AFEM for the Laplace-Beltrami Operator: Convergence Rates

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Joint work with

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Outline

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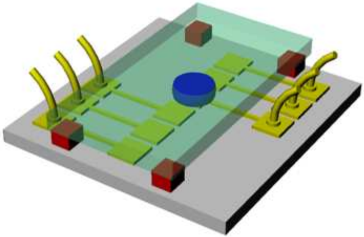
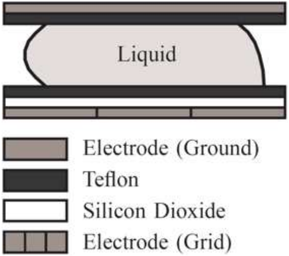
Electrowetting on Dielectric: Modeling (w. A. Bonito and S. Walker)





Mixed Formulation (u velocity, p pressure, H curvature, λ multiplier)

$$\alpha \frac{\partial u}{\partial t} + \beta u + \nabla p = 0 \qquad \text{in } \Omega$$

$$\text{div } u = 0 \qquad \text{in } \Omega$$

$$p = H + \underbrace{E}_{\text{electric actuation}} + \underbrace{P_0 \text{sign } (u \cdot \nu)}_{\lambda \text{(contact line pinning)}} + \underbrace{D_{visc} u \cdot \nu}_{\text{viscous damping}} \qquad \text{on } \Gamma$$

Interface Motion

$$u \cdot \nu = \partial_t X \cdot \nu \quad \text{on } \Gamma$$

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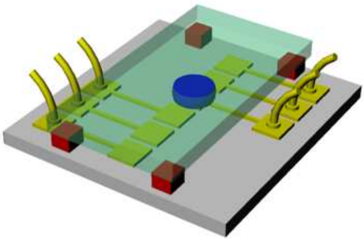

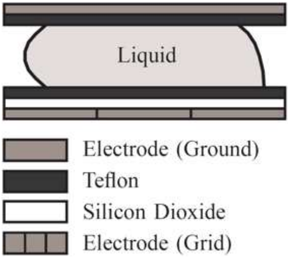

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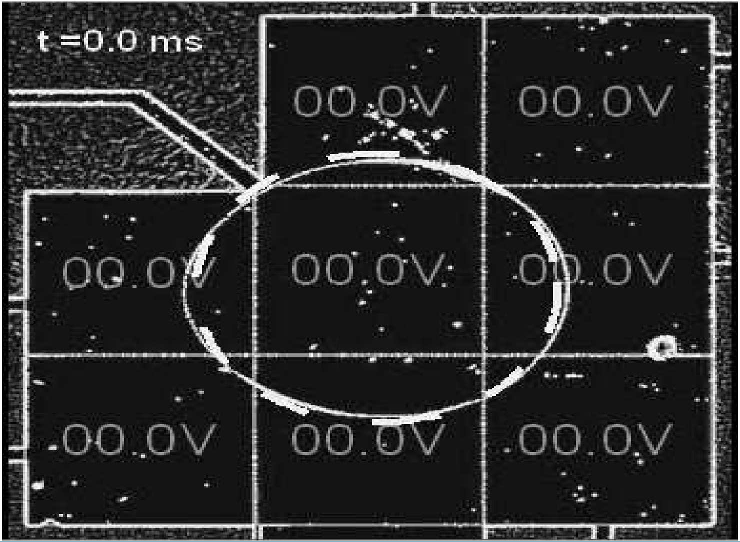
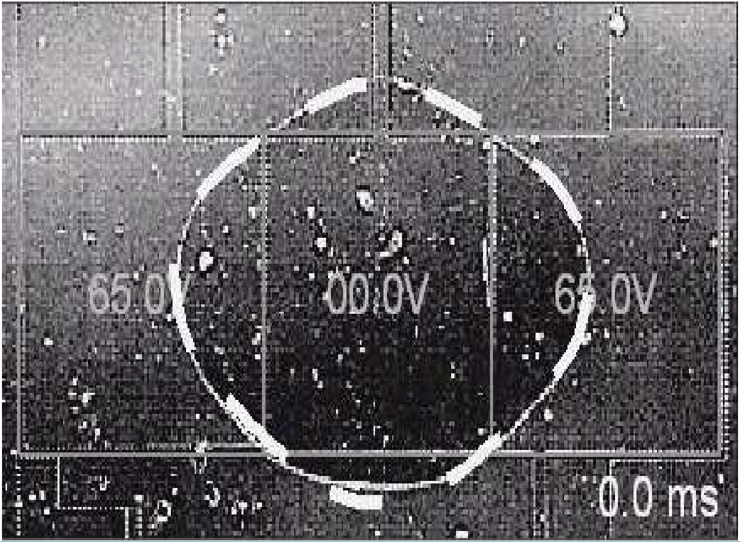
Electrowetting on Dielectric: Experiments vs Simulations

Moving droplet stirred around by varying voltages



Electrowetting on Dielectric: Experiments vs Simulations

Splitting of glycerin droplet due to voltage actuation



Biomembranes: Modeling (w. A. Bonito and M.S. Pauletti)

- Bending (Willmore) energy: $J(\Gamma) = \frac{1}{2}\int_\Gamma H^2$, H mean curvature

- Geometric Gradient Flow (with area and volume constraint):

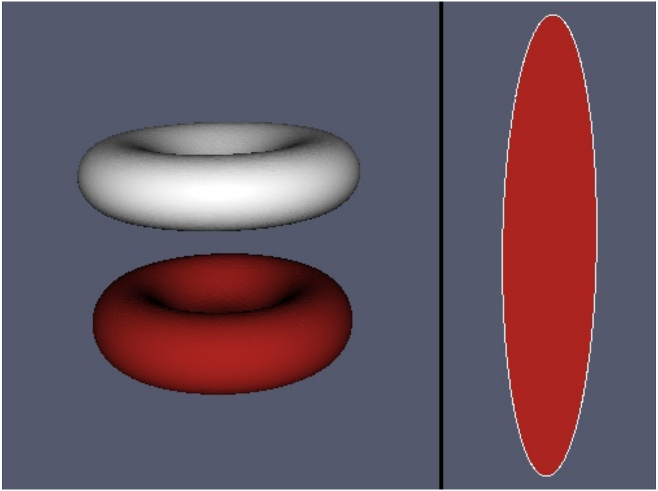
$$\mathbf{v} = -\delta_\Gamma J = -\Big(\Delta_\Gamma H + \frac{1}{2}H^3 - 2\kappa H\Big)\boldsymbol{\nu} - \Big(\lambda H\boldsymbol{\nu} + p\boldsymbol{\nu}\Big)$$

  where Δ_Γ is the Laplace-Beltrami operator on Γ.

- Fluid-Membrane Interaction (with area constraint):

$$\begin{aligned} \rho D_t\mathbf{v} - \text{div}\,\underbrace{\big(-p\mathbf{I} + \mu D(\mathbf{v})\big)}_{\Sigma} &= \mathbf{b} && \text{in } \Omega_t, \\ \text{div}\,\mathbf{v} &= 0 && \text{in } \Omega_t, \\ [\boldsymbol{\Sigma}]\boldsymbol{\nu} &= \delta_\Gamma J && \text{on } \Gamma_t \end{aligned}$$

Biomembrane: Geometric vs Fluid Red Blood Cell



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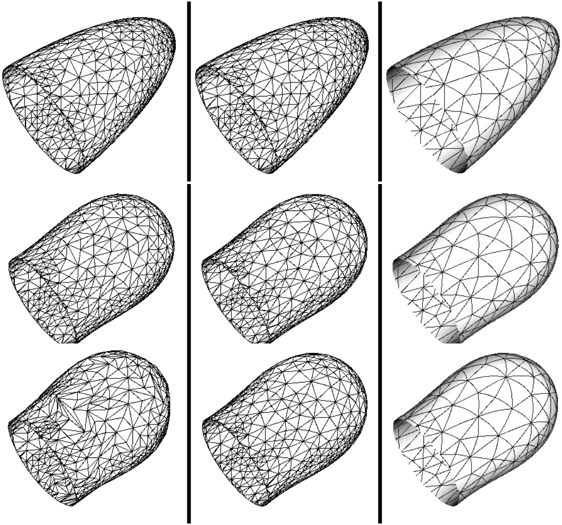
Mesh Smoothing: Comparison between P^1 and P^2 Elements



The Laplace-Beltrami Operator and Curvature

- Vector curvature: $\mathbf{H} = H\boldsymbol{\nu} = -\Delta_\Gamma \mathbf{X}$, $\quad \mathbf{X} =$ identity on Γ $\quad$ (Dziuk' 91)
- Semi-implicit Time Discretization $(t_n \to t_{n+1})$: explicit geometry $(\Gamma = \Gamma_n, \quad \nabla_\Gamma = \nabla_{\Gamma^n}, \quad \boldsymbol{\nu} = \boldsymbol{\nu}^n)$

$$\int_{\Gamma^n} \mathbf{H}^{n+1} \cdot \boldsymbol{\Psi} = \int_{\Gamma^n} \nabla_{\Gamma^n} \mathbf{X}^{n+1} : \nabla_{\Gamma^n} \boldsymbol{\Psi}, \qquad \mathbf{X}^{n+1} = \mathbf{X}^n + \tau^n \mathbf{V}^{n+1}$$

- $\int_{\Gamma^n} \mathbf{H}^{n+1} \cdot \boldsymbol{\Psi} - \tau^n \int_{\Gamma^n} \nabla_{\Gamma^n} \mathbf{V}^{n+1} : \nabla_{\Gamma^n} \boldsymbol{\Psi} = \int_{\Gamma^n} \nabla_{\Gamma^n} \mathbf{X}^n : \nabla_{\Gamma^n} \boldsymbol{\Psi}$
- Mixed Method: operator splitting
  - Velocity (gradient flow or Navier-Stokes)
  - Curvature (Laplace-Beltrami)

Outline

Motivation: Geometric PDE

Parametric Surfaces: Representation and Approximation

The Laplace-Beltrami Operator

AFEM for the Laplace-Beltrami Operator

Convergence Rates of AFEM

Discontinuous Coefficients

Comments and Conclusions

Surface Representation

- **Polyhedral surface:** $\overline{\Gamma}_0 = \bigcup_{i=1}^{M} \overline{\Gamma}_0^i$ is made of M (closed) facets $\overline{\Gamma}_0^i$;
- **Globally Lipschitz homeomorphism:** $P_0 : \overline{\Gamma}_0 \to \gamma \subset \mathbb{R}^{d+1}$, $\gamma^i := P_0^i(\overline{\Gamma}_0^i)$;
- **Local parametric domain** $\Omega \subset \mathbb{R}^d$**:** $\overline{X}_0^i : \Omega \to \overline{\Gamma}_0^i$ affine map;
- **Non-overlapping parametrization:** $\chi^i := P_0^i \circ \overline{X}_0^i : \Omega \to \gamma^i$

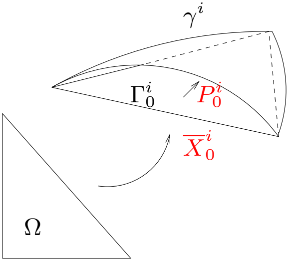


Figure: Representation of each component γ^i when $d = 2$ as a parametrization from a flat triangle $\overline{\Gamma}_0^i \subset \mathbb{R}^3$ as well as from the master simplex $\Omega \subset \mathbb{R}^2$. The map $\overline{X}_0^i : \Omega \to \overline{\Gamma}_0^i$ is affine.

Surface Approximation

- **Conforming graded bisection meshes:** $\overline{\mathcal{T}} \in \overline{\mathbb{T}} = \overline{\mathbb{T}}(\overline{\mathcal{T}}_0)$, $\widehat{\mathcal{T}}^i = \widehat{\mathcal{T}}^i(\Omega)$;
- **Finite element space:** $\widehat{\mathbb{V}}^i := \widehat{\mathbb{V}}(\widehat{\mathcal{T}}^i)$ space C^0-elements of degree $n \geq 1$;
- **Surface interpolant:** $X^i_{\widehat{\mathcal{T}}^i} := I_{\widehat{\mathcal{T}}^i}\chi^i$ Lagrange interpolant of χ^i in $\widehat{\mathbb{V}}^i$, $\Gamma^i := X^i_{\widehat{\mathcal{T}}^i}(\Omega)$ is the piecewise polynomial interpolation of γ_i

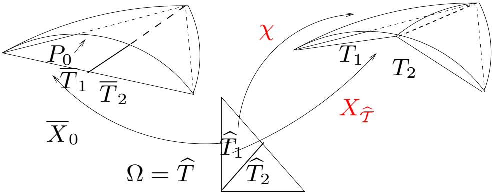


Figure: Effect of one bisection of the macro-element $\overline{X}_0(\Omega)$ when $d = 2$ and $n = 1$; the superscript i is omitted for simplicity. (Left) A triangle $\overline{T} \in \overline{\mathcal{T}}_0$ is split into two triangles $\overline{T}_1$, $\overline{T_2} \subset \mathbb{R}^3$. (Bottom) Equivalently, via the affine map $\overline{X}_0^{-1}$, the corresponding triangle $\widehat{T} \in \widehat{\mathcal{T}}$ is split into two triangles $\widehat{T}_1$, $\widehat{T}_2 \subset \mathbb{R}^2$, whereas (Right) γ is interpolated by a new piecewise linear surface $\Gamma := X_{\widehat{\mathcal{T}}}(\Omega)$, with $X_{\widehat{\mathcal{T}}} = \mathcal{I}_{\widehat{\mathcal{T}}}\chi$ the piecewise linear interpolant of the parametrization χ defined in Ω and subordinate to the new triangulation $\widehat{\mathcal{T}}$. The images via $X_{\widehat{\mathcal{T}}}$ of $\widehat{T}_1$ and $\widehat{T}_2$ are denoted T_1 and T_2 respectively; they are affine when $n = 1$.

Geometric Estimator

- **Geometric indicator:** for $1 \le i \le M$ and $\widehat{T} \in \widehat{\mathcal{T}}^i$

$$\lambda_{\widehat{\mathcal{T}}^i}(\widehat{T}) := \|\widehat{\nabla}(\chi^i - X^i_{\widehat{\mathcal{T}}^i})\|_{L_\infty(\widehat{T})};$$

- **Geometric estimator:** $\lambda_{\widehat{\mathcal{T}}} := \max_{i=1,\dots,M} \max_{\widehat{T} \in \widehat{\mathcal{T}}^i} \lambda_{\widehat{\mathcal{T}}^i}(\widehat{T})$;
- **Quasi-monotonicity:** There exists a constant $\Lambda_0 > 1$, solely depending on $\overline{\mathcal{T}}_0$, the polynomial degree n, and dimension d, such that

$$\lambda_{\widehat{\mathcal{T}}_*} \le \Lambda_0 \lambda_{\widehat{\mathcal{T}}}$$

  for any $\widehat{\mathcal{T}}, \widehat{\mathcal{T}}_* \in \widehat{\mathbb{T}}$ with $\widehat{\mathcal{T}}_* \ge \widehat{\mathcal{T}}$.
- **Shape-regularity:** The forest $\mathbb{T} := \mathbb{T}(\mathcal{T}_0)$ is shape-regular provided

$$\lambda_{\widehat{\mathcal{T}}_0} \le \frac{1}{2\Lambda_0 L},$$

  where $L > 1$ is the non-degeneracy constant in

$$L^{-1}|\hat{\mathbf{x}} - \hat{\mathbf{y}}| \le |\chi^i(\hat{\mathbf{x}}) - \chi^i(\hat{\mathbf{y}})| \le L|\hat{\mathbf{x}} - \hat{\mathbf{y}}|, \qquad \forall \hat{\mathbf{x}}, \hat{\mathbf{y}} \in \Omega,$$

  and $\mathcal{T}_0 \in \mathbb{T}$ is the subdivision corresponding to $\widehat{\mathcal{T}}_0 \in \widehat{\mathbb{T}}$.

Outline

Basic Differential Geometry

- **Tangent vectors:** The matrix of tangent vectors has rank d

$$\mathbf{T} := [\widehat{\partial}_1\chi, \ldots, \widehat{\partial}_d\chi] \in \mathbb{R}^{(d+1)\times d};$$

- **First fundamental form:**

$$\mathbf{G} = \big(g_{\gamma,ij}\big)_{1\le i,j\le d} := \big(\widehat{\partial}_i\chi^T\widehat{\partial}_j\chi\big)_{1\le i,j\le d} = \mathbf{T}^T\mathbf{T};$$

- **Area element:**

$$q := \sqrt{\det \mathbf{G}};$$

- **Tangent gradient:**

$$\widehat{\nabla}\hat{v} = \nabla_\gamma v\,\mathbf{T};$$

- **Weak form of Laplace-Beltrami operator:**

$$\int_\gamma \nabla_\gamma u \nabla_\gamma^T v = \sum_{i=1}^M \int_\Omega \widehat{\nabla}\hat{u}\mathbf{G}_i^{-1}\widehat{\nabla}\hat{v}^T q;$$

- **Strong form of Laplace-Beltrami operator:**

$$\Delta_\gamma v = \frac{1}{q}\widehat{\mathsf{div}}\big(q\widehat{\nabla}\hat{v}\mathbf{G}^{-1}\big).$$

Variational Formulation

- **Data assumptions:** $f \in L^2(\gamma)$, $\gamma \in W^1_\infty$
- **Weak formulation:** $H^1_\#(\gamma) = H^1$ space with zero meanvalue functions

$$u \in H^1_\#(\gamma): \quad \underbrace{\int_\gamma \nabla_\gamma u \; \nabla_\gamma^T v}_{=\sum_{i=1}^M \int_\Omega \widehat{\nabla}\hat{u} \;\; \mathbf{G}_i^{-1} \;\; \widehat{\nabla} v^T q^i} \quad = \quad \underbrace{\int_\gamma f \;\; v}_{=\sum_{i=1}^M \int_\Omega \hat{f} \;\; \hat{v} q^i} \qquad \forall v \in H^1_\#(\gamma);$$

- **Discrete geometric quantities:** $\mathbf{T}_\Gamma, \mathbf{G}_\Gamma, q_\Gamma;$
- **Galerkin formulation:** seek $U : \Gamma \to \mathbb{R}$

$$U \in \mathbb{V}(\mathcal{T}): \quad \underbrace{\int_\Gamma \nabla_\Gamma U \;\; \nabla_\Gamma^T V}_{=\sum_{i=1}^M \int_\Omega \widehat{\nabla}\hat{U} \;\; \mathbf{G}_{\Gamma^i}^{-1} \;\; \widehat{\nabla} V^T \;\; q_{\Gamma^i}} \quad = \quad \underbrace{\int_\Gamma F \;\; V}_{=\sum_{i=1}^M \int_\Omega \hat{F} \;\; \hat{V} \;\; q_{\Gamma^i}} \qquad \forall V \in \mathbb{V}(\mathcal{T}).$$

Multiplicative structure $q_{\Gamma^i} \widehat{\nabla} \widehat{U} \mathbf{G}_{\Gamma^i}^{-1}$: this quantity is piecewise constant for polynomial degree $n = 1$ but is rational for $n > 1$.

A Posteriori Error Analysis: Geometric Estimates

- **References:**
  - **A posteriori estimates:** Demlow-Dziuk ($n = 1$, 2007), Demlow ($n > 1$, 2009);
  - **Convergence and optimality** $n = 1$**:** Mekchay-Morin-Nochetto (2011), Bonito-Cascón-Morin-Nochetto (2013);

- **Consistency error:** $\mathbf{E}_\Gamma := \frac{1}{q}\mathbf{T}(q_\Gamma \mathbf{G}_\Gamma^{-1} - q\mathbf{G}^{-1})\mathbf{T}^T$ error matrix

$$\int_\Gamma \nabla_\Gamma v \nabla_\Gamma^T w - \int_\gamma \nabla_\gamma v \nabla_\gamma^T w = \int_\gamma \nabla_\gamma v \mathbf{E}_\Gamma \nabla_\gamma^T w;$$

- **Estimate of $\mathbf{E}_\Gamma$:** If $\lambda_{\widehat{\mathcal{T}}_0} \le \frac{1}{6\Lambda_0 L^3}$, then

$$\|\mathbf{E}_\Gamma\|_{L_\infty(\widehat{T})} \lesssim \lambda_{\widehat{\mathcal{T}}^i}(\widehat{T}) \qquad \forall\, \widehat{T} \in \widehat{\mathcal{T}}^i, \quad 1 \le i \le M;$$

- **Estimate of $q, \mathbf{G}, \boldsymbol{\nu}$:** If $\lambda_{\widehat{\mathcal{T}}_0} \le \frac{1}{6\Lambda_0 L^3}$, then for all $\mathcal{T} \in \mathbb{T}$

$$\max_{\widehat{T} \in \widehat{\mathcal{T}}} \Big(\|q - q_\Gamma\|_{L_\infty(\widehat{T})} + \|\mathbf{G} - \mathbf{G}_\Gamma\|_{L_\infty(\widehat{T})} + \|\boldsymbol{\nu} - \boldsymbol{\nu}_\Gamma\|_{L_\infty(\widehat{T})} \Big) \lesssim \lambda_{\mathcal{T}}.$$

A Posteriori Error Analysis: Upper and Lower Bounds

- **Interior and jump residuals:**

$$\mathcal{R}(U) := F_\Gamma|_T + \Delta_\Gamma U|_T \quad \forall\, T \in \mathcal{T},$$
$$\mathcal{J}(U) := \nabla_\Gamma U^+|_S \cdot \mathbf{n}_S^+ + \nabla_\Gamma U^-|_S \cdot \mathbf{n}_S^- \quad \forall\, S \in \mathcal{S}_\mathcal{T};$$

- **A posteriori error estimator:**

$$\eta_\mathcal{T}(U,T)^2 := h_T^2 \|\mathcal{R}(U)\|^2_{L^2(T)} + h_T \|\mathcal{J}(U)\|^2_{L^2(\partial T)} \quad \forall\, T \in \Gamma;$$

- **Upper bound:** there exist constants $C_1, \Lambda_1 > 0$ such that

$$\|\nabla_\gamma (u - U)\|^2_{L_2(\gamma)} \le C_1 \eta_\mathcal{T}(U)^2 + \Lambda_1 \lambda^2_{\widehat{\mathcal{T}}};$$

- **Lower bound:** there exists constant $C_2 > 0$ such that

$$C_2 \eta_\mathcal{T}(U)^2 \le \|\nabla_\gamma (u - U)\|^2_{L_2(\gamma)} + \mathrm{osc}_{\widehat{\mathcal{T}}}(U, f)^2 + \Lambda_1 \lambda^2_{\widehat{\mathcal{T}}};$$

Note that $\Delta_\Gamma U = 0$ for $n = 1$ but is a rational function for $n > 1$;

- **Localized upper bound:** for $\mathcal{T}_* \ge \mathcal{T}$ there holds

$$\|\nabla_\gamma (U_* - U)\|^2_{L_2(\gamma)} \le C_1 \eta_\mathcal{T}(U, \mathcal{R})^2 + \Lambda_1 \lambda^2_{\widehat{\mathcal{T}}}.$$

Outline

Standard AFEM (for flat domains)

$\boxed{\text{SOLVE}}$: Compute the solution $U_k \in \mathbb{V}_k := \mathbb{V}(\mathcal{T}_k)$ of the discrete problem.

$\boxed{\text{ESTIMATE}}$: Compute a local estimator $\eta_k(U_k, K)$, $K \in \mathcal{T}_k$, for the error in terms of the discrete solution U_k and given data.

$\boxed{\text{MARK}}$: Use the estimator to mark a subset $\mathcal{M}_k \subset \mathcal{T}_k$ for refinement $\eta_k(U_k, \mathcal{M}_k)^2 \geq \theta^2 \eta_k(U_k, \mathcal{T}_k)^2$ (Dörfler marking).

$\boxed{\text{REFINE}}$: Refine the marked subset $\mathcal{M}_k$ to obtain $\mathcal{T}_{k+1}$, conforming or with hanging nodes, increment k and go to step SOLVE.

Quasi-Optimal Algorithm: If f is piecewise polynomial and the decay rate for the best approximation of u is

$$\inf_{\#\mathcal{T} - \#\mathcal{T}_0 \leq N} \quad \inf_{V \in \mathbb{V}(\mathcal{T})} \|\nabla_\gamma (u - V)\|_{L^2(\gamma)} \leq C_1 N^{-s} \qquad 0 < s \leq n/d,$$

then the finite element method delivers the same rate

$$\|\nabla_\gamma (u - U_k)\|_{L^2(\gamma)} \leq C_2 (\#\mathcal{T}_k)^{-s}.$$

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Quasi-optimality of AFEM

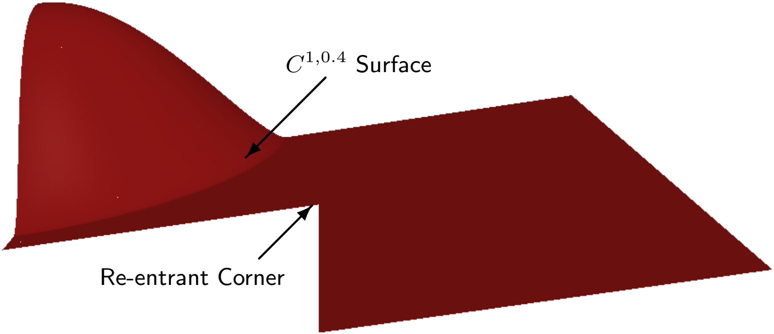
- **Main references:**
  - Babuska, Vogelius, 1986 (1D problem).
  - Binev, Dahmen, DeVore, 2004 (2D problem, coarsening).
  - Stevenson, 2006 (marking by oscillation).
  - Kreuzer, Cascon, Nochetto, Siebert, 2008.
  - Bonito, Nochetto, 2010 (dG).
  - Cohen, DeVore, Nochetto, 2011 (H^{-1} data and approximation classes).
  - Diening, Kreuzer, Stevenson, 2013 (maximum strategy).

- **Sufficient Condition:** for best approximation of $u \in B_p^{1+sd}(L^p(\Omega))$ with graded meshes to decay with rate $N^{-s/d}$ with $0 < s \le n/d$ ($n \ge 1$ polynomial degree)

$$s > \frac{1}{p} - \frac{1}{2} \qquad \Rightarrow \qquad B_p^{1+sd}(L^p(\Omega)) \hookrightarrow H^1(\Omega)$$

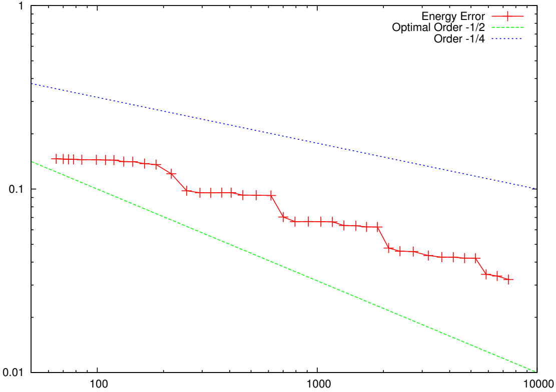
  - Binev, Dahmen, DeVore, Petrushev, 2002 ($n = 1$)
  - Gaspoz, Morin, 2014 ($n \ge 1$).

Standard AFEM: $C^{1,0.4}$ Surface with Lipschitz Boundary



Standard AFEM: Decay Rates for $n = 1$

- **Error indicator:** $\eta^2_{\mathcal{T}}(U,T) = h_T^2 ||f||^2_{L^2(T)} + h_T ||\mathcal{J}(U)||^2_{L^2(\partial T)}$
- **Dörfler parameter in MARK:** $\theta = 10\%$ (quite conservative)
- **Decay rate:**



$\Rightarrow$ suboptimal decay rate

AFEM for the Laplace-Beltrami Operator

- **AFEM:** Given $\overline{\mathcal{T}_0}$, maps $\{P_0^i\}_{i=1}^L$ parametrizing the surface γ from $\overline{\mathcal{T}_0}$, and parameters $\varepsilon_0 > 0$, $0 < \rho < 1$, and $\omega > 0$, set $k = 0$.

  1. $\mathcal{T}_k^+ = \text{ADAPT\_SURFACE}(\mathcal{T}_k, \omega\varepsilon_k)$
  2. $\mathcal{T}_{k+1} = \text{ADAPT\_PDE}(\mathcal{T}_k^+, \varepsilon_k)$
  3. $\varepsilon_{k+1} = \rho\varepsilon_k$; $k = k + 1$; goto 1.

- $\mathcal{T}^+ = \text{ADAPT\_SURFACE}(\mathcal{T}, \tau)$

  while $\mathcal{M} := \{T \in \mathcal{T}^i,\ 1 \le i \le M \,|\, \lambda_{\widehat{\mathcal{T}}^i}(\widehat{T}) > \tau\} \neq \emptyset$

  $\quad\mathcal{T} = \text{REFINE}(\mathcal{T}, \mathcal{M})$

  end while

  return $\mathcal{T}$

- $[\mathcal{T}, U] = \text{ADAPT\_PDE}(\mathcal{T}, \varepsilon)$

  do

  $\quad U = \text{SOLVE}(\mathcal{T})$

  $\quad\{\eta_{\mathcal{T}}(U, T)\}_{T\in\mathcal{T}} = \text{ESTIMATE}(\mathcal{T}, U)$

  $\quad\mathcal{M} := \text{MARK}(\mathcal{T}, \{\eta_{\mathcal{T}}(U, T)\}_{T\in\mathcal{T}})$

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  while $\eta_{\mathcal{T}}(U) > \varepsilon$

  return$(\mathcal{T}, U)$

AFEM for the Laplace-Beltrami Operator

- **AFEM:** Given $\overline{\mathcal{T}_0}$, maps $\{P_0^i\}_{i=1}^L$ parametrizing the surface γ from $\overline{\mathcal{T}_0}$, and parameters $\varepsilon_0 > 0$, $0 < \rho < 1$, and $\omega > 0$, set $k = 0$.

  1. $\mathcal{T}_k^+ = \text{ADAPT\_SURFACE}(\mathcal{T}_k, \omega\varepsilon_k)$
  2. $\mathcal{T}_{k+1} = \text{ADAPT\_PDE}(\mathcal{T}_k^+, \varepsilon_k)$
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Conditional Contraction Property of Module PDE

- **Theorem.** If $\lambda(\mathcal{T}_j) \leq \Lambda_0 \omega \eta(U_j, \mathcal{T}_j)$ for $\omega \leq \omega_*$, then there exists constants $0 < \alpha < 1$ and $\beta > 0$ such that the inner iterates of the module PDE satisfy

$$\|\nabla_\gamma(u - U_{j+1})\|^2_{L^2(\gamma)} + \beta\eta(U_{j+1}, \mathcal{T}_{j+1})^2 \leq \alpha^2 \Big(\|\nabla_\gamma(u - U_j)\|^2_{L^2(\gamma)} + \beta\eta(U_j, \mathcal{T}_j)^2 \Big).$$

  Moreover, the number J of inner iterates of PDE is uniformly bounded.

- **Idea of proof:** it proceeds as in Cascón, Kreuzer, Nochetto, and Siebert (2008) and Bonito and Nochetto (2010), with the additional information

$$\lambda(\mathcal{T}) \leq 2\omega\eta(U, \mathcal{T})$$

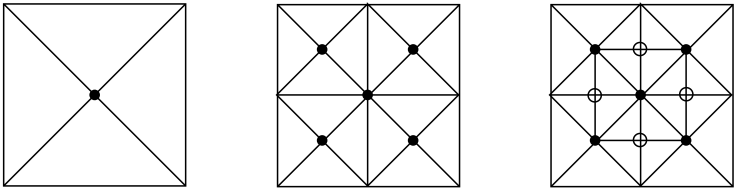
  in the inner loops of PDE for $\omega \leq \omega_*$ sufficiently small.

- **Reduction of error estimator:** there exist constants $0 < \xi < 1$ and $\Lambda_2, \Lambda_3 > 0$ such that for all $\delta > 0$ and $\mathcal{T}_*$ conforming refinement of $\mathcal{T}$

$$\eta(U_*, \mathcal{T}_*)^2 \leq (1+\delta)\big(\eta(U, \mathcal{T})^2 - \xi\eta(U, \mathcal{M})^2\big) + (1+\delta^{-1})\big(\Lambda_3 \|\nabla_\gamma(U_* - U)^2\|_{L^2(\gamma)} + \Lambda_2 \lambda(\mathcal{T})^2\big).$$

Contracting Quantities of AFEM (in flat domains)

- **Energy error:** $\|U_k - u\|_\Omega$ is monotone, but not strictly monotone (e.g. $U_{k+1} = U_k$).



$$\Omega = (0,1)^2, A = I, f = 1 \quad \Rightarrow \quad U_0 = U_1 = \frac{1}{12}\phi_0, \quad U_2 \neq U_1$$

- **Residual estimator**: $\eta_k(U_k, \mathcal{T}_k)$ is not reduced by AFEM, and is not even monotone. But, if $U_{k+1} = U_k$, then $\eta_k(U_k, \mathcal{T}_k)$ decreases strictly

$$\eta_{k+1}^2(U_{k+1}, \mathcal{T}_{k+1}) = \eta_{k+1}^2(U_k, \mathcal{T}_{k+1}) \leq \eta_k^2(U_k, \mathcal{T}_k) - \xi\eta_k^2(U_k, \mathcal{M}_k)$$

- **Heuristics:** the quantity $\|U_k - u\|_\Omega^2 + \beta\eta_k^2(U_k, \mathcal{T}_k)$ might contract!

- **Laplace-Beltrami:** additional term $\lambda(\mathcal{T}_k)$ but $\lambda(\mathcal{T}_k) \leq 2\omega\eta_k(U_k, \mathcal{T}_k)$.

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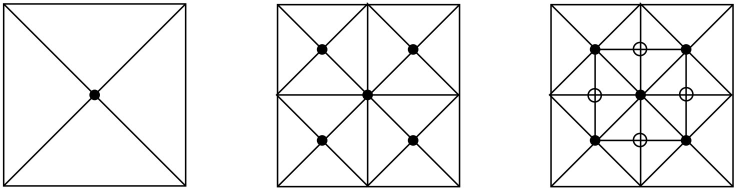

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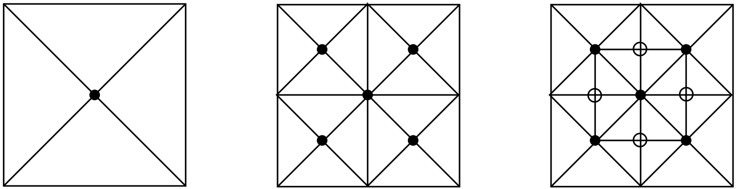

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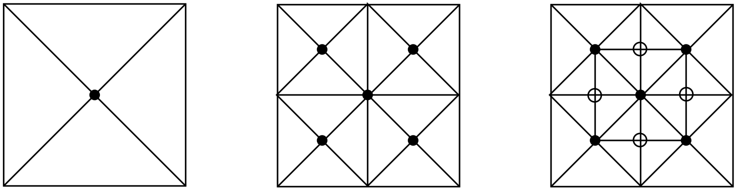

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Outline

Approximation Class

- **Total error:** Given $\gamma \in W^1_\infty, v \in H^1_\#(\gamma), f \in L^2(\gamma)$ and $V \in \mathbb{V}(\mathcal{T})$ let

$$E_{\mathcal{T}}(V; v, f, \gamma) := \|\nabla_\gamma (v - V)\|_{L^2(\gamma)} + \operatorname{osc}_{\widehat{\mathcal{T}}}(V, f) + \omega^{-1} \lambda_{\widehat{\mathcal{T}}}.$$

- **Approximation class** $\mathbb{A}_s$**:** We say that $(u, f, \gamma) \in \mathbb{A}_s$ for $0 < s \le n/d$ if given $\epsilon > 0$ there exists a conforming partition $\mathcal{T}_\varepsilon$ with $\mathcal{T}_\varepsilon \ge \mathcal{T}_0$ and a discrete function $U_\varepsilon \in \mathbb{V}(\mathcal{T}_\varepsilon)$ so that

$$E_{\mathcal{T}_\epsilon}(U_\epsilon; u, f, \gamma) \le \varepsilon, \qquad \text{and} \qquad \#\mathcal{T}_\varepsilon - \#\mathcal{T}_0 \le C(u, f, \gamma, s)\varepsilon^{-\frac{1}{s}}.$$

- **Besov regularity: sufficient conditions for** $(u, f, \gamma) \in \mathbb{A}_s$**?**

Besov Regularity of γ, u and Greedy Algorithm

- **Greedy algorithm:**

  $\overline{\mathcal{T}}^+ = \text{GREEDY}(\{g^i\}_{i=1}^M, \overline{\mathcal{T}}, \delta)$

  while $\overline{\mathcal{M}} := \{\overline{T} \in \overline{\mathcal{T}}^i,\ 1 \le i \le M \mid \zeta_{\widehat{\mathcal{T}}^i}(g^i, \widehat{T}) > \delta\} \neq \emptyset$

  $\overline{\mathcal{T}} :=$ REFINE$(\overline{\mathcal{T}}, \overline{\mathcal{M}})$

  end while

  return$(\overline{\mathcal{T}})$

- **Constructive approximation of γ:** Let γ be of class $B_q^{1+td}(L_q(\Omega))$, with $tq > 1, 0 < q \le \infty$ and $td \le n$, and globally of class W_∞^1. Then GREEDY with $\zeta_{\widehat{\mathcal{T}}^i}(g^i, \widehat{T}) = \lambda_{\widehat{\mathcal{T}}^i}(\widehat{T})$ implies

$$\#\mathcal{M}^+ \lesssim |\gamma|_{B_p^{1+td}(L_p(\Omega))}^{1/t} \tau^{-1/t}.$$

- **Constructive approximation of u:** Let $u \in H^1(\gamma)$ be such that, for $i = 1, ..., M$, $u^i := u|_{\gamma^i} \circ \chi^i \in B_p^{1+sd}(L_p(\Omega))$ with $s - 1/p + 1/2 > 0$, $0 < p \le \infty$ and $0 < sd \le n$. Then GREEDY with $\zeta_{\widehat{\mathcal{T}}^i}(g, \widehat{T}) = \|\widehat{\nabla}(u^i - \Pi_n u^i)\|_{L_2(\widehat{T})}$ implies

$$\inf_{V \in \mathbb{V}(\mathcal{T})} \|\nabla_\gamma (u - V)\|_{L^2(\gamma)} \lesssim |u|_{B_p^{1+sd}(L_p(\Omega))} (\#\mathcal{T} - \#\mathcal{T}_0)^{-s}.$$

Besov Regularity of γ, f and Greedy Algorithm

- **Constructive approximation of f:** Let the oscillation of f be

$$\mathrm{osc}_{\widehat{\mathcal{T}}^i}(f,\widehat{T})^2 := h_T^2 \|(\mathsf{id} - \Pi_{2n-2})(\widehat{f}^i q_\Gamma)\|_{L_2(\widehat{T})}.$$

If $f \in L_2(\gamma)$ is such that, for $i = 1, \dots, M$, $f^i := f|_{\gamma^i} \circ \chi^i \in B_p^{sd}(L_p(\Omega))$ with $s - 1/p + 1/2 > 0$, $0 < p \le \infty$ and $sd \le 2n-1$, then GREEDY gives

$$\mathrm{osc}_{\widehat{\mathcal{T}}}(f) \lesssim |f|_{B_p^{sd}(L^p(\Omega))} \big(\#\mathcal{T} - \#\mathcal{T}_0\big)^{-\left(s+\frac{1}{d}\right)};$$

- **Decay rate of oscillation:** Let the oscillation associated to U be

$$\begin{aligned}\mathrm{osc}_{\widehat{\mathcal{T}}^i}(U,\widehat{T})^2 &:= h_T^2 \|(\mathsf{id} - \Pi_{2n-2})\widehat{\mathsf{div}}\big(q_\Gamma \widehat{\nabla}\widehat{U}^i \mathbf{G}_\Gamma^{-1}\big)\|_{L_2(\widehat{T})}^2 \\ &+ h_T \|(\mathsf{id} - \Pi_{2n-1})\left(q_\Gamma\big(\widehat{\nabla}(\widehat{U}^i)^+ (\mathbf{G}_\Gamma^+)^{-1} - \widehat{\nabla}(\widehat{U}^i)^- (\mathbf{G}_\Gamma^-)^{-1}\big)\widehat{\mathbf{n}}\right)\|_{L_2(\partial\widehat{T})}^2.\end{aligned}$$

If γ is of class $B_q^{1+td}(L_q(\Omega))$, with $tq > 1$, $td \le n$, and globally of class W_∞^1, then GREEDY with tolerance $\delta > 0$ gives for $t \le t' < 2t$

$$\mathrm{osc}_{\widehat{\mathcal{T}}}(U) \lesssim \delta \|\nabla_\gamma U\|_{L_2(\gamma)}, \quad \#\mathcal{T} - \#\mathcal{T}_0 \lesssim C(\gamma,t,q)\delta^{-\frac{1}{t'}}.$$

Multiplicative structure: $\mathrm{osc}_{\widehat{\mathcal{T}}^i}(U) \neq 0$ for $n > 1$.

Optimal Decay Rates

- **Membership in $\mathbb{A}_s$:** Let $\gamma \in B_q^{1+td}(L_q(\Omega))$ with $tq > 1, 0 < q \leq \infty$ and $td \leq n$, and globally of class W^1_∞. Let $u \in H^1_\#(\gamma)$ and $f \in L_2(\gamma)$ such that $u^i := u \circ \chi^i \in B_p^{1+sd}(L_p(\Omega))$ and $f^i := f \circ \chi^i \in B_p^{sd}(L_p(\Omega))$ for $i = 1, ..., M$, with $ds - d/p + d/2 > 0$, $0 < p \leq \infty$ and $0 < sd \leq n$. Then,

  $$(u, f, \gamma) \in \mathbb{A}_{min\{s,t\}},$$

  In addition,

  $$\sup_{V \in \mathbb{V}(\mathcal{T}_N)} \frac{\mathrm{osc}_{\widehat{\mathcal{T}}_N}(V)}{\|\nabla V\|_{L_2(\mathcal{T})}} \quad \text{and} \quad \mathrm{osc}_{\widehat{\mathcal{T}}}(f)$$

  decay faster than $N^{-\min(s,t)}$ and $\mathrm{osc}_{\widehat{\mathcal{T}}_N}(V, f) \leq \mathrm{osc}_{\widehat{\mathcal{T}}_N}(V) + \mathrm{osc}_{\widehat{\mathcal{T}}_N}(f)$ can be asymptotically discarded in the definition of $E_{\widehat{\mathcal{T}}_N}(V; u, f, \gamma)$.

- **Theorem.** If $(u, \gamma, f) \in \mathbb{A}_s$ for $0 < s \leq n/d$, and $0 < \theta \leq \theta_*$ and $0 < \omega \leq \omega_*$, then the sequence of iterates $(\Gamma_k, \mathcal{T}_k, U_k\}_{k \geq 0}$ generated by AFEM satisfy

  $$e(U_k) + \mathrm{osc}_{\widehat{\mathcal{T}}_k}(U_k, f) + \omega^{-1} \lambda_{\widehat{\mathcal{T}}_k} \leq C(u, f, \gamma, s)\big(\#\mathcal{T}_k - \#\mathcal{T}_0\big)^{-s}.$$

Ingredients of the Proof

- Localized upper bound (to the refined set)
- Minimality of set $\mathcal{M}$ in Dörfler marking
- Explicit restriction of Dörfler parameter $\theta < \theta_* < 1$
- Explicit restriction of surface parameter $\omega \leq \omega_* < 1$
- Conditional contraction property of PDE
- Complexity of REFINE (Binev-Dahmen-DeVore $(d = 2)$, Stevenson $(d > 2)$, for conforming meshes, and Bonito-Nochetto for non-conforming meshes $(d \geq 2)$.

The Role of ω for Convergence Rates

- **Example:** Let

$$-\Delta_\gamma u = 1, \quad \text{in } \gamma, \qquad u = 0, \quad \text{on } \partial\gamma,$$

where γ is the graph of class $C^{1,\alpha}$ given by

$$\chi(x,y) = \left(0.75 - x^2 - y^2\right)_+^{1+\alpha},$$

over the flat domain $\Omega = (0,1)^2$.

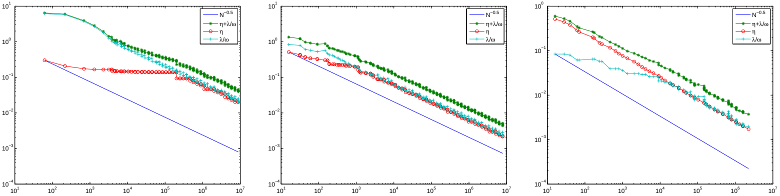
- **Besov regularity:** It turns out that $t = \frac{1}{2}$, $d = 2$, $td = 1$ and

$$\alpha = 3/5: \qquad \Rightarrow \qquad \chi \in B_q^2(L_q(\Omega)) \backslash W_\infty^2(\Omega) \quad q > 2;$$

- **Polynomial degree and decay rate:**

$$n = 1 \quad \Rightarrow \quad s = t = \frac{1}{2}.$$

The Role of ω for Convergence Rates: Case $\alpha = 3/5$



$\omega = 0.1$ $\qquad$ $\omega = 1$ $\qquad$ $\omega = 10$

Figure: η_k, λ_k/ω and $\eta_k + \lambda_k/\omega$ for $\omega = 0.1$ (left) $\omega = 1$ (middle) and $\omega = 10$ (right).

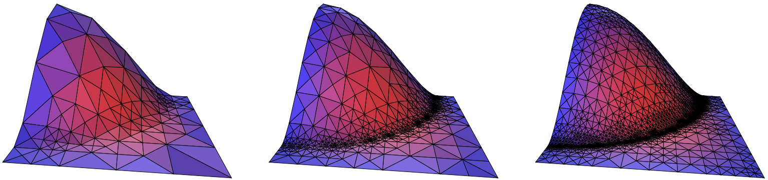


Figure: Meshes after 10, 20 and 30 refinements have been performed, $C^{1,0.6}$-surface, with $\omega = 1$. They are composed of 192, 1216 and 5564 elements, respectively.

Outline

Discontinuous Coefficients (w. A. Bonito and R. DeVore)

Motivation: Lipschitz surfaces with kinks not matched by the partitions.

Model problem: consider elliptic PDE of the form $-\operatorname{div}(A\nabla u) = f$ with

- $A = (a_{ij}(x))_{i,j=1}^d$ uniformly positive definite and bounded

$$\lambda_{\min}(A)|y|^2 \le y^t A(x) y \le \lambda_{\max}(A)|y|^2 \quad \forall\, x \in \Omega,\ y \in \mathbb{R}^d;$$

- The discontinuities of A are not match by the sequence of meshes $\mathcal{T}$;
- The forcing $f \in W_p^{-1}(\Omega)$ for some $p > 2$.

Goal: Design and study an AFEM able to handle such an A.

Difficulty: PDE perturbation results hinge on approximation of A in L^∞

$$\|u - \widehat{u}\|_{H_0^1(\Omega)} \le \lambda_{\min}^{-1}(\widehat{A})\Big(\|f - \widehat{f}\|_{H^{-1}(\Omega)} + \|A - \widehat{A}\|_{L_\infty(\Omega)}\|f\|_{H^{-1}(\Omega)}\Big)$$

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Perturbation Argument

Theorem (perturbation). Let $p \geq 2, q = 2p/(p-2) \in [2, \infty]$ and $\nabla u \in L^p(\Omega)$. Then

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Proposition (Meyers). Let $\widetilde{K} > 0$ be so that the solution $\widetilde{u}$ of the Laplacian satisfies

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DISC: AFEM for Discontinuous Diffusion Matrices

Given $\omega > 0$ explicit and $\beta < 1$, let

DISC$(\mathcal{T}_0, \epsilon_1)$
    $k = 1$
    LOOP
        $[\mathcal{T}_k(f), f_k] =$ **RHS**$(\mathcal{T}_{k-1}, f, \omega\varepsilon_k)$
        $[\mathcal{T}_k(A), A_k] =$ **COEFF**$(\mathcal{T}_k(f), A, \omega\varepsilon_k)$
        $[\mathcal{T}_k, U_k] =$ **PDE**$(\mathcal{T}_k(A), A_k, f_k, \varepsilon_k/2)$
        $\epsilon_{k+1} = \beta\epsilon_k$
        $k \leftarrow k+1$
    END LOOP
END **DISC**

- $[\mathcal{T}_k(f), f_k] =$ **RHS**$(\mathcal{T}_{k-1}, f, \omega\varepsilon_k)$ gives a mesh $\mathcal{T}_k(f) \geq \mathcal{T}_{k-1}$ and a pw polynonial approximation f_k of f on $\mathcal{T}_k(f)$ such that $\|f - f_k\|_{H^{-1}(\Omega)} \leq \omega\epsilon_k$;
- $[\mathcal{T}_k(A), A_k] =$ **COEFF**$(\mathcal{T}_k(f), A, \omega\varepsilon_k)$ gives a mesh $\mathcal{T}_k(A) \geq \mathcal{T}_k(f)$ and a pw polynomial approximation A_k of A on $\mathcal{T}_k(A)$ such that $\|A - A_k\|_{L^q(\Omega)} \leq \omega\epsilon_k$ and its eigenvalues satisfy uniformly in k

$$C^{-1}\lambda_{\min}(A) \leq \lambda(A_k) \leq C\lambda_{\max}(A).$$

DISC: AFEM for Discontinuous Diffusion Matrices

Given $\omega > 0$ explicit and $\beta < 1$, let

DISC$(\mathcal{T}_0, \epsilon_1)$
  $k = 1$
  LOOP
    $[\mathcal{T}_k(f), f_k] =$ **RHS**$(\mathcal{T}_{k-1}, f, \omega\varepsilon_k)$
    $[\mathcal{T}_k(A), A_k] =$ **COEFF**$(\mathcal{T}_k(f), A, \omega\varepsilon_k)$
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  END LOOP
END **DISC**

- $[\mathcal{T}_k(f), f_k] =$ **RHS**$(\mathcal{T}_{k-1}, f, \omega\varepsilon_k)$ gives a mesh $\mathcal{T}_k(f) \geq \mathcal{T}_{k-1}$ and a pw polynonial approximation f_k of f on $\mathcal{T}_k(f)$ such that $\|f - f_k\|_{H^{-1}(\Omega)} \leq \omega\epsilon_k$;
- $[\mathcal{T}_k(A), A_k] =$ **COEFF**$(\mathcal{T}_k(f), A, \omega\varepsilon_k)$ gives a mesh $\mathcal{T}_k(A) \geq \mathcal{T}_k(f)$ and a pw polynomial approximation A_k of A on $\mathcal{T}_k(A)$ such that $\|A - A_k\|_{L^q(\Omega)} \leq \omega\epsilon_k$ and its eigenvalues satisfy uniformly in k

$$C^{-1}\lambda_{\min}(A) \leq \lambda(A_k) \leq C\lambda_{\max}(A).$$

Optimality of DISC

Theorem (optimality). Assume that the right side f is in $\mathcal{B}^{s_f}(H^{-1}(\Omega))$ with $0 < s_f \leq S$, and that the diffusion matrix A is positive definite, in $L_\infty(\Omega)$ and in $\mathcal{M}^{s_A}(L_q(\Omega))$ for $q := \frac{2p}{p-2}$ and $0 < s_A \leq S$. Let $\mathcal{T}_0$ be the initial subdivision and $U_k \in \mathbb{V}(\mathcal{T}_k)$ be the Galerkin solution obtained at the kth iteration of the algorithm. Then, whenever $u \in \mathcal{A}^{s_u}(H_0^1(\Omega))$ for $0 < s_u \leq S$, we have for $k \geq 1$

$$\|u - U_k\|_{H_0^1(\Omega)} \leq \epsilon_k,$$

and

$$\#\mathcal{T}_k - \#\mathcal{T}_0 \lesssim \Big(|A|^{1/s_*}_{\mathcal{M}^{s_*}(L_q(\Omega))} + |f|^{1/s_*}_{\mathcal{B}^{s_*}(H^{-1}(\Omega))} + |u|^{1/s_*}_{\mathcal{A}^{s_*}(H_0^1(\Omega))} \Big) \epsilon_k^{-1/s_*},$$

with $s_* = \min(s_u, s_A, s_f)$.

Counterexample: s_u cannot be achieved if $s_A, s_f < s_u$.

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and

$$\#\mathcal{T}_k - \#\mathcal{T}_0 \lesssim \Big(|A|_{\mathcal{M}^{s_*}(L_q(\Omega))}^{1/s_*} + |f|_{\mathcal{B}^{s_*}(H^{-1}(\Omega))}^{1/s_*} + |u|_{\mathcal{A}^{s_*}(H_0^1(\Omega))}^{1/s_*} \Big) \epsilon_k^{-1/s_*},$$

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Checkerboard Example: $u \approx r^{1.25}$

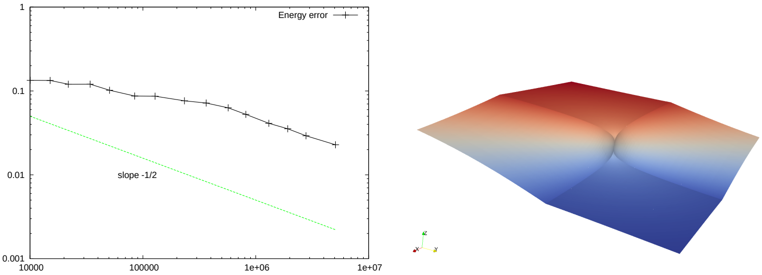

Figure: Checkerboard: The parameters are chosen so that the solution $u \in H^{1+s}(\Omega)$, $s < 0.25$. (Left) Energy error versus number of degrees of freedom. The optimal rate of convergence ≈ -0.5 is recovered. (Right) The Galerkin solution together with the underlying partition after 6 iterations of the algorithm **DISC**. The discontinuity of A is never captured by the partitions and the singularities of both A and ∇u drive the refinements.

Checkerboard Example: $u \approx r^{1.25}$

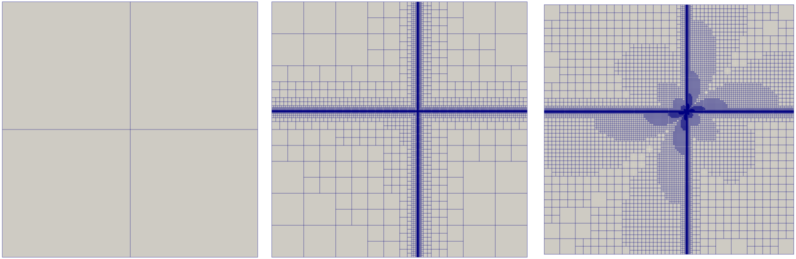

Figure: Checkerboard: Sequence of partitions (from left to right) generated by **DISC** with $\omega = 0.8$. The initial partition (first) is made of four quadrilaterals, The algorithm refines at early stages only to capture the discontinuity in the diffusion coefficient (second). Later the singularity of u comes into play and, together with that of A, drives the refinement (third). The corresponding subdivision consists of 5 million degrees of freedom. The smallest cell has a diameter of 2^{-8} which illustrates the strongly graded mesh constructed by **DISC**.

Outline

Comments and Conclusions

- **Coupling PDE-Geometry:** This is a new feature in adaptivity and leads to separate handling of geometry and PDE resolution with specific relative tolerances.
- **Convergence rates:** We show optimal convergence rates in the energy norm
$$\|\nabla(u - U_k)\|_{L^2(\gamma)} \lesssim (\#\mathcal{T}_k)^{-s}$$
provided this is the rate of the best approximation of u in H^1 and that of γ in W^1_∞.
- **Weaker conditions on f:** We refer to Cohen, DeVore, Nochetto (2011) for convergence rates of elliptic PDE in flat domains with $f \in H^{-1}$ and A piecewise constant:
$$\mathrm{div}(A\nabla u) = f.$$
We show that approximability of u is sufficient for a complete theory.
- **Weaker conditions on γ:** We assume γ is W^2_p with $p > d$, which implies γ is C^1. In the flat case, this corresponds to piecewise continuous A. We would like to extend to surfaces the results of Bonito, DeVore, Nochetto (2013) for convergence rates with weaker assumptions on A.