Combinatorial Algebraic Geometry

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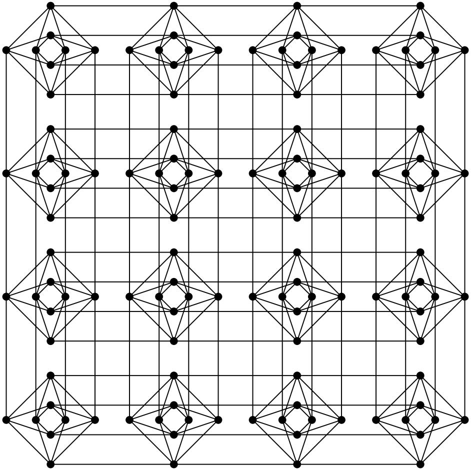
Outline

Theme of the Talk

Combinatorial algebraic geometry is an emerging area of algebraic geometry. This lecture is my attempt to give you a sense of the field.

A Quantum Computer

Here is a diagram from a D-Wave adiabatic quantum computer:



Polytopes are Involved

A Polytope

The complete bipartite graph $K_{n,n}$ has vertices $q_1, \ldots, q_{2n}$ and edges $q_i q_j$, $i = 1, \ldots, n$, $j = n+1, \ldots, 2n$. Consider the 2^{2n} points

$$(q_1, \ldots, q_{2n}, q_1 q_{n+1}, \ldots, q_n q_{2n}) \in \mathbb{R}^{2n+n^2} = \mathbb{R}^{n(n+2)}$$

as $q_1, \ldots, q_{2n}$ independently range over $\{0, 1\}$. The convex hull of these points is a polytope with 2^{2n} vertices.

When $n = 4$, each facet of this polytope gives an Ising model that can be implemented on the D-Wave computer. The idea is to sample from states associated to the Ising model and check if the states returned are ground states of roughly comparble probability.

How Many Facets?

36,391,264, as computed by Mathieu Dutour

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A (Slight) Toric Connection

Toric Varieties

These are algebraic varieties with strong connections to lattice polytopes and polyehdral fans. Toric geometry is an important part of combinatorial algebraic geometry.

Denny Dahl, a physicist at D-Wave, knew about toric varieties from the book I wrote with John Little and Hal Schenck. So he asked me about these polytopes. With help from Greg Smith, I was able to connect Denny with people like Mathieu Dutour, Komei Fukuda and David Avis.

Email from Denny Dahl to David Avis

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An Interesting Secant Variety

- The *d*-th Veronese map $v_d : \mathbb{P}^n \to \mathbb{P}^{N-1}$ is the projective embedding given by all $N = \binom{n+d}{n}$ monomials of degree d.
- The *k*-th secant variety of $v_d(\mathbb{P}^n) \subseteq \mathbb{P}^{N-1}$, denoted $\sigma_k(v_d(\mathbb{P}^n))$, is the Zariski closure of the union of all $(k-1)$-planes formed by k points of $v_d(\mathbb{P}^n)$.
- In 1995, Alexander and Hirschowitz computed the codimension of $\sigma_k(v_d(\mathbb{P}^n))$ in $\mathbb{P}^{N-1}$.

$\sigma_7(v_3(\mathbb{P}^4)) \subseteq \mathbb{P}^{34}$

This secant variety has codimension 1 and degree 15, so it is given by a single equation of degree 15. There are

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monomials of degree 15 in 35 variables.

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16,051 Formulas

In 2009, Ottaviani showed the defining equation $F = 0$ of $\sigma_7(v_3(\mathbb{P}^4)) \subseteq \mathbb{P}^{34}$ is SL_5-invariant, so the number of monomials reduces to

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He also showed that F^3 is the determinant of a 45×45 determinant with linear entries.

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16,051 formulas for Ottaviani's invariant of cubic threefolds by Abdesselam, Ikenmeyer and Royle gives many descriptions of F. They use certain bipartite graphs to encode the information needed to compute the polynomial F.

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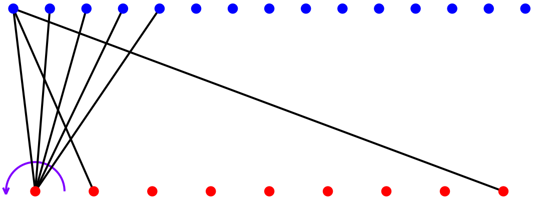
The Bipartite Graph

The Veronese map $v_3 : \mathbb{P}^4 \to \mathbb{P}^{34}$ uses the 35 monomials $y_\alpha = x^\alpha = x_1^{a_1} \cdots x_5^{a_5}$ with $|\alpha| = 3$. Fix a bipartite graph

15 vertices, all degree 3



9 vertices, all degree 5

with oriented edges on the bottom. Color the edges using $1, \ldots, 5$. A vertex $\color{blue}\bullet$ with edges colored i_1, i_2, i_3 gives $x^{\alpha(\color{blue}\bullet)} = x_{i_1} x_{i_2} x_{i_3}$, hence a coordinate $y_{\alpha(\color{blue}\bullet)}$. A vertex $\color{red}\bullet$ with edges colored i_1, i_2, i_3, i_4, i_5 gives

$$\epsilon(\color{red}\bullet) = \begin{cases} 0 & i_j \text{ not distinct} \\ \pm 1 & i_j \text{ distinct} \end{cases}$$

The Formula

Then a 5-coloring of the edges gives a term of degree 15

$$\prod_{9\,\color{red}\bullet} \epsilon(\color{red}\bullet) \cdot \prod_{15\,\color{blue}\bullet} y_{\alpha(\color{blue}\bullet)}$$

The sum of these terms for all 5-colorings of the edges gives a polynomial F of degree 15 in the 35 variables y_α.

Theorem (Abdesselam, Ikenmeyer and Royle)

Fix a bipartite graph as above with no 7-coloring of the vertices and $F \neq 0$. Then:

- *$F = 0$ is the defining equation of $\sigma_7(v_3(\mathbb{P}^4))$.*
- *There are 16,051 such bipartite graphs.*

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Final Comments

Question

$\sigma_7(v_3(\mathbb{P}^4))$ is combinatorial (it is a secant variety of the toric variety of the polytope $3\Delta_4$). The same is true for the bipartite graphs used to create the defining equation $F = 0$. How are these related?

Two Remarks:

- For a parametrized hypersurface, methods from tropical geometry can be used to describe the Newton polytope (i.e., exponent vectors) of the defining equation.
- The methods used by Abdesselam, Ikenmeyer and Royle involve the representation theory of SL_5. The bipartite graphs encode contractions of tensors from $\bigotimes^{45} \mathbb{C}^5$ to $\bigotimes^{15}(Sym^3(\mathbb{C}^5))$.

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The Strand Symmetric Model

The SSM

This phylogenetic model encodes the symmetry of the double-stranded structure of DNA.

Let the phylogenetic tree be the complete bipartite graph $K_{1,3}$ (a claw ). Each of the three edges has four Fourier coordinates. This gives coordinates x_j^i, y_l^k, z_n^m for $i, j, k, l, m, n \in \{0, 1\}$.

The Parametrization

The SSM model is parametrized by

$$q_{jln}^{ikm} = x_{0j}^i y_{0l}^k z_{0n}^m + x_{1j}^i y_{1l}^k z_{1n}^m$$

for all $i, j, k, l, m, n \in \{0, 1\}$ with $i + k + m \equiv 0 \bmod 2$.

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Translate to Algebraic Geometry

The x^i_j, y^k_l, z^m_n are homogenoeous coordinates on 3 copies of $\mathbb{P}^3$, and

$$\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3 \longrightarrow \mathbb{P}^{63}, \quad (x^i_j, y^k_l, z^m_n) \longmapsto (x^i_j y^k_l z^m_n)$$

is the Segre map (the toric variety of $\Delta_3 \times \Delta_3 \times \Delta_3$). Then

$$q^{ikm}_{jln} = x^i_{0j} y^k_{0l} z^m_{0n} + x^i_{1j} y^k_{1l} z^m_{1n} \tag{1}$$

parametrizes the secant variety of the Segre variety of $(\mathbb{P}^3)^3$.

The SSM

The SSM uses (1) for the 32 coordinates that satisfy the congruence $i + k + m \equiv 0 \bmod 2$. Thus the SSM is a coordinate projection of the secant variety of Segre variety of $(\mathbb{P}^3)^3$!

The SSM Claw Equations

The SSM claw variety lives in $\mathbb{P}^{31}$ and has the parametrization

$$q^{ikm}_{jln} = x^i_{0j} y^k_{0l} z^m_{0n} + x^i_{1j} y^k_{1l} z^m_{1n}, \quad i + k + m \equiv 0 \bmod 2.$$

Well-know Gröbner basis algorithms will compute the defining equations of this variety. However, the time required for the computation would be enormous.

- Casanellas and Sullivant, 2005: Found 32 degree 3 equations and 18 degree 4 equations that vanish on the SSM claw model. Do these equations define the variety?
- Long and Sullivant, October 2014: Proved that these 50 equations define the variety. The proof uses a mix of computational and theoretical methods.
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Likelihood Geometry

In June 2013, Bernd Sturmfels lectured on Likelihood Geometry at the CIME-CIRM summer course on Combinatorial Algebraic Geometry. The lecture notes, written with June Huh, are available on arXiv.

From the Abstract

We study the critical points of monomial functions over an algebraic subset of the probability simplex. The number of critical points on the Zariski closure is a topological invariant of that embedded projective variety, known as its maximum likelihood degree.

We present an introduction to this theory and its statistical motivations. Many favorite objects from combinatorial algebraic geometry are featured: toric varieties, *A*-discriminants, hyperplane arrangements, Grassmannians, and determinantal varieties. Several new results are included, especially on the likelihood correspondence and its bidegree.

$M_{0,n}$ and $\overline{M}_{0,n}$

$M_{0,n}$

$M_{0,n}$ is the moduli space of rational curves with $n \geq 3$ marked points. This is an irreducible variety of dimension $n-3$. It is not compact.

We compactify $M_{0,n}$ using the following curves:

Stable Curves of Genus 0 with n Marked Points

Such a curve consists of a tree of $\mathbb{P}^1$'s such that

- No three $\mathbb{P}^1$'s intersect.
- When two $\mathbb{P}^1$'s meet, they intersect transversely.
- For each $\mathbb{P}^1$, the number of marked points plus the number of intersection points is ≥ 3.

This gives the compact moduli space $\overline{M}_{0,n}$. The boundary $\overline{M}_{0,n} \setminus M_{0,n}$ has a wonderful combinatorial stratification.

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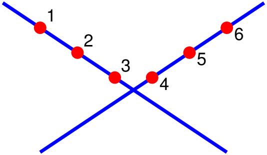
Two Examples when $n = 6$

Here are two genus 0 stable curves with $n = 6$, together with their dual graphs:



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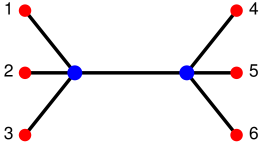

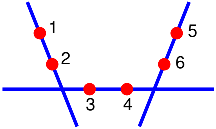

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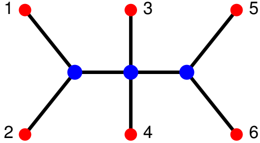


Metric Trees and a Fan

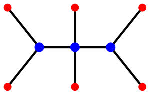
Consider a tree with n labeled leaves and internal vertices have degree ≥ 3. Edges are ends if connected to a leaf and bounded otherwise.

Metric Trees

In a metric tree C, the n ends have length ∞ and bounded edges have length in $\mathbb{R}_{>0}$. $T_{0,n}^{\text{trop}}$ is the set of such trees up to isomorphism.

Let C be a metric tree. Given leaves i, j, let

$$d_{ij} = \text{distance between vertices adjacent to } i, j.$$



Then C gives a point $(d_{ij}) \in \mathbb{R}^{\binom{n}{2}}/L$, $L = \{(x_i + x_j) : (x_1, \dots, x_n) \in \mathbb{R}^n\}$.

$T_{0,n}^{\text{trop}}$ is a Fan

Via the map $C \mapsto (d_{ij})$, we can regard $T_{0,n}^{\text{trop}}$ as a fan Δ in $\mathbb{R}^{\binom{n}{2}}/L$. Each cone of Δ consists of combinatorially equivalent trees.

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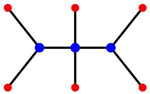

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Constructing $\overline{M}_{0,n}$

The fan Δ in $\mathbb{R}^{\binom{n}{2}}/L \simeq \mathbb{R}^{\binom{n}{2}-n}$ gives the toric variety X_Δ with torus

$$(\mathbb{C}^*)^{\binom{n}{2}-n} \subseteq X_\Delta.$$

Torus orbits stratify X_Δ and correspond to cones of Δ.

Theorem (Gibney-Maclagan)

There is an embedding $M_{0,n} \subseteq (\mathbb{C}^*)^{\binom{n}{2}-n}$ such that:

1. *$\overline{M}_{0,n}$ is the Zariski closure of $M_{0,n}$ in X_Δ.*
2. *The stratification of $\overline{M}_{0,n} \setminus M_{0,n}$ is induced by the torus orbit stratification of X_Δ.*

Furthermore, if a stratum $Y \subseteq \overline{M}_{0,n} \setminus M_{0,n}$ is induced by the torus orbit corresponding to $\sigma \in \Delta$, then:

$C \in Y \Rightarrow$ trees $\in \sigma$ are combinatorially equivalent to dual graph of C.

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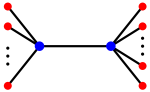
Consequences

The toric variety X_Δ has global coordinates determined by rays of the fan Δ. Using these coordinates, Gibney and Maclagan give explicit equations for $\overline{M}_{0,n} \subseteq X_\Delta$.

However, there are lots of variables since

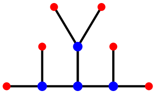
variables $\longleftrightarrow$ rays of Δ $\longleftrightarrow$ graphs



Taking labels into account, there are $2^{n-1} - n - 1$ *variables*.

One can also work locally, since X_Δ is covered by affine toric varieties U_σ for $\sigma \in \Delta$ a maximal cone. However,

maximal cones of Δ $\longleftrightarrow$ phylogenetic trees



Taking labels into account, there are

$$(2n-5)!! = (2n-5)(2n-3)\cdots 5\cdot 3\cdot 1 \text{ maximal cones.}$$

Tropical Geometry

In tropical geometry, one uses $\mathbb{R} \cup \{\infty\}$ with operations

$$\text{addition} : a \oplus b = \min(a, b)$$
$$\text{multiplication} : a \odot b = a + b.$$

$\mathbb{R} \cup \{\infty\}$ is a semiring under these operations with ∞ as additive identity and 0 as multiplicative identity

Example

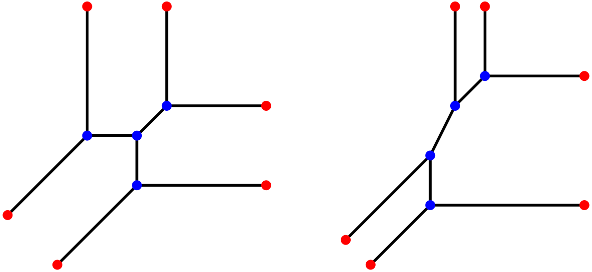
If $p = a \odot x^2 \oplus b \odot xy \oplus c \odot y^2 \oplus d \odot x \oplus e \odot y \oplus f$, then $p = \mathbb{R}^2 \to \mathbb{R}$ is given by

$$p = \min(a + 2x, b + x + y, c + 2y, d + x, e + y, f).$$

The tropical curve $\mathbf{V}(p) \subseteq \mathbb{R}^2$ is the locus of all points in $\mathbb{R}^2$ where p fails to be linear.

Tropical Varieties

For various choices of coefficients $a, \ldots, f$ in the previous example, the tropical curve $\mathbf{V}(p) \subseteq \mathbb{R}^2$ looks like



The edges ending in red go to infinity, i.e., have infinite length, and the other edges are bounded. Look familiar?

Tropicalization

- Given a field K with a non-archimedean valuation and a variety $Y \subseteq (K^*)^n$, one gets $\mathrm{trop}(V) \subseteq \mathbb{R}^n$.
- If $Y \subseteq (\mathbb{C}^*)^n$ and $\mathbb{C}$ has the trivial valuation, then $\mathrm{trop}(V) \subseteq \mathbb{R}^n$ is the underlying space of a fan.
- $M_{0,n} \subseteq (\mathbb{C}^*)^{\binom{n}{2}-n}$ tropicalizes to

$$\mathrm{trop}(M_{0,n}) = T_{0,n}^{\mathrm{trop}}, \qquad (1)$$

  where $T_{0,n}^{\mathrm{trop}}$ is the set of metric trees with n ends defined earlier.
- $T_{0,n}^{\mathrm{trop}}$ is the moduli space of abstract rational tropical curves with n marked infinite edges.
- The fan Δ coming from (1) satisfies

$$\overline{M}_{0,n} = \text{Zariski closure of } M_{0,n} \text{ in } X_\Delta.$$

- This is an example of tropical compactification, a.k.a geometric tropicalization.

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Definition

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This definition is incomplete (what is a "combinatorial structure"?). The whole subject is a work-in-progress. In this talk, we have seen:

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- some themes that run deep.

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