Zeros (of some polynomials) prefer curves

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Let us start with 3 objects to play with

PADÉ APPROXIMANTS

Let

$$f(z) = \sum_{k=0}^{\infty} \frac{a_k}{z_k}$$

represent an analytic germ at infinity.

We can approximate it by a (Laurent) polynomial,

$$P_n(1/z) = \sum_{k=0}^{n} \frac{a_k}{z_k}$$

or we can do much better using rational functions: find P_n, Q_n, of degree $\leq n$, such that

> the expansion of $\dfrac{P_n}{Q_n}(z)$ matches the expansion of $f(z)$
>
> to the highest possible order.

This is a non-linear problem on the coefficients of P_n and Q_n. Existence of a solution?

PADÉ APPROXIMANTS

Let

$$f(z) = \sum_{k=0}^{\infty} \frac{a_k}{z_k}$$

represent an analytic germ at infinity.

The linearized version is:

Find P_n and Q_n, $Q_n \not\equiv 0$, such that

$$(Q_n f - P_n)(z) = \mathcal{O}\left(\frac{1}{z^{n+1}}\right), \quad z \to \infty.$$

Then $\pi_n = P_n/Q_n$ is the (unique) $[n/n]$ Padé approximant to f at ∞.

The analytic theory of these approximants (and their generalizations) has blossomed in the 1980-ies.

In particular, what is the behavior (as $n \to \infty$) of the poles of π_n = obstacles for convergence = zeros of Q_n?

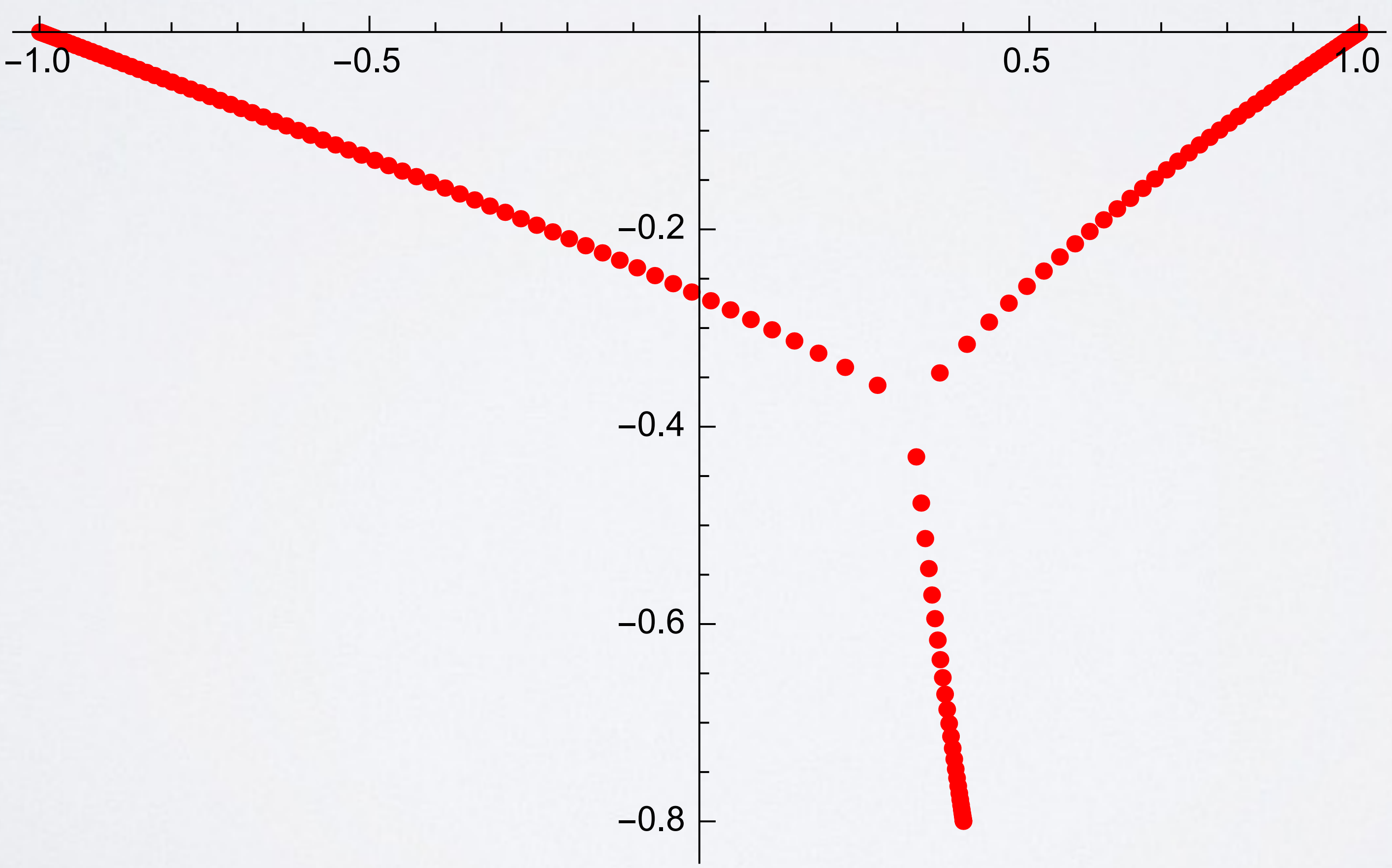
PADÉ APPROXIMANTS

The most interesting case is when f is a germ of a multivalued (algebraic) function.

Example:

$$f(z) = \frac{(1-z^2)^{1/3}}{(z-0.4+0.8i)^{2/3}}$$

Here are the poles of π_{150}:



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Example:

$$f(z) = \frac{(1-z^2)^{1/5}(z+0.8+0.4i)^{1/5}}{(z-0.4+0.8i)^{3/5}}$$

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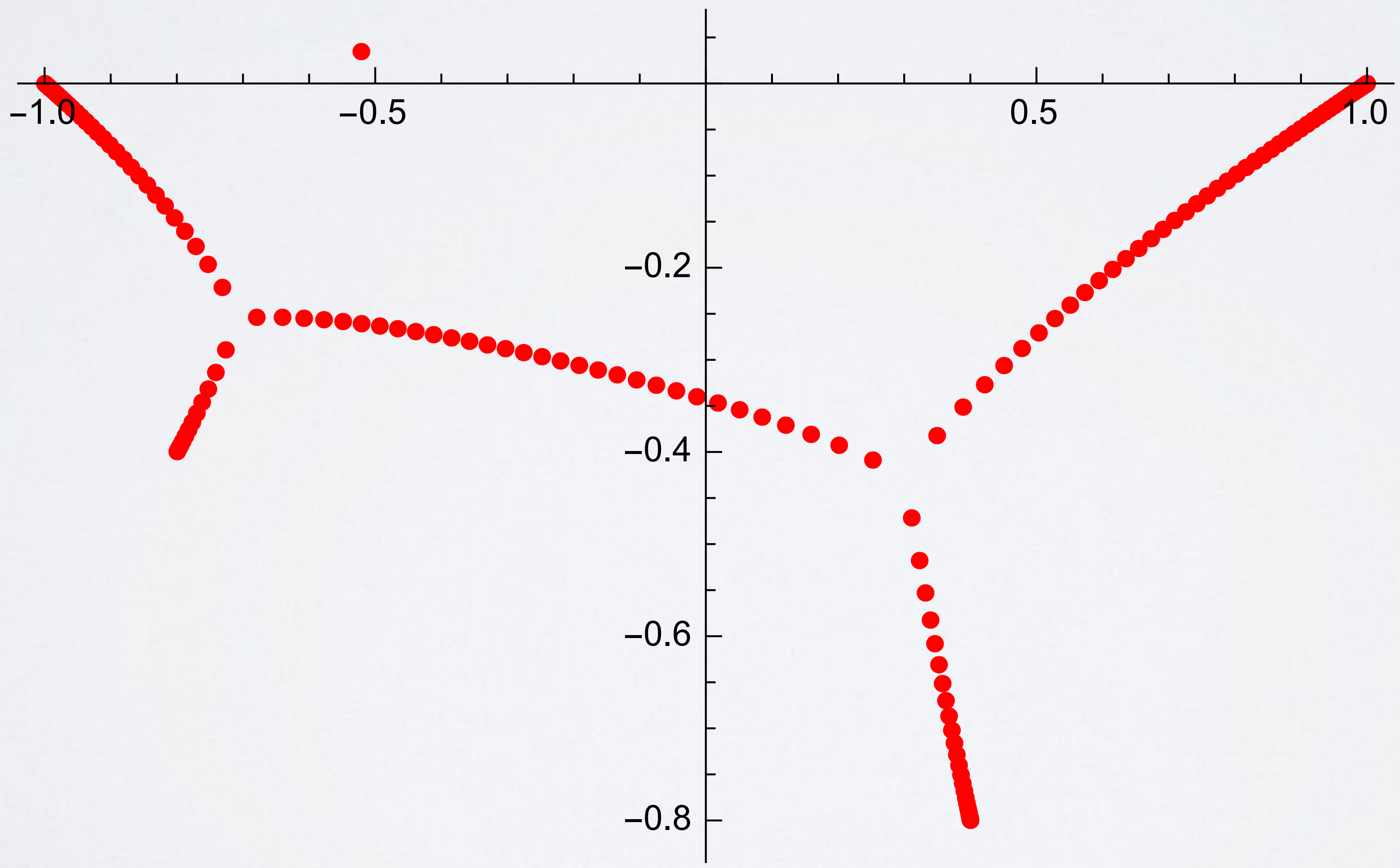
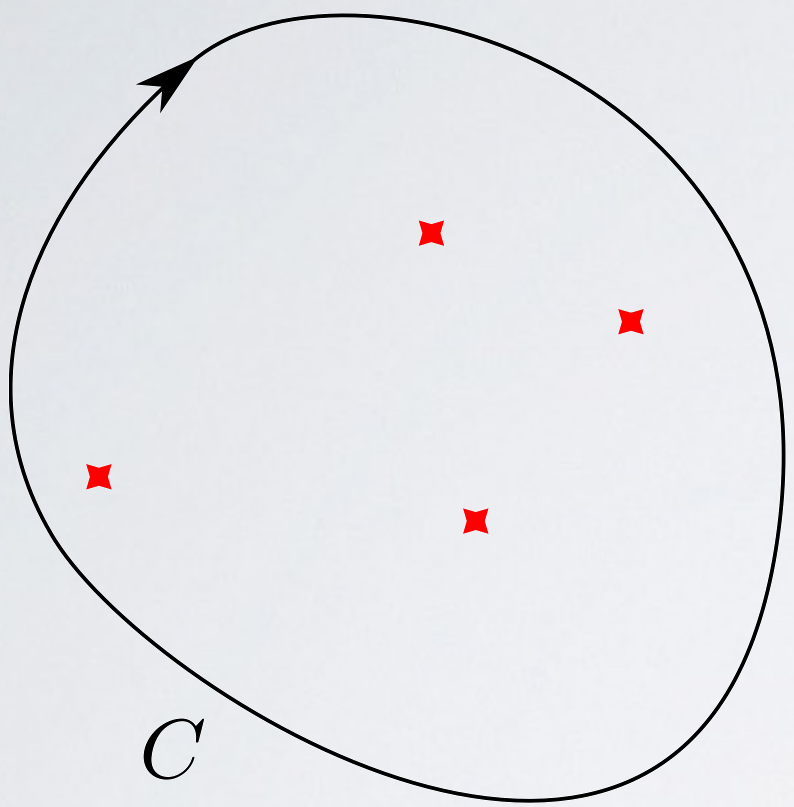


PADÉ APPROXIMANTS

Key observation: use the Cauchy formula and the definition of the residue,



$$(Q_n f - P_n)(z) = \mathcal{O}\left(\frac{1}{z^{n+1}}\right), \quad z \to \infty.$$

$$\Downarrow$$

$$\oint_C z^k (Q_n f - P_n)(z)\, dz = 0$$

$$k = 0, 1, \dots, n-1$$

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$$\oint_C z^k Q_n(z) f(z)\, dz = 0$$

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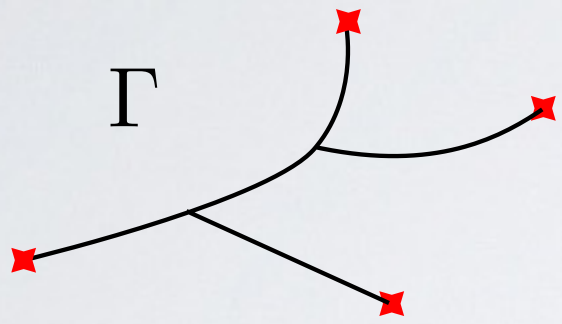

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or even $\displaystyle\int_\Gamma z^k Q_n(z) \Delta f(z)\, dz = 0$

Non-hermitian orthogonality

CLASSICAL POLYNOMIALS: LAGUERRE

$$L_n^{(\alpha)}(z) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-z)^k}{k!} = \frac{(-1)^n}{n!} z^{-\alpha} e^{z} \left(\frac{d}{dz}\right)^n \left[z^{n+\alpha} e^{-z}\right]$$

For $\alpha > -1$ they form a well-known family of orthogonal polynomials on $[0, +\infty)$:

$$\int_0^{+\infty} L_n^{(\alpha)}(x) x^{k+\alpha} e^{-x} dx = 0, \quad \text{for } k = 0, 1, \ldots, n-1.$$

In consequence, all zeros of $L_n^{(\alpha)}$ for $\alpha > -1$ are positive and simple.

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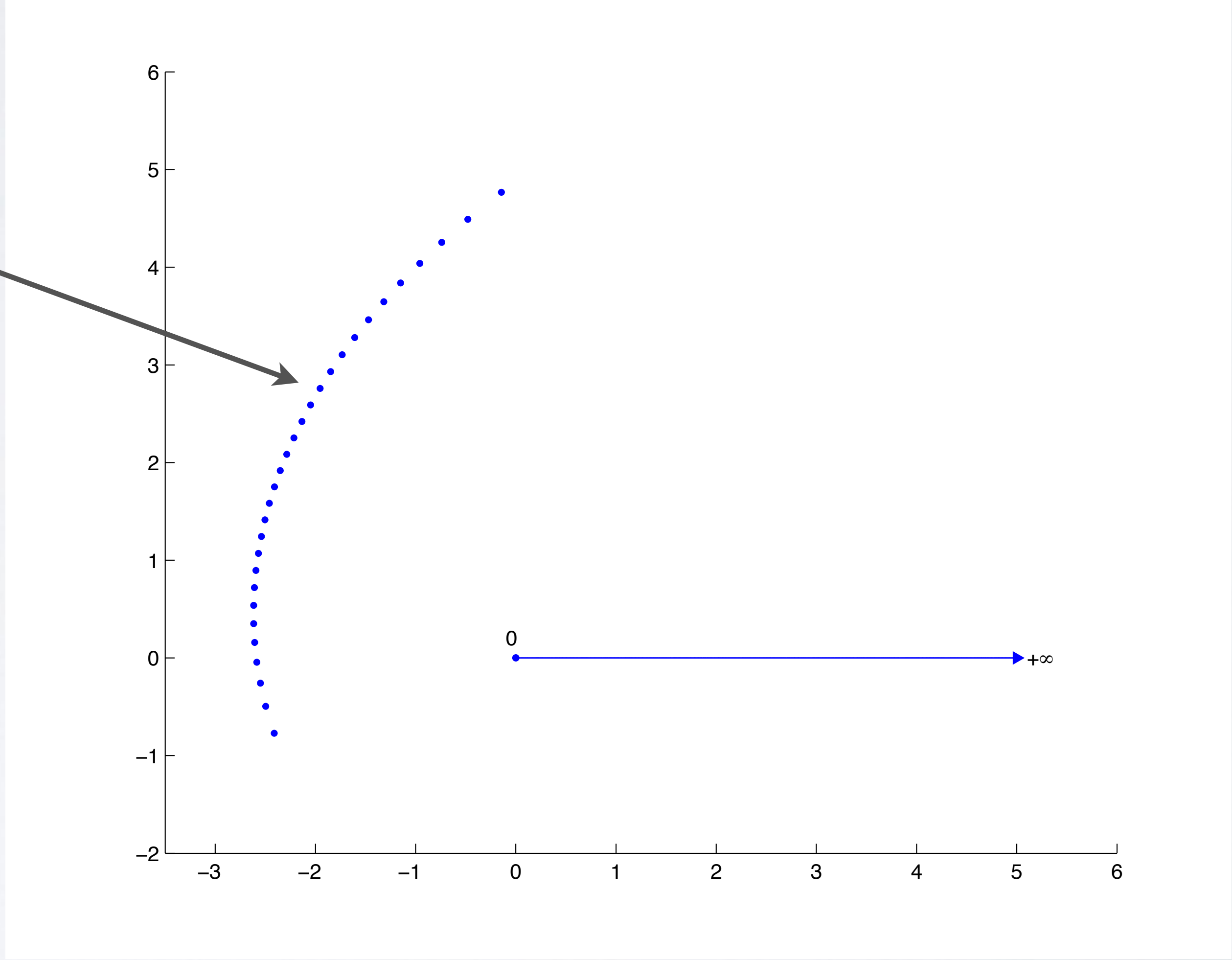
In this definition, α can be complex.

Zeros of $L_{30}^{(-3+2i)}$



In particular, if

$$\lim_n \frac{\alpha_n}{n} = A \in \mathbb{C}$$

$$p_n(z) = L_n^{(\alpha_n)}(nz) \sim \ ?$$

[J. Math. Anal. Appl. 416, 52–80]

$L_n^{(\alpha)}$ also satisfy non-hermitian orthogonality relations, this time on the plane. Besides, they are polynomial solutions of degree n of

$$zy''(z) + (\alpha + 1 - z)y'(z) + ny(z) = 0$$

LAMÉ ODE

Generalized Lamé (or Heun) ODE (in an algebraic form):

$$y''(x)+\left(\sum_{i=0}^{p}\frac{\rho_i}{x-a_i}\right)y'(x)-\frac{V(x)}{A(x)}\,y(x)=0\,,\qquad A(x)=\prod_{i=0}^{p}(x-a_i)\,,$$

where V is a polynomial of degree $\leq p-1$.

Heine (1878): for every $N\in\mathbb{N}$ there exist at most $\binom{N+p-1}{N}$ different polynomials V (Van Vleck polynomials) such that this equation has a polynomial solution of degree N (Heine-Stieltjes polynomial).

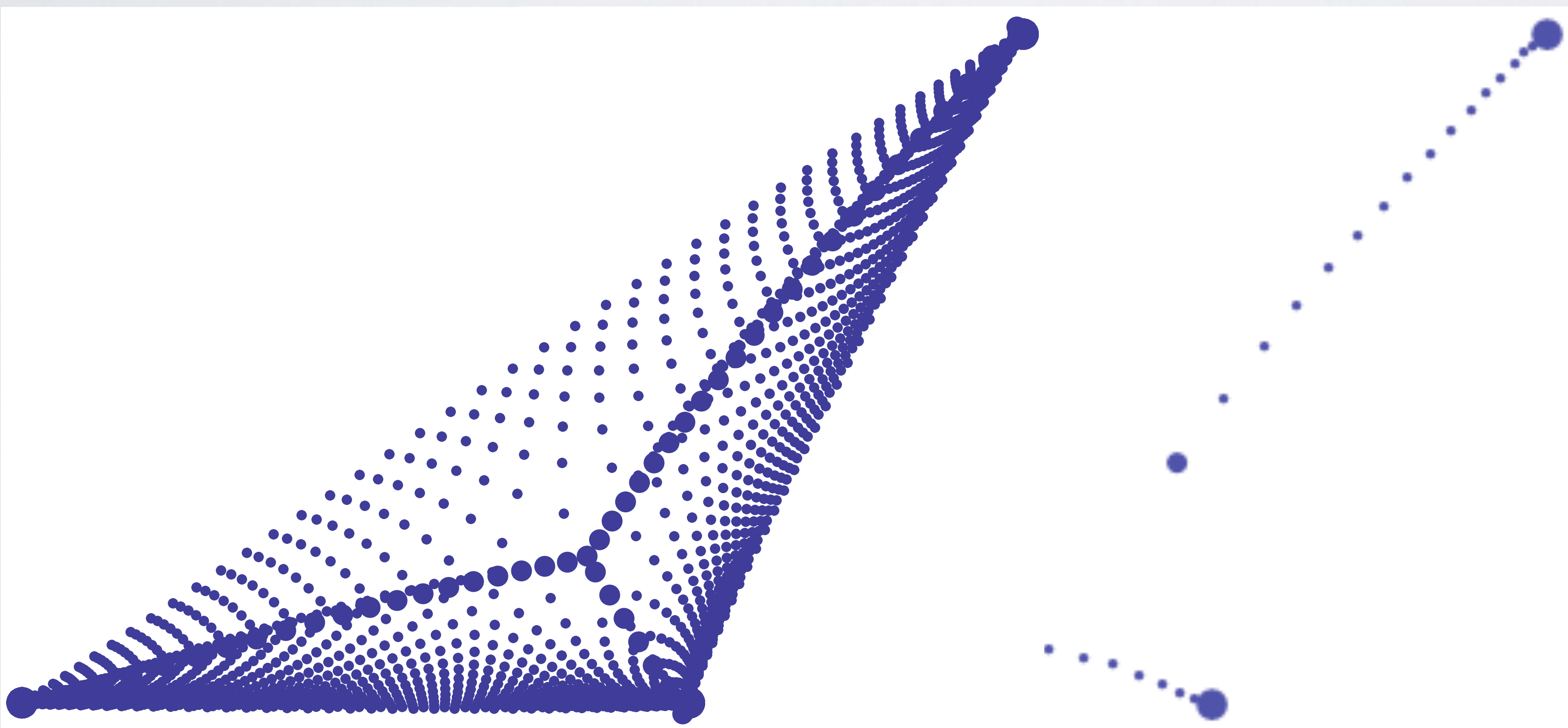
Stieltjes, 1885: electrostatic interpretation for $a_0<a_1<\cdots<a_p$ and all $\rho_j>0$:

LAMÉ ODE (P=2)

Zeros of H-S polynomials for p=2 (three finite singularities):

$$A(z)y''(z) + B(z)y'(z) - V(z)y(z) = 0$$

$$A(z) = \prod_{k=0}^{2}(z - a_k), \qquad V(z) = c(z - \bullet), \qquad y(z) = \prod(z - \bullet)$$



Explanation and asymptotics?

LOGARITHMIC POTENTIAL

How can we study the behavior of polynomials?

Trivial observation: if $P(z) = (z - a_1)(z - a_2)\dots(z - a_n)$, then

$$-\log|P(z)| = \sum_{k=1}^{n} \log \frac{1}{|z - a_k|}$$

Since $u(z) = \log(1/|z|)$ satisfies

$$-2\pi\Delta u(z) = \delta_0(z)$$

we can say that

$$-\log|P(z)| = \sum_{k=1}^{n} \log \frac{1}{|z - a_k|}$$

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Hence, we should consider zeros as charged particles interacting according to the logarithmic law!

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Continuous analogue: given a (in general, signed) measure μ, its logarithmic potential is

$$V^{\mu}(z) = \int \log \frac{1}{|z-t|}\, d\mu(t)\,.$$

orthogonality $\Updownarrow$ L^2 norm minimization $\Updownarrow$ an extremal problem for potentials (equilibrium)

ODE $\Updownarrow$ gradient of a potential $= 0$ $\Updownarrow$

Warning!
Intuitive arguments

Proceed at your own risk

EQUILIBRIUM FOR A LOGARITHMIC POTENTIAL

The (continuous) logarithmic energy of a measure μ is defined as

$$I(\mu) = \iint \log \frac{1}{|z-t|}\, d\mu(z)d\mu(t) = \int V^{\mu}(z)\, d\mu(z)$$

For $K \subset \mathbb{C}$ compact let $\mathcal{M}_1(K) = \{$probability measures with $supp \subset K\}$. The Robin constant of K is

$$\kappa = \inf_{\mu \in \mathcal{M}_1(K)} I(\mu) \in (-\infty, +\infty],$$

and $\mathrm{cap}(K) = e^{-\kappa}$ is the logarithmic capacity of K.

The unique $\mu_K \in \mathcal{M}_1(K)$ such that $I(\mu_K) = \kappa$ is the equilibrium measure of K.

Characterization: $V^{\mu_K} = \kappa$ on K (equilibrium condition).

Also, $V^{\mu_K}(z) = \kappa - g_D(z, \infty)$, where $g(\cdot, K)$ is the Green function of $D = \mathbb{C} \setminus K$ with pole at ∞.

ASYMPTOTICS FOR STANDARD OP

In the case of the "standard" (Hermitian) orthogonality,

$$\int_K \overline{Q_n(z)} z^k \, d\mu(z) = 0, \quad k = 0, 1, \quad n - 1,$$

our intuition can be rigorously justified:

- when K has no interior, and measure μ is reasonable;
- when K has non-empty interior, and measure μ is very nice ("Bergman polynomials"),...

More precisely, if

$$\nu_n = \nu(Q_n) = \frac{1}{n} \sum_{Q_n(x)=0} \delta_x$$

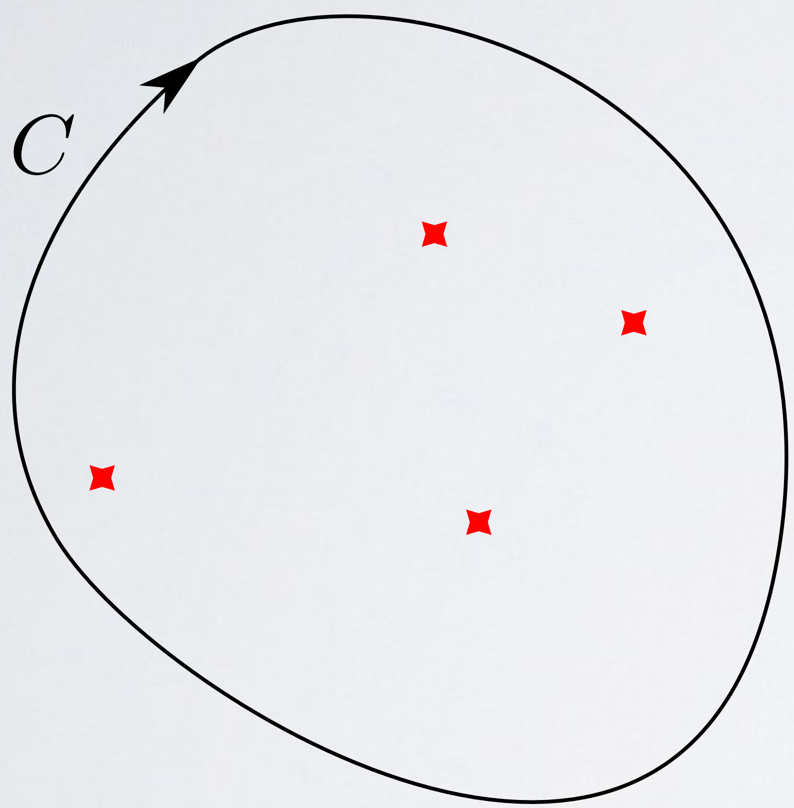
is the (normalized) zero-counting measure for Q_n, then $\nu_n \to \mu_K$ in the weak-* sense.

Observe that conjugation "fixes" the geometry.

A THEOREM OF H. STAHL (1985-1986)

Assume that f is an algebraic function, with branch points at $E = \{a_k\}$, and Q_n is a polynomial of degree n such that

$$\oint_C z^k Q_n(z) f(z)\, dz = 0 \qquad k = 0, 1, \ldots, n-1$$



Where do the zeros of Q_n go when $n \to \infty$?

In other words, if $\nu(Q_n) \to \mu$, as $n \to \infty$, who is μ?

By the Padé-based intuition, μ is such that f has a holomorphic branch in $\mathbb{C} \setminus \mathrm{supp}(\mu)$.

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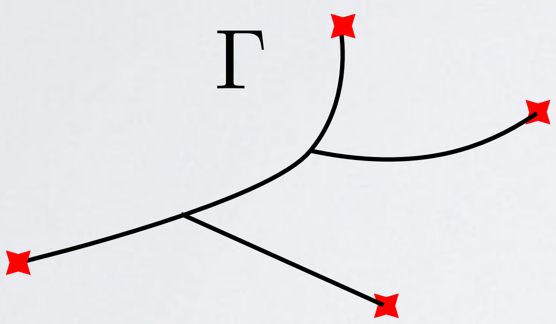


By the Padé-based intuition, μ is such that f has a holomorphic branch in $\mathbb{C} \setminus \operatorname{supp}(\mu)$.

Answer: take the set K of minimal capacity such that f is single-valued in $D = \mathbb{C} \setminus K$. This is the attractor of the zeros of Q_n.

Moreover, $\nu(Q_n) \to \mu_K$ as $n \to \infty$.

SETS OF MINIMAL CAPACITY

Observe that

$$\min cap(K) \quad \Leftrightarrow \quad \max \kappa$$

where κ is both the equilibrium constant AND the equilibrium energy of K, so that $\min cap(K)$ is equivalent to

$$\boxed{\max_K \min_{\mu \in \mathcal{M}_1(K)} I(\mu),} \quad \text{or}$$

$$\max_K \max_{\mu \in \mathcal{M}_1(K)} \min_{z \in K} V^{\mu}(z).$$

Characterization (Stahl): on the extremal compact K,

S-property

$$\frac{\partial g_{\mathbb{C}\setminus K}(z,\infty)}{\partial n_-} = \frac{\partial g_{\mathbb{C}\setminus K}(z,\infty)}{\partial n_+}$$

or equivalently,

$$\frac{\partial V^{\mu_K}}{\partial n_-}(z) = \frac{\partial V^{\mu_K}}{\partial n_+}(z)$$

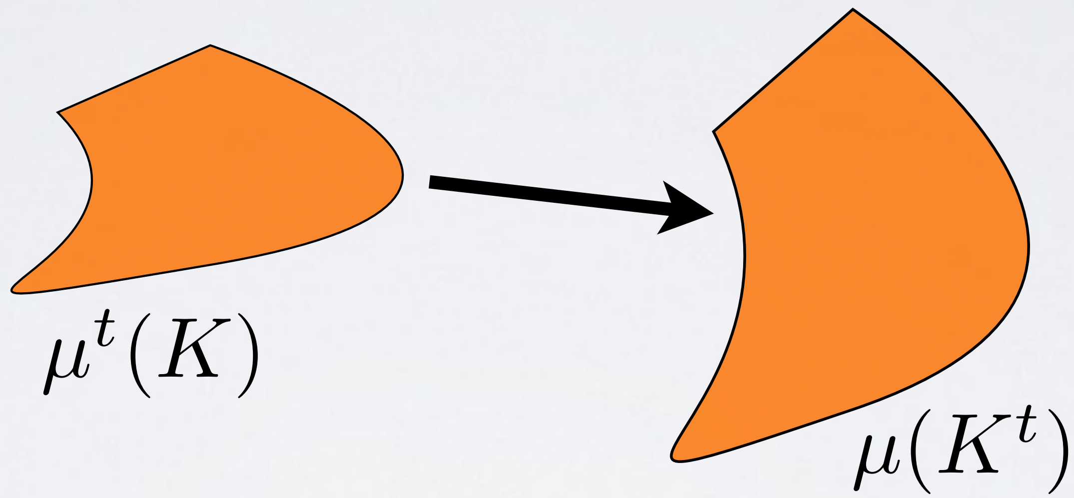
WHAT ABOUT THE LAMÉ ODE?

We need to generalize the notion of the set of minimal capacity…

CRITICAL MEASURES

Any $h : \mathbb{C} \to \mathbb{C} \in C^1$ and $t \in \mathbb{C}$ create a local variation $\mu \to \mu^t$ by

$$\int f(z) d\mu^t(z) = \int f(z + th(z)) d\mu(z).$$



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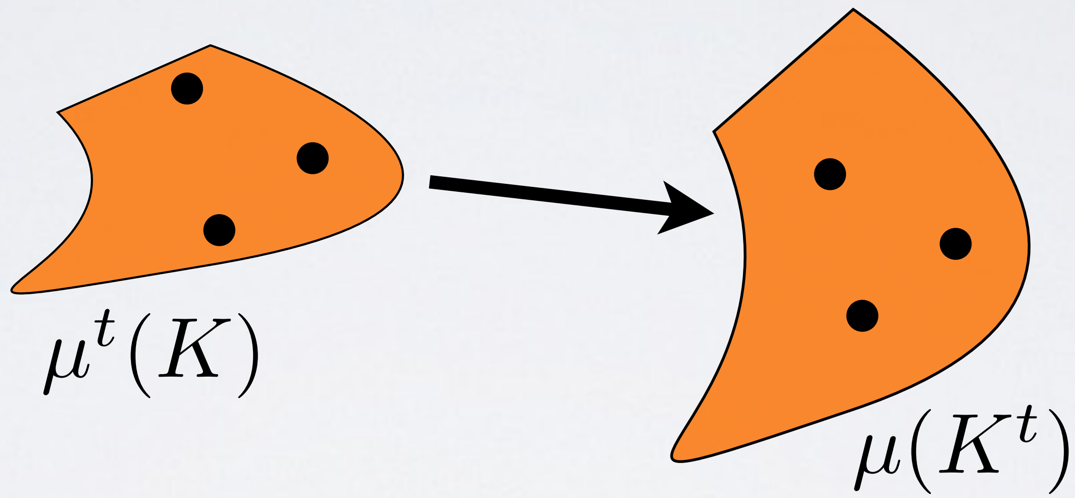


If $\mathcal{A} = \{a_0, a_1, \ldots, a_p\}$, a measure μ is called $\mathcal{A}$-critical if

$$\frac{d}{dt} I(\mu^t)\big|_{t=0} = 0, \quad \forall\, h\big|_{\mathcal{A}} = 0$$

$$\Rightarrow \qquad \frac{\partial g_{\mathbb{C}\setminus K}(z,\infty)}{\partial n_-} = \frac{\partial g_{\mathbb{C}\setminus K}(z,\infty)}{\partial n_+} \quad (S\text{-property})$$

LAMÉ ODE

AMF & E.A. Rakhmanov, Commun. Math. Phys. 302, 53–111 (2011):

Any weak limit of a zero counting measure of the polynomial solutions of the generalized Lamé ODE

$$A(z)y''(z) + B(z)y'(z) - V_n(z)y(z) = 0, \quad A(z) = \prod_{j=0}^{p}(z - a_j)$$

is an $\mathcal{A}$-critical measure, and viceversa.

THE S-PROPERTY

Assume that K is given by analytic arcs and $D = \mathbb{C} \setminus K$ is connected.

Let $G(z)$ be the complex Green function, $\operatorname{Re} G = g_D(\cdot, \infty)$. Since $g_D(\cdot, \infty) \equiv 0$ on K, we see that

$$\frac{\partial g_D(z,\infty)}{\partial n_+} = \frac{\partial g_D(z,\infty)}{\partial n_-} \quad \Leftrightarrow \quad \left(G'(z)\right)_+(z) = -\left(G'(z)\right)_-(z)$$

and we conclude that $H = (G')^2$ is analytic.

In other words,

$$G(z) = \int^z \sqrt{H(t)}dt$$

Trajectory of a quadratic differential

and K lies on the level line

$$\boxed{\operatorname{Re} \int^z \sqrt{H(t)}dt = \text{const}}$$

This is just the first level…

Non-hermitian OP and Lamé ODE with fixed A and B

In order to describe more complex constructions (Lamé ODE with varying coefficients, OP with the weight depending on the degree,. . .), we must expand the notion of equilibrium.

EQUILIBRIUM IN AN EXTERNAL FIELD

We can add to the picture an external field $\psi : K \to \mathbb{R}$ and consider the extremal problem $\inf_{\mathcal{M}_1(K)} I_\psi(\mu;\psi)$, with

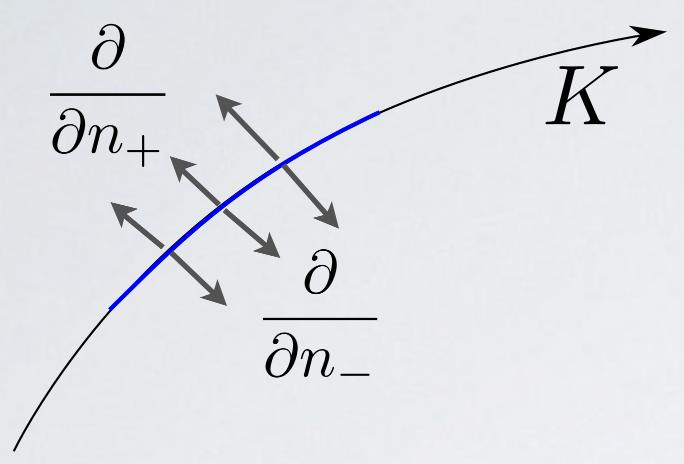
$$I(\mu;\psi) = I(\mu) + 2\int \psi(z)d\mu(z)$$

The unique solution $\mu_K(\psi)$ is characterized by

$$V^{\mu_K(\psi)} + \psi \begin{cases} = \omega(K) = \text{const} & \text{on } \operatorname{supp}(\mu_K(\psi)), \\ \geq \omega(K) = \text{const} & \text{on } K. \end{cases}$$

S-PROPERTY FOR THE LOG ENERGY

Let K be a compact set, $D = \mathbb{C} \setminus K$ connected.



Assume that K is made of analytic arcs, so that we can define normal derivatives on each side.

Let $\mu = \mu_K(\psi)$ be again the logarithmic equilibrium measure on K in the external field ψ.

K has the S-property if

$$\frac{\partial\left(V^\mu + \psi\right)}{\partial n_+} = \frac{\partial\left(V^\mu + \psi\right)}{\partial n_-} \quad \text{on } \operatorname{supp}\mu$$

If ψ is harmonic, $\psi = \operatorname{Re}\Psi$, then μ satisfies a variational identity on $\mathbb{C}$ of the form

$$\boxed{\left(\int \frac{d\mu(x)}{x-z} + \Psi'(z)\right)^2 = R(z), \quad \text{a.e. in } \mathbb{C}.}$$

VARIATIONAL IDENTITY

Let μ be a measure, $\mathcal{A} = \{a_0, a_1, \dots, a_p\}$ a set of points on $\mathbb{C}$, $\psi = \operatorname{Re}\Psi$, such that there exists an analytic function R such that

$$\boxed{\left(\int \frac{d\mu(x)}{x-z} + \Psi'(z) \right)^2 = R(z), \quad \text{a.e. in } \mathbb{C}.}$$

We assume that points in $\mathcal{A}$ are poles of R, and μ satisfies the equilibrium conditions.

What can we say about μ?

- $\operatorname{supp}(\mu)$ is a union of of analytic arcs, satisfying

$$\operatorname{Re} \int^z \sqrt{R(t)}\,dt = \text{const}$$

Trajectory of a quadratic differential

- Under suitable conditions on ψ (e.g., $\psi = V^\sigma$, $\sigma \geq 0$), μ is uniquely determined, and $D = \mathbb{C} \setminus \operatorname{supp}(\mu)$ is connected.
- The S-property holds,

$$\frac{\partial\,(V^\mu + \psi)}{\partial n_+} = \frac{\partial\,(V^\mu + \psi)}{\partial n_-} \quad \text{on } \operatorname{supp}\mu$$

VARIATIONAL IDENTITY

Example: two fixed points $(-1 \pm 2i)$, external field $\psi(z) = \operatorname{Re} z$.

Variational identity:

$$\left(\int \frac{d\mu(x)}{x-z} + 1\right)^2 = R(z)$$

Properties of R:

- its only poles are $-1 \pm 2i$
- $\sqrt{R}$ is holomorphic in $\mathbb{C} \setminus K$, K joins $-1 \pm 2i$
- $\displaystyle\int \frac{d\mu(x)}{x-z} = \sqrt{R(z)} - 1 \Rightarrow \sqrt{R(z)} = 1 - \tfrac{1}{z} + \mathcal{O}(\tfrac{1}{z^2})$ as $z \to \infty$

Conclusion: $\displaystyle R(z) = \frac{(z-c)^2}{(z+1-2i)(z+1+2i)} = 1 - \frac{2(c+1)}{z} + \dots$

$$\Rightarrow \quad c = -\frac{1}{2} \qquad \Rightarrow \quad \int \frac{d\mu(x)}{x-z} = -1 + \frac{z+1/2}{\sqrt{(z+1-2i)(z+1+2i)}}$$

We can recover the measure μ if we know its support: the trajectory $\operatorname{Re} \int^z \sqrt{R(t)}dt = \text{const}$

THEOREM OF GONCHAR-RAKHMANOV

Assume that Q_n of degree n satisfy

$$\oint_C z^k Q_n(z) f_n(z)\, dz = 0, \quad k = 0, 1, \dots, n-1,$$

where f_n are analytic,

$$\frac{1}{2n} \log \frac{1}{|f_n(z)|} \to \psi(z)$$

with ψ harmonic.

Let also K be such that the support of $\mu_K(\psi)$ has the S-property in the external field ψ.

If $\mathbb{C} \setminus \operatorname{supp} \mu_K(\psi)$ is connected, then

$$\nu(Q_n) = \frac{1}{n} \sum_{Q_n(z)=0} \delta_z \to \mu_K(\psi)$$

It has been mentioned that such measures live on trajectories of quadratic differentials

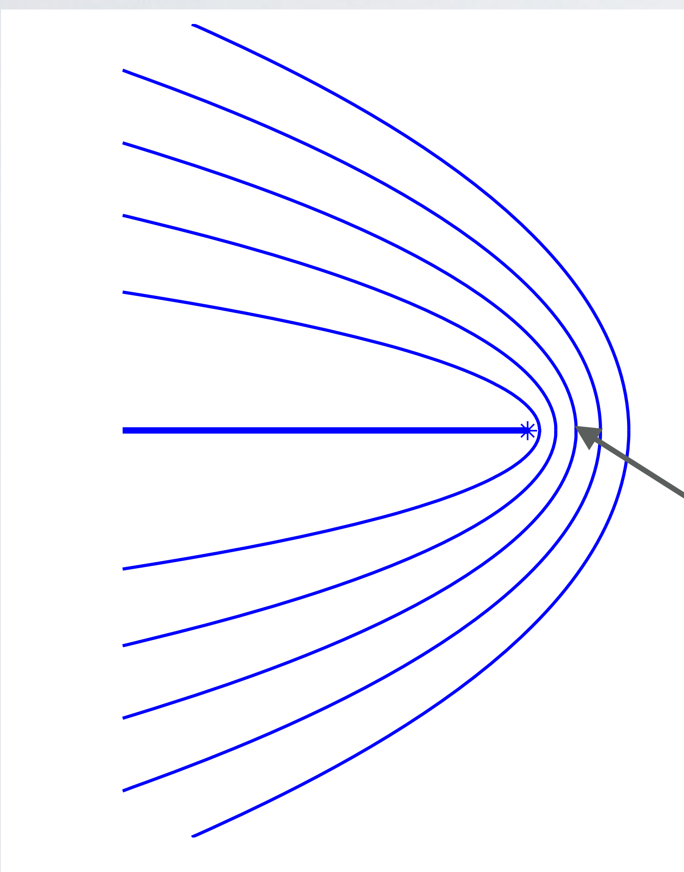
Time to discuss quadratic differentials

QUADRATIC DIFFERENTIALS

To keep it simple, let speak only about trajectories of a quadratic differential (q.d.) associated to an analytic (meromorphic) function Q.

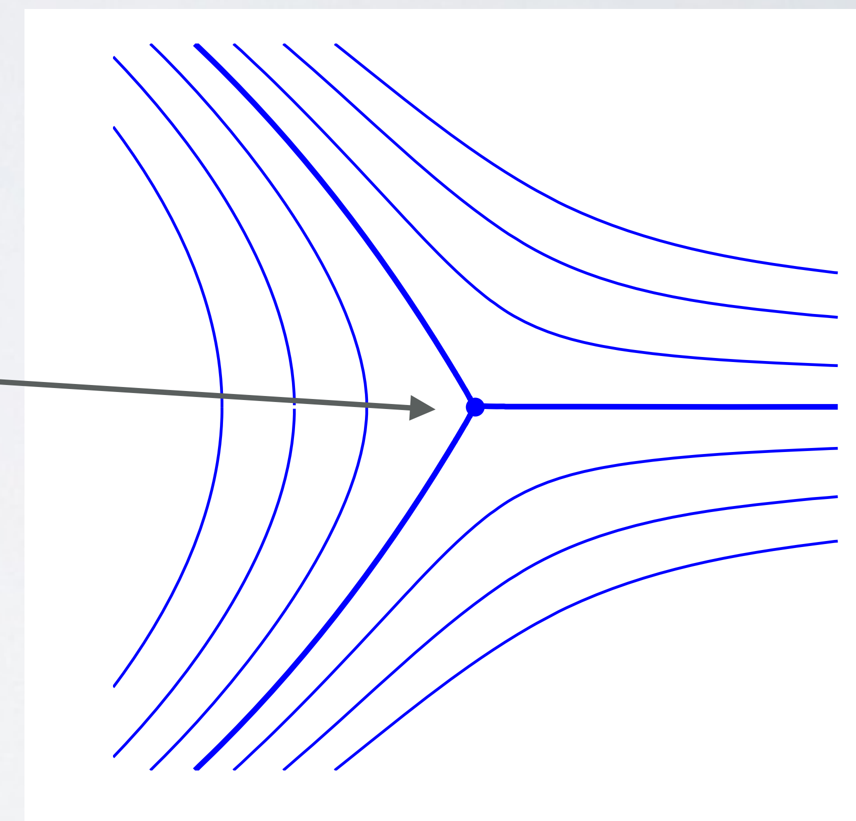
A trajectory arc of a q.d. $Q(z)dz^2$ is a curve $\gamma : (a, b) \mapsto D$ that satisfies

$$\operatorname{Re} \int^{z} \sqrt{Q}(t)\, dt \equiv \text{const}$$



Simple pole

Simple zero



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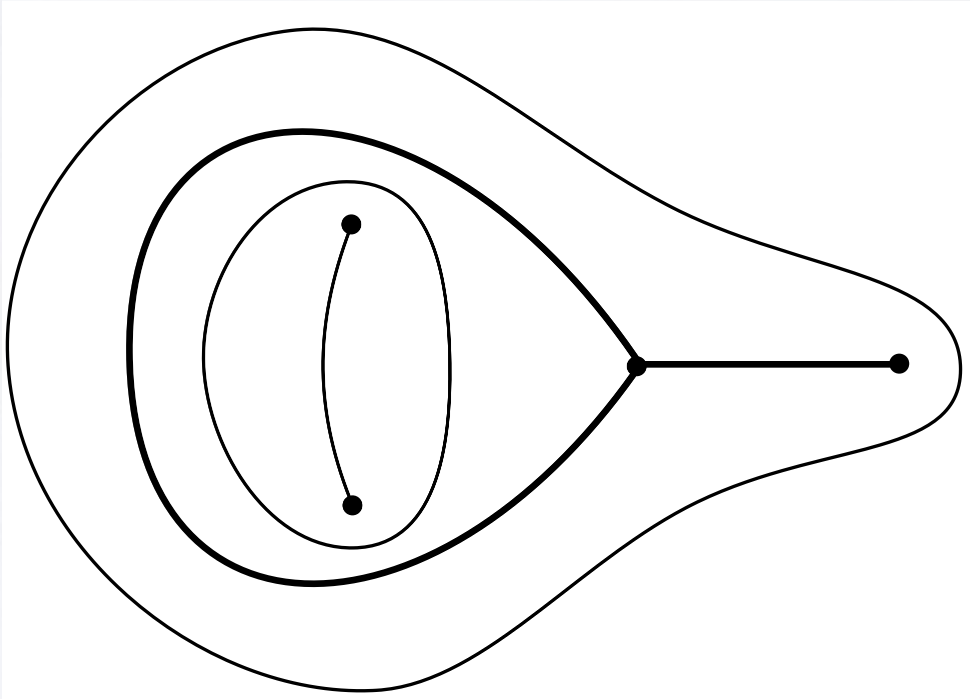
The q.d. is closed if all its trajectories are either closed or critical.

Example: $\dfrac{z-c}{z^3-1}\, dz^2,\ c \in (-2,2).$



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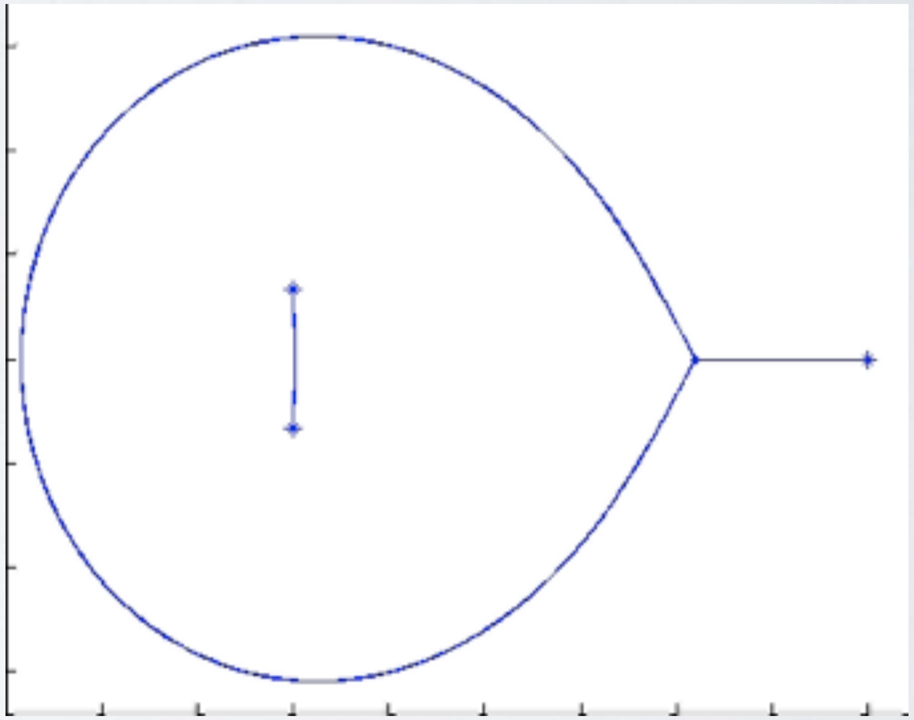

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QUADRATIC DIFFERENTIALS

The relation of the symmetric measures with the trajectories shows that we are interested in closed differentials, and in their short or critical trajectories, namely those either closed or starting and ending at critical points (= zeros and poles) of the q.d.

We need tools to study the global structure of the trajectories.

There are not so many tools:

- the local structure of trajectories
- Jenkins' 3 pole theorem
- teichmüller's lemma
- possibility to associate the trajectories with level curves of a harmonic function on a Riemann surface

Also valid for quadratic differentials on an algebraic curve!

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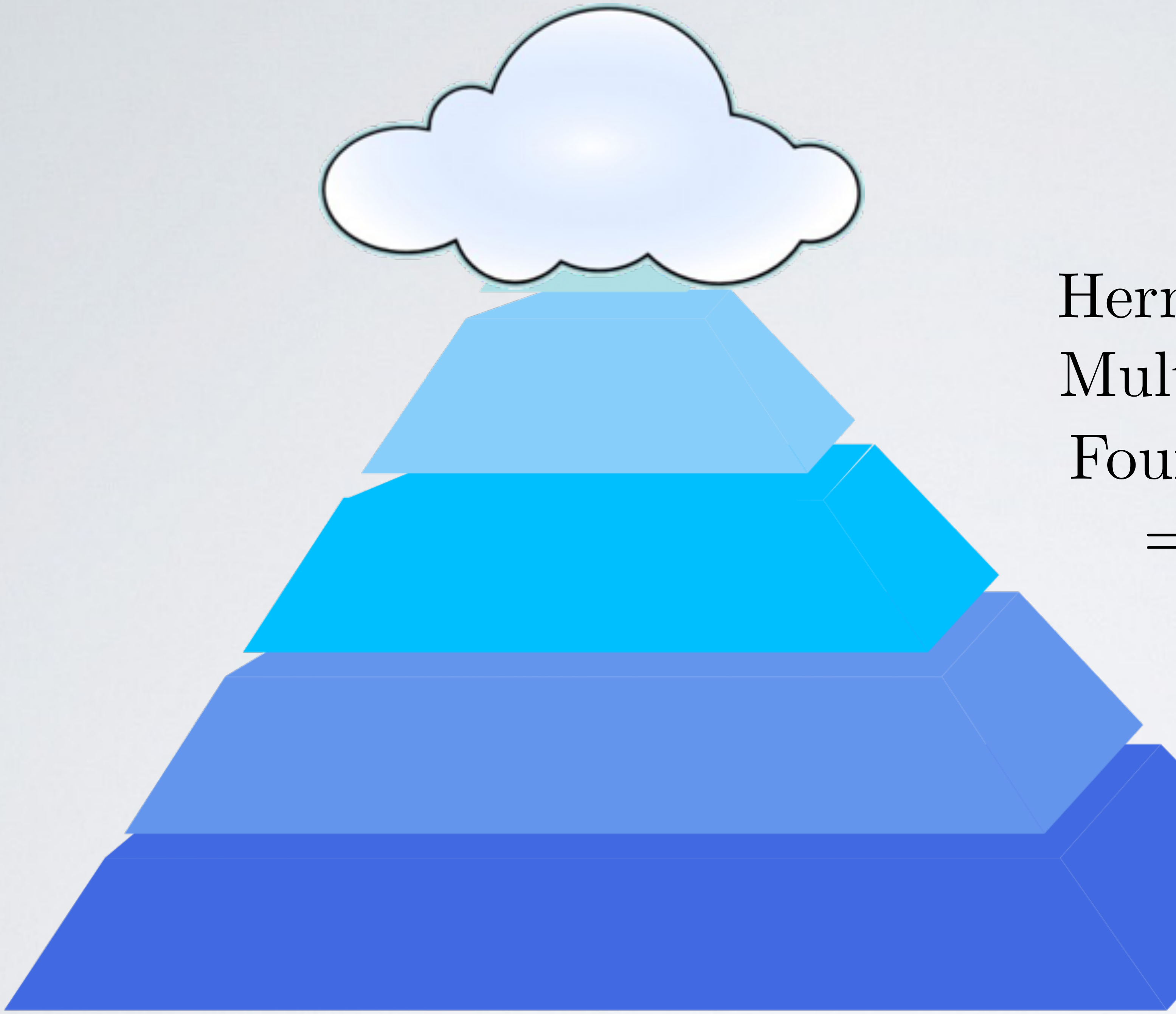
The study of the global structure of trajectories of quadratic differentials on compact Riemann surfaces is an ongoing project, with ramifications also in the geometric function theory, random matrix models, dynamical systems. . .



Hermite-Padé approximation
Multiple orthogonality
Fourier-Padé approximation
$\Rightarrow$ Green equilibrium...

Varying non-hermitian orthogonality

Non-hermitian OP and Lamé ODE with fixed A and B

Other approximation schemes or problems in inverse scattering require more sophisticated equilibria...

HERMITE-PADÉ APPROXIMANTS

Now we have two analytic germs at infinity,

$$f_1(z) = \sum_{k=0}^{\infty} \frac{a_k}{z_k}, \qquad f_2(z) = \sum_{k=0}^{\infty} \frac{b_k}{z_k},$$

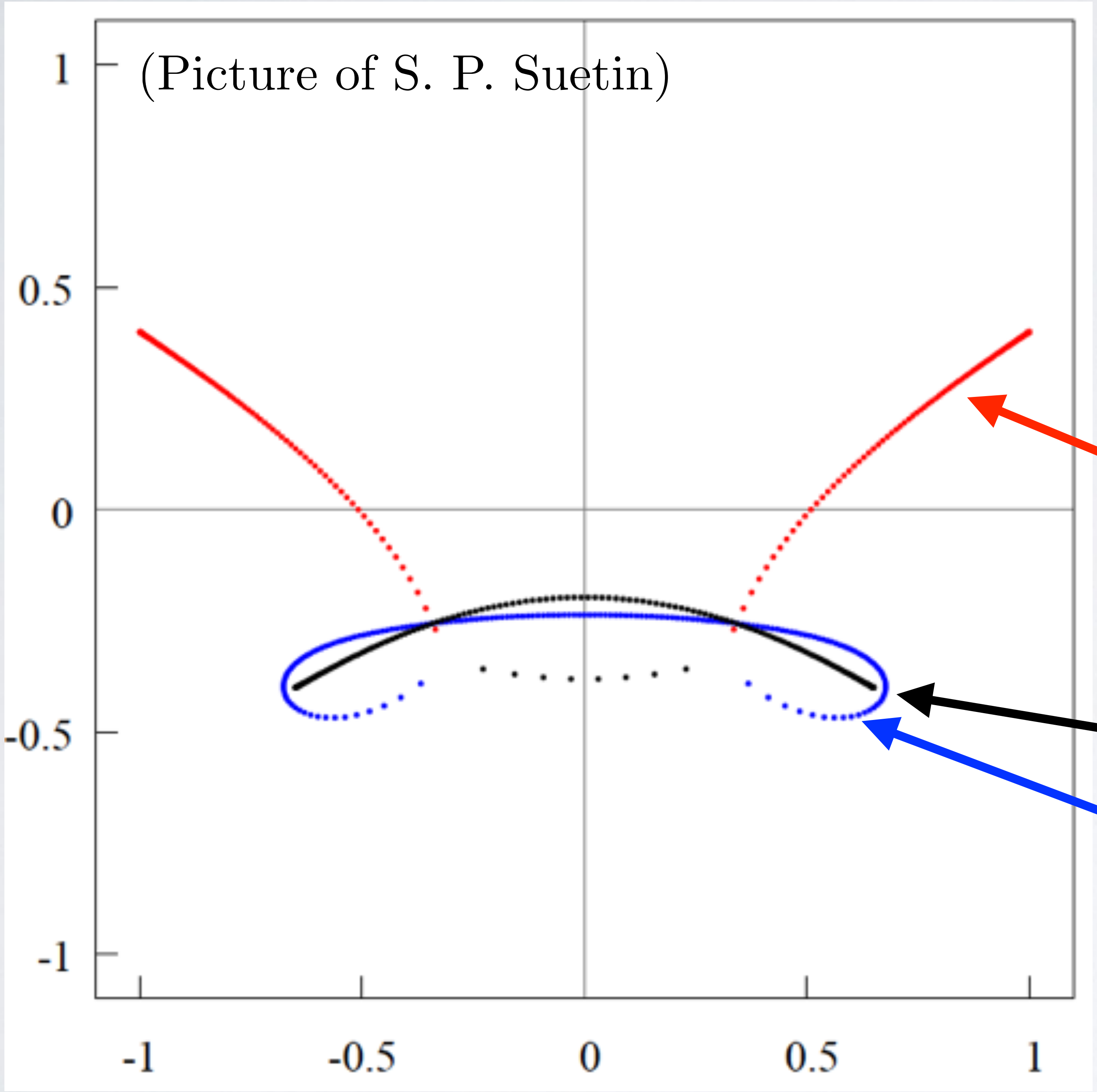
For $n \in \mathbb{N}$ we seek a vector of Hermite–Padé polynomials of the first kind, $Q_{n,0}$, $Q_{n,1}$, and $Q_{n,2}$, such that for $z \to \infty$,

$$R_n(z) = (Q_{n,0} + Q_{n,1} f_1 + Q_{n,2} f_2)(z) = \mathcal{O}\left(\frac{1}{z^{2n+2}}\right)$$

Again, for algebraic f_j we can derive several non-hermitian orthogonality relations, now involving both f_1 and f_2 (multiple orthogonality).

HERMITE-PADÉ APPROXIMANTS

One example: with $a_\pm = \pm 1 + 0.4i$, $b_\pm = -0.65 - 0.4i$,



$$f_1(z) = \frac{1}{\sqrt{(z-a_-)(z-a_+)}}$$

$$f_2(z) = \frac{1}{\sqrt{(z-b_-)(z-b_+)}}$$

Zeros of $Q_{180,1}$

Zeros of $Q_{180,2}$

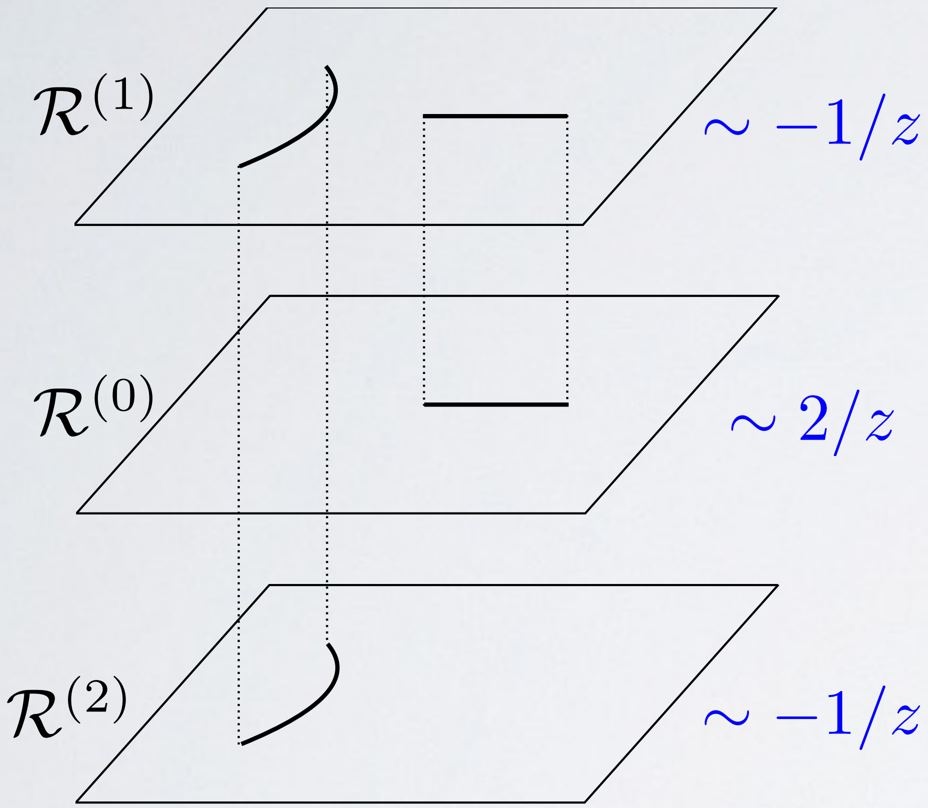
Zeros of $Q_{180,0}$

$$R_n(z) = (Q_{n,0} + Q_{n,1} f_1 + Q_{n,2} f_2)(z) = \mathcal{O}\left(\frac{1}{z^{2n+2}}\right)$$

ANALOGUE OF GONCHAR-RAKHMANOV?

Main ingredients:

- a compact 3-sheeted Riemann surface $\mathcal{R}$ associated with the problem



- a meromorphic differential $u(z)dz$ on $\mathcal{R}$ with prescribed behavior at $\infty^{(j)}$ and such that

$$\phi(z) = \operatorname{Re} \int^z u(z)dz$$

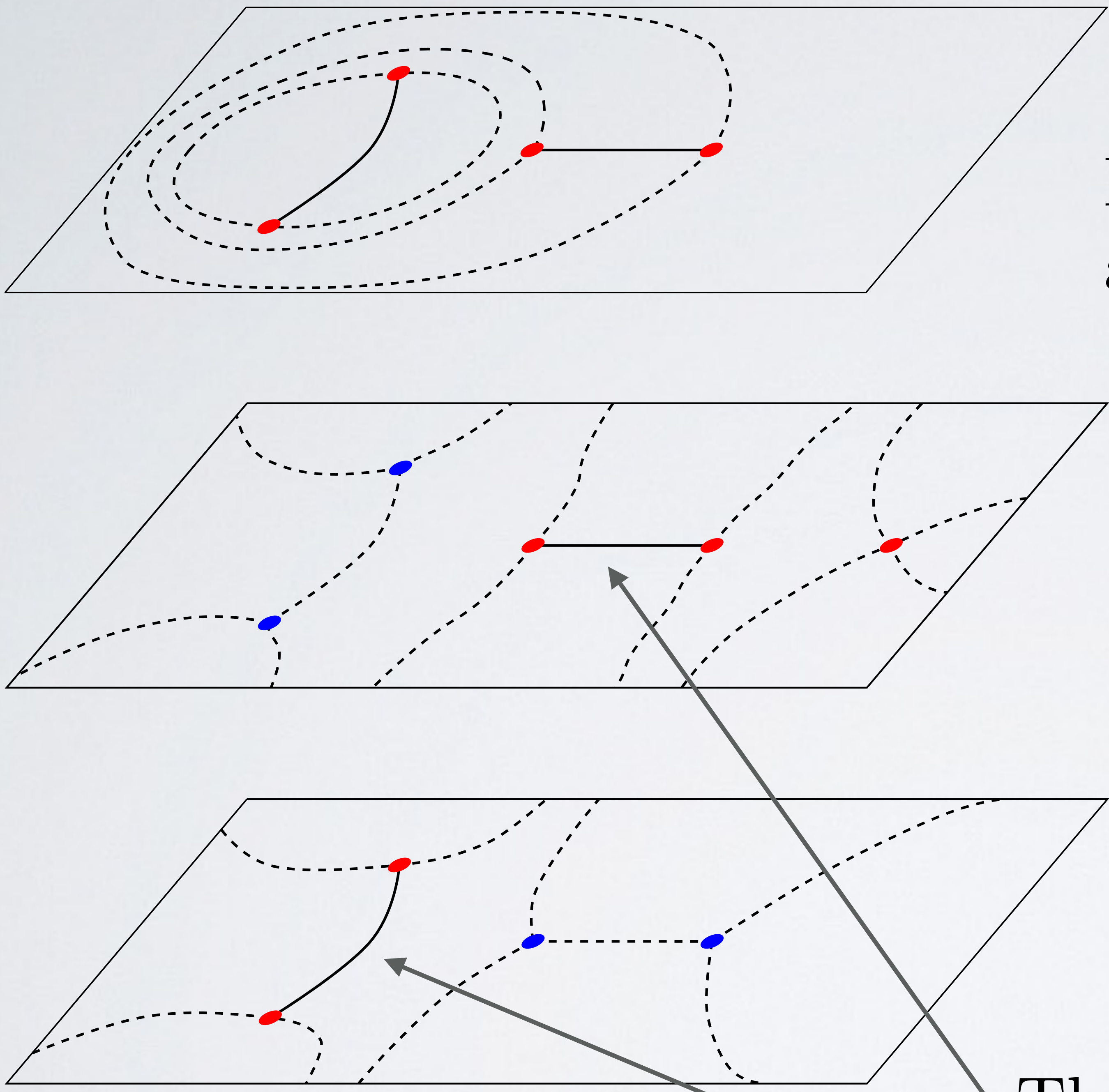
is single-valued on $\mathcal{R}$,

- a natural ordering of the sheets,

$$\phi(z^{(0)}) > \phi(z^{(1)}) > \phi(z^{(2)})$$

Then we look at the curves on $\mathcal{R}$ where $\phi(z^{(i)}) = \phi(z^{(j)})$, something like this:

ANALOGUE OF GONCHAR-RAKHMANOV?



Projections of these curves on $\mathbb{C}$ are our analogues of S-curves.

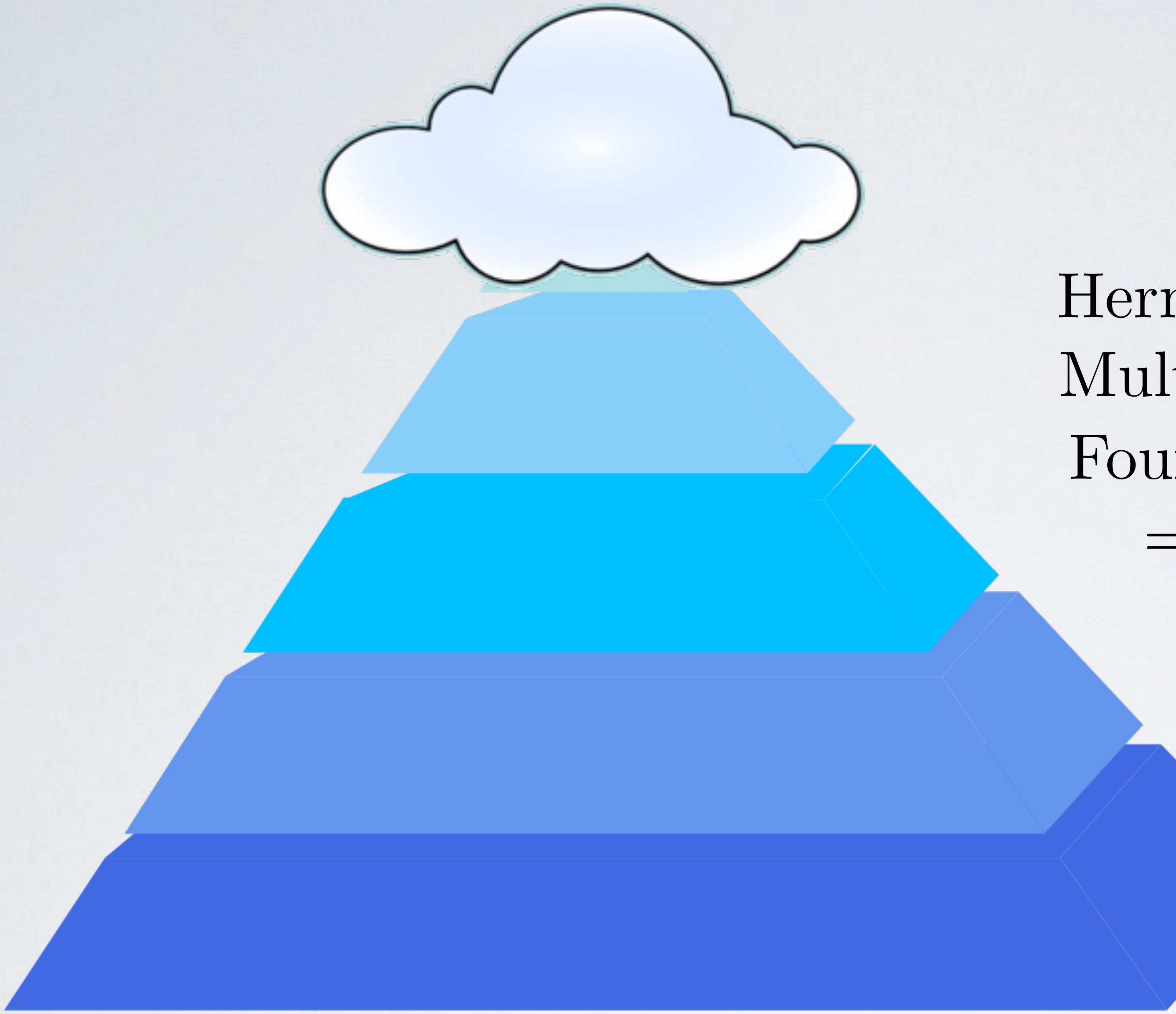
Again, they are trajectories of a quadratic differential on $\mathcal{R}$.

We can use them for the asymptotic analysis of some specific cases.

But we still don't have any analogue of Gonchar-Rakhmanov theorem!

Then we look at the curves on $\mathcal{R}$ where $\phi(z^{(i)}) = \phi(z^{(j)})$, something like this:

[Picture from a work in progress with G. Silva]



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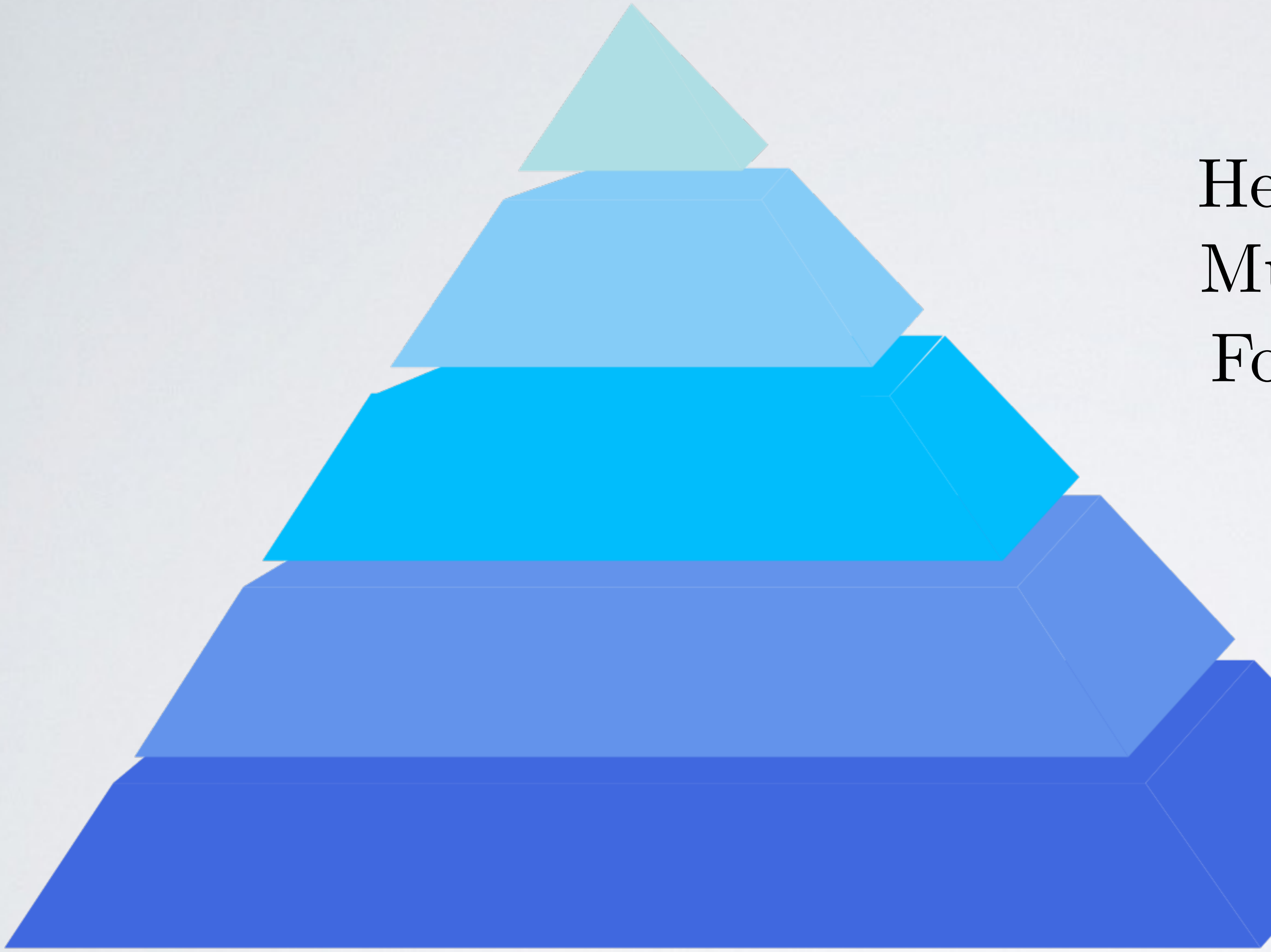

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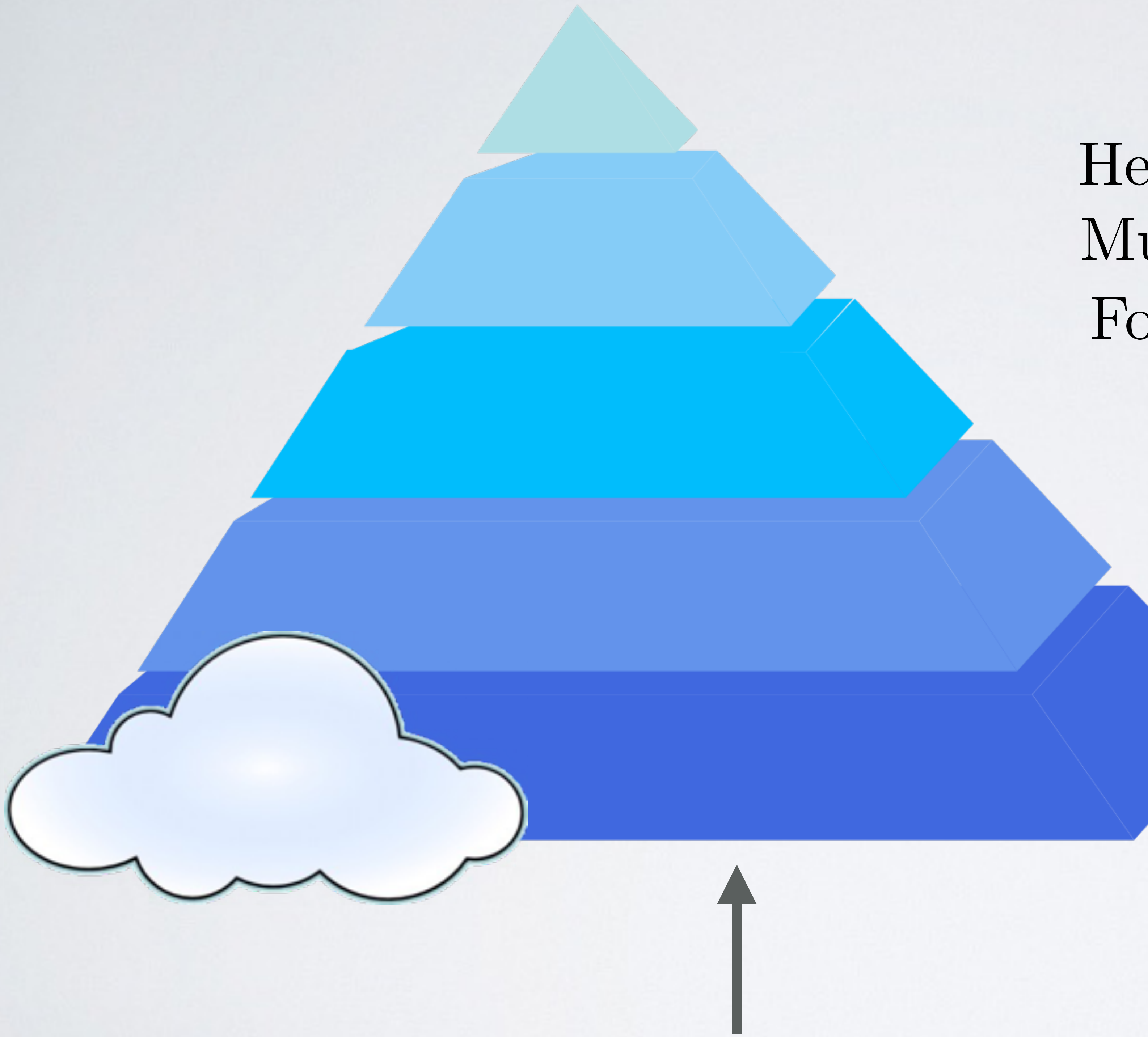
Let us look more carefully here, at the bottom



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A SIMPLE PROBLEM

Let us consider a final example: orthogonal polynomials Q_n satisfying

$$\int_C z^k Q_n(z) e^{-n\alpha z} dz = 0, \quad k = 0, 1, \dots, n-1,$$

where C is an arc joining points $a_1 = -1 + 2i$ and $a_2 = -1 - 2i$, and $\alpha \geq 0$.

The zeros of Q_n will accumulate at the curve with the S-property for the log potential in the external field $\psi(z) = \alpha \operatorname{Re} z$.

We can use the Gonchar-Rakhmanov theorem

Clearly, for $\alpha = 0$ the S-curve is just the segment $[a_1, a_2]$.

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$$\frac{(z-\beta)^2}{(z-a_1)(z-a_2)} dz^2, \quad \beta = -1 + \frac{1}{\alpha}$$

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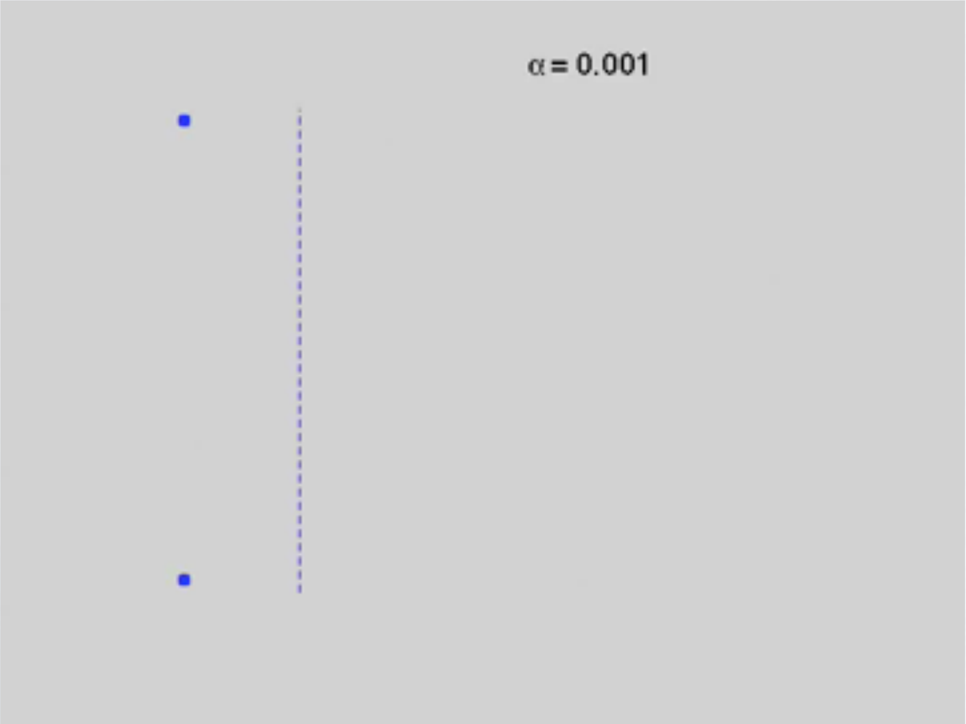

As α grows, the double zero β moves towards the curve.

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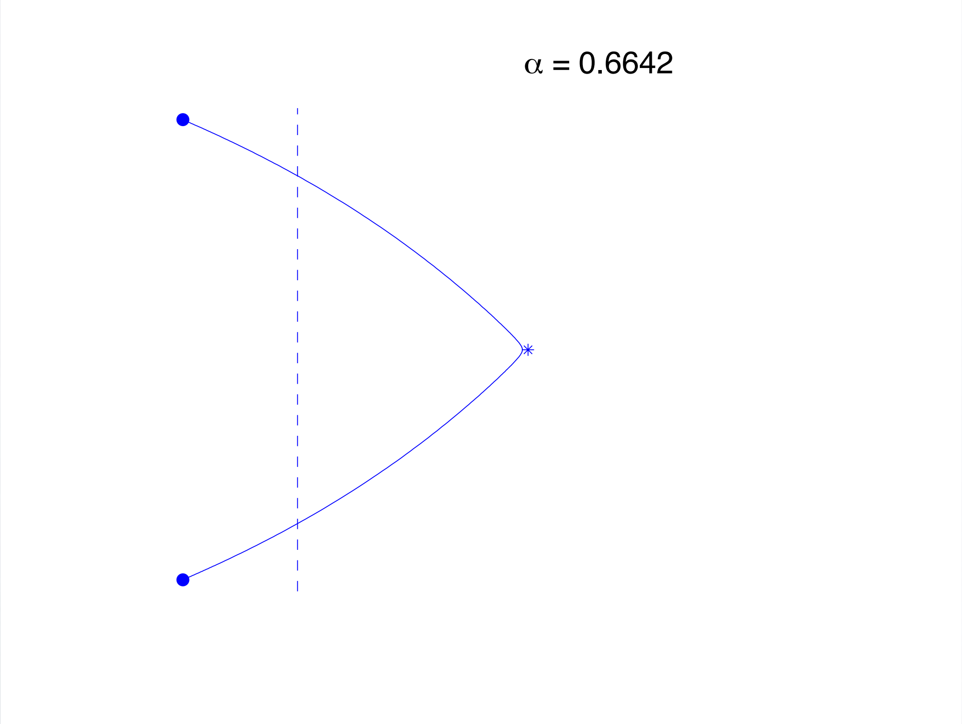

There is a critical value of α^* for which C crosses the imaginary axis, and another critical value α^{**} for which β collides with C!

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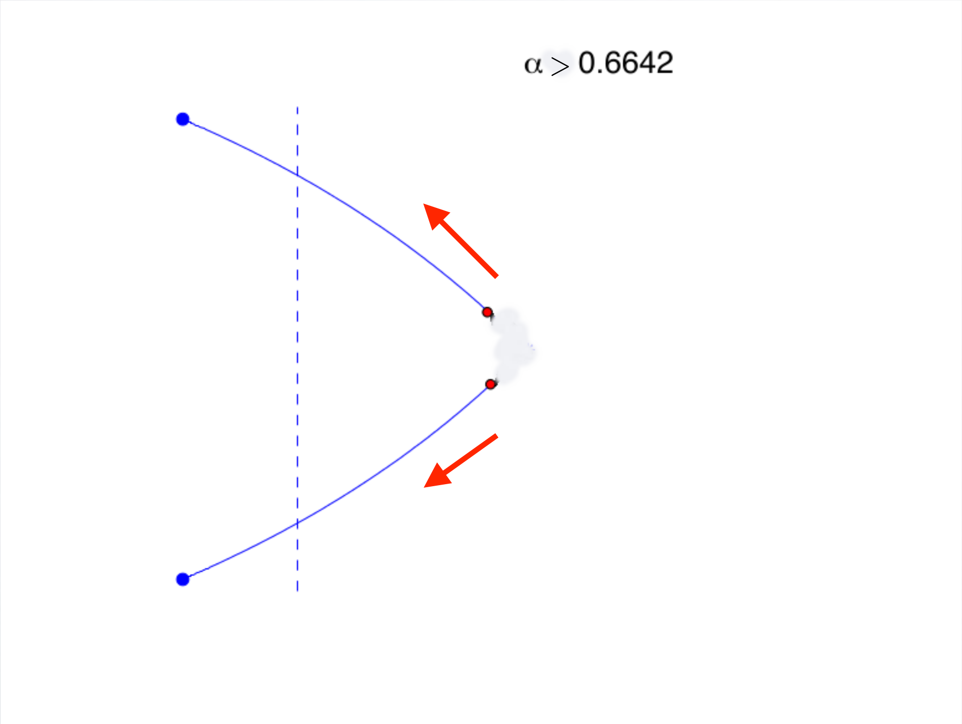

After the collision, the support of μ_K splits into 2 pieces, shrinking towards a_j as $\alpha \to \infty$.

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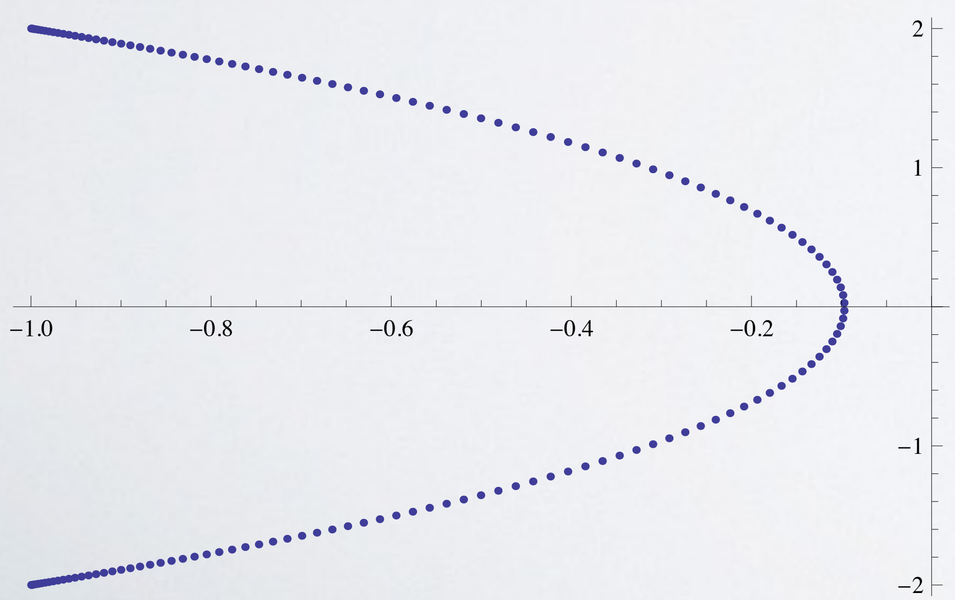

The zeros of Q_n comply.

$$\alpha = 0.4$$

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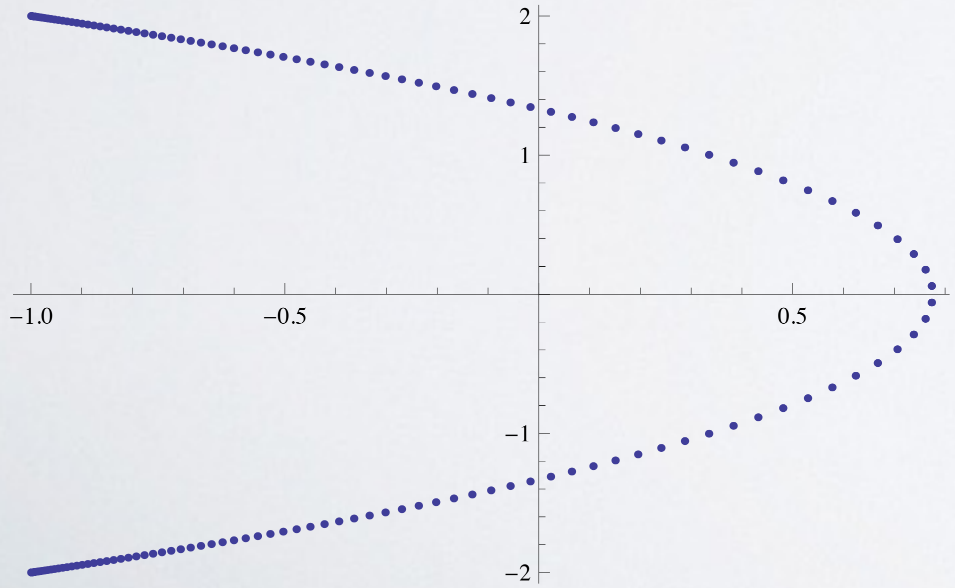

The zeros of Q_n comply.

$$\alpha = 0.6$$

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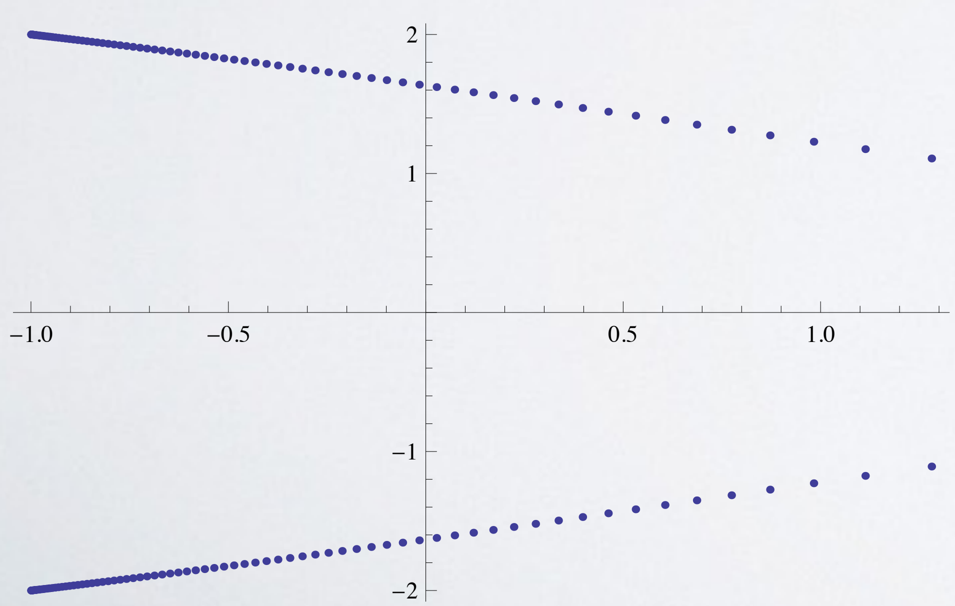

The zeros of Q_n comply.

$\alpha = 0.8$

Everything is clear, life is nice.

AN APPARENTLY SIMPLE PROBLEM

Let us modify the problem slightly: orthogonal polynomials Q_n satisfying

$$\int_C z^k Q_n(z)\left(1+e^{-n\alpha z}\right)dz = 0, \quad k = 0, 1, \dots, n-1,$$

where C is an arc joining points $a_1 = -1+2i$ and $a_2 = -1-2i$, and $\alpha \geq 0$.

This orthogonality is connected with the logarithmic equilibrium in a piecewise-harmonic external field

$$\psi(z) = \begin{cases} \alpha \operatorname{Re} z, & \operatorname{Re} z \leq 0, \\ 0, & \operatorname{Re} z > 0. \end{cases}$$

Gonchar-Rakhmanov theorem does not apply always!

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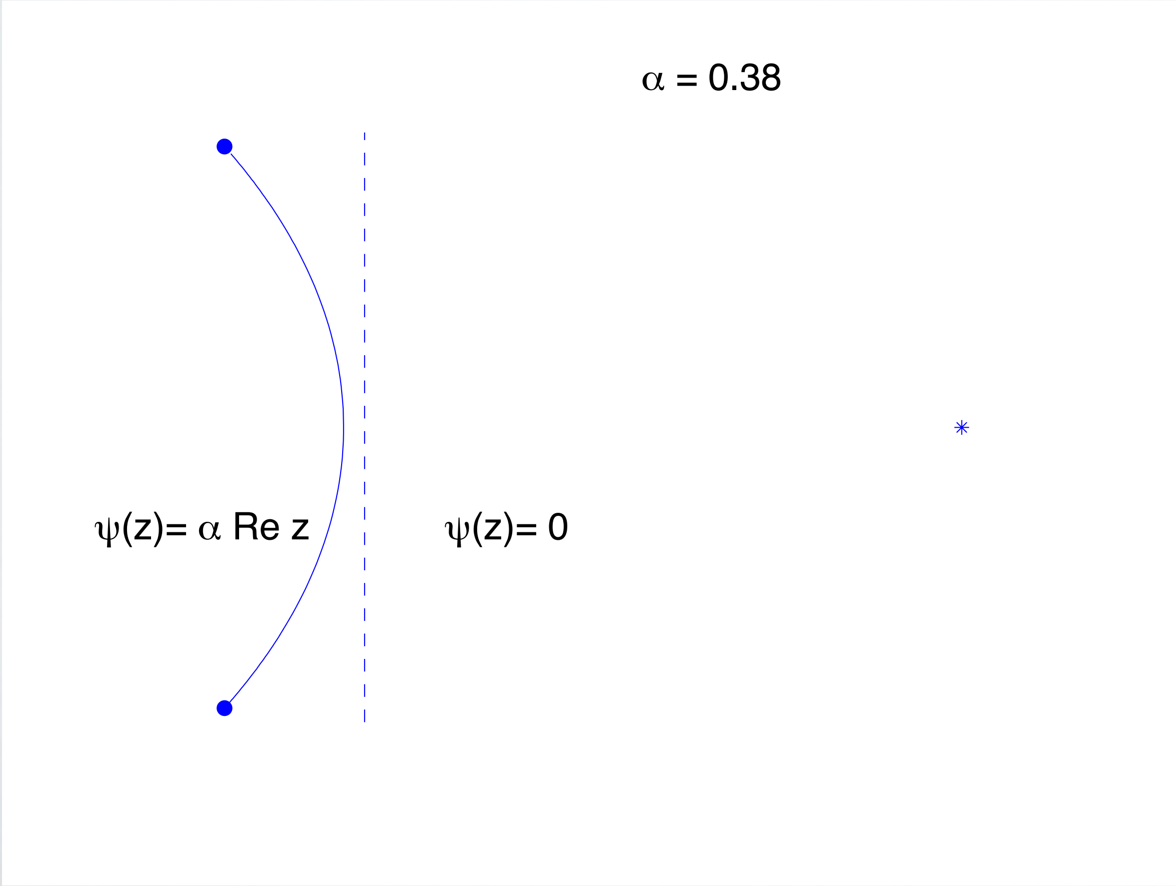
For small values of α, the min-max curve C is still the short trajectory of the quadratic differential



$$\frac{(z-\beta)^2}{(z-a_1)(z-a_2)}dz^2, \quad \beta = -1+\frac{1}{\alpha}$$

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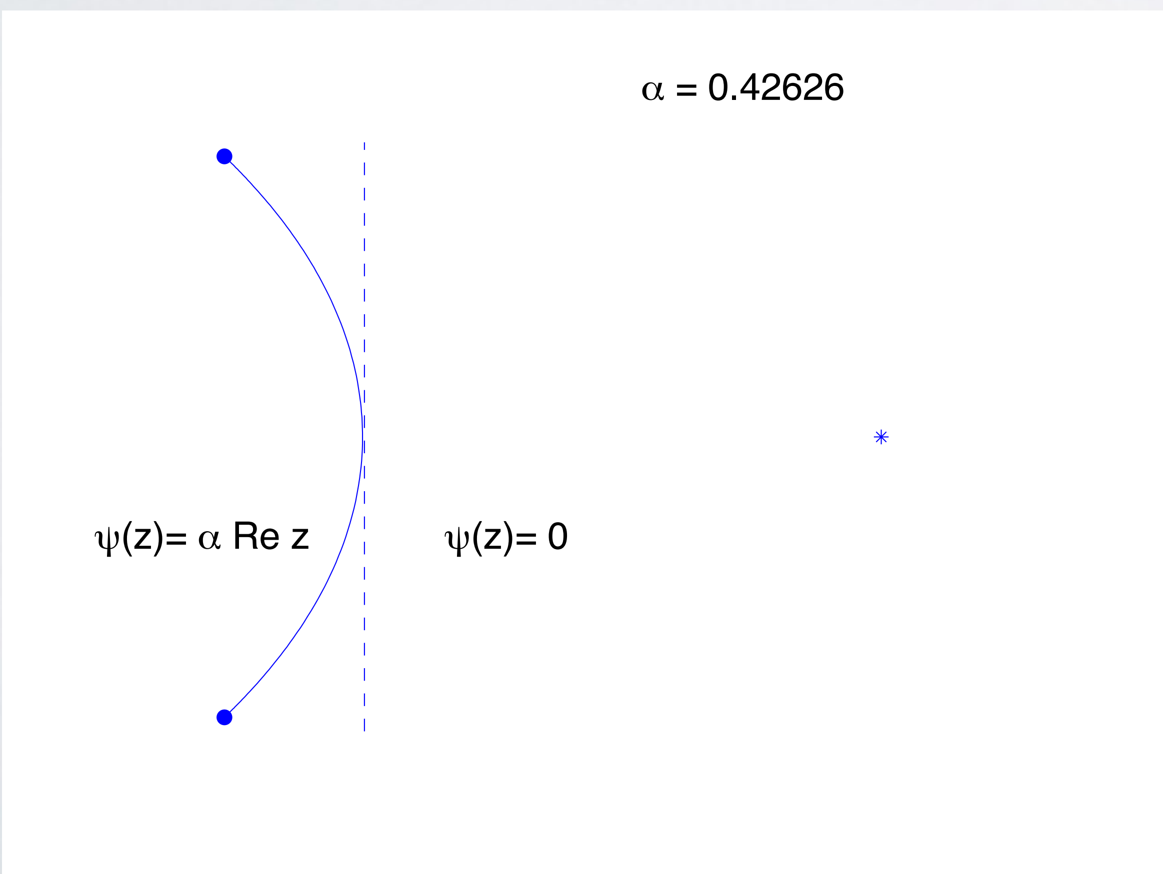

$$\frac{(z-\beta)^2}{(z-a_1)(z-a_2)}dz^2, \quad \beta = -1+\frac{1}{\alpha}$$

Up to the critical value of α.

Observe: no S-property at the vertex:

$$\frac{\partial}{\partial n_+}\left(V^\mu+\psi\right) - \frac{\partial}{\partial n_-}\left(V^\mu+\psi\right) = \alpha$$

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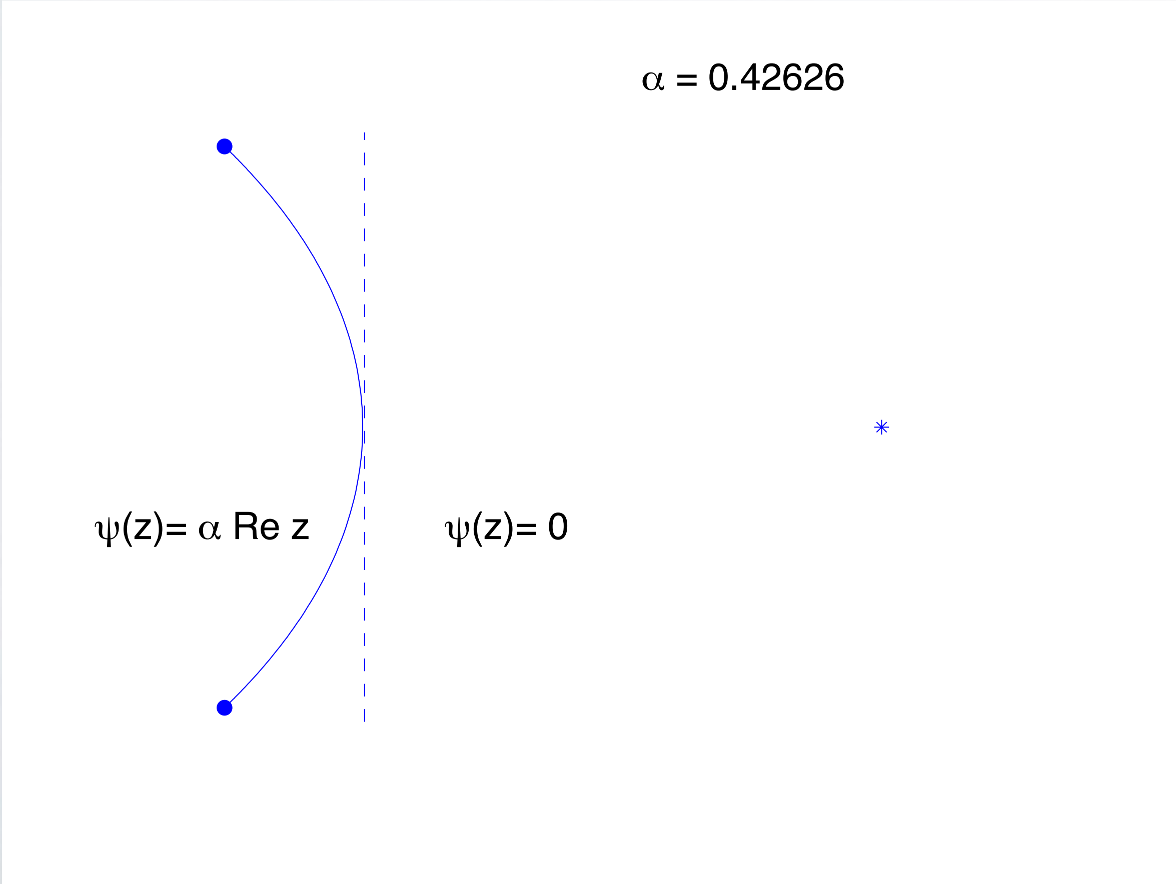

$$\frac{(z-\beta)^2}{(z-a_1)(z-a_2)}dz^2, \quad \beta = -1+\frac{1}{\alpha}$$

What happens beyond this α?

What is the asymptotic zeros distribution of such polynomials?

No curve with S-property in this case?

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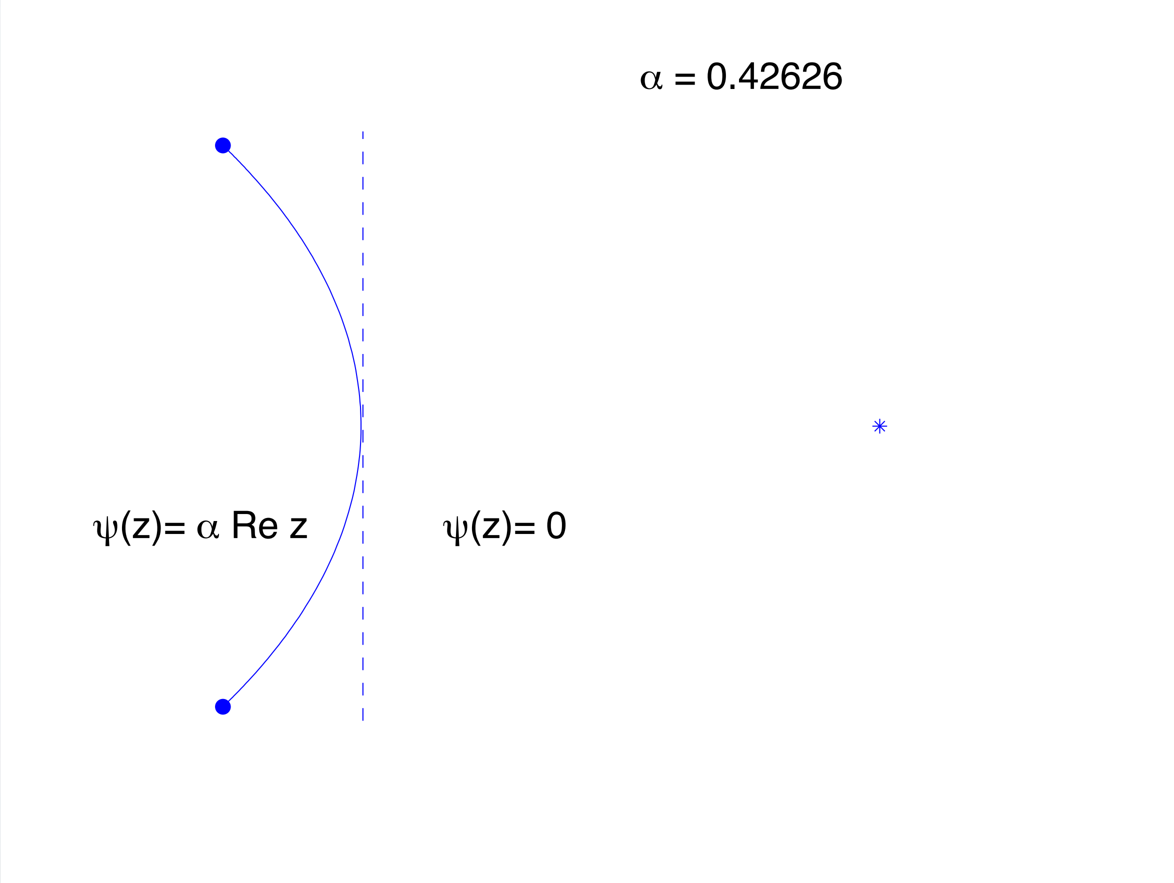

$$\frac{(z-\beta)^2}{(z-a_1)(z-a_2)}dz^2, \quad \beta = -1 + \frac{1}{\alpha}$$

If the S-property no longer rules, then what does?

Maybe we should go back to the origin and recall the max-min property of the energy?

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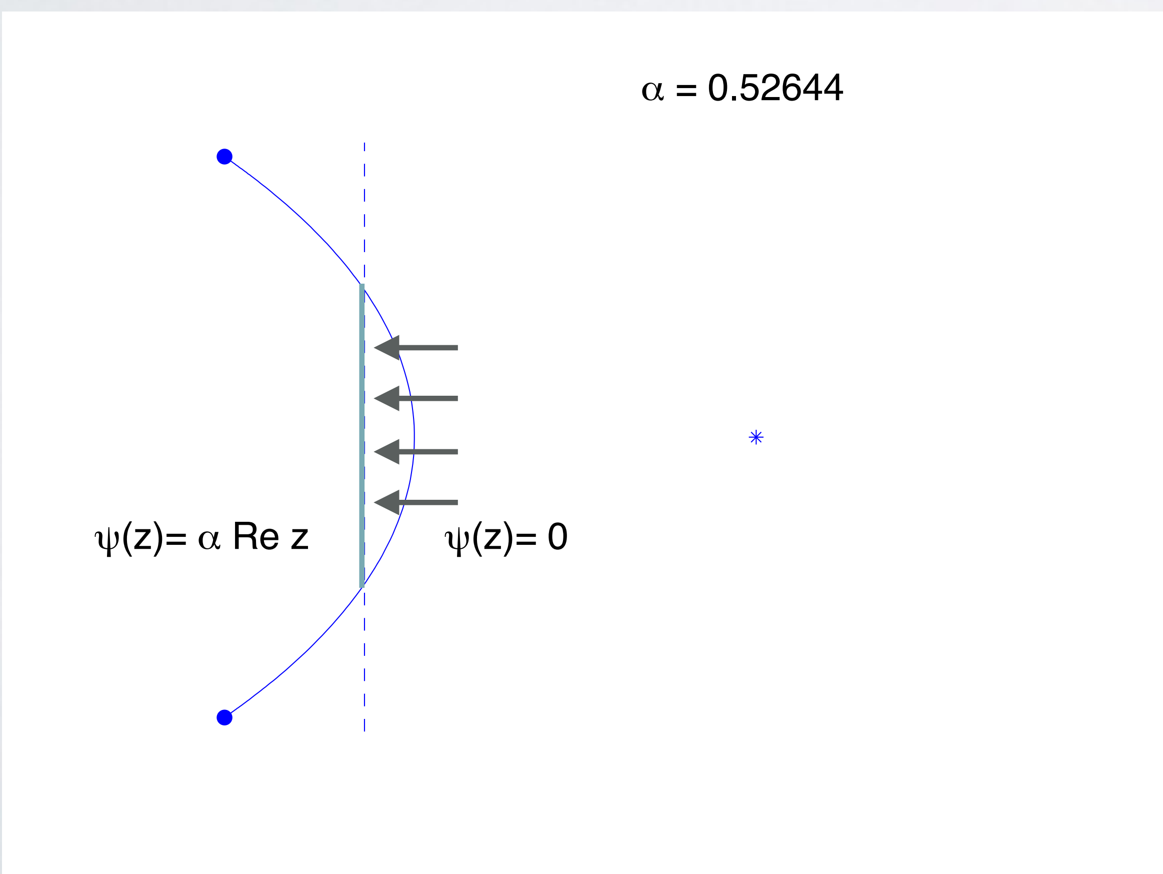

$$\frac{(z-\beta)^2}{(z-a_1)(z-a_2)} dz^2, \quad \beta = -1 + \frac{1}{\alpha}$$

For larger α's, the max-min set will still lie in the left half-plane.

The trajectories are no longer associated with one quadratic differential.

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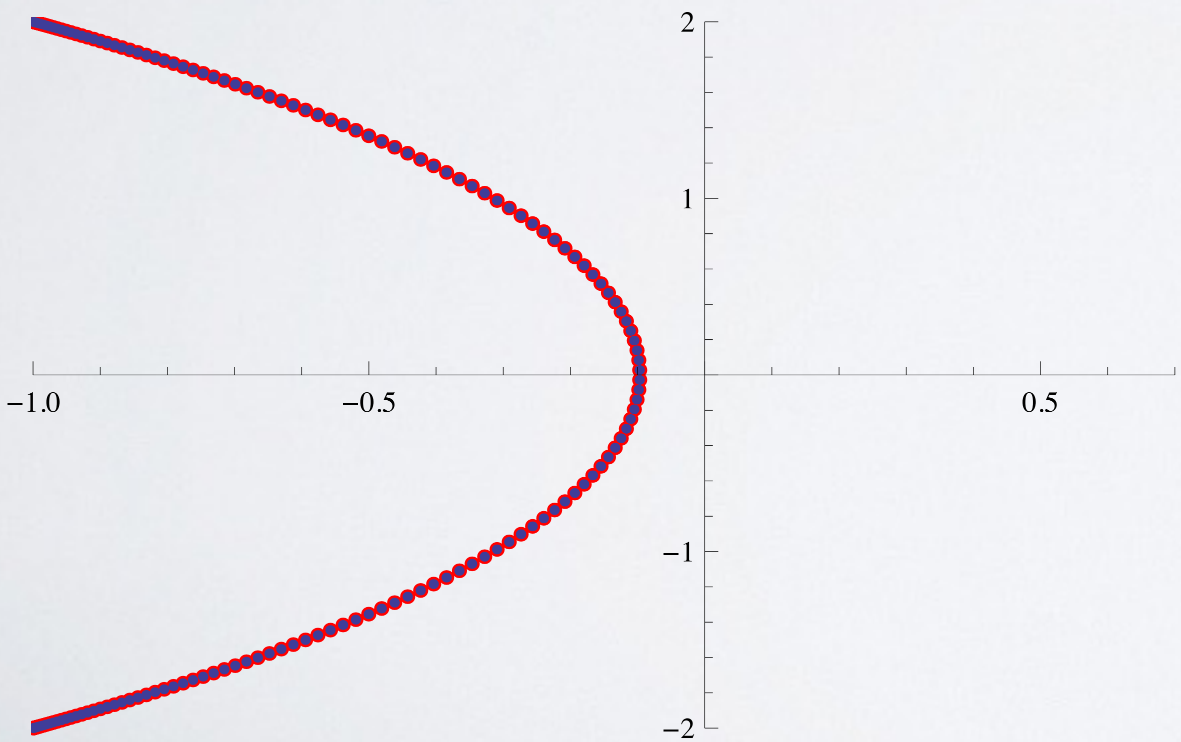

Zeros of Q_{150} for $w(z) = e^{-150\alpha z}$ (small blue dots) and for $w(z) = 1 + e^{-150\alpha z}$ (medium red dots), in the pre-critical case,

$$\alpha = 0.4$$

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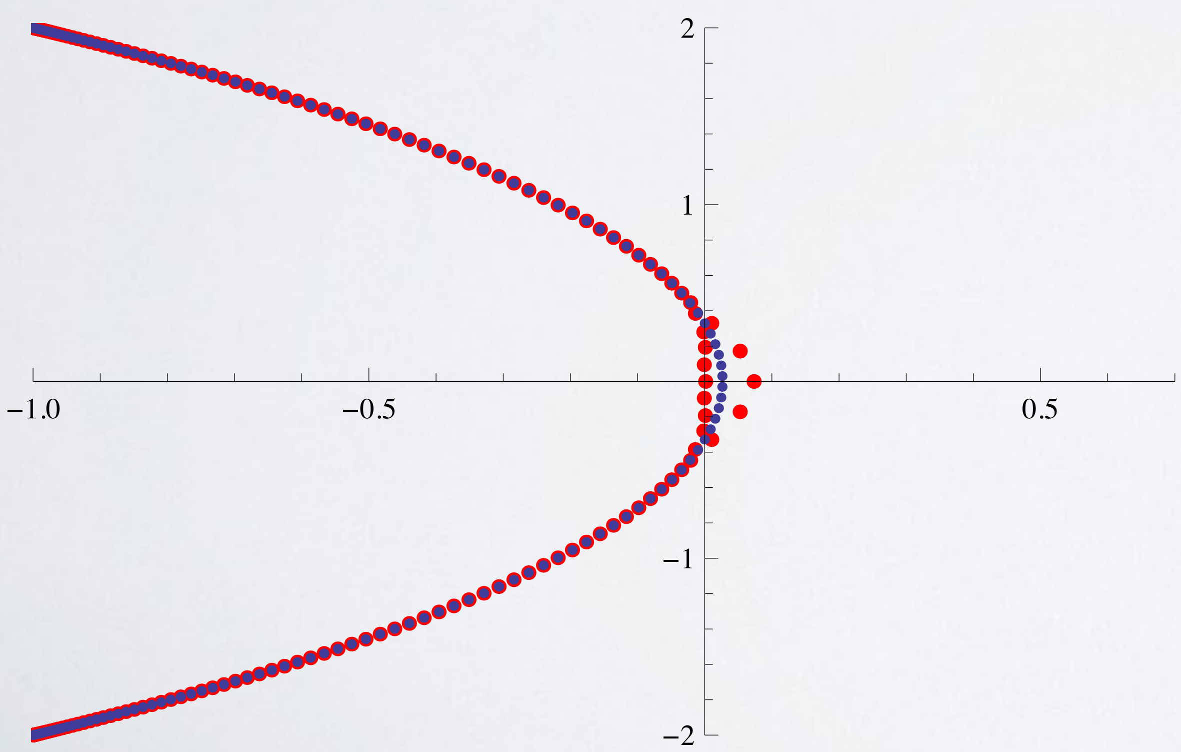

Zeros of Q_{150} for $w(z) = e^{-150\alpha z}$ (small blue dots) and for $w(z) = 1+e^{-150\alpha z}$ (medium red dots), immediately after α^*,

$$\alpha = 0.44$$

AN APPARENTLY SIMPLE PROBLEM

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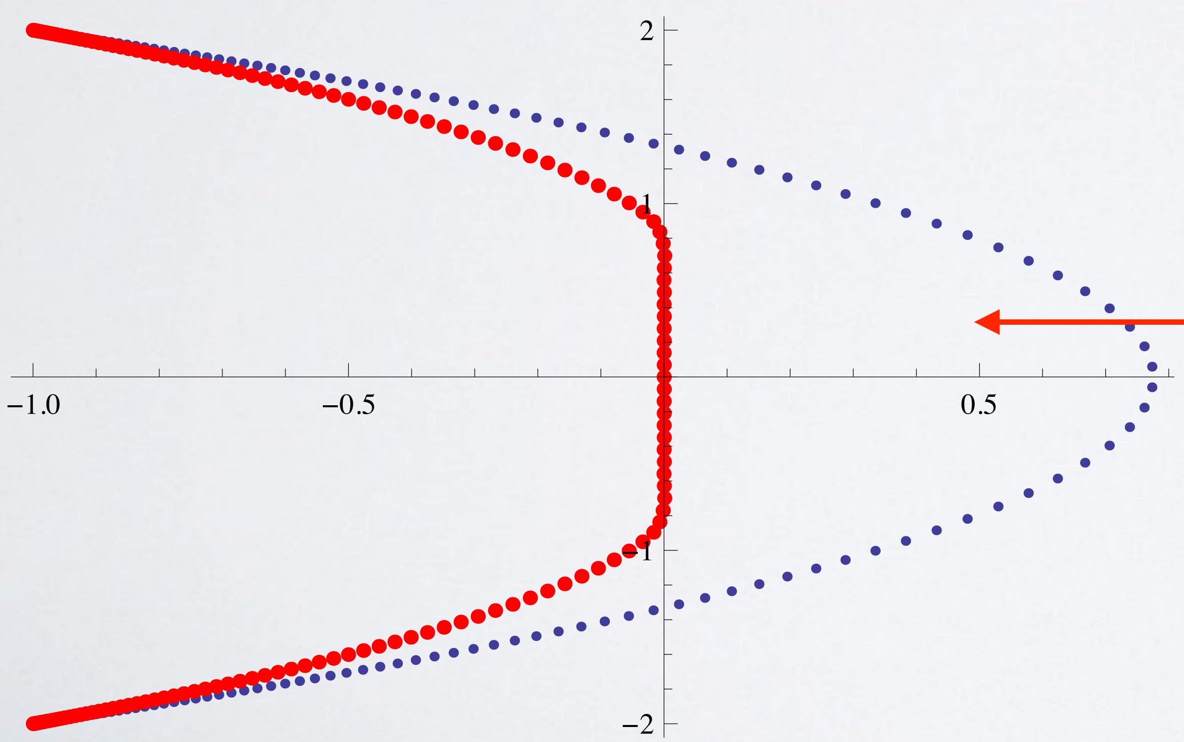

Isn't that beautiful?
max-min rules!

Let α grow a little more:

$$\alpha = 0.6$$

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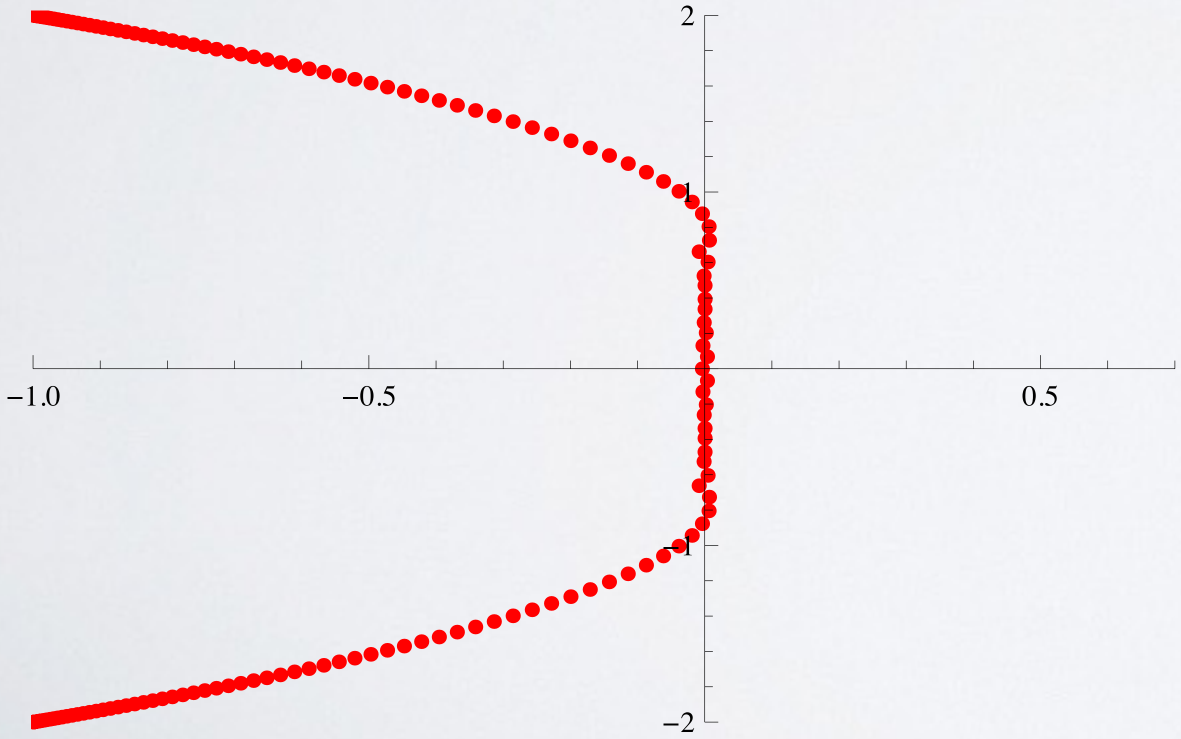
Let us concentrate on the zeros of Q_{150} for $w(z) = 1 + e^{-150\alpha z}$.



What happens next?

$\alpha = 0.63$

Numerical instability? A riot?

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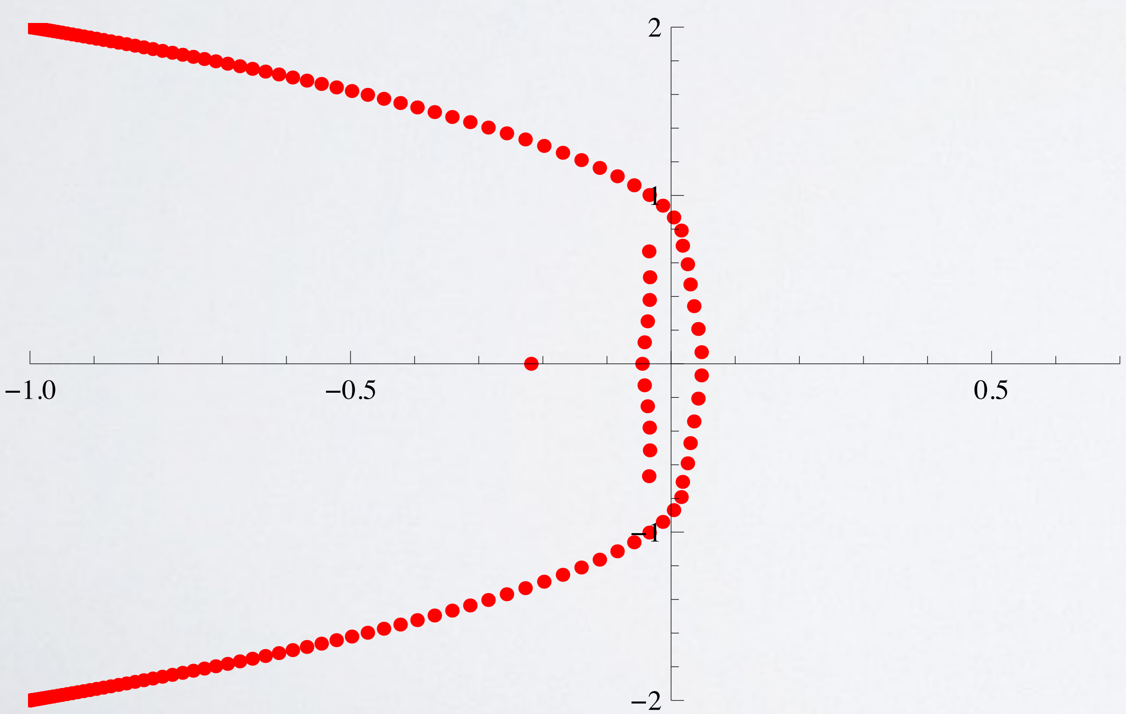

What happens next?

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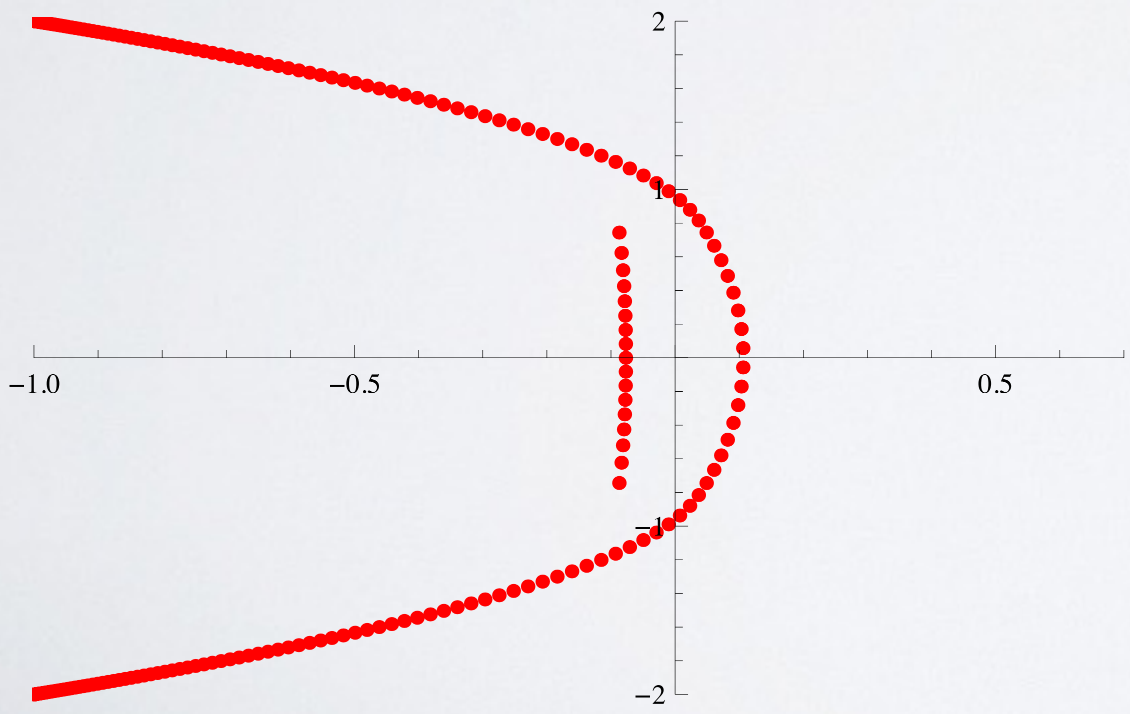
where C is an arc joining points $a_1 = -1 + 2i$ and $a_2 = -1 - 2i$, and $\alpha \geq 0$.

Let us concentrate on the zeros of Q_{150} for $w(z) = 1 + e^{-150\alpha z}$.



What happens next?

$\alpha = 0.66$

Numerical instability? A riot?

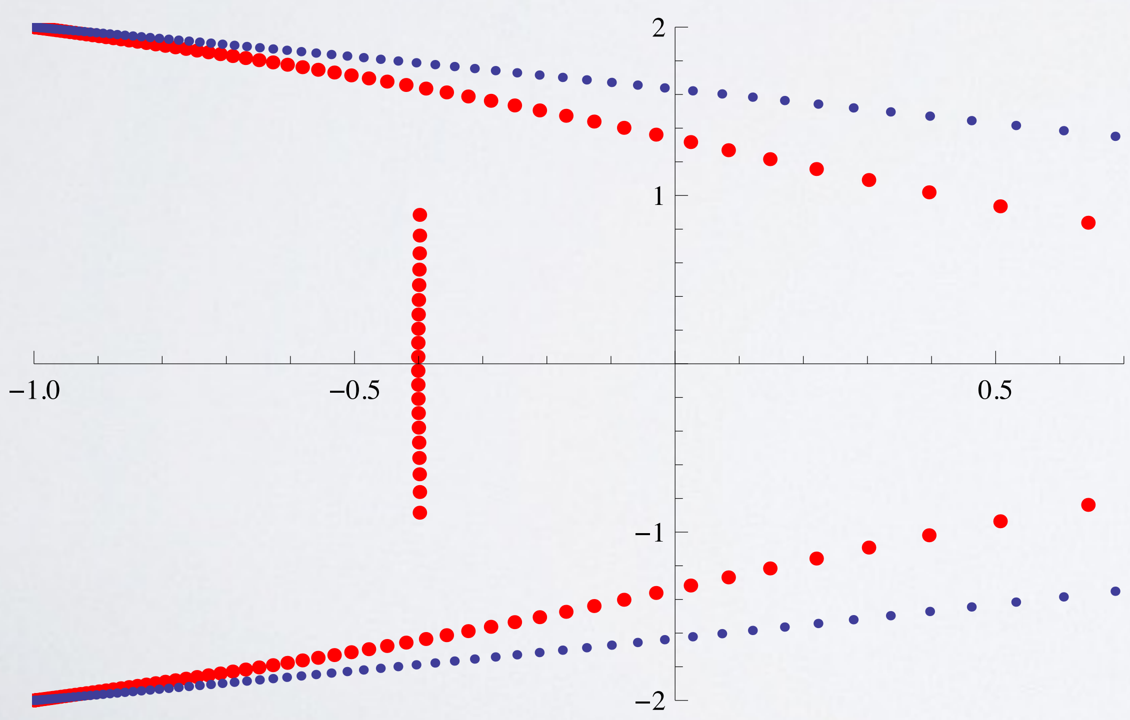
AN APPARENTLY SIMPLE PROBLEM

Let us modify the problem slightly: orthogonal polynomials Q_n satisfying

$$\int_C z^k Q_n(z)\left(1+e^{-n\alpha z}\right)dz = 0, \quad k = 0, 1, \ldots, n-1,$$

where C is an arc joining points $a_1 = -1+2i$ and $a_2 = -1-2i$, and $\alpha \geq 0$.

Let us concentrate on the zeros of Q_{150} for $w(z) = 1 + e^{-150\alpha z}$.



After the second phase transition:

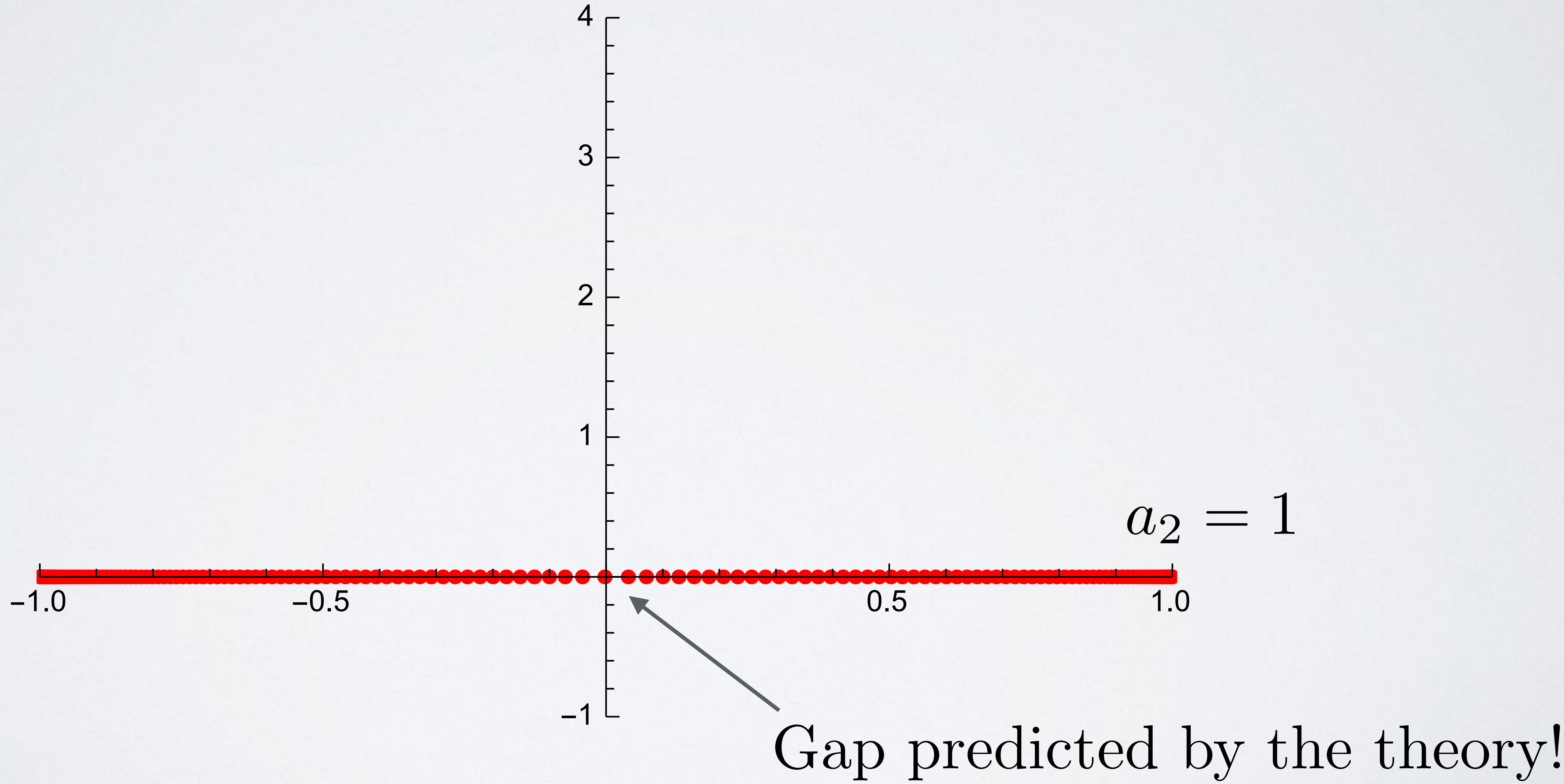
$$\alpha = 0.8$$

The freedom is complete!

AN APPARENTLY SIMPLE PROBLEM

$$\int_C z^k Q_n(z)\left(1+e^{-n\alpha z}\right)dz = 0, \quad k = 0,1,\ldots,n-1,$$

The situation is even more puzzling if we see how the distribution evolves with a fixed parameter $\alpha = 0.8$, but with C joining -1 with a_2, where a_2 goes from 1 to $-1+4i$.



AN APPARENTLY SIMPLE PROBLEM

$$\int_C z^k Q_n(z)\left(1+e^{-n\alpha z}\right)dz = 0, \quad k = 0, 1, \ldots, n-1,$$

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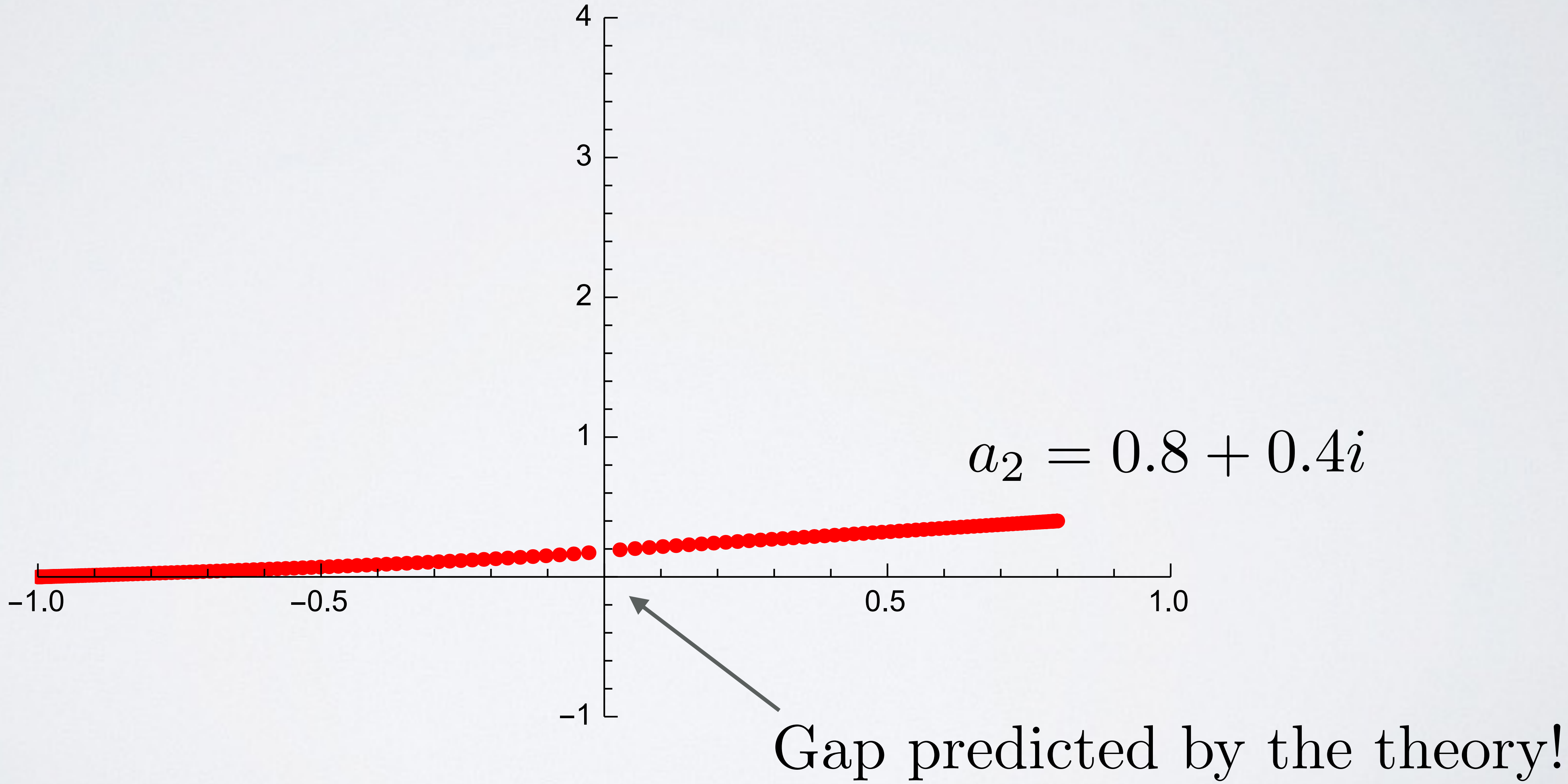

AN APPARENTLY SIMPLE PROBLEM

$$\int_C z^k Q_n(z)\left(1 + e^{-n\alpha z}\right) dz = 0, \quad k = 0, 1, \ldots, n-1,$$

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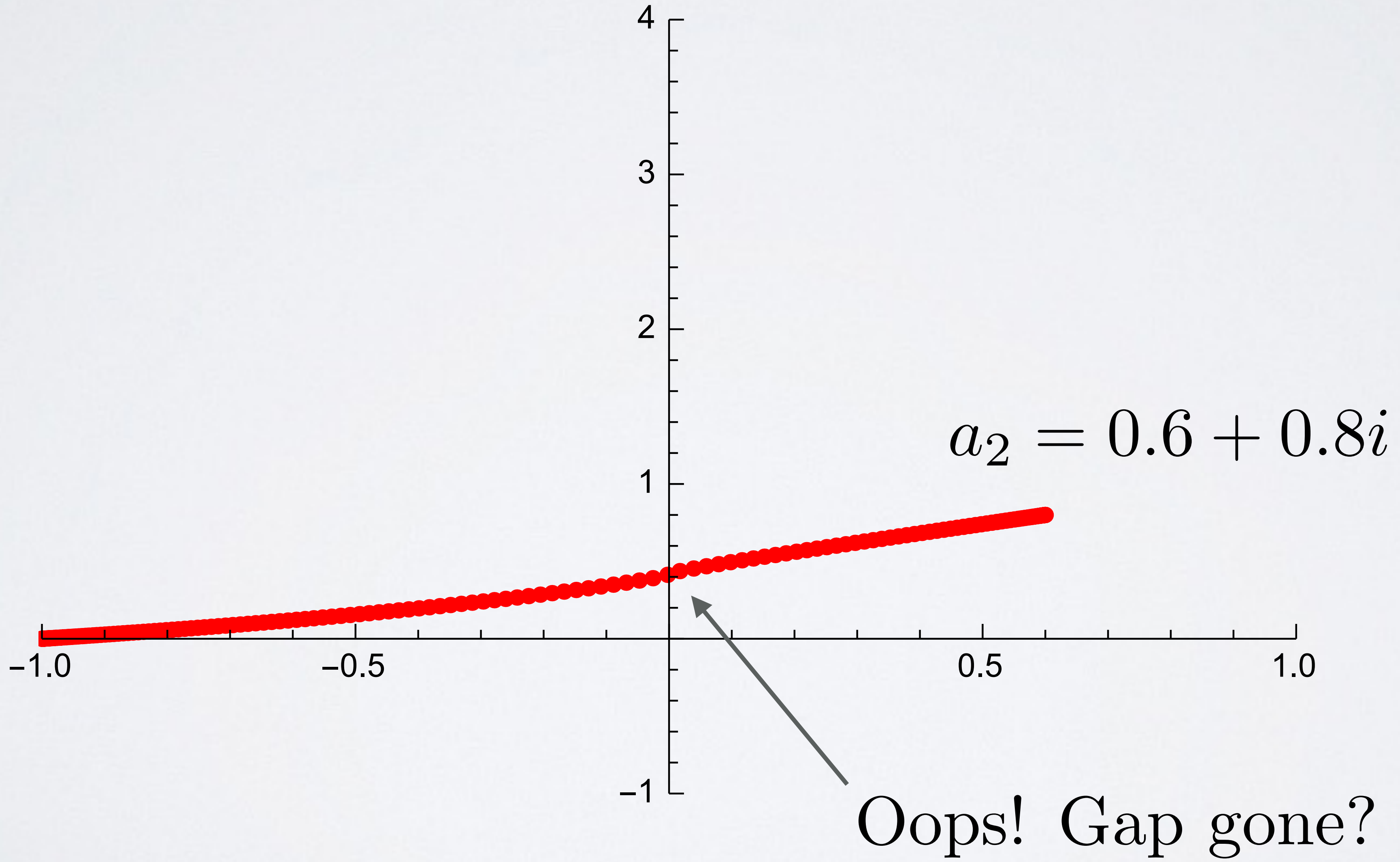

AN APPARENTLY SIMPLE PROBLEM

$$\int_C z^k Q_n(z)\left(1+e^{-n\alpha z}\right)dz = 0, \quad k = 0, 1, \ldots, n-1,$$

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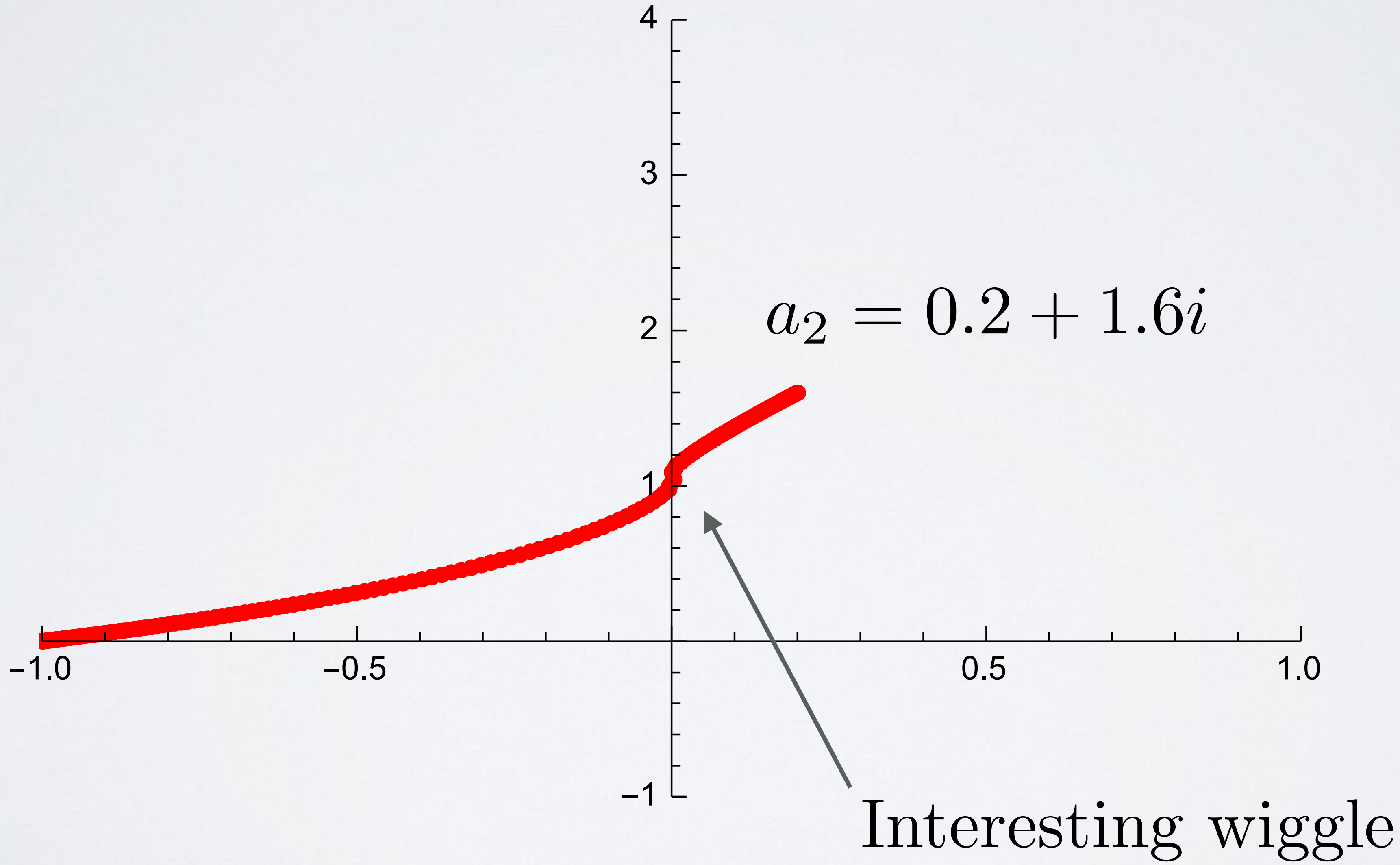
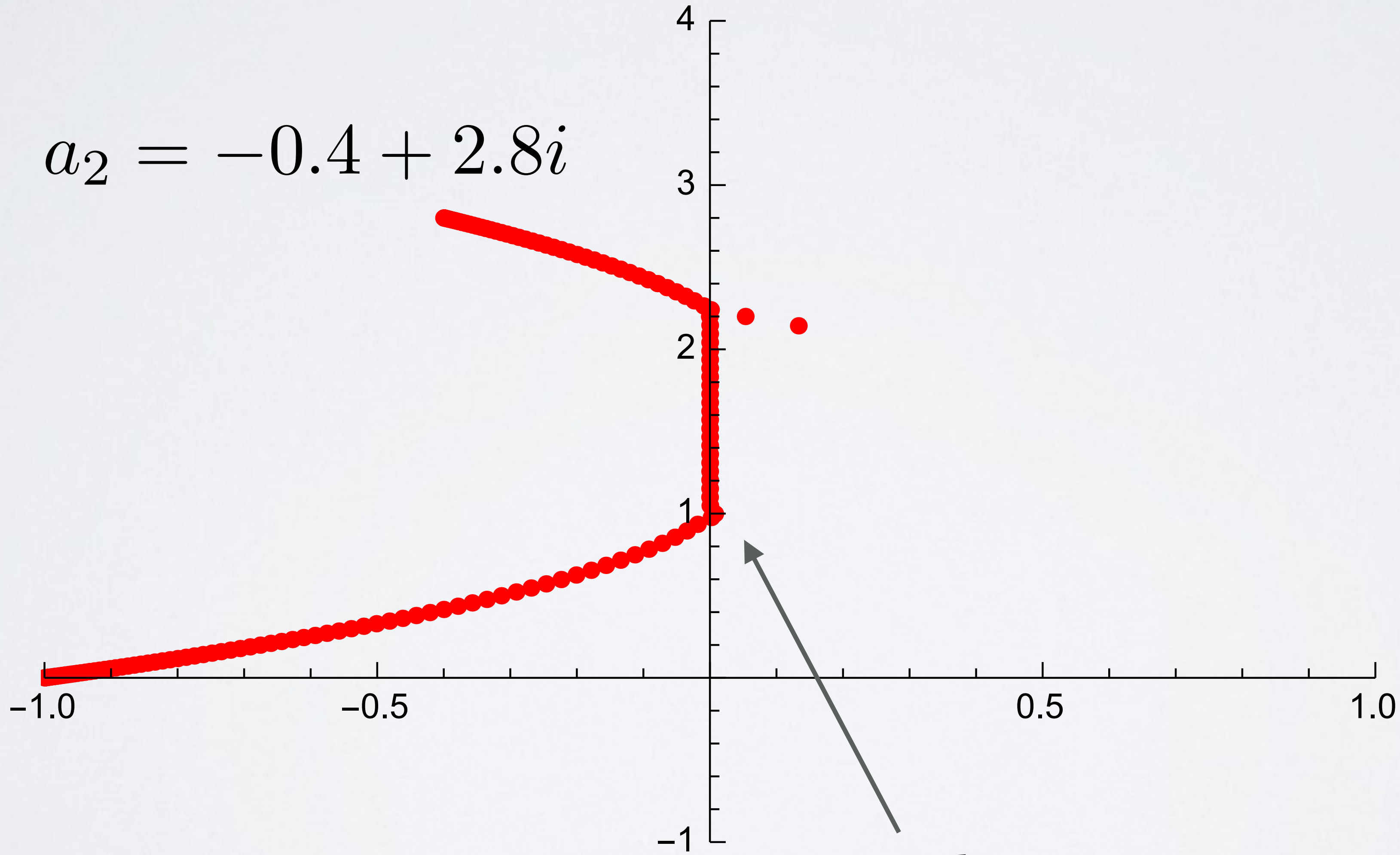


AN APPARENTLY SIMPLE PROBLEM

$$\int_C z^k Q_n(z)\big(1+e^{-n\alpha z}\big)dz = 0, \quad k = 0, 1, \ldots, n-1,$$

The situation is even more puzzling if we see how the distribution evolves with a fixed parameter $\alpha = 0.8$, but with C joining -1 with a_2, where a_2 goes from 1 to $-1+4i$.

$a_2 = -0.4 + 2.8i$



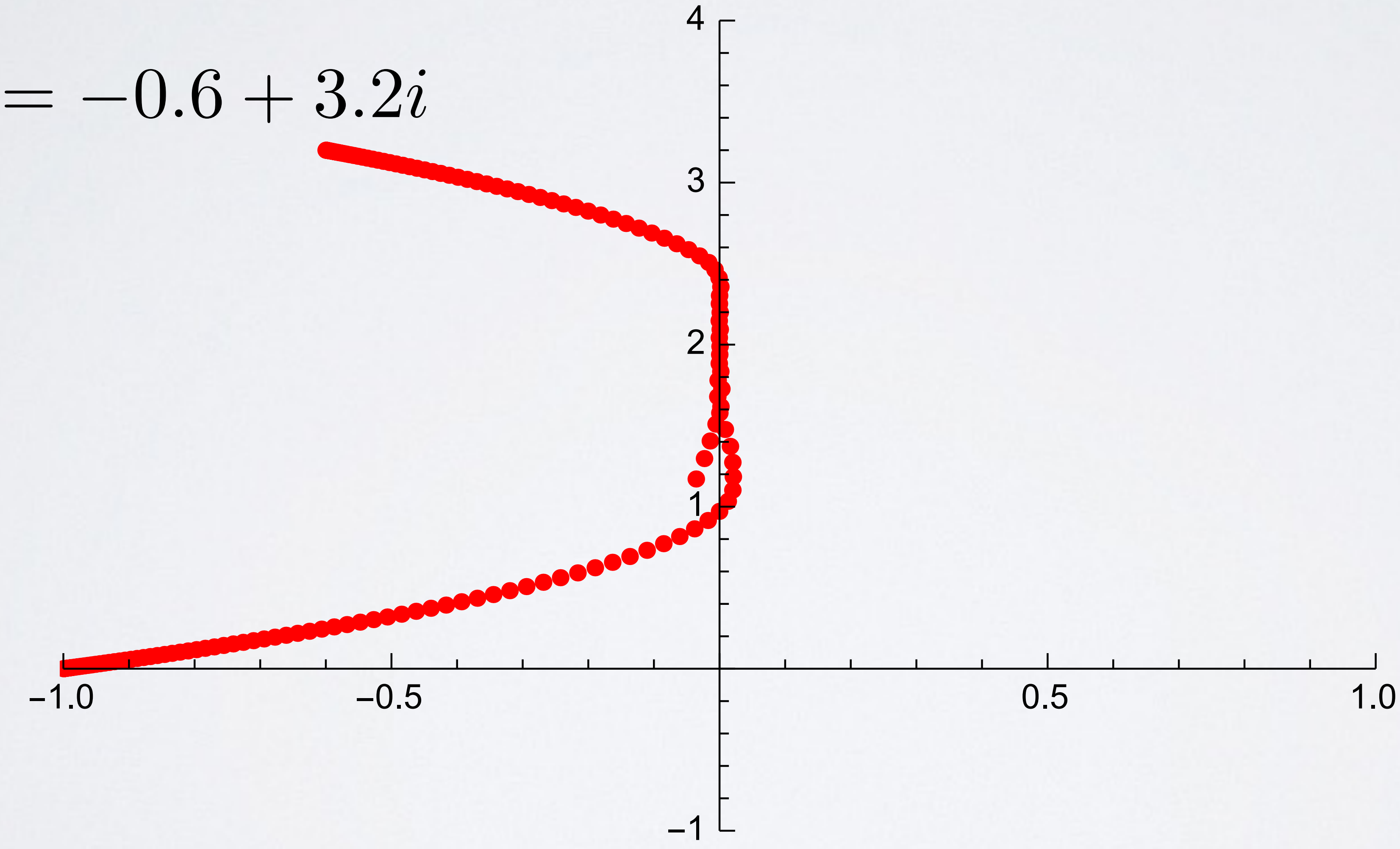
The max-min solution

AN APPARENTLY SIMPLE PROBLEM

$$\int_C z^k Q_n(z)\left(1+e^{-n\alpha z}\right)dz = 0, \quad k=0,1,\ldots,n-1,$$

The situation is even more puzzling if we see how the distribution evolves with a fixed parameter $\alpha = 0.8$, but with C joining -1 with a_2, where a_2 goes from 1 to $-1+4i$.

$a_2 = -0.6 + 3.2i$

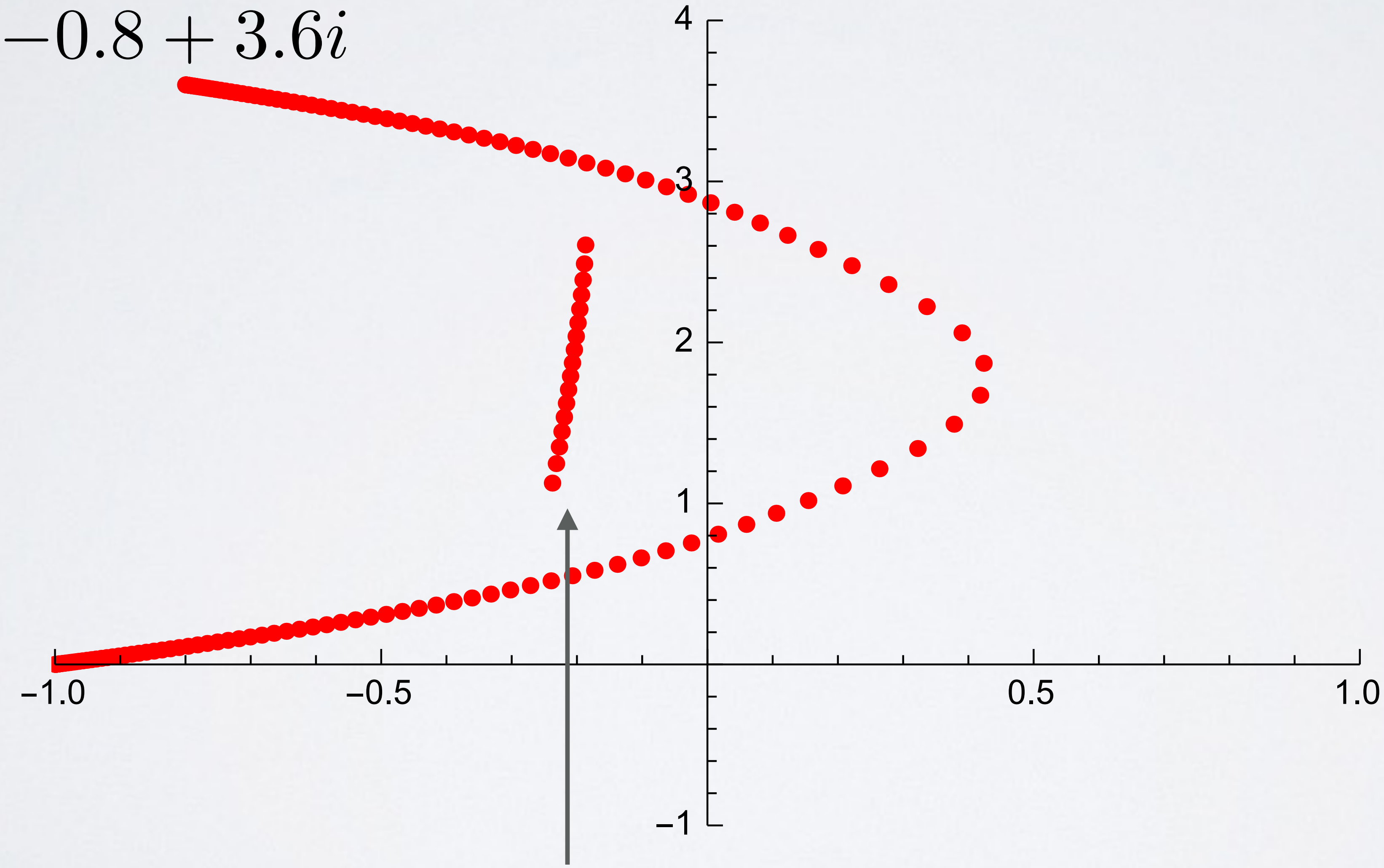


AN APPARENTLY SIMPLE PROBLEM

$$\int_C z^k Q_n(z)\left(1 + e^{-n\alpha z}\right) dz = 0, \quad k = 0, 1, \ldots, n-1,$$

The situation is even more puzzling if we see how the distribution evolves with a fixed parameter $\alpha = 0.8$, but with C joining -1 with a_2, where a_2 goes from 1 to $-1 + 4i$.

$a_2 = -0.8 + 3.6i$

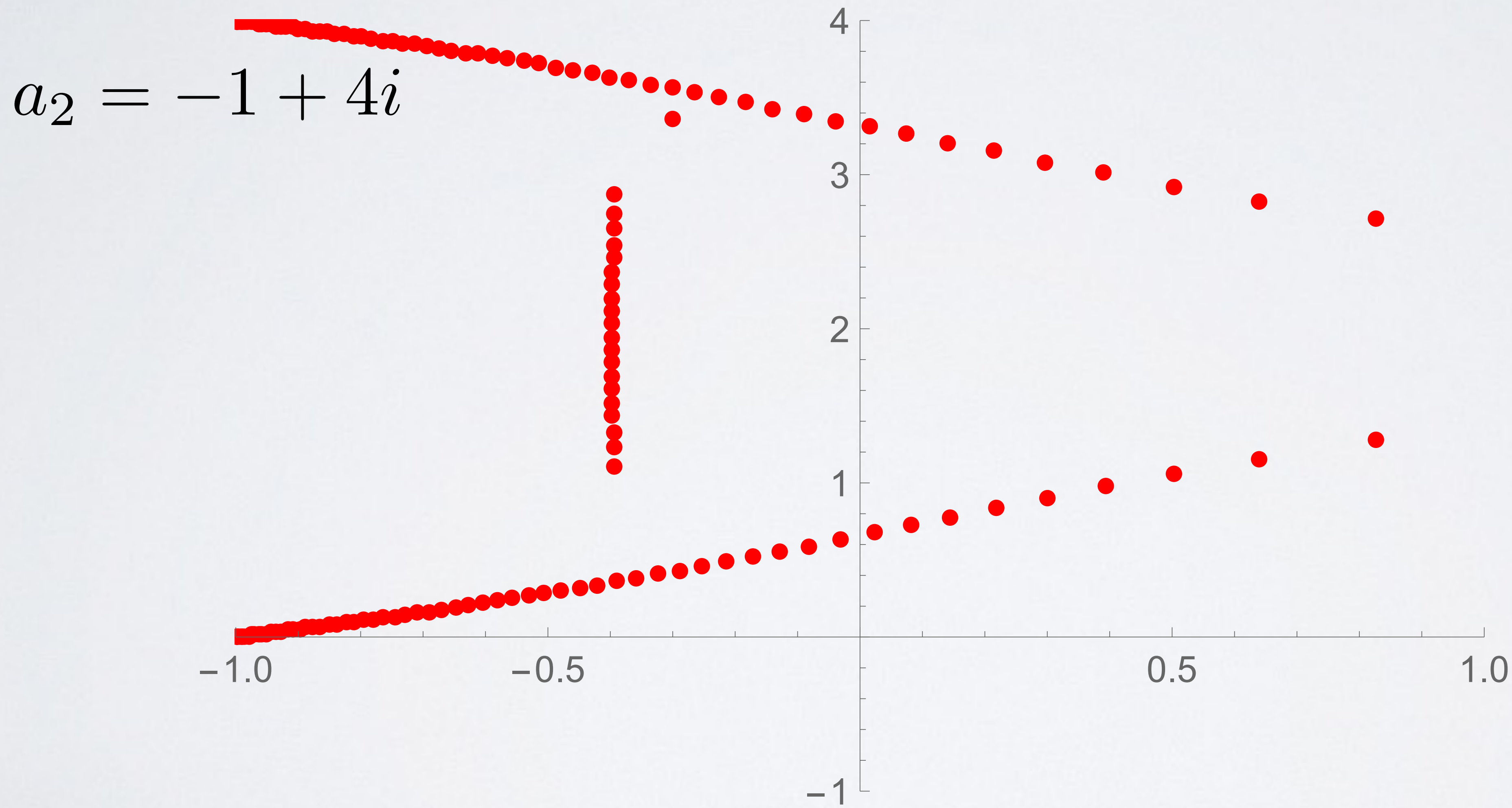


Generates additional external field!

AN APPARENTLY SIMPLE PROBLEM

$$\int_C z^k Q_n(z)\left(1+e^{-n\alpha z}\right)dz = 0, \quad k = 0, 1, \ldots, n-1,$$

The situation is even more puzzling if we see how the distribution evolves with a fixed parameter $\alpha = 0.8$, but with C joining -1 with a_2, where a_2 goes from 1 to $-1+4i$.



AN APPARENTLY SIMPLE PROBLEM

$$\int_C z^k Q_n(z)\left(1+e^{-n\alpha z}\right)dz = 0, \quad k = 0, 1, \ldots, n-1,$$

The situation is even more puzzling if we see how the distribution evolves with a fixed parameter $\alpha = 0.8$, but with C joining -1 with a_2, where a_2 goes from 1 to $-1+4i$.

To-do list for the next FoCM:

- understand the electrostatic model explaining these pictures
- find the mechanism behind the obvious phase transitions
- extend the Gonchar-Rakhmanov theorem to the piece-wise harmonic external fields
- use the Riemann-Hilbert steepest descent method to find the strong asymptotics of these polynomials
- exercise more and eat healthy

Thank you!