On the characterization of approximation spaces in Nonlinear Approximation

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Outline

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Objectives

Characterize the rate of approximation (Approximation spaces) in various settings in Nonlinear Approximation

Nonlinear *n*-term approximation from orthonormal bases in Hilbert spaces

Suppose H is a separable Hilbert space and $\{\psi_m\}_{m\ge 1}$ is an orthonormal basis for H.
Denote by Ω_n is the set of all functions g of the form

$$g = \sum_{\nu=1}^{n} a_\nu \psi_{m_\nu},$$

Let

$$\sigma_n(f) := \inf_{g\in\Omega_n} \|f - g\|_H.$$

"Besov" spaces. Let $s > 0$ and $1/\tau := s + 1/2$. The Besov space B^s_τ is defined as the set of all $f \in H$ s.t.

$$\|f\|_{B^s_\tau} := \Big(\sum_{m=1}^{\infty} |\langle f, \psi_m\rangle|^\tau \Big)^{1/\tau} < \infty.$$

Nonlinear n-term approximation from bases (Cont.)

Jackson estimate: If $f \in B^s_\tau$, then

$$\sigma_n(f) \le n^{-s} \|f\|_{B^s_\tau}, \quad n \ge 1.$$

Proof. Given $f \in H$ let $|\langle f, \psi_{m_1} \rangle| \ge |\langle f, \psi_{m_2} \rangle| \ge \cdots$
Then

$$\sigma_n(f) = \Big\| f - \sum_{\nu=1}^{n} \langle f, \psi_{m_\nu} \rangle \psi_{m_\nu} \Big\|_H = \Big(\sum_{\nu=n+1}^{\infty} |\langle f, \psi_{m_\nu} \rangle|^2 \Big)^{1/2} \le n^{-s} \|f\|_{B^s_\tau}.$$

Bernstein estimate: If $g \in \Omega_n$, $n \ge 1$, then

$$\|g\|_{B^s_\tau} \le n^s \|g\|_H.$$

These estimates allow to characterize the rates of nonlinear n-term approximation from $\{\psi_m\}$ in H.

Approximation spaces

Approximation spaces: $\|f\|_{A^s_q(H)} := \|f\|_H + |f|_{A^s_q(H)}$, where

$$|f|_{A^s_q(H)} := \begin{cases} \left(\sum_{n=1}^{\infty} [n^s \sigma_n(f)]^q \frac{1}{n} \right)^{1/q}, & 0 < q < \infty, \\ \sup_{n \ge 1} n^s \sigma_n(f), & q = \infty. \end{cases}$$

***K*-functional:** Suppose $Y \hookrightarrow X$. Then the K-functional for $f \in X$ is defined by

$$K(f,t) := \inf_{g \in Y} \{\|f - g\|_X + t|g|_Y\}, \quad t \ge 0.$$

Interpolation spaces: For $0 < q \le \infty$ and $0 < \theta < 1$, the interpolation space $(X, Y)_{\theta,q}$ consists of all $f \in X$ for which

$$|f|_{(X,Y)_{\theta,q}} := \begin{cases} \left(\int_0^\infty [t^{-\theta} K(f,t)]^q \frac{dt}{t} \right)^{1/q}, & 0 < q < \infty \\ \sup_{0 \le t < \infty} t^{-\theta} K(f,t), & q = \infty. \end{cases}$$

Norm: $\|\cdot\|_{(X,Y)_{\theta,q}} := \|\cdot\|_X + |\cdot|_{(X,Y)_{\theta,q}}$.

Claim: The Jackson inequality implies

$$\sigma_n(f) \le cK(f, n^{-s}), \quad f \in H, \quad K(f,t) := K(f,t; H, B^s_\tau)$$

Claim: The Bernstein inequality implies (if $\tau \le 1$)

$$K(f, 2^{-ms}) \le c2^{-ms}\left(\sum_{j=0}^{m}[2^{js}\sigma_{2^j}(f)]^\tau + \|f\|_H^\tau\right)^{1/\tau}, \quad f \in H, \ m \ge 0.$$

Characterization: $A^\gamma_q(H) = (H, B^s_\tau)_{\gamma/s,q}$, $0 < \gamma < s$, $0 < q \le \infty$, with equivalent norms.

In particular, $\sigma_n(f) = O(n^{-\gamma})$ iff $K(f,t) = O(t^{\gamma/s})$, $0 < \gamma < s$.

Nonlinear Spline Approximation in dimension $d = 1$

Nonlinear approximation from **piecewise poynomials** in $L^p(\mathbb{R})$ or $L^p(a,b)$, $1 \le p < \infty$ (Free knot spline approximation)
Denote by $S(k,n)$, $k \ge 1$, the set of all functions S of the form

$$S = \sum_{\nu=1}^{n} P_\nu \mathbb{1}_{I_\nu},$$

where P_ν is a polynomial of degree $\le k-1$ and $\mathbb{1}_{I_\nu}$ is the characteristic functiuon of the compact interval I_ν. Assume that $\{I_\nu\}$ have disjoint interiors.
Denote

$$S_n^k(f)_p := \inf_{S \in S(k,n)} \|f - S\|_{L^p}.$$

The goal is to characterize the associated approximation spaces.

Let $s > 0$, $1 \le p < \infty$, and $1/\tau = s + 1/p$.

Besov space:

$$|f|_{B^{s,k}_\tau} := \Big(\int_0^\infty \big(t^{-s}\omega_k(f,t)_\tau\big)^\tau \frac{dt}{t} \Big)^{1/\tau}, \quad \omega_k(f,t)_\tau := \sup_{|h|\le t} \|\Delta_h^k f(\cdot)\|_{L^\tau(\Omega)}.$$

Jackson estimate: If $f \in B^s_\tau$, $s > 0$, then

$$S_n^k(f)_p \le cn^{-s}|f|_{B^{s,k}_\tau}.$$

Bernstein estimate: If $S \in S(k,n)$, $s > 0$, then

$$|S|_{B^{s,k}_\tau} \le cn^s\|S\|_{L^p}.$$

Nonlinear Spline Approximation in dimension $d = 1$

Approximation spaces: $\|f\|_{A^s_q(L^p)} := \|f\|_{L^p} + |f|_{A^s_q(L^p)}$, where

$$|f|_{A^s_q(L^p)} := \begin{cases} \left(\sum_{n=1}^{\infty} [n^s S^k_n(f)_p]^q \frac{1}{n}\right)^{1/q}, & 0 < q < \infty, \\ \sup_{n \ge 1} n^\alpha S^k_n(f)_p, & q = \infty. \end{cases}$$

Characterization: $A^\gamma_q(L^p) = (L^p, B^{s,k}_\tau)_{\gamma/s,q}$, $0 < \gamma < s$, $0 < q \le \infty$.

In particular: For $f \in L^p$ we have

$$S^k_n(f)_p = O(n^{-\gamma}) \quad \text{iff} \quad K(f,t) = O(t^{\gamma/s}), \ \ 0 < \gamma < s.$$

Here $K(f,t) := \inf_{g \in B^{s,k}_\tau} \left\{ \|f - g\|_{L^p} + t|g|_{B^{s,k}_\tau} \right\}$.

Nonlinear Spline Approximation in dimensions $d \geq 2$

Nonlinear Spline Approximation in dimensions $d = 2$ over **multilevel nested triangulations.**

Besov type spaces are introduced and companien Jackson and Bernstein estimates are established, which allow to characterize the associated approximation spaces.

B. Karaivanov, PP, Nonlinear piecewise polynomial approximation beyond Besov spaces. Appl. Comput. Harmon. Anal. 15 (2003), no. 3, 177-223

O. Davydov, PP, Nonlinear approximation from differentiable piecewise polynomials. SIAM J. Math. Anal. 35 (2003), no. 3, 708-758.

Nonlinear approximation from **piecewise constants** on a polygonal domain $\Omega \subset \mathbb{R}^2$ or $\Omega = \mathbb{R}^2$ **(the nonnested case)**. Denote by Σ_n the set of all functions S of the form

$$S = \sum_{\nu=1}^{n} a_\nu \mathbb{1}_{R_\nu},$$

where $R_\nu = Q_\nu \setminus \tilde{Q}_\nu$, $\tilde{Q}_\nu \subset Q_\nu$, Q_ν, $\tilde{Q}_\nu$ are convex isotropic polygonal subdomains ($\tilde{Q}_\nu = \emptyset$ is Okay). We assume that $\{R_\nu\}$ are with disjoint interiors and no “thin” rings R_ν are allowed.

Example. Q_ν, $\tilde{Q}_\nu$ can be rectangles with sides parallel to the coordinate axes (or triangles), but no “thin” rectangles or rings are allowed.
Denote

$$\sigma_n(f)_p := \inf_{S \in \Sigma_n} \|f - S\|_{L^p(\Omega)}.$$

Nonlinear piecewise constant approximation ($d \geq 2$)

Let $s > 0$, $1 \leq p < \infty$, and $1/\tau = s/2 + 1/p$.

Besov space:

$$|f|_{B^s_\tau} := \Big(\int_0^\infty \big(t^{-s}\omega(f,t)_\tau\big)^\tau \frac{dt}{t}\Big)^{1/\tau}, \quad \omega(f,t)_\tau := \sup_{|h| \leq t} \|\Delta_h f(\cdot)\|_{L^\tau(\Omega)}.$$

Jackson estimate: If $f \in B^s_\tau$, where $0 < s < 2/p$, then

$$\sigma_n(f)_p \leq c n^{-s/2} |f|_{B^s_\tau}.$$

In general, $S_1 - S_2 \notin \Sigma_{cn}$ if $S_1, S_2 \in \Sigma_n$, and

$$|S_1 - S_2|_{B^s_\tau} \not\leq c n^{s/2} \|S_1 - S_2\|_{L^p}, \quad S_1, S_2 \in \Sigma_n.$$

Example: $|f|_{B^s_\tau} \sim \varepsilon^{-s/2} \|f\|_{L^p}$ for $f = \mathbb{1}_{[0,\varepsilon]\times[0,1]}$, $0 < s < 2/p$.

New Bernstein estimate: If $0 < s < 2/p$ and $\tau \le 1$, then

$$|S_1|^\tau_{B^s_\tau} \le |S_2|^\tau_{B^s_\tau} + cn^{s\tau/2}\|S_1 - S_2\|^\tau_{L^p}, \quad S_1, S_2 \in \Sigma_n.$$

Inverse estimate: If $1 \le p < \infty$, $0 < s < 2/p$, $\tau \le 1$ and $f \in L^p$, then

$$K(f, 2^{-\frac{ms}{2}}) \le c2^{-\frac{ms}{2}}\Big(\sum_{k=0}^{m}\big(2^{\frac{ks}{2}}\sigma_{2^k}(f)_p\big)^\tau + \|f\|_p^\tau\Big)^{1/\tau}, \quad m \ge 0,$$

where $K(f,t) := \inf_{g \in B^s_\tau}\big\{\|f-g\|_{L^p} + t|g|_{B^s_\tau}\big\}$.

Characterization: $A^\gamma_q(L^p) = (L^p, B^s_\tau)_{2\gamma/s,q}$, $0 < \gamma < s/2$, $0 < q \le \infty$.

Corollary. For $f \in L^p$ we have

$$\sigma_n(f)_p = O(n^{-\gamma}) \quad \text{iff} \quad K(f,t) = O(t^{2\gamma/s}), \ \ 0 < \gamma < s/2 < 1/p.$$

Nonlinear approximation from **piecewise linear polynomials** on a polygonal domain Ω or $\Omega = \mathbb{R}^2$ **(the nonnested case)**. Denote by Σ_n^1 the set of all functions S of the form

$$S = \sum_{\nu=1}^{n} a_\nu \varphi_{\theta_\nu},$$

where φ_{θ_ν} is the Courant element supported on the polygonal cell θ_ν. The minimal angle condition is imposed on the underlying triangles. No "thin" rings are allowed.
Denote

$$\sigma_n(f)_p := \inf_{S \in \Sigma_n^1} \|f - S\|_{L^p(\Omega)}.$$

Let $s > 0$, $1 \leq p < \infty$, and $1/\tau = s/2 + 1/p$.

Besov space:

$$|f|_{B^s_\tau} := \Big(\int_0^\infty \big(t^{-s}\omega_2(f,t)_\tau\big)^\tau \frac{dt}{t} \Big)^{1/\tau}, \quad \omega_2(f,t)_\tau := \sup_{|h| \leq t} \|\Delta_h f(\cdot)\|_{L^\tau(\Omega)}.$$

Jackson estimate: If $f \in B^s_\tau$, where $0 < s/2 < 1/p + 1$, then

$$\sigma_n(f)_p \leq c n^{-s/2} |f|_{B^s_\tau}.$$

In general, $S_1 - S_2 \notin \Sigma^1_{cn}$ if $S_1, S_2 \in \Sigma^1_n$, and

$$|S_1 - S_2|_{B^s_\tau} \not\leq c n^{s/2} \|S_1 - S_2\|_{L^p}, \quad S_1, S_2 \in \Sigma^1_n.$$

Bernsten estimate

Bernstein estimate: If $0 < s/2 < 1/p + 1$ and $\tau \le 1$, then

$$|S_1|^\tau_{B^s_\tau} \le |S_2|^\tau_{B^s_\tau} + cn^{s\tau/2}\|S_1 - S_2\|^\tau_{L^p}, \quad S_1, S_2 \in \Sigma^1_n.$$

Inverse estimate: If $1 \le p < \infty$, $0 < s/2 < 1/p + 1$, $\tau \le 1$ and $f \in L^p$, then

$$K(f, 2^{-\frac{ms}{2}}) \le c2^{-\frac{ms}{2}} \Big(\sum_{k=0}^{m} (2^{\frac{ks}{2}} \sigma_{2^k}(f)_p)^\tau + \|f\|^\tau_p \Big)^{1/\tau}, \quad m \ge 0,$$

where $K(f, t) := \inf_{g \in B^s_\tau} \big\{ \|f - g\|_{L^p} + t|g|_{B^s_\tau} \big\}$.

Characterization: $A^\gamma_q(L^p) = (L^p, B^s_\tau)_{2\gamma/s,q}$, $0 < \gamma < s/2 < 1/p + 1$, $0 < q \le \infty$.

Corollary. For $f \in L^p$ we have

$$\sigma_n(f)_p = O(n^{-\gamma}) \quad \text{iff} \quad K(f, t) = O(t^{2\gamma/s}), \ \ 0 < \gamma < s/2 < 1/p + 1.$$

Nonlinear n-term approximation from dilates and shifts of smooth localized functions

Suppose $\Phi \in C^\infty(\mathbb{R}^d)$ and Φ is well localized. For example,

$$\Phi(x) = \frac{1}{(1+|x|^2)^N} \quad \text{or} \quad \Phi(x) = \exp\{-|x|^2\} \quad \text{or} \quad \ldots$$

Denote by Ω_n the set of all functions of the form

$$g(x) = \sum_{\nu=1}^{n} c_\nu \Phi(a_\nu x + b_\nu), \quad a_\nu, c_\nu \in \mathbb{R}, b_\nu \in \mathbb{R}^d.$$

Consider

$$\sigma_n(f)_p := \inf_{g \in \Omega_n} \|f - g\|_{L^p}, \quad f \in L^p(\mathbb{R}^d).$$

The goal is to characterize the associated approximation spaces.

Jackson estimate

Theorem. Let $s > 0$, $1 < p < \infty$, and $1/\tau = s/d + 1/p$. If $f \in B^s_{\tau\tau}$, then

$$\sigma_n(f)_p \le cn^{-s/d}\|f\|_{B^s_{\tau\tau}}.$$

Here

$$\|f\|_{B^s_{pq}} := \Big(\int_0^\infty \big(t^{-s}\omega_k(f,t)_p\big)^q \frac{dt}{t}\Big)^{1/q}, \quad k > s > 0.$$

Theorem. (d=1) Let $s > 0$, $1 < p < \infty$. There exists K and a function θ of the form

$$\theta(x) = \sum_{\nu=1}^{K} c_\nu \Phi(ax + b_\nu), \quad a, c_\nu \in \mathbb{R}, b_\nu \in \mathbb{R}^d,$$

s.t. if $\theta_{j\ell}(x) := 2^{j/2}\theta(2^j x + \ell)$ then $\mathcal{B} := \{\theta_{j\ell}\}$ is unconditional basis for $L^p(\mathbb{R})$ which characterizes the Besov norm $B^s_{\tau\tau}$.

Jackson estimate

Namely,

$$\|f\|_{B^s_{\tau\tau}} \sim \Big(\sum_{j,\ell} \|\langle\tilde{\theta}_{j\ell}, f\rangle\theta_{j\ell}\|_p^\tau\Big)^{1/\tau}.$$

Denote by $\sigma_n(f, \mathcal{B})_p$ the best n-term appromation in L^p from $\mathcal{B}$.

Theorem. Let $s > 0$, $1 < p < \infty$, and $1/\tau = s/d + 1/p$. If $f \in B^s_{\tau\tau}$, then

$$\sigma_n(f, \mathcal{B})_p \le cn^{-s/d}\|f\|_{B^s_{\tau\tau}}.$$

PP, Bases consisting of rational functions of uniformly bounded degrees or more general functions. J. Funct. Anal. 174 (2000), no. 1, 18-75.

G. Kyriazis, PP, New bases for Triebel-Lizorkin and Besov spaces. Trans. Amer. Math. Soc. 354 (2002), no. 2, 749-776

Rational approximation on $\mathbb{R}$

Denote

$$R_n(f)_p := \inf_{g \in R_n} \|f - g\|_{L^p(\mathbb{R})}, \quad \{R_n : g = P/Q, \deg P, Q \le n\}.$$

Theorem. [Pekarskii] If $f \in B^s_{\tau\tau}$, $s > 0$, $1 < p < \infty$, $1/\tau = s + 1/p$, then

$$R_n(f)_p \le cn^{-s/d} \|f\|_{B^s_{\tau\tau}}.$$

Let $A^s_{pq}(R)$ be the approximation spaces assoc. with $\{R_n(f)_p\}$:

$$|f|_{A^s_{pq}(R)} := \begin{cases} \left(\sum_{n=1}^{\infty} [n^s R_n(f)_p]^q \frac{1}{n} \right)^{1/q}, & 0 < q < \infty, \\ \sup_{n \ge 1} n^s R_n(f)_p, & q = \infty. \end{cases}$$

Theorem. [Pekarski] If $1 < p < \infty$, $s > 0$, $1/\tau = s + 1/p$, then

$$A^\gamma_{pq}(R) = (L^p, B^s_{\tau\tau})_{\gamma/s,q}, \quad 0 < \gamma < s, 0 < q \le \infty.$$

Open problem

Prove or disprove the **Bernstein inequality**:

$$\|g\|_{B^s_{\tau\tau}} \le cn^{s/d}\|g\|_{L^p},$$

where $s > 0$, $1 < p < \infty$, $1/\tau = s + 1/p$, for functions of the form

$$g(x) = \sum_{\nu=1}^{n} c_\nu \Phi(a_\nu x + b_\nu), \quad a_\nu, c_\nu \in \mathbb{R}, b_\nu \in \mathbb{R}^d,$$

where

$$\Phi(x) = \exp\{-|x|^2\}$$

or another $\Phi \in C^\infty(\mathbb{R}^d)$ with fast decay.

The idea is to use a “small perturbation argument” method.

The setting: Let H be a separable Hilbert space of functions and

$$\mathcal{S} \subset H \subset \mathcal{S}',$$

where $\mathcal{S}$ is a linear space of test functions and $\mathcal{S}'$ is the associated space of distributions. Suppose

$$\mathcal{B} \subset \mathcal{S}'$$

is a quasi-Banach space of distributions with associated sequence space $b(\mathcal{X})$.

For instance, $\mathcal{B}$ can be a Besov or Triebel-Lizorkin space.

Construction of bases

The old basis. Assume $\Psi := \{\psi_\xi : \xi \in \mathcal{X}\} \subset \mathcal{S}$ is an orthonormal basis for H and Ψ is a basis for the space $\mathcal{B}$ in the following sense:
(a) Every $f \in \mathcal{B}$ has a unique representation in terms of $\{\psi_\xi\}_{\xi \in \mathcal{X}}$ and

$$f = \sum_{\xi \in \mathcal{X}} \langle f, \psi_\xi \rangle \psi_\xi \quad \text{in } \mathcal{B}.$$

(b) The operator $S_\Psi f := (\langle f, \psi_\xi \rangle)_{\xi \in \mathcal{X}}$ is bounded as an operator from $\mathcal{B}$ to $b(\mathcal{X})$.
(c) For any sequence $h \in b(\mathcal{X})$ the operator $T_\Psi h := \sum_{\xi \in \mathcal{X}} h_\xi \psi_\xi$ is well defined and bounded as an operator from $b(\mathcal{X})$ to $\mathcal{B}$.
Consequently, for any $f \in \mathcal{B}$

$$c_1 \|f\|_{\mathcal{B}} \le \|(\langle f, \psi_\xi \rangle)\|_{b(\mathcal{X})} \le c_2 \|f\|_{\mathcal{B}}$$

for some constants $c_1, c_2 > 0$.

Construction of bases (Cont.)

Construction of a new basis. The idea is by perturbing Ψ to construct a new basis $\Theta = \{\theta_\xi : \xi \in \mathcal{X}\}$ for H and $\mathcal{B}$.
Since Ψ is a basis for H, we have

$$\theta_\xi = \sum_{\eta \in \mathcal{X}} \langle \theta_\xi, \psi_\eta \rangle \psi_\eta \quad \text{in } H.$$

Denote by A the transformation matrix

$$A := (a_{\xi,\eta})_{\xi,\eta \in \mathcal{X}}, \quad a_{\xi,\eta} := \langle \theta_\xi, \psi_\eta \rangle.$$

The key assumption is that the operator A with matrix A is bounded and invertible on $\ell^2(\mathcal{X})$ and A^{-1} is bounded on $\ell^2(\mathcal{X})$.
Observe that if

$$D = (d_{\xi,\eta})_{\xi,\eta \in \mathcal{X}} := (\langle \psi_\xi - \theta_\xi, \psi_\eta \rangle)_{\xi,\eta \in \mathcal{X}},$$

then $A = Id - D$ and, therefore, A^{-1} exists and is bounded on $\ell^2(\mathcal{X})$ if

$$\|D\|_{\ell^2(\mathcal{X}) \mapsto \ell^2(\mathcal{X})} < 1.$$

Construction of bases (Cont.)

If $A^{-1} =: (b_{\xi,\eta})_{\xi,\eta\in\mathcal{X}}$ we define the dual by

$$\tilde{\theta}_\xi := \sum_{\eta\in\mathcal{X}} \overline{b_{\eta,\xi}}\psi_\eta \quad \text{and set} \quad \tilde{\Theta} := \{\tilde{\theta}_\xi : \xi \in \mathcal{X}\}.$$

Theorem. Assume in addition that the operators A^T and $(A^{-1})^T$ with matrices A^T and $(A^{-1})^T$ are bounded on $b(\mathcal{X})$. Then Θ (with dual $\tilde{\Theta}$) is a basis for $\mathcal{B}$ in the following sense:

(a) Every $f \in \mathcal{B}$ has a unique representation in terms of $\{\theta_\xi\}_{\xi\in\mathcal{X}}$ and

$$f = \sum_{\xi\in\mathcal{X}} \langle f, \tilde{\theta}_\xi\rangle \theta_\xi,$$

where by definition $\langle f, \tilde{\theta}_\xi\rangle := \sum_{\eta\in\mathcal{X}} \langle f, \psi_\eta\rangle\langle\psi_\eta, \tilde{\theta}_\xi\rangle$ and the series converges unconditionally in $\mathcal{B}$.

(b) There exist constants $c_1, c_2 > 0$ such that

$$c_1\|f\|_{\mathcal{B}} \le \|(\langle f, \tilde{\theta}_\xi\rangle)_\xi\|_{b(\mathcal{X})} \le c_2\|f\|_{\mathcal{B}} \quad \text{for} \quad f \in \mathcal{B}.$$

Construction of holomorphic rational bases

Goal: Apply the above scheme for the construction of a unconditional basis

$$\{R_{j,k}\} \quad \text{with dual} \quad \{\tilde{R}_{j,k}\}$$

for the Hardy spaces H^p, $0 < p < \infty$, on the unit disc $D := \{z \in \mathbb{C} : |z| < 1\}$ which characterizes holomorphic Besov spaces $B^s_{pq}(A)$ on D. Here each

$$R_{j,k} \in \mathrm{R}_K(D) \quad \text{with} \quad K < \infty \quad \text{fixed,}$$

where $\mathrm{R}_K(D)$ is the set of all rational functions of degree $\leq K$ with poles outside $\bar{D}$.

Rational approximation in H^p. Pekarski's results

Let A be the set of all holomorphic functions on $D := \{|z| < 1\}$ and for $f \in A$ set

$$\|f(r\cdot)\|_{L^p} := \Big(\int_{|z|=1} |f(rz)|^p |dz|\Big)^{1/p} \quad 0 < p < \infty.$$

The Hardy space H^p, $0 < p \le \infty$, is the set of all $f \in A$ s.t.

$$\|f\|_{H^p} := \lim_{r\to 1^-} \|f(r\cdot)\|_{L^p} < \infty.$$

If $f(z) = \sum_{n\ge 0} \hat{f}(n) z^n$ we set

$$J^\beta f(z) := \sum_{n\ge 0} (n+1)^\beta \hat{f}(n) z^n, \quad \beta \in \mathbb{R}.$$

Besov space $B^s_{pq} := B^s_{pq}(A)$, $s \in \mathbb{R}$, $0 < p, q \le \infty$, is defined by

$$\|f\|_{B^s_{pq}} := \Big(\int_0^1 (1-r)^{(\beta-s)q-1} \|J^\beta f(r\cdot)\|^q_{L^p} dr\Big)^{1/q}, \ \ \beta > s, \quad \text{if} \ \ q < \infty.$$

Pekarski's results

Denote

$$R_n(f, H^p) := \inf_{g \in \mathrm{R}_n(D)} \|f - g\|_{H^p},$$

where $\mathrm{R}_n(D)$ is the set of rational functions of degree $\leq n$ on D.

Theorem (A. Pekarski)
(a) If $f \in B^s_{\tau\tau}$, $s > 0$, $\frac{1}{\tau} = s + \frac{1}{p}$, $0 < p < \infty$, then

$$R_n(f, H^p) \leq c n^{-s} \|f\|_{B^s_{\tau\tau}}, \quad n \geq 1 \quad (\textit{Jackson})$$

(b) If $g \in \mathrm{R}_n(D)$, $n \geq 1$ and $s > 0$, $\frac{1}{\tau} = s + \frac{1}{p}$, $1 < p \leq \infty$, then

$$\|g\|_{B^s_{\tau\tau}} \leq c n^{-s} \|g\|_{H^p} \quad (\textit{Bernstein})$$

A. Pekarskii, Classes of analytic functions defined by best rational approximation in H_p, Mat. Sb. 127 (1985), 3–20.

Two hump holomorphic wavelet basis of Y. Meyer

Let $\Psi := \{2^{j/2}\psi(2^j x - k), j, k \in \mathbb{Z}\}$ be Meyer's orthonormal wavelet basis for $L^2(\mathbb{R})$. Recall that ψ is a real-valued function with the properties: $\psi \in \mathcal{S}(\mathbb{R})$,

$$\operatorname{supp}\hat{\psi} \subset \Big\{\xi : \frac{2\pi}{3} \le |\xi| \le \frac{8\pi}{3}\Big\}, \quad \sum_{j\in\mathbb{Z}} |\hat{\psi}(\xi 2^{-j})|^2 = 1,\ \xi \neq 0.$$

The 1-periodic Meyer's wavelets are defined by

$$g_{j,k}(x) := 2^{j/2}\sum_{\ell\in\mathbb{Z}} \psi(2^j(x+\ell) - k), \quad 0 \le k < 2^j,\ j \ge 0.$$

Using the Poisson summation formula it readily follows that

$$g_{j,k}(x) = 2^{-j/2}\sum_{\nu\in\mathbb{Z}} \hat{\psi}(2\pi\nu 2^{-j}) e^{2\pi i\nu(x-k2^{-j})}.$$

For $0 \le k < 2^{j-1}$, $j \ge 0$, and $k^* = 2^j - k - 1$, set

$$G_{j,k}(x) := 2^{-j/2}\sum_{\nu\ge 0} \hat{\psi}(2\pi\nu 2^{-j})\big(e^{2\pi i\nu(x-k2^{-j})} + e^{2\pi i\nu(x-k^*2^{-j})}\big).$$

Two hump holomorphic wavelet basis of Y. Meyer

Note that $G_{0,0}(x) = -e^{2\pi ix}$. In addition, let $G_{-1,0}(x) := 1$. Then

$$\{G_{j,k} : 0 \le k < 2^{j-1}, j \ge -1\}$$

is an orthonormal basis for H^2.
Given $s \in \mathbb{R}$ and $0 < p, q \le \infty$ the space b^s_{pq} is defined as the set of all complex-valued sequences $h := \{h_{jk}\}$ s.t.

$$\|h\|_{b^s_{pq}} := \Big(\sum_{j=-1}^{\infty} 2^{j(s-\frac{1}{p}+\frac{1}{2})q} \Big(\sum_{0 \le k < 2^{j-1}} |h_{jk}|^p \Big)^{\frac{q}{p}} \Big)^{1/q} < \infty$$

with the usual modification for $q = \infty$.

Theorem. Each $f \in B^s_{pq}$ has a unique representation

$$f = \sum_{jk} c_{jk}(f) G_{j,k}, \quad \text{where} \quad c_{jk}(f) := \langle f, G_{j,k} \rangle.$$

Moreover,

$$\|f\|_{B^s_{pq}} \sim \|(c_{jk}(f))\|_{b^s_{pq}}.$$

Holomorphic rational bases

Let

$$\Phi(x) := \frac{1}{(1+x^2)^n}, \quad x \in \mathbb{R}, \ n \in \mathbb{N},$$

and denote

$$\Theta_K := \Big\{\theta : \theta(x) = \sum_{\nu=1}^{K} c_\nu \Phi(ax+b_\nu), \ c_\nu, b_\nu \in \mathbb{R}, \ a>0\Big\}.$$

Obviously, $\Theta_K \subset \mathrm{R}_{2nK}(\mathbb{R})$, where $\mathrm{R}_n(\mathbb{R})$ is the set of all rational functions of degree (order) $\le n$ on $\mathbb{R}$ with real coefficients.

Proposition. Given $N, n \in \mathbb{N}$, $M > 0$ with $2n > M$, and $\varepsilon > 0$ there exists $K \ge 1$ and $\theta \in \Theta_K$ such that

$$(i) \qquad |\psi^{(r)}(x) - \theta^{(r)}(x)| \le \varepsilon(1+|x|)^{-M}, \quad 0 \le r \le N+2,$$

$$(ii) \qquad \int_{\mathbb{R}} x^r \theta(x)\, dx = 0, \quad 0 \le r \le N.$$

Holomorphic rational bases

For $j \geq 0$, $0 \leq k < 2^{j-1}$, with $k^* = 2^j - k - 1$, we define

$$R_{j,k}(x) := 2^{-j/2} \sum_{\nu \geq 0} \hat{\theta}(2\pi\nu 2^{-j})\big(e^{2\pi i\nu(x-k2^{-j})} + e^{2\pi i\nu(x-k^*2^{-j})}\big),$$

and $R_{-1,0}(x) := 1$. Write $\mathcal{R} := \{R_{j,k}\}$.

Theorem. Let above $N > \max\{s, \mathcal{J}, \mathcal{J} - s - 1\}$ with $\mathcal{J} := 1/\min\{1, p\}$ and $M > N + 1$. Then
(a) Each $R_{j,k}$ extends to a rational function in $\mathrm{R}_K(D)$ for some fixed $K < \infty$.
(b) If $\varepsilon > 0$ is sufficiently small $\mathcal{R}$ has a dual system $\tilde{\mathcal{R}}$ s.t. $(\mathcal{R}, \tilde{\mathcal{R}})$ is a unconditional basis for H^p and any $f \in B^s_{pq}$ has a unique representation

$$f = \sum_{j,k} d_{jk}(f) R_{j,k}, \quad d_{jk}(f) := \langle f, \tilde{R}_{j,k} \rangle,$$

Furthermore, if $f \in B^s_{pq}$ we have $\|f\|_{B^s_{pq}} \sim \|(d_{jk}(f))\|_{b^s_{pq}}$.

Proof of Pekarski's direct estimate

n-term approximation in H^p from the rational basis $\mathcal{R} := \{R_{jk}\}$.
Let Σ_n be the nonlinear set of all functions g of the form

$$g = \sum_{\nu=1}^{n} a_\nu R_\nu, \quad R_\nu \in \mathcal{R}.$$

Denote

$$\sigma_n(f, H^p) := \inf_{g \in \Sigma_n} \|f - g\|_{H^p}.$$

Theorem. If $f \in B^s_{\tau\tau}$, $s > 0$, $0 < p < \infty$, $\frac{1}{\tau} = s + \frac{1}{p}$, then

$$\sigma_n(f, H^p) \le cn^{-s} \|f\|_{B^s_{\tau\tau}}, \quad n \ge 1.$$

Corollary. [Pekarski]

$$R_n(f, H^p) \le cn^{-s} \|f\|_{B^s_{\tau\tau}}, \quad n \ge 1.$$

G. Kyriazis, PP, Rational bases for spaces of holomorphic functions in the disc, J. Lond. Math. Soc. (2) 89 (2014), 434–460.

Nonlinear n-term approximation from frames

Goal:

Characterize the approximation spaces associated with nonlinear n-term approximation in L^p from frames with smooth and localized elements in $\mathbb{R}^d$, on the sphere, interval, ball and simplex with weights as well as in general Dirichlet spaces

Let $\psi_0, \psi \in \mathcal{S}$, $\operatorname{supp} \hat{\psi}_0 \subset [-2, 2]$, $\operatorname{supp} \hat{\psi} \subset [-2, -1/2] \cup [1/2, 2]$, and

$$|\hat{\psi}_0(\xi)|^2 + \sum_{\nu \ge 1} |\hat{\psi}(2^{-\nu}\xi)|^2 = 1 \quad \text{for } \xi \in \mathbb{R}.$$

Set

$$\psi_{0k}(x) := \psi_0(x-k) =: \psi_I(x), \quad I := [k, k+1],$$

$$\psi_{jk}(x) := 2^{j/2}\psi(2^j x - k) =: \psi_I(x), \quad I := \Big[\frac{k}{2^j}, \frac{k+1}{2^j}\Big], \ j \ge 1.$$

Denote $D_j := \{[\frac{k}{2^j}, \frac{k+1}{2^j}] : k \in \mathbb{Z}\}$ and set $D := \cup_{j \ge 0} D_j$.
It is easy to see that $\{\psi_I\}_{I \in D}$ is a tight frame for $L^2(\mathbb{R})$:

$$f = \sum_{I \in D} \langle f, \psi_I \rangle \psi_I \quad \text{and} \quad \|f\|_{L^2} = \Big(\sum_{I \in D} |\langle f, \psi_I \rangle|^2\Big)^{1/2} \quad \text{for } f \in L^2.$$

Definition of inhomogeneous Besov spaces.

Let $\varphi_0, \varphi \in \mathcal{S}$, $\operatorname{supp} \hat{\psi}_0 \subset [-2, 2]$, $\operatorname{supp} \hat{\psi} \subset [-2, -1/2] \cup [1/2, 2]$, and

$$|\hat{\varphi}_0(\xi)| + \sum_{\nu \geq 1} |\hat{\varphi}(2^{-\nu}\xi)| \geq c > 0 \quad \text{for} \;\; \xi \in \mathbb{R}.$$

Set $\varphi_j(x) := 2^j \varphi(2^j x)$, $j \geq 1$.

Let $s \in \mathbb{R}$, $0 < p, q \leq \infty$. The space B^s_{pq} is defined as the set of all $f \in \mathcal{S}'$ such that

$$\|f\|_{B^s_{pq}} := \Big(\sum_{j \geq 0} \Big(2^{js} \|\varphi_j * f\|_{L^p} \Big)^q \Big)^{1/q} < \infty$$

with the usual modification when $p, q = \infty$.

Frame decomposition of Besov spaces

Theorem (Frazier-Jawert). Let $s \in \mathbb{R}$, $0 < p, q \le \infty$. Then for any $f \in B^s_{pq}$

$$\|f\|_{B^s_{pq}} \sim \Big(\sum_{j\ge 0}\Big(\sum_{I\in D_j}\big(|I|^{s/d+1/p-1/2}|\langle f,\psi_I\rangle|\big)^p\Big)^{q/p}\Big)^{1/q}$$

with the usual modification when $p, q = \infty$.

Frame decomposition of L^p, $1 < p < \infty$, and H^p, $0 < p \le 1$, spaces:

$$f = \sum_{I\in D}\langle f,\psi_I\rangle\psi_I \quad \text{and} \quad \|f\|_{L^p} \sim \Big\|\Big(\sum_{j\ge 0}[|\langle f,\psi_I\rangle\psi_I(\cdot)|]^2\Big)^{1/2}\Big\|_{L^p}$$

Kernels on the unit sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$, $d \geq 2$

Let $\mathcal{H}_n$ be the set of all spherical harmonics of degree n on $\mathbb{S}^d \subset \mathbb{R}^{d+1}$. Then

$$P_n(x \cdot y) = \frac{n+\lambda}{\lambda \omega_d} C_n^\lambda(x \cdot y) \quad \text{with} \quad \lambda := \frac{d-1}{2}, \;\; \omega_d := \int_{\mathbb{S}} 1 d\sigma,$$

is the kernel of the orthogonal projector onto $\mathcal{H}_n$.
Here $C_n^\lambda(t)$ is the nth degree Gegenbauer polynomial
The construction of frames on $\mathbb{S}^d$ relies on kernels

$$K_n(x \cdot y) = \sum_{j=0}^{\infty} \varphi\left(\frac{j}{n}\right) P_j(x \cdot y), \quad \varphi \in C^\infty(\mathbb{R}_+), \; \operatorname{supp} \varphi \subset [1/2, 2].$$

Localization: Here $\rho(x, y)$ is the geodesic distance on the sphere

$$|K_n(x \cdot y)| \leq \frac{c_\sigma n^d}{(1 + n\rho(x, y))^\sigma} \quad \forall \sigma > 0.$$

Construction of spherical needlets

Let $\varphi \in C^\infty[0,\infty)$, $\operatorname{supp}\varphi \subset [1/2,2]$ and $|\varphi(t)|^2 + |\varphi(t/2)|^2 = 1$ for $t \in [1,2]$. Hence $\sum_{j\ge 0} |\varphi(2^{-j}t)|^2 = 1$, $t \in [1,\infty)$. Set

$$\Psi_j(x\cdot y) := \sum_{\nu=0}^{\infty} \varphi\Big(\frac{\nu}{2^{j-1}}\Big) P_\nu(x\cdot y), \quad j \ge 1, \quad \Psi_0(x\cdot y) := P_0(x\cdot y).$$

Let the cubature $\quad \int_{\mathbb{S}^d} f(x)d\sigma(x) \sim \sum_{\xi\in\mathcal{X}_j} c_\xi f(\xi)$

be exact for spherical harmonics of degree $\le 2^{j+2}$ and $c_\xi \sim 2^{-jd}$.

Needlets: $\psi_\xi(x) := c_\xi^{-1/2}\Psi_j(\xi\cdot x)$, $\xi \in \mathcal{X}_j$. Set $\mathcal{X} := \cup_{j=0}^\infty \mathcal{X}_j$

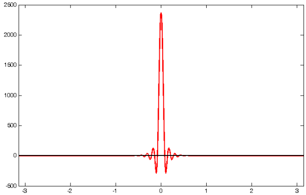
Needlet decomposition: $f = \sum_{\xi\in\mathcal{X}} \langle f, \psi_\xi\rangle \psi_\xi$ for $f \in L^2$, and

$$\|f\|_{L^2} = \Big(\sum_{\xi\in\mathcal{X}} |\langle f, \psi_\xi\rangle|^2\Big)^{1/2}$$

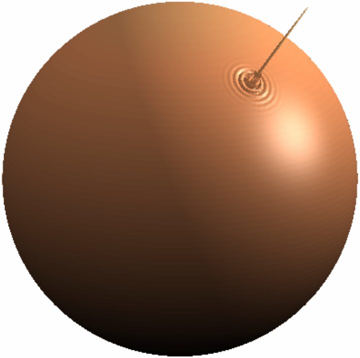
Localization:

$$|\psi_\xi(x)| \le \frac{c_\sigma 2^{jd/2}}{(1+2^j\rho(\xi,x))^\sigma}, \quad \xi \in \mathcal{X}_j,\ \forall \sigma > 0.$$

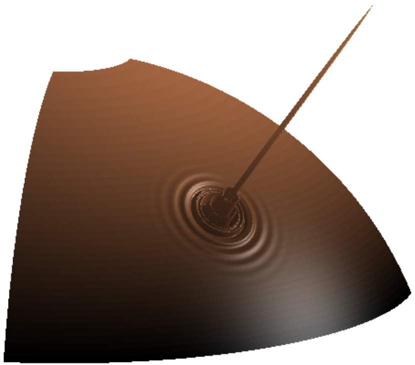
Needlets on $\mathbb{S}^2$:



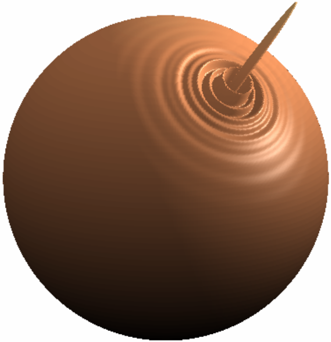
Needlet cross section, degree 32



Needlet of degree 512



Needlet of degree 512



Needlet of degree 128

Definition of Besov spaces

Distributions on $\mathbb{S}^d$. Test functions: $\mathcal{D} := C^\infty(\mathbb{S}^d)$. Claim: $\phi \in \mathcal{D}$ iff

$$\mathcal{N}_k(\phi) := \sup_{n\ge 0} (n+1)^k \|(\mathrm{Proj}_n)\phi\|_2 < \infty \quad \forall k = 0, 1, \dots$$

The topology on $\mathcal{D}$ is defined by the seminorms $\mathcal{N}_k(\phi)$.
The space $\mathcal{D}' := \mathcal{D}'(\mathbb{S}^d)$ of distributions on $\mathbb{S}^d$ is defined as the set of all continuous linear functionals on $\mathcal{D}$.

Kernels: Consider the kernels $\{\Phi_j\}$ defined by

$$\Phi_0(x\cdot y) := 1 \quad \text{and} \quad \Phi_j(x\cdot y) := \sum_{\nu=0}^{\infty} \varphi\Big(\frac{\nu}{2^{j-1}}\Big) P_\nu(x\cdot y), \quad j \ge 1,$$

where φ obeys the conditions

$$\begin{aligned} &\varphi \in C^\infty[0,\infty), && \mathrm{supp}\ \varphi \subset [1/2, 2], \\ &|\varphi(t)| > c > 0, && \text{if } t \in [3/5, 5/3]. \end{aligned}$$

Frame decomposition of Besov spaces

Definition of Besov spaces.
Let $s \in \mathbb{R}$ and $0 < p, q \leq \infty$. The Besov space $B^s_{pq} = B^s_{pq}(\mathbb{S}^d)$ is defined as the set of all distributions $f \in \mathcal{D}'$ s.t.

$$\|f\|_{B^s_{pq}} := \Big(\sum_{j \geq 0} \Big(2^{sj} \|\Phi_j f(\cdot)\|_{L^p} \Big)^q \Big)^{1/q} < \infty.$$

Here $\Phi_j f(x) := \int_{\mathbb{S}^d} \Phi_j(x \cdot y) f(y) d\sigma(y)$.

Theorem. Let $s \in \mathbb{R}$ and $0 < p, q \leq \infty$. Then for any $f \in B^s_{pq}$

$$\|f\|_{B^s_{pq}} \sim \Big(\sum_{j \geq 0} 2^{jsq} \Big[\sum_{\xi \in \mathcal{X}_j} \|\langle f, \psi_\xi \rangle \psi_\xi\|_p^p \Big]^{q/p} \Big)^{1/q}.$$

Localized frames in other settings

Similar frames have been constructed

- on the interval with Jacobi weights
- on the ball with weights
- on the simplex with weights
- on product spaces

Compactly supported frames with small shrinking supports have also been constructed on the sphere, interval, ball, simplex.

Frames in a general setting

(a) (M, ρ, μ) is a metric measure space with doubling measure:

$$0 < \mu(B(x, 2r)) \leq c\mu(B(x, r)) < \infty, \quad x \in M, r > 0,$$

which implies $\quad \mu(B(x, \lambda r)) \leq c\lambda^d \mu(B(x, r)), \; r > 0, \lambda > 1.$

(b) L is a self-adjoint positive operator on $L^2(M, d\mu)$ s.t. the (heat) kernel $p_t(x, y)$ of the associated semigroup

$$P_t = e^{-tL}$$

obeys

$$|p_t(x, y)| \leq \frac{C \exp\{-\frac{c\rho^2(x,y)}{t}\}}{\sqrt{\mu(B(x, \sqrt{t}))\mu(B(y, \sqrt{t}))}} \quad \text{for} \;\; x, y \in M, \, 0 < t \leq 1.$$

Setting

(c) Hölder continuity: There exists a constant $\alpha > 0$ s.t

$$\big|p_t(x,y) - p_t(x,y')\big| \le C\Big(\frac{\rho(y,y')}{\sqrt{t}}\Big)^{\alpha} \frac{\exp\{-\frac{c\rho^2(x,y)}{t}\}}{\sqrt{\mu(B(x,\sqrt{t}))\mu(B(y,\sqrt{t}))}}$$

for $x, y, y' \in M$ and $0 < t \le 1$, whenever $\rho(y,y') \le \sqrt{t}$.

(d) Markov property:

$$\int_M p_t(x,y)d\mu(y) \equiv 1 \quad \text{for } t > 0.$$

Realization of the setting in Dirichlet spaces

A natural effective realization of the above setting appears in the general framework **of strictly local regular Dirichlet spaces with a complete intrinsic metric**
where it only suffices to verify

the **local Poincaré inequality** and
the **doubling condition on the measure**

then the above general setting applies in full.

Examples

- Classical setting on $\mathbb{R}^d$ with $L = -\Delta$
- Uniformly elliptic divergence form operators on $\mathbb{R}^d$.
- Uniformly elliptic divergence form operators on subdomains of $\mathbb{R}^d$ with boundary conditions.
- Riemannian manifolds and Lie groups. In particular, Compact Riemannian manifolds, Riemannian manifold with non-negative Ricci curvature, Compact Lie groups, Lie groups with polynomial growth and their homogeneous spaces, ...
- Heat kernel on $[-1, 1]$ associated with the Jacobi operator, Heat kernel on the sphere, ball and simplex

Functional calculus in the general setting

Theorem. Suppose $f \in C_0^\infty(\mathbb{R})$ and let f be even. Then $f(\delta\sqrt{L})$, $\delta > 0$, is an integral operator with kernel $f(\delta\sqrt{L})(x,y)$ satisfying

$$\left|f(\delta\sqrt{L})(x,y)\right| \le c_\sigma \frac{\left(1+\frac{\rho(x,y)}{\delta}\right)^{-\sigma}}{\sqrt{|B(x,\delta)||B(y,\delta)|}} \quad \forall\sigma > 0,$$

and if $\rho(y,y') \le \delta$

$$\left|f(\delta\sqrt{L})(x,y) - f(\delta\sqrt{L})(x,y')\right| \le c_\sigma \left(\frac{\rho(y,y')}{\delta}\right)^\alpha \frac{\left(1+\frac{\rho(x,y)}{\delta}\right)^{-\sigma}}{\sqrt{|B(x,\delta)||B(y,\delta)|}}$$

for some $\alpha > 0$.
Here $B(x,\delta)$ is the ball with center x and radius δ.

Spectral spaces

Let E_λ, $\lambda \geq 0$, be the spectral resolution associated with L. Then

$$L = \int_0^\infty \lambda dE_\lambda$$

Let F_λ, $\lambda \geq 0$, be the spectral resolution associated with $\sqrt{L}$, i.e. $F_\lambda = E_{\lambda^2}$ and hence $\sqrt{L} = \int_0^\infty \lambda dF_\lambda$.
The spectral space Σ_λ is defined by

$$\Sigma_\lambda := \{f \in L^2 : F_\lambda f = f\}.$$

This can be extended to define Σ_λ^p, $1 \leq p \leq \infty$:

$$\Sigma_\lambda^p := \{f \in L^p : \theta(\sqrt{L})f = f \text{ for all } \theta \in C_0^\infty(\mathbb{R}_+),\ \theta \equiv 1 \text{ on } [0,\lambda]\}.$$

Suppose

$$\Sigma_\lambda^2 \cdot \Sigma_\lambda^2 \subset \Sigma_{2\lambda}^1, \quad \text{i.e.} \quad f, g \in \Sigma_\lambda^2 \Longrightarrow fg \in \Sigma_{2\lambda}^1.$$

Choose $\Psi_0, \Psi \in C^\infty(\mathbb{R}_+)$ s.t.
$\operatorname{supp} \Psi_0 \subset [0, b], \quad \operatorname{supp} \Psi \subset [b^{-1}, b], \quad 0 \le \Psi_0, \Psi \le 1, b > 1,$

$$\Psi_0^2(u) + \sum_{j \ge 1} \Psi^2(b^{-j}u) = 1, \quad u \in \mathbb{R}_+$$

Set $\Psi_j(u) := \Psi(b^{-j}u)$. Then $\sum_{j\ge 0} \Psi_j^2(u) = 1$, $u \in \mathbb{R}_+$ and

$$f = \sum_{j \ge 0} \Psi_j^2(\sqrt{L}) f, \quad f \in L^p.$$

Construction of frames (Con.)

Cubature: Let $\mathcal{X}_j$ be a maximal δ-net with $\delta = O(b^{-j})$ s.t.

$$\int_M f(x)d\mu(x) = \sum_{\xi\in\mathcal{X}_j} w_{j\xi} f(\xi) \quad \text{for} \quad f \in \Sigma^1_{2b^{j+1}},$$

Discretization:

$$\begin{aligned}\Psi_j^2(\sqrt{L})(x,y) &= \int_M \Psi_j(\sqrt{L})(x,u)\Psi_j(\sqrt{L})(u,y)d\mu(u)\\ &= \sum_{\xi\in\mathcal{X}_j} w_{j\xi}\Psi_j(\sqrt{L})(x,\xi)\Psi_j(\sqrt{L})(\xi,y).\end{aligned}$$

Definition: $\psi_\xi(x) := \sqrt{w_{j\xi}}\Psi_j(\sqrt{L})(x,\xi),\ \ \xi \in \mathcal{X}_j$.
Then

$$f = \sum_{\xi\in\mathcal{X}} \langle f, \psi_\xi\rangle \psi_\xi \quad \text{in } L^p \quad \text{and} \quad \|f\|_2^2 = \sum_{\xi\in\mathcal{X}} |\langle f, \psi_\xi\rangle|^2 \quad \text{for } f \in L^2.$$

Frames in the general setting: Properties

Frames: $\{\psi_\xi\}_{\xi\in\mathcal{X}}$, $\{\tilde{\psi}_\xi\}_{\xi\in\mathcal{X}}$, $\mathcal{X} = \cup_{j\ge 0}\mathcal{X}_j$

Representation: for any $f \in L^p$, $1 \le p \le \infty$, with $L^\infty := \mathrm{UCB}$

$$f = \sum_{\xi\in\mathcal{X}} \langle f, \tilde{\psi}_\xi\rangle \psi_\xi = \sum_{\xi\in\mathcal{X}} \langle f, \psi_\xi\rangle \tilde{\psi}_\xi \quad \text{in} \;\; L^p.$$

Frame: The system $\{\tilde{\psi}_\xi\}$ is a frame for L^2:

$$c^{-1}\|f\|_2^2 \le \sum_{\xi\in\mathcal{X}} |\langle f, \tilde{\psi}_\xi\rangle|^2 \le c\|f\|_2^2, \quad \forall f \in L^2.$$

The same is true for $\{\psi_\xi\}$.

Frames Properties

Space localization: For any $\xi \in \mathcal{X}_j$, $j \geq 0$,

$$|\psi_\xi(x)| \leq c|B(\xi, b^{-j})|^{-1/2} \exp\{-\kappa(b^j\rho(x,\xi))^\beta\},$$

and if $\rho(x,y) \leq b^{-j}$

$$|\psi_\xi(x) - \psi_\xi(y)| \leq c|B(\xi, b^{-j})|^{-1/2}(b^j\rho(x,y))^\alpha \exp\{-\kappa(b^j\rho(x,\xi))^\beta\}.$$

Here $0 < \kappa < 1$ and $b > 1$ are constants. Same holds for $\tilde{\psi}_\xi$.

Spectral localization: $\psi_\xi, \tilde{\psi}_\xi \in \Sigma_b^p$ if $\xi \in \mathcal{X}_0$ and

$$\psi_\xi, \tilde{\psi}_\xi \in \Sigma^p_{[b^{j-2}, b^{j+2}]} \text{ if } \xi \in \mathcal{X}_j,\; j \geq 1,\; 0 < p \leq \infty.$$

Norms:

$$\|\psi_\xi\|_p \sim \|\tilde{\psi}_\xi\|_p \sim |B(\xi, b^{-j})|^{\frac{1}{p}-\frac{1}{2}} \quad \text{for } 0 < p \leq \infty.$$

Definition of Besov spaces

Let $\varphi_0, \varphi \in C^\infty(\mathbb{R}_+)$, $\operatorname{supp} \varphi_0 \subset [0,2]$, $\varphi_0^{(\nu)}(0) = 0$ for $\nu \geq 1$, $\operatorname{supp} \varphi \subset [1/2, 2]$, and $|\varphi_0(\lambda)| + \sum_{j\geq 1} |\varphi(2^{-j}\lambda)| \geq c > 0$, $\lambda \in \mathbb{R}_+$.

Set $\varphi_j(\lambda) := \varphi(2^{-j}\lambda)$ for $j \geq 1$.

Let $s \in \mathbb{R}$ and $0 < p, q \leq \infty$.

(i) The "classical" Besov space $B^s_{pq} = B^s_{pq}(L)$ is defined by

$$\|f\|_{B^s_{pq}} := \Big(\sum_{j\geq 0} \Big(2^{sj}\|\varphi_j(\sqrt{L})f(\cdot)\|_{L^p}\Big)^q\Big)^{1/q}.$$

(ii) The "nonclassical" Besov space $\tilde{B}^s_{pq} = \tilde{B}^s_{pq}(L)$ is defined by

$$\|f\|_{\tilde{B}^s_{pq}} := \Big(\sum_{j\geq 0} \Big(\||B(\cdot, 2^{-j})|^{-s/d}\varphi_j(\sqrt{L})f(\cdot)\|_{L^p}\Big)^q\Big)^{1/q}.$$

Theorem. Let $s \in \mathbb{R}$ and $0 < p, q \leq \infty$. Then for any $f \in B^s_{pq}$

$$\|f\|_{B^s_{pq}} \sim \Big(\sum_{j \geq 0} b^{jsq} \Big[\sum_{\xi \in \mathcal{X}_j} \|\langle f, \tilde{\psi}_\xi \rangle \psi_\xi\|_p^p \Big]^{q/p} \Big)^{1/q}$$

and for $f \in \tilde{B}^s_{pq}$

$$\|f\|_{\tilde{B}^s_{pq}} \sim \Big(\sum_{j \geq 0} \Big[\sum_{\xi \in \mathcal{X}_j} \Big(|B(\xi, b^{-j})|^{-s/d} \|\langle f, \tilde{\psi}_\xi \rangle \psi_\xi\|_p \Big)^p \Big]^{q/p} \Big)^{1/q}.$$

Here $b > 1$ is from the definition of the frames.

Publications

1. T. Coulhon, G. Kerkyacharian, P. Petrushev, Heat kernel generated frames in the setting of Dirichlet spaces, J. Fourier Anal. Appl. 18 (2012), no. 5, 995-1066.

2. G. Kerkyacharian, P. Petrushev, Heat kernel based decomposition of spaces of distributions in the framework of Dirichlet spaces, Trans. Amer. Math. Soc. 367 (2015), no. 1, 121-189.

Denote by Ω_n is the set of all functions g of the form

$$g = \sum_{\xi \in \Lambda_n} a_\xi \psi_\xi,$$

where $\Lambda_n \subset \mathcal{X}$, $\#\Lambda_n \le n$, and Λ_n may vary with g. Let

$$\sigma_n(f)_p := \inf_{g \in \Omega_n} \|f - g\|_{L^p}.$$

The approximation will take place in L^p, $1 \le p < \infty$.
Suppose $s > 0$ and let $1/\tau := s/d + 1/p$. The Besov spaces

$$\tilde{B}^s_\tau := \tilde{B}^s_{\tau\tau}$$

play a prominent role. Observe that

$$\|f\|_{\tilde{B}^s_\tau} \sim \Big(\sum_{\xi \in \mathcal{X}} \|\langle f, \tilde{\psi}_\xi \rangle \psi_\xi\|_p^\tau \Big)^{1/\tau}$$

For any $f \in L^p$, $1 \le p < \infty$,

$$f = \sum \langle f, \tilde{\psi}_\xi \rangle \psi_\xi \quad \text{in } L^p.$$

Nonlinear n-term approximation (Cont.)

Proposition. If $f \in \tilde{B}^s_\tau$, then f can be identified as a function $f \in L^p$ and

$$\|f\|_{L^p} \le \Big\| \sum_{\xi \in \mathcal{X}} |\langle f, \tilde{\psi}_\xi \rangle \psi_\xi(\cdot)| \Big\|_{L^p} \le c\|f\|_{\tilde{B}^s_\tau}.$$

Theorem. If $f \in \tilde{B}^s_\tau$, then

$$\sigma_n(f)_p \le cn^{-s/d}\|f\|_{\tilde{B}^s_\tau}, \quad n \ge 1.$$

Open problem: Prove or disprove the Bernstein estimate:

$$\|g\|_{\tilde{B}^s_\tau} \le cn^{s/d}\|g\|_{L^p} \quad \text{for} \quad g \in \Omega_n, \quad 1 < p < \infty.$$

This estimate would allow to characterize the rates of nonlinear n-term approximation from $\{\psi_\xi\}_{\xi \in \mathcal{X}}$ in L^p ($1 < p < \infty$).

Thank you!