Liberating the Dimension

Quasi Monte Carlo Methods
for High Dimensional Integration

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Outline

- Motivating example – a flow through a porous medium

- **Quasi-Monte Carlo (QMC) methods**
- **Component-by-component (CBC) construction**

- **Application of QMC to PDEs with random coefficients**
  - uniform, lognormal
  - single level, multi-level
  - first order, higher order

- Other applications
  - maximum likelihood estimation
  - option pricing

Motivating example

Uncertainty in groundwater flow

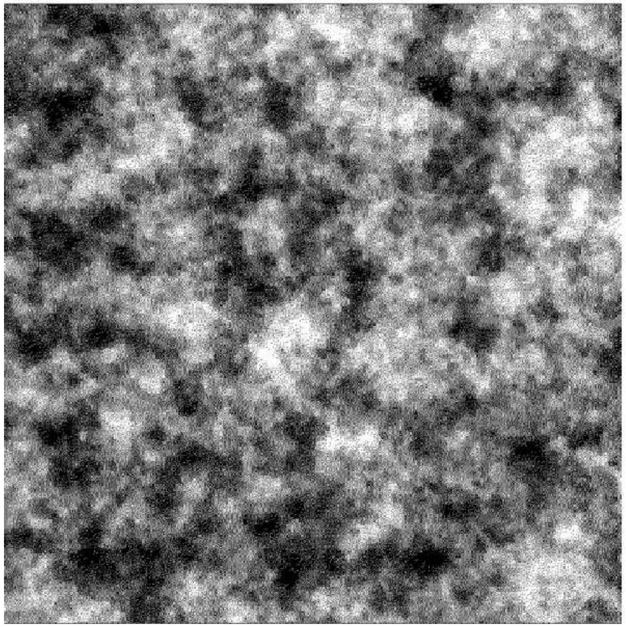
eg. risk analysis of radwaste disposal or CO_2 sequestration

Darcy's law $\quad \boldsymbol{q} + \boldsymbol{a}\,\vec{\nabla}\boldsymbol{p} = \boldsymbol{f}$

mass conservation law $\quad \nabla \cdot \boldsymbol{q} = 0$

in $D \subset \mathbb{R}^d$, $d = 1, 2, 3$

together with boundary conditions



[Cliffe, *et. al.* (2000)]

Uncertainty in $a(\boldsymbol{x}, \omega)$ leads to uncertainty in $q(\boldsymbol{x}, \omega)$ and $p(\boldsymbol{x}, \omega)$

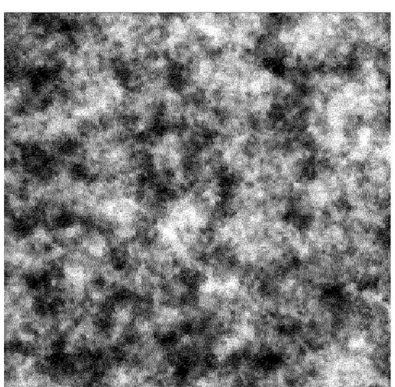
Expected values of quantities of interest

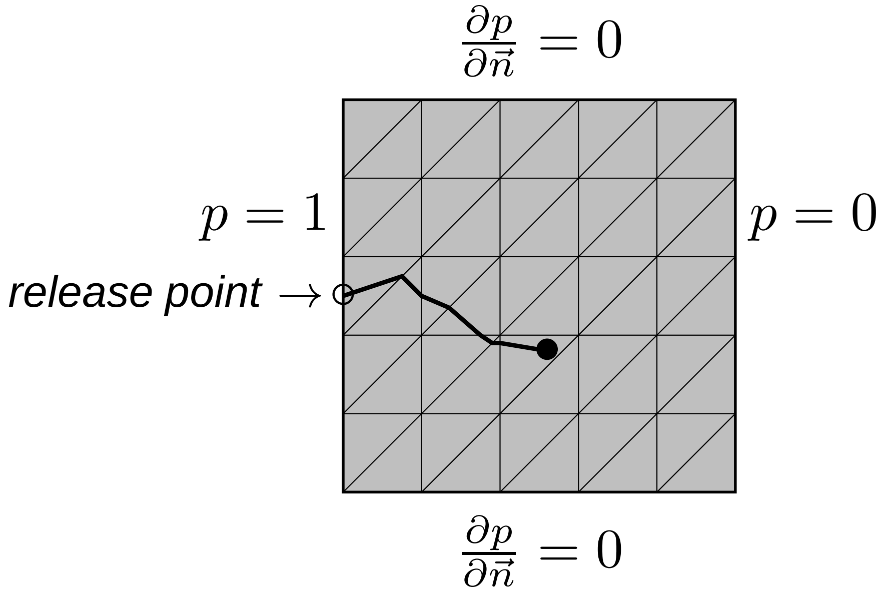
To compute the expected value of some quantity of interest:

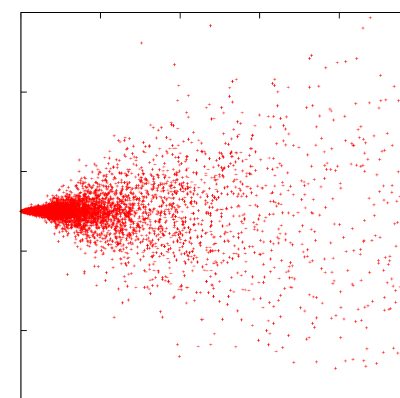
1. Generate a number of realizations of the random field
   (Some approximation may be required)
2. For each realization, solve the PDE using e.g. FEM / FVM / mFEM
3. Take the average of all solutions from different realizations

This describes Monte Carlo simulation.

Example: particle dispersion







Expected values of quantities of interest

To compute the expected value of some quantity of interest:

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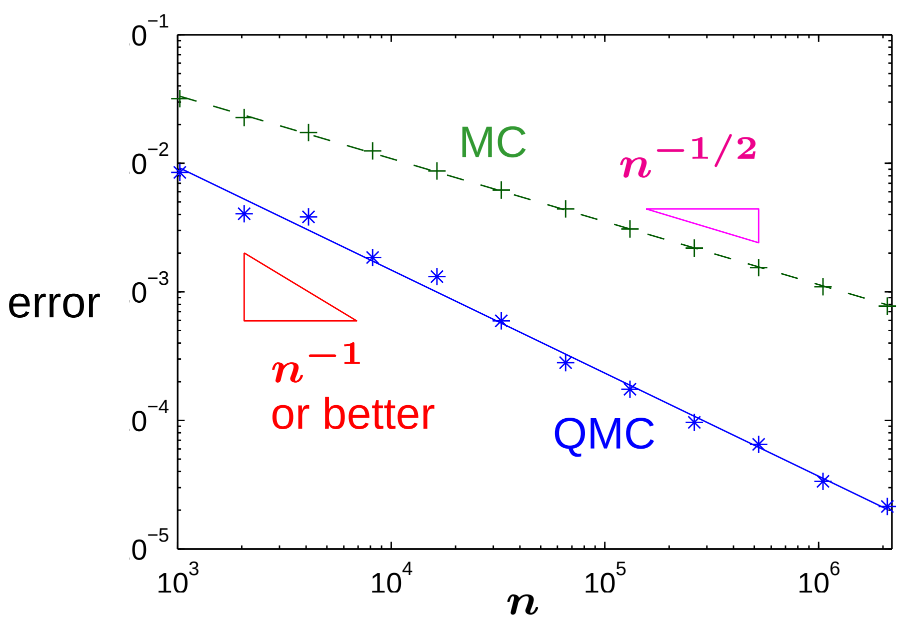
NOTE: **expected value** $=$ (high dimensional) **integral**



$\rightarrow$ use **quasi-Monte Carlo methods**

s $=$ stochastic dimension

Curse of dimensionality

s = the number of variables = the dimension

How large is s in practical applications?

- Collateralized Mortgage Obligations (CMO)

  30 years $\times$ 12 monthly repayment calculations $=$ 360 dimensions

- Daily counts of asthma patients seeking hospital treatments

  5 years $\times$ 365 days $=$ 1825 dimensions

- Macquarie Bank ALPS series (a.k.a. CEO)

  5 years $\times$ 250 trading days $\times$ 80 stocks $=$ one million dimensions

- Porous flow with permeability modeled as a random field

  501 by 501 mesh with circulant embedding $=$ one million dimensions

  Karhunen-Loève expansion of covariance function $=$ ∞ dimensions

MC v.s. QMC in the unit cube

$$\int_{[0,1]^s} F(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} \;\approx\; \frac{1}{n}\sum_{i=1}^{n} F(\boldsymbol{t}_i)$$

Monte Carlo method

$\boldsymbol{t}_i$ random uniform

$n^{-1/2}$ convergence

order of variables irrelevant

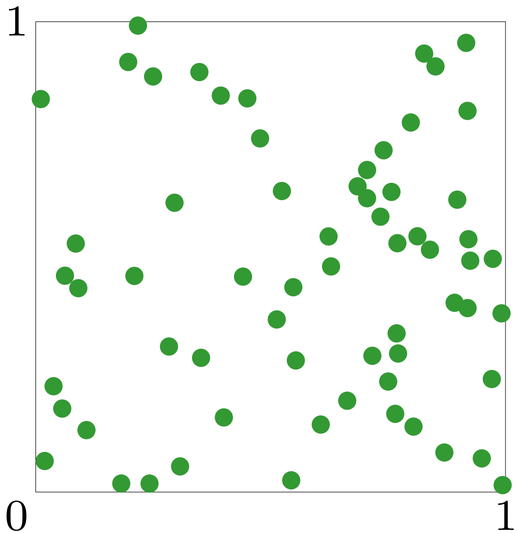
Quasi-Monte Carlo methods

$\boldsymbol{t}_i$ deterministic

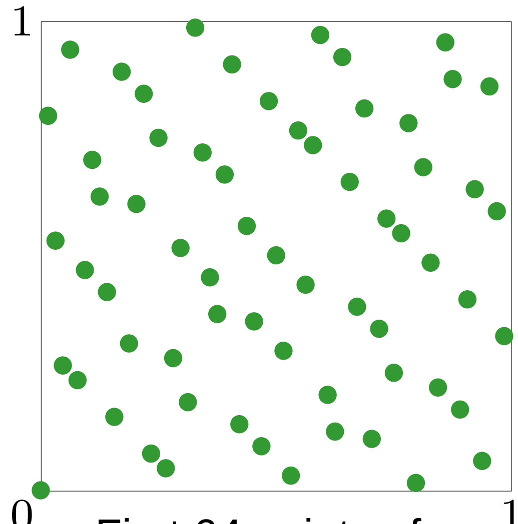
close to n^{-1} convergence or better

more effective for earlier variables and lower-order projections

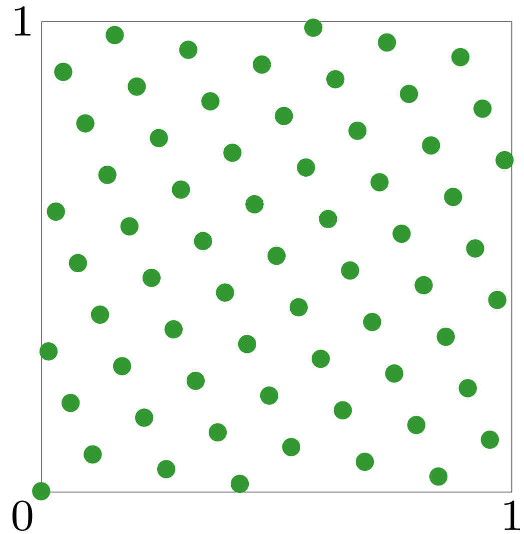
order of variables very important



64 random points



First 64 points of a 2D Sobol′ sequence



A lattice rule with 64 points

use *randomized* QMC methods for error estimation

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QMC

Two main families of QMC methods:

- **(t,m,s)-nets** and **(t,s)-sequences**
- **lattice rules**

Niederreiter book (1992)

Sloan and Joe book (1994)

Dick and Pillichshammer book (2010)

Dick, Kuo, Sloan Acta Numerica (2013)

Important developments:

- ***component-by-component (CBC) construction***
- ***higher order digital nets***

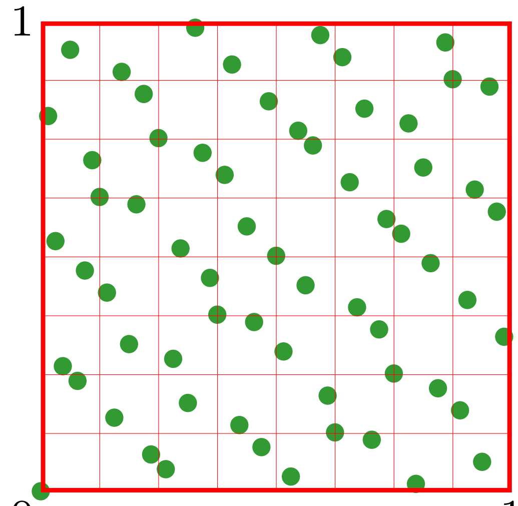
Contribute to:

- **Information-Based Complexity**
- **Tractability**

Traub, Wasilkowski, Woźniakowski book (1988)

Novak and Woźniakowski books (2008, 2010, 2012)

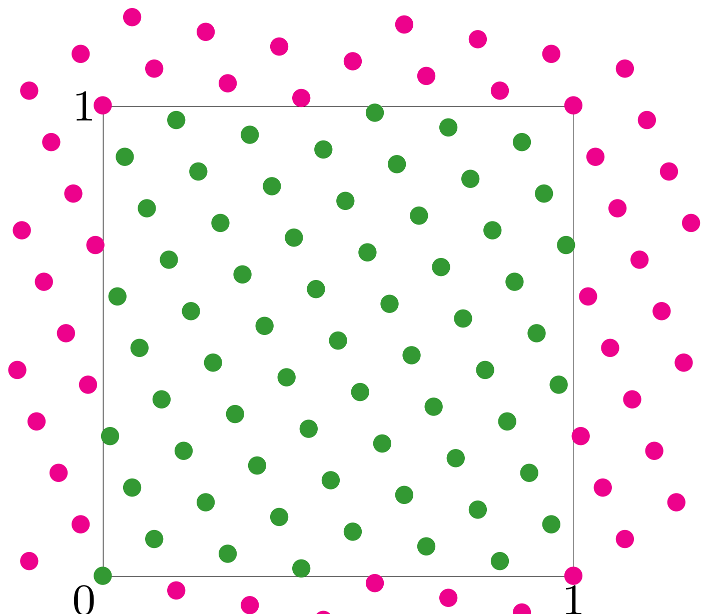
Having the right number of points in various sub-cubes



First 64 points of a 2D Sobol′ sequence

(0,6,2)-net

A group under addition modulo $\mathbb{Z}$ and includes the integer points



A lattice rule with 64 points

Lattice rules

Rank-1 lattice rules have points

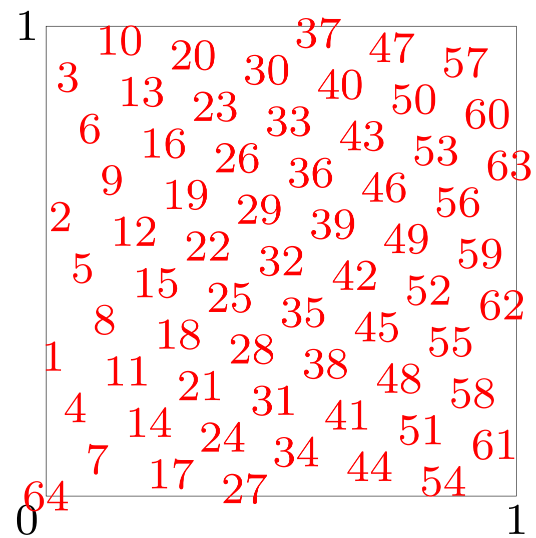
$$\boxed{\boldsymbol{t}_i = \operatorname{frac}\left(\frac{i}{n}\,\boldsymbol{z}\right), \qquad i = 1, 2, \ldots, n}$$

$\boldsymbol{z} \in \mathbb{Z}^s$ – the generating vector, with all components *coprime* to n

$\operatorname{frac}(\cdot)$ – means to take the fractional part of all components

$\sim$ quality determined by the choice of $\boldsymbol{z}$ $\sim$

$$n = 64 \quad \boldsymbol{z} = (1, 19) \quad \boldsymbol{t}_i = \operatorname{frac}\left(\frac{i}{64}(1, 19)\right)$$



A lattice rule with 64 points

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Component-by-component construction

- Want to find $\boldsymbol{z}$ with (shift-averaged) "*worst case error*" as small possible.

  $\sim$ Exhaustive search is practically impossible - too many choices! $\sim$

- **CBC algorithm** [Sloan, K., Joe (2002);. . .]
  1. Set $z_1 = 1$.
  2. With z_1 fixed, choose z_2 to minimize the worst case error in 2D.
  3. With z_1, z_2 fixed, choose z_3 to minimize the worst case error in 3D.
  4. etc.

  - Cost of algorithm is only $\mathcal{O}(n \log n\, s)$ using FFTs. [Nuyens, Cools (2005)]
  - Optimal rate of convergence $\mathcal{O}(n^{-1+\delta})$ in "weighted Sobolev space", with the implied constant independent of s under an appropriate condition on the weights. [K. (2003); Dick (2004)]

    $\sim$ Averaging argument: there is always one choice as good as average! $\sim$

  - Extensible/embedded variants. [Cools, K., Nuyens (2006); Dick, Pillichshammer, Waterhouse (2007)]

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http://www.maths.unsw.edu.au/~fkuo/lattice/
http://www.maths.unsw.edu.au/~fkuo/sobol/
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Application of QMC theory

Worst case error bound

$$\left| \int_{[0,1]^s} F(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} - \frac{1}{n}\sum_{i=1}^{n} F(\boldsymbol{t}_i) \right| \le e^{\mathrm{wor}}_{\boldsymbol{\gamma}}(\boldsymbol{t}_1,\ldots,\boldsymbol{t}_n)\,\|F\|_{\boldsymbol{\gamma}}$$

Weighted Sobolev space

[Sloan & Woźniakowski, 1998]

$$\|F\|_{\boldsymbol{\gamma}}^2 = \underbrace{\sum_{\mathfrak{u}\subseteq\{1:s\}}}_{2^s \text{ subsets}} \frac{1}{\gamma_{\mathfrak{u}}} \int_{[0,1]^{|\mathfrak{u}|}} \left| \frac{\partial^{|\mathfrak{u}|} F}{\partial \boldsymbol{y}_{\mathfrak{u}}}(\boldsymbol{y}_{\mathfrak{u}};0) \right|^2 \mathrm{d}\boldsymbol{y}_{\mathfrak{u}}$$

2^s subsets

"weights"

"anchor" at 0 (also "unanchored")

Mixed first derivatives are square integrable

Small weight $\gamma_{\mathfrak{u}}$ means that F depends weakly on the variables $\boldsymbol{y}_{\mathfrak{u}}$

Choose weights to minimize the error bound

$$\underbrace{\left(\frac{2}{n}\sum_{\mathfrak{u}\subseteq\{1:s\}} \gamma_{\mathfrak{u}}^{\lambda} a_{\mathfrak{u}}\right)^{1/(2\lambda)}}_{\text{bound on worst case error (CBC)}} \underbrace{\left(\sum_{\mathfrak{u}\subseteq\{1:s\}} \frac{b_{\mathfrak{u}}}{\gamma_{\mathfrak{u}}}\right)^{1/2}}_{\text{bound on norm}} \Rightarrow \gamma_{\mathfrak{u}} = \left(\frac{b_{\mathfrak{u}}}{a_{\mathfrak{u}}}\right)^{1/(1+\lambda)}$$

Construct points (CBC) to minimize the worst case error

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PDEs with random coefficients

Many important results using various methods by

- Babuska, Nobile, Tempone
- Babuska, Tempone, Zouraris
- Barth, Schwab, Zollinger
- Charrier
- Charrier, Scheichl, Teckentrup
- Cliffe, Giles, Scheichl, Teckentrup
- Cliffe, Graham, Scheichl, Stals
- Cohen, Chkifa, Schwab
- Cohen, De Vore, Schwab
- Graham, Scheichl, Ullmann
- Hansen, Schwab
- Harbrecht, Peters, Siebenmorgen
- Hoang, Schwab
- Kunoth, Schwab
- Nobile, Tempone, Webster
- Schwab, Todor
- Schillings, Schwab
- Teckentrup, Scheichl, Giles, Ullmann

⋮

See also `http://people.maths.ox.ac.uk/~gilesm/mlmc_community.html`

Application of QMC to PDEs with random coefficients

- Graham, K., Nuyens, Scheichl, Sloan (J. Comput. Physics, 2011)
  - Application of QMC to the lognormal case, no error analysis
  - Use circulant embedding to avoid truncation of KL expansion
- ✓ K., Schwab, Sloan (SIAM J. Numer. Anal., 2012)
  - Application of QMC to the uniform case [cf. Cohen, De Vore, Schwab, 2010]
  - First complete error analysis: how to choose “weights”
- ✓ K., Schwab, Sloan (J. FoCM, to appear) [cf. Heinrich 1998; Giles 2008]
  - A multi-level version of the analysis for the uniform case
- Schwab (Proceedings of MCQMC 2012)
  - Application of QMC to the uniform case for affine operator equations
- Le Gia (Proceedings of MCQMC 2012)
  - Application of QMC to the uniform case for PDE over the sphere

Application of QMC to PDEs with random coefficients

- ✓ <u>Dick, K., Le Gia, Nuyens, Schwab</u> (SIAM J. Numer. Anal., 2014)

  - Application of higher order QMC to the uniform case

- ✓ <u>Dick, K., Le Gia, Schwab</u> (submitted)

  - A multi-level version of the higher order analysis for the uniform case

- <u>Dick, Le Gia, Schwab</u> (submitted)

  - Holomorphic extension of the higher order analysis for the uniform case

- ✓ <u>Graham, K., Nichols, Scheichl, Schwab, Sloan</u> (Numer. Math., 2014)

  - Application of QMC to the lognormal case, full analysis and numerics

- <u>Graham, K., Scheichl, Schwab, Sloan, Ullmann</u> (in progress)

  - A multi-level version of the analysis for the lognormal case

⋮

Also QMC works by Harbrecht, Peters, Siebenmorgen

The uniform case

[K., Schwab, Sloan (2012)]

$$-\nabla \cdot (a(\boldsymbol{x},\boldsymbol{y})\,\nabla u(\boldsymbol{x},\boldsymbol{y})) = f(\boldsymbol{x}) \text{ in } D, \quad u(\boldsymbol{x},\boldsymbol{y}) = 0 \text{ on } \partial D$$

$$a(\boldsymbol{x},\boldsymbol{y}) = \bar{a}(\boldsymbol{x}) + \sum_{j=1}^{\infty} y_j\,\psi_j(\boldsymbol{x}), \quad \boldsymbol{y} \in (-\tfrac{1}{2},\tfrac{1}{2})^{\infty}$$

Assumptions:

- There exists $a_{\min}$ and $a_{\max}$ such that $0 < a_{\min} \le a(\boldsymbol{x},\boldsymbol{y}) \le a_{\max}$
- There exists $p_0 \in (0,1]$ such that $\sum_{j=1}^{\infty} \|\psi_j\|_{L^{\infty}(D)}^{p_0} < \infty$
- $\cdots$

Goal: for $F(\boldsymbol{y}) = G(u(\cdot,\boldsymbol{y}))$ a linear functional of u, we want to approximate

$$\int_{(-\frac{1}{2},\frac{1}{2})^{\infty}} F(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = \lim_{s\to\infty} \int_{(-\frac{1}{2},\frac{1}{2})^{s}} F(y_1,\ldots,y_s,0,0,\ldots)\,\mathrm{d}y_1\cdots\mathrm{d}y_s$$

Our strategy:

- Truncate the infinite sum to s terms (truncation error)
- Solve the PDE using FEM (discretization error)
- Approximate the integral using QMC (quadrature error)

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Error analysis – a sum of three terms

$$I(G(u)) \approx Q_{s,n}(G(u_h^s))$$
$$= \frac{1}{n}\sum_{i=1}^{n} G(u_h^s(\cdot, \boldsymbol{t}_i))$$

- $\sum_{j=1}^{\infty} \|\psi_j\|_{L^\infty(D)}^{p_0} < \infty$
- $f \in H^{-1+t}(D)$
- $G \in H^{-1+t'}(D)$

- Truncation error $= \mathcal{O}(s^{-2(1/p_0-1)})$ use "integration"
- Discretization error $= \mathcal{O}(h^{t+t'})$ Aubin-Nitsche duality trick
- Quadrature error $= \mathcal{O}(n^{-\min(1/p_0-1/2,1-\delta)})$ constant independent of s
  - Differentiate the PDE w.r.t. $\boldsymbol{y}$ [Cohen, De Vore, Schwab, 2010]

$$\boxed{\|\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} u(\cdot,\boldsymbol{y})\|_{H_0^1(D)} \le C\,|\boldsymbol{\nu}|!\prod_{j\ge1}\|\psi_j\|_{L^\infty(D)}^{\nu_j}}$$

  - Weighted Sobolev space with square-integrable mixed first derivatives
  - Fast CBC construction of randomly shifted lattice rules
  - Minimize error bound: get POD weights (product and order dependent)

$$\gamma_{\mathfrak{u}} = \Big(c^{|\mathfrak{u}|}\,|\mathfrak{u}|!\prod_{j\in\mathfrak{u}}\|\psi_j\|_{L^\infty(D)}\Big)^b$$

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Multi-level version

[K., Schwab, Sloan (to appear)]

$$I(G(u)) \approx Q_*^L(G(u))$$

$$= \sum_{\ell=0}^{L} Q_{s_\ell,n_\ell}\left(G(u_{h_\ell}^{s_\ell} - u_{h_{\ell-1}}^{s_{\ell-1}})\right) \qquad \text{cost} = \mathcal{O}\left(\sum_{\ell=0}^{L} s_\ell n_\ell h_\ell^{-d}\right)$$

Key estimate (challenge):

$$\boxed{\|G(u_{h_\ell}^{s_\ell} - u_{h_{\ell-1}}^{s_{\ell-1}})\|_\gamma \le \|G(u_{h_\ell}^{s_\ell} - u_{h_{\ell-1}}^{s_\ell})\|_\gamma + \|G(u_{h_{\ell-1}}^{s_\ell} - u_{h_{\ell-1}}^{s_{\ell-1}})\|_\gamma}$$

RMS error with randomly shifted lattice rules:

$$h_L^{t+t'} + s_L^{-2(1/p_0-1)} + \sum_{\ell=0}^{L} n_\ell^{-\min(1/p_1-1/2,1-\delta)}\left(h_{\ell-1}^{t+t'} + s_{\ell-1}^{-(1/p_0-1/p_1)}\right)$$

New Assumption: there exists $p_1 \in [p_0, 1]$ s.t.

$$\sum_{j=1}^{\infty} \|\psi_j\|_{W^{1,\infty}(D)}^{p_1} < \infty$$

- differentiate PDE w.r.t. $\boldsymbol{y}$
- need stronger regularity w.r.t. $\boldsymbol{x}$
- use duality argument

- Constants are independent of truncation dimension s_L
- Minimize some error bound $\Rightarrow$ get different POD weights
- Lagrange multipliers $\Rightarrow$ choose s_ℓ, n_ℓ, h_ℓ

Higher order QMC rules

- Classical polynomial lattice rule [Niederreiter, 1992]
  - $n = b^m$ points with prime b
  - An irreducible polynomial with degree m
  - A generating vector of s polynomials with degree $< m$
- Interlaced polynomial lattice rule [Goda and Dick, 2012]
  - Interlacing factor α
  - An irreducible polynomial with degree m
  - A generating vector of αs polynomials with degree $< m$
  - Digit interlacing function $\mathscr{D}_\alpha : [0,1)^\alpha \to [0,1)$

    $(0.x_{11}x_{12}x_{13}\cdots)_b, (0.x_{21}x_{22}x_{23}\cdots)_b, \ldots, (0.x_{\alpha 1}x_{\alpha 2}x_{\alpha 3}\cdots)_b$

    becomes $(0.x_{11}x_{21}\cdots x_{\alpha 1}x_{12}x_{22}\cdots x_{\alpha_2}x_{13}x_{23}\cdots x_{\alpha_3}\cdots)_b$

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Non-Hilbert, fast CBC, SPOD weights

$\|F\|_{q,r} =$ [Dick, K., Le Gia, Nuyens, Schwab (2014)]

$$\left[\sum_{\mathfrak{u}\subseteq\{1:s\}} \left(\frac{1}{\gamma_{\mathfrak{u}}{}^q} \sum_{\mathfrak{v}\subseteq\mathfrak{u}} \sum_{\substack{\boldsymbol{\tau}_{\mathfrak{u}\setminus\mathfrak{v}}\in \\ \{1:\alpha\}^{|\mathfrak{u}\setminus\mathfrak{v}|}}} \int_{[0,1]^{|\mathfrak{v}|}} \left| \int_{[0,1]^{s-|\mathfrak{v}|}} (\partial_{\boldsymbol{y}}^{(\boldsymbol{\alpha}_{\mathfrak{v}},\boldsymbol{\tau}_{\mathfrak{u}\setminus\mathfrak{v}},\boldsymbol{0})} F)(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}_{\{1:s\}\setminus\mathfrak{v}} \right|^q \mathrm{d}\boldsymbol{y}_{\mathfrak{v}} \right)^{r/q}\right]^{1/r}$$

- Combine $\underline{L^q \text{ norm}}$ ($\uparrow$ as $q\uparrow$) for functions and $\underline{\ell^r \text{ norm}}$ ($\uparrow$ as $r\downarrow$) for vectors
- Recover Hilbert space setting with $q = r = 2$
- Obtain worst case error bound for digital nets
- Take $r = \infty$ and any q. Choose $\gamma_{\mathfrak{u}}$ to make $\|F\|_{q,\infty} \le c$.
- Get **SPOD weights** (smoothness-driven product and order dependent)

$$\gamma_{\mathfrak{u}} = \sum_{\boldsymbol{\nu}_{\mathfrak{u}}\in\{1:\alpha\}^{|\mathfrak{u}|}} |\boldsymbol{\nu}_{\mathfrak{u}}|! \prod_{j\in\mathfrak{u}} \left(c\,\|\psi_j\|_{L^\infty(D)}^{\nu_j}\right)$$

- Fast (using FFT) CBC construction with SPOD weights
  - Work with s blocks of α dimensions
  - Cost $\mathcal{O}(\alpha s n \log n + \alpha^2 s^2 n)$ operations

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Higher order multi-level analysis

First order single level [K., Schwab, Sloan (2012)]

$$s^{-2(1/p_0-1)} + h^{t+t'} + n^{-\min(1/p_0-1/2,1-\delta)}$$

First order multi-level [K., Schwab, Sloan (to appear)]

$$s_L^{-2(1/p_0-1)} + h_L^{t+t'} + \sum_{\ell=0}^{L} n_\ell^{-\min(1/p_1-1/2,1-\delta)} \left(s_{\ell-1}^{-(1/p_0-1/p_1)} + h_{\ell-1}^{t+t'}\right)$$

Higher order single level [Dick, K., Le Gia, Nuyens, Schwab (2014)]

$$s^{-2(1/p_0-1)} + h^{t+t'} + n^{-1/p_0}$$

Higher order multi-level [Dick, K., Le Gia, Schwab (submitted)]

$$s_L^{-2(1/p_0-1)} + h_L^{t+t'} + \sum_{\ell=0}^{L} n_\ell^{-1/p_t} \left(s_{\ell-1}^{-(1/p_0-1/p_t)} + h_{\ell-1}^{t+t'}\right)$$

- New assumption: there exists $p_t \in [p_0, 1]$ s.t. $\sum_{j=1}^{\infty} \|\psi_j\|_{\mathcal{X}_t}^{p_t} < \infty$
- In many examples $p_t = p_0/(1 - tp_0)$
- Interlacing factor $\alpha = \lfloor 1/p_t \rfloor + 1 \geq 2$

QMC theory for integration over $\mathbb{R}^s$

Change of variables

ϕ - any pdf, Φ - cdf, Φ^{-1} - icdf

$$\int_{\mathbb{R}^s} q(\boldsymbol{z})\,\rho(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z} = \int_{\mathbb{R}^s} F(\boldsymbol{y}) \prod_{j=1}^{s} \phi(y_j)\,\mathrm{d}\boldsymbol{y} = \int_{[0,1]^s} F(\Phi^{-1}(\boldsymbol{w}))\,\mathrm{d}\boldsymbol{w}$$

$F \circ \Phi^{-1}$ rarely belongs to weighted Sobolev space

Norm with weight function [Wasilkowski & Woźniakowski (2000)]

$$\|F\|_{\boldsymbol{\gamma}}^2 = \sum_{\mathfrak{u} \subseteq \{1:s\}} \frac{1}{\gamma_{\mathfrak{u}}} \int_{\mathbb{R}^{|\mathfrak{u}|}} \left| \frac{\partial^{|\mathfrak{u}|} F}{\partial \boldsymbol{y}_{\mathfrak{u}}}(\boldsymbol{y}_{\mathfrak{u}}; \boldsymbol{0}) \right|^2 \prod_{j=1}^{s} \psi^2(y_j)\,\mathrm{d}\boldsymbol{y}_{\mathfrak{u}}$$

weight function

Nichols & K. (2014) cf. [K., Sloan, Wasilkowski, Waterhouse (2010)]

- Also unanchored variant, coordinate dependent ψ_j
- Randomly shifted lattice rules CBC error bound for general weights $\gamma_{\mathfrak{u}}$
- Convergence rate depends on the relationship between ϕ and ψ
- Fast CBC for POD weights $\gamma_{\mathfrak{u}} = \Gamma_{|\mathfrak{u}|} \prod_{j \in \mathfrak{u}} \gamma_j$

Important for applications: $\boxed{\phi \text{ and } \psi \text{ and } \gamma_{\mathfrak{u}} \text{ are up to us to choose (tune)}}$

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Application: the lognormal case

Elliptic PDE with lognormal random coefficient

$$-\nabla \cdot (a(\boldsymbol{x},\boldsymbol{y})\,\nabla u(\boldsymbol{x},\boldsymbol{y})) = f(\boldsymbol{x}), \qquad \boldsymbol{x} \in D \subseteq \mathbb{R}^d,\ d = 1,2,3$$

$$a(\boldsymbol{x},\boldsymbol{y}) = \exp\left(\sum_{j=1}^{\infty} \sqrt{\mu_j}\,\xi_j(\boldsymbol{x})\,y_j\right), \qquad y_j \sim \text{i.i.d. normal}$$

$$\int_{\mathbb{R}^\infty} G(u(\cdot,\boldsymbol{y})) \prod_{j=1}^{\infty} \phi_{\rm nor}(y_j)\,\mathrm{d}\boldsymbol{y} = \int_{[0,1]^\infty} G(u(\cdot,\boldsymbol{\Phi}_{\rm nor}^{-1}(\boldsymbol{x})))\,\mathrm{d}\boldsymbol{x}$$

Graham, K., Nichols, Scheichl, Schwab, Sloan (2014)

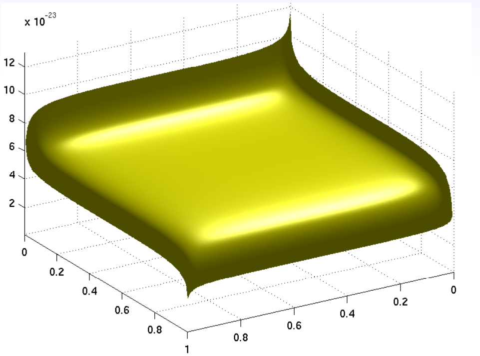
- Differentiate PDE to obtain bound on the norm
- Choose $\phi \equiv \phi_{\rm nor}$
- Choose $\psi_j(y_j) = \exp(-\alpha_j y_j)$, $\alpha_j > 0$
- Choose POD weights $\gamma_{\mathfrak{u}} = \Gamma_{|\mathfrak{u}|} \prod_{j\in\mathfrak{u}} \gamma_j$

Application: maximum likelihood

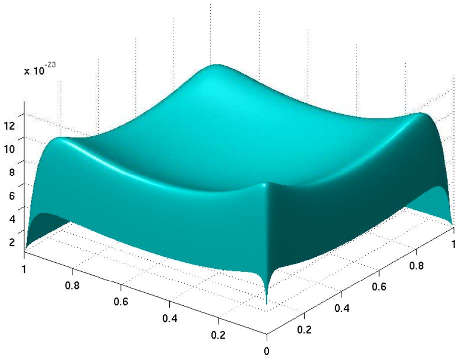
Generalized linear mixed model

[K., Dunsmuir, Sloan, Wand, Womersley (2008)]

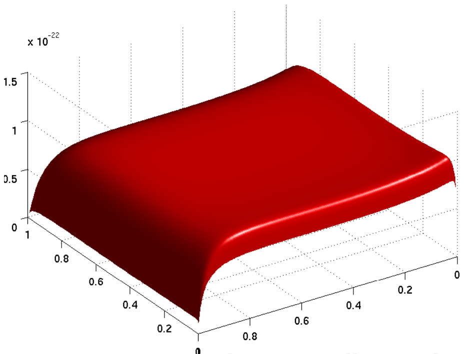
$$\int_{\mathbb{R}^s} \left(\prod_{j=1}^{s} \frac{\exp(\tau_j(\beta + z_j) - e^{\beta + z_j})}{\tau_j!} \right) \frac{\exp(-\frac{1}{2}\boldsymbol{z}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{z})}{\sqrt{(2\pi)^s \det(\boldsymbol{\Sigma})}} \, \mathrm{d}\boldsymbol{z}$$



ϕ normal (good)



ϕ logistic (better)



ϕ Student-t (best)

Sinescu, K., Sloan (2013)

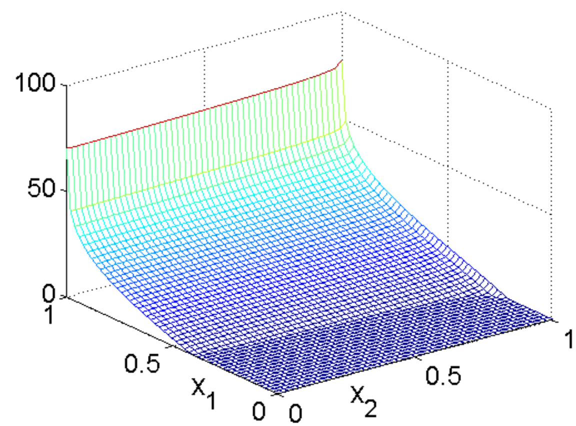
- Differentiate integrand to obtain bound on the norm
- Choose $\phi \equiv \phi_{\mathbf{nor}}$ or $\phi_{\mathbf{logit}}$ or $\phi_{\mathbf{stud}}$
- Choose $\psi \equiv 1$
- Choose POD weights $\gamma_{\mathfrak{u}} = \Gamma_{|\mathfrak{u}|} \prod_{j\in\mathfrak{u}} \gamma_j$

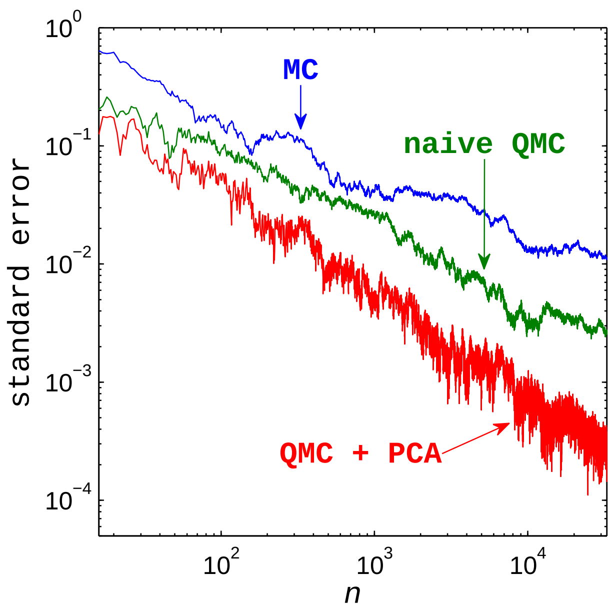
Application: option pricing

Black-Scholes model

$$\int_{\mathbb{R}^s} \max\left(\frac{1}{s}\sum_{j=1}^{s} S_j(\boldsymbol{z}) - K, 0\right) \frac{\exp(-\frac{1}{2}\boldsymbol{z}^{\mathrm{T}}\Sigma^{-1}\boldsymbol{z})}{\sqrt{(2\pi)^s \det(\Sigma)}}\,\mathrm{d}\boldsymbol{z}$$

$$S_j(\boldsymbol{z}) = \exp(\cdots \sum a_j z_j)$$





Griebel, K., Sloan (2010, 2013, 2014)

- ANOVA decomposition: $g = \sum_{\mathfrak{u}\subseteq\{1:s\}} g_{\mathfrak{u}}$ in $[0,1]^s$ or $f = \sum_{\mathfrak{u}} f_{\mathfrak{u}}$ in $\mathbb{R}^s$
- RW/BB: all $g_{\mathfrak{u}}$ with $|\mathfrak{u}| \le \frac{s+1}{2}$ belong to Sobolev space; PCA: similar
- RW/BB: all $f_{\mathfrak{u}}$ with $\mathfrak{u} \ne \{1:s\}$ are smooth; PCA: similar
- ANOVA decomposition for ∞-variate function f: all terms are smooth
- Remains to see how to apply the theory of Nichols & K. (2014)