Differential Groups and the Gamma Function

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Theorem: (Hölder, 1887) The Gamma function defined by

$$\Gamma(x+1) - x\Gamma(x) = 0$$

satisfies no polynomial differential equation that is, there is no nonzero polynomial

$$P(x, y, y', y'', \ldots, y^{(n)}) \in \mathbb{C}[x, y, y', \ldots, y^{(n)}, \ldots]$$

such that $P(x, \Gamma(x), \Gamma'(x), \ldots \Gamma^{(n)}(x)) = 0$.

Group Theory

$\Downarrow$ Galois Theory

Form of Functional Dependencies

- Galois Theory of Linear Differential Equations
- Galois Theory of Linear Differential Equations with Continuous Parameters
- Galois Theory of Linear Difference Equations with Continuous Parameters
- Galois Theory of Linear Differential/Difference Equations with Discrete Parameters

Galois Theory of Polynomial Equations

$$P(y) = y^n + a_{n-1}y^{n-1} + \ldots + a_1 y + a_0 = 0, \quad a_i \in \mathbb{Q}$$

Galois group = the group of transformations of the roots $\alpha_1, \ldots, \alpha_n$ that preserve all algebraic relations among them.

- Splitting Field E = $\mathbb{Q}(\alpha_1, \ldots, \alpha_n)$.
- Galois Group Gal($E/\mathbb{Q}$) = $\{\sigma \mid \sigma = \mathbb{Q}$-autom. of $E\}$.

$$\forall \sigma \in \mathrm{Gal}(E/\mathbb{Q}),\ \sigma(\alpha_i) \text{ is again a root of } P(y)$$

$$\Rightarrow\ \mathrm{Gal}(E/\mathbb{Q}) \hookrightarrow \mathrm{Symm}_n$$

- Galois Correspondence: Subgroups $\Leftrightarrow$ Subfields
- Galois group measures the amount of interaction among the roots:

$$|\mathrm{Gal}(E/\mathbb{Q})| = [E : \mathbb{Q}]$$

e^x is not an algebraic function.

Picard-Vessiot (PV) Theory

$$\frac{dY}{dx} = A(x)Y \quad A \in \mathrm{M}_n(\mathbb{C}(x))$$

- Splitting Field: $Y = (y_{i,j}), \quad y_{i,j}$ analytic near $x = x_0$, det $Y \neq 0$,
  $K = \mathbb{C}(x) \subset \mathbb{C}(x)(y_{1,1}, \ldots, y_{n,n}) = E$, PV-Extension
  Note: E is closed under $\partial = \frac{d}{dx}$
- Galois Group $\mathrm{Gal}_\partial(E/K)$ = $\{\sigma \mid \sigma = K\text{-autom. of } E, \sigma\partial = \partial\sigma\}$
  $\forall\sigma \in \mathrm{Gal}_\partial(E/K),\ \partial(\sigma Y) = A(\sigma Y) \Rightarrow \exists\ C_\sigma \in \mathrm{GL}_n(\mathbb{C})\sigma Y = Y \cdot C_\sigma$
  $$\mathrm{Gal}_\partial(K/\mathbb{Q}) \hookrightarrow \mathrm{GL}_n(\mathbb{C})$$
- $\mathrm{Gal}_\partial(E/K) \subset \mathrm{GL}_n(\mathbb{C})$ is *Zariski closed*, a linear algebraic group
- Galois Correspondence:
  $$H^{\text{Zariski closed } \subset \mathrm{Gal}_\partial(E/K)} \Leftrightarrow F^{\text{Diff. field, } k \subset F \subset E}$$
- Measuring the amount of relations:
  $$\dim_{\mathbb{C}} \mathrm{Gal}_\partial(E/K) = \text{ tr. deg.}_K E$$

e^x is not an algebraic function

$$\frac{dy}{dx} = y \qquad k = \mathbb{C}(x)$$

- PV-extension $E = k(e^x)$, $\mathrm{Gal}_\partial(E/k) \subset \mathrm{GL}_1(\mathbb{C}) = (\mathbb{C}^*, \times)$
- Closed subgps of $\mathrm{GL}_1(\mathbb{C}) = \begin{cases} \mathrm{GL}_1(\mathbb{C}) \\ \mathbb{Z}/n\mathbb{Z} = \{\zeta \mid \zeta^n = 1\},\ n \in \mathbb{N}_{>0} \end{cases}$
- e^x algebraic over $\mathbb{C}(x) \Rightarrow \mathrm{Gal}_\partial(E/k) = \{\zeta \mid \zeta^n = 1\}$ for some n.
- e^x algebraic over $\mathbb{C}(x) \Rightarrow (e^x)^n \in \mathbb{C}(x)$ for some $n \in \mathbb{N}_{>0}$.
- e^x algebraic over $\mathbb{C}(x) \Rightarrow \frac{dz}{dx} = nz$ has a soln $z = e^{nx}$ in $\mathbb{C}(x)$.

  $$z = c\prod(x - a_i)^{m_i} \Rightarrow n = \frac{dz/dx}{z} = \sum \frac{m_i}{x - a_i} \neq n.$$

$$\mathrm{Gal}_\partial = \{\zeta \mid \zeta^n = 1\} \Rightarrow y^n \in \mathbb{C}(x)$$

Defining eqns. of group $\Rightarrow$ Algebraic relations for solutions

More Examples

Ex. $\frac{dy}{dx} = \frac{t}{x} y$ $\quad E = \mathbb{C}(x)(x^t)$

$$\mathrm{Gal}(E/\mathbb{C}(x)) = \begin{cases} \mathbb{Z}/n\mathbb{Z} & \text{if } t = m/n, (m,n) = 1 \\ \mathbb{C}^* = \mathrm{GL}_1(\mathbb{C}) & \text{if } t \notin \mathbb{Q} \end{cases}$$

Ex. $\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} + (1 - \frac{\nu^2}{x^2})y = 0$ $\quad E = \mathbb{C}(x)(Y_\nu, Y_\nu', J_\nu, J_\nu')$

$$\mathrm{Gal}(E/K) = \mathrm{SL}_2(\mathbb{C}) \Leftrightarrow \nu \notin \mathbb{Z} + \frac{1}{2}$$

$\dim_{\mathbb{C}} \mathrm{SL}_2(\mathbb{C}) = 3 \Rightarrow$ tr. deg.$_{\mathbb{C}(x)} E = 3$

$Y_\nu J_\nu' - Y_\nu' J_\nu = \frac{\pi}{x}$ and Y_ν, Y_ν', J_ν alg. indep. over $\mathbb{C}(x)$.

Galois Theory of Linear Differential Equations with Continuous Parameter

$$\partial_x Y = A(t,x) Y \quad A \in \mathrm{M}_n(\mathbb{C}(t,x)), \;\; \partial_x = \frac{\partial}{\partial x}$$

Galois group = the group of transformations of Y that preserve all algebraic relations among x, Y, and the $\{\partial_x, \partial_t\}$-derivatives of Y.

∂_t-PV-Extension: $Y = (y_{i,j}),\; y_{i,j}$ analytic near (t_0, x_0), det $Y \neq 0$,
$K = \mathbb{C}(t,x) \subset \mathbb{C}(t,x)(y_{1,1}, \ldots, y_{n,n}, \partial_t y_{1,1}, \ldots, \partial_t^r \partial_x^s y_{i,j,\ldots}) = E$

∂_t-Gal(E/K) $= \{\sigma \mid \sigma = k\text{-autom.}, \sigma\partial = \partial\sigma \;\; \partial \in \{\partial_x, \partial_t\}\}$

$\forall \sigma \in \partial_t\text{-Gal}(E/K), \exists\; C_\sigma \in \mathrm{GL}_n(\mathbb{C}(t))$ s. t. $\sigma Y = Y \cdot C_\sigma$

$$\partial_t\text{-Gal}(E/K) \hookrightarrow \mathrm{GL}_n(\mathbb{C}(t))$$

Ex. $\frac{\partial y}{\partial x} = \frac{t}{x} y$, $\quad K = \mathbb{C}(x,t), \{\partial_x = \frac{\partial}{\partial x}, \partial_t = \frac{\partial}{\partial t}\}$

$$\partial_t\text{-PV-Ext.} = E \quad = \quad \mathbb{C}(x)(x^t, \partial_t(x^t), \ldots) = \mathbb{C}(x, x^t, \log x)$$

$$\partial_t\text{-Gal}(E/\mathbb{C}(x,t)) = G \quad = \quad \{c \in \mathbb{C}(t)^* \mid \partial_t(\frac{\partial_t c}{c}) = 0\}$$

G is a Linear Differential Algebraic Group, a group of matrices whose entries satisfy a system of differential equations.

$$\partial_t(\frac{\partial_t c}{c}) = 0 \quad \Rightarrow \quad c(t) = ae^{bt}, \quad a, b \in \mathbb{C}$$

If $c(t) \in \mathbb{C}(t)$ then $c(t) = a \in \mathbb{C}$! Need more functions of t.

Replace $\mathbb{C}(t)$ with a “large” field k of meromorphic functions of t.

Parameterized PV-Theory for Differential Eqns

Cassidy/Singer 2006

$k =$ a sufficiently large field of functions of t
$K = k(x)$ with derivations ∂_x, ∂_t.

$$\partial_x Y = A(x,t)Y, \quad A \in \mathrm{M}_n(k(x,t))$$

One then has:

- Parameterized PV-extension $\mathrm{E} = K(Y, \partial_t(Y), \partial_t^2(Y), \ldots)$
- Parameterized PV-Galois group $\mathrm{G} = \mathrm{Aut}_{\partial_x,\partial_t}(E)$ - linear differential algebraic group
- Galois Correspondence: differential subfields $\leftrightarrow$ closed subgroups
- The size of G measures the amount of differential dependence

defining differential equations for $G \Rightarrow \partial_t$-differential relations among solutions

Linear Differential Algebraic Groups (LDAGs)

k - a ∂_t-differential field, $C = k^{\partial_t}$ = constants in k.

$G \subset \mathrm{GL}_n(k)$, entries satisfy algebraic DEs with respect to t

- Any Linear Algebraic Group and the constant pts of such a group.

  eg, $\mathrm{SL}_n(C) \subset \mathrm{SL}_n(k)$

- $G \subsetneq \mathbb{G}_a(k_0) = (k, +) = \{\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \mid \alpha \in k\}$

  $\Rightarrow \;\; G = \{\alpha \mid L(\alpha) = 0\}$ where $L \in k[\frac{\partial}{\partial t}]$

- $G \subsetneq \mathbb{G}_m(k) = (k^*, \times) = \mathrm{GL}_1(k)$

  $$\Rightarrow \;\; G = \begin{cases} \mathbb{Z}/n\mathbb{Z} \\ \{\alpha \neq 0 \mid L(\frac{\partial_t(\alpha)}{\alpha}) = 0\} \text{ where } L \in k[\frac{\partial}{\partial t}] \end{cases}$$

  $$\ell\partial_t : \mathbb{G}_m \to \mathbb{G}_a \quad z \mapsto \frac{\partial_t z}{z}$$

Ex. The Incomplete Gamma Function:

$$\gamma(t,x) = \int_0^x s^{t-1} e^{-s} ds$$

satisfies

$$\frac{\partial^2 \gamma}{\partial x^2} - \frac{t-1-x}{x}\frac{\partial \gamma}{\partial x} = 0$$

but satisfies no differential equation of the form

$$p(x, t, \gamma, \gamma_t, \gamma_{tt}, \ldots) = 0.$$

- Algorithms for 2^{nd} order param. LDE - Dreyfus 2012, Arreche 2013
- Partial Algorithms for n^{th} order - Minchenko/Ovchinnikov/Singer 2014

Ex. Integrable Systems/Isomonodromy:

Def. The system

$$\frac{\partial Y}{\partial x} = A(x,t)Y, \quad A(x,t) \in \mathrm{M}_n(\mathbb{C}(x,t))$$

is integrable if $\exists B(x,t) \in \mathbb{C}(x,t)$ such that

$$A_t - B_x = AB - BA.$$

i.e., if the systems $\frac{\partial Y}{\partial x} = A(x,t)Y, \quad \frac{\partial Y}{\partial t} = B(x,t)Y$ are consistent.

Thm. A parameterized system is integrable if and only if its parameterized PV-group is conjugate to a group of constant matrices.

The param. PV-Group measures how far a system is from integrable.

Thm. (Mitschi-Singer, 2013) The Darboux-Halphen V system is associated with a projectively integrable system.

Galois Theory of Linear Difference Equations with Continuous Parameter

Hardouin/Singer 2008

Ex. Hermite polynomials:

$$H(n,t) = n! \sum_{m=0}^{[n/2]} \frac{(-1)^m (2t)^{n-2m}}{m!(n-2m)!}$$

If $Y(n,t) = (H(n,t), H(n+1,t))^T$, then

$$Y(n+1,t) = \begin{pmatrix} 0 & 1 \\ -2n & 2t \end{pmatrix} Y(n,t)$$

Is there a differential dependence on t? What is it?

$$\frac{\partial Y(n,t)}{\partial t} = \begin{pmatrix} 2t & -1 \\ 2n & 0 \end{pmatrix} Y(n,t).$$

Ex. Gamma Function:

$$\Gamma(x+1) = x\Gamma(x)$$

$\Gamma(x)$ is a meromorphic function of x.

What are the differential properties of Γ?

Algebraic Setting

K = a field with automorphism σ and derivation ∂ such that $\sigma\partial = \partial\sigma$.

= a $\sigma\partial$-field

Ex. $K = \mathbb{C}(x,t)$

$$\left\{ \begin{array}{cc} \sigma(x) = x+1 & \sigma(t) = t \\ \partial(x) = 0 & \partial(t) = 1 \end{array} \right\} \quad \text{or} \quad \left\{ \begin{array}{cc} \sigma(x) = qx & \sigma(t) = t \\ \partial(x) = 0 & \partial(t) = 1 \end{array} \right\}$$

Ex. $K = \mathbb{C}(x)$

$$\left\{ \begin{array}{c} \sigma(x) = x+1 \\ \partial(x) = 1 \end{array} \right\} \quad \text{or} \quad \left\{ \begin{array}{c} \sigma(x) = qx \\ \partial(x) = x \end{array} \right\}$$

Given

$$\sigma(Y) = AY, \quad A \in \mathrm{GL}_n(K)$$

describe the behavior of solutions with respect to ∂.

Parameterized PV-Theory for Difference Eqns

$K =$ a $\sigma\partial$-field

$$\sigma(Y) = AY, \quad A \in \mathrm{M}_n(K)$$

One then has:

- Parameterized PV-extension E
- Parameterized PV-Galois group $\mathrm{Gal}_{\sigma\partial}(E/K) = \mathrm{Aut}_{\sigma\partial}(E/K)$
  - linear differential algebraic group
- Galois Correspondence
- The size of G measures the amount of differential dependence

defining differential equations for $G \Rightarrow$ ∂-differential relations among solutions

Ex. Gamma Function:

$$\Gamma(x+1) = x\Gamma(x)$$

- Galois Group $\subset \mathrm{GL}_1$
- Differential Subgroups of $\mathrm{GL}_1 = \begin{cases} \{\zeta \mid \zeta^n = 1\}, \text{ or} \\ \{c \mid L(\frac{c'}{c}) = 0\} \leftarrow \end{cases}$
- Galois Theory $\Rightarrow$ $L(\frac{\Gamma'(x)}{\Gamma(x)}) = g(x) \in \mathbb{C}(x)$
- Can assume L has constant coefficients, apply the shift σ and subtract:

$$L(\frac{1}{x}) = g(x+1) - g(x)$$

  - $L(\frac{1}{x})$ has exactly one pole
  - $g(x+1) - g(x)$ has at least two poles

Ex. First order q-difference equations:

Thm:(Ishizaki, 1998) If $a(x), b(x) \in \mathbb{C}(x)$ and $z(x) \notin \mathbb{C}(x)$ satisfies

$$z(qx) = a(x)z(x) + b(x), \ |q| \neq 1 \tag{1}$$

and is meromorphic on $\underline{\mathbb{C}}$, then $z(x)$ is not differentially algebraic over q-periodic functions.

$z(x)$ meromorphic on $\underline{\mathbb{C}\backslash\{0\}}$ and satisfies (1):
Assume distinct zeroes and poles of $a(x)$ are not q-multiples of each other.

Thm: (H-S) $z(x)$ is differentially algebraic iff $a(x) = cx^n$ and

- $b = f(qx) - a(x)f(x)$ for some $f \in \mathbb{C}(x)$, when $a \neq q^r$, or
- $b = f(qx) - af(x) + dx^r$ for some $f \in \mathbb{C}(x), d \in \mathbb{C}$ when $a = q^r, r \in \mathbb{Z}$.

Ex. q-Hypergeometric Functions:

Thm: (Roques, 2007) Let $y_1(x), y_2(x)$ be lin. indep. solutions of

$$y(q^2x) - \frac{2ax-2}{a^2x-1}y(qx) + \frac{x-1}{a^2x-q^2}y(x) = 0 \qquad (2)$$

with $a \notin q^{\mathbb{Z}}$ and $a^2 \in q^{\mathbb{Z}}$.

Then $y_1(x), y_2(x), y_1(qx)$ are algebraically independent.

(Hardouin/Singer): $y_1(x), y_2(x), y_1(qx)$ are differentially independent.

In general, we give necessary and sufficient *computable* conditions for a large class of linear difference equations for differential dependence of solutions and give differential relations if these are differentially dependent.

Other recurrences

The series

$$B(x) = \sum_{n=0}^{\infty} B_n x^n = 1 + x + 2x^2 + 5x^3 + 15x^4 + \dots$$

$B_n =$ no. of ways to partition $\{1, 2 \dots, n\}$ = Bell numbers

$$B\left(\frac{x}{1+x}\right) - xB(x) = 1$$

$$M(x) = \sum_{n=0}^{\infty} x^{k^n}, \quad k \geq 2, \quad \text{Mahler function}$$

$$M(x^k) - M(x) = x$$

are not differentially algebraic.

Galois Theory of Linear Differential Equations with Discrete Parameters

Hardouin/Di Vizio/Wibmer 2014

Galois groups are Difference Groups.

Ex. $K = \mathbb{C}(x,\alpha),\ \delta = \frac{d}{dx},\ \sigma(\alpha) = \alpha + 1$. The Bessel function $J_\alpha(x)$ satisfies

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \alpha^2)y = 0$$

How does J_α depend on α?

$$xJ_{\alpha+2}(x) - 2(\alpha+1)J_{\alpha+1}(x) + xJ_\alpha(x) = 0.$$

Ex. $K = \mathbb{C}(x),\ \delta = \frac{d}{dx},\ \sigma(x) = x + 1$. Ai$(x)$,Bi$(x)$ are lin. indep. solns. of

$$\frac{d^2y}{dx^2} - xy = 0$$

then $\mathrm{Ai}(x), \mathrm{Bi}(x), \frac{d\mathrm{Ai}(x)}{dx}, \mathrm{Ai}(x+1), \mathrm{Bi}(x+1), \frac{d\mathrm{Ai}(x+1)}{dx}, \ldots$ are alg. ind. over $\mathbb{C}(x)$.

Galois Theory of Linear Difference Equations with Discrete Parameters

Ovchinnikov/Wibmer 2014

Galois groups are Difference Groups.

Thm: The Gamma Function satisfies no difference equation with respect to $x \mapsto x + c, c \notin \mathbb{Q}$ over $M_{<1}$, the field of meromorphic functions f whose Nevanlinna characteristic satisfies $T(f, r) = o(r)$.