ gives a flat deformation of $G - 0$ towards $\operatorname{As}(G_0)$. We keep the notations of the preceding paragraphs.

PROPOSITION 8 *Let G_0 be a semisimple group. Then* $\operatorname{Env}(G_0)$ *is a very flat monoid.*

PROOF. Recall that reductive monoids are Cohen-Macaulay varieties ([13]), and that the abelianisation of $\operatorname{Env}(G_0)$ is $\pi : \operatorname{Env}(G_0) \to \mathbb{A}^l$, which is a smooth variety. Let $\mathcal{P}$ be a prime ideal of $k[x_1, \ldots, x_l] = k[\mathbb{A}^l]$ and $\mathcal{P}k[\mathbb{A}^l] \subset \mathcal{Q} \subset k[\operatorname{Env}(G_0)]$ be a prime ideal in the preimage of $\mathcal{P}$ by $\pi : \operatorname{Env}(G_0) \to \mathbb{A}^l$; we consider $A = k[\operatorname{Env}(G_0)]_{\mathcal{Q}}$ and $R = k[\mathbb{A}^l]_{\mathcal{P}}$. From [4, thm. 18.16], it follows that A is a flat R-module if and only if $\dim A = \dim R + \dim A/\mathcal{M}_R A$, where $\mathcal{M}_R = \mathcal{P}R$ is the unique maximal ideal of R.

By Corollary 2 and Theorem 7, π is equidimensional with reducible (as schemes) and irreducible fibres, so if V is the subvariety associated to $\mathcal{Q}$, then $\dim A - \dim R = \dim \pi^{-1}\pi(V) = \dim A/\mathcal{P}A$. □

THEOREM 9 *Let G_0 be a semisimple group. Let S be a reductive monoid such that its unit group G' has commutator G_0. Let Z' be the connected center of G', and $Z_0' = T_0 \cap Z'$ (so $G' \cong (G_0 \times Z')/Z_0'$). Then:*

i) If S is very flat, then there exists $\theta : Z' \to T_0$ such that $Z'|_{Z_0'} = \mathrm{Id}_{Z_0'}$ and

$$\mathcal{C}(S) = \mathbb{Q}^+ \left\{(-\theta_*(f), f) \in \mathcal{Q}(G_0) \times \mathcal{Q}(Z') \ : \ f \in \mathcal{C}(S) \cap \mathcal{Q}(Z')\right\} + \mathbb{Q}^+\mathcal{F} ,$$

where $\mathcal{F} \subset \mathcal{D}(G')$ is the set of colors of of S. Moreover, $\mathcal{C}(A_S) = \mathcal{C}(S) \cap \mathcal{Q}(Z')$ and $\theta_*(\mathcal{C}(A_S)) \subset C_0$, where C_0 denotes as before the Weyl chamber of G_0. In particular, $\mathbb{Q}^+\mathcal{B}(S) = \mathbb{Q}^+ \left\{(-\theta_*(f), f) \ : \ f \in \mathcal{C}(S) \cap \mathcal{Q}(Z')\right\}$).

ii) Conversely, suppose that there exists a morphism $\theta : Z' \to T_0$ such that the restriction to Z'_0 is the identity, and such that

$$\mathcal{C}(S) = \mathbb{Q}^+ \left\{ (-\theta_*(f), f) \ : \ f \in \mathcal{C}(S) \cap \mathcal{Q}(Z') \right\} + \mathbb{Q}^+ \mathcal{F} \, ,$$

with $\mathcal{C}(A_S) = \mathcal{C}(S) \cap \mathcal{Q}(Z')$, and $\theta_*(\mathcal{C}(A_S)) \subset C_0$. Then S is a very flat monoid.

In particular, S is very flat if and only if $S \cong A_S \times_{\mathbb{A}^l} \mathrm{Env}(G_0)$.

PROOF. The proof of i) is carried on straightforward from [15, §4].

In order to prove ii), we observe that if $\varphi : A_S \to \mathbb{A}^l$ is the morphism induced by $\theta_* : \mathcal{C}(A_S) \to C_0 = \mathcal{C}(\mathbb{A}^l)$, then by Propositions 2 and 1, $\widehat{S} = (A_S \times_{\mathbb{A}^l} \mathrm{Env}(G_0))$ is a very flat monoid, with abelianisation A_S (recall that $\pi : \mathrm{Env}(G_0) \to \mathbb{A}^l$ has normal irreducible fibres). We claim that $\widehat{S} \cong S$. In particular, the last assertion of the theorem is proved.

In order to prove the assertion, first we show that $\widehat{G}$, the unit group of $\widehat{S}$, is isomorphic to G'.

We know that $G = G(\mathrm{Env}(G_0)) \cong (G_0 \times T_0)/Z_0$. In order to avoid confusion, we note Z the connected center of G, and $\kappa : Z \to T_0$ the induced isomorphism. By Proposition 1, $\widehat{G} = G(A_S \times_{\mathbb{A}^l} \mathrm{Env}(G_0)) = (Z'/Z'_0) \times_{Z/Z_0} G$. Consider

$$\gamma : (G_0 \times Z')/Z'_0 \to (Z'/Z'_0) \times_{Z/Z_0} G = \widehat{G} \quad , \quad \overline{(g_0, z')} \mapsto \left(\overline{z'}, \overline{(g_0, \kappa^{-1}\theta(z'))} \right) \, .$$

The morphism γ is well defined. Indeed, for all $z' \in Z'$, $z'_0 \in Z'_0$ and for all $g_0 \in G_0$ (recall that $Z'_0 \subset Z_0$),

$$\pi \left(\overline{(z'_0 g_0, \kappa^{-1}\theta(z' {z'_0}^{-1}))} \right) = \overline{\kappa^{-1}\theta(z' {z'_0}^{-1})} = \overline{\kappa^{-1}\theta(z')} \, .$$

It follows that

$$\left(\overline{z'}, \overline{(g_0, \kappa^{-1}\theta(z'))} \right) = \left(\overline{z' {z'_0}^{-1}}, \overline{(z'_0 g_0, \kappa^{-1}\theta(z' {z'_0}^{-1}))} \right) \, .$$

We claim that γ is an isomorphism. Indeed,

$$\left(\overline{z'}, \overline{(g_0, \kappa^{-1}\theta(z'))} \right) = \left(\overline{1}, \overline{(1, 1)} \right)$$

if and only if $z' \in Z'_0$ and $\overline{(1, 1)} = \overline{(g_0, \kappa^{-1}\theta(z'))}$. Then, $\kappa^{-1}\theta(z') = z'$, and $\overline{(1, 1)} = \overline{(g_0, z')} = \overline{(g_0 {z'}^{-1}, 1)}$. It follows that $g_0 {z'}^{-1} = 1$, and $\overline{(g_0, z')} = \overline{(1, 1)}$; i.e., γ is injective.

Consider an element $\left(\overline{z'}, \overline{(g_0, z)}\right) \in (Z'/Z'_0) \times_{Z/Z_0} G$; then $\overline{\kappa^{-1}\theta(z')} = \gamma(\overline{z'}) = \pi\left(\overline{(g_0, z)}\right) = \overline{z}$, that is, there exists $z_0 \in Z_0$ such that $\kappa^{-1}\theta(z') = zz_0$, so

$$\left(z', \overline{(g_0, z)}\right) = \left(z', \overline{(g_0 z_0^{-1}, zz_0)}\right) = \gamma(g_0 z_0^{-1}, z') .$$

It follows that γ is surjective, so it is an isomorphism.

By Proposition 1, $\widehat{Z} = (Z'/Z'_0) \times_{Z/Z_0} Z = (Z'/Z'_0) \times_{Z/Z_0} \kappa^{-1}\theta(Z')$, which is isomorphic to Z'. This isomorphism is given by $Z' \ni z' \mapsto \left(\overline{z'}, \overline{(1, \kappa^{-1}\theta(z'))}\right)$. Moreover,

$$\mathbb{Q}^+\mathcal{B}(\widehat{S}) = \left\{(-\widehat{\kappa}_*(f), f) \in \mathcal{Q}(G_0) \times \mathcal{Q}(\widehat{Z}) \ : \ f \in \mathcal{C}(A_S)\right\}$$

where $\widehat{\kappa}(r, z) = \kappa(z)$. Then, $\widehat{\kappa}\circ\gamma(z') = \kappa(\kappa^{-1}\theta(z')) = \theta(z')$ for all $z' \in Z'$, so $\mathbb{Q}^+\mathcal{B}(S) = \mathbb{Q}^+\mathcal{B}(\widehat{S})$. It follows that $S \cong \widehat{S}$. □

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Alvaro Rittatore
alvaro@cmat.edu.uy
Facultad de Ciencias
Universidad de la República
Montevideo
Uruguay

El 2-subgrupo de Sylow del grupo de clases de ideales de un orden cuadrático real.

Gonzalo Tornaría

RESUMEN

Presentamos un algoritmo que permite calcular efectivamente una base del 2-subgrupo de Sylow del grupo de clases de ideales de un orden cuadrático real.

1 Introducción

En [5], Shanks propone el siguiente problema: dado un cuerpo cuadrático imaginario K de discriminante $\Delta < 0$, donde se conoce completamente la factorización de Δ, calcular el 2-subgrupo de Sylow de su grupo de clases de ideales. Para estudiar este problema, es posible trabajar con clases de formas cuadráticas binarias enteras en vez de clases de ideales, dado que los grupos correspondientes son isomorfos.

El famoso Teorema de Duplicación de Gauss [2] afirma que una forma cuadrática tiene raíz cuadrada si y sólo si está en el género principal. La demostración dada por Gauss es constructiva, utiliza formas cuadráticas ternarias, y la eficiente reducción de tales formas. Basándose en estas ideas Shanks describe un algoritmo para la extracción de raíces cuadradas de formas cuadráticas, y con él resuelve el problema planteado de forma eficiente para el caso en que el grupo es cíclico o de la forma $\mathsf{C}(2) \times \mathsf{C}(2^n)$. En el caso general propone utilizar el algoritmo de raíz cuadrada para construir explícitamente todas las formas del 2-subgrupo de Sylow, pero observa que esto es ineficiente, y deja planteada la cuestión de resolver el problema con el mínimo número posible de operaciones de raíz cuadrada.

Lagarias [3] describe un algoritmo para construir una base de un p-grupo abeliano finito en el que podemos extraer raíces p-ésimas. Aplicando este algoritmo al 2-subgrupo de Sylow del grupo de clases de formas cuadráticas binarias enteras de discriminante $\Delta < 0$, se resuelve el problema general de manera eficiente.

Es posible aplicar el algoritmo cuando $\Delta > 0$, obteniéndose una base del 2-subgrupo de Sylow del grupo de clases de formas cuadráticas de discriminante Δ. Esto está estrechamente vinculado con el grupo de clases de un orden cuadrático real, pero en este caso no siempre hay un isomorfismo. Cuando la norma de la unidad fundamental es positiva, el grupo de clases del orden cuadrático será un cociente del grupo de clases de formas cuadráticas por un subgrupo de orden 2.

Morton [4] describe también el algoritmo de la base, y lo aplica directamente al 2-subgrupo del grupo de clases de ideales de un orden cuadrático cualquiera, pero tomando las clases de ideales en el sentido restringido, y por lo tanto trabajando efectivamente con el grupo de clases de formas cuadráticas.

Además, el método utilizado para calcular raíces cuadradas de clases de ideales depende de resolver ciertas ecuaciones ternarias, para lo que no se conocen algoritmos eficientes.

En este trabajo presentaremos un algoritmo para construir una base del cociente de un p-grupo abeliano finito por un subgrupo de orden p dado por un generador, y mostraremos como aplicar este algoritmo al 2-subgrupo de Sylow del grupo de clases de formas cuadráticas binarias enteras de discriminante $\Delta > 0$, con lo que se determina el 2-subgrupo de Sylow del grupo de clases de un orden cuadrático real cualquiera, de manera eficiente.

2 El algoritmo de la base

Sea $\mathfrak{H}$ un p-grupo abeliano finito con elemento identidad 1, cuya operación de grupo denotaremos multiplicativamente.

DEFINICIÓN 2.1 Decimos que un conjunto $\{b_1, b_2, \ldots, b_g\}$ de elementos de $\mathfrak{H}$, de órdenes $\{p^{s_1}, p^{s_2}, \ldots, p^{s_g}\}$ respectivamente, es una *base* de $\mathfrak{H}$ si todo elemento $h \in \mathfrak{H}$ puede ser expresado de forma única como

$$h = \prod_{j=1}^{g} (b_j)^{\lambda_j}, \qquad\qquad 0 \leq \lambda_j < p^{s_j}.$$

Si además se cumple que $s_1 \leq s_2 \leq \cdots \leq s_g$, decimos que $\{b_1, b_2, \ldots, b_g\}$ es una *base ordenada*.

En lo que sigue utilizaremos la siguiente notación:

$$\mathfrak{H}^p = \{h^p \mid h \in \mathfrak{H}\},$$
$$\mathfrak{H}_l = \left\{h \in \mathfrak{H} \mid h^{p^l} = 1\right\},$$

y denotaremos $\mathfrak{X}$ al grupo de carácteres de $\mathfrak{H}$ de orden p, es decir, los homomorfismos $\chi : \mathfrak{H} \to \mathbb{F}_p$. Al menor l para el cual $\mathfrak{H}_l = \mathfrak{H}$ lo llamaremos el *exponente* de $\mathfrak{H}$.

El algoritmo de la base permite calcular una base ordenada de un p grupo abeliano finito $\mathfrak{H}$ cuando conocemos:

- Una base $\{\chi_1, \chi_2, \ldots, \chi_g\}$ del grupo $\mathfrak{X}$ de carácteres de $\mathfrak{H}$ de orden p, y un algoritmo para evaluar $\chi_j(h)$, cualquiera sea j, y cualquiera sea $h \in \mathfrak{H}$.

- Un conjunto $\{t_1, t_2, \ldots, t_n\}$, generador del subgrupo $\mathfrak{H}_1$ de elementos de orden p en $\mathfrak{H}$.

- Un algoritmo que dado $h \in \mathfrak{H}^p$, encuentre un elemento $k \in \mathfrak{H}$ tal que $k^p = h$. Es decir, que calcule raíces p-ésimas en $\mathfrak{H}$.

Este algoritmo será útil cuando $\mathfrak{H}$ no es dado explícitamente, como es el caso cuando se trata del 2-subgrupo de Sylow de un grupo de clases de formas cuadráticas. Destacamos que el algoritmo no efectúa ninguna comparación entre los elementos del grupo. Esto es de suma importancia pues permite trabajar con clases de equivalencia para las que no tenemos representantes canónicos, como es el caso de las clases de formas cuadráticas indefinidas.

Observemos que el grupo $\mathfrak{H}/\mathfrak{H}^p$ puede ser considerado como un espacio vectorial sobre el cuerpo finito $\mathbb{F}_p$, ya que todos sus elementos tienen orden p. Definimos entonces los invariantes

$$r_l = \dim \pi(\mathfrak{H}_l),$$

donde $\pi : \mathfrak{H} \longrightarrow \mathfrak{H}/\mathfrak{H}^p$ es la proyección canónica.

El siguiente teorema nos da un criterio para reconocer una base ordenada de $\mathfrak{H}$:

TEOREMA 2.2 (LAGARIAS [3, THEOREM 3.3]) *Sea $\mathfrak{H}$ un p-grupo abeliano y sea $\pi : \mathfrak{H} \longrightarrow \mathfrak{H}/\mathfrak{H}^p$ la proyección canónica. Entonces $\{b_1, b_2, \ldots, b_g\}$ es una base ordenada de $\mathfrak{H}$ si y sólo si, para todo l,*

1. $\{b_1, b_2, \ldots, b_{r_l}\} \subseteq \mathfrak{H}_l$,
2. $\{\pi(b_1), \pi(b_2), \ldots, \pi(b_{r_l})\}$ *es una base de* $\pi(\mathfrak{H}_l)$. □

La clave para poder aplicar este teorema es que la base dada para el grupo $\mathfrak{X}$ induce una base dual para $\mathfrak{H}/\mathfrak{H}^p$. En coordenadas con respecto a esta base, la proyección canónica $\pi : \mathfrak{H} \to \mathfrak{H}/\mathfrak{H}^p$ se puede calcular fácilmente como

$$\pi(h) = (\chi_1(h), \chi_2(h), \ldots, \chi_g(h)).$$

Comenzando con $b_j = t_j$ para $j = 1, 2, \ldots, n$, bastará con aplicar el método de eliminación de Gauss a $\{\pi(b_1), \pi(b_2), \ldots, \pi(b_n)\}$ para obtener una base de

$\pi(\mathfrak{H}_1)$, y $n - r_1$ vectores nulos, que corresponderán a elementos en $\mathfrak{H}^p$. Calculando las raíces p-ésimas de estos últimos elementos, obtendremos un conjunto generador de $\mathfrak{H}_2$, que cumplirá el criterio del teorema 2.2 con $l = 1$. Repitiendo este procedimiento tantas veces como el exponente de $\mathfrak{H}$, obtendremos una base ordenada.

DEFINICIÓN 2.3 Decimos que una matriz $g \times n$,

$$\mathbf{M} = (m_{i,j}),$$

es *escalonada por columnas* si existe $r \leq n$, al que llamamos *rango* de $\mathbf{M}$, tal que

1. Para $1 \leq j \leq r$, $m_{j,j} \neq 0$, y $m_{i,j} = 0$ si $i < j$.
2. Las últimas $n - r$ columnas de $\mathbf{M}$ son iguales a 0.

El siguiente algoritmo tiene la propiedad de no modificar las primeras columnas de $\mathbf{M}$ si ya están escalonadas:

ALGORITMO 2.4 (ESCALONAR POR COLUMNAS)
Dada una matriz $g \times n$ sobre un cuerpo,

$$\mathbf{M} = (m_{i,j}),$$

este algoritmo devuelve una matriz $\mathbf{T}$ escalonada por columnas tal que $\mathbf{T} = \mathbf{PMR}$ con $\mathbf{P}$ una matriz $g \times g$ de permutación y $\mathbf{R}$ una matriz $n \times n$ de determinante 1.

1. Hacer $k \leftarrow 1$.
2. Si $m_{k,k} = 0$, buscar un índice (i, j) con $k \leq i \leq g$, $k \leq j \leq n$, tal que $m_{i,j} \neq 0$, e intercambiar las filas k e i y las columnas k y j.
   Si un tal índice no existe, devolver $\mathbf{M}$, de rango $k - 1$.
3. Para $j > k$, sumar $-\frac{m_{k,j}}{m_{k,k}}$ veces la columna k a la columna j.
4. Hacer $k \leftarrow k + 1$ y volver al paso 2.

Ahora podemos describir el algoritmo de la base, que coincide esencialmente con el dado por Lagarias en [3, pp. 494–495].

ALGORITMO 2.5 (BASE DE UN p-GRUPO ABELIANO FINITO)
Sea $\mathfrak{H}$ un p-grupo abeliano finito. Dados una base de $\mathfrak{X}$, un conjunto generador de $\mathfrak{H}_1$, y un algoritmo para calcular raíces p-ésimas en $\mathfrak{H}^p$, este algoritmo devuelve una base ordenada de $\mathfrak{H}$.

1. Para $j = 1, 2, \ldots, n$, hacer $b_j \leftarrow t_j$, $s_j \leftarrow 1$, donde $\{t_1, t_2, \ldots, t_n\}$ es el conjunto generador de $\mathfrak{H}_1$.
   Calcular la matriz $g \times n$ dada por
   $$\mathbf{M} = (\chi_i(b_j)), \tag{2.1}$$
   donde $\{\chi_1, \chi_2, \ldots, \chi_g\}$ es la base de $\mathfrak{X}$.

2. Usando el algoritmo 2.4 sobre el cuerpo $\mathbb{F}_p$, calcular a partir de $\mathbf{M}$ una matriz $\mathbf{T}$ escalonada por columnas, de rango r.
   Al hacerlo, cada vez que se intercambian las filas k e i se intercambian también χ_k con χ_i, cada vez que se intercambian las columnas k y j se intercambian b_k con b_j, y al sumar d veces la columna k a la columna j se multiplica b_j por $(b_k)^d$.

3. Si $r = g$, devolver $\{b_1, b_2, \ldots, b_g\}$, de órdenes $\{p^{s_1}, p^{s_2}, \ldots, p^{s_g}\}$ respectivamente.

4. Para $j = r+1, \ldots, n$, remplazar b_j por una de sus raíces p-ésimas y hacer $s_j \leftarrow s_j + 1$.
   Recalcular $\mathbf{M}$ como en (2.1), observando que las primeras r columnas coinciden con las de $\mathbf{T}$, y volver al paso 2.

TEOREMA 2.6 (LAGARIAS [3, THEOREM 3.4]) *En el algoritmo 2.5:*

1. *Después de la l-ésima pasada por el paso 2, $\{b_1, b_2, \ldots, b_n\}$ genera $\mathfrak{H}_l$, y $\{\pi(b_1), \pi(b_2), \ldots, \pi(b_r)\}$ es una base de $\pi(\mathfrak{H}_l)$.*
   El orden de b_j es p^{s_j} para $j = 1, 2, \ldots, r$, y estos elementos quedan fijos hasta el final del algoritmo. Además $s_{r+1} = s_{r+2} = \cdots = s_n = l$.
   En particular $b_j \in \mathfrak{H}_{s_j}$ para todo j.

2. *El algoritmo termina después de pasar s veces por el paso 2, donde s es el exponente de $\mathfrak{H}$.*

3. *Al terminar, $\{b_1, b_2, \ldots, b_g\}$ es una base ordenada de $\mathfrak{H}$.* □

3 El cociente por un subgrupo de orden p.

Si $\{b_1, b_2, \ldots, b_g\}$ es una base ordenada del p-grupo abeliano $\mathfrak{H}$, y $\mathfrak{N} \subseteq \mathfrak{H}$ es un subgrupo, no es cierto en general que $\{b_1\mathfrak{N}, b_2\mathfrak{N}, \ldots, b_g\mathfrak{N}\}$ sea una base ordenada de $\mathfrak{H}/\mathfrak{N}$.

Mostraremos aquí como modificar el algoritmo 2.5 cuando $\mathfrak{N}$ es un grupo de orden p generado por

$$t = \prod_{j=1}^{n} (t_j)^{\lambda_j},$$

para que la proyección sobre $\mathfrak{N}$ de la base obtenida sea una base ordenada de $\mathfrak{H}/\mathfrak{N}$. Observemos que no hay pérdida de generalidad en suponer conocida una expresión de t en términos del conjunto generador $\{t_1, t_2, \ldots, t_n\}$, pues en caso contrario bastará con agregar el propio t al conjunto generador.

A estos efectos, tendremos una matriz $n \times n$ con coeficientes en $\mathbb{F}_p$, que llevará la cuenta de las combinaciones efectuadas con los b_j en el paso 2. Como resultado adicional se obtendrá, al finalizar el algoritmo, un sistema completo de $n - g$ relaciones independientes en el conjunto generador de $\mathfrak{H}_1$.

Por el teorema 2.6, durante el algoritmo siempre tenemos que $b_j \in \mathfrak{H}_{s_j}$, por lo que

$$(b_j)^{p^{s_j - 1}} \in \mathfrak{H}_1.$$

Los coeficientes de la matriz

$$\mathbf{U} = (u_{i,j}),$$

verificarán

$$b_j^{p^{s_j-1}} = \prod_{i=1}^{n} (t_i)^{u_{i,j}}. \tag{3.1}$$

La matriz $\mathbf{U}$ tendrá rango n, y podremos, al finalizar el algoritmo, encontrar coeficientes $\alpha_i \in \mathbb{F}_p$ tales que

$$\mathbf{U} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix},$$

o lo que es lo mismo, que

$$\prod_{j=1}^{n} (b_j)^{\alpha_j p^{s_j - 1}} = \prod_{j=1}^{n} (t_j)^{\lambda_j} = t \in \mathfrak{N}.$$

Sea i el menor índice para el que $\alpha_i \neq 0$. Si $i > g$, esto significa que $t = 1$, y por lo tanto el subgrupo $\mathfrak{N}$ es trivial. En otro caso podemos asumir que $s_{i-1} < s_i$, permutando algunos b_j si fuera necesario, sin que la base deje de estar ordenada.

Remplazamos entonces b_i por

$$b'_i = \prod_{j=i}^{g} (b_j)^{\alpha_j p^{s_j - s_i}},$$

que cumple $(b'_i)^{s_i - 1} \in \mathfrak{N}$.

LEMA 3.1 *El conjunto* $\{b_1, b_2, \ldots, b_g\}$ *así obtenido es una base ordenada de* $\mathfrak{H}$, *y* $\{b_1\mathfrak{N}, b_2\mathfrak{N}, \ldots, b_g\mathfrak{N}\}$ *es una base ordenada de* $\mathfrak{H}/\mathfrak{N}$.

El orden de b_j *es* p^{s_j}, *mientras que el orden de* $b_j\mathfrak{N}$ *es* p^{s_j} *cuando* $j \neq i$, *y el orden de* $b_i\mathfrak{N}$ *es* $p^{s_i - 1}$.

Demostración. Observemos que

$$\pi(b'_i) = \prod_{j=i}^{r_{s_i}} \pi(b_j)^{\alpha_j},$$

y como $\alpha_i \neq 0$, la condición 2 del teorema 2.2 no se ve alterada al sustituir b_i por b'_i.

Además, es claro que $\{b_1\mathfrak{N}, b_2\mathfrak{N}, \ldots, b_g\mathfrak{N}\}$ así obtenido es un conjunto generador de $\mathfrak{H}/\mathfrak{N}$. Como ahora $(b_i)^{s_i - 1} = t$, el orden de $b_i\mathfrak{N}$ es $p^{s_i - 1}$, y por coincidir el orden de $\mathfrak{H}/\mathfrak{N}$ con el producto de los órdenes de los $b_j\mathfrak{N}$, se concluye que éstos forman una base de $\mathfrak{H}/\mathfrak{N}$. Como asumimos que $s_{i-1} < s_i$, se sigue que se trata de una base ordenada. □

Para calcular la matriz $\mathbf{U}$, comenzamos en el paso 1 con $\mathbf{U} = \mathbf{Id}_n$, y en el paso 2 recalculamos las columnas de $\mathbf{U}$ de la siguiente manera:

- Toda vez que se intercambia b_k con b_j, se intercambian las columnas k y j de $\mathbf{U}$.
- Toda vez que se multiplica b_j por $(b_k)^d$, si $s_j = s_k$ se suma d veces la columna k a la columna j de $\mathbf{U}$.

LEMA 3.2 *En todo momento del algoritmo 2.5, la matriz* $\mathbf{U}$ *así calculada verifica las ecuaciones* (3.1), *y tiene rango* n.

Demostración. Es claro que las ecuaciones (3.1) valen después del paso 1. Cuando en el paso 2 se intercambia b_k con b_j y se intercambian las columnas k y j de $\mathbf{U}$, siguen valiendo. Por otra parte, si $b'_j = b_j(b_k)^d$, tenemos que

$$(b'_j)^{p^{s_j - 1}} = (b_j)^{p^{s_j - 1}} (b_k)^{dp^{s_j - 1}} = \prod_{i=1}^{n} (t_i)^{u_{i,j} + \delta d u_{i,k}},$$

donde $\delta = \delta(s_j, s_k)$ vale 1 si $s_j = s_k$ y 0 si $s_j \neq s_k$. Observando que, en cualquier caso $s_j \geq s_k$, es que se sigue que

$$(b_k)^{dp^{s_j-1}} = (b_k)^{\delta dp^{s_k-1}},$$

pues si $s_j > s_k$ entonces el exponente dp^{s_j-1} anula a b_k. Finalmente, al cambiar b_j por una raíz p-ésima e incrementar s_j en el paso 4, el valor de $(b_j)^{p^{s_j-1}}$ no cambia.

La segunda afirmación vale trivialmente al comienzo del algoritmo, y ninguno de los pasos 2 o 4 altera este hecho. □

ALGORITMO 3.3 (BASE DE UN p-GRUPO ABELIANO FINITO, CON COCIENTE) Sea $\mathfrak{H}$ un p-grupo abeliano finito. Dados una base de $\mathfrak{X}$, un conjunto generador de $\mathfrak{H}_1$, un algoritmo para calcular raíces p-ésimas en $\mathfrak{H}^p$, y dado $t = (t_1)^{\lambda_1}(t_2)^{\lambda_2}\cdots(t_n)^{\lambda_n}$, este algoritmo calcula una base ordenada de $\mathfrak{H}$, que al proyectar es base ordenada del cociente $\mathfrak{H}/\langle t\rangle$, devuelve los órdenes respectivos, y determina si $t = 1$ o no.

1. Para $j = 1, 2, \ldots, n$, hacer $b_j \leftarrow t_j$ y $s_j \leftarrow 1$, donde $\{t_1, t_2, \cdots, t_n\}$ es el conjunto generador de $\mathfrak{H}_1$.
   Calcular la matriz $g \times n$ dada por
   $$\mathbf{M} = (\chi_i(b_j)), \tag{3.2}$$
   donde $\{\chi_1, \chi_2, \ldots, \chi_g\}$ es la base de $\mathfrak{X}$.
   Hacer $\mathbf{U} \leftarrow \mathbf{Id}_n$ (coeficientes en $\mathbb{F}_p$).

2. Usando el algoritmo 2.4 sobre el cuerpo $\mathbb{F}_p$, calcular a partir de $\mathbf{M}$ una matriz $\mathbf{T}$ escalonada por columnas, de rango r.
   Al hacerlo, cada vez que se intercambian las filas k e i se intercambian también χ_k con χ_i, cada vez que se intercambian las columnas k y j se intercambian b_k con b_j y las columnas k y j de $\mathbf{U}$, y al sumar d veces la columna k a la columna j se multiplica b_j por $(b_k)^d$, y si $s_j = s_k$ se suma d veces la columna k a la columna j de la matriz $\mathbf{U}$.

3. Si $r = g$, ir al paso 5.

4. Para $j = r+1, \ldots, n$, remplazar b_j por una de sus raíces p-ésimas, y hacer $s_j \leftarrow s_j + 1$.
   Recalcular $\mathbf{M}$ como en (3.2), observando que las primeras r columnas coinciden con las de $\mathbf{T}$, y volver al paso 2.

5. Resolver el sistema lineal

$$\mathbf{U}\begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_n\end{pmatrix}=\begin{pmatrix}\lambda_1\\ \lambda_2\\ \vdots\\ \lambda_n\end{pmatrix}.$$

6. Sea i el menor índice tal que $\alpha_i \neq 0$ Si $i > g$, devolver $\{b_1, b_2, \ldots, b_g\}$, de órdenes $\{p^{s_1}, p^{s_2}, \ldots, p^{s_g}\}$ respectivamente, afirmando que $t = 1$.

7. En otro caso, sea k el menor índice tal que $s_k = s_i$, y hacer

$$(b_k, b_i) \leftarrow \left(\prod_{j=i}^{g} b_j^{\alpha_j p^{s_j - s_i}}, b_k\right).$$

   Devolver $\{b_1, b_2, \ldots, b_g\}$, de órdenes $\{p^{s_1}, p^{s_2}, \ldots, p^{s_g}\}$ respectivamente, y afirmar que $t \neq 1$, y que $\{b_1\langle t\rangle, b_2\langle t\rangle, \ldots, b_g\langle t\rangle\}$ es base ordenada de $\mathfrak{H}/\langle t\rangle$, con órdenes $\{p^{s_1}, p^{s_2}, \ldots, p^{s_k - 1}, \ldots, p^{s_g}\}$.

LEMA 3.4 *Al terminar el algoritmo, las últimas $n - g$ columnas de la matriz* $\mathbf{U}$ *serán los coeficientes de un sistema completo de relaciones independientes en el conjunto* $\{t_1, t_2, \ldots, t_n\}$.

Demostración. Al finalizar tendremos que $\pi(b_{g+1}) = \cdots = \pi(b_n) = 0$, y $s_{g+1} = \cdots = s_n = s$, donde s es el exponente de $\mathfrak{H}$, por lo que

$$(b_j)^{p^{s_j - 1}} = 1,$$

para $j = g + 1, \ldots, n$. Por las ecuaciones (3.1), las últimas $n - g$ columnas de la matriz $\mathbf{U}$ nos dan entonces $n - g$ relaciones en el conjunto $\{t_1, t_2, \ldots, t_n\}$, que serán independientes por el lema 3.2. Como $\mathfrak{H}_1$ tiene dimensión g, estas relaciones generan todas las posibles. □

4 El 2-subgrupo de Sylow para un orden cuadrático real

El algoritmo 2.5 puede aplicarse directamente al 2-subgrupo de Sylow del grupo $\mathcal{F}(\Delta)$ de clases de formas cuadráticas binarias enteras de discriminante Δ. En efecto, la teoría de géneros de Gauss [2] nos permite dar explícitamente una base del grupo de carácteres cuadráticos de $\mathcal{F}(\Delta)$ y un conjunto generador del subgrupo de clases de orden 2 [3, pp. 499-500], y un algoritmo para extracción de raíces cuadradas [5]. Las referencias citadas consideran solamente el caso de formas cuadráticas binarias enteras en el sentido clásico; para el caso general ver [6].

Podemos relacionar el grupo de clases de ideales $\mathbf{C}(\mathcal{O})$ de un orden cuadrático $\mathcal{O}$ de discriminante Δ con el grupo $\mathcal{F}(\Delta)$:

TEOREMA 4.1 *Sea $\mathcal{O}$ el orden cuadrático de discriminante Δ.*

1. *Si $f = [a, b, c]$ es una forma cuadrática primitiva de discriminante Δ, entonces*
$$\mathcal{I}(f) = \left\langle a, \frac{-b+\sqrt{\Delta}}{2} \right\rangle_{\mathbb{Z}}$$
es un ideal propio de $\mathcal{O}$.

2. *La aplicación $f \mapsto \mathcal{I}(f)$ induce un epimorfismo*
$$\mathcal{I} : \mathcal{F}(\Delta) \longrightarrow \mathbf{C}(\mathcal{O}).$$
Cuando $\Delta < 0$, se trata de un isomorfismo, mientras que si $\Delta > 0$, su núcleo tiene orden 1 o 2, generado por
$$f_{-1} = \begin{cases} [-1, -1, \frac{\Delta-1}{4}] & si\ \Delta \equiv 1 \pmod 4, \\ [-1, 0, \frac{\Delta}{4}] & si\ \Delta \equiv 0 \pmod 4. \end{cases}$$

Demostración. Ver Cox [1, Theorem 7.7] □

Se sigue del teorema que el cálculo del 2-subgrupo de Sylow para un orden cuadrático imaginario de discriminante Δ es equivalente al cálculo del 2-subgrupo de Sylow para $\mathcal{F}(\Delta)$, pero en el caso de un orden cuadrático real no siempre es así.

En este caso, el algoritmo 3.3 nos permitirá decidir si $\mathcal{F}(\Delta)$ y $\mathbf{C}(\mathcal{O})$ son isomorfos, y en caso que no lo sean calcular el 2-subgrupo de Sylow de este último como el cociente del 2-subgrupo de Sylow de $\mathcal{F}(\Delta)$ por $\langle f_{-1} \rangle$.

EJEMPLO 4.2 Sea $\mathrm{K} = \mathbb{Q}\left(\sqrt{\Delta}\right)$ el cuerpo cuadrático real de discriminante $\Delta = 3110728 = 8 \cdot 17 \cdot 89 \cdot 257$. Determinaremos una base ordenada del 2-subgrupo de Sylow del grupo $\mathbf{C}_{\mathrm{K}}$ de clases de ideales de su orden maximal.

Para esto, trabajaremos con el grupo $\mathcal{F}(\Delta)$ de clases de formas cuadráticas binarias enteras de discriminante Δ. En primer lugar, una base del grupo de carácteres cuadráticos de $\mathcal{F}(\Delta)$ está dada por $\{\chi_{17}, \chi_{89}, \chi_{257}\}$, donde, identificando el grupo aditivo de $\mathbb{F}_2$ con el grupo multiplicativo $\{\pm\}$,
$$\chi_p(f) = \begin{cases} + & \text{si } f \text{ representa residuos cuadráticos módulo } p, \\ - & \text{si } f \text{ representa no residuos cuadráticos módulo } p. \end{cases}$$

					χ_{17}	χ_{89}	χ_{257}
t_1	=	f_{-1}	=	$[-1, 0, 777682]$	+	+	+
t_2	=	f_{17}	=	$[17, 0, -45746]$	+	+	+
t_3	=	f_{89}	=	$[89, 0, -8738]$	+	+	+
t_4	=	f_{257}	=	$[257, 0, -3026]$	+	+	+
t_5	=	$\sqrt{t_1}$	=	$[449, 1518, -449]$	−	+	−
t_6	=	$\sqrt{t_2}$	=	$[446, 1356, -713]$	+	+	+
t_7	=	$\sqrt{t_3}$	=	$[198, 1492, -1117]$	−	+	+
t_8	=	$\sqrt{t_4}$	=	$[97, 1710, -481]$	−	+	−
t_9	=	t_7t_5	=	$[106, 1560, -1597]$	+	+	−
t_{10}	=	t_8t_5	=	$[-121, 1522, 1641]$	+	+	+
t_{11}	=	$\sqrt{t_6}$	=	$[66, 1712, -681]$	+	−	−
t_{12}	=	$\sqrt{t_{10}}$	=	$[93, 1744, -186]$	+	+	−
t_{13}	=	$t_{11}t_9$	=	$[-442, 1020, 1171]$	+	−	+
t_{14}	=	$t_{12}t_9$	=	$[222, 1472, -1063]$	+	+	+

Tabla 4.1: Formas cuadráticas para el ejemplo 4.2.

Un conjunto generador para el subgrupo de clases de orden 2 está dado por $\{t_1, t_2, t_3, t_4\}$ como en la tabla 4.1.

Al inicio del algoritmo, calculamos **M**:

M	t_1	t_2	t_3	t_4
χ_{17}	+	+	+	+
χ_{89}	+	+	+	+
χ_{257}	+	+	+	+
s	1	1	1	1

U	t_1	t_2	t_3	t_4
t_1	1	0	0	0
t_2	0	1	0	0
t_3	0	0	1	0
t_4	0	0	0	1

La matriz **M** ya está escalonada por columnas, con rango $r_1 = 0$. Calculamos entonces las raíces cuadradas de t_1, t_2, t_3, y t_4 (ver tabla 4.1), y recalculamos **M**:

M	t_5	t_6	t_7	t_8
χ_{17}	−	+	−	−
χ_{89}	+	+	+	+
χ_{257}	−	+	+	−
s	2	2	2	2

U	t_5^2	t_6^2	t_7^2	t_8^2
t_1	1	0	0	0
t_2	0	1	0	0
t_3	0	0	1	0
t_4	0	0	0	1

Debemos ahora escalonar **M**. Para esto basta con "sumar" (multiplicar) la primera columna a la tercera y a la cuarta, e intercambiar las filas segunda y tercera y las columnas segunda y tercera. Las matrices $\mathbf{M}$ y $\mathbf{U}$ resultan:

$\mathbf{M}$	t_5	t_9	t_6	t_{10}
χ_{17}	$-$	$+$	$+$	$+$
χ_{257}	$-$	$-$	$+$	$+$
χ_{89}	$+$	$+$	$+$	$+$
s	2	2	2	2

$\mathbf{U}$	t_5^2	t_9^2	t_6^2	t_{10}^2
t_1	1	1	0	1
t_2	0	0	1	0
t_3	0	1	0	0
t_4	0	0	0	1

La matriz $\mathbf{M}$ tiene rango $r_2 = 2$. Calculamos ahora las raíces cuadradas de t_6 y t_{10}, y recalculamos $\mathbf{M}$:

$\mathbf{M}$	t_5	t_9	t_{11}	t_{12}
χ_{17}	$-$	$+$	$+$	$+$
χ_{257}	$-$	$-$	$-$	$-$
χ_{89}	$+$	$+$	$-$	$+$
s	2	2	3	3

$\mathbf{U}$	t_5^2	t_9^2	t_{11}^4	t_{12}^4
t_1	1	1	0	1
t_2	0	0	1	0
t_3	0	1	0	0
t_4	0	0	0	1

La matriz $\mathbf{M}$ se escalona multiplicando la segunda columna a la tercera y a la cuarta. La matriz $\mathbf{U}$ no se ve afectada puesto que $s_2 \neq s_3$ y $s_2 \neq s_4$.

$\mathbf{M}$	t_5	t_9	t_{13}	t_{14}
χ_{17}	$-$	$+$	$+$	$+$
χ_{257}	$-$	$-$	$+$	$+$
χ_{89}	$+$	$+$	$-$	$+$
s	2	2	3	3

$\mathbf{U}$	t_5^2	t_9^2	t_{13}^4	t_{14}^4
t_1	1	1	0	1
t_2	0	0	1	0
t_3	0	1	0	0
t_4	0	0	0	1

Ahora $\mathbf{M}$ tiene rango $r_3 = 3$, y concluimos que el 2-subgrupo de Sylow de $\mathcal{F}(\Delta)$ es

$$\mathsf{C}(4) \times \mathsf{C}(4) \times \mathsf{C}(8),$$

con generadores t_5, t_9, y t_{13}.

Por otra parte, la última columna de $\mathbf{U}$ nos da la relación $t_1 t_4 = t_{14}^4$; como t_{14} es un cuadrado, t_{14}^4 es equivalente a la forma principal, y concluimos que $t_1 = t_4$ es la (única) relación del conjunto $\{t_1, t_2, t_3, t_4\}$. En particular, $t_1 = [-1, 0, 777682]$ no es equivalente a la forma principal $[1, 0, -777682]$. Deducimos por lo tanto que la ecuación de Pell $X^2 - 777682Y^2 = -1$ no tiene solución o, lo que es lo mismo, la norma de la unidad fundamental de K es positiva.

Por el teorema 4.1, debemos hacer el cociente por el subgrupo generado por $t = t_1$. Para esto resolvemos el sistema

$$\mathbf{U}\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{pmatrix} = \begin{pmatrix}1\\ 0\\ 0\\ 0\end{pmatrix},$$

cuya solución es claramente $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (1, 0, 0, 0)$. Concluimos por lo tanto que el 2-subgrupo de Sylow de $\mathbf{C}_K$ es

$$\mathsf{C}(2) \times \mathsf{C}(4) \times \mathsf{C}(8),$$

con generadores

$$\left\langle 449, \frac{-1518 + \sqrt{\Delta}}{2} \right\rangle_{\mathbb{Z}},$$
$$\left\langle 106, \frac{-1560 + \sqrt{\Delta}}{2} \right\rangle_{\mathbb{Z}},$$
$$\left\langle -442, \frac{-1020 + \sqrt{\Delta}}{2} \right\rangle_{\mathbb{Z}},$$

de órdenes 2, 4, y 8, respectivamente.

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Gonzalo Tornaría
tornaria@math.utexas.edu
Department of Mathematics
University of Texas at Austin
Austin, Texas 78712, USA

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