Multipliers and contraints for spline-based methods

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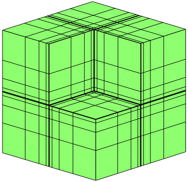
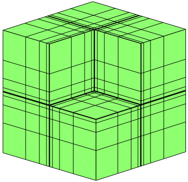
The framework of applications



$$
\begin{aligned}
\nabla \cdot (\mathbf{u} \otimes \mathbf{u}) - \nu \Delta \mathbf{u} + \nabla p &= \mathbf{f} \\
\operatorname{div} \mathbf{u} &= 0
\end{aligned}
$$

$+$ boundary conditions

The framework of applications



$$\boldsymbol{\sigma} = 2\,\mu\,\boldsymbol{\varepsilon} + \lambda\,\nabla\cdot\mathbf{u}\,\mathbf{1}$$
$$\boldsymbol{\varepsilon} = \nabla^s\mathbf{u}$$
$$\mathrm{div}\,(\boldsymbol{\sigma}) = \mathbf{f}$$

$\lambda \to \infty$ incompressible limit

$+$ boundary conditions

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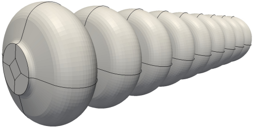


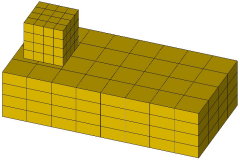
$$\textbf{curl}\,\mathbf{H} = i\,\omega\,\mathbf{D} + \mathbf{J} \qquad \textbf{curl}\,\mathbf{E} = -i\,\omega\,\mathbf{B}$$

$$\mathbf{B} = \mu\mathbf{H} \qquad \mathbf{D} = \varepsilon\mathbf{E}$$

$$\mathrm{div}\,(\mathbf{B}) = 0 \qquad \mathrm{div}\,(\mathbf{D}) = 0$$

+ boundary conditions

The framework of applications



$$\operatorname{div}(\boldsymbol{\sigma}) = \mathbf{f} \qquad \boldsymbol{\sigma} = \mathbb{C}\varepsilon(\mathbf{u})$$
$$\operatorname{div}(\mathbf{u}) = 0 \qquad \mathfrak{g}(\mathbf{u}) \geq 0 \quad \text{contact}$$

$+$ boundary conditions

Constraints and Multipliers

Find $u \in V$ and $p \in M$ such that:

$$\begin{aligned} a(u,v) + b(v,p) &= \langle f, v\rangle && \forall u \in V \\ b(u,q) - \varepsilon(p,q) &= 0 && \forall q \in M \end{aligned}$$

where the bilinear form $\varepsilon(p,q)$ is “small” when the constraint is almost verified as in the case of quasi-incompressible materials.

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Variational framework: Galerkin methods

Find $u_h \in V_h$ and $p_h \in M_h$ such that:

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It is well known (Brezzi 1974 ...) that the stability depends upon

- $a_h(u_h, u_h) \geq \|u_h\|_V^2 \quad \forall\, u_h \in \mathrm{Ker}(B_h)$
- $\displaystyle \inf_{p_h \in M_h} \sup_{u_h \in V_h} \frac{b(u_h, p_h)}{\|u_h\|_V \|p_h\|_M} \geq \alpha > 0$

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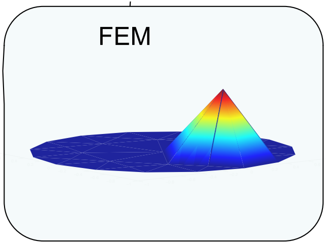
There is a **huge** literature for finite elements!!



- triangulation of the domain
- piecewise polynomials
- C^0 global continuity

What happens for methods based on splines?

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B-Splines are defined by the Cox-DeBoor formulae:

$$N_{i,0}(\zeta) = \begin{cases} 1 & \text{if } \xi_i \leq \zeta < \xi_{i+1}, \\ 0 & \text{otherwise}, \end{cases}$$

$$N_{i,p}(\zeta) = \frac{\zeta - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\zeta) + \frac{\xi_{i+p+1} - \zeta}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\zeta).$$

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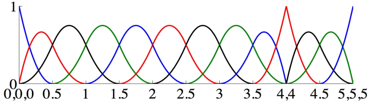
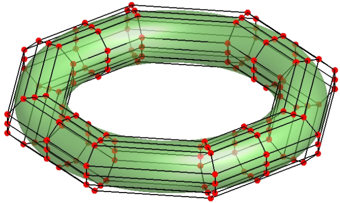
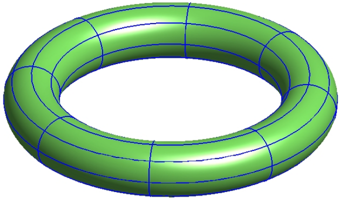
$\mathbf{F}(\xi) = \sum_{\mathbf{i}} \mathbf{C}_i N_{i,p}(\xi)$:

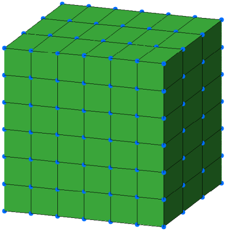




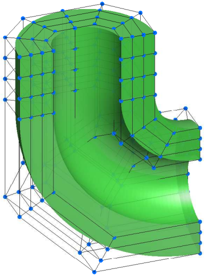


Isogeometric methods

Thomas J.R. Hughes et al 2005 + 650 papers since then



$\xrightarrow{\mathbf{F}}$



- The geometry Ω and its splines parametrization $\mathbf{F}$ is "given" by CAD **general geometry:** unstructured collection of "patches".

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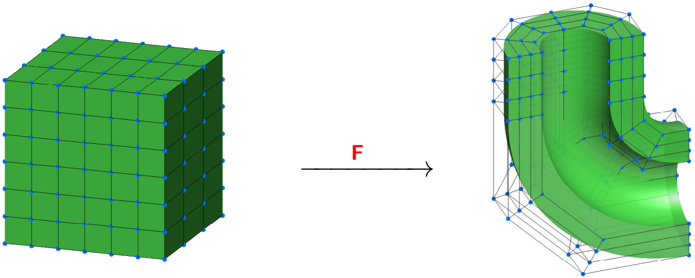
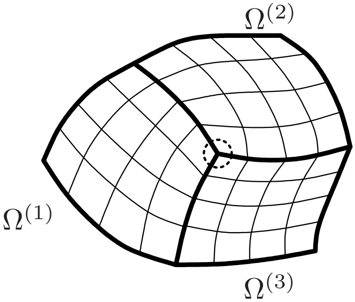


- The geometry Ω and its splines parametrization **F** is "given" by CAD
  general geometry: unstructured collection of "patches".
- The discrete space on Ω is the *push-forward* of Splines

General geometries are multi-patch



Globally unstructured
Locally structured

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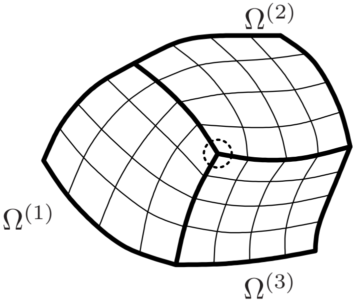

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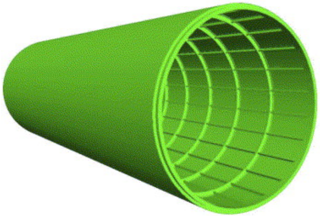
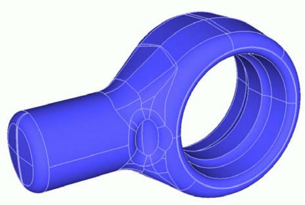


A. Cottrell



V. Skytt

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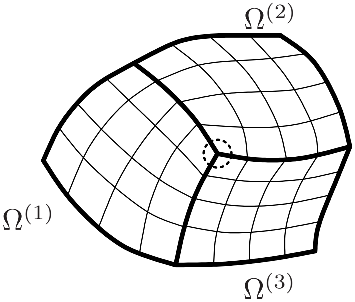

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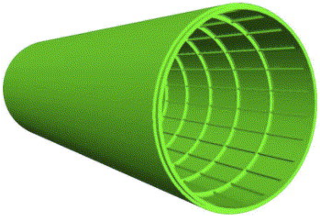

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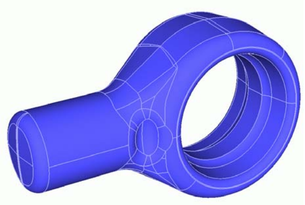


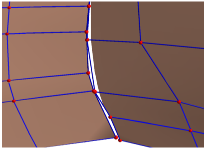
V. Skytt

Question: How to enhance flexibility?
Question: Can these methods be applied in the engineering practise?

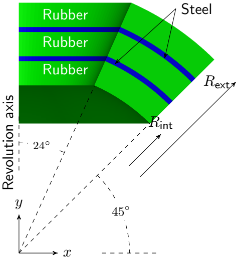
Three main methodologies are needed

Break tensor product structure: Hierarchical splines, T-splines, LR splines ... [Workshop A5]

Mortar Method: gluing subdomains with non-matching grids



Treatment of quasi incompressibility : to simulate e.g., rubber.



Mortar Method

in the spirit of the mortar method by Bernardi, Maday and Patera '91

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Let Ω be a computational domain in $\mathbb{R}^n$, we want to solve the Laplace problem (or linear elasticity with minor changes)

$$-\operatorname{div}(\mathbf{A}\nabla u) = f$$

with boundary conditions $\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_N$.

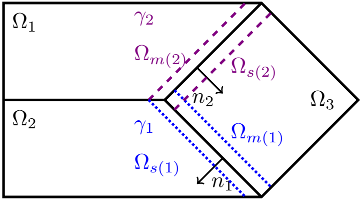
$$u = 0 \text{ on } \Gamma_D \text{ and } (\mathbf{A}\nabla u)\cdot\mathbf{n} = h \text{ on } \Gamma_N$$

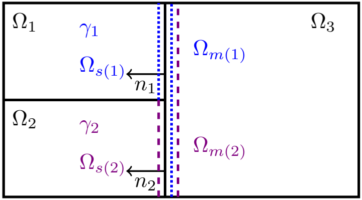
We suppose that

$\Omega = \bigcup_i^N \Omega_i$, $\Omega_i = \mathbf{F}_i(\widehat{\Omega})$, $\Gamma_{ij} = \partial\Omega_i \cap \Omega_j$,

- $\mathbf{F}_i$ are splines
- non compatible meshes at the interfaces Γ_{ij}

About the admissible partition of the domain





- Decomposition can be conforming or non-conforming
- The interface Γ_{ij} is a face of either Ω_i or Ω_j.

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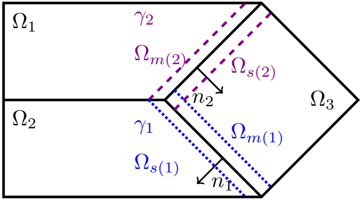

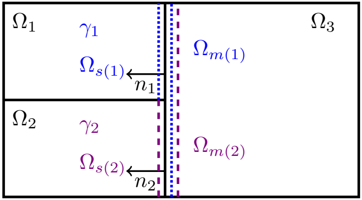

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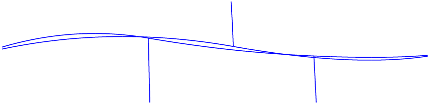
- Non compatible geometries interfaces



Non conforming interfaces and mortaring

Let $S_p(\widehat{\mathcal{T}}_j)$ be the space of tensor product splines of degree p, on the knot mesh $\widehat{\mathcal{T}}_j$.

- in each subdomain Ω_j,

$$V_j = \{v_j \in H^1(\Omega_j) \ : \ v \circ \mathbf{F}_j \in S_p(\widehat{\mathcal{T}}_j)\}$$

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$$V = \{v \in L^2(\Omega) \ : v_{|\Omega_j} \in V_j \,,\ v_{|\Gamma_D} = 0\} \qquad \|v\|_V^2 = \sum_{i=1}^{N} \|v\|^2_{H^1(\Omega_j)}.$$

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Interface numbering and spaces

$$\Sigma_0 = \bigcup_{\ell=1}^{n_I} \Gamma_\ell \,, \qquad \forall \ell \quad \exists (i_\ell, j_\ell) \ : \Gamma_\ell = \partial\Omega_{i_\ell} \cap \Omega_{j_\ell}.$$

Continuity across Σ_0 imposed via Lagrange multipliers:

$$M = \{\lambda \in L^2(\Sigma_0) \ : \ \lambda_\ell = \lambda_{|\Gamma_\ell} \in M_\ell\}$$

M_ℓ to be chosen properly!

Variational formulation of the problem

Find $u_h \in V_h$, $\lambda_h \in M_h$ such that

$$\begin{aligned} a(u_h, v_h) + b(\lambda_h, v_h) &= R(v_h) \qquad && \forall v_h \in V_h \\ b(\mu_h, u_h) &= 0 && \forall \mu_h \in M_h \end{aligned}$$

where

$$a(u,v) = \sum_i \int_{\Omega_i} \mathbf{A} \nabla u \cdot \nabla v \qquad b(\lambda, v) = \sum_\ell \int_{\Gamma_\ell} \lambda_\ell [u] \qquad [u] = u_{i_\ell} - u_{j_\ell}$$

$R(v)$ is the RHS taking into account also Neumann BC...

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Choice of the Langrange multiplier space

- ... I want to have the **largest possible set of multipliers** such that the form $b(\lambda, v) = \int_{\Gamma_\ell} \lambda_\ell [u]$ remains uniformly stable

Favorite choice: if i_ℓ is the one side, we want $M_\ell \sim V_{i_\ell}|_{\Gamma_\ell}$!
It contraints **all** functions on one side !

But .. stability fails! We need:

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Cross point reduction (Bernardi Maday Patera 91)

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Each Γ_ℓ is a face of a subdomain Ω_i (the slave side)

- Γ_ℓ inherits a spline mapping $\mathbf{F}_\ell : (0,1)^{d-1} \to \Gamma_\ell$
- and a parametric mesh on $\widehat{\Gamma} = (0,1)^{d-1}$ denoted as $\widehat{\mathcal{T}}_\ell$.

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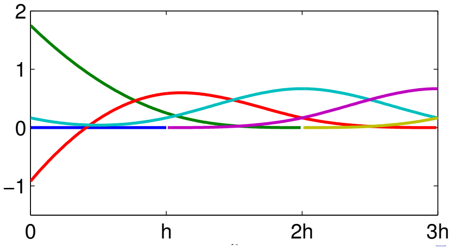
Choice 1: same degree, cross point reduction

$$\widehat{M}^1_\ell = \widetilde{S}_p(\widehat{\mathfrak{T}}_\ell)$$



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Choice 2: degree reduction

$$\widehat{M}_\ell^2 = S_{p-2}(\widehat{\mathcal{T}}_\ell)$$

Indeed, it is true that

$$\dim(\widehat{M}_\ell^2) = \{\widehat{v} \in S_p(\widehat{\mathcal{T}}_{i_\ell})|_{\Gamma_\ell} \ : \ \widehat{v}|_{\partial\Gamma_\ell} = 0\}$$

- No need for degree reduction or other manipulation
- If stable, it will deliver a slightly suboptimal order : 1/2 suboptimal

Stability: Proof of the inf-sup condition

the $p/p-2$ case

We consider $\widehat{M}_\ell^2$ and can prove that:

$$\inf_{\widehat{\mu}\in S_{p-2}} \sup_{\widehat{v}\in S_p\cap H_0^1} \frac{\int_{\widehat{\Gamma}} \widehat{\mu}\,\widehat{v}}{\|\widehat{v}\|_{L^2}\|\widehat{\mu}\|_{L^2}} \geq \alpha_0$$

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Proof

In 2D:

- $S_p \cap H_0^1 \xrightarrow{\partial_x} S_{p-1} \cap L_0^2 \xrightarrow{\partial_x} S_{p-2}$ is exact
- choose $\widehat{v} \in S_p \cap H_0^1$ such that $\partial_{xx}^2 \widehat{v} = \widehat{\mu}$ and the work with Sobolev norms.

In 3D, basically the same applies...

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Proof

In 2D:

- $S_p \cap H_0^1 \xrightarrow{\partial_x} S_{p-1} \cap L_0^2 \xrightarrow{\partial_x} S_{p-2}$ is exact
- choose $\widehat{v} \in S_p \cap H_0^1$ such that $\partial^2_{xx}\widehat{v} = \widehat{\mu}$ and the work with Sobolev norms.

In 3D, basically the same applies...

It is stable! ... we need now to go to physical space

Stability in the physical space

the $p/p-2$ case

$$\inf_{\widehat{\mu}\in S_{p-2}} \sup_{\widehat{v}\in S_p\cap H^1_0} \frac{\int_{\widehat{\Gamma}} \widehat{\mu}\,\widehat{v}}{\|\widehat{v}\|_{L^2}\|\widehat{\mu}\|_{L^2}} \geq \alpha_0$$

$$\Downarrow$$

$$\inf_{\mu\in M_\ell} \sup_{v\in V_{i_\ell}:v\in H^1_0(\Gamma_\ell)} \frac{\int_{\Gamma_\ell} \mu\, v}{\|v\|_{L^2}\|\mu\|_{L^2}} \geq \alpha_0$$

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$\int_{\Gamma_\ell} \mu\, v = \int_{\widehat{\Gamma}} \rho\, \widehat{\mu}\,\widehat{v}$ $\quad \rho =$ weight, area change..

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and by super-convergence results à la Wahlbin:

$$\Pi : L^2(\widehat{\Gamma}) \to \widehat{M}^2_\ell \quad \Rightarrow \quad \|\rho\widehat{\mu} - \Pi(\rho\widehat{\mu})\|_{L^2(\widehat{\Gamma})} \leq Ch\|\widehat{\mu}\|_{L^2(\widehat{\Gamma})}$$

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For h small enough the stability holds in physical space!

Back to our variational problem

Find $u_h \in V_h$, $\lambda_h \in M_h$ such that

$$\begin{aligned} a(u_h, v_h) + b(\lambda_h, v_h) &= R(v_h) & \forall v_h \in V_h \\ b(\mu_h, u_h) &= 0 & \forall \mu_h \in M_h \end{aligned}$$

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It is well-posed and verifies the following error estimate: if $u \in H^r(\Omega)$:

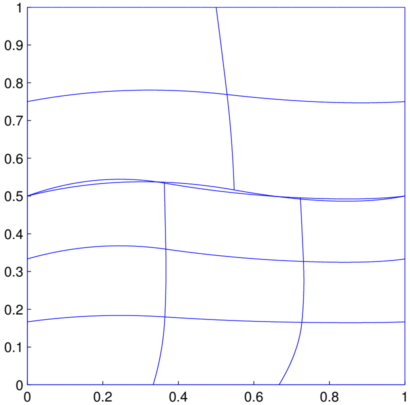
$$\begin{aligned} \|u - u_h\|_V &\le C \inf_{v_h \in V} \|u - v_h\|_V + \inf_{\mu_h \in M} \|\lambda - \mu_h\|_M && (1) \\ &\le Ch^t + Ch^s \qquad t = \min\{p, r-1\} && (2) \end{aligned}$$

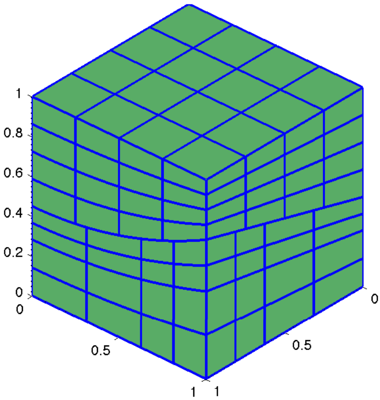
- $s = \min\{p + 1/2, r - 1\}$ for **Choice 1: same degree,**
- $s = \min\{p - 1/2, r - 1\}$ for **Choice 2: degree reduction**

Or, indeed:

$$\|u - u_h\|_V \le C \inf_{v_h \in V \cap \mathrm{Ker}(B)} \|u - v_h\|_V \le C \ldots$$

Numerical validation: problem 1





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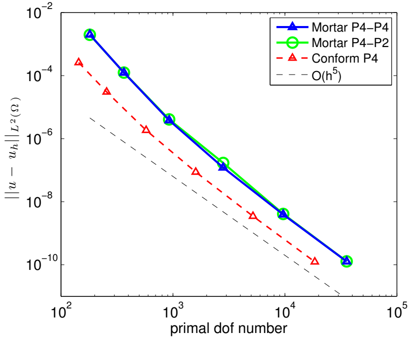
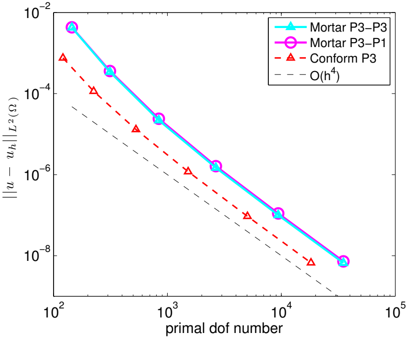
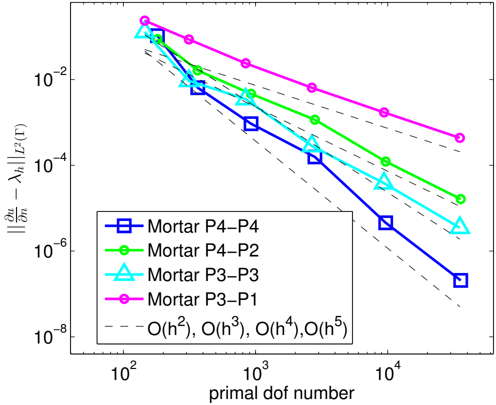




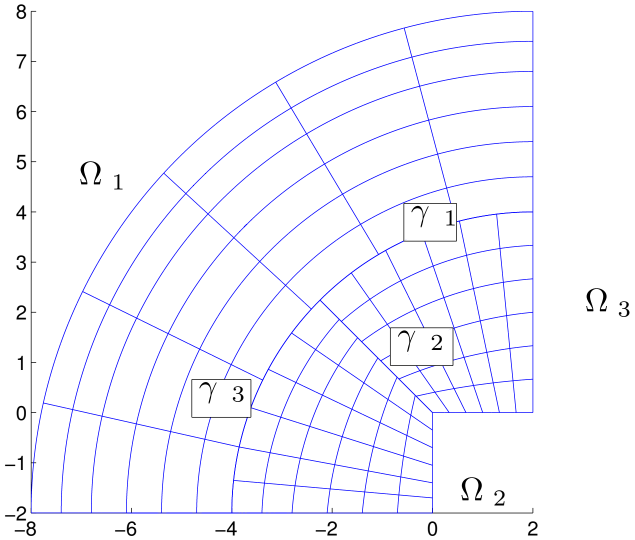
Multipliers' degree **does not** affect the order for the primal unknown!

Numerical validation: problem 1

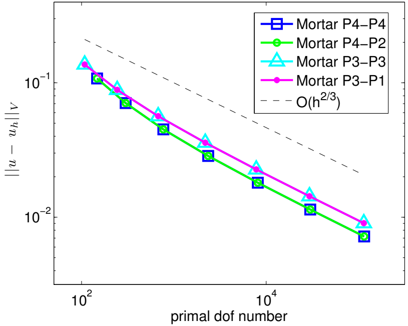


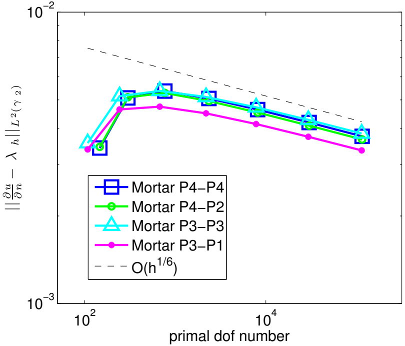
but it affects the convergence of the multiplier!

Numerical validation: problem 2



Numerical validation: problem 2





$$1/6 + 1/2 = 2/3$$

Treatment of incompressibility

Linear elasticity

Strong form problem

$$
\begin{aligned}
\nabla \cdot \boldsymbol{\sigma} + \mathbf{f} &= \mathbf{0} \quad \text{in}\, \Omega \\
\mathbf{u} &= \bar{\mathbf{u}} \quad \text{on}\, \Gamma_D \\
\boldsymbol{\sigma} \cdot \mathbf{n} &= \mathbf{t} \quad \text{on}\, \Gamma_N
\end{aligned}
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Isotropic linear elasticity

$$\boldsymbol{\sigma} = 2\,\mu\,\boldsymbol{\varepsilon} + \lambda\,\nabla \cdot \mathbf{u}\,\mathbf{1}$$

$$\boldsymbol{\varepsilon} = \nabla^s \mathbf{u}$$

$$\lambda = \frac{\nu\, E}{(1+\nu)\,(1-2\,\nu)}$$

$$\mu = \frac{E}{2\,(1+\nu)}$$

$$\nu \to 1/2, \quad \lambda \to \infty$$

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Weak form

$$\underbrace{\int_\Omega \mu\,\nabla^s \mathbf{w} : \nabla^s \mathbf{u}\, d\Omega + \int_\Omega \lambda\,\nabla \cdot \mathbf{w}\,\nabla \cdot \mathbf{u}\, d\Omega}_{a(\mathbf{w},\mathbf{u})} = L\,(\mathbf{w})$$

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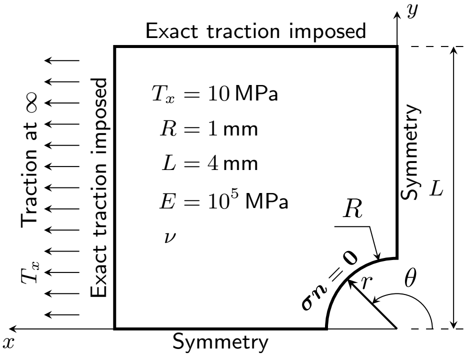
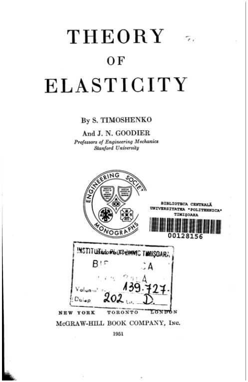
The effect of quasi-incompressibility

Exact versus computed solution





THEORY
OF
ELASTICITY

By S. TIMOSHENKO
And J. N. GOODIER
Professors of Engineering Mechanics
Stanford University

ENGINEERING SOCIETIES MONOGRAPHS

NEW YORK TORONTO LONDON
McGRAW-HILL BOOK COMPANY, Inc.
1951

$$\sigma_{rr}(r,\,\theta) = \frac{T_x}{2}\left[1 - \frac{R^2}{r^2} + \left(1 - 4\frac{R^2}{r^2} + 3\frac{R^4}{r^4}\right)\cos 2\theta\right],$$

$$\sigma_{\theta\theta}(r,\,\theta) = \frac{T_x}{2}\left[1 + \frac{R^2}{r^2} - \left(1 + 3\frac{R^4}{r^4}\right)\cos 2\theta\right],$$

$$\sigma_{r\theta}(r,\,\theta) = -\frac{T_x}{2}\left(1 + 2\frac{R^2}{r^2} - 3\frac{R^4}{r^4}\right)\sin 2\theta\,,$$

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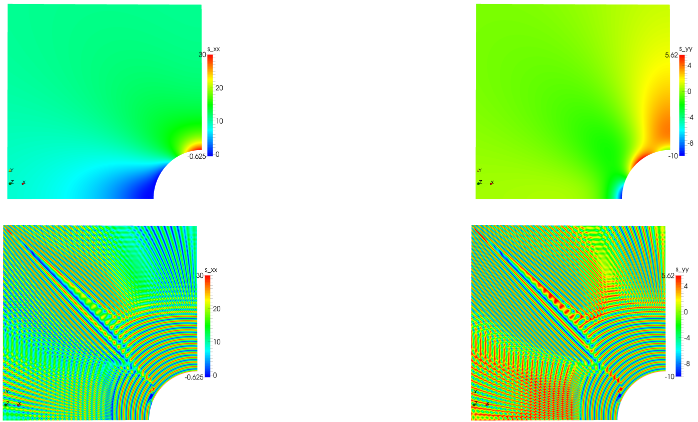

Mixed formulation

Mixed formulation

$$\int_\Omega \mu \, \nabla^s \mathbf{w} : \nabla^s \mathbf{u} \mathrm{d}\Omega + \int_\Omega \nabla \cdot \mathbf{w} \, p \, \mathrm{d}\Omega = \mathbf{L}(\mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{V}$$

$$\lambda \int_\Omega q \, \nabla \cdot \mathbf{u} \mathrm{d}\Omega - \int_\Omega p \, q \, \mathrm{d}\Omega = 0 \qquad \forall q \in M$$

where we can solve for p: $p = \lambda \, \Pi_M (\nabla \cdot \mathbf{u})$

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Primal formulation

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Unlocked solution (M "small") + incompressible (M "large")

Mixed formulation

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Unlocked solution (M "small") + incompressible (M "large")
Sparse (M "discontinuous" or Π_M modified)

Choice of multipliers

Primal formulation

$$a_h(\mathbf{w},\mathbf{u}) = \int_\Omega \mu\, \nabla^s \mathbf{w} : \nabla^s \mathbf{u} d\Omega + \int_\Omega \lambda\, \Pi_M(\nabla \cdot \mathbf{w})\, \Pi_M(\nabla \cdot \mathbf{u})\, d\Omega = L(\mathbf{w})$$

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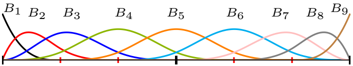
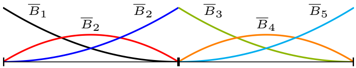
Sparse (M "discontinuous" or Π_M modified)

$\mathbf{u}, \mathbf{w} \in$ 

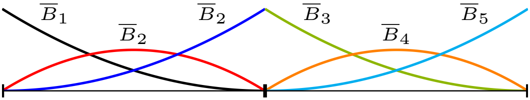
$\Pi_M(\nabla\cdot\mathbf{u}), \Pi_M(\nabla\cdot\mathbf{w}) \in$ 

$$\mathbf{V} = S^p_{p-1}(\mathcal{Q}_h),\ M = S^{p-1}_{-1}(\mathcal{Q}_{p*h})\ \text{(subgrid pressure)}.$$

The matrix representing $a_h(\cdot,\cdot)$ is "almost" as sparse as the one representing $a(\cdot,\cdot)$

Choice of multipliers

$$\overline{B} \in \mathcal{S}^{p-1}_{-1} \quad \Pi_M(\mathbf{x}) = \sum_{i=1}^{N} \overline{B}_i(\mathbf{x}) \left[\sum_{j=1}^{N} \overline{\mathbf{M}}^{-1}_{ij} \int_\Omega \overline{B}_j(\mathbf{x})\, f(\mathbf{x}) \mathrm{d}\Omega \right]$$



$\mathbf{M} =$

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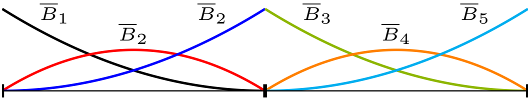
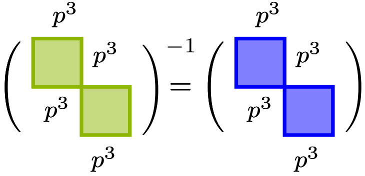


$\mathbf{M} =$



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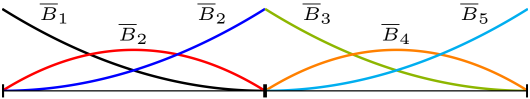
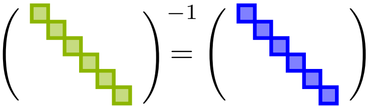


$\mathbf{M} =$



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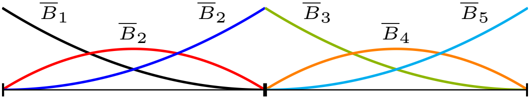
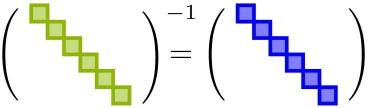


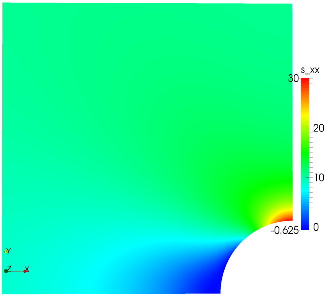
$\mathbf{M} =$



- the proof of inf-sup when the mesh for pressure is coarse enough
  Bressan-Sangalli 2010
- the method is used with the richest possible pressures i.e. $S^{p-1}_{-1}(\mathcal{Q}_{p*h})$.

Numerics

σ_{xx} for with and without subgrids :



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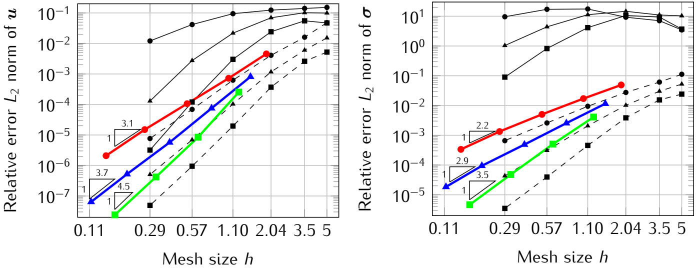
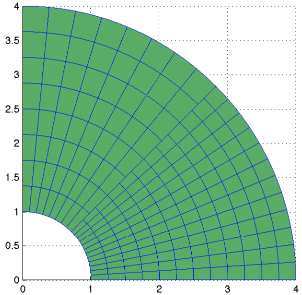


dashed = reference solution with dense matrices

continuous = plain primal formulation

color = our solution for degree $2, 3, 4$.

Incompressibility treatment and Mortar



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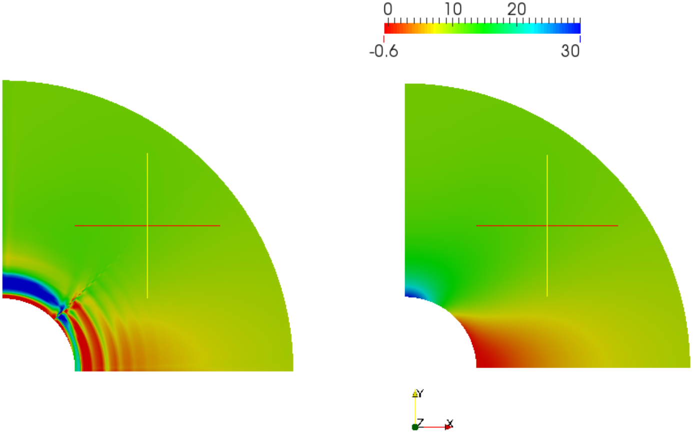


Mortar

Mortar + Subgrid

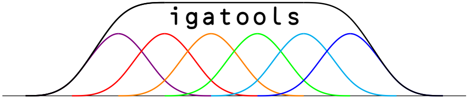
Final remarks

Surveys and Codes

- New **Acta Numerica** survey paper with several math results:

> L. Beirão Da Veiga, A. Buffa, G. Sangalli, R. Vázquez,
> *Mathematical analysis of variational isogeometric methods*

- We have two codes available to public :
  - GeoPDEs library is a GNU licensed software available here: `www.imati.cnr.it/geopdes` R. Vázquez
  - IGATools is a C++ dimension independent library `http://code.google.com/p/igatools` S. Pauletti, M. Martinelli ...



THANKS!

the support of ERC StG 205004 (Buffa), 259229 (Reali), ERC CoG 616563 (Sangalli), of Total SD, Hutchinson SA, Michelin RD, EU 284981 Call FP7-2011-NMP-ICT-FoF is gratefully acknowledged