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RELATION ENTRE LES STRUCTURES ABÉLIENNES DE MODÈLES ET LES DIMENSIONS HOMOLOGIQUES (DE GORENSTEIN)

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In Memoriam,

dedicated to the fallen students in Venezuela.

dédiée aux étudiants tombés au Venezuela.

dedicada los estudiantes caídos en Venezuela.

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GLOSSARY OF SYMBOLS

Categories and universal constructions

$\mathrm{Ob}(\mathcal{C})$	The class of objects of a category $\mathcal{C}$.
$\mathbf{Mor}(\mathcal{C})$	The class of morphisms of a category $\mathcal{C}$.
$\mathrm{Hom}_{\mathcal{C}}(X, Y)$	The set of morphisms (also called arrows or maps) $X \to Y$ of a category $\mathcal{C}$.
$X \cong Y$	The objects X and Y are isomorphic.
$\mathcal{C}^{\mathrm{op}}$	The opposite category of the category $\mathcal{C}$.
$\mathrm{Lim}_{s\in S}X_s$	The limit of a family of objects $\{X_s : s \in S\} \subseteq \mathrm{Ob}(\mathcal{C})$.
$\mathrm{CoLim}_{s\in S}X_s$	The colimit of a family of objects $\{X_s : s \in S\} \subseteq \mathrm{Ob}(\mathcal{C})$.
$\prod_{s\in S} X_s$	The product of a family of objects $\{X_s : s \in S\} \subseteq \mathrm{Ob}(\mathcal{C})$.
$\coprod_{s\in S} X_s$	The coproduct of a family of objects $\{X_s : s \in S\} \subseteq \mathrm{Ob}(\mathcal{C})$.
$Y \times_X Z$	The pullback object of two maps $Y \to X$ and $Z \to X$ in $\mathcal{C}$.
$Y \coprod_X Z$	The pushout object of two maps $X \to Y$ and $X \to Z$ in $\mathcal{C}$.
$\mathrm{Ker}(f)$	The kernel object of a map f in $\mathcal{C}$.
$\mathrm{CoKer}(f)$	The cokernel object of a map f in $\mathcal{C}$.
$\mathrm{Im}(f)$	The image of a map f in $\mathcal{C}$.
$Z_m(X)$	The mth cycle of the chain complex X.
$B_m(X)$	The mth boundary of the chain complex X.

Index of categories

Set	The category of sets and functions.
Grp	The category of groups and homomorphisms.
Ab	The full subcategory of **Grp** consisting of abelian groups.
Rings	The category of rings and homomorphisms.
Top	The category of topological spaces and continuous functions.
$\mathbf{Mod}_R$	The category of right R-modules and right R-homomorphisms.
${}_R\mathbf{Mod}$	The category of left R-modules and left R-homomorphisms.
${}_A\mathbf{Mod}$	The category of graded left A-modules and graded A-homomorphisms, where A is the graded ring $A := R[x]/(x^2)$.
$\mathbf{Ch}(\mathcal{C})$	The category of chain complexes and chain maps over an abelian category $\mathcal{C}$.
$\mathbf{Map}(\mathcal{C})$	The category of maps of $\mathcal{C}$, where the objects are the maps $X \to Y$ and the morphisms are given by commutative squares

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] & Y' \end{tikzcd}$$

Weak factorization systems and model structures

$(\mathcal{L}, \mathcal{R})$	The weak factorization system formed by the classes of morphisms $\mathcal{L}$ and $\mathcal{R}$.
${}^{\pitchfork}\mathcal{M}$ (resp. $\mathcal{M}^{\pitchfork}$)	The class of morphisms having the left (resp. right) lifting property with respect to the morphisms in $\mathcal{M}$.
$\xrightarrow{\sim}$	Weak equivalences.
$\mathcal{W}_{eak}$	The class of weak equivalences.
$\hookrightarrow$	Cofibrations.
$\mathcal{C}_{of}$	The class of cofibrations.
$\twoheadrightarrow$	Fibrations.
$\mathcal{F}_{ib}$	The class of fibrations.
$\overset{\sim}{\hookrightarrow}$	Trivial cofibrations.
$\mathcal{C}_{of} \cap \mathcal{W}_{eak}$	The class of trivial cofibrations.
$\overset{\sim}{\twoheadrightarrow}$	Trivial fibrations.
$\mathcal{F}_{ib} \cap \mathcal{W}_{eak}$	The class of trivial fibrations.

Homological algebra

$\mathcal{P}_0(\mathcal{C})$	The class of projective objects of an abelian category $\mathcal{C}$.
	We shall use the color for projective objects.
$\mathcal{P}_n(\mathcal{C})$	The class of objects having a left projective resolution of length at most n.
$\mathrm{pd}(X)$	The projective dimension of an object X in an abelian category $\mathcal{C}$.
$\mathcal{I}_0(\mathcal{C})$	The class of injective objects of an abelian category $\mathcal{C}$.
	We shall use the color for injective objects.
$\mathcal{I}_n(\mathcal{C})$	The class of objects having a right injective resolution of length at most n.
$\mathrm{id}(X)$	The injective dimension of an object X in an abelian category $\mathcal{C}$.
$\mathcal{F}_0$	The class of flat modules.
	We shall use the color for flat objects.
$\mathcal{F}_n$	The class of modules having a left flat resolution of length at most n.
$\mathrm{fd}(M)$	The flat dimension of a module M.
$\mathrm{Ext}^i_{\mathcal{C}}(X,Y)$	The ith right derived functor of $\mathrm{Hom}_{\mathcal{C}}(X,Y)$.
$\overline{\mathrm{Ext}}^i(X,Y)$	The ith right derived functor of the internal hom $\overline{\mathrm{Hom}}_{\mathbf{Ch}({}_R\mathbf{Mod})}(X,Y)$.
$- \otimes_R -$	Tensor product of modules.
$\mathrm{Tor}^R_i(M,N)$	The ith left derived functor of $M \otimes N$, where $M \in \mathrm{Ob}(\mathbf{Mod}_R)$ and $N \in \mathrm{Ob}({}_R\mathbf{Mod})$.
$- \otimes -$	Usual tensor product of chain complexes.
$\mathrm{Tor}_i(X,Y)$	The ith left derived functor of $X \otimes Y$, where $X \in \mathrm{Ob}(\mathbf{Ch}(\mathbf{Mod}_R))$ and $Y \in \mathrm{Ob}(\mathbf{Ch}({}_R\mathbf{Mod}))$.
$-\overline{\otimes}-$	Bar tensor product of chain complexes.
$\overline{\mathrm{Tor}}_i(X,Y)$	The ith left derived functor of $X\overline{\otimes}Y$, where $X \in \mathrm{Ob}(\mathbf{Ch}(\mathbf{Mod}_R))$ and $Y \in \mathrm{Ob}(\mathbf{Ch}({}_R\mathbf{Mod}))$.

Gorenstein homological algebra

$\mathcal{GP}_0(\mathcal{C})$	The class of Gorenstein-projective objects of an abelian category $\mathcal{C}$.
	We shall use the color for Gorenstein-projective objects.
$\mathcal{GP}_r(\mathcal{C})$	The class of objects having a left Gorenstein-projective resolution of length at most r.
$\mathrm{Gpd}(X)$	The Gorenstein-projective dimension of an object X in a Gorenstein category $\mathcal{C}$.
$\mathcal{GI}_0(\mathcal{C})$	The class of Gorenstein-injective objects of an abelian category $\mathcal{C}$.
	We shall use the color for Gorenstein-injective objects.
$\mathcal{GI}_r(\mathcal{C})$	The class of objects having a right Gorenstein-injective resolution of length at most r.
$\mathrm{Gid}(X)$	The Gorenstein-injective dimension of an object X in a Gorenstein category $\mathcal{C}$.
$\mathcal{GF}_0$	The class of Gorenstein-flat modules.
	We shall use the color for Gorenstein-flat objects.
$\mathcal{GF}_r$	The class of modules having a left Gorenstein-flat resolution of length at most r.
$\mathrm{Gfd}(M)$	The Gorenstein-flat dimension of a module M.
$\mathcal{W}(\mathcal{C})$	The class of objects with finite projective dimension in a Gorenstein category $\mathcal{C}$.
$\mathrm{GExt}^i_{\mathcal{C}}(X,Y)$	The ith right derived functor of $\mathrm{Hom}_{\mathcal{C}}(X,Y)$.

Cotorsion pairs

$(\mathcal{A}, \mathcal{B})$	The cotorsion pair in an abelian category $\mathcal{C}$ formed by $\mathcal{A} \subseteq \mathrm{Ob}(\mathcal{C})$ and $\mathcal{B} \subseteq \mathrm{Ob}(\mathcal{C})$. The classes $\mathcal{A}$ and $\mathcal{B}$ are called the left and right halves of $(\mathcal{A}, \mathcal{B})$, respectively.
${}^{\perp}\mathcal{X}$ (resp. $\mathcal{X}^{\perp}$)	The class of modules which are right (resp. left) orthogonal to the class of modules $\mathcal{X}$, with respect to the functor $\mathrm{Ext}^1(-,-)$.
$(\mathcal{A} \mid \mathcal{B})$	The bar-cotorsion pair in the category of complexes formed by $\mathcal{A} \subseteq \mathrm{Ob}(\mathbf{Ch}({}_R\mathbf{Mod}))$ and $\mathcal{B} \subseteq \mathrm{Ob}(\mathbf{Ch}({}_R\mathbf{Mod}))$.
${}^{\overline{\perp}}\mathcal{X}$ (resp. $\mathcal{X}^{\overline{\perp}}$)	The class of chain complexes which are right (resp. left) orthogonal to the class of complexes $\mathcal{X}$, with respect to the functor $\mathrm{Ext}^1(-,-)$.

RÉSUMÉ

Nous étudions la relation entre les paires de cotorsion et les structures Abéliennes de modèles au sens de Hovey. L'idée est d'utiliser la correspondance de Hovey pour obtenir des nouvelles structures de modèles sur les catégories de modules ${}_R\mathbf{Mod}$ et de complexes de chaînes $\mathbf{Ch}({}_R\mathbf{Mod})$, à partir des paires de cotorsion compatibles et complètes. Nous généralisons la procédure de zig-zag pour démontrer que les classes $\widetilde{\mathcal{P}_n}$ de complexes n-projectifs (ceux dont la dimension projective est bornée par n) et $\mathrm{dg}\widetilde{\mathcal{P}_n}$ de complexes n-projectifs différentiels gradués sont les moités à gauches de deux paires de cotorsion compatibles et complètes, qui sont induites par la classe $\mathcal{P}_n$ de modules n-projectifs. Nous présentons aussi une version pour modules n-projectifs d'un résultat par Kaplansky sur modules projectifs, pour obtenir deux paires de cotorsion compatibles et complètes $(\mathrm{dw}\widetilde{\mathcal{P}_n}, (\mathrm{dw}\widetilde{\mathcal{P}_n})^\perp)$ et $(\mathrm{ex}\widetilde{\mathcal{P}_n}, (\mathrm{ex}\widetilde{\mathcal{P}_n})^\perp)$, où $\mathrm{dw}\widetilde{\mathcal{P}_n}$ est la classe de complexes de chaînes dont les termes sont n-projectifs, et $\mathrm{ex}\widetilde{\mathcal{P}_n}$ est donnée par les complexes appartenant à $\mathrm{dw}\widetilde{\mathcal{P}_n}$ qui sont aussi exactes. D'autres applications de la procédure de zig-zag produisent deux paires de cotorsion compatibles et complètes, à savoir $(\widetilde{\mathcal{F}_n}, (\widetilde{\mathcal{F}_n})^\perp)$ et $(\mathrm{dg}\widetilde{\mathcal{F}_n}, (\mathrm{dg}\widetilde{\mathcal{F}_n})^\perp)$, et aussi $(\mathrm{dw}\widetilde{\mathcal{F}_n}, (\mathrm{dw}\widetilde{\mathcal{F}_n})^\perp)$ et $(\mathrm{ex}\widetilde{\mathcal{F}_n}, (\mathrm{ex}\widetilde{\mathcal{F}_n})^\perp)$. En relation à l'algèbre homologique de Gorenstein, nous donnons un ensemble qui cogénère le paire de cotorsion $(\mathcal{GP}_r, (\mathcal{GP}_r)^\perp)$ en ${}_R\mathbf{Mod}$ (R un anneau de Gorenstein), où $\mathcal{GP}_r$ est la classe de modules r-projective de Gorenstein. Nous donnons aussi un résultat analogue pour complexes de chaînes. Dans une catégorie de Gorenstein et localement Noethérienne, nous trouvons un ensemble que cogénère le paire $({}^\perp(\mathcal{GI}_r(\mathcal{C})), \mathcal{GI}_r(\mathcal{C}))$, où $\mathcal{GI}_r(\mathcal{C})$ est la classe d'objets r-injective de Gorenstein. Les paires $(\mathcal{GP}_r, (\mathcal{GP}_r)^\perp)$ et $({}^\perp(\mathcal{GI}_r(\mathcal{C})), \mathcal{GI}_r(\mathcal{C}))$ s'avèrent être compatibles respectivement avec $(\mathcal{P}_r, (\mathcal{P}_r)^\perp)$ et $({}^\perp(\mathcal{I}_r(\mathcal{C})), \mathcal{I}_r(\mathcal{C}))$. Pour la dimension plate de Gorenstein, nous restreignons le concept de submodules purs pour obtenir une paire de cotorsion complète $(\mathcal{GF}_r, (\mathcal{GF}_r)^\perp)$. Afin d'avoir un résultat similaire pour complexes de chaînes, nous modifions la notion habituelle d'une paire de cotorsion en remplaçant le foncteur $\mathrm{Ext}^1(-,-)$ par le foncteur $\overline{\mathrm{Ext}}^1(-,-)$.

Mots-clés : Paires de cotorsion, structures Abéliennes de modèles, dimensions homologiques (de Gorenstein), correspondence de Hovey, procédure de zig-zag.

ABSTRACT

We study the relationship between (Gorenstein) homological dimensions and Abelian model structures in the sense of Hovey. The idea is to use Hovey's correspondence in order to get new model structures on the categories of modules ${}_R\mathbf{Mod}$ and chain complexes $\mathbf{Ch}({}_R\mathbf{Mod})$, from compatible and complete cotorsion pairs. We use a generalization of the zig-zag procedure to prove that the classes $\widetilde{\mathcal{P}_n}$ of n-projective (i.e. projective dimension bounded by n) and $\mathrm{dg}\widetilde{\mathcal{P}_n}$ of differential graded n-projective chain complexes are the left halves of two compatible and complete cotorsion pairs in $\mathbf{Ch}({}_R\mathbf{Mod})$ induced by the class $\mathcal{P}_n$ of n-projective modules. We also present a version of a result by Kaplansky on projective modules for n-projective modules, to obtain two compatible and complete cotorsion pairs $(\mathrm{dw}\widetilde{\mathcal{P}_n}, (\mathrm{dw}\widetilde{\mathcal{P}_n})^\perp)$ and $(\mathrm{ex}\widetilde{\mathcal{P}_n}, (\mathrm{ex}\widetilde{\mathcal{P}_n})^\perp)$, where $\mathrm{dw}\widetilde{\mathcal{P}_n}$ is the class of chain complexes whose terms are n-projective, and $\mathrm{ex}\widetilde{\mathcal{P}_n}$ is given by the complexes in $\mathrm{dw}\widetilde{\mathcal{P}_n}$ which are also exact. Other applications of the zig-zag procedure, along with a construction of small pure subresolutions for n-flat modules, yield two pairs of compatible and complete cotorsion pairs, namely $(\widetilde{\mathcal{F}_n}, (\widetilde{\mathcal{F}_n})^\perp)$ and $(\mathrm{dg}\widetilde{\mathcal{F}_n}, (\mathrm{dg}\widetilde{\mathcal{F}_n})^\perp)$, and $(\mathrm{dw}\widetilde{\mathcal{F}_n}, (\mathrm{dw}\widetilde{\mathcal{F}_n})^\perp)$ and $(\mathrm{ex}\widetilde{\mathcal{F}_n}, (\mathrm{ex}\widetilde{\mathcal{F}_n})^\perp)$. Concerning Gorenstein homological algebra, we give a cogenerating set for the cotorsion pair $(\mathcal{GP}_r, (\mathcal{GP}_r)^\perp)$ in ${}_R\mathbf{Mod}$ (with R a Gorenstein ring), where $\mathcal{GP}_r$ is the class of Gorenstein-r-projective modules. We present an analogous result for chain complexes. If we work in a locally Noetherian Gorenstein category, we give a cogenerating set for the pair $({}^\perp(\mathcal{GI}_r(\mathcal{C})), \mathcal{GI}_r(\mathcal{C}))$, where $\mathcal{GI}_r(\mathcal{C})$ is the class of Gorenstein-r-injective objects. The pairs $(\mathcal{GP}_r, (\mathcal{GP}_r)^\perp)$ and $({}^\perp(\mathcal{GI}_r(\mathcal{C})), \mathcal{GI}_r(\mathcal{C}))$ turn out to be compatible with $(\mathcal{P}_r, (\mathcal{P}_r)^\perp)$ and $({}^\perp(\mathcal{I}_r(\mathcal{C})), \mathcal{I}_r(\mathcal{C}))$, respectively. For the Gorenstein-flat dimension, we restrict the concept of pure submodules to obtain a complete cotorsion pair $(\mathcal{GF}_r, (\mathcal{GF}_r)^\perp)$. In order to have a similar result for chain complexes, we modify the usual definition of a cotorsion pair by considering orthogonality with respect to $\overline{\mathrm{Ext}}^1(-,-)$ instead of $\mathrm{Ext}^1(-,-)$.

Keywords: Cotorsion pairs, Abelian model structures, (Gorenstein) homological dimensions, Hovey's correspondence, zig-zag procedure.

INTRODUCTION

Some history, from Salce to Hovey

Nowadays, probably among the most important objects in the realm of homological algebra are the cotorsion pairs. First introduced by Luigi Salce in the category of groups, they where rediscovered by Edgar E. Enochs for the category of modules in the decade of 90s. Colloquially, two classes of modules form a cotorsion pair if they are orthogonal to each other with respect to the first extension functor $\mathrm{Ext}^1_R(-,-)$. This definition, which seems to be very simply at first sight, turns out to have very deep applications in several branches of mathematics, being probably the Representation Theory of Algebras the most favoured.

Two of the most important episodes representing the impact of cotorsion pairs are linked to some homological conjectures. For instance, the flat cover conjecture explicitly first stated in 1981 by Enochs in his paper *Injective and flat covers, envelopes and resolvents*, remained open for 20 years. It asserts the existence of a flat cover for every module. This was proven to be true in 2001 by Enochs, and simultaneously and independently by L. Bican and R. El Bashir. Thanks to some contributions by Paul Eklof and Jan Trlifaj, Enochs settled the conjecture by proving that the class of flat modules is the left half of a cotorsion pair cogenerated by a set. The famous Eklof and Trlifaj's Theorem states that every cotorsion pair cogenerated by a set is complete. It follows that every module has a special flat pre-cover. Once this is known, flat covers are constructed by using the fact that the class of flat modules is closed under direct limits.

The theory of cotorsion pairs was also used by Lidia Angeleri-Hügel and Octavio Mendoza in (6) to establish a validity criterion for the second finitistic dimension conjecture, which states that the little finitistic dimension of every finite Artin algebra is finite. This has been proved to be true in some particular cases, such as for finite dimensional monomial algebras. The proof of the cited criterion uses the fact, proved by S. T. Aldrich, E. E. Enochs, O. M. G. Jenda and L. Oyonarte, that the class of modules with projective dimension at most n (with n some positive integer) is the left half of a complete cotorsion pair.

Recently in 2002, Mark Hovey established a correspondence between the theories of cotorsion pairs and model structures. Namely, Hovey proved that given an Abelian model structure on a bicomplete Abelian category, it is possible to construct two complete cotorsion pairs from the classes of cofibrant, fibrant and trivial objects of the given model. Moreover, the converse is also true, that is if we are given three classes of objects in such a category forming two compatible and complete cotorsion pairs, then it is possible to obtain a unique Abelian model structure such that the classes of cofibrant, fibrant and trivial objects coincide with the given classes.

Hovey's results provide an easy method to construct model structures on categories such as modules or chain complexes. Concerning complexes over a ring or a ringed space, James Gillespie introduced the notions of differential graded chain complexes with respect to a class of modules. One important example of an Abelian model structure obtained from Hovey's correspondence is given by him in the paper *The flat model structure on* $\mathrm{Ch}(R)$, published in 2004, by proving that the classes of flat and dg-flat chain complexes are the left halves of two compatible complete cotorsion pairs.

Another interesting example of a model structure on chain complexes was given by D. Bravo, E. Enochs, A. Iacob, O. Jenda and J. Rada in their article *Cotorsion pairs in C(RMod)*, published in 2012. There they proved that the classes of degreewise and exact degreewise projective chain complexes are the left halves of two compatible complete cotorsion pairs.

The goal of this work is to construct new model structures in homological algebra. The idea is to construct a pair of compatible complete cotorsion pairs $(\mathcal{A} \cap \mathcal{W}, \mathcal{B})$ and $(\mathcal{A}, \mathcal{B} \cap \mathcal{W})$ related to a specific homological dimension, and then apply Hovey's correspondence to obtain an Abelian model structure where the classes of (trivially) cofibrant, (trivially) fibrant, and trivial objects coincide with $\mathcal{A}$ (resp. $\mathcal{A} \cap \mathcal{W}$), $\mathcal{B}$ (resp. $\mathcal{B} \cap \mathcal{W}$) and $\mathcal{W}$.

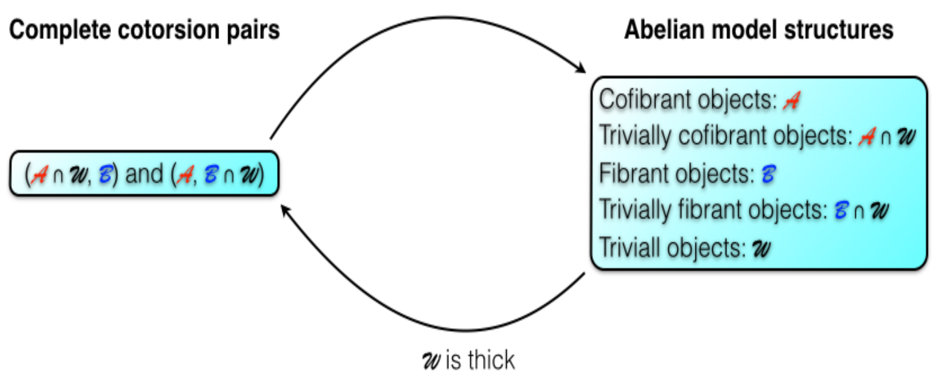

Figure I.1: Hovey's correspondence.

The contributions presented in this work can be split into two parts. In this first one, we study the projective, injective and flat dimensions of objects in the category $\mathbf{Ch}({}_R\mathbf{Mod})$ of chain complexes of modules, in order to obtain pairs of compatible and complete cotorsion pairs for each homological dimension, via the application of techniques such as the zig-zag argument. We recall several model structures on the category of chain complexes, obtained from the classes of projective, injective and flat modules, such as the flat and the degreewise projective model structures just mentioned, and then we shall present their corresponding generalizations to any homological dimension.

In the second part, we restrict our attention to the case where R is a Gorenstein ring, in which another type of homological algebra appears in ${}_R\mathbf{Mod}$ and $\mathbf{Ch}({}_R\mathbf{Mod})$, described by the notions of Gorenstein-projective, Gorenstein-injective and Gorenstein-flat dimensions. M. Hovey and J. Gillespie constructed model structures on ${}_R\mathbf{Mod}$ having the classes of Gorenstein-projective, Gorenstein-injective and Gorenstein-flat modules among the cofibrant and fibrant objects. We shall see how to generalize Hovey's arguments to get new Abelian model structures on ${}_R\mathbf{Mod}$ and $\mathbf{Ch}({}_R\mathbf{Mod})$ from the notions of Gorenstein-projective and Gorenstein-injective dimensions. With respect to the Gorenstein-flat case, we shall need to introduce the notions of $\mathcal{W}$-pure submodules and bar-cotorsion pairs to realize and prove that the class of modules (resp. complexes) with bounded Gorenstein-flat dimension constitutes the left half of a new complete cotorsion pairs in ${}_R\mathbf{Mod}$ (resp. $\mathbf{Ch}({}_R\mathbf{Mod})$).

Summary of principal results

In homological algebra. The first point in our work on homological dimensions and Abelian model structures concerns to the study of the projective dimension of chain complexes. The first model structure we shall construct, named the **n-projective model structure** on $\mathbf{Ch}({}_R\mathbf{Mod})$, is obtained after proving several results involving the class $\widetilde{\mathcal{P}_n}$ of chain complexes whose projective dimension is bounded by a nonnegative integer n.

Theorem 3.2.2

There is a unique Abelian model structure on $\mathbf{Ch}({}_R\mathbf{Mod})$ where the (trivial) cofibrations are the monomorphisms with cokernel in $\mathrm{dg}\widetilde{\mathcal{P}_n}$ (resp. $\widetilde{\mathcal{P}_n}$), the (trivial) fibrations are the epimorphisms with kernel in $(\widetilde{\mathcal{P}_n})^{\perp}$ (resp. $(\mathrm{dg}\widetilde{\mathcal{P}_n})^{\perp}$), and the trivial objects are the exact chain complexes.

The above result is motivated by the investigations developed by M. Hovey in the case $n = 0$. The projective model structure described in (36, Section 2.3) is obtained from sets of generating cofibrations and generating trivial cofibrations.

It turns out that the class $\widetilde{\mathcal{P}_n}$, whose elements are also called n-projective complexes, is the left half of a complete cotorsion pair. This result was proven by S. T. Aldrich, E. E. Enochs, O. M. G. Jenda, and L. Oyonarte in (5, Proposition 4.1) for the category of left R-modules. Specifically, they use a technique called the zig-zag argument to show that every $M \in \mathcal{P}_n$ is a transfinite extension of the set of κ-small n-projective modules (By a κ-small module S we mean that $\mathrm{Card}(S) \leq \kappa$, where κ is a fixed regular cardinal with $\mathrm{Card}(R) < \kappa$). Their arguments can be generalized to the category of left modules over a ringoid $\mathfrak{R}$.

Lemma 3.1.11

Let M be a n-projective $\mathfrak{R}$-module. Then for every homogeneous element $x \in M(a)$ there exists a κ-small submodule $N \hookrightarrow M$ such that:

(1) $x \in N(a)$.

(2) The $\mathfrak{R}$-modules N and M/N are n-projective.

The previous lemma is a key result to show that $(\mathcal{P}_n(\mathbf{Mod}(\mathfrak{R})), (\mathcal{P}_n(\mathbf{Mod}(\mathfrak{R})))^{\perp})$ is a complete cotorsion pair, where $\mathcal{P}_n(\mathbf{Mod}(\mathfrak{R}))$ is the class of left modules over

$\mathfrak{R}$. The categories ${}_R\mathbf{Mod}$ and $\mathbf{Ch}({}_R\mathbf{Mod})$ of modules and chain complexes over a ring are particular cases of $\mathbf{Mod}(\mathfrak{R})$, obtained by putting $\mathfrak{R} = R$ and $\mathfrak{R} = \mathbb{Z} \otimes R$, respectively. So the pair $(\widetilde{\mathcal{P}_n}, (\widetilde{\mathcal{P}_n})^\perp)$ is complete in $\mathbf{Ch}({}_R\mathbf{Mod})$. On the other hand, the class of differential graded chain complexes $\mathrm{dg}\widetilde{\mathcal{P}_n}$ is the left half of another complete cotorsion pair. The two pairs $(\widetilde{\mathcal{P}_n}, (\widetilde{\mathcal{P}_n})^\perp)$ and $(\mathrm{dg}\widetilde{\mathcal{P}_n}, (\mathrm{dg}\widetilde{\mathcal{P}_n})^\perp)$ are compatible in the sense that $\widetilde{\mathcal{P}_n} = \mathrm{dg}\widetilde{\mathcal{P}_n} \cap \mathcal{E}$ and $(\mathrm{dg}\widetilde{\mathcal{P}_n})^\perp = (\widetilde{\mathcal{P}_n})^\perp \cap \mathcal{E}$, where $\mathcal{E}$ is the class of exact chain complexes. This compatibility allows us to deduce the completeness of $(\mathrm{dg}\widetilde{\mathcal{P}_n}, (\mathrm{dg}\widetilde{\mathcal{P}_n})^\perp)$ from that of $(\widetilde{\mathcal{P}_n}, (\widetilde{\mathcal{P}_n})^\perp)$. In fact, we show that if we are given two compatible cotorsion pairs, then one of them if complete if, and only if, the other one is (see Proposition 2.3.6). Applying Hovey's result mentioned above, we obtain the Abelian model structure described in the previous theorem.

With respect to the flat case, we previously mentioned the flat model structure on $\mathbf{Ch}({}_R\mathbf{Mod})$ constructed by J. Gillespie. The cofibrant and trivial objects are given by the dg-flat and the exact chain complexes, respectively. In order to obtain a generalization of Gillespie's result, we first show that $(\mathcal{F}_n, (\mathcal{F}_n)^\perp)$ is a complete cotorsion pair, where $\mathcal{F}_n$ is the class of n-flat modules. The key thing at this point is to construct κ-small pure subresolutions for each module in $\mathcal{F}_n$. This shall be a consequence of the following lemma.

Lemma 3.1.21

Let $M \in \mathcal{F}_n$ with a flat resolution

$$(1) = (0 \to F_n \overset{f_n}{\to} F_{n-1} \to \cdots \to F_1 \overset{f_1}{\to} F_0 \overset{f_0}{\to} M \to 0)$$

and N be a small submodule of M. Then there exists a flat subresolution (i.e. a subcomplex)

$$0 \to S'_n \to \cdots \to S'_1 \to S'_0 \to N' \to 0$$

of (1) such that S'_k is a small and pure submodule of F_k, for every $0 \leq k \leq n$, and such that $N \subseteq N'$. In this case, we shall say that N' is a n-pure submodule of M. Moreover, if N has a subresolution of (1),

$$0 \to S_n \to \cdots \to S_1 \to S_0 \to N \to 0$$

where S_k is a small and pure submodule of F_k, for every $0 \leq k \leq n$, then the above resolution of N' can be constructed in such a way that it contains the resolution of N.

Let $\widetilde{\mathcal{F}_n}$ denote the class of n-flat chain complexes. In (4, Proposition 3.1), the authors construct a cogenerating set for the cotorsion pair $(\widetilde{\mathcal{F}_0}, (\widetilde{\mathcal{F}_0})^{\perp})$, by using a slightly modified version of the zig-zag procedure. This method, combined with the previous lemma, can be applied to prove the following theorem.

Theorem 3.6.5

For every left n-flat complex $X \in \widetilde{\mathcal{F}_n}$ and every element $x \in X$ (i.e. $x \in X_k$ for some $k \in \mathbb{Z}$), there exists a small n-flat subcomplex $L \subseteq X$ such that $x \in L$ and $X/L \in \widetilde{\mathcal{F}_n}$.

As a consequence, we have the existence of the following model structure on $\mathbf{Ch}({}_R\mathbf{Mod})$, that we call the **$n$-flat model structure**.

Theorem 3.6.2

There exists a unique Abelian model structure on $\mathbf{Ch}({}_R\mathbf{Mod})$ where the (trivial) cofibrations are the monomorphisms with cokernel in $\mathrm{dg}\widetilde{\mathcal{F}_n}$ (resp. $\widetilde{\mathcal{F}_n}$), the (trivial) fibrations are the epimorphisms with kernel in $(\widetilde{\mathcal{F}_n})^\perp$ (resp. $(\mathrm{dg}\widetilde{\mathcal{F}_n})^\perp$), and the weak equivalences are the quasi-isomorphisms.

The model structures just mentioned are not the only structures that can be obtained from the notion of homological dimensions. In the paper (12), J. Rada and coauthors construct an Abelian model structure on $\mathbf{Ch}({}_R\mathbf{Mod})$ where the class of cofibrant objects is given by the class $\mathrm{dw}\widetilde{\mathcal{P}_0}$ of degreewise projective complexes (i.e. complexes which are projective at each degree). The methods used there can be generalized to any projective dimension, in order to prove that:

(1) If R is a Noetherian ring, every n-projective chain complex is a transfinite extension of the set $\mathrm{dw}\widetilde{(\mathcal{P}_n)^{\aleph_0}}$ of complexes whose terms are in the set $(\mathcal{P}_n)^{\aleph_0}$, where a module is in $(\mathcal{P}_n)^{\aleph_0}$ if, and only if, it has a projective resolution of length n, where each projective term is written as a direct sum, over a countable set, of countably generated projective modules.

(2) Every exact n-projective chain complex (i.e. a complex in $\mathrm{ex}\widetilde{\mathcal{P}_n} = \mathrm{dw}\widetilde{\mathcal{P}_n} \cap \mathcal{E}$) is a transfinite extension of the set $\mathrm{ex}\widetilde{(\mathcal{P}_n)^{\leq\kappa}}$ of exact complexes whose terms are in the set $(\mathcal{P}_n)^{\leq\kappa}$, where a module is in $(\mathcal{P}_n)^{\leq\kappa}$ if, and only if, it has a projective resolution of length n, where each projective term is written as a direct sum, over a κ-small set, of countably generated projective modules.

As a consequence of **(1)** and **(2)**, we obtain the following model structure on $\mathbf{Ch}({}_R\mathbf{Mod})$, that we call the **degreewise n-projective model structure**.

Theorem 3.3.2

There exists a unique Abelian model structure on $\mathbf{Ch}({}_R\mathbf{Mod})$ where the (trivial) cofibrations are the monomorphisms with cokernel in $\mathrm{dw}\widetilde{\mathcal{P}_n}$ (resp. $\mathrm{ex}\widetilde{\mathcal{P}_n}$), the (trivial) fibrations are the epimorphisms with kernel in $(\mathrm{ex}\widetilde{\mathcal{P}_n})^\perp$ (resp. $(\mathrm{dw}\widetilde{\mathcal{P}_n})^\perp$), and the weak equivalences are the quasi-isomorphisms.

In the paper (4) mentioned above, the authors also prove that the classes of degreewise and exact degreewise flat complexes are the left halves of two cotorsion pairs cogenerated by sets. There they apply a modified zig-zag procedure to construct for every (exact) degreewise flat complex a transfinite extension of small (exact) degreewise flat complexes. We call this method the stairway zig-zag procedure, which we can combine with Lemma 3.1.20 to prove that for every $X \in \mathrm{ex}\widetilde{\mathcal{F}_n}$ and every $x \in X$, there exists a κ-small complex $Y \in (\mathrm{ex}\widetilde{\mathcal{F}_n})^{\leq \kappa}$ such that $x \in Y$ and $X/Y \in \mathrm{ex}\widetilde{\mathcal{F}_n}$. A similar result holds for the class $\mathrm{dw}\widetilde{\mathcal{F}_n}$ of degreewise n-flat complexes. It follows that every (exact) degreewise n-flat complexes is a transfinite extension of the set $Y \in (\mathrm{dw}\widetilde{\mathcal{F}_n})^{\leq \kappa}$ (resp. $Y \in (\mathrm{ex}\widetilde{\mathcal{F}_n})^{\leq \kappa}$).As a consequence, we shall obtain the **degreewise n-flat model structure** on $\mathbf{Ch}({}_R\mathbf{Mod})$ described as follows.

Theorem 3.5.2

There exists a unique Abelian model structure on $\mathbf{Ch}({}_R\mathbf{Mod})$ there the (trivial) cofibrations are the monomorphisms with cokernel in $\mathrm{dw}\widetilde{\mathcal{F}_n}$ (resp. $\mathrm{ex}\widetilde{\mathcal{F}_n}$), the (trivial) fibrations are the epimorphisms with kernel in $(\mathrm{ex}\widetilde{\mathcal{F}_n})^\perp$ (resp. $(\mathrm{dw}\widetilde{\mathcal{F}_n})^\perp$), and the weak equivalences are the quasi-isomorphisms.

In Gorenstein homological algebra. If we assume that R is a Gorenstein ring, another sort of homological algebra can be developed in the categories of modules and chain complexes. It turns out that the classes of modules with finite projective, injective and flat dimension coincide. This class, denoted by $\mathcal{W}$, appears as the right and left half of two complete cotorsion pairs $({}^{\perp}(\mathcal{W}), \mathcal{W})$ and $(\mathcal{W}, (\mathcal{W})^{\perp})$. This fact was proven in different ways by M. Hovey in (35) for the category of left R-modules, and independently by E. E. Enochs, S. Estrada and J. R. García Rozas for any Gorenstein category. Moreover, the latter authors also proved that ${}^{\perp}(\mathcal{W})$ and $(\mathcal{W})^{\perp}$ turn out to be the classes of Gorenstein-projective and Gorenstein-injective modules, denoted $\mathcal{GP}_0$ and $\mathcal{GI}_0$, respectively. On the other hand, Hovey noted the equalities $\mathcal{P}_0 = \mathcal{GP}_0 \cap \mathcal{W}$ and $\mathcal{I}_0 = \mathcal{GI}_0 \cap \mathcal{W}$, so his correspondence implies the existence of two Abelian model structures on $\mathbf{Ch}({}_R\mathbf{Mod})$, known as the **Gorenstein-projective** and the **Gorenstein-injective** model structures on ${}_R\mathbf{Mod}$. Our first little contribution for the theory of model categories in the context of Gorenstein homological algebra, is to rewrite the previous two model structures in the language of Gorenstein categories.

The completeness of the cotorsion pairs $(\mathcal{GP}_0, \mathcal{W})$ and $(\mathcal{W}, \mathcal{GI}_0)$ allows us to define Goresntein homological dimensions of modules. Moreover, this notion makes sense in any Gorenstein category $\mathcal{C}$. We consider the class $\mathcal{GP}_r(\mathcal{C})$ of objects in $\mathcal{C}$ whose Gorenstein-projective dimension is bounded by r (with $0 \leq r \leq \sup\{\mathrm{pd}(X) : X$ has finite projective dimension$\}$). We shall prove that this class appears as the left half of a complete cotorsion pair $(\mathcal{GP}_r(\mathcal{C}), (\mathcal{GP}_r(\mathcal{C}))^{\perp})$ in the cases where $\mathcal{C}$ is the category of modules or chain complexes over a Gorenstein ring. After showing that the pairs $(\mathcal{GP}_r(\mathcal{C}), (\mathcal{GP}_r(\mathcal{C}))^{\perp})$ and $(\mathcal{P}_r(\mathcal{C}), (\mathcal{P}_r(\mathcal{C}))^{\perp})$ are compatible, we shall obtain what we call the **Gorenstein-r-projective model structure** on $\mathcal{C}$.

Theorems 4.4.2 and 4.7.4

If $\mathcal{C}$ is either the category ${}_R\mathbf{Mod}$ of left R-modules or $\mathbf{Ch}({}_R\mathbf{Mod})$ of complexes over an n-Gorenstein ring, then for each $0 \leq r \leq n$ there exists a unique Abelian model structure on ${}_R\mathbf{Mod}$ where the (trivial) cofibrations are the monomorphisms with cokernel in $\mathcal{GP}_r(\mathcal{C})$ (resp. $\mathcal{P}_r(\mathcal{C})$), the (trivial) fibrations are the epimorphisms with kernel in $(\mathcal{P}_r(\mathcal{C}))^\perp$ (resp. $(\mathcal{GP}_r(\mathcal{C}))^\perp$), and $\mathcal{W}$ is the class of trivial objects.

With respect to the cotorsion pair $(\mathcal{W}(\mathcal{C}), \mathcal{GI}_0(\mathcal{C}))$, Hovey provided a cogenerating set $\mathcal{S}$ formed by the i-cosyzygies $\Omega^i(J)$ with $i \geq 0$ and J running over the set of indecomposable injective objects J of $\mathcal{C}$, provided $\mathcal{C}$ is locally Noetherian. If $\mathcal{S}(r)$ denotes the subset of $\mathcal{S}$ where $i \geq r$, then we shall show that $({}^\perp(\mathcal{GI}_r(\mathcal{C})), \mathcal{GI}_r(\mathcal{C}))$ is cogenerated by $\mathcal{S}(r)$. Since this pair turns out to be compatible with $({}^\perp(\mathcal{I}_r(\mathcal{C})), \mathcal{I}_r(\mathcal{C}))$, by Hovey's correspondence we obtain the **Gorenstein-r-injective model structure** on $\mathcal{C}$.

Theorem 4.5.1

Let $\mathcal{C}$ be a locally Noetherian Gorenstein category. Then there exists a unique Abelian model structure on $\mathcal{C}$, where the (trivial) fibrations are the epimorphisms with kernel in $\mathcal{GI}_r(\mathcal{C})$ (resp. $\mathcal{I}_r(\mathcal{C})$), the (trivial) cofibrations are the monomorphisms with cokernel in ${}^\perp(\mathcal{I}_r(\mathcal{C}))$ (resp. in ${}^\perp(\mathcal{GI}_r(\mathcal{C}))$), and $\mathcal{W}(\mathcal{C})$ is the class of trivial objects.

The Gorenstein-flat model structure on ${}_R\mathbf{Mod}$ is constructed by Hovey and Gillespie in (30), where the class $\mathcal{GF}_0$ of Gorenstein-flat modules is the class of cofibrant objects, and the trivial objects are given by $\mathcal{W}$. We construct the same model on the category of complexes, by applying the following two results.

Proposition 4.7.14

The classes $\widehat{\mathcal{GF}_0}$ and $(\widehat{\mathcal{GF}_0})^{\overline{\perp}}$ form a bar-cotorsion pair $(\widehat{\mathcal{GF}_0} \mid (\widehat{\mathcal{GF}_0})^{\overline{\perp}})$.

Proposition 4.7.15

Let E be a Gorenstein-flat complex and $x \in E$. Then there exists a Gorenstein-flat subcomplex $E' \subseteq E$ with $\mathrm{Card}(E') \leq \kappa$, such that $x \in E'$ and E/E' is also Gorenstein-flat.

By bar-cotorsion pair $(\mathcal{A} \mid \mathcal{B})$ in $\mathbf{Ch}({}_R\mathbf{Mod})$ we mean two classes of complexes $\mathcal{A}$ and $\mathcal{B}$ orthogonal to each other with respect to bar-extension functor $\overline{\mathrm{Ext}}^1(-,-)$, the first right derived functor of the internal hom $\overline{\mathrm{Hom}}(-,-)$. From these two results we deduce that the class $\widehat{\mathcal{GF}_0}$ of Gorenstein-flat complexes is the left half of a complete cotorsion pair. As a consequence of this and the completeness of the pair $(\mathcal{GF}_0, (\mathcal{GF}_0)^{\perp})$, we can define the Gorenstein-flat dimension for both modules and chain complexes. For the class $\mathcal{GF}_r$ of Gorenstein-r-flat modules, we have the following "Gorenstein version" of Lemma 3.1.21.

Lemma 4.6.12

Let $M \in \mathcal{GF}_r$ with a Gorenstein-flat resolution

$$(1) = (0 \to E_r \xrightarrow{f_r} E_{r-1} \to \cdots \to E_1 \xrightarrow{f_1} E_0 \xrightarrow{f_0} M \to 0)$$

and N be a submodule of M with $\mathrm{Card}(N) \leq \kappa$. Then there exists a Gorenstein-flat subresolution

$$0 \to S'_r \to S'_{r-1} \to \cdots \to S'_1 \to S'_0 \to N' \to 0$$

of (1) such that S'_k is a $\mathcal{W}$-pure submodule of E_k and $\mathrm{Card}(S'_k) \leq \kappa$, for every $0 \leq k \leq r$, and such that $N \subseteq N'$. Moreover, if N has a subresolution of (1)

$$0 \to S_r \to S_{r-1} \to \cdots \to S_1 \to S_0 \to N \to 0$$

where S_k is a $\mathcal{W}$-pure submodule of E_k with $\mathrm{Card}(S_k) \leq \kappa$, for every $0 \leq k \leq r$, then the above resolution of N' can be constructed in such a way that it contains the given resolution of N.

This result is a tool to construct a cogenerating set of $(\mathcal{GF}_r, (\mathcal{GF}_r)^\perp)$. This pair and $(\mathcal{F}_r, (\mathcal{F}_r)^\perp)$ are compatible and so we get a generalization to any Gorenstein-flat dimension of the Gorenstein-flat model structure constructed by Hovey and Gillespie. We call it the **Gorenstein-r-flat model structure** on ${}_R\mathbf{Mod}$.

Theorem 4.6.2

If R is an n-Gorenstein ring, then for each $0 \leq r \leq n$ there exists a unique Abelian model structure on ${}_R\mathbf{Mod}$ where the (trivial) cofibrations are the monomorphisms with cokernel in $\mathcal{GF}_r$ (resp. in $\mathcal{F}_r$), the (trivial) fibrations are the epimorphisms with kernel in $(\mathcal{F}_r)^\perp$ (resp. $(\mathcal{GF}_r)^\perp$), and $\mathcal{W}$ is the class of trivial objects.

Applying the techniques used to show that $(\mathrm{dw}\widetilde{\mathcal{F}_n}, (\mathrm{dw}\widetilde{\mathcal{F}_n})^\perp)$ is complete, along with some results concerning bar-cotorsion pairs, we show that the class $\widehat{\mathcal{GF}_r}$ of Gorenstein-r-flat complexes is the left half of a complete cotorsion pair. From this point, as in the case of modules, we shall deduce the existence of the Gorenstein-r-flat model structure on $\mathbf{Ch}({}_R\mathbf{Mod})$, with the difference that we need to apply the notion of bar-cotorsion pairs.

Theorem 4.7.22

If R is an n-Gorenstein ring, then for every $0 \leq r \leq n$ there exists a unique Abelian model structure on ${}_R\mathbf{Mod}$ where the (trivial) cofibrations are the monomorphisms with cokernel in $\widehat{\mathcal{GF}_r}$ (resp. in $\widetilde{\mathcal{F}_r}$), the (trivial) fibrations are the epimorphisms with kernel in $(\widetilde{\mathcal{F}_r})^\perp$ (resp. in $(\widehat{\mathcal{GF}_r})^\perp$), and $\widetilde{\mathcal{W}}$ is the class of trivial objects.

The projective, injective and flat model structures are somehow connected to the Gorenstein model structures just mentioned. It is not hard to construct an equivalence between the categories of complexes over R and A-modules, say $\mathbf{Ch}({}_R\mathbf{Mod}) \xrightarrow{\Psi} {}_A\,\mathbf{Mod} \xleftarrow{\Phi} \mathbf{Ch}({}_R\mathbf{Mod})$, where $A = R[x]/(x^2)$ is a $\mathbb{Z}$-graded ring. If R is a Gorenstein ring, then so is A. Under this assumption, Hovey and Gillespie proved that differential graded projective complexes over R are actually Gorenstein-projective A-modules, and vice versa. This interpretation can be seen via the mentioned equivalence between $\mathbf{Ch}({}_R\mathbf{Mod})$ and ${}_A\mathbf{Mod}$. This assertion is also true for the injective and flat cases. After establishing a similar equivalence between exact complexes over R and graded A-modules with finite projective dimension (i.e. the class $\mathcal{W}$), we are capable to state and prove the following result.

Theorem 4.8.5

The functor $\Psi : \mathbf{Ch}({}_R\mathbf{Mod}) \to {}_A\,\mathbf{Mod}$ maps:

(1) dg-r-projective complexes into Gorenstein-r-projective A-modules,

(2) dg-r-injective complexes into Gorenstein-r-injective A-modules, and

(3) dg-r-flat complexes into Gorenstein-r-flat A-modules.

If R is a left and right Noetherian ring of finite global dimension, then the inverse functor $\Phi : {}_A\,\mathbf{Mod} \to \mathbf{Ch}({}_R\mathbf{Mod})$ maps:

(1') Gorenstein-r-projective A-modules into dg-r-projective complexes,

(2') Gorenstein-r-injective A-modules into dg-r-injective complexes, and

(3') Gorenstein-r-flat A-modules into dg-r-flat complexes.

How to read this thesis

In the next paragraphs we describe the outline we shall be following throughout this thesis, which is divided into four chapters plus two appendices. The figure below represents and explains the logical dependence of the chapters forming the thesis. Later we shall give diagrams explaining a more detailed dependence between the sections of each chapter.

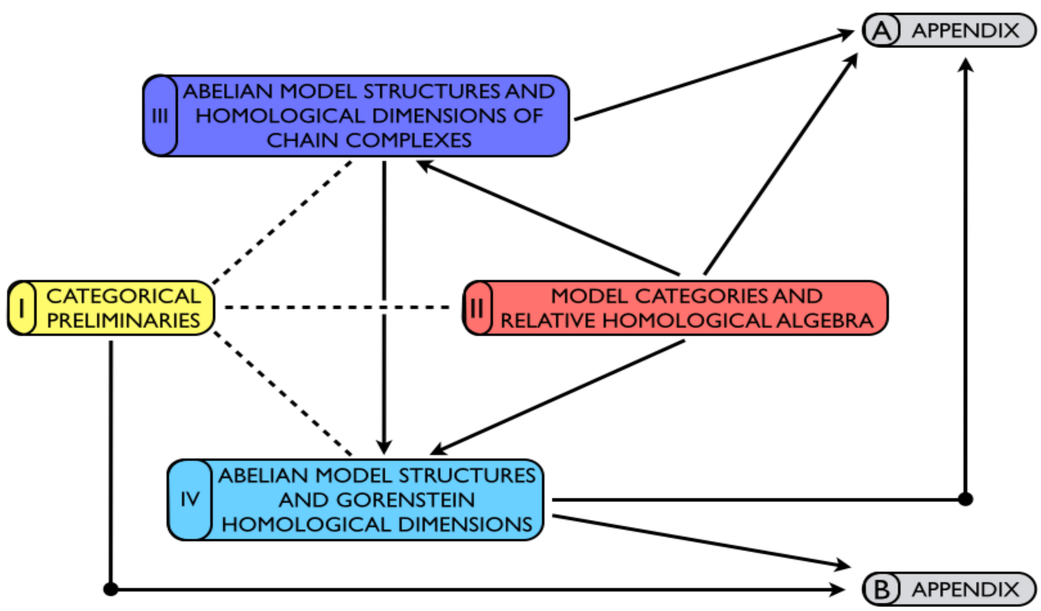

Figure I.2: Logical dependence.

Chapter 1. We introduce some categorical preliminaries and notations. One of the purposes of this thesis is to present most of the definitions and results in a categorical setting. So it is necessary to recall the definitions and notations of the universal constructions most used in category theory. We present each construction along with a diagram so it will be easier to understand and recall the concept for the reader who is not familiar with category theory.

This chapter represents a review of Abelian and Grothendieck categories. Sections 1.1, 1.2 and 1.9 represent the basic background on this matter. We basically work in two categories throughout this thesis, namely modules and chain complexes over a ring. As we said before, much of the contents are presented in a categorical setting, that means results and definitions given for an Abelian category $\mathcal{C}$ and for $\mathbf{Ch}(\mathcal{C})$, the category of chain complexes over $\mathcal{C}$.

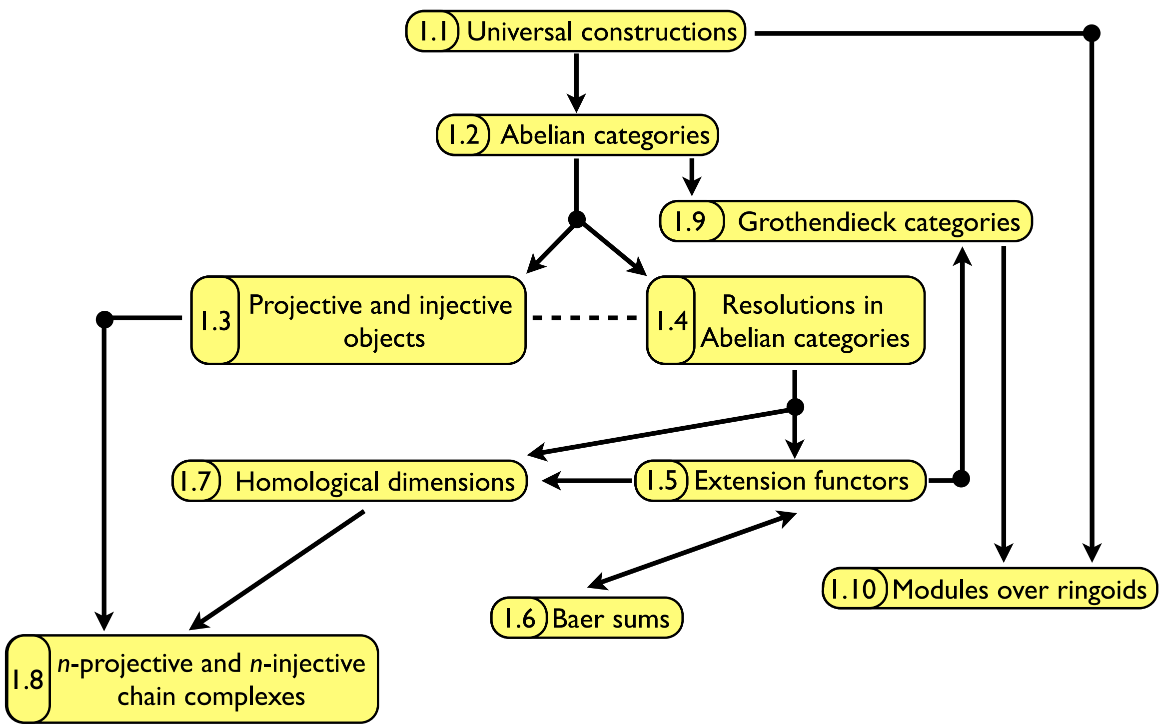

Figure I.3: Logical dependence in Chapter 1.

We present some notions known in Relative Homological Algebra, such as left and right resolutions with respect to a class of objects, (pre-)covers and (pre-) envelopes, and left and right homological dimensions. It is known that using projective or injective resolutions we can compute the extension functors $\mathrm{Ext}^i(-,-)$

for every $i \geq 0$. For the case $i = 1$, $\mathrm{Ext}^1(X, Y)$ can be described as the set of classes of short exact sequences of the form $0 \to Y \to Z \to X \to 0$ under certain equivalence relation, being a group when it is equipped with an operation called the Baer sum. This description of first extensions allows us to prove some natural isomorphisms for Ext involving disk and spheres complexes, namely $\mathrm{Ext}^1_{\mathbf{Ch}(\mathcal{C})}(D^m(C), Y) \cong \mathrm{Ext}^1_{\mathcal{C}}(C, Y_m)$ and $\mathrm{Ext}^1_{\mathbf{Ch}(\mathcal{C})}(X, D^{m+1}(C)) \cong \mathrm{Ext}^1_{\mathcal{C}}(X_m, C)$. These isomorphisms were proven by J. Gillespie in (27). If in addition, the complexes X and Y are exact, then $\mathrm{Ext}^1_{\mathbf{Ch}(\mathcal{C})}(S^m(C), Y) \cong \mathrm{Ext}^1_{\mathcal{C}}(C, Z_m(Y))$ and $\mathrm{Ext}^1_{\mathbf{Ch}(\mathcal{C})}(X, S^m(C)) \cong \mathrm{Ext}^1_{\mathcal{C}}(X_m/B_m(X), C)$ (see the paper (25) by the same author). This constitutes sections from 1.3 to 1.7. Section 1.8 shows how complexes with bounded projective dimension can be expressed as exact complexes whose cycles have projective dimension with the same bound. The same characterization is valid for complexes with bounded injective dimension. Finally, section 1.10 represents a brief reminder of the notion of left modules over a ringoid. This background is very important to generalize some results presented in Section 3.1.

Chapter 2. We present the investigations done by M. Hovey that connects the theories of cotorsion pairs and model categories. We begin by giving the definition of weak factorization systems, as the core notion of that of a model structure. Roughly speaking, a weak factorization system is given by two classes of morphisms in a category $\mathcal{C}$ such that they have a lifting property with respect to each other and satisfy a certain factorization axiom. A morphism $X \xrightarrow{f} Y$ lifts with respect to a morphism $W \xrightarrow{g} Z$ in a commutative square

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & W \arrow[d, "g"] \\ Y \arrow[r] & Z \end{tikzcd}$$

if there exists a morphism $Y \overset{d}{\to} W$ such that the resulting inner triangles

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & W \arrow[d, "g"] \\ Y \arrow[r] \arrow[ur, "d"'] & Z \end{tikzcd}$$

commute. On the other hand, the equality $\mathrm{Ext}^1_{\mathcal{C}}(X,Y) = 0$ means that every short exact sequence of the form $0 \to Y \overset{\alpha}{\to} Z \overset{\beta}{\to} X \to 0$ splits, that is there is a morphism $X \overset{\beta'}{\to} Z$ such that $\beta \circ \beta' = \mathrm{id}_X$. In other words, we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & Z \arrow[d, "\beta"] \\ X \arrow[r, equal] \arrow[ur, "\beta'"] & X \end{tikzcd}$$

meaning that $0 \to X$ lifts with respect to β. Hovey noticed this particular behavior, and established a correspondence for constructing a certain type of model structure from a pair of cotorsion pairs satisfying a compatibility condition (see the above diagram of Hovey's correspondence). Since this correspondence is of vital importance in this work, we think pertinent to present a proof, although in a particular way via the concept of cotorsion factorization systems, introduced after defining model structures from weak factorization systems. Every cotorsion pair gives rise to a cotorsion factorization system. Since every model structure is formed by two weak factorization systems, namely (trivial cofibrations, fibrations) and (cofibrations, trivial fibrations), the idea is that from two compatible cotorsion pairs we can obtain two cotorsion factorization systems forming an Abelian model structure. We assume factorizations in model structures to be functorial. We add to Hovey's proof that functorially complete cotorsion pairs yield functorial factorizations.

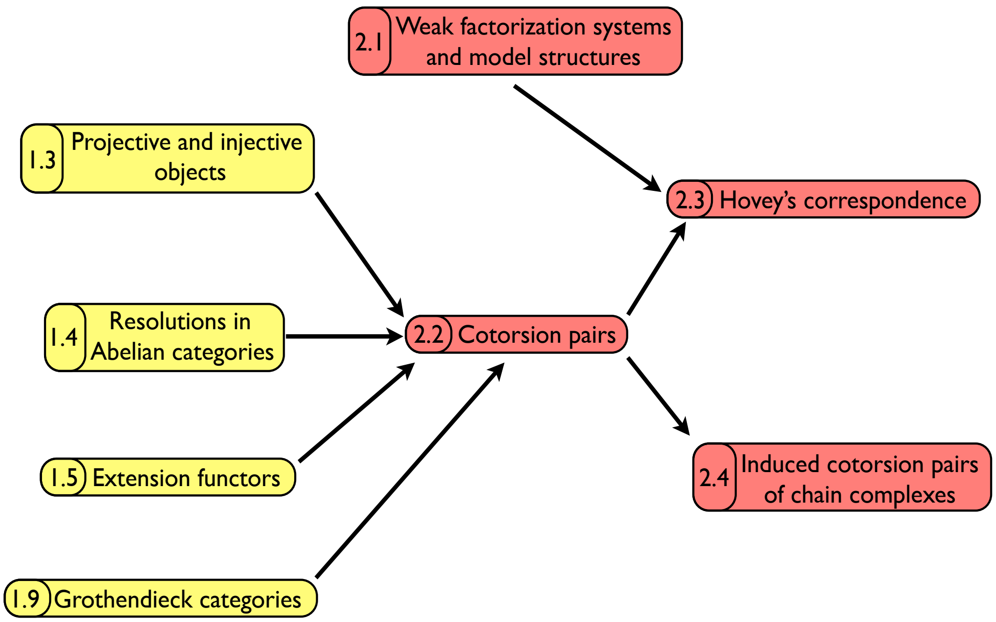

Figure I.4: Logical dependence in Chapter 2.

Concerning cotorsion pairs, we present a proof of Eklof and Trlifaj's Theorem in Grothedieck categories. This result was originally proven in the category of modules, and it is mentioned in some of the literature that it is also valid in any Grothedieck category. Although this seems to be folklore, we give a complete proof of this result.

In the last section, we present some methods obtained by J. Gillespie to induce certain cotorsion pairs in chain complexes from a cotorsion pair in an Abelian category $\mathcal{C}$. Given an hereditary complete cotorsion pair $(\mathcal{A}, \mathcal{B})$, the classes of differential graded $\mathcal{A}$-complexes and $\mathcal{B}$-complexes dg$\widetilde{\mathcal{A}}$ and dg$\widetilde{\mathcal{B}}$ are the left and right halves of the cotorsion pairs (dg$\widetilde{\mathcal{A}}$, dg$\widetilde{\mathcal{B}} \cap \mathcal{E}$) and (dg$\widetilde{\mathcal{A}} \cap \mathcal{E}$, dg$\widetilde{\mathcal{B}}$), where $\mathcal{E}$ is the class of exact complexes. In the end, we study other results by Gillespie that allow us to obtain cotorsion pairs from degreewise $\mathcal{A}$-complexes $\mathrm{dw}\widetilde{\mathcal{A}}$ and $\mathcal{B}$-complexes $\mathrm{dw}\widetilde{\mathcal{B}}$, namely $(\mathrm{dw}\widetilde{\mathcal{A}}, (\mathrm{dw}\widetilde{\mathcal{A}})^{\perp})$ and $({}^{\perp}(\mathrm{dw}\widetilde{\mathcal{B}}), \mathrm{dw}\widetilde{\mathcal{B}})$. We shall see that under certain hypothesis we can express the classes $(\mathrm{dw}\widetilde{\mathcal{A}})^{\perp}$ and ${}^{\perp}(\mathrm{dw}\widetilde{\mathcal{B}})$ as $(\mathrm{dw}\widetilde{\mathcal{A}} \cap \mathcal{E})^{\perp} \cap \mathcal{E}$ and ${}^{\perp}(\mathrm{dw}\widetilde{\mathcal{B}} \cap \mathcal{E}) \cap \mathcal{E}$, respectively.

Chapter 3. This chapter is devoted to study the relationship between model structures and homological dimensions. We construct six model structures on the category of chain complexes, namely: the n-projective, n-injective, n-flat, degreewise n-projective, degreewise n-injective and degreewise n-flat model structures, mentioned in the previous section.

We start with the projective dimension in the category of chain complexes. In Section 3.1 we work with the category $\mathbf{Mod}(\mathfrak{R})$ of modules over a ringoid $\mathfrak{R}$. We prove that $\mathcal{P}_n(\mathbf{Mod}(\mathfrak{R}))$ of n-projective modules over $\mathfrak{R}$ is the left half of a cotorsion pair cogenerated by a set. We shall see how to adapt Enochs' zig-zag argument in $\mathbf{Mod}(\mathfrak{R})$ in order to construct transfinite extension of κ-small n-projective modules. We shall define free modules over $\mathfrak{R}$, and then we shall show how to construct a free resolution of length n for every n-projective module over $\mathfrak{R}$. In Section 3.2, we deduce that $(\widetilde{\mathcal{P}_n}, (\widetilde{\mathcal{P}_n})^{\perp})$ is a complete cotorsion pair, applying the results from Section 3.1 in the particular case where $\mathfrak{R} = \mathbb{Z} \otimes R$.

In Section 3.3, we present and construct the degreewise n-projective model structure mentioned before. The case $n = 0$ proved by J. Rada and coauthors is based on a famous theorem by Kaplansky on projective modules: Every projective module can be written as a direct sum of countably generated projective modules. So every element in $\mathcal{P}_n$ has a projective resolution where each projective term is written as such a direct sum. If we work over a Noetherian ring, we prove that every module in $\mathcal{P}_n$ is a transfinite extension of the set $(\mathcal{P}_n)^{\aleph_0}$ mentioned before.

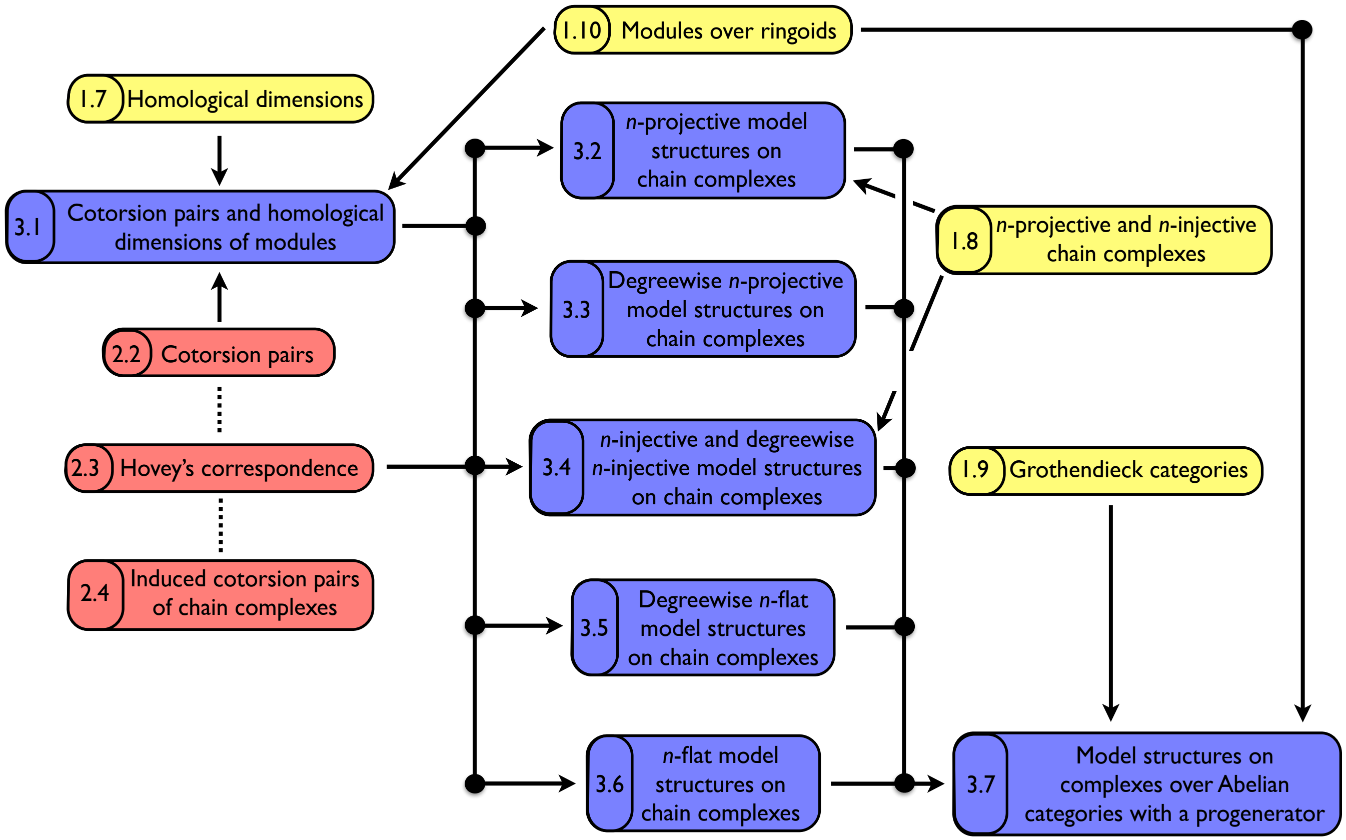

Figure I.5: Logical dependence in Chapter 3.

This fact help us to construct the respective cogenerating sets of $(\mathrm{dw}\widetilde{\mathcal{P}_n}, (\mathrm{dw}\widetilde{\mathcal{P}_n})^{\perp})$ and $(\mathrm{ex}\widetilde{\mathcal{P}_n}, (\mathrm{ex}\widetilde{\mathcal{P}_n})^{\perp})$.

The induced cotorsion pairs $(\widetilde{\mathcal{P}_n}, (\widetilde{\mathcal{P}_n})^{\perp})$, $(\mathrm{dg}\widetilde{\mathcal{P}_n}, (\mathrm{dg}\widetilde{\mathcal{P}_n})^{\perp})$, $(\mathrm{dw}\widetilde{\mathcal{P}_n}, (\mathrm{dw}\widetilde{\mathcal{P}_n})^{\perp})$ and $(\mathrm{ex}\widetilde{\mathcal{P}_n}, (\mathrm{ex}\widetilde{\mathcal{P}_n})^{\perp})$ are obtained by Gillespie's results given in section 2.4. The compatibility of the first two is a consequence of another result by Gillespie, while the compatibility of the last two is deduced from a general result given at the end of section 2.4.

Using properties of injective objects in Grothendieck categories and the theory of induced cotorsion pairs, we obtain two compatible and complete cotorsion pairs $({}^{\perp}(\widetilde{\mathcal{I}_n}), \widetilde{\mathcal{I}_n})$ and $({}^{\perp}(\mathrm{dg}\widetilde{\mathcal{I}_n}), \mathrm{dg}\widetilde{\mathcal{I}_n})$, as well as $({}^{\perp}(\mathrm{dw}\widetilde{\mathcal{I}_n}), \mathrm{dw}\widetilde{\mathcal{I}_n})$ and $({}^{\perp}(\mathrm{dw}\widetilde{\mathcal{I}_n}), \mathrm{dw}\widetilde{\mathcal{I}_n})$, in the category of chain complexes $\mathbf{Ch}({}_R\mathbf{Mod})$.

Sections 3.5 and 3.6 are devoted to construct the n-flat and degreewise n-flat model structures on $\mathbf{Ch}({}_R\mathbf{Mod})$. Moreover, the classes $\widetilde{\mathcal{F}_n}$, $\mathrm{dw}\widetilde{\mathcal{F}_n}$ and $\mathrm{ex}\widetilde{\mathcal{F}_n}$ have the interesting property of being closed under direct limits. It follows the cotorsion pairs $(\widetilde{\mathcal{F}_n}, (\widetilde{\mathcal{F}_n})^{\perp})$, $(\mathrm{dw}\widetilde{\mathcal{F}_n}, (\mathrm{dw}\widetilde{\mathcal{F}_n})^{\perp})$ and $(\mathrm{ex}\widetilde{\mathcal{F}_n}, (\mathrm{ex}\widetilde{\mathcal{F}_n})^{\perp})$ are perfect and so the existence of n-flat and (exact) degreewise n-flat covers of every complex is guaranteed. In other words, we have an extension of Enochs' flat cover conjecture in the category of complexes to any flat dimension.

We shall finish this chapter presenting the projective model structures obtained so far in a more general context of Abelian categories. It turns out to be that every Abelian category $\mathcal{C}$ equipped with a progenerator is equivalent to the category of right R-modules, for a certain ring R. This result is known as the Mitchell equivalence. As a consequence, we obtain the n-projective and the degreewise n-projective model structures on $\mathbf{Ch}(\mathcal{C})$.

CHAPTER 4. We focus our attention in a special type of Grothendieck categories known as Gorenstein categories. They are a sort of generalization of some situations that occur for modules over a Gorenstein ring R. In such categories we can construct left Gorenstein-projective and right Gorenstein-injective resolutions for every object, yielding a new theory of homological algebra in terms of Gorenstein-projective and injective dimensions. We present some properties in this matter, and then we construct cogenerating sets of the cotorsion pairs $(\mathcal{GP}_r, (\mathcal{GP}_r)^\perp)$ and $(\widehat{\mathcal{GP}}_r, (\widehat{\mathcal{GP}}_r)^\perp)$. For the Gorenstein-injective dimension, we present our results in any locally Noetherian Gorenstein category.

In our study of Gorenstein-flat modules, we replace the notion of pure submodules by that of $\mathcal{W}$-pure submodules, i.e. submodules $N \subseteq M$ of a module M such that the inclusion $0 \to N \to M$ remains exact after tensoring with every module in $\mathcal{W}$. Among the properties we shall see for this particular type of pure submodules, we have the possibility to express every Gorenstein-flat module as a transfinite extension of κ-small $\mathcal{W}$-pure submodules. Concerning homological dimensions, the previous fact can be used to construct κ-small $\mathcal{W}$-pure subresolutions for every Goresntein r-flat module. In the category of complexes over a Gorenstein ring, some tools are required before studying the Gorenstein-flat dimension in the category of complexes. If we replace the bifunctor $\operatorname{Ext}^1(-,-)$ by $\overline{\operatorname{Ext}}^1(-,-)$ in the definition of cotorsion pairs in chain complexes, we get the notion of bar-cotorsion pairs. There are some cases in which it is easier to prove that two classes of complexes form a (complete) bar-cotorsion pair, rather than a (complete) cotorsion pair. That is the case of the classes $\widehat{\mathcal{GF}}_r$ and $(\widehat{\mathcal{GF}}_r)^\perp$. We give some properties of bar-cotorsion pairs and show the connection they have with the standard cotorsion pairs.

We finish this chapter proving the equivalence given in Theorem 4.8.5.

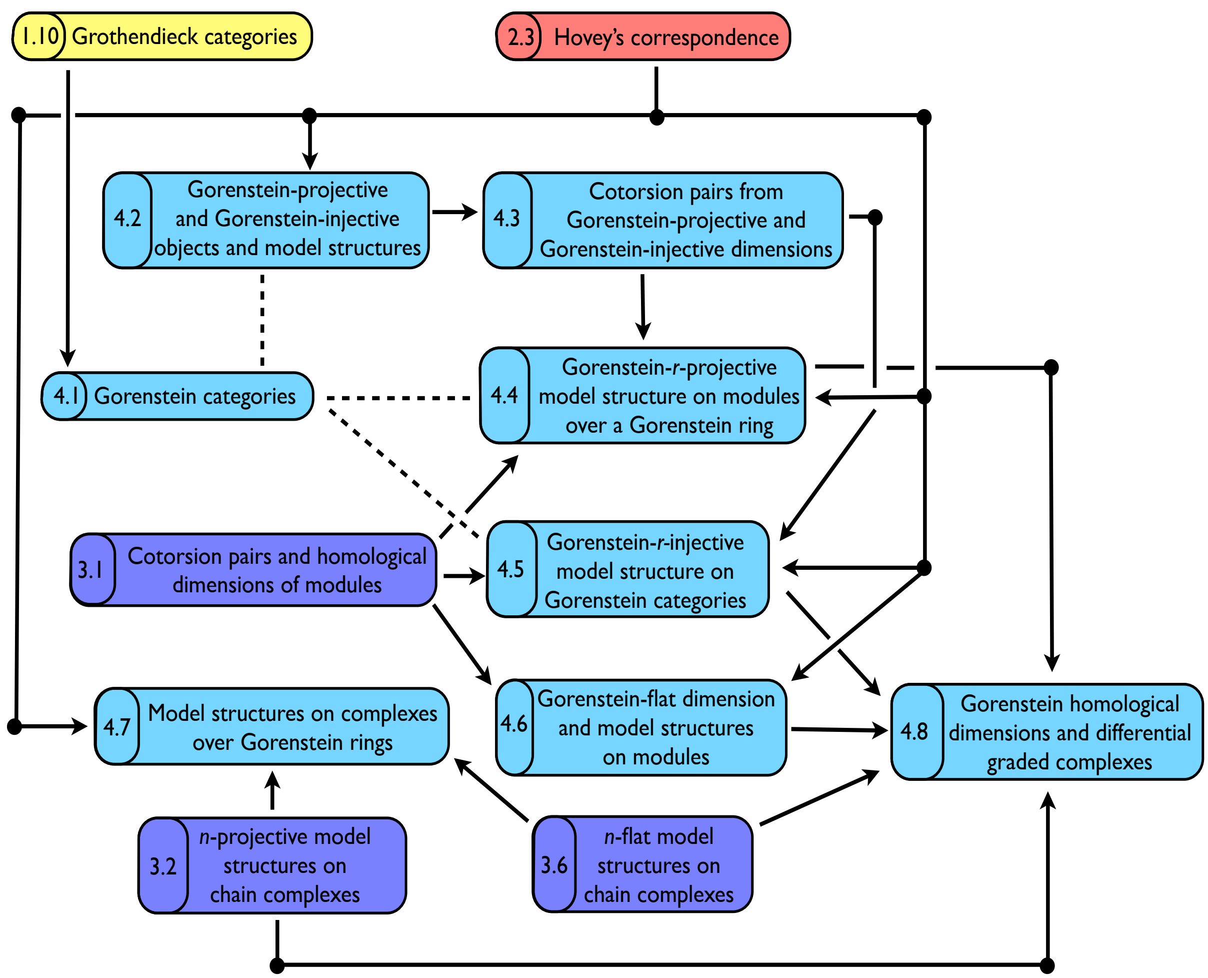

Figure I.6: Logical dependence in Chapter 4.

Appendices. When a model category is equipped with a monoidal structure, a natural question that comes to us is if the given model structure is monoidal. The homological model structures we obtain are not monoidal when the homological dimension in question is greater than zero. Besides proving this, we study the monoidality of the Gorenstein-projective and flat model structures. Probably the most interesting fact proven in Appendix A is that the degreewise flat model structure is monoidal if the ground ring has finite weak dimension.

Appendix B is devoted to give certain applications of Gorenstein homological algebra to the theory of derived functors. As extension functors are obtained from left projective or right injective resolutions, Gorenstein-extension functors are obtained in the same way by using left Gorenstein-projective or right Gorenstein-injective resolutions. For every pair of objects X and Y in a Gorenstein category $\mathcal{C}$, we give a Baer-like description of the first Gorenstein-extension functor $\mathrm{GExt}^1(X, Y)$. This help us to establish some natural isomorphisms between the Gorenstein-extension functors on $\mathcal{C}$ and $\mathbf{Ch}(\mathcal{C})$, involving disk and sphere complexes. Specifically, we prove that $\mathrm{GExt}^1_{\mathbf{Ch}(\mathcal{C})}(D^m(C), Y) \cong \mathrm{GExt}^1_{\mathcal{C}}(C, Y_m)$ and $\mathrm{GExt}^1_{\mathbf{Ch}(\mathcal{C})}(X, D^{m+1}(C)) \cong \mathrm{GExt}^1_{\mathcal{C}}(X_m, C)$. If in addition, the complexes X and Y are exact, we also get $\mathrm{GExt}^1_{\mathbf{Ch}(\mathcal{C})}(S^m(C), Y) \cong \mathrm{GExt}^1_{\mathcal{C}}(C, Z_m(Y))$ and $\mathrm{GExt}^1_{\mathbf{Ch}(\mathcal{C})}(X, S^m(C)) \cong \mathrm{GExt}^1_{\mathcal{C}}(X_m/B_m(X), C)$.

The coloration we have chosen for the classes of homological dimensions and Gorenstein homological dimensions can be compared to a diagram of subtractive color mixing. For example, the class of projective objects in a Gorenstein category can be written as the intersection of the class of Gorenstein-projective objects with the class of objects with finite projective dimension. This equality is represented by the fact that is obtained after mixing and .

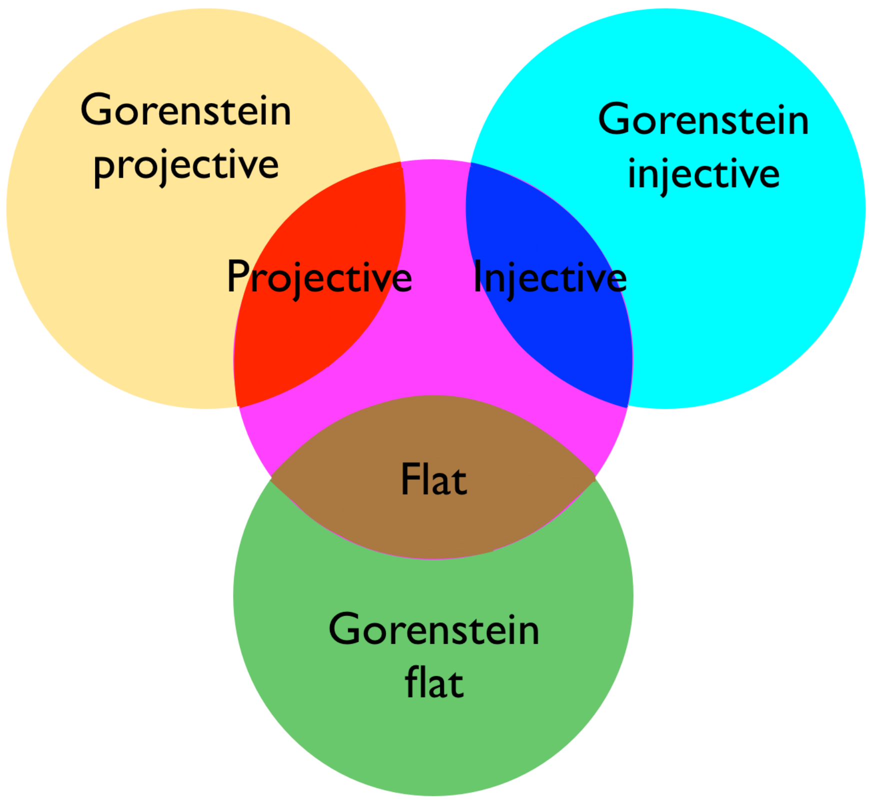

Figure I.7: Subtractive color mixing and Gorenstein homological algebra.

CHAPTER I

CATEGORICAL PRELIMINARIES

> *"The saddest aspect of life right now is that science gathers knowledge faster than society gathers wisdom."*
>
> Isaac Asimov.

This chapter consists of a review of some categorical notions and constructions. Recall that a category $\mathcal{C}$ is given by a collection of objects and arrows between objects such that composition of arrows is associative and such that for each object there is an identity arrow. We shall refer to arrows as morphisms or maps. The class of objects of $\mathcal{C}$ shall be denoted by $\mathrm{Ob}(\mathcal{C})$, and for each pair of objects X and Y, $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ shall denote the class of arrows from X to Y. We shall work with locally small categories, i.e. categories $\mathcal{C}$ such that for each pair $X, Y \in \mathrm{Ob}(\mathcal{C})$, the class $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is a set, called a homset. Given an arrow $X \xrightarrow{f} Y$, the object X is called the domain of f, and Y the codomain of f. The opposite category of $\mathcal{C}$ is the category $\mathcal{C}^{\mathrm{op}}$ whose objects are labeled as X^{op}, where X is an object of $\mathcal{C}$, and whose homsets are given by $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X^{\mathrm{op}}, Y^{\mathrm{op}}) = \mathrm{Hom}_{\mathcal{C}}(Y,X)$, i.e. an arrow $X^{\mathrm{op}} \xrightarrow{f^{\mathrm{op}}} Y^{\mathrm{op}}$ is given by an arrow $Y \xrightarrow{f} X$ of $\mathcal{C}$. The composition $\circ^{\mathrm{op}}$ is defined by $g^{\mathrm{op}} \circ^{\mathrm{op}} f^{\mathrm{op}} = (f \circ g)^{\mathrm{op}}$. We shall consider statements defined for $\mathcal{C}$ along with its dual. Roughly speaking, the dual of a statement P is formed by making the following replacements throughout in P: "domain" by "codomain", "codomain" by "domain", and "h is the composite of g with f" by "h is the composite of f with g". In other words, arrows and compositions of arrows are reversed. We shall denote the dual of a statement P by P^{op}.

Duality principle: P is valid for some category $\mathcal{C}$ if, and only if, P^{op} is valid for $\mathcal{C}^{\mathrm{op}}$.

Some definitions and results in this work are given along with their dual statements. We shall use a two-column environment to present the statement in the left column, and its dual in the right one.

1.1 Universal constructions

Definition 1.1.1. Let $X : \Sigma \to \mathcal{C}$ be a functor, where Σ is a small category, i.e. the class $\mathrm{Ob}(\Sigma)$ is a set. Such a functor is called a diagram of type Σ in $\mathcal{C}$. Some authors call Σ a scheme.

A cone of X is given by an object L of $\mathcal{C}$ and a family of morphisms

$$(L \stackrel{\pi_s}{\to} X_s \,:\, s \in \mathrm{Ob}(\Sigma))$$

such that $X_{s\to j} \circ \pi_s = \pi_t$ for every map $s \to t$.

A limit of X is defined as a cone

$$(L \stackrel{\pi_s}{\to} X_s \,:\, s \in \mathrm{Ob}(\Sigma))$$

such that if $(L' \stackrel{\pi'_s}{\to} X_s : s \in \mathrm{Ob}(\Sigma))$ is another cone of X, then there is a unique map $L' \stackrel{h}{\to} L$ such that the fo-

A cocone of X is given by an object C of $\mathcal{C}$ and a family of morphisms

$$(X_s \stackrel{\mu_s}{\to} C \,:\, s \in \mathrm{Ob}(\Sigma))$$

such that $\mu_t \circ X_{s\to t} = \mu_s$ for every map $s \to t$.

A colimit of X is defined as a cocone

$$(X_s \stackrel{\mu_s}{\to} C \,:\, s \in \mathrm{Ob}(\Sigma))$$

such that if $(X_s \stackrel{\mu'_s}{\to} C' : s \in \mathrm{Ob}(\Sigma))$ is another cocone of X, then there is a unique map $C \stackrel{h}{\to} C'$ such that the following diagram commutes for every map $s \to t$ in Σ:

$$\begin{tikzcd}
 & L' \arrow[ddl, "\pi'_s"'] \arrow[ddr, "\pi'_t"] \arrow[d, dotted, "\exists ! h"] & \\
 & L \arrow[dl, "\pi_s"] \arrow[dr, "\pi_t"'] & \\
X_s \arrow[rr, "X_{s\to t}"'] & & X_t
\end{tikzcd}$$

llowing diagram commutes for every map $s \to t$ in Σ:

$$\begin{tikzcd}
X_s \arrow[rr, "X_{s\to t}"] \arrow[dr, "\mu_s"] \arrow[ddr, "\mu'_s"'] & & X_t \arrow[dl, "\mu_t"'] \arrow[ddl, "\mu'_t"] \\
 & C \arrow[d, dotted, "\exists ! h"] & \\
 & C' &
\end{tikzcd}$$

Figure 1.1: Limits and colimits.

Remark 1.1.1. Using the universal property above, one can show that the limit of a diagram $X : \Sigma \to \mathcal{C}$ is unique up to isomorphisms, i.e. if $(L \xrightarrow{\pi_s} X_s : s \in \mathrm{Ob}(\Sigma))$ and $(L' \xrightarrow{\pi'_s} X_s : s \in \mathrm{Ob}(\Sigma))$ are both limits of $\mathcal{C}$, then $L \cong L'$. Similarly, the colimit of X is also unique up to isomorphisms.

Definition 1.1.2. A category $\mathcal{C}$ is called:

(finitely) complete if every (finite) diagram $\Sigma \xrightarrow{X} \mathcal{C}$ (resp. with $\mathrm{Ob}(\Sigma)$ a finite set) has a limit.

(finitely) cocomplete if every (finite) diagram $\Sigma \xrightarrow{X} \mathcal{C}$ (resp. with $\mathrm{Ob}(\Sigma)$ a finite set) has a colimit.

Some of the universal constructions in category theory are particular examples of limits and colimits.

(1) Products and coproducts: Suppose the scheme Σ is a discrete category, i.e. $\mathrm{Hom}_\Sigma(X, X) = \{\mathrm{id}_X\}$ and $\mathrm{Hom}_\mathcal{C}(X, Y) = \emptyset$ if $Y \neq X$. So Σ can be represented as a diagram

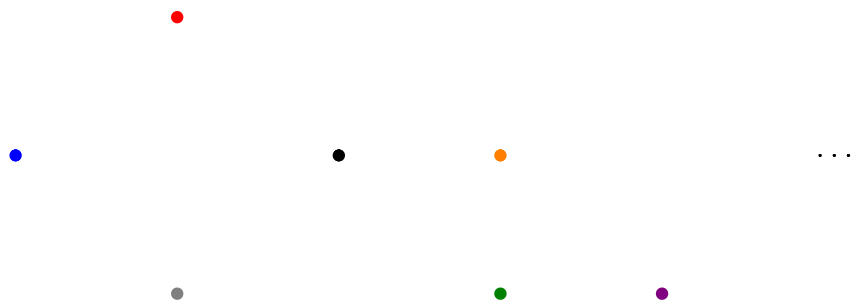

Figure 1.2: Scheme for products and coproducts.

The limit of a diagram $\Sigma \xrightarrow{X} \mathcal{C}$ is called the product of the family $(X_s : s \in \mathrm{Ob}(\Sigma))$, and it is denoted by $L = \prod\{X_s : s \in \mathrm{Ob}(\Sigma)\}$.

$$\begin{tikzcd}
\prod_{s\in \mathrm{Ob}(\Sigma)} X_s \arrow[r, "\pi_s"] & X_s \\
L' \arrow[u, dotted, "\exists!\ h"] \arrow[ur, "\pi'_s"'] &
\end{tikzcd}$$

The colimit of a diagram $\Sigma \xrightarrow{X} \mathcal{C}$ is called the coproduct of the family $(X_s : s \in \mathrm{Ob}(\Sigma))$, and it is denoted by $C = \coprod\{X_s : s \in \mathrm{Ob}(\Sigma)\}$.

$$\begin{tikzcd}
X_s \arrow[r, "\mu_s"] \arrow[dr, "\mu'_s"'] & \coprod_{s\in \mathrm{Ob}(\Sigma)} X_s \arrow[d, dotted, "\exists!\ h"] \\
& C'
\end{tikzcd}$$

Figure 1.3: Universal property of products and coproducts.

(2) Equalizers and coequalizers: Let Σ be the following scheme:

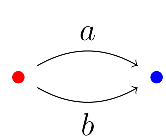

Figure 1.4: Scheme for equalizers and coequalizers.

The limit of a diagram $\Sigma \xrightarrow{X} \mathcal{C}$ is called the equalizer of $f = X(a)$ and $g = X(b)$.

$$\begin{tikzcd}
 & & X(\textcolor{red}{\bullet}) \arrow[r, "f"] & X(\textcolor{blue}{\bullet}) \arrow[dd, equal] \\
K' \arrow[r, dotted, "\exists!"] \arrow[urr, "\pi'"] \arrow[drr, "\pi'"'] & K \arrow[ur, "\pi"'] \arrow[dr, "\pi"] & & \\
 & & X(\textcolor{red}{\bullet}) \arrow[r, "g"'] & X(\textcolor{blue}{\bullet})
\end{tikzcd}$$

The colimit of a diagram $\Sigma \xrightarrow{X} \mathcal{C}$ is called the coequalizer $f = X(a)$ and $g = X(b)$.

$$\begin{tikzcd}
X(\textcolor{red}{\bullet}) \arrow[r, "f"] \arrow[dd, equal] & X(\textcolor{blue}{\bullet}) \arrow[dr, "\mu"'] \arrow[drr, "\mu'"] & & \\
 & & C \arrow[r, dotted, "\exists!"] & C' \\
X(\textcolor{red}{\bullet}) \arrow[r, "g"'] & X(\textcolor{blue}{\bullet}) \arrow[ur, "\mu"] \arrow[urr, "\mu'"'] & &
\end{tikzcd}$$

Figure 1.5: Universal property of equalizers and coequalizers.

(3) **Pullbacks and pushouts:** Let Σ be the following scheme:

$$\begin{tikzcd}
 & \textcolor{blue}{\bullet} \arrow[d, "a"] \\
\textcolor{red}{\bullet} \arrow[r, "b"'] & \bullet
\end{tikzcd}$$

Figure 1.6: Scheme for pullbacks.

The limit of the diagram $\Sigma \xrightarrow{X} \mathcal{C}$ is called the pullback of $f = X(a)$ and $g = X(b)$.

$$\begin{tikzcd}
P' \arrow[dr, dotted, "\exists!\, h"] \arrow[drr, bend left, "\pi'_{\textcolor{blue}{\bullet}}"] \arrow[ddr, bend right, "\pi'_{\textcolor{red}{\bullet}}"'] & & \\
 & P \arrow[r, "\pi_{\textcolor{blue}{\bullet}}"] \arrow[d, "\pi_{\textcolor{red}{\bullet}}"'] & X(\textcolor{blue}{\bullet}) \arrow[d, "f"] \\
 & X(\textcolor{red}{\bullet}) \arrow[r, "g"'] & X(\bullet)
\end{tikzcd}$$

The colimit of a diagram $\Sigma^{\mathrm{op}} \xrightarrow{X} \mathcal{C}$ is called the pushout $f = X(a^{\mathrm{op}})$ and $g = X(b^{\mathrm{op}})$.

$$\begin{tikzcd}
X(\bullet) \arrow[r, "f"] \arrow[d, "g"'] & X(\textcolor{red}{\bullet}) \arrow[d, "\mu_{\textcolor{red}{\bullet}}"] \arrow[ddr, bend left, "\mu'_{\textcolor{red}{\bullet}}"] & \\
X(\textcolor{blue}{\bullet}) \arrow[r, "\mu_{\textcolor{blue}{\bullet}}"'] \arrow[drr, bend right, "\mu'_{\textcolor{blue}{\bullet}}"'] & C \arrow[dr, dotted, "\exists!\, h"] & \\
 & & C'
\end{tikzcd}$$

Figure 1.7: Universal property of pullbacks and pushouts.

There are some interesting equivalences involving the previous concepts. You can see the proof of the following result in (54, Theorem 1.26) or (47, Chapter 2, Lemma 1 & Proposition 2). This reference is also an excellent source of examples of the universal constructions above in the categories **Set**, **Top**, **Gpr**, **Rings**, ${}_R$**Mod**, etc.

Proposition 1.1.1

The following conditions are equivalent for any category $\mathcal{C}$:

(1) $\mathcal{C}$ is (finitely) complete.

(2) $\mathcal{C}$ has (finite) products and equalizers.

(3) $\mathcal{C}$ has (finite) products and pullbacks.

(1') $\mathcal{C}$ is (finitely) cocomplete.

(2') $\mathcal{C}$ has (finite) coproducts and coequalizers.

(3') $\mathcal{C}$ has (finite) coproducts and pushouts.

1.2 Abelian categories

Before explaining the notion of Abelian categories, we need to recall the concepts of initial, terminal and zero objects. Then we give the definition of additive categories, a special type of category in which we can add morphisms. Finally, we present the notion of an Abelian category, which is an additive category where kernels and cokernels of morphisms exist. Roughly speaking, an Abelian category is somehow a generalization of the category of Abelian groups **Ab**.

An object X in a category $\mathcal{C}$ is called <u>initial</u> (resp. <u>terminal</u>) if $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ (resp. $\mathrm{Hom}_{\mathcal{C}}(Y, X)$) is a singleton for all $Y \in \mathrm{Ob}(\mathcal{C})$. An object X is a <u>zero object</u> if it is both an initial and a terminal object. Initial, terminal and zero objects are unique up to isomorphisms. We shall say that a category $\mathcal{C}$ is pointed if the zero object exists. We shall denote the zero object of a pointed category by 0.

For example, in **Set**, $\emptyset$ is the initial object, and any singleton is the terminal object. Similarly, the empty space is the initial object of **Top**, and every one-point space is terminal. In **Grp**, the trivial group is the zero object. Similarly, the zero module 0 is the zero object of ${}_R\mathbf{Mod}$ and $\mathbf{Mod}_R$.

A category with zero morphisms is one where, for every two objects X and Y in $\mathcal{C}$, there is a fixed morphism $0_{XY} : X \to Y$ such that for every object Z in $\mathcal{C}$ and every pair of morphisms $f : X \to Y$ and $g : Y \to Z$, the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "0_{XY}"] \arrow[d, "f"'] \arrow[dr, "0_{XZ}"] & Y \arrow[d, "g"] \\ Y \arrow[r, "0_{YZ}"'] & Z \end{tikzcd}$$

Figure 1.8: Zero morphisms.

The collection $\{0_{XY} : X, Y \in \mathrm{Ob}(\mathcal{C})\}$ is unique. Every pointed category is a category with zero objects, where the previous family is defied by the compositions $0_{XY} : X \to 0 \to Y$.

Definition 1.2.1. Given a finite family of objects $X_1, \ldots, X_n$ in a category $\mathcal{C}$, the finite biproduct of this family is an object $X_1 \oplus \cdots \oplus X_n$, together with a family of morphisms $X_1 \oplus \cdots \oplus X_n \xrightarrow{\pi_i} X_i$ and $X_j \xrightarrow{\mu_j} X_1 \oplus \cdots \oplus X_n$, with $i, j = 1, \ldots, n$, called projections and injections, respectively, satisfying the following conditions:

(1) $\pi_i \circ \mu_j = \begin{cases} \mathrm{id}_{X_i} & \text{if } j = i, \\ 0 & \text{if } j \neq i. \end{cases}$

(2) $(X_1 \oplus \cdots \oplus X_n, \pi_1, \ldots, \pi_n)$ is the product for the family $(X_1, \ldots, X_n)$.

(3) $(X_1 \oplus \cdots \oplus X_n, \mu_1, \ldots, \mu_n)$ is the coproduct for the family $(X_1, \ldots, X_n)$.

Definition 1.2.2. A category $\mathcal{C}$ is said to be pre-additive if every set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is equipped with a binary operation $\mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{+} \mathrm{Hom}_{\mathcal{C}}(X, Y)$ that makes $(\mathrm{Hom}_{\mathcal{C}}(X, Y), +)$ into an Abelian group, where the zero element is denoted by 0_{XY}, such that the composition $\circ$ of morphisms is distributive with respect to $+$.

A pre-additive category $\mathcal{C}$ is called additive if it has a zero object and finite biproducts. That is if $(X_1, \ldots, X_n)$ is a finite family of objects of $\mathcal{C}$, then the biproduct $X_1 \oplus \cdots \oplus X_n$ exists.

Definition 1.2.3. Let $X \xrightarrow{f} Y$ be a morphism in a additive category $\mathcal{C}$.

The kernel of f is defined as the equalizer of f and the zero map 0_{XY}.	The cokernel of f is defined as the coequalizer of f and the zero map 0_{XY}.

Recall that a morphism $X \xrightarrow{f} Y$ is a monomorphism (or is monic) if for every pair of morphisms $h, h' : Z \to X$ satisfying $f \circ h = f \circ h'$, then $h = h'$. Dually, f is an epimorphism (or is epic) if for every pair of morphisms $h, h' : Y \to Z$ satisfying $h \circ f = h' \circ f$, then $h = h'$.

Definition 1.2.4. A additive category $\mathcal{C}$ is said to be pre-Abelian if for each morphism f in $\mathcal{C}$, the kernel and cokernel of f exist. An Abelian category is a pre-Abelian category $\mathcal{C}$ such that the following two conditions are satisfied for every pair of objects X and Y:

(1) $\mathcal{C}$ is Ab-monic: If $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a monomorphism, then there exists an object $Z \in \mathrm{Ob}(\mathcal{C})$ and a morphism $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$ such that f is the kernel of g.

(2) <u>$\mathcal{C}$ is Ab-epic</u>: If $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ is an epimorphism, then there exists an object $Z \in \mathrm{Ob}(\mathcal{C})$ and a morphism $g \in \mathrm{Hom}_{\mathcal{C}}(Z,X)$ such that f is the cokernel of g.

Proposition 1.2.1 (see (10, Proposition 1.5.4))

Let $f : X \to Y$ be a morphism in a pre-Abelian category $\mathcal{C}$. The following conditions are equivalent:

(1) f is a monomorphism.

(2) $\mathrm{Ker}(f) = 0$.

(3) For every object $Z \in \mathrm{Ob}(\mathcal{C})$ and every map $h : Z \to X$, if $f \circ h = 0$ then $h = 0$.

(1') f is an epimorphism.

(2') $\mathrm{CoKer}(f) = 0$.

(3') For every object $Z \in \mathrm{Ob}(\mathcal{C})$ and every map $h : Y \to Z$, if $h \circ f = 0$ then $h = 0$.

Example 1.2.1.

(1) The categories ${}_R\mathbf{Mod}$ and $\mathbf{Mod}_R$ are Abelian.

(2) <u>The category of chain complexes over an Abelian category $\mathcal{C}$</u>: Let $\mathcal{C}$ be an abelian category. A <u>chain complex</u> X over $\mathcal{C}$ is given by a family of objects $(X_m : m \in \mathbb{Z})$ of $\mathcal{C}$ together with a family $(X_m \xrightarrow{\partial_m^X} X_{m-1} : m \in \mathbb{Z})$ of morphisms of $\mathcal{C}$, called <u>boundary maps</u>, such that $\partial_m^X \circ \partial_{m+1}^X = 0$, for every $m \in \mathbb{Z}$. We shall write X as a sequence $X = \cdots \to X_{m+1} \xrightarrow{\partial_{m+1}^X} X_m \xrightarrow{\partial_m^X} X_{m-1} \to \cdots$. For every $m \in \mathbb{Z}$, the objects $Z_m(X) = \mathrm{Ker}(\partial_m^X)$ and $B_m(X) = \mathrm{Im}(\partial_{m+1}^X)$ are called the <u>mth cycle</u> and the <u>mth boundary</u> of X.

Given two chain complexes X and Y, a <u>chain map</u> $X \xrightarrow{f} Y$ is given by a family of morphisms $(f_m : m \in \mathbb{Z})$ such that $\partial_m^Y \circ f_m = f_{m-1} \circ \partial_n^X$ for every $m \in \mathbb{Z}$.

$$\begin{tikzcd} X_m \arrow[r, "\partial_m^X"] \arrow[d, "f_m"'] & X_{m-1} \arrow[d, "f_{m-1}"] \\ Y_m \arrow[r, "\partial_m^Y"'] & Y_{m-1} \end{tikzcd}$$

Figure 1.9: Chain maps.

Let $\mathbf{Ch}(\mathcal{C})$ denote the category whose objects are the chain complexes over $\mathcal{C}$, and whose morphisms are given by the chain maps. One can show that $\mathbf{Ch}(\mathcal{C})$ is an Abelian category, where the zero object is given by the complex $\cdots \to 0 \xrightarrow{0} 0 \xrightarrow{0} 0 \to \cdots$, and the chain maps are added componentwise. Finite biproducts are also computed componentwise, as well as kernel and cokernel of chain maps.

In any Abelian category $\mathcal{C}$, one of the most important notions is that of an exact sequence. Before stating a formal definition, we need to recall the notions of subobjects and images.

Definition 1.2.5. Let $X \xrightarrow{f} Z$ and $Y \xrightarrow{g} Z$ be two monomorphisms sharing the same codomain. We shall say that $f \leq g$ if f factors through g, i.e. there exists $X \xrightarrow{h} Y$ such that $f = g \circ h$. We define the following relation on the set of all monomorphisms with codomain Z: $f \sim g$ if, and only if, $f \leq g$ and $g \leq f$. This relation is an equivalence relation, and the equivalence classes are called the <u>subobjects</u> of Z.

Example 1.2.2.

(1) If $\mathcal{C} = \mathbf{Set}$, $\mathbf{Grp}$, ${}_R\mathbf{Mod}$ or $\mathbf{Mod}_R$, the notion of a subobject of an object X coincides with that of a subset, subgroup and submodule, respectively.

(2) In the category of chain complexes $\mathbf{Ch}(\mathcal{C})$ over an Abelian category $\mathcal{C}$, the subobjects of a complex X are called subcomplexes of X. It is not hard to see that Y is a subcomplex of X if, and only if, each Y_m is a subobject of X_m.

If we work with groups, for example, the image of a homomorphism $G \xrightarrow{f} H$ is a subgroup of H such that g can be factored uniquely as the composition $G \xrightarrow{h} \mathrm{Im}(g) \xrightarrow{i} H$, where h is the restriction $g \mapsto f(g)$ and i is the inclusion. Then it is natural to think that the image of a morphism in a category is a subobject of the codomain, satisfying certain universal property.

Definition 1.2.6. An image of a morphism $X \xrightarrow{f} Y$ in a category $\mathcal{C}$ is a subobject $Y' \xrightarrow{i} Y$ of Y to which there exists a morphism $X \xrightarrow{g} Y'$ with $i \circ g = f$, satisfying the following universal property: If $Y'' \xrightarrow{i'} Y$ is another subobject of Y and if $X \xrightarrow{g'} Y''$ is another morphism such that $i' \circ g' = f$, then there exists a unique morphism $Y' \xrightarrow{h} Y''$ such that $h \circ g = g'$ and $i' \circ h = i$.

The image of a morphism f is unique up to isomorphisms. Note that g is uniquely determined since i is a monomorphism. If $\mathcal{C}$ is a category with equalizers, then g is epic (47, Lemma 2, page 35). Moreover, Y' is unique up to isomorphisms. So we shall denote the image of f by $\mathrm{Im}(f)$.

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "g"] \arrow[ddr, "g'"'] & & Y \\
& Y' \arrow[ur, "i"] \arrow[d, dotted, "\exists ! h"] & \\
& Y'' \arrow[uur, "i'"'] &
\end{tikzcd}$$

Figure 1.10: Image of a morphism.

Definition 1.2.7. A sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

in $\mathcal{C}$ is called <u>exact in Y</u> if $\mathrm{Ker}(g) = \mathrm{Im}(f)$.

Throughout the rest of this work, sometimes we shall denote monomorphisms and epimorphsims by $X \hookrightarrow Y$ and $W \twoheadrightarrow Z$, respectively. Note that $X \to Y$ is a monomorphism if, and only if, $0 \to X \to Y$ is exact in X. Similarly, $W \to Z$ is an epimorphism if, and only if, $W \to Z \to 0$ is exact in Z.

Definition 1.2.8. A long sequence $\cdots \to X_{m+1} \to X_m \to X_{m-1} \to \cdots$ is called <u>exact</u> if every sequence $X_{m+1} \to X_m \to X_{m-1}$ is exact in X_m. A sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is called <u>short exact</u> if f is monic, g is epic, and $\mathrm{Ker}(g) = \mathrm{Im}(f)$.

Note that for every monomorphism $X \xrightarrow{f} Y$ there exists an exact sequence

$$X \overset{f}{\hookrightarrow} Y \overset{\mu}{\twoheadrightarrow} \mathrm{CoKer}(f).$$

Dually, from every epimorphism $X \xrightarrow{f} Y$ we can get an exact sequence

$$\mathrm{Ker}(f) \overset{\pi}{\hookrightarrow} X \overset{f}{\twoheadrightarrow} Y.$$

Every Abelian category has pullpacks and pushouts by Proposition 1.1.1. Concerning those constructions, we shall use the following property repeatedly throughout this thesis.

Proposition 1.2.2 (see (43, Proposition 2, page 203))

If we are given two epimorphisms $Y \overset{f}{\twoheadrightarrow} X$ and $Z \overset{g}{\twoheadrightarrow} X$ in an Abelian category $\mathcal{C}$, then there exists a commutative diagram

$$\begin{tikzcd}
 & \mathrm{Ker}(f) \arrow[r, equal] \arrow[d, hook] & \mathrm{Ker}(f) \arrow[d, hook] \\
\mathrm{Ker}(g) \arrow[r, hook] \arrow[d, equal] & Y \times_X Z \arrow[r, two heads] \arrow[d, two heads] & Y \arrow[d, two heads, "f"] \\
\mathrm{Ker}(g) \arrow[r, hook] & Z \arrow[r, two heads, "g"'] & X
\end{tikzcd}$$

with exact rows and columns, where the bottom right square is a pullback square.

If we are given two monomorphisms $X \overset{f}{\hookrightarrow} Y$ and $X \overset{g}{\hookrightarrow} Z$ in an Abelian category $\mathcal{C}$, then there exists a commutative diagram

$$\begin{tikzcd}
X \arrow[r, hook, "g"] \arrow[d, hook, "f"'] & Z \arrow[r, two heads] \arrow[d, hook] & \mathrm{CoKer}(g) \arrow[d, equal] \\
Y \arrow[r, hook] \arrow[d, two heads] & Y \coprod_X Z \arrow[r, two heads] \arrow[d, two heads] & \mathrm{CoKer}(g) \\
\mathrm{CoKer}(f) \arrow[r, equal] & \mathrm{CoKer}(f) &
\end{tikzcd}$$

with exact rows and columns, where the top left square is a pushout square.

1.3 Projective and injective objects

Projective objects are probably among the most interesting concepts in homological algebra. They are the categorical version of the notion of projective modules, which were introduced in 1956 by Henri Cartan and Samuel Eilenberg in their book *Homological algebra* (13) in order to provide a generalization of the notion of free modules. In some cases, projective objects allow us to construct for every object certain exact sequences called projective resolutions, which are basic to define extension groups.

Definition 1.3.1. An object X in a category $\mathcal{C}$ is called:

projective if for every epimorphism $Y \overset{f}{\twoheadrightarrow} Z$ and every morphism $X \overset{g}{\to} Z$, there exists a morphism $X \overset{h}{\to} Y$ such that $f \circ h = g$.

$$\begin{tikzcd} & X \arrow[d, "g"] \arrow[dl, dotted, "\exists h"'] \\ Y \arrow[r, two heads, "f"'] & Z \end{tikzcd}$$

injective if for every monomorphism $Y \overset{f}{\hookrightarrow} Z$ and every morphism $Y \overset{g}{\to} X$, there exists a morphism $Z \overset{h}{\to} X$ such that $h \circ f = g$.

$$\begin{tikzcd} Y \arrow[r, hook, "f"] & Z \arrow[dl, dotted, "\exists h"] \\ X \arrow[u, "g"] & \end{tikzcd}$$

Figure 1.11: Projective and injective objects.

Remark 1.3.1. We shall denote by $\mathcal{P}_0(\mathcal{C})$ and $\mathcal{I}_0(\mathcal{C})$ the classes of projective and injective objects of $\mathcal{C}$, respectively. It is known that $\mathcal{P}_0(\mathcal{C})$ is closed under coproducts, i.e. the coproduct $\coprod_{s \in S} P_s$, of a family $(P_s : s \in S)$ of projective objects, is a projective object. Dually, $\mathcal{I}_0(\mathcal{C})$ are closed under products. The **projective** objects shall be colored **red**, while the **injective** objects **blue**. This particular choice of colours can be checked in the introduction.

Definition 1.3.2. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and $F, G : \mathcal{C} \to \mathcal{D}$ be two functors from $\mathcal{C}$ to $\mathcal{D}$. A natural transformation $\eta : F \to G$ from F to G is a collection of morphisms $(F(X) \overset{\eta_X}{\to} G(X))_{X \in \mathrm{Ob}(\mathcal{C})}$ in $\mathcal{D}$ such that for every morphism $f : X \to Y$ in $\mathcal{C}$, the equality $G(f) \circ \eta_X = \eta_Y \circ F(f)$ holds.

A natural transformation $\eta : F \to G$ is called:

(1) A natural monomorphism if η_X is a monomorphism for every $X \in \mathrm{Ob}(\mathcal{C})$.

(2) A natural epimorphism if η_X is an epimorphism for every $X \in \mathrm{Ob}(\mathcal{C})$.

(3) A natural isomorphism if η_X is an isomorphism for every $X \in \mathrm{Ob}(\mathcal{C})$.

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"'] & G(Y) \end{tikzcd}$$

Figure 1.12: Natural transformation.

Given an Abelian category $\mathcal{C}$, we know $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is an Abelian group for every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$.

$\mathrm{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \to \mathbf{Ab}$ is the covariant functor defined as follows: for every morphism $Y \xrightarrow{f} Z$, $f_* := \mathrm{Hom}_{\mathcal{C}}(X, f)$ is the group homomorphism $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$ given by $h \mapsto f \circ h$, for every $h \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$.

$\mathrm{Hom}_{\mathcal{C}}(-, Y) : \mathcal{C} \to \mathbf{Ab}$ is the contravariant functor defined as follows: for every morphism $X \xrightarrow{f} Z$, $f^* := \mathrm{Hom}_{\mathcal{C}}(f, Y)$ is the group homomorphism $\mathrm{Hom}_{\mathcal{C}}(Z, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Y)$ given by $h \mapsto h \circ f$, for every $h \in \mathrm{Hom}_{\mathcal{C}}(Z, Y)$.

Definition 1.3.3. Let $\mathcal{C}$ be an Abelian category.

$\mathcal{C}$ has enough projective objects if for every $X \in \mathrm{Ob}(\mathcal{C})$ there is an epimorphism $P \twoheadrightarrow X$, for some projective object P.

Moreover, $\mathcal{C}$ has functorially enough projective objects if there is a functor $\mathcal{C} \xrightarrow{P} \mathcal{C}$ along with a natural epimorphism $P \to \mathrm{id}_{\mathcal{C}}$ such that $P(X)$ is projective for every $X \in \mathrm{Ob}(\mathcal{C})$.

$\mathcal{C}$ has enough injective objects if for every $X \in \mathrm{Ob}(\mathcal{C})$ there is a monomorphism $X \hookrightarrow I$, for some injective object I.

Moreover, $\mathcal{C}$ has functorially enough injective objects if there is a functor $\mathcal{C} \xrightarrow{I} \mathcal{C}$ along with a natural monomorphism $\mathrm{id}_{\mathcal{C}} \to G$ such that $I(X)$ is injective for every $X \in \mathrm{Ob}(\mathcal{C})$.

Example 1.3.1. Assuming the Axiom of Choice, the following categories have functorially enough projective and injective objects.

(1) ${}_R\mathbf{Mod}$ and $\mathbf{Mod}_R$: Let M be a left R-module. It is well known that M is the epimorphic image of a free module. We present a construction of such an epimorphism to show that it is functorial. Let S be the set of all nonzero elements of M, and $\langle S\rangle$ be the free module generated by S. Define an epimorphism $\langle S\rangle \xrightarrow{\beta} M$ by $\beta\left(\sum r_i(s_i)\right) = \sum r_i \cdot s_i$, where (s_i) denotes the element in $\langle S\rangle$ corresponding to $s_i \in M$. Now denote $F(M) := \langle S\rangle$. Given a morphism $M_1 \xrightarrow{f} M_2$, we have an induced morphism $F(M_1) \xrightarrow{F(f)} F(M_2)$ given by $F(f)\left(\sum_{i\in I} r_i(x_i)\right) = \sum_{j\in J} r_j\left(f(x_j)\right)$, where $J = \{j \in I : f(x_j) \neq 0\}$ (with $F(f)\left(\sum_{i\in I} r_i(x_i)\right) = 0$ if $f(x_i) = 0$ for every $i \in I$). It is not hard to see that F is a functor and β defines a natural epimorphism $F \to \mathrm{id}_{{}_R\mathbf{Mod}}$.

Now we show how to embed every module into an injective module. First, consider the case $R = \mathbb{Z}$. So ${}_R\mathbf{Mod}$ (or $\mathbf{Mod}_R$) is the category of Abelian groups. So consider M as an Abelian group. We have an epimorphism $F(M) \xrightarrow{\beta} M$, where $F(M)$ is a free group. We know $\mathbb{Z}$ can be embedded into $\mathbb{Q}$, which is a divisible group. Recall that an Abelian group G is said to be <u>divisible</u> if $nG = G$, for every nonzero $n \in \mathbb{Z}$. It follows that every free group is embedded into a divisible group, and this can be done functorially via some functor G. So we have an embedding $F(M) \xrightarrow{\alpha} G(F(M))$, where $G(F(M))$ is a divisible group. Then we get an embedding $M \cong F(M)/\mathrm{Ker}(\beta) \xrightarrow{\alpha} G(F(M))/\mathrm{Ker}(\beta)$, where $G(F(M))/\mathrm{Ker}(\beta)$ is divisible and so injective (An Abelian group is injective if, and only if, it is divisible). This embedding is functorial in M. The general case follows by the inclusion $M \cong \mathrm{Hom}_R(R, M) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, M) \cong \mathrm{Hom}_{\mathbb{Z}}(R, \beta(M)) \subseteq \mathrm{Hom}_{\mathbb{Z}}(R, G(F(M))/\mathrm{Ker}(\beta))$.

(2) $\mathbf{Ch}({}_R\mathbf{Mod})$ and $\mathbf{Ch}(\mathbf{Mod}_R)$ (a consequence of Proposition 1.3.2 below).

Definition 1.3.4. Let $m \in \mathbb{Z}$ and C be an object in an Abelian category $\mathcal{C}$.

(1) The <u>m-th sphere complex</u> $S^m(C)$ is defined by

$$(S^m(C))_k = \begin{cases} C & \text{if } k = m, \\ 0 & \text{otherwise.} \end{cases}$$

whose boundary maps are all zero.

(2) The <u>m-th disk complex</u> $D^m(C)$ is defined by

$$(D^m(C))_k = \begin{cases} C & \text{if } k = m, m-1, \\ 0 & \text{otherwise.} \end{cases}$$

whose boundary maps are all zero except for $\partial_m^{D^m(C)} = \mathrm{id}_C$.

Proposition 1.3.1

For every $X, Y \in \mathrm{Ob}(\mathbf{Ch}(\mathcal{C}))$ and every $C \in \mathrm{Ob}(\mathcal{C})$, there exist natural isomorphisms:

$\mathrm{Hom}(D^m(C), Y) \underset{\Phi}{\overset{\mathbf{(1)}}{\longrightarrow}} \mathrm{Hom}(C, Y_m)$	$\mathrm{Hom}(X, D^m(C)) \underset{\Phi'}{\overset{\mathbf{(1')}}{\longrightarrow}} \mathrm{Hom}(X_{m-1}, C)$
$\mathrm{Hom}(S^m(C), Y) \underset{\Psi}{\overset{\mathbf{(2)}}{\longrightarrow}} \mathrm{Hom}(C, Z_m(Y))$	$\mathrm{Hom}(X, S^m(C)) \underset{\Psi'}{\overset{\mathbf{(2')}}{\longrightarrow}} \mathrm{Hom}(\frac{X_m}{B_m(X)}, C)$

Proof.

To define **(1)**, let f be a chain map in $\mathrm{Hom}_{\mathbf{Ch}(\mathcal{C})}(D^m(C), Y)$. It suffices to set $\Phi(f) = f_m$. To define the inverse, if $g \in \mathrm{Hom}_{\mathcal{C}}(C, Y_m)$ then set $\overline{g} : D^m(C) \to Y$ as the chain map given by

$$\overline{g}_k = \begin{cases} g & \text{if } k = m, \\ \partial_m^Y \circ g & \text{if } k = m-1, \\ 0 & \text{otherwise.} \end{cases}$$

The mapping $g \mapsto \overline{g}$ defines the inverse of Φ. The map Φ' can be defined dually.

For **(2)**, let $f \in \mathrm{Hom}_{\mathbf{Ch}(\mathcal{C})}(S^m(C), Y)$. Then $\partial_m^Y \circ f_m = 0$. Since $Z_m(Y) = \mathrm{Ker}(\partial_m^Y)$, there exists a unique morphism $\Psi(f) : C \to Z_m(Y)$ such that the following diagram commutes:

$$\begin{tikzcd}
Z_m(C) \arrow[r, hook, "i_{Z_m(Y)}"] & Y_m \arrow[r, two heads, "\partial_m^Y"] & Y_{m-1} \\
C \arrow[u, "\exists!\ \Psi(f)"] \arrow[ur, "f_m"'] & &
\end{tikzcd}$$

To define the inverse of Ψ, set $g \mapsto i_{Z_m(Y)} \circ g$ for every $g \in \mathrm{Hom}_{\mathcal{C}}(C, Z_m(Y))$.

The map Ψ' can be defined dually since $\frac{X_m}{B_m(X)}$ is the cokernel of the map ∂_{m+1}^X. $\square$

Definition 1.3.5. Let $\mathcal{C}$ and $\mathcal{D}$ be Abelian categories and

$$S = 0 \to Y \to Z \to X \to 0$$

be a short exact sequence in $\mathcal{C}$.

A covariant functor $F : \mathcal{C} \to \mathcal{D}$ is said to be:

(1) <u>half exact</u> if

$$F(Y) \to F(Z) \to F(X)$$

is exact for every S.

(2) <u>left exact</u> if

$$0 \to F(Y) \to F(Z) \to F(X)$$

is exact for every S.

A contravariant functor $G : \mathcal{C} \to \mathcal{D}$ is said to be:

(1') <u>half exact</u> if

$$F(X) \to F(Z) \to F(Y)$$

is exact for every S.

(2') <u>left exact</u> if

$$0 \to F(X) \to F(Z) \to F(Y)$$

is exact for every S.

(3) <u>right exact</u> if

$$F(Y) \to F(Z) \to F(X) \to 0$$

is exact for every S.

(4) <u>exact</u> if

$$0 \to F(Y) \to F(Z) \to F(X) \to 0$$

is exact for every S.

(3') <u>right exact</u> if

$$F(X) \to F(Z) \to F(Y) \to 0$$

is exact for every S.

(4') <u>exact</u> if

$$0 \to F(X) \to F(Z) \to F(Y) \to 0$$

is exact for every S.

Example 1.3.2. If P is a projective object of $\mathcal{C}$, then $D^m(P)$ is a projective object of $\mathbf{Ch}(\mathcal{C})$, for every $m \in \mathbb{Z}$. Recall that an object X of an Abelian category $\mathcal{C}$ is projective (resp. injective) if, and only if, the functor $\mathrm{Hom}_{\mathcal{C}}(X, -)$ (resp. $\mathrm{Hom}_{\mathcal{C}}(-, X)$) is exact. Then $D^m(P)$ is a projective since $\mathrm{Hom}_{\mathbf{Ch}(\mathcal{C})}(D^m(P), -)$ is exact (being naturally isomorphic to the exact functor $\mathrm{Hom}_{\mathcal{C}}(P, (-)_m)$). Dually, disk complexes of injective objects are injective in $\mathbf{Ch}(\mathcal{C})$. However, sphere complexes of projective (resp. injective) objects are not projective (resp. injective) in $\mathbf{Ch}(\mathcal{C})$.

Proposition 1.3.2 (see (52, Theorem 10.43))

Let $\mathcal{C}$ be an Abelian category.

If $\mathcal{C}$ has (functorially) enough projective objects, then so does $\mathbf{Ch}(\mathcal{C})$.

If $\mathcal{C}$ has (functorially) enough injective objects, then so does $\mathbf{Ch}(\mathcal{C})$.

Proof.

We only prove the functorial case of the left statement. The proof of the non-functorial case appears in the cited reference.

In the projective case, suppose there is a functor $P : \mathcal{C} \to \mathcal{C}$ and a natural epimorphism $b : P \to \mathrm{id}_{\mathcal{C}}$ such that $P(C)$ is projective for every $C \in \mathrm{Ob}(\mathcal{C})$. We can collect all the natural maps $D^m(P(X_m)) \to D^m(X_m) \to X$ defined for chain complex X to obtain a natural surjection $\bigoplus_{m\in\mathbb{Z}} D^{m+1}(P(X_m)) \to X$. It is not hard to see that the object $\bigoplus_{m\in\mathbb{Z}} D^{m+1}(P(X_m))$ is functorial in X. $\square$

We can say more about the relationship between the projective objects of $\mathcal{C}$ and $\mathbf{Ch}(\mathcal{C})$. In (52, Theorem 10.42) it is proven that a chain complex X in $\mathbf{Ch}(\mathcal{C})$ is projective if, and only if, it is split and X_m is a projective object in $\mathcal{C}$, for every $m \in \mathbb{Z}$.

Recall that a complex X over $\mathcal{C}$ is split if the exact sequences

$$Z_m(X) \hookrightarrow X_m \twoheadrightarrow B_{m-1}(X) \text{ and } B_m(X) \hookrightarrow Z_m(X) \twoheadrightarrow H_m(X)$$

are split for every $m \in \mathbb{Z}$. Let P be a projective chain complex, i.e. P is split and P_m is a projective object of $\mathcal{C}$, for every $m \in \mathbb{Z}$. We show that P is exact. First, we need to recall the notions of cones of chain maps, and suspensions of complexes.

Definition 1.3.6. The cone of a chain map $X \xrightarrow{f} Y$ is the chain complex $\mathrm{cone}(f)$ such that $\mathrm{cone}(f)_m = X_{m-1} \oplus Y_m$ and whose differential maps $\partial_m^{\mathrm{cone}(f)} : \mathrm{cone}(f)_m \to \mathrm{cone}(f)_{m-1}$ are given by

$$\begin{pmatrix} -\partial_{m-1}^X & 0 \\ -f_{m-1} & \partial_m^Y \end{pmatrix}.$$

Definition 1.3.7. The nth suspension of a chain complex X is the complex $\Sigma^n(X)$ such that $(\Sigma^n(X))_m = X_{m-n}$ and whose differential morphisms $\partial_m^{\Sigma^n(X)} : (\Sigma^n(X))_m \to (\Sigma^n(X))_{m-1}$ are given by $(-1)^n \partial_{m-n}^X : X_{m-n} \to X_{m-n-1}$.

It is not hard to see that there is a short exact sequence $P \hookrightarrow \mathrm{cone}(\mathrm{id}_P) \twoheadrightarrow \Sigma^1(P)$. By (52, Theorem 10.42), the first suspension $\Sigma^1(P)$ is projective. So the previous sequence splits, i.e. P is a direct summand of $\mathrm{cone}(\mathrm{id}_P)$. According to (52, Lemma 10.40), $\mathrm{cone}(\mathrm{id}_X)$ is an exact complex, for every complex X. So P is a direct summand of an exact complex. Since the class of exact complexes is closed under direct summands, we conclude that P is also exact. On the other hand, $Z_m(P)$ is a direct summand of a projective object in $\mathcal{C}$ (namely P_m), and so it is projective (the class of projective objects is also closed under direct summands).

Now suppose P is an exact chain complex such that $Z_m(P)$ is a projective object in $\mathcal{C}$, for every $m \in \mathbb{Z}$. The exact sequences $Z_m(P) \hookrightarrow P_m \twoheadrightarrow B_{m-1}(P)$ and $B_m(P) \hookrightarrow Z_m(P) \twoheadrightarrow H_m(P)$ are split since $Z_m(P) = B_m(P)$ and $H_m(P) = 0$, for every $m \in \mathbb{Z}$. Hence, P is a projective chain complex according to (52, Theorem 10.42). We have obtained the following equivalence.

Proposition 1.3.3

Let $\mathcal{C}$ be an Abelian category.

A chain complex X in $\mathbf{Ch}(\mathcal{C})$ is projective if, and only if, it is exact and $Z_m(X)$ is a projective object in $\mathcal{C}$, for every $m \in \mathbb{Z}$.	A chain complex Y in $\mathbf{Ch}(\mathcal{C})$ is injective if, and only if, it is exact and $Z_m(Y)$ is an injective object in $\mathcal{C}$, for every $m \in \mathbb{Z}$.

1.4 Resolutions in Abelian categories

In this section we recall the notions of left and right $\mathcal{F}$-resolutions, where $\mathcal{F}$ is a class of objects in an Abelian category $\mathcal{C}$. First, we study how to construct left and right derived functors from a given class $\mathcal{F}$. Then we present the particular examples of $\mathrm{Ext}^i_{\mathcal{C}}(X, -)$ and $\mathrm{Ext}^i_{\mathcal{C}}(-, X)$, the ith extension functors, for an object

$X \in \mathrm{Ob}(\mathcal{C})$. There exists an interesting characterization of these functors in terms of classes of exact sequences under certain equivalence relation. We present this characterization for the case $i = 1$. This provides us an easier method to deal with elements in $\mathrm{Ext}^1_{\mathcal{C}}(X, Y)$. One of its advantages can be noticed in the construction of natural isomorphisms

$$\mathrm{Ext}^1_{\mathbf{Ch}(\mathcal{C})}(D^m(C), Y) \overset{\cong}{\to} \mathrm{Ext}^1_{\mathcal{C}}(C, Y_m) \text{ and } \mathrm{Ext}^1_{\mathcal{C}}(X, D^{m+1}(C)) \overset{\cong}{\to} \mathrm{Ext}^1_{\mathcal{C}}(X_m, C),$$

where X and Y are chain complexes, and C is an object of $\mathcal{C}$. These isomorphisms were stablished by J. Gillespie in (27), for any Abelian category $\mathcal{C}$. The same author also shows in (25) that, if in addition X and Y are exact, it is possible to construct natural isomorphisms

$$\mathrm{Ext}^1_{\mathbf{Ch}(\mathcal{C})}(S^m(C), Y) \to \mathrm{Ext}^1_{\mathcal{C}}(C, Z_m(Y)) \text{ and } \mathrm{Ext}^1_{\mathbf{Ch}(\mathcal{C})}(X, S^m(C)) \to \mathrm{Ext}^1_{\mathcal{C}}\left(\frac{X_m}{B_m(X)}, C\right).$$

One of the reasons to studying these isomorphisms is that we repeatedly consider disk and sphere complexes in our constructions of model structures. Another reason is that we are interested in establishing similar isomorphisms for Gorenstein-extension functors, but we shall wait until the appendix to focus on this topic.

Definition 1.4.1. A functor $F : \mathcal{C} \to \mathcal{D}$ between additive categories $\mathcal{C}$ and $\mathcal{D}$ is said to be additive if it preserves finite biproducts, that is if:

(1) F maps the zero object of $\mathcal{C}$ to the zero object of $\mathcal{D}$.

(2) For every pair of objects X and Y, there is an isomorphism φ from $F(X \oplus Y)$ to $F(X) \oplus F(Y)$ which respects the inclusion and projection maps of the biproducts, i.e. the following diagram commutes.

$$\begin{tikzcd}
 & F(X) \arrow[dl, "F(i_X)"'] \arrow[dr, "i_{F(X)}"] & & F(X) & \\
F(X \oplus Y) \arrow[rr, "\varphi"] & & F(X) \oplus F(Y) \arrow[ur, "\pi_{F(X)}"] \arrow[dr, "\pi_{F(Y)}"'] & & F(X \oplus Y) \arrow[ll, "\varphi"'] \arrow[ul, "F(\pi_X)"'] \arrow[dl, "F(\pi_Y)"] \\
 & F(Y) \arrow[ul, "F(i_Y)"] \arrow[ur, "i_{F(Y)}"'] & & F(Y) &
\end{tikzcd}$$

Figure 1.13: Additive functor.

Definition 1.4.2. Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be Abelian categories and $\mathcal{C} \times \mathcal{D} \xrightarrow{T} \mathcal{E}$ be a functor contravariant in the first variable, covariant in the second, and additive in both.

If $\mathcal{F}$ is a class of objects of $\mathcal{C}$, we say that a complex

$$\cdots \to D_1 \to D_0 \to D^0 \to D^1 \to \cdots$$

in $\mathcal{D}$ is <u>$T(\mathcal{F}, -)$-exact</u> if for every $F \in \mathcal{F}$ the complex

$$\cdots \to T(F, D_0) \to T(F, D^0) \to \cdots$$

is an exact sequence in $\mathcal{E}$.

If $\mathcal{G}$ is a class of objects of $\mathcal{D}$, we say that a complex

$$\cdots \to C_1 \to C_0 \to C^0 \to C^1 \to \cdots$$

in $\mathcal{C}$ is <u>$T(-, \mathcal{G})$-exact</u> if for every $G \in \mathcal{G}$ the complex

$$\cdots \to T(C^0, G) \to T(C_0, G) \to \cdots$$

is an exact sequence in $\mathcal{E}$.

Definition 1.4.3. Let $\mathcal{F}$ and $\mathcal{G}$ be classes of objects of $\mathcal{C}$ and $X, Y \in \mathrm{Ob}(\mathcal{C})$.

A <u>left $\mathcal{F}$-resolution</u> of X is a $\mathrm{Hom}(\mathcal{F}, -)$-exact complex

$$\cdots \to F_1 \to F_0 \to X \to 0$$

with each $F_i \in \mathcal{F}$.

A <u>right $\mathcal{G}$-resolution</u> of Y is a $\mathrm{Hom}(-, \mathcal{G})$-exact complex

$$0 \to Y \to G^0 \to G^1 \to \cdots$$

with each $G^i \in \mathcal{G}$.

One of the most important problems in Representation Theory of Algebras is the existence of covers and envelopes for a given class of modules. Covers and envelopes were first introduced by M. Auslander and S. O. Smalø, using the terminology of minimal left and minimal right approximations.

Definition 1.4.4. Let $\mathcal{F}$ and $\mathcal{G}$ be classes of objects in an Abelian category $\mathcal{C}$ and $X, Y \in \mathrm{Ob}(\mathcal{C})$.

A morphism $F \xrightarrow{f} X$ with $F \in \mathcal{F}$ is said to be an $\mathcal{F}$-cover if:

(1) Given another morphism $F' \xrightarrow{f'} X$ with $F' \in \mathcal{F}$, there exists a morphism $F' \xrightarrow{\varphi} F$ such that the following triangle commutes:

$$\begin{tikzcd} F \arrow[r, "f"] & X \\ F' \arrow[u, dotted, "\exists\, \varphi"] \arrow[ur, "f'"'] & \end{tikzcd}$$

(2) If $F' = F$, the above diagram can only be completed by automorphisms of F.

If f satisfies **(1)** but may be not **(2)**, then it is called an $\mathcal{F}$-pre-cover. The class $\mathcal{F}$ is called a (pre-)covering class if every object of $\mathcal{C}$ has an $\mathcal{F}$-(pre-) cover.

A morphism $Y \xrightarrow{g} G$ with $G \in \mathcal{G}$ is said to be a $\mathcal{G}$-envelope if:

(1') Given another morphism $Y \xrightarrow{g'} G'$ with $G' \in \mathcal{G}$, there exists a morphism $G \xrightarrow{\varphi} G'$ such that the following triangle commutes:

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[dr, "g'"'] & G \arrow[d, dotted, "\exists\, \varphi"] \\ & G' \end{tikzcd}$$

(2') If $G' = G$, the above diagram can only be completed by automorphisms of G.

If g satisfies **(1')** but may be not **(2')**, then it is called an $\mathcal{G}$-pre-envelope. The class $\mathcal{G}$ is called a (pre-)enveloping class if every object of $\mathcal{C}$ has a $\mathcal{G}$-(pre-)envelope.

The main classes of objects we shall be considering in this work are the projective and injective objects. For instance, in the category ${}_R\mathbf{Mod}$ of left R-modules, projective covers are rare, i.e. not every module has a projective cover. Moreover,

$\mathcal{P}_0$ is a covering class if, and only if, R is a perfect ring (i.e. a ring R where projective and flat modules over R coincide). On the other hand, flat covers of modules always exist. This is not easy to prove at all. This was an open problem known as the *Flat Cover Conjecture*, until it was proven by E. E. Enochs and (independently) by L. Bican and R. El Bashir in 2001 (see (8)).

We know that in the category of modules or chain complexes over a ring, it is always possible to find epic projective pre-covers and monic injective pre-envelopes. However, (pre-)covers are not epic in general. Flat covers of modules constitute an example of covers which are epic. This is a consequence of the following result.

Proposition 1.4.1 (see (24, Page 8))

Let $\mathcal{F}$ and $\mathcal{G}$ be two classes of objects in an Abelian category $\mathcal{C}$.

If $\mathcal{C}$ has enough projective objects and $\mathcal{F} \supseteq \mathcal{P}_0(\mathcal{C})$, then every $\mathcal{F}$-cover is epic.	If $\mathcal{C}$ has enough injective objects and $\mathcal{G} \supseteq \mathcal{I}_0(\mathcal{C})$, then every $\mathcal{G}$-envelope is monic.

Definition 1.4.5. Let $\mathcal{C} \xrightarrow{T} \mathcal{D}$ be a covariant functor between Abelian categories.

Let $\mathcal{F}$ be a pre-covering class of $\mathcal{C}$. Consider a left $\mathcal{F}$-resolution

$$\cdots \to F_1 \to F_0 \to X \to 0$$

of an object X of $\mathcal{C}$. Denote by $\mathbf{F}_\bullet$ the deleted complex $\cdots \to F_1 \to F_0$. The homology groups of $T(\mathbf{F}_\bullet)$ give the left derived functors of T:

$$L_nT : X \mapsto (L_nT)(X)$$

Let $\mathcal{G}$ be a pre-enveloping class of $\mathcal{C}$. Consider a right $\mathcal{G}$-resolution

$$0 \to Y \to G^0 \to G^1 \to \cdots$$

of an object Y of $\mathcal{C}$. Denote by $\mathbf{G}^\bullet$ the deleted complex $G^0 \to G^1 \to \cdots$. The cohomology groups of $T(\mathbf{G}^\bullet)$ give the right derived functors of T:

$$R^nT : Y \mapsto (R^nT)(Y)$$

If T is contravariant, then the left (right) derived functors can be computed using right $\mathcal{G}$-resolutions (left $\mathcal{F}$-resolutions).

Theorem 1.4.2 (see (21, Theorems 8.2.3 & 8.2.5))

Let $\mathcal{C} \xrightarrow{T} \mathcal{D}$ be a functor between Abelian categories, and $\mathcal{F}$ and $\mathcal{G}$ be classes of objects in $\mathcal{C}$ closed under finite biproducts.

Suppose $0 \to X' \to X \to X'' \to 0$ is a $\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, -)$-exact sequence of objects in $\mathcal{C}$ and $\mathcal{F}$ is a pre-covering class.

(1) If T is covariant, there exists a long exact sequence

$$\begin{tikzcd} & & \cdots \arrow[r] & (L_2T)(X'') \arrow[dll] \\ (L_1T)(X') \arrow[r] & (L_1T)(X) \arrow[r] & (L_1T)(X'') \arrow[dll] & \\ (L_0T)(X') \arrow[r] & (L_0T)(X) \arrow[r, two heads] & (L_0T)(X'') & \end{tikzcd}$$

(2) It T is contravariant, there exists a long exact sequence

$$\begin{tikzcd} (R^0T)(X'') \arrow[r, hook] & (R^0T)(X) \arrow[r] & (R^0T)(X') \arrow[dll] \\ (R^1T)(X'') \arrow[r] & (R^1T)(X) \arrow[r] & (R^1T)(X') \arrow[dll] \\ (R^2T)(X'') \arrow[r] & \cdots & \end{tikzcd}$$

Suppose $0 \to Y' \to Y \to Y'' \to 0$ is a $\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{G})$-exact sequence of objects in $\mathcal{C}$ and $\mathcal{G}$ is a pre-enveloping class.

(1') If T is covariant, there exists a long exact sequence

$$\begin{tikzcd} (R^0T)(Y') \arrow[r, hook] & (R^0T)(Y) \arrow[r] & (R^0T)(Y'') \arrow[dll] \\ (R^1T)(Y') \arrow[r] & (R^1T)(Y) \arrow[r] & (R^1T)(Y'') \arrow[dll] \\ (R^2T)(Y') \arrow[r] & \cdots & \end{tikzcd}$$

(2') If T is contravariant, there exists a long exact sequence

$$\begin{tikzcd} & \cdots \arrow[r] & (L_2T)(Y') \arrow[dll] \\ (L_1T)(Y'') \arrow[r] & (L_1T)(Y) \arrow[r] & (L_1T)(Y') \arrow[dll] \\ (L_0T)(Y'') \arrow[r] & (L_0T)(Y) \arrow[r, two heads] & (L_0T)(Y') \end{tikzcd}$$

1.5 The extension functor

Let $\mathcal{C}$ be an Abelian category with enough projective and injective objects, and $X \in \mathrm{Ob}(\mathcal{C})$. Then there exists a short exact sequence $K_0 \overset{g_1}{\hookrightarrow} P_0 \overset{f_0}{\twoheadrightarrow} X$ where P_0 is a projective object in $\mathcal{C}$. Applying the same argument for K_0, we have a short exact sequence $K_1 \overset{g_2}{\hookrightarrow} P_1 \overset{h_1}{\twoheadrightarrow} K_0$, where P_1 is projective. If we set $f_1 := g_1 \circ h_1$, we get a commutative diagram

$$\begin{tikzcd}
K_1 \arrow[dr, hook, "g_2"'] & & & & \\
& P_1 \arrow[rr, "f_1"] \arrow[dr, two heads, "h_1"'] & & P_0 \arrow[r, two heads, "f_0"] & X \\
& & K_0 \arrow[ur, hook, "g_1"'] & &
\end{tikzcd}$$

Proceeding this way, we get a commutative diagram

$$\begin{tikzcd}
& & K_1 \arrow[dr, hook, "g_2"] & & & & \\
\cdots \arrow[r] & P_2 \arrow[ur, "h_2"] \arrow[rr, "f_2"'] & & P_1 \arrow[rr, "f_1"] \arrow[dr, two heads, "h_1"'] & & P_0 \arrow[r, two heads, "f_0"] & X \\
K_2 \arrow[ur, hook, "g_3"'] & & & & K_0 \arrow[ur, hook, "g_1"'] & &
\end{tikzcd}$$

Set $f_i := g_i \circ h_i$ for every $i > 0$. To show the central row is exact, for every $i > 0$ we have $\mathrm{Ker}(f_i) = \mathrm{Im}(g_{i+1})$. Since h_{i+1} is epic, $\mathrm{Im}(g_{i+1}) = \mathrm{Im}(g_{i+1} \circ h_{i+1}) = \mathrm{Im}(f_{i+1})$. Hence $\mathrm{Ker}(f_i) = \mathrm{Im}(f_{i+1})$. Therefore, $\cdots \to P_2 \overset{f_2}{\to} P_1 \overset{f_1}{\to} P_0 \overset{f_0}{\to} X \to 0$ is an exact sequence. Since this sequence is $\mathrm{Hom}_{\mathcal{C}}(\mathcal{P}_0(\mathcal{C}), -)$-exact, we conclude that every object has an exact left projective resolution.

Therefore, if $\mathcal{C}$ is an Abelian category with enough projective (resp. injective) objects, then every object has an exact left projective (resp. right injective) resolution.

It is known that $\mathcal{P}_0(\mathcal{C})$ is pre-covering and $\mathcal{I}_0(\mathcal{C})$ is pre-enveloping. Also, it is easy to see that they are closed under finite biproducts. So we can compute the right derived functors of $\mathrm{Hom}_{\mathcal{C}}(X,-)$ and $\mathrm{Hom}_{\mathcal{C}}(-,Y)$, called the extension functors, for any pair of objects X and Y of $\mathcal{C}$.

Let X and Y be objects in $\mathcal{C}$. If $\mathcal{C}$ has enough projective objects, we can consider an exact left projective resolution of $\cdots \to P_1 \to P_0 \to X \to 0$. The ith extension group $\mathrm{Ext}^i_{\mathcal{C}}(X,Y)$ is defined as the ith cohomology group of the complex $\mathrm{Hom}_{\mathcal{C}}(\mathbf{P}_\bullet, Y)$. If we assume instead that $\mathcal{C}$ has enough injective objects, consider an exact right injective resolution of Y, say $0 \to Y \to I^0 \to I^1 \to \cdots$. The ith extension group $\mathrm{Ext}^i_{\mathcal{C}}(X,Y)$ in this case is defined as the ith cohomology group of the complex $\mathrm{Hom}_{\mathcal{C}}(X, I_\bullet)$. If $\mathcal{C}$ has both enough projective and injective objects, it is known that the definition of $\mathrm{Ext}^i_{\mathcal{C}}(X,Y)$ does not depend on the choice of an exact left projective resolution of X or an exact right injective resolution of Y. A good reference to check this fact is (46).

Corollary 1.5.1

For every object X and every exact sequence $0 \to Y'' \to Y \to Y' \to 0$ in an Abelian category $\mathcal{C}$ with enough injective objects, there exists a long exact sequence

$$\begin{tikzcd}
\mathrm{Hom}(X,Y'') \arrow[d, hook] & \mathrm{Ext}^1(X,Y') \arrow[r, bend left] & \mathrm{Ext}^2(X,Y'') \arrow[d] \\
\mathrm{Hom}(X,Y) \arrow[d] & \mathrm{Ext}^1(X,Y) \arrow[u] & \vdots \\
\mathrm{Hom}(X,Y') \arrow[r, bend right] & \mathrm{Ext}^1(X,Y'') \arrow[u] &
\end{tikzcd}$$

For every object Y and every exact sequence $0 \to X' \to X \to X'' \to 0$ in an Abelian category $\mathcal{C}$ with enough projective objects, there exists a long exact sequence

$$\begin{tikzcd}
\mathrm{Hom}(X'',Y) \arrow[d, hook] & \mathrm{Ext}^1(X',Y) \arrow[r, bend left] & \mathrm{Ext}^2(X'',Y) \arrow[d] \\
\mathrm{Hom}(X,Y) \arrow[d] & \mathrm{Ext}^1(X,Y) \arrow[u] & \vdots \\
\mathrm{Hom}(X',Y) \arrow[r, bend right] & \mathrm{Ext}^1(X'',Y) \arrow[u] &
\end{tikzcd}$$

Remark 1.5.1. Let $\mathcal{C}$ be an Abelian category. It is not hard to show that $X \in \mathrm{Ob}(\mathcal{C})$ is projective if, and only if, $\mathrm{Ext}^1_{\mathcal{C}}(X, Y) = 0$ for every $Y \in \mathrm{Ob}(\mathcal{C})$. Dually, X is injective if, and only if, $\mathrm{Ext}^1_{\mathcal{C}}(Y, X) = 0$ for every $Y \in \mathrm{Ob}(\mathcal{C})$.

Example 1.5.1. In $\mathcal{C} = {}_R\,\mathbf{Mod}$ or $\mathbf{Mod}_R$, a module M is projective if, and only if, it is a direct summand of a free module. In other words, there is an index set I and a module N such that $M \oplus N = R^{(I)}$. If R is a principal ideal domain, then M is projective if, and only if, M is free.

A left R-module N is injective if, and only if, $\mathrm{Ext}^1_R(R/I, N) = 0$, for every left ideal $I \subseteq R$. The same characterization applies in $\mathbf{Mod}_R$. This result is known as the Baer Criterion. In the end of this chapter, we shall see a categorical version of this result.

Homology and cohomology are not the only tools to compute extension functors. The next section is devoted to constructing an isomorphism from $\mathrm{Ext}^1_{\mathcal{C}}(X, Y)$ to an Abelian group of classes of short exact sequences of the form $Y \hookrightarrow Z \twoheadrightarrow X$, under a certain equivalence relation. This is known as the Baer description of $\mathrm{Ext}^1_{\mathcal{C}}(X, Y)$.

1.6 Baer sums

Given two exact sequences $S_1 = Y \overset{f_1}{\hookrightarrow} Z_1 \overset{g_1}{\twoheadrightarrow} X$ and $S_2 = Y \overset{f_2}{\hookrightarrow} Z_2 \overset{g_2}{\twoheadrightarrow} X$ in an Abelian category $\mathcal{C}$, we shall say that they are (Baer) equivalent if there is a morphism $\varphi : Z_1 \to Z_2$ such that the following diagram commutes:

$$\begin{tikzcd}
Y \arrow[r, hook, "f_1"] \arrow[d, equal] & Z_1 \arrow[r, two heads, "g_1"] \arrow[d, "\varphi"] & X \arrow[d, equal] \\
Y \arrow[r, hook, "f_2"'] & Z_2 \arrow[r, two heads, "g_2"'] & X
\end{tikzcd}$$

We shall denote by $[S]$ the equivalence class of a sequence $S = Y \hookrightarrow Z \twoheadrightarrow X$, and by $E^{\mathcal{C}}(X,Y)$ the set of all the equivalence classes $[S]$. We shall equip this set with an operation called the <u>Baer sum</u>, which shall turn $E^{\mathcal{C}}(X,Y)$ into an Abelian group.

Consider two short exact sequences S_1 and S_2 as above. Taking the pullback of g_1 and g_2, by Proposition 1.2.2, we get the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & Y \arrow[r, equal] \arrow[d, hook, "\hat{f}_2"'] & Y \arrow[d, hook, "f_2"] \\
Y \arrow[r, hook, "\hat{f}_1"] \arrow[d, equal] & Z_1 \times_X Z_2 \arrow[r, two heads, "\rho_{Z_2}"] \arrow[d, two heads, "\rho_{Z_1}"'] & Z_2 \arrow[d, two heads, "g_2"] \\
Y \arrow[r, hook, "f_1"'] & Z_1 \arrow[r, two heads, "g_1"'] & X
\end{tikzcd}$$

Consider the map $Y \overset{\hat{f}_1-\hat{f}_2}{\to} Z_1 \times_X Z_2$ and set $Z_1 +_B Z_2 := \mathrm{CoKer}(\hat{f}_1 - \hat{f}_2)$. By the universal property of cokernels, there is a unique map $Z_1 +_B Z_2 \overset{g_1 +_B g_2}{\to} X$ such that the following diagram commutes:

$$\begin{tikzcd}
Y \arrow[r, hook, "\hat{f}_1 - \hat{f}_2"] & Z_1 \times_X Z_2 \arrow[r, two heads, "\pi"] \arrow[dr, "g_1 \circ \rho_{Z_1} = g_2 \circ \rho_{Z_2}"'] & Z_1 +_B Z_2 \arrow[d, dotted, "\exists!\ g_1 +_B g_2"] \\
 & & X
\end{tikzcd}$$

On the other hand, set $f_1 +_B f_2 := \pi \circ \hat{f}_1 = \pi \circ \hat{f}_2$. We get a short exact sequence

$$S_1 +_B S_2 := 0 \to Y \overset{f_1 +_B f_2}{\to} Z_1 +_B Z_2 \overset{g_1 +_B g_2}{\to} X \to 0.$$

Define a binary operation $+_B : E^{\mathcal{C}}(X,Y) \times E^{\mathcal{C}}(X,Y) \to E^{\mathcal{C}}(X,Y)$, called the Baer sum, by setting $[S_1] +_B [S_2] := [S_1 +_B S_2]$. It is not hard to check that $+_B$ is a well defined and makes $E^{\mathcal{C}}(X,Y)$ into an Abelian group.

We know the category of chain complexes $\mathbf{Ch}(\mathcal{C})$ over an Abelian category $\mathcal{C}$ is also Abelian. So Baer sums can be defined for $\mathbf{Ch}(\mathcal{C})$. Actually, the Baer sum of two exact sequences $Y \overset{f^1}{\hookrightarrow} Z^1 \overset{g^1}{\twoheadrightarrow} X$ and $Y \overset{f^2}{\hookrightarrow} Z^2 \overset{g^2}{\twoheadrightarrow} X$ can be computed componentwise from the Baer sums of $Y_m \overset{f^1_m}{\hookrightarrow} Z^1_m \overset{g^1_m}{\twoheadrightarrow} X_m$ and $Y_m \overset{f^2_m}{\hookrightarrow} Z^2_m \overset{g^2_m}{\twoheadrightarrow} X_m$.

The following result gives a description of $\operatorname{Ext}^1_{\mathcal{C}}(X, Y)$ in the sense of Baer. One can construct similar isomorphisms for the case $i > 1$, but for our purposes we just need the case $i = 1$.

Theorem 1.6.1

The groups $E^{\mathcal{C}}(X, Y)$ and $\operatorname{Ext}^1_{\mathcal{C}}(X, Y)$ are isomorphic if $\mathcal{C}$ has enough projective or injective objects.

The previous theorem is a well known result in the category of left R-modules. It is not hard to give a category theoretical proof, but we are going to skip it since in Appendix B we provide a generalization of this proposition.

As we said before, sometimes it is better to consider the description of $\operatorname{Ext}^1_{\mathcal{C}}(-,-)$ as the Abelian group $E^{\mathcal{C}}(-,-)$. For instance, the connecting homomorphisms $\operatorname{Hom}_{\mathcal{C}}(X, Y') \xrightarrow{\delta} \operatorname{Ext}^1_{\mathcal{C}}(X, Y'')$ and $\operatorname{Hom}_{\mathcal{C}}(X', Y) \xrightarrow{\delta'} \operatorname{Ext}^1_{\mathcal{C}}(X'', Y)$ in Corollary 1.5.1 can be constructed in terms of this description.

Consider a morphism $X \to Y'$. Taking its pullback with $Y \twoheadrightarrow Y'$, we get the following commutative diagram with exact rows:

$$\begin{tikzcd}
Y'' \arrow[r, hook] \arrow[d] & Y \times_{Y'} X \arrow[r, two heads] \arrow[d] & X \arrow[d] \\
Y'' \arrow[r, hook] & Y \arrow[r, two heads] & Y'
\end{tikzcd}$$

The homomorphism δ is defined by $(X \to Y') \mapsto [Y'' \hookrightarrow Y \times_{Y'} X \twoheadrightarrow X]$. Dually, δ' is defined by taking pushouts instead of pullbacks. A proof of Corollary 1.5.1 using these description of δ is given in (38, Theorems 25.8 & 25.10).

The following results, due to J. Gillespie, were originally stated for $\mathrm{Ext}^1_{\mathcal{C}}(-,-)$ and $\mathrm{Ext}^1_{\mathbf{Ch}(\mathcal{C})}(-,-)$ and proven by using the Baer description of Ext.

Proposition 1.6.2 (see (27, Lemma 3.1))

Let C be an object of an Abelian category $\mathcal{C}$, and X and Y be chain complexes. There are natural isomorphisms

$E^{\mathbf{Ch}(\mathcal{C})}(D^m(C), Y) \to E^{\mathcal{C}}(C, Y_m)$. | $E^{\mathbf{Ch}(\mathcal{C})}(X, D^{m+1}(C)) \to E^{\mathcal{C}}(X_m, C)$.

Proposition 1.6.3 (see (25, Lemma 4.2))

Let C be an object of an Abelian category $\mathcal{C}$ and X and Y be exact chain complexes. There are natural isomorphisms

$E^{\mathbf{Ch}(\mathcal{C})}(S^m(C), Y) \to E^{\mathcal{C}}(C, Z_m(Y))$. | $E^{\mathbf{Ch}(\mathcal{C})}(X, S^m(C)) \to E^{\mathcal{C}}(\frac{X_m}{B_m(X)}, C)$.

We do not give the construction of these isomorphisms. We shall recall them in Appendix B, where some generalizations of the previous propositions are given. If $\mathcal{C}$ is an Abelian category with enough projective and injective objects, we can replace $E^{\mathbf{Ch}(\mathcal{C})}(-,-)$ and $E^{\mathcal{C}}(-,-)$ by $\mathrm{Ext}^1_{\mathbf{Ch}(\mathcal{C})}(-,-)$ and $\mathrm{Ext}^1_{\mathcal{C}}(-,-)$, respectively. Under this hypothesis, it is possible to give proofs of the previous two results simpler than those appearing in (27) and (25), as noted in Remarks B.1 and B.3.

1.7 Homological dimensions

Definition 1.7.1. Let $\mathcal{F}$ and $\mathcal{G}$ be classes of objects in an Abelian category $\mathcal{C}$.

(1) An object X of $\mathcal{C}$ is called a <u>left n-$\mathcal{F}$-object</u> if there exists an exact sequence of length n, say

$$0 \to F_n \to \cdots \to F_0 \to X \to 0,$$

with $F_i \in \mathcal{F}$ for every $0 \leq i \leq n$. If n is the smallest integer for which such a sequence exists, we say that X has <u>left $\mathcal{F}$-dimension</u> equal to n. If such an integer n does not exist, we say that X has <u>infinite</u> left $\mathcal{F}$-dimension.

(2) Given a left $\mathcal{F}$-resolution of $X \in \mathrm{Ob}(\mathcal{C})$,

$$\mathbf{F} = \cdots \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} X \to 0,$$

we shall say that $\mathrm{Im}(f_i)$ is the <u>i-th $\mathcal{F}$-syzygy of X in $\mathbf{F}$</u>. By $\Omega^i_{\mathcal{F}}(X)$ we denote the class of all i-th $\mathcal{F}$-syzygies occurring in all left $\mathcal{F}$-resolutions of X. We use the notation $\Omega^i(X)$ for the class of all projective syzygies of X.

(1') An object Y of $\mathcal{C}$ is called a <u>right n-$\mathcal{G}$-object</u> if there exists an exact sequence of length n, say

$$0 \to Y \to G^0 \to \cdots \to G^n \to 0,$$

with $G^i \in \mathcal{G}$ for every $0 \leq i \leq n$. If n is the smallest integer for which such a sequence exists, we say that Y has <u>right $\mathcal{G}$-dimension</u> equal to n. If such an integer n does not exist, we say that Y has <u>infinite</u> right $\mathcal{G}$-dimension.

(2') Given a right $\mathcal{G}$-resolution of $Y \in \mathrm{Ob}(\mathcal{C})$,

$$\mathbf{G} = 0 \to Y \xrightarrow{g^0} G^0 \xrightarrow{g^1} G^1 \to \cdots,$$

we shall say that $\mathrm{Ker}(g^i)$ is the <u>i-th $\mathcal{G}$-cosyzygy of Y in $\mathbf{G}$</u>. By $\Omega^{-i}_{\mathcal{G}}(Y)$ we denote the class of all i-th $\mathcal{G}$-cosyzygies occurring in all right $\mathcal{G}$-resolutions of Y. We use the notation $\Omega^{-i}(Y)$ for the class of all injective cosyzygies of Y.

Remark 1.7.1. Let X be an object of an Abelian category $\mathcal{C}$.

We denote the (left) **projective dimension** of an object X by $\mathrm{pd}(X)$. Note X is n-projective if, and only if, $\mathrm{pd}(X) \leq n$. We denote the class of n-projective objects of $\mathcal{C}$ by $\mathcal{P}_n(\mathcal{C})$.

We denote the (right) **injective dimension** of an object Y by $\mathrm{id}(Y)$. Note Y is n-injective if, and only if, $\mathrm{id}(Y) \leq n$. We denote the class of n-injective objects of $\mathcal{C}$ by $\mathcal{I}_n(\mathcal{C})$.

Proposition 1.7.1 (Dimension Shifting. See (46, Proposition 4.2))

Let X and Y be two objects of an Abelian category $\mathcal{C}$. For every $i \geq 0$:

If $\mathcal{C}$ has enough projective objects and $A \in \Omega^i(X)$, then $\mathrm{Ext}^1_{\mathcal{C}}(A, Y) \cong \mathrm{Ext}^{i+1}_{\mathcal{C}}(X, Y)$.

If $\mathcal{C}$ has enough injective objects and $B \in \Omega^{-i}(Y)$, then $\mathrm{Ext}^1_{\mathcal{C}}(X, B) \cong \mathrm{Ext}^{i+1}_{\mathcal{C}}(X, Y)$.

Proposition 1.7.2

Let $X \in \mathrm{Ob}(\mathbf{Ch}(\mathcal{C}))$. For every $m \in \mathbb{Z}$ and $i \in \mathbb{Z}_{\geq 0}$:

Suppose $\mathcal{C}$ has enough projective objects. If $Y \in \Omega^i(X)$ then $Y_m \in \Omega^i(X_m)$.

Suppose $\mathcal{C}$ has enough injective objects. If $Y \in \Omega^{-i}(X)$ then $Y_m \in \Omega^{-i}(X_m)$.

Proof.

Let $Y \in \Omega^i(X)$ and $\cdots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} X \to 0$ be an exact left projective resolution of X with $Y = \mathrm{Im}(f_i)$. Since each P_n is a projective complex, we have for every $m \in \mathbb{Z}$ a short exact sequence

$$Z_m(P_n) \hookrightarrow (P_n)_m \twoheadrightarrow Z_{m-1}(P_n),$$

where $Z_{m-1}(P_n)$ and $Z_m(P_n)$ are projective objects. Since $\mathcal{P}_0(\mathcal{C})$ is closed under extensions, $(P_n)_m$ is also projective. Then we have an exact left projective resolution of X_m:

$$\cdots \to (P_1)_m \overset{(f_1)_m}{\to} (P_0)_m \overset{(f_0)_m}{\to} X_m \to 0$$

in $\mathcal{C}$. Hence $Y_m = \mathrm{Im}((f_i)_m) \in \Omega^i(X_m)$. $\square$

Proposition 1.7.3

Let X and Y be objects in an Abelian category $\mathcal{C}$:

If $\mathcal{C}$ has enough projective objects, then the following conditions are equivalent:

(1) $X \in \mathcal{P}_n(\mathcal{C})$.

(2) X has an exact and finite left projective resolution of length n.

(3) $\mathrm{Ext}^i_{\mathcal{C}}(X, Y) = 0$ for every $Y \in \mathrm{Ob}(\mathcal{C})$ and every $i > n$.

(4) $\mathrm{Ext}^{n+1}_{\mathcal{C}}(X, Y) = 0$ for every $Y \in \mathrm{Ob}(\mathcal{C})$.

(5) Every $K \in \Omega^n(X)$ is projective.

If $\mathcal{C}$ has enough injective objects, then the following conditions are equivalent:

(1') $Y \in \mathcal{I}_n(\mathcal{C})$.

(2') Y has an exact and finite right injective resolution of length n.

(3') $\mathrm{Ext}^i_{\mathcal{C}}(X, Y) = 0$ for every $X \in \mathrm{Ob}(\mathcal{C})$ and every $i > n$.

(4') $\mathrm{Ext}^{n+1}_{\mathcal{C}}(X, Y) = 0$ for every $X \in \mathrm{Ob}(\mathcal{C})$.

(5') Every $C \in \Omega^{-n}(Y)$ is injective.

1.8 n-Projective and n-injective chain complexes

In this section we characterize the n-projective and n-injective chain complexes over an Abelian category $\mathcal{C}$, for every positive integer n. Specifically, we shall show that X is an n-projective complex if, and only if, it is exact and every cycle of X is a n-projective object of $\mathcal{C}$. We have a dual equivalence for the injective case.

Definition 1.8.1. Given an object X in a category $\mathcal{C}$, we shall say that an object Y is a retract of X if there exist morphisms $Y \xrightarrow{r} X$ and $p : X \xrightarrow{p} Y$ such that $p \circ r = \mathrm{id}_Y$, i.e. r has a left inverse. The map r is called a retraction.

Definition 1.8.2. Let $S = X' \hookrightarrow X \twoheadrightarrow X''$ denote a short exact sequence in an Abelian category $\mathcal{C}$. A class $\mathcal{X} \subseteq \mathrm{Ob}(\mathcal{C})$ is said to be:

(1) closed under extensions if for every S, $X', X'' \in \mathcal{X}$ implies $X \in \mathcal{X}$;

(2) closed under taking kernels of epimorphisms if for every S, $X, X'' \in \mathcal{X}$ implies $X' \in \mathcal{X}$;

(3) closed under taking cokernels of monomorphisms if for every S, $X', X \in \mathcal{X}$ implies $X'' \in \mathcal{X}$;

(4) resolving if it contains the projective objects of $\mathcal{C}$ and it satisfies **(1)** and **(2)**;

(5) coresolving if it contains the injective objects of $\mathcal{C}$ and it satisfies **(1)** and **(3)**;

(6) thick if it satisfies **(1)**, **(2)** and **(3)**, and if it is closed under retracts.

Example 1.8.1. The class $\mathcal{E}$ of exact chain complexes over an Abelian category $\mathcal{C}$ is thick.

First, we show $\mathcal{E}$ is closed under retracts. Recall that a complex E is exact if, and only if, the nth homology group $H_n(E) = \frac{Z_n(E)}{B_n(E)}$ of E is zero for every $n \in \mathbb{Z}$. Suppose we are given a sequence $E' \xrightarrow{r} E \xrightarrow{p} E'$ such that E is exact and $p \circ r = \mathrm{id}_{E'}$. For every $n \in \mathbb{Z}$, after applying the functor $H_n(-)$ to the previous sequence, we have that $H_n(E')$ is a retract of $H_n(E) = 0$. It follows $H_n(E') = 0$ and hence E' is exact.

Now suppose we are given a short exact sequence $E' \hookrightarrow E \twoheadrightarrow E''$. By (46, Theorem 7.48), there exist connecting homomorphisms $\delta : H_n(E'') \to H_{n-1}(E')$ such that the following sequence is exact:

$$\cdots \to H_{n+1}(E'') \xrightarrow{\delta} H_n(E') \to H_n(E) \to H_n(E'') \xrightarrow{\delta} H_{n-1}(E') \to \cdots$$

This sequence is called the long exact homology sequence. Then it is clear that $\mathcal{E}$ satisfies **(1)**, **(2)** and **(3)**.

Lemma 1.8.1 (see (49, Lemma 4.2))

Let $\mathcal{C}$ be an Abelian category.

If $0 \to A_n \to \cdots \to A_0 \xrightarrow{f_0} X \to 0$ is an exact sequence in $\mathbf{Ch}(\mathcal{C})$ such that A_i is exact for every $0 \leq i \leq n$, then so is X.	If $0 \to Y \xrightarrow{g^0} B^0 \to \cdots \to B^n \to 0$ is an exact sequence in $\mathbf{Ch}(\mathcal{C})$ such that B^i is exact for every $0 \leq i \leq n$, then so is Y.

Proof.

We only prove the left statement, by using induction on n. The case $n = 0$ is trivial, while the case $n = 1$ follows by the previous example. For the general case, we note first by the induction hypothesis that $\mathrm{Im}(f_1)$ is exact since the sequence $0 \to A_n \to A_{n-1} \to \cdots \to A_1 \to \mathrm{Im}(f_1) \to 0$ is exact. It follows that X is exact by using the short exact sequence $\mathrm{Im}(f_1) \hookrightarrow A_0 \twoheadrightarrow X$ and the case $n = 1$. $\square$

Lemma 1.8.2

Consider a short exact sequence $S = Y \overset{f}{\hookrightarrow} Z \overset{g}{\twoheadrightarrow} X$ in $\mathbf{Ch}(\mathcal{C})$.

If Y is exact, then S gives rise to a short exact sequence $$Z_m(Y) \hookrightarrow Z_m(Z) \twoheadrightarrow Z_m(X).$$	If X is exact, then S gives rise to a short exact sequence $$\frac{Y_m}{B_m(Y)} \hookrightarrow \frac{Z_m}{B_m(Z)} \twoheadrightarrow \frac{X_m}{B_m(X)}.$$

Proof.

We only show the left statement for the category ${}_R\mathbf{Mod}$. The general proof follows by (43, Theorem 3, page 204). Note we have the following commutative diagrams:

$$\begin{tikzcd}
Z_m(Z) \arrow[r, hook, "i_m^Z"] & Z_m \arrow[r, "\partial_m^Z"] & Z_{m-1} \\
Z_m(Y) \arrow[u, dotted, "\exists!\ Z_m(f)"] \arrow[ur, "f_m \circ i_m^Y"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
Z_m(X) \arrow[r, hook, "i_m^X"] & X_m \arrow[r, "\partial_m^X"] & X_{m-1} \\
Z_m(Z) \arrow[u, dotted, "\exists!\ Z_m(g)"] \arrow[ur, "g_m \circ i_m^Z"'] & &
\end{tikzcd}$$

It is easy to check that $Z_m(f)$ is monic and that $\mathrm{Ker}(Z_m(g)) = \mathrm{Im}(Z_m(f))$. These facts do not depend on the exactness of Y. Let $x \in Z_m(X)$. There exists $z \in Z_m$ such that $x = g_m(z)$. We have $g_{m-1} \circ \partial_m^Z(z) = \partial_m^X \circ g_m(z) = 0$. Since the sequence $Y_{m-1} \hookrightarrow Z_{m-1} \twoheadrightarrow X_{m-1}$ is exact, there exists $y \in Y_{m-1}$ such that $\partial_m^Z(z) = f_{m-1}(y)$. Then $f_{m-2} \circ \partial_{m-1}^Y(y) = \partial_{m-1}^Z \circ f_{m-1}(y) = 0$ and so $\partial_{m-1}^Y(y) = 0$ since f_{m-2} is monic. By the exactness of Y, there exists $y' \in Y_m$ such that $y = \partial_m^Y(y')$. Hence $\partial_m^Z(z - f_m(y')) = 0$ and $g_m(z - f_m(y')) = x$. $\square$

Lemma 1.8.3 (see (49, Lemma 4.3))

Let $0 \to A_n \xrightarrow{f_n} A_{n-1} \to \cdots \to A_1 \xrightarrow{f_1} A_0 \to 0$ be an exact sequence in $\mathbf{Ch}(\mathcal{C})$ of exact chain complexes. Then the mth cycles $Z_m(A_i)$ form the following exact sequence in $\mathcal{C}$, for every $m \in \mathbb{Z}$:

$$0 \to Z_m(A_n) \to Z_m(A_{n-1}) \to \cdots \to Z_m(A_1) \to Z_m(A_0) \to 0.$$

Proof.

The case $n = 1$ is trivial. The case $n = 2$ follows by the previous lemma. Now suppose the statement is true for $n - 1$. Then we have an exact sequence of the form $0 \to A_n \to A_{n-1} \to \cdots \to A_2 \to \mathrm{Im}(f_2) \to 0$, where $\mathrm{Im}(f_2)$ is an exact complex by Lemma 1.8.1. By the induction hypothesis, we have an exact sequence of mth cycles

$$0 \to Z_m(A_n) \to Z_m(A_{n-1}) \to \cdots \to Z_m(A_2) \to Z_m(\mathrm{Im}(f_2)) \to 0.$$

We get the following commutative diagram with exact rows and columns

$$\begin{tikzcd}
0 \arrow[r] & Z_m(A_n) \arrow[r] & Z_m(A_{n-1}) \arrow[r] & \cdots \arrow[r] & Z_m(A_2) \arrow[r] \arrow[dr, "h"'] & Z_m(\mathrm{Im}(f_2)) \arrow[r] \arrow[d, hook] & 0 \\
& & & & & Z_m(A_1) \arrow[d, two heads] & \\
& & & & & Z_m(A_0) &
\end{tikzcd}$$

It is not hard to show that

$$0 \to Z_m(A_n) \to Z_m(A_{n-1}) \to \cdots \to Z_m(A_2) \xrightarrow{h} Z_m(A_1) \to Z_m(A_0) \to 0$$

is an exact sequence in $\mathcal{C}$, for every $m \in \mathbb{Z}$. □

The right statement of the following proposition is proven in (24, Theorem 3.1.3). We give a different argument.

Proposition 1.8.4 (see (49, Proposition 4.4))

Let $\mathcal{C}$ be an Abelian category and n be a positive integer.

Assume $\mathcal{C}$ has enough projective objects. A chain complex X is n-projective if, and only if, X is exact and $Z_m(X)$ is an n-projective object of $\mathcal{C}$ for every $m \in \mathbb{Z}$.	Assume $\mathcal{C}$ has enough injective objects. A chain complex Y is n-injective if, and only if, Y is exact and $Z_m(Y)$ is an n-injective object of $\mathcal{C}$ for every $m \in \mathbb{Z}$.

$\boxed{\textbf{Proof}}$.

($\Longleftarrow$) Let X be an exact complex such that $Z_m(X)$ is n-projective for every $m \in \mathbb{Z}$. Consider a partial projective resolution

$$0 \to K \to P_{n-1} \to \cdots \to P_1 \to P_0 \to X \to 0.$$

We shall see K is also projective by proving that K is exact with projective cycles (Proposition 1.3.3). Note K is exact by Lemma 1.8.1. It follows by Lemma 1.8.3 that the mth cycles give rise to an exact sequence

$$0 \to Z_m(K) \to Z_m(P_{n-1}) \to \cdots \to Z_m(P_1) \to Z_m(P_0) \to Z_m(X) \to 0$$

in $\mathcal{C}$, where each $Z_m(P_i)$ is projective, and so $Z_m(K) \in \Omega^n(Z_m(X))$. Hence, $Z_m(K) \in \Omega^n(Z_m(X)) \subseteq \mathcal{P}_0(\mathcal{C})$ since $Z_m(X)$ is n-projective.

($\Longrightarrow$) Now suppose that X has a projective resolution

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to X \to 0$$

of length n. The complex X is exact by Lemma 1.8.1. Then by Lemma 1.8.3, we get for every $m \in \mathbb{Z}$ an exact sequence

$$0 \to Z_m(P_n) \to Z_m(P_{n-1}) \to \cdots \to Z_m(P_1) \to Z_m(P_0) \to Z_m(X) \to 0$$

in $\mathcal{C}$, where $Z_m(P_k)$ is projective, for every $0 \leq k \leq n$. Hence $Z_m(X)$ is n-projective. $\square$

1.9 Grothendieck categories

In this section we recall the notion of Grothendieck categories. They are a special type of Abelian categories introduced by Alexander Grothendieck in 1957 in order to generalize the machinery of homological algebra known in the category of modules. Some of the results used in this thesis are known in the categories of modules and chain complexes of modules, but they can be presented in terms of Grothendieck categories. Since one of our objectives is to present our results in a categorical setting, if possible, it would be good if the reader took some minutes to review this section.

Definition 1.9.1. A family of objects $(G_i : i \in I)$ in a category $\mathcal{C}$, indexed by a set I, is called

a set of generators if for each pair of different morphisms $f, g : X \to Y$, there is a G_i and a map $G_i \xrightarrow{h} X$ with $f \circ h \neq g \circ h$. If the set of generators is a singleton $\{G\}$, then G is called a generator.

a set of cogenerators if for each pair of different morphisms $f, g : X \to Y$, there is a G_i and a map $Y \xrightarrow{h} G_i$ with $h \circ f \neq h \circ g$. If the set of cogenerators is a singleton $\{G\}$, then G is called a cogenerator.

Remark 1.9.1. If $\mathcal{C}$ is a pointed cocomplete category with a set of generators $(G_i : i \in I)$, then by $\prod_{i \in I} \mathrm{Hom}_{\mathcal{C}}(G_i, -) \cong \mathrm{Hom}_{\mathcal{C}}(\coprod_{i \in I} G_i, -)$ the coproduct $\coprod_{i \in I} G_i$ of the G_i is a generator.

Example 1.9.1.

(1) In **Set**, each nonempty set is a generator, and each set with at least two elements is a cogenerator.

(2) In **Top**, each discrete and nonempty topological space is a generator, and each topological space with at least two elements and equipped with the indiscrete topology is a cogenerator.

(3) The category **Grp** does not have any cogenerator. In the full subcategory **Ab**, $\mathbb{Q}/\mathbb{Z}$ is a cogenerator. In both cases, the group $\mathbb{Z}$ is a generator.

(4) In ${}_R\mathbf{Mod}$ or $\mathbf{Mod}_R$, R is a generator.

(5) If G is a generator of $\mathcal{C}$, then the direct sum $\oplus_{m\in\mathbb{Z}} D^m(G)$ is a generator of $\mathbf{Ch}(\mathcal{C})$.

Proposition 1.9.1 (see (47, Lemma 2, page 111))

Let $\mathcal{C}$ be a category with coproducts. An object G is a generator if, and only if, to each object $X \in \mathrm{Ob}(\mathcal{C})$ there is an epimorphism $G^{(I)} \to X$, where $I = \mathrm{Hom}_{\mathcal{C}}(G, X)$ and $G^{(I)}$ denotes the coproduct of copies of G over I.

Definition 1.9.2. A direct limit is a colimit of a diagram $\Sigma \xrightarrow{F} \mathcal{C}$ where $\mathrm{Ob}(\Sigma)$ is a directed set, i.e. if $\mathrm{Ob}(\Sigma)$ has a reflexive and transitive binary relation $\leq$ such that each pair of elements has an upper bound.

Definition 1.9.3. A Grothendieck category is a cocomplete Abelian category $\mathcal{C}$ with a generator such that:

(1) $\mathcal{C}$ is AB3: $\mathcal{C}$ has arbitrary direct sums.

(2) $\mathcal{C}$ is AB5: $\mathcal{C}$ is AB3 and direct limits of short exact sequences are exact.

Definition 1.9.4. Let Y be an object in a category $\mathcal{C}$ and $(X_i \xrightarrow{f_i} Y : i \in I)$ be a family of subojects of Y indexed by a set I.

(1) A subobject $X \xrightarrow{f} Y$ is called the intersection of the X_i's if the following two conditions are satisfied:

  (a) X is a subobject of X_i, for every $i \in I$.

  (b) For each object $Z \in \mathrm{Ob}(\mathcal{C})$ and each morphism $Z \xrightarrow{g} Y$ which may be factored through all X_i, there exists a morphism $Z \xrightarrow{h} X$ such that $f \circ h = g$.

In pictures, we have:

$$\forall\, i \in I: \quad \begin{tikzcd} Z \arrow[rr, "g"] \arrow[dr, dotted, "\exists h_i"'] & & Y \\ & X_i \arrow[ur, "f_i"'] & \end{tikzcd} \quad \to \quad \begin{tikzcd} Z \arrow[rr, "g"] \arrow[dr, dotted, "\exists h"'] & & Y \\ & X \arrow[ur, "f"'] & \end{tikzcd}$$

Figure 1.14: Intersection of objects.

(2) A subobject $X \xrightarrow{f} Y$ is called the <u>union</u> of the X_i's if the following two conditions are satisfied:

(a) X_i is a subobject of X, for every $i \in I$.

(b) For each object $Z \in \mathrm{Ob}(\mathcal{C})$ and each morphism $Y \xrightarrow{g} Z$ such that every $g \circ f_i$ may be factored through a morphism $Z' \xrightarrow{k} Z$, there exists a morphism $X \xrightarrow{h} Z'$ such that $g \circ f = k \circ h$.

In pictures, we have:

$$\forall\, i \in I: \quad \begin{tikzcd} X_i \arrow[r, "f_i"] \arrow[d, dotted, "\exists h_i"'] & Y \arrow[d, "g"] \\ Z' \arrow[r, "k"'] & Z \end{tikzcd} \quad \to \quad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, dotted, "\exists h"'] & Y \arrow[d, "g"] \\ Z' \arrow[r, "k"'] & Z \end{tikzcd}$$

Figure 1.15: Union of objects.

Remark 1.9.2.

(1) One can show that the intersection and union of the X_i's are uniquely determined. So we shall denote these constructions by $\bigcap_{i\in I} X_i$ and $\bigcup_{i\in I} X_i$, respectively.

(2) In any Grothendieck category, the intersection and union of any family of objects exist. Moreover, $(\bigcup_{i\in I} X_i) \cap Y = \bigcup_{i\in I} X_i \cap Y$ for every subobject $Y \subseteq X$ and every chain of subobjects $(X_i : i \in I)$ of X. This equality is known as the <u>Grothendieck condition</u>. Moreover, this equality also holds for every directed family of subobjects of X (47, Lemma 2, page 182).

Example 1.9.2.

(1) If $\mathcal{C} = \textbf{Set}$, $\textbf{Grp}$, ${}_R\textbf{Mod}$, or $\textbf{Mod}_R$, then the intersection and the union of a family of subobjects $(X_i : i \in I)$ of an object Y are given by the objects

$$\bigcap_{i\in I} X_i = \{x \in Y : x \in X_i \text{ for every } i \in I\} \text{ and}$$
$$\bigcup_{i\in I} X_i = \{x \in Y : x \in X_i \text{ for some } i \in I\},$$

respectively.

(2) Union of an ascending chain of chain complexes: Let $(X^i : i \in I)$ be a family of chain complexes such that X^i is a subcomplex X^j whenever $i \leq j$, then the union of this family is given by the chain complex

$$\bigcup_{i\in I} X^i = \cdots \to \bigcup_{i\in I} X^i_{n+1} \overset{\partial_{n+1}}{\to} \bigcup_{i\in I} X^i_n \overset{\partial_n}{\to} \bigcup_{i\in I} X^i_{n-1} \to \cdots,$$

where the boundary maps $\partial_n : \bigcup_{i\in I} X^i_n \to \bigcup_{i\in I} X^i_{n-1}$ are defined by $\partial_n(x) := \partial_n^{X^i}(x)$ if $x \in X_i$.

In (20), the notion of cardinality is given for Grothendieck categories. This shall allow us to extend some results in the context of the set theoretical homological algebra to any Grothendieck Category.

Definition 1.9.5. Let $\mathcal{C}$ be a Grothendieck category with a fixed generator G. For each object $X \in \text{Ob}(\mathcal{C})$, we shall define the cardinality of X by $\text{Card}(X) := |\text{Hom}_{\mathcal{C}}(G, X)|$, where $|\text{Hom}_{\mathcal{C}}(G, X)|$ denotes the cardinal number of $\text{Hom}_{\mathcal{C}}(G, X)$.

Lemma 1.9.2 (see (20, Lemma 2.5))

If X is a subobject of an object Y in a Grothendieck category $\mathcal{C}$, then $\text{Card}(X) \leq \text{Card}(Y)$.

Example 1.9.3.

(1) If $\mathcal{C} =_R \mathbf{Mod}$ or $\mathbf{Mod}_R$, then the cardinality of a module M coincides with $|M|$. In this case, R is a generator of $\mathcal{C}$. The mapping $f \mapsto f(1)$ defines an isomorphism $\mathrm{Hom}_R(R, M) \to M$.

(2) The cardinal of a chain complex $X = (X_m, \partial_m^X)_{m\in\mathbb{Z}} \in \mathrm{Ob}(\mathbf{Ch}(_R\mathbf{Mod}))$ is usually defined as the cardinal number of the disjoint union $\bigsqcup_{m\in\mathbb{Z}} X_m$. On the other hand, since $\mathbf{Ch}(_R\mathbf{Mod})$ is a Grothendieck category with a generator $G = \bigoplus_{m\in\mathbb{Z}} D^m(R)$, the cardinality of X is given by $\mathrm{Card}(X) = |\mathrm{Hom}(G, X)|$. For every $m \in \mathbb{Z}$, the mapping $f \mapsto f_m(1)$ defines an isomorphism of modules from $\mathrm{Hom}(D^m(R), X)$ to X_m. Then $|\mathrm{Hom}(D^m(R), X)| = |X_m|$ and since $\mathrm{Hom}(G, X) \cong \prod_{m\in\mathbb{Z}} \mathrm{Hom}(D^m(R), X)$, we have $\mathrm{Card}(X) \geq |\bigsqcup_{m\in\mathbb{Z}} X_m|$.

Now we study a special type of generators, called progenerator, that allows to construct an equivalence between any Abelian category $\mathcal{C}$ equipped with a progenerator, and $\mathbf{Mod}_R$ for some ring R. Colloquially, this represents a method to translate results in $\mathbf{Mod}_R$ to $\mathcal{C}$.

Definition 1.9.6. If $\mathcal{C}$ is a cocomplete Abelian category, we say that a projective object P of $\mathcal{C}$ is finite if the functor $\mathrm{Hom}_{\mathcal{C}}(P, -) : \mathcal{C} \to \mathbf{Ab}$ preserves coproducts. A progenerator of $\mathcal{C}$ is a finite projective generator.

According to (47, Corollary 1, page 213), every cocomplete Abelian category with a progenerator is a Grothendieck category with an injective cogenerator. So the results we have given so far for Grothendieck categories hold for this particular class of Abelian categories.

Definition 1.9.7. An equivalence of categories between two categories $\mathcal{C}$ and $\mathcal{D}$ is given by a pair of functors $\mathcal{C} \xrightarrow{F} \mathcal{D} \xleftarrow{G} \mathcal{C}$ such that $G \circ F$ and $F \circ G$ are naturally isomorphic to $\mathrm{id}_{\mathcal{C}}$ and $\mathrm{id}_{\mathcal{D}}$, respectively.

Theorem 1.9.3 (see (47, Theorem 1, page 211))

Let $\mathcal{C}$ be an Abelian category. There exists an equivalence of categories $F : \mathcal{C} \to \mathbf{Mod}_R$ for some ring R if, and only if, $\mathcal{C}$ contains a progenerator P and arbitrary coproducts of copies of P. If F is an equivalence, then P may be chosen such that $\mathrm{Hom}_{\mathcal{C}}(P, P) \cong R$ and $F = \mathrm{Hom}_{\mathcal{C}}(P, -)$.

The equivalence F of the previous theorem is called Mitchell's equivalence.

To conclude this section, we present the existence of enough injective objects in every Grothendieck category. This can be found in the famous Grothendieck's Tohoku paper *Sur quelques points d'algèbre homologique*.

Theorem 1.9.4 (A. Grothendieck, see (32, Theorem 1.10.1))

If $\mathcal{C}$ is an Abelian category satisfying AB5 and admitting a generator, then for every object $Y \in \mathrm{Ob}(\mathcal{C})$ there exists a monomorphism $Y \hookrightarrow I$, for some injective object I.

The injective object I can be constructed to be functorial in Y. This is due to the fact that every Grothendieck category has an injective cogenerator, as mentioned in (1, page 379).

We know that a left R-module I is injective if, and only if, every morphism $J \to I$ defined on a left ideal J of R can be extended to all of R (Baer's Criterion). Another important result in the just cited Grothendieck's paper is a generalization of Baer's Criterion.

Theorem 1.9.5 (A. Grothendieck, see (32, Lemme 1, page 136))

If $\mathcal{C}$ is an AB5 Abelian category with a generator G, then an object Y of $\mathcal{C}$ is injective if, and only if, for every subobject U of G and for every morphism $U \to Y$, there exists a morphism $G \to Y$ such that $(G \to Y) \circ (U \hookrightarrow G) = U \to Y$.

The following result follows.

Corollary 1.9.6

Let $\mathcal{C}$ be a Grothendieck category with a generator G, and Y be an object of $\mathcal{C}$. The following are equivalent:

(1) Y is n-injective.

(2) $\mathrm{Ext}^{n+1}_{\mathcal{C}}(G/J, Y) = 0$ for every subobject J of G.

1.10 Modules over ringoids

Throughout this thesis, we shall be working mainly with the categories of left R-modules and complexes over left R-modules. These categories are particular examples of a categorical notion known as modules over ringoids. Some results concerning the relationship between projective dimensions and model structures are presented in the context of modules over ringoids, so we devote the last lines of this chapter to recall this concept.

Definition 1.10.1. We shall say that a small pre-additive category $\mathfrak{R}$ is a ring with many objects, or a ringoid.

If $\mathfrak{R}$ is a ringoid, then we have a composition law

$$\mathrm{Hom}_{\mathfrak{R}}(b,c) \otimes \mathrm{Hom}_{\mathfrak{R}}(a,b) \to \mathrm{Hom}_{\mathfrak{R}}(a,c)$$
$$y \otimes x \mapsto y \circ x$$

for every $a, b, c \in \mathrm{Ob}(\mathfrak{R})$, and a unit $\mathrm{id}_a \in \mathrm{Hom}_{\mathfrak{R}}(a,a)$ for every $a \in \mathrm{Ob}(\mathfrak{R})$. The composition law defines a ring structure on $\mathrm{Hom}_{\mathfrak{R}}(a,a)$ for every $a \in \mathrm{Ob}(\mathfrak{R})$. Moreover, the Abelian group $\mathrm{Hom}_{\mathfrak{R}}(a,b)$ has the structure of a bimodule, with a left action by $\mathrm{Hom}_{\mathfrak{R}}(b,b)$ and a right action by $\mathrm{Hom}_{\mathfrak{R}}(a,a)$.

Example 1.10.1.

(1) Every ring R can be regarded as a ringoid $\mathfrak{R}$ having a single object $\star$ if we put $\mathrm{Hom}_{\mathfrak{R}}(\star,\star) = R$.

(2) We shall denote by $\mathfrak{S}$ the ringoid generated by the following infinite graph

$$\begin{tikzcd} \cdots \arrow[r] & 2 \arrow[r, "\partial_2"] \arrow[loop above, "e_2"] & 1 \arrow[r, "\partial_1"] \arrow[loop above, "e_1"] & 0 \arrow[r, "\partial_0"] \arrow[loop above, "e_0"] & -1 \arrow[r, "\partial_{-1}"] \arrow[loop above, "e_{-1}"] & -2 \arrow[r] \arrow[loop above, "e_{-2}"] & \cdots \end{tikzcd}$$

together with the relation $\partial_n \circ \partial_{n+1} = 0$ for $n \in \mathbb{Z}$. We have

(i) $\mathrm{Ob}(\mathfrak{S}) = \mathbb{Z}$.

(ii) $\mathrm{Hom}_{\mathfrak{S}}(i,j) = \begin{cases} \langle e_i \rangle := \{m \cdot e_i \,:\, m \in \mathbb{Z}\} & \text{if } j = i, \\ \langle \partial_i \rangle := \{m \cdot \partial_i \,:\, m \in \mathbb{Z}\} & \text{if } j = i+1, \\ 0 & \text{otherwise.} \end{cases}$

(3) The <u>opposite</u> $\mathfrak{R}^{op}$ of a ringoid $\mathfrak{R}$ is defined by putting $\mathrm{Ob}(\mathfrak{R}^{\mathrm{op}}) = \mathrm{Ob}(\mathfrak{R})$ and $\mathrm{Hom}_{\mathfrak{R}^{\mathrm{op}}}(a,b) = \mathrm{Hom}_{\mathfrak{R}}(b,a)$.

We shall denote the category of additive functors between two pre-additive categories $\mathcal{C}$ and $\mathcal{D}$ by $[\mathcal{C},\mathcal{D}]$. It is known that if $\mathcal{D}$ is Abelian, complete or cocomplete, then so is $[\mathcal{C},\mathcal{D}]$.

Definition 1.10.2. A (left) module M over a ringoid $\mathfrak{R}$ is an additive functor $M : \mathfrak{R} \to \mathbf{Ab}$. A map of (left) $\mathfrak{R}$-modules is a natural transformation $M \xrightarrow{f} N$.

A right module over $\mathfrak{R}$ is defined to be a left $\mathfrak{R}^{op}$-module. In other words, a right $\mathfrak{R}$-module is a contravariant additive functor $\mathfrak{R} \to \mathbf{Ab}$.

Example 1.10.2. Given a ringoid $\mathfrak{R}$ and $a \in \mathrm{Ob}(\mathfrak{R})$, the covariant functor $\mathrm{Hom}_{\mathfrak{R}}(a, -) : \mathfrak{R} \to \mathbf{Ab}$ is a (left) $\mathfrak{R}$-module and the contravariant functor $\mathrm{Hom}_{\mathfrak{R}}(-, a) : \mathfrak{R} \to \mathbf{Ab}$ is a right $\mathfrak{R}$-module.

We shall denote the category of left $\mathfrak{R}$-modules by $\mathbf{Mod}(\mathfrak{R})$. Note that this category is Abelian, complete and cocomplete, since $\mathbf{Ab}$ is.

Remark 1.10.1. A sequence of $\mathfrak{R}$-modules $\cdots \to M_1 \to M_0 \to M_{-1} \to \cdots$ is exact if the sequence of Abelian groups $\cdots \to M_1(a) \to M_0(a) \to M_{-1}(a) \to \cdots$ is exact for every $a \in \mathrm{Ob}(\mathfrak{R})$.

Definition 1.10.3. If M is a $\mathfrak{R}$-module, we say that the direct sum $\bigoplus_{a \in \mathrm{Ob}(\mathfrak{R})} M(a)$ is the total group of M. We shall say that an element $x \in M(a)$ is homogenous of grade a and we shall write $a = |x|$.

If M is a left module over a ringoid $\mathfrak{R}$, then the map $\mathrm{Hom}_{\mathfrak{R}}(a, b) \to \mathrm{Hom}_{\mathbf{Ab}}(M(a), M(b))$ of Abelian groups defined by M induces a multiplication

$$\begin{aligned} \mathrm{Hom}_{\mathfrak{R}}(a, b) \otimes M(a) &\to M(b) \\ (r, x) &\mapsto r \cdot x := M(r)(x) \end{aligned}$$

for every $a, b \in \mathrm{Ob}(\mathfrak{R})$. The product of $r \in \mathrm{Hom}_{\mathfrak{R}}(a, b)$ by $x \in M(a)$ is an element $r \cdot x \in M(b)$.

Definition 1.10.4. We shall say that a linear combination of homogenous elements $y = \sum_{i \in I} r_i \cdot x_i$ is admissible if y is homogenous and $r_i \in \mathrm{Hom}_{\mathfrak{R}}(|x_i|, |y|)$ for every $i \in I$. We will accept infinite combinations in the case where $r_i = 0$ for all but finitely many $i \in I$.

Definition 1.10.5. If M is a left $\mathfrak{R}$-module we shall say that a family $N = \{N(a) : a \in \mathrm{Ob}(\mathfrak{R})\}$ of subgroups $N(a) \subseteq M(a)$ is a submodule if $x \in N(a)$ implies $r \cdot x \in N(b)$ for every $r \in \mathrm{Hom}_{\mathfrak{R}}(a, b)$.

Remark 1.10.2. Note that the family $N = \{N(a) : a \in \mathrm{Ob}(\mathfrak{R})\}$ in the previous definition defines a functor $N : \mathfrak{R} \to \mathbf{Ab}$. Let $r : a \to b$ be a map in $\mathfrak{R}$. For every $x \in N(a)$, we have $M(r)(x) = r \cdot x \in N(b)$. Then we define $N(r)$ as the restriction $M(r)|_{N(a)}$. Conversely, if N is a subfunctor of M, then $r \cdot x \in N(b)$ for every $r \in \mathrm{Hom}_{\mathfrak{R}}(a, b)$ and every $x \in N(a)$.

We finish this section by presenting a generalized version of Theorem 1.9.3, which states that any Abelian category satisfying certain conditions is equivalent to the category of right modules over certain ringoid.

Let $\mathcal{G}$ be a set of finite projective generators of a cocomplete Abelian category $\mathcal{C}$. Let $\mathrm{End}_{\mathcal{C}}(\mathcal{G})$ denote the full subcategory of $\mathcal{C}$ with $\mathrm{Ob}(\mathrm{End}_{\mathcal{C}}(\mathcal{G})) = \mathcal{G}$. Note that $\mathrm{End}_{\mathcal{C}}(\mathcal{G})$ is a ringoid. Let $L : \mathcal{C} \to [\mathrm{End}_{\mathcal{C}}(\mathcal{G})^{\mathrm{op}}, \mathbf{Ab}]$ be the functor defined as follows:

(1) If $X \in \mathrm{Ob}(\mathcal{C})$, then $L(X)(G) = \mathrm{Hom}_{\mathcal{C}}(G, X)$, for every $G \in \mathcal{G}$.

(2) If $X \xrightarrow{f} Y$ is an arrow in $\mathcal{C}$, then $L(f) : L(X) \to L(Y)$ is the natural transformation defined by putting

$$L(f)_G = \mathrm{Hom}_{\mathcal{C}}(G, f) : \mathrm{Hom}_{\mathcal{C}}(G, X) \to \mathrm{Hom}_{\mathcal{C}}(G, Y)$$

for every $G \in \mathcal{G}$.

Theorem 1.10.1 (P. Freyd. See (18, Theorem 7.1))

Let $\mathcal{C}$ be a cocomplete Abelian category with a set of finite projective generators $\mathcal{G}$. Then the functor L defined above is an equivalence of categories.

CHAPTER II

MODEL CATEGORIES AND RELATIVE HOMOLOGICAL ALGEBRA

"Monsters are real, and ghosts are real too. They live inside us, and sometimes, they win."

Stephen King.

This chapter is devoted to study the connection between model categories and cotorsion theories. The notion of model categories was introduced by Daniel Quillen in 1967 (51). Roughly speaking, these are categories equipped with three distinguished classes of morphisms called weak equivalences, cofibrations and fibrations, which allow us to do homotopy theory. The triplet formed by these three classes is known as a model structure on the given category. Problably the most well-known example of a model category is the category of topological spaces **Top**, where the weak equivalences are given by weak homotopy equivalences of continuous functions (Details can be found in (51)).

In this work we deal with a particular class of model categories, called Abelian model categories. In 2002, Mark Hovey established criteria to construct Abelian model structures from two compatible and functorially complete cotorsion pairs, and viceversa (35). Roughly speaking, cotorsion pairs are given by two classes of objects in an Abelian category which are orthogonal to each other with respect to the 1st extension functor $\mathrm{Ext}^1_{\mathcal{C}}(-,-)$. The theory of complete cotorsion pairs was first introduced by Luigi Salce in 1977 is his paper *Cotorsion Theories for Abelian Groups* (53). This notion was rediscovered by Edgar E. Enochs in 2000 in the category of modules over a ring, turning out to be an important tool to prove the existence of a flat cover for every module. About 2008, James Gillespie started to develop several methods to induce cotorsion pairs in chain complexes over an Abelian category $\mathcal{C}$, from a cotorsion pair in $\mathcal{C}$ satisfying certain conditions (see (25), (27) and (26)). We shall present proofs of these results, based on Gillespie arguments, but some of them with slight modifications and remarks.

2.1 Weak factorization systems and model structures

We shall give the formal definition of a model category and present some examples. We shall use the concept stated by Hovey in (36), which differs slightly to the one originally given by Quillen in (51). In order to understand better the notion of a model structure, we first present the concept of a weak factorization system. The purpose for doing so is to give the proof of Hovey's Correspondence in a more understandable way.

Definition 2.1.1. Let $X \xrightarrow{f} Y$ and $W \xrightarrow{g} Z$ be two morphisms in a category $\mathcal{C}$. We shall say that f has the left lifting property with respect to g if for every equality $g \circ u = v \circ f$, there exists a map $Y \xrightarrow{d} W$ such that $d \circ f = u$ and $g \circ d = v$. One may also say that g has the right lifting property with respect to f.

The commutative square below is known as a lifting problem and d is called a solution or a diagonal filler. We write $f \pitchfork g$ if f has the left lifting property with respect to g.

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & W \arrow[d, "g"] \\ Y \arrow[r, "v"'] & Z \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & W \arrow[d, "g"] \\ Y \arrow[r, "v"'] \arrow[ur, dotted, "\exists\, d"] & Z \end{tikzcd}$$

Figure 2.1: Left lifting property.

If $\mathcal{M}$ is a class of morphisms of $\mathcal{C}$, we shall say that f has the left lifting property with respect to $\mathcal{M}$ if it has the left lifting property with respect to every $g \in \mathcal{M}$. We shall denote by ${}^{\pitchfork}\mathcal{M}$ the class of all morphisms having the left lifting property with respect to $\mathcal{M}$. Similarly, $\mathcal{M}^{\pitchfork}$ shall denote the class of all morphisms having the right lifting property with respect to $\mathcal{M}$.

Definition 2.1.2. A weak factorization system on a category $\mathcal{C}$ is a pair $(\mathcal{L}, \mathcal{R})$ formed by two classes of morphisms $\mathcal{L}$ and $\mathcal{R}$ of $\mathcal{C}$ such that:

(1) Lifting axiom: $\mathcal{L} = {}^{\pitchfork}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\pitchfork}$.

(2) Factorization axiom: For every morphism $X \xrightarrow{f} Y$ in $\mathcal{C}$, there exists $l \in \mathcal{L}$ and $r \in \mathcal{R}$ such that the following triangle commutes:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "l"'] & & Y \\ & Z \arrow[ur, "r"'] & \end{tikzcd}$$

Figure 2.2: Factorization axiom.

Condition **(1)** in the previous definition can be difficult to check in several cases. The notions of domain and codomain retracts provide an easier way to verify that a given pair of classes of morphisms in $\mathcal{C}$ form a weak factorization system.

Definition 2.1.3.

A map $X \xrightarrow{u} Z$ is a <u>domain retract</u> of $Y \xrightarrow{v} Z$ if there exist maps $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} X$ such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[rr, bend left, equal] \arrow[dr, "u"'] & Y \arrow[r, "g"] \arrow[d, "v"] & X \arrow[dl, "u"] \\
& Z &
\end{tikzcd}$$

A map $Z \xrightarrow{u} X$ is a <u>codomain retract</u> of $Z \xrightarrow{v} Y$ if there exist maps $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} X$ such that the following diagram commutes:

$$\begin{tikzcd}
& Z \arrow[dl, "u"'] \arrow[d, "v"] \arrow[dr, "u"] & \\
X \arrow[r, "f"'] \arrow[rr, bend right, equal] & Y \arrow[r, "g"'] & X
\end{tikzcd}$$

Figure 2.3: Domain and codomain retracts.

A class $\mathcal{M}$ of maps in $\mathcal{C}$ is said to be:

<u>closed under domain retracts</u> if every domain retract of a map in $\mathcal{M}$ belongs to $\mathcal{M}$.

<u>closed under codomain retracts</u> if every codomain retract of a map in $\mathcal{M}$ belongs to $\mathcal{M}$.

Theorem 2.1.1 (see (39, Weak factorization systems, Theorem 2))

A pair $(\mathcal{L}, \mathcal{R})$ of classes of maps in a category $\mathcal{C}$ is a weak factorization system if, and only if, the following conditions are satisfied:

(1) $l \pitchfork r$ for every $l \in \mathcal{L}$ and $r \in \mathcal{R}$.

(2) The class $\mathcal{L}$ is closed under codomain retracts, and the class $\mathcal{R}$ under domain retracts.

(3) Every map $X \xrightarrow{f} Y$ admits a factorization $f = r \circ l$ with $l \in \mathcal{L}$ and $r \in \mathcal{R}$.

In some cases, factorizations in a weak factorization system are assumed to be functorial.

Definition 2.1.4. Given a category $\mathcal{C}$, let $\mathbf{Map}(\mathcal{C})$ denote the category whose objects are the morphisms of $\mathcal{C}$, and whose maps $(u,v) : (X \xrightarrow{f} Y) \to (W \xrightarrow{g} Z)$ are commutative squares

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & W \arrow[d, "g"] \\ Y \arrow[r, "v"'] & Z \end{tikzcd}$$

A functorial factorization is an ordered pair (F,G) of functors $\mathbf{Map}(\mathcal{C}) \to \mathbf{Map}(\mathcal{C})$ such that every morphism $X \xrightarrow{f} Y$ of $\mathcal{C}$ can be written as $f = G(f) \circ F(f)$.

Definition 2.1.5. A functorial weak factorization system on a category $\mathcal{C}$ is a pair $(\mathcal{L},\mathcal{R})$ formed by two classes of morphisms $\mathcal{L}$ and $\mathcal{R}$ of $\mathcal{C}$ such that:

(1) Lifting axiom: $\mathcal{L} = {}^{\pitchfork}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\pitchfork}$.

(2) Factorization axiom: $\mathcal{C}$ is equipped with a functorial factorization (L,R) such that for every morphism $X \xrightarrow{f} Y$, $L(f) \in \mathcal{L}$ and $R(f) \in \mathcal{R}$.

In most of the definitions appearing in the literature, factorizations do not need to be functorial. However, we shall work with the previous definition in order to establish Hovey's correspondence.

Example 2.1.1. The following are examples of weak factorization systems:

(1) In **Set**: $\mathcal{L} = \mathbf{Surj} :=$ {surjective functions} and $\mathcal{R} = \mathbf{Inj} :=$ {injective functions}.

(2) In **Grp**: $\mathcal{L} = \mathbf{Epi} :=$ {epimorphisms} and $\mathcal{R} = \mathbf{Mono} :=$ {monomorphisms}.

(3) In any category $\mathcal{C}$: $(\mathbf{Mor}(\mathcal{C}), \mathbf{Iso}(\mathcal{C}))$ and $(\mathbf{Iso}(\mathcal{C}), \mathbf{Mor}(\mathcal{C}))$ are weak factorization systems, where $\mathbf{Iso}(\mathcal{C})$ is the class of isomorphisms.

The three examples above are in fact factorization systems.

(4) In **Set**, the classes $\mathcal{L} := \mathbf{Inj}$ and $\mathcal{R} := \mathbf{Surj}$ form a weak factorization system where factorizations of functions are not necessarily unique. This is mentioned in (7, Example 3.-1.).

Definition 2.1.6. Two classes of morphism $\mathcal{L}$ and $\mathcal{R}$ in a category $\mathcal{C}$ form a <u>factorization system</u> $(\mathcal{L}, \mathcal{R})$ if the following two conditions are satisfied:

(1) The classes $\mathcal{L}$ and $\mathcal{R}$ contain $\mathbf{Iso}(\mathcal{C})$ and are closed under compositions.

(2) <u>Unique factorizations</u>: Every map $X \xrightarrow{f} Y$ admits a factorization $f = r \circ l$ with $l \in \mathcal{L}$ and $r \in \mathcal{R}$ such that this factorization is unique up to isomorphisms. That is, if $f = r' \circ l'$ is another factorization with $l' \in \mathcal{L}$ and $r' \in \mathcal{R}$, then there exists an isomorphism φ such that the following diagram commutes:

$$\begin{tikzcd}
 & Z \arrow[dr, "r"] \arrow[dd, "\varphi", "\cong"'] & \\
X \arrow[ur, "l"] \arrow[dr, "l'"'] & & Y \\
 & Z' \arrow[ur, "r'"'] &
\end{tikzcd}$$

Definition 2.1.7. Let $X \xrightarrow{f} Y$ be a morphism in a category $\mathcal{C}$.

If pullbacks exist in $\mathcal{C}$, the <u>diagonal</u> of f is the only morphism $\delta(f) : X \to X \times_Y X$ such that the diagram

$$\begin{tikzcd}
X \arrow[dr, "\delta(f)"] \arrow[drr, bend left, equal] \arrow[ddr, bend right, equal] & & \\
 & X \times_Y X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & X \arrow[d, "f"] \\
 & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

commutes.

A class of maps $\mathcal{M}$ of $\mathcal{C}$ is said to be <u>closed under diagonals</u> if $f \in \mathcal{M} \Rightarrow \delta(f) \in \mathcal{M}$.

If pushouts exist in $\mathcal{C}$, the <u>codiagonal</u> of f is the only morphism $\delta^o(f) : Y \coprod_X Y \to Y$ such that the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "f"'] & Y \arrow[d, "\iota_1"] \arrow[ddr, bend left, equal] & \\
Y \arrow[r, "\iota_2"'] \arrow[drr, bend right, equal] & Y \coprod_X Y \arrow[dr, "\delta^o(f)"] & \\
 & & Y
\end{tikzcd}$$

commutes.

A class of maps $\mathcal{M}$ of $\mathcal{C}$ is said to be <u>closed under codiagonals</u> if $f \in \mathcal{M} \Rightarrow \delta^o(f) \in \mathcal{M}$.

Proposition 2.1.2 (see (39, Factorization systems, Propositions 3 and 4))

In a category $\mathcal{C}$ with pullbacks, a weak factorization system $(\mathcal{L}, \mathcal{R})$ on $\mathcal{C}$ is a factorization system if, and only if, the class $\mathcal{R}$ is closed under diagonals.

In a category $\mathcal{C}$ with pushouts, a weak factorization system $(\mathcal{L}, \mathcal{R})$ on $\mathcal{C}$ is a factorization system if, and only if, the class $\mathcal{L}$ is closed under codiagonals.

Definition 2.1.8. Let $X \xrightarrow{f} Y$ and $W \xrightarrow{g} Z$ be morphisms in a category $\mathcal{C}$.

If pullbacks exist in $\mathcal{C}$, the base change of g along a map $Y \xrightarrow{v} Z$ is the map $Y \times_Z W \to Y$ in the pullback square

$$\begin{tikzcd} Y \times_Z W \arrow[r] \arrow[d] & W \arrow[d, "g"] \\ Y \arrow[r, "v"'] & Z \end{tikzcd}$$

A class of maps $\mathcal{M}$ of $\mathcal{C}$ is said to be closed under base change if the base change of each map in $\mathcal{M}$ belongs to $\mathcal{M}$.

If pushouts exist in $\mathcal{C}$, the cobase change of f along a map $X \xrightarrow{u} W$ is the map $W \xrightarrow{g} Y \coprod_X W$ in the pushout square

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & W \arrow[d] \\ Y \arrow[r] & Y \coprod_X W \end{tikzcd}$$

A class of maps $\mathcal{M}$ of $\mathcal{C}$ is said to be closed under cobase change if the cobase change of each map in $\mathcal{M}$ belongs to $\mathcal{M}$.

Definition 2.1.9. We shall say that a class of maps $\mathcal{M}$ in a category $\mathcal{C}$ is closed under retracts if the following implication is true: f is a retract of g in the category $\mathrm{Map}(\mathcal{C})$ and $g \in \mathcal{M} \implies f \in \mathcal{M}$.

The following result gives some closure properties of the classes $\mathcal{M}^{\pitchfork}$ and ${}^{\pitchfork}\mathcal{M}$. The details can be found in (39, Weak Factorization Systems, Lemma 2).

Proposition 2.1.3

Let $\mathcal{M}$ be a class of morphisms in a category $\mathcal{C}$.

If $\mathcal{C}$ has products and pullbacks (i.e. $\mathcal{C}$ is complete), then $\mathcal{M}^{\pitchfork}$ contains the isomorphisms and is closed under composition, retractions, products and base changes.

If $\mathcal{C}$ has coproducts and pushouts (i.e. $\mathcal{C}$ is cocomplete), then ${}^{\pitchfork}\mathcal{M}$ contains the isomorphisms and is closed under composition, retractions, coproducts and cobase changes.

Lemma 2.1.4

Suppose $\mathcal{C}$ is an Abelian category with $f : X \to Y$ monic and $g : W \to Z$ epic. If $\mathrm{Ext}^1_{\mathcal{C}}(\mathrm{CoKer}(f), \mathrm{Ker}(g)) = 0$ then $f \pitchfork g$.

$\boxed{\textbf{Proof}}$.

Suppose we are given a commutative diagram

$$\begin{tikzcd}
 & X \arrow[r, hook, "f"] \arrow[d, "\alpha"] & Y \arrow[r, two heads, "c"] \arrow[d, "\beta"] & \mathrm{CoKer}(f) \\
\mathrm{Ker}(g) \arrow[r, hook, "k"'] & W \arrow[r, two heads, "g"'] & Z &
\end{tikzcd}$$

We want to find a diagonal filler $d : Y \to W$ such that $d \circ f = \alpha$ and $g \circ d = \beta$.

We have the following commutative grid (Corollary 1.5.1):

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Hom}(\mathrm{CoKer}(f),\mathrm{Ker}(g)) \arrow[r] \arrow[d] & \mathrm{Hom}(Y,\mathrm{Ker}(g)) \arrow[r] \arrow[d] & \mathrm{Hom}(X,\mathrm{Ker}(g)) \arrow[r] \arrow[d] & \cancelto{0}{\mathrm{Ext}^1(\mathrm{CoKer}(f),\mathrm{Ker}(g))} \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \mathrm{Hom}(\mathrm{CoKer}(f),W) \arrow[r] \arrow[d] & \mathrm{Hom}(Y,W) \arrow[r] \arrow[d] & \mathrm{Hom}(X,W) \arrow[r, "\delta"] \arrow[d] & \mathrm{Ext}^1(\mathrm{CoKer}(f),W) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \mathrm{Hom}(\mathrm{CoKer}(f),Z) \arrow[r] \arrow[d] & \mathrm{Hom}(Y,Z) \arrow[r] \arrow[d] & \mathrm{Hom}(X,Z) \arrow[r] \arrow[d] & \mathrm{Ext}^1(\mathrm{CoKer}(f),Z) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \cancelto{0}{\mathrm{Ext}^1(\mathrm{CoKer}(f),\mathrm{Ker}(g))} \arrow[r] \arrow[d] & \mathrm{Ext}^1(Y,\mathrm{Ker}(g)) \arrow[r] \arrow[d] & \mathrm{Ext}^1(X,\mathrm{Ker}(g)) \arrow[r] \arrow[d] & \mathrm{Ext}^2(\mathrm{CoKer}(f),\mathrm{Ker}(g)) \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

We begin with $\alpha \in \mathrm{Hom}_{\mathcal{C}}(X,W)$. We have a commutative diagram

$$\begin{tikzcd}
 & \alpha \arrow[r, mapsto] \arrow[d, mapsto] & \delta(\alpha) \arrow[d, mapsto] \\
\beta \arrow[r, mapsto] & g\circ\alpha = \beta\circ f \arrow[r, mapsto] & 0
\end{tikzcd}$$

Since the map $\mathrm{Ext}^1_{\mathcal{C}}(\mathrm{CoKer}(f),W) \to \mathrm{Ext}^1_{\mathcal{C}}(\mathrm{CoKer}(f),Z)$ is monic, we have $\delta(\alpha) = 0$. Then by exactness there exists a map $d^0 : Y \to W$ such that $\alpha = d^0 \circ f$. Consider the map $\beta - g\circ d^0 : Y \to Z$. Then $\beta - g\circ d^0 \mapsto \beta\circ f - g\circ d^0\circ f = \beta\circ f - g\circ\alpha = 0$, and so by exactness there exists a map $l : \mathrm{CoKer}(f) \to Z$ such that $\beta - g\circ d^0 = l\circ c$. Since $\mathrm{Hom}_{\mathcal{C}}(\mathrm{CoKer}(f),W) \to \mathrm{Hom}_{\mathcal{C}}(\mathrm{CoKer}(f),Z)$ is epic, there exists a map $l' : \mathrm{CoKer}(f) \to W$ such that $l = g\circ l'$. Set $d := d^0 + l'\circ c : Y \to W$, we have:

$$d\circ f = d^0\circ f + l'\circ c\circ f = \alpha + 0 = \alpha,$$
$$g\circ d = g\circ d^0 + g\circ l'\circ c = g\circ d^0 + l\circ c = g\circ d^0 + \beta - g\circ d^0 = \beta.$$

Hence, $f \pitchfork g$. $\square$

We introduce the following notation for a weak factorization system $(\mathcal{L}, \mathcal{R})$:

$$\mathrm{Ker}(\mathcal{R}) := \{\mathrm{Ker}(r) : r \in \mathcal{R}\} \text{ and } \mathrm{CoKer}(\mathcal{L}) := \{\mathrm{CoKer}(l) : l \in \mathcal{L}\}.$$

Lemma 2.1.5

Let $\mathcal{C}$ be an Abelian category equipped with a weak factorization system $(\mathcal{L}, \mathcal{R})$ such that $\mathcal{L} \subseteq \mathbf{Mono}(\mathcal{C})$ and $\mathcal{R} \subseteq \mathbf{Epi}(\mathcal{C})$. The following conditions are equivalent:

(1) l is monic and $\mathrm{CoKer}(l) \in \mathrm{CoKer}(\mathcal{L}) \implies l \in \mathcal{L}$.

(2) r is epic and $\mathrm{Ker}(r) \in \mathrm{Ker}(\mathcal{R}) \implies r \in \mathcal{R}$.

(3) $\mathrm{Ext}^1_{\mathcal{C}}(A, X) = 0$ for every $A \in \mathrm{CoKer}(\mathcal{L})$ and $X \in \mathrm{Ker}(\mathcal{R})$.

Proof.

We only prove **(1)** $\iff$ **(3)**, since **(2)** $\iff$ **(3)** is dual.

(1) $\implies$ **(3)**: Let $A \in \mathrm{CoKer}(\mathcal{L})$ and $X \in \mathrm{Ker}(\mathcal{R})$. Consider a representative $0 \to X \to Y \to A \to 0$ of a class in $\mathrm{Ext}^1_{\mathcal{C}}(A, X)$. We show this sequence splits. On the one hand, by **(1)**, $f : X \to Y$ is in $\mathcal{L}$. On the other hand, there exists $r \in \mathcal{R}$ such that $X = \mathrm{Ker}(r)$. Since r is epic, we have a short exact sequence $0 \to X \to W \xrightarrow{r} Z \to 0$. It is not hard to see that X is the pullback of the maps $0 \to Z$ and r. We have a pullback square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & W \arrow[d, "r \in \mathcal{R}"] \\ 0 \arrow[r] & Z \end{tikzcd}$$

So $X \to 0$ is the base change of r along $0 \to Z$. By Proposition 2.1.3, we have $(X \to 0) \in \mathcal{R}$. It follows we have a commutative square

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "\mathcal{L} \ni f"'] & X \arrow[d, "\in \mathcal{R}"] \\ Y \arrow[r] & 0 \end{tikzcd}$$

So there is a map $d : Y \to X$ such that $d \circ f = \mathrm{id}_X$, i.e. the short exact sequence $0 \to X \to Y \to A \to 0$ splits.

(1) $\Longleftarrow$ **(3)**: Let f be a monic map such that $\mathrm{CoKer}(f) \in \mathrm{CoKer}(\mathcal{L})$. Let $r \in \mathcal{R}$. Then r is epic by hypothesis, with $\mathrm{Ker}(r) \in \mathrm{Ker}(\mathcal{R})$. Since $\mathrm{Ext}^1_{\mathcal{C}}(\mathrm{CoKer}(l), \mathrm{Ker}(r))$ is 0 by **(3)**, we have $l \pitchfork r$ by Lemma 2.1.4. Since $l \pitchfork r$ for every $r \in \mathcal{R}$, we have $l \in {}^{\pitchfork}\mathcal{R} = \mathcal{L}$. $\square$

Definition 2.1.10. Let $\mathcal{C}$ be a category and $\mathcal{C}_{of}$, $\mathcal{F}_{ib}$ and $\mathcal{W}_{eak}$ be three classes of morphisms called **cofibrations** ($\hookrightarrow$), **fibrations** ($\twoheadrightarrow$), and **weak equivalences** ($\xrightarrow{\sim}$), respectively. We also call $\mathcal{C}_{of} \cap \mathcal{W}_{eak}$ the class of trivial or acyclic cofibrations and $\mathcal{F}_{ib} \cap \mathcal{W}_{eak}$ the class of trivial or acyclic fibrations. The triple $(\mathcal{C}_{of}, \mathcal{F}_{ib}, \mathcal{W}_{eak})$ is said to be a model structure on $\mathcal{C}$ if the following conditions are satisfied:

(1) 3 × 2 axiom [1]: Let f and g be two morphisms such that the composition $g \circ f$ makes sense. If two out of three of the morphisms f, g and $g \circ f$ are weak equivalences, then so is the third.

$$\begin{tikzcd} X \arrow[r, "\sim"'] \arrow[rr, bend left, "\sim"] & Y \arrow[r] & Z \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} X \arrow[r, "\sim"'] \arrow[rr, bend left, "\sim"] & Y \arrow[r, "\sim"'] & Z \end{tikzcd}$$

Figure 2.4: One of the possible cases of the 3×2 axiom.

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1. Better known as the two-out-of-three axiom in the literature (for instance (36)). We prefer to call this axiom as A. Joyal does in (39, Model categories): "*This is like getting three apples for the price of two in a food store*".

(2) $(\mathcal{C}_{of} \cap \mathcal{W}_{eak}, \mathcal{F}_{ib})$ and $(\mathcal{C}_{of}, \mathcal{F}_{ib} \cap \mathcal{W}_{eak})$ are weak factorization systems.

Definition 2.1.11. A model category is a bicomplete[2] category equipped with a model structure.

The original definition given by Daniel Quillen in (51) only requires finite limits and colimits to exist, and it also drops the adjective functorial in the factorization axiom. At a first glance, the difference between finite and infinite (co)limits is not very important. The examples considered in this work correspond to bicomplete categories. However, we stress the functoriality in the factorization axiom, since this is a key thing to establish Hovey's Correspondence.

Some references, for example (36), add a third axiom in the previous definition, known as the retract axiom, which states that the classes $\mathcal{C}_{of}$, $\mathcal{F}_{ib}$, and $\mathcal{W}_{eak}$ are closed under retracts. However, this condition is true for $\mathcal{C}_{of}$ and $\mathcal{F}_{ib}$ since $(\mathcal{C}_{of}, \mathcal{F}_{ib} \cap \mathcal{W}_{eak})$ and $(\mathcal{C}_{of} \cap \mathcal{W}_{eak}, \mathcal{F}_{ib})$ are weak factorization systems. The fact that $\mathcal{W}_{eak}$ is closed under retracts is a result known as Tierney's Lemma (due to Myles Tierney), and the reader can check the details in (39, Model categories, Lemma 1).

Proposition 2.1.6

Let $\mathcal{C}$ be a bicomplete category equipped with a model structure $(\mathcal{C}_{of}, \mathcal{F}_{ib}, \mathcal{W}_{eak})$. A morphism $X \xrightarrow{f} Y$ is a weak equivalence if, and only if, it is the composition of a trivial cofibration followed by a trivial fibration.

Proof.

Suppose f is a weak equivalence. By the factorization axiom, f can be factored

2. A category $\mathcal{C}$ is bicomplete if it is complete and cocomplete.

as $f = r \circ l$ where l is a trivial cofibration and r is a fibration. By the 3×2 axiom, we have r is a trivial fibration. Conversely, if f is the composition of a trivial cofibration followed by a trivial fibration, then f is a weak equivalence by the 3×2 axiom. □

Remark 2.1.1. Note that in a model category $\mathcal{C}$, any of the two classes $\mathcal{C}_{of}$ and $\mathcal{F}_{ib}$ is determined by the remaining two. For example, $\mathcal{C}_{of} = {}^{\pitchfork}(\mathcal{F}_{ib} \cap \mathcal{W}_{eak})$.

Example 2.1.2. We give some classical examples of model categories.

(1) Every bicomplete category is equipped with a model structure. Namely, $\mathcal{W}_{eak} := \mathbf{Iso}(\mathcal{C})$ and $\mathcal{C}_{of} = \mathcal{F}_{ib} := \mathbf{Mor}(\mathcal{C})$.

(2) If $\mathcal{C}$ is equipped with a model structure, then so is $\mathcal{C}^{\mathrm{op}}$. The cofibrations, fibrations and weak equivalences of $\mathcal{C}^{\mathrm{op}}$ are the fibrations, cofibrations and weak equivalences of $\mathcal{C}$, respectively.

(3) Two model structures on **Top**: Let I be the unit interval $I = [0,1]$ and $I^n = [0,1]\times\cdots\times[0,1]$ (n times). For $n = 0$, I^0 is a point. Recall that two continuous functions $f, g : X \to Y$ between topological spaces are said to be homotopic (denoted $f \sim g$) if there exists a continuous function $H : X \times I \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$, for every $x \in X$. It is not hard to see that $\sim$ defines an equivalence relation. Let ∂I^n denote the boundary of I^n. Fix a point x in a topological space X. By a continuous map $f : (I^n, \partial I^n) \to (X, x)$ we mean $f(s) = x$ for every $s \in \partial I^n$. The nth homotopy group of X at x is defined by

$$\pi_n(X,x) := \{f : (I^n, \partial I^n) \to (X,x) : f \text{ is continuous}\}/\sim .$$

Every continuous function $X \xrightarrow{g} Y$ induces a group homomorphism

$$\pi_n(X,x) \xrightarrow{\pi_n(g,x)} \pi_n(Y, g(x))$$

given by $f \mapsto g \circ f$, for every $x \in X$. A morphism g in **Top** is called a weak homotopy equivalence if $\pi_n(g, x)$ is a group isomorphism for every $n \geq 0$ and every $x \in X$ (Note $\pi_0(X)$ is the set of connected components of X). The category **Top** is equipped with the following model structure:

- The weak equivalences are given by the weak homotopy equivalences.
- The class of fibrations is formed by the Serre fibrations, i.e. maps which have the right lifting property with respect to inclusions of the form $D^n \hookrightarrow D^n \times I$ that include the n-disk D^n as $D^n \times \{0\}$.
- The class of cofibrations is given by retract of relative CW complexes.

This model structure is known as the Quillen model structure or q-model structure on **Top**. Details can be found in (36, Section 2.4) or in (51, Chapter II, Section 3).

There is another model structure on Top, known as the Hurewicz model structure or Strøm model structure (see (55) for details):

- Weak equivalences are given by the homotopy equivalences. Recall that a continuous function $f : X \to Y$ is said to be a homotopy equivalence if there exists a continuous function $g : Y \to X$ such that $g \circ f \sim \mathrm{id}_X$ and $f \circ g \sim \mathrm{id}_Y$.
- Fibrations are given by the Hurewicz fibrations, i.e. continuous maps which have the right lifting property with respect to all inclusions of the form $A \hookrightarrow A \times I$ for any space A.
- The class of cofibrations is the class of continuous functions having the left lifting property with respect to Hurewicz fibrations which are also homotopy equivalences.

2.2 Cotorsion pairs

Throughout this section, $\mathcal{C}$ shall denote an Abelian category.

Definition 2.2.1. Let $\mathcal{A}$ be a class of objects in an Abelian category $\mathcal{C}$ with enough projective or injective objects. The classes

$$\begin{aligned} {}^{\perp}\mathcal{A} &:= \{X \in \mathrm{Ob}(\mathcal{C}) : \mathrm{Ext}^1_{\mathcal{C}}(X, A) = 0, \text{ for every } A \in \mathcal{A}\}, \text{ and} \\ \mathcal{A}^{\perp} &:= \{Y \in \mathrm{Ob}(\mathcal{C}) : \mathrm{Ext}^1_{\mathcal{C}}(A, Y) = 0, \text{ for every } A \in \mathcal{A}\} \end{aligned}$$

are known as the left and right orthogonal classes of $\mathcal{A}$, respectively. Two classes $\mathcal{A}, \mathcal{B} \subseteq \mathrm{Ob}(\mathcal{C})$ form a cotorsion pair $(\mathcal{A}, \mathcal{B})$ if $\mathcal{A} = {}^{\perp}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp}$.

The main purpose of this section is to prove Eklof and Trlijaf's Theorem. It states that from every set of modules one can construct a complete cotorsion pair. The author does not know a reference to find a proof of this result for any Grothendieck category. We shall give a proof in this categorical setting, based on some arguments appearing in (21). One of the results we need to prove this theorem is called Eklof's Lemma, which is proven in (21) for the category of modules.

Example 2.2.1.

(1) If $\mathcal{P}_0(\mathcal{C})$ and $\mathcal{I}_0(\mathcal{C})$ denote the classes of projective and injective objects in $\mathcal{C}$, respectively, then $(\mathcal{P}_0(\mathcal{C}), \mathrm{Ob}(\mathcal{C}))$ and $(\mathrm{Ob}(\mathcal{C}), \mathcal{I}_0(\mathcal{C}))$ are cotorsion pairs, known as the trivial cotorsion pairs.

(2) Consider the class $\mathcal{F}_0$ of flat left R-modules. We shall see later that $(\mathcal{F}_0, (\mathcal{F}_0)^{\perp})$ is a cotorsion pair, known as the Enochs cotorsion pair.

(3) For every class $\mathcal{S}$ of objects of $\mathcal{C}$, it is not hard to see that $\mathcal{S}^{\perp} = ({}^{\perp}(\mathcal{S}^{\perp}))^{\perp}$ and ${}^{\perp}\mathcal{S} = {}^{\perp}(({}^{\perp}\mathcal{S})^{\perp})$. It follows $({}^{\perp}(\mathcal{S}^{\perp}), \mathcal{S}^{\perp})$ and $({}^{\perp}\mathcal{S}, ({}^{\perp}\mathcal{S})^{\perp})$ are cotorsion pairs in $\mathcal{C}$ (see (21, Page 152)).

Remark 2.2.1. Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in $\mathcal{C}$:

(1) The classes $\mathcal{A}$ and $\mathcal{B}$ are closed under extensions. For if $A' \hookrightarrow A \twoheadrightarrow A''$ is a short exact sequence with $A', A'' \in \mathcal{A}$ and $B \in \mathcal{B}$, then after deriving $\mathrm{Hom}_{\mathcal{C}}(-, B)$ we obtain a long exact sequence of groups

$$\cdots \to \cancelto{0}{\mathrm{Ext}^1_{\mathcal{C}}(A'', B)} \to \mathrm{Ext}^1_{\mathcal{C}}(A, B) \to \cancelto{0}{\mathrm{Ext}^1_{\mathcal{C}}(A', B)} \to \cdots$$

It follows $\mathrm{Ext}^1(A, B) = 0$ for every $B \in \mathcal{B}$. Similarly, $\mathcal{B}$ is also closed under extensions.

(2) Note that $\mathcal{A}$ contains the class of projective objects. Since $\mathcal{A}$ is closed under extensions by the remark above, we have $\mathcal{A}$ is resolving if, and only if, $\mathcal{A}$ is closed under taking kernels of epimorphisms. Similarly, $\mathcal{B}$ is coresolving if, and only if, $\mathcal{B}$ is closed under taking cokernels of monomorphisms.

(3) The classes $\mathcal{A}$ and $\mathcal{B}$ are closed under retractions. For if A' is a retract of $A \in \mathcal{A}$, then we have a sequence $A' \xrightarrow{r} A \xrightarrow{p} A'$ such that $p \circ r = \mathrm{id}_{A'}$. Consider the contravariant functor $\mathrm{Ext}^1_{\mathcal{C}}(-, B)$ with $B \in \mathcal{B}$. We get a sequence $\mathrm{Ext}^1_{\mathcal{C}}(A', B) \xrightarrow{\mathrm{Ext}^1_{\mathcal{C}}(p,B)} \mathrm{Ext}^1_{\mathcal{C}}(A, B) \xrightarrow{\mathrm{Ext}^1_{\mathcal{C}}(r,B)} \mathrm{Ext}^1_{\mathcal{C}}(A', B)$ such that $\mathrm{id}_{\mathrm{Ext}^1_{\mathcal{C}}(A',B)} = \mathrm{Ext}^1_{\mathcal{C}}(r, B) \circ \mathrm{Ext}^1_{\mathcal{C}}(p, B)$, i.e. $\mathrm{Ext}^1_{\mathcal{C}}(A', B)$ is a retract of $\mathrm{Ext}^1_{\mathcal{C}}(A, B)$. Since $\mathrm{Ext}^1_{\mathcal{C}}(A, B) = 0$, it follows $\mathrm{Ext}^1_{\mathcal{C}}(A', B) = 0$.

(4) The classes $\mathcal{A}$ and $\mathcal{B}$ are closed under direct summands. This follows by **(3)**, since if A' is a direct summand of A, then A' is a retract of A.

If we are given a cotorsion pair $(\mathcal{A}, \mathcal{B})$, it is not necessarily true that $\mathrm{Ext}^i_{\mathcal{C}}(A, B) = 0$ for every $A \in \mathcal{A}$, $B \in \mathcal{B}$, and $i > 1$. There is a special type of cotorsion pairs whose left and right classes are orthogonal with respect to every $\mathrm{Ext}^i_{\mathcal{C}}(-, -)$.

Definition 2.2.2. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ in an Abelian category $\mathcal{C}$ is said to be <u>hereditary</u> if $\mathcal{A}$ is a resolving[3] class.

---

3. See Definition 1.8.2.

Proposition 2.2.1

Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in an Abelian category $\mathcal{C}$. Consider the following conditions:

(1) $\mathcal{A}$ is resolving.

(2) $\mathcal{B}$ is coresolving.

(3) $\mathrm{Ext}^i_{\mathcal{C}}(A, B) = 0$ for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$, and $i > 1$.

If $\mathcal{C}$ has enough projective objects, then **(1)** and **(3)** are equivalent.

If $\mathcal{C}$ has enough injective objects, then **(2)** and **(3)** are equivalent.

$\boxed{\textbf{Proof}}$.

We only prove the case where $\mathcal{C}$ has enough projective objects. Notice that a cotorsion pair $(\mathcal{A}, \mathcal{B})$ is hereditary if, and only if, $\mathcal{A}$ is closed under taking kernels of epimorphisms. Let $A \in \mathcal{A}$ and $B \in \mathcal{B}$ and suppose **(1)**. Take a short exact sequence $K \hookrightarrow P \twoheadrightarrow A$, with P projective. Then we have a long exact sequence $\cdots \to \mathrm{Ext}^{i-1}(K, B) \to \mathrm{Ext}^i(A, B) \to \mathrm{Ext}^i(P, B) \to \cdots$. We use induction on i. Since $\mathcal{A}$ is resolving, we have $K \in \mathcal{A}$ and so $\mathrm{Ext}^1(K, B) = 0$. On the other hand, $\mathrm{Ext}^2(P, B) = 0$. It follows $\mathrm{Ext}^2(A, B) = 0$. If the result is true for $i - 1$, then $\mathrm{Ext}^{i-1}(K, B) = 0$. Also, $\mathrm{Ext}^i(P, B) = 0$. Hence $\mathrm{Ext}^i(A, B) = 0$. The converse **(3)** $\Longrightarrow$ **(1)** follows similarly. □

Throughout this thesis, we shall only work with hereditary cotorsion pairs. All of the examples of cotorsion pairs we have given so far are hereditary, and so will be those introduced in the next chapters. We present an example of a cotorsion pair which is not hereditary.

Example 2.2.2. Let R be an integral domain and let Q denote the field of quotients of R. A left R-module M is said to be a <u>Matlis cotorsion module</u> if

$\mathrm{Ext}^1_R(Q, M) = 0$. The class $\mathcal{MC}$ of Matlis cotorsion modules is the right half of a cotorsion pair $({}^{\perp}(\mathcal{MC}), \mathcal{MC})$ which is not hereditary. Details about this pair can be found in (45, Example 1.20).

For the rest of this section, we focus our attention in the study of a special type of cotorsion pairs, called complete cotorsion pairs. Their importance resides in the fact that it is possible to obtain certain pre-covers and pre-envelopes from them. Moreover, if the left class $\mathcal{A}$ of a complete cotorsion pair $(\mathcal{A}, \mathcal{B})$ in an Abelian category $\mathcal{C}$ is closed under direct limits, then it is possible to construct $\mathcal{A}$-covers for every object in $\mathcal{C}$. These cotorsion pairs from which it is possible to obtain $\mathcal{A}$-covers are called perfect. We shall recall this notion in Chapter 3.

Definition 2.2.3. Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in an Abelian category $\mathcal{C}$ (with enough projective or injective objects).

We say that $(\mathcal{A}, \mathcal{B})$ is <u>left complete</u> if for every object $X \in \mathrm{Ob}(\mathcal{C})$ there is a short exact sequence $B \hookrightarrow A \twoheadrightarrow X$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$.	We say that $(\mathcal{A}, \mathcal{B})$ is <u>right complete</u> if for every object $X \in \mathrm{Ob}(\mathcal{C})$ there is a short exact sequence $X \hookrightarrow B \twoheadrightarrow A$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$.
We say $(\mathcal{A}, \mathcal{B})$ is <u>functorially left complete</u> if there exists a functor $F : \mathcal{C} \to \mathcal{C}$ along with a natural epimorphism $F \to \mathrm{id}_{\mathcal{C}}$ such that $F(X) \in \mathcal{A}$ and $\mathrm{Ker}(F(X) \twoheadrightarrow X) \in \mathcal{B}$ for every object $X \in \mathrm{Ob}(\mathcal{C})$.	We say $(\mathcal{A}, \mathcal{B})$ is <u>functorially right complete</u> if there exists a functor $G : \mathcal{C} \to \mathcal{C}$ along with a natural monomorphism $\mathrm{id}_{\mathcal{C}} \to G$ such that $G(X) \in \mathcal{B}$ and $\mathrm{CoKer}(X \hookrightarrow G(X)) \in \mathcal{A}$ for every object $X \in \mathrm{Ob}(\mathcal{C})$.

A pair $(\mathcal{A}, \mathcal{B})$ is <u>(functorially) complete</u> if it is both (functorially) left and right complete.

Proposition 2.2.2 (Salce's Lemma)

(1) [Original Salce's Lemma] A cotorsion pair $(\mathcal{A}, \mathcal{B})$ in an Abelian category $\mathcal{C}$ with enough projective and injective objects is left complete if, and only if, it is right complete.

(2) [An extension of Salce's Lemma] Let $(\mathcal{A}, \mathcal{B})$ be a left complete and $(\mathcal{A}', \mathcal{B}')$ be a right complete cotorsion pair in an Abelian category $\mathcal{C}$. If $\mathcal{A} \subseteq \mathcal{A}'$, then the two cotorsion pairs are complete.

(3) [Functorial Salce's Lemma] Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in $\mathcal{C}$.

If $(\mathcal{A}, \mathcal{B})$ is functorially left complete and $\mathcal{C}$ has functorially enough injective objects, then $(\mathcal{A}, \mathcal{B})$ is functorially complete.

If $(\mathcal{A}, \mathcal{B})$ is functorially right complete and $\mathcal{C}$ has functorially enough projective objects, then $(\mathcal{A}, \mathcal{B})$ is functorially complete.

Proof.

(2) We prove $(\mathcal{A}, \mathcal{B})$ is right complete. Let $X \in \mathrm{Ob}(\mathcal{C})$. Since $(\mathcal{A}', \mathcal{B}')$ is right complete, there exists a short exact sequence $X \hookrightarrow B' \twoheadrightarrow A'$ where $B' \in \mathcal{B}'$ and $A' \in \mathcal{A}'$. Since $\mathcal{A} \subseteq \mathcal{A}'$, we have $\mathcal{B}' = (\mathcal{A}')^{\perp} \subseteq (\mathcal{A})^{\perp} = \mathcal{B}$, and so $B' \in \mathcal{B}$. Since $(\mathcal{A}, \mathcal{B})$ is left complete, we have a short exact sequence $B \hookrightarrow A \twoheadrightarrow A'$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Taking the pullback of $B' \twoheadrightarrow A'$ and $A \twoheadrightarrow A'$, we get a commutative diagram

$$\begin{tikzcd}
 & B \arrow[r, equal] \arrow[d, hook] & B \arrow[d, hook] \\
X \arrow[r, hook] \arrow[d, equal] & B' \times_{A'} A \arrow[r, two heads] \arrow[d, two heads] & A \arrow[d, two heads] \\
X \arrow[r, hook] & B' \arrow[r, two heads] & A'
\end{tikzcd}$$

Since $B, B' \in \mathcal{B}$ and $\mathcal{B}$ is closed under extensions, we have $B' \times_{A'} A \in \mathcal{B}$. Hence $(\mathcal{A}, \mathcal{B})$ is complete.

Note that **(1)** follows from the inclusion $\mathcal{P}_0(\mathcal{C}) \subseteq \mathcal{A}$, and from the fact that $(\mathcal{P}_0(\mathcal{C}), \mathrm{Ob}(\mathcal{C}))$ and $(\mathrm{Ob}(\mathcal{C}), \mathcal{I}_0(\mathcal{C}))$ are left and right complete, respectively, if $\mathcal{C}$ has enough projective and injective objects.

(3) We only prove the left statement, by showing the above pullback diagram is functorial if $(\mathcal{A}, \mathcal{B})$ is functorially left complete and if there exists a functor $I : \mathcal{C} \to \mathcal{C}$ along with a natural monomorphism $\mathrm{id}_{\mathcal{C}} \to I$ such that $I(X)$ injective for every $X \in \mathrm{Ob}(\mathcal{C})$. As above, we have a commutative diagram:

$$\begin{tikzcd}
 & B_{C_X} \arrow[r, equal] \arrow[d, hook] & B_{C_X} \arrow[d, hook] \\
X \arrow[r, hook, "j_X"] \arrow[d, equal] & I(X) \times_{C_X} F(C_X) \arrow[r, two heads, "q_X"] \arrow[d, two heads, "\pi_{I(X)}"] & F(C_X) \arrow[d, two heads, "\varphi_X"] \\
X \arrow[r, hook, "i_X"'] & I(X) \arrow[r, two heads, "p_X"'] & C_X
\end{tikzcd}$$

We construct a functor $\mathcal{C} \xrightarrow{G} \mathcal{C}$ as follows:

- Set $G(X) := I(X) \times_{C_X} F(C_X)$ for each $X \in \mathrm{Ob}(\mathcal{C})$. Note $I(X) \in \mathcal{B}$ since $\mathcal{B}$ contains the class of injective objects. Then $I(X) \times_{C_X} F(C_X) \in \mathcal{B}$.
- Consider a morphism $X \xrightarrow{f} Y$. We have a pullback diagram

$$\begin{tikzcd}
Y \arrow[r, hook, "j_Y"] \arrow[d, equal] & I(Y) \times_{C_Y} F(C_Y) \arrow[r, two heads, "q_Y"] \arrow[d, "\pi_{I(Y)}"] & F(C_Y) \arrow[d, "\varphi_Y"] \\
Y \arrow[r, hook, "i_Y"'] & I(Y) \arrow[r, two heads, "p_Y"'] & C_Y
\end{tikzcd}$$

  Since $I(X)$ is injective, there is a morphism $I(X) \xrightarrow{l_f} I(Y)$ such that $l_f \circ i_X = i_Y \circ f$. By the universal property of cokernels, there exists a unique morphism $C_X \xrightarrow{r_f} C_Y$ such that $r_f \circ p_X = p_Y \circ l_f$. Since $\varphi : F \to \mathrm{id}_{\mathcal{C}}$ is a natural transformation, we have

$$\varphi_Y \circ F(r_f) \circ q_X = r_f \circ \varphi_X \circ q_X = r_f \circ p_X \circ \pi_{I(X)} = p_Y \circ l_f \circ \pi_{I(X)}.$$

Then there exists a unique map $I(X) \times_{C_X} F(C_X) \overset{G(f)}{\to} I(Y) \times_{C_Y} F(C_Y)$ such that the following diagram commutes:

$$
\begin{tikzcd}
I(X) \times_{C_X} F(C_X) \arrow[dr, dotted, "G(f)"] \arrow[drr, bend left, "F(r_f) \circ q_X"] \arrow[ddr, bend right, "l_f \circ \pi_{I(X)}"'] & & \\
& I(Y) \times_{C_Y} F(C_Y) \arrow[r, "q_Y"] \arrow[d, "\pi_{I(Y)}"'] & F(C_Y) \arrow[d, "\varphi_Y"] \\
& I(Y) \arrow[r, "p_Y"'] & C_Y
\end{tikzcd}
$$

Consider $G(f) \circ j_X, j_Y \circ f : X \to I(Y) \times_{C_Y} F(C_Y)$. We have

$$q_Y \circ G(f) \circ j_X = F(r_f) \circ q_X \circ j_X = F(r_f) \circ 0 = 0.$$

$$\pi_{I(Y)} \circ G(f) \circ j_X = l_f \circ \pi_{I(X)} \circ j_X = l_f \circ i_X = i_Y \circ f.$$

$$q_Y \circ j_Y \circ f = 0 \circ f = 0.$$

$$\pi_{I(Y)} \circ j_Y \circ f = i_J \circ f.$$

We also have $G(f) \circ j_X = j_Y \circ f$ since the following diagram commutes.

$$
\begin{tikzcd}
X \arrow[dr, bend left=15, "j_Y \circ f"] \arrow[dr, bend right=15, "G(f) \circ j_X"'] \arrow[drr, bend left, "0"] \arrow[ddr, bend right, "i_Y \circ f"'] & & \\
& I(Y) \times_{C_Y} F(C_Y) \arrow[r, "q_Y"] \arrow[d, "\pi_{I(Y)}"'] & F(C_Y) \arrow[d, "\varphi_Y"] \\
& I(Y) \arrow[r, "p_Y"'] & C_Y
\end{tikzcd}
$$

Summarizing, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[rr] \arrow[dr, "f"] \arrow[dd, equal] & & G(X) \arrow[rr] \arrow[dr, "G(f)"] \arrow[dd] & & F(C_X) \arrow[rr] \arrow[dr] \arrow[dd] & & 0 & \\
& 0 \arrow[r] & Y \arrow[rr] \arrow[dd, equal] & & G(Y) \arrow[rr] \arrow[dd] & & F(C_Y) \arrow[rr] \arrow[dd] & & 0 \\
0 \arrow[r] & X \arrow[rr] \arrow[dr] & & I(X) \arrow[rr] \arrow[dr] & & C_X \arrow[rr] \arrow[dr] & & 0 & \\
& 0 \arrow[r] & Y \arrow[rr] & & I(Y) \arrow[rr] & & C_Y \arrow[rr] & & 0
\end{tikzcd}$$

It is clear that $G(\mathrm{id}_X) = \mathrm{id}_{G(X)}$. The equality $G(g \circ f) = G(g) \circ G(f)$ can be proven by using the universal property of pullbacks and the fact that F and I are functors. $\square$

Example 2.2.3. We know that for every left R-module M there exists a projective module P along with an epimorphism $P \twoheadrightarrow M$, with P functorial in M. Dually, M can be (functorially) embedded into an injective module I. By the Salce's Lemma, $(\mathcal{P}_0, {}_R\mathbf{Mod})$ and $({}_R\mathbf{Mod}, \mathcal{I}_0)$ are (functorially) complete cotorsion pairs.

Definition 2.2.4. Let $\mathcal{C}$ be an Abelian category.

An $\mathcal{F}$-pre-cover $F \to X$ is said to be <u>special</u> if it is epi and $\mathrm{Ker}(f) \in {}^{\perp}\mathcal{F}$. The class $\mathcal{F}$ is a <u>special pre-covering class</u> if every object has a special $\mathcal{F}$-pre-covering.

A $\mathcal{G}$-pre-envelope $X \to G$ is said to be <u>special</u> if it is monic and $\mathrm{CoKer}(f) \in \mathcal{G}^{\perp}$. The class $\mathcal{G}$ is a <u>special pre-enveloping class</u> if every object has a special $\mathcal{G}$-pre-envelope.

Note that we can rewrite Salce's Lemma using the previous definition. Specifically: if $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair in an Abelian category $\mathcal{C}$ with enough projective and injective objects, then $\mathcal{A}$ is a special pre-covering class if, and only if, $\mathcal{B}$ is special pre-enveloping.

Definition 2.2.5. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ in an Abelian category $\mathcal{C}$ (with enough projective or injective objects) is said to be cogenerated by a set of objetcs $\mathcal{S} \subseteq \mathcal{A}$ if $\mathcal{B} = \mathcal{S}^{\perp}$.

The method used by Enochs to prove the existence of flat covers of modules was to give a set of generators for the cotorsion pair $(\mathcal{F}_0, (\mathcal{F}_0)^{\perp})$. In this thesis we shall construct cotorsion pairs and sometimes we shall need to prove they are complete. Here is the importance of Eklof and Trlifaj's Theorem. As we stated before, we shall give a proof of this result for any Grothendieck category with enough projective objects. The first step is to show a result known as Eklof's Lemma. The proof we are giving is based on the arguments appearing in (21) and (31) for the category of left R-modules. Most of these arguments carry over perfectly to any Abelian category, but at some points we need to impose extra conditions on the given category, in order to translate these arguments from modules to objects.

Definition 2.2.6. A transfinite composition in a cocomplete Abelian category $\mathcal{C}$[4] is a morphism of the form $f : F_0 \to \mathrm{CoLim}_{\alpha<\lambda}(F_\alpha)$, where $F : [\lambda] \to \mathcal{C}$ is a colimit preserving functor and λ is an ordinal[5]. The morphism f is also known as the transfinite composition of the morphisms $F_\alpha \to F_{\alpha+1}$ for every $\alpha + 1 < \lambda$. If in addition, all the morphisms $F_\alpha \to F_{\alpha+1}$ are monic with cokernel in some class $\mathcal{S}$, the $F_0 \xrightarrow{f} \mathrm{CoLim}_{\alpha<\lambda}(F_\alpha)$ is called a transfinite extension of F_0 by $\mathcal{S}$. If $F_0 \in \mathcal{S}$ as well, the colimit $\mathrm{CoLim}_{\alpha<\lambda}(F_\alpha)$ is called a transfinite extension of $\mathcal{S}$.

4. For example, every Grothendieck category admits colimits.

5. Every ordinal number λ can be considered as a category $[\lambda]$, where the objects are given by the ordinals $\alpha \leq \lambda$, and the relation $\alpha \leq \alpha'$ is the only morphism from α to α'. In $[\lambda]$, every limit ordinal $\beta < \lambda$ is the colimit of the family $(\alpha \in \mathrm{Ob}([\lambda]) : \alpha < \beta)$.

Example 2.2.4. A transfinite composition of a family of modules $(M_\alpha : \alpha < \lambda)$ is called a continuous chain of $M = \bigcup_{\alpha<\lambda} M_\alpha$ indexed by λ. To be clear, M can be written as $M = \bigcup_{\alpha<\lambda} M_\alpha$, where $M_\alpha \subseteq M_{\alpha'}$ whenever $\alpha \leq \alpha'$, and $M_\beta = \bigcup_{\alpha<\beta} M_\alpha$ for every limit ordinal β.

Lemma 2.2.3 (Eklof's Lemma)

Let X and Y be two objects of a cocomplete Abelian category $\mathcal{C}$, and suppose $X = \mathrm{CoLim}_{\alpha<\lambda}(X_\alpha)$, where $(X_\alpha : \alpha < \lambda)$ is a transfinite extension of ${}^\perp\{Y\}$. Then $X \in {}^\perp\{Y\}$.

Proof.

We use transfinite induction on $\alpha < \lambda$.

(i) Initial case: Immediate from the definition of transfinite extensions.

(ii) Successor case: Suppose $\alpha < \lambda$ is not a limit ordinal and that $\mathrm{Ext}^1_{\mathcal{C}}(X_\alpha, Y) = 0$. We want to show $\mathrm{Ext}^1_{\mathcal{C}}(X_{\alpha+1}, Y) = 0$. Since $X_\alpha \to X_{\alpha+1}$ is monic with cokernel $X_{\alpha+1}/X_\alpha$ in ${}^\perp\{Y\}$, we have an exact sequence $X_\alpha \hookrightarrow X_{\alpha+1} \twoheadrightarrow X_{\alpha+1}/X_\alpha$, where the two ends are in ${}^\perp\{Y\}$. Since ${}^\perp\{Y\}$ is closed under extensions, we have $X_{\alpha+1} \in {}^\perp\{Y\}$.

(iii) Limit ordinal case: Suppose $\beta < \lambda$ is a limit ordinal and that $\mathrm{Ext}^1_{\mathcal{C}}(X_\alpha, Y) = 0$ for every non limit ordinal $\alpha < \beta$. We show that $\mathrm{Ext}^1_{\mathcal{C}}(X_\beta, Y) = 0$, i.e. that every short exact sequence $Y \overset{f}{\hookrightarrow} Z \overset{g}{\twoheadrightarrow} X_\beta$ splits, where $X_\beta = \mathrm{CoLim}_{\alpha<\beta}(X_\alpha)$. Let $\alpha < \beta$ and take the pullback of $X^\beta_\alpha : X_\alpha \to X_\beta$ and $g : Z \to X_\beta$ in order to get the following commutative diagram with exact rows:

$$\begin{tikzcd} Y \arrow[r, hook, "f_\alpha"] \arrow[d, equal] & Z \times_{X_\beta} X_\alpha \arrow[r, two heads, "g_\alpha"] \arrow[d, "\rho_\alpha"] & X_\alpha \arrow[d, "X^\beta_\alpha"] \\ Y \arrow[r, hook, "f"] & Z \arrow[r, two heads, "g"] & X_\beta \end{tikzcd}$$

Since $\mathrm{Ext}^1_{\mathcal{C}}(X_\alpha, Y) = 0$, there exists a section $s_\alpha : X_\alpha \to Z \times_{X_\beta} X_\alpha$ of g_α, i.e. $g_\alpha \circ s_\alpha = \mathrm{id}_{Z_\alpha}$. Our goal is to construct a section of g by using the family $(s_\alpha)_{\alpha<\beta}$ and the universal property of colimits. For every pair of non limit ordinals $\gamma \leq \alpha$, we have a diagram

$$\begin{tikzcd}
X_\gamma \arrow[rr, "X_\gamma^\alpha"] \arrow[dr, "X_\gamma^\beta"'] \arrow[ddr, bend right, "\rho_\gamma \circ s_\gamma"'] & & X_\alpha \arrow[dl, "X_\alpha^\beta"] \arrow[ddl, bend left, "\rho_\alpha \circ s_\alpha"] \\
& X_\beta & \\
& Z &
\end{tikzcd}$$

where the inner triangle is a cocone. However, it is not necessarily true that the outer triangle is a cocone. The idea is to construct section maps $s_\alpha : X_\alpha \to Z \times_{X_\beta} X_\alpha$ such that $\rho_\alpha \circ s_\alpha \circ X_\gamma^\alpha = \rho_\gamma \circ s_\gamma$. We say that s_α and $s_{\alpha'}$ are <u>compatible</u> if the previous equality holds. Suppose we have construct a family of compatible sections $s_\gamma : X_\gamma \to Z \times_{X_\beta} X_\gamma$ with $\gamma \leq \alpha$ a non limit ordinal. We construct $s_{\alpha+1} : X_{\alpha+1} \to Z \times_{X_\beta} X_{\alpha+1}$ compatible with every s_γ.

By the universal property of pullbacks, for every $\gamma \leq \alpha$ there is a unique morphism $l_\gamma^\alpha : Z \times_{X_\beta} X_\gamma \to Z \times_{X_\beta} X_\alpha$ such that the following diagram commutes:

$$\begin{tikzcd}
Z \times_{X_\beta} X_\gamma \arrow[dr, dotted, "\exists!\, l_\gamma^\alpha"] \arrow[drr, bend left, "X_\gamma^\alpha \circ g_\gamma"] \arrow[ddr, bend right, "\rho_\gamma"'] & & \\
& Z \times_{X_\beta} X_\alpha \arrow[r, "g_\alpha"] \arrow[d, "\rho_\alpha"'] & X_\alpha \arrow[d, "X_\alpha^\beta"] \\
& Z \arrow[r, "g"'] & X_\beta
\end{tikzcd}$$

We have a section $t : X_{\alpha+1} \to Z \times_{X_\beta} X_{\alpha+1}$ of $g_{\alpha+1}$ since $\mathrm{Ext}^1_{\mathcal{C}}(X_{\alpha+1}, Y) = 0$ (note $\alpha + 1 < \beta$ since $\alpha < \beta$ and β is a limit ordinal). Consider the morphism

$l_\alpha^{\alpha+1} \circ s_\alpha - t \circ X_\alpha^{\alpha+1} : X_\alpha \to Z \times_{X_\beta} X_{\alpha+1}$. We have $g_{\alpha+1} \circ (l_\alpha^{\alpha+1} \circ s_\alpha - t \circ X_\alpha^{\alpha+1}) = X_\alpha^{\alpha+1} \circ g_\alpha \circ s_\alpha - X_\alpha^{\alpha+1} = X_\alpha^{\alpha+1} - X_\alpha^{\alpha+1} = 0$.

There exists a unique morphism $r : X_\alpha \to Y$ such that $l_\alpha^{\alpha+1} \circ s_\alpha - t \circ X_\alpha^{\alpha+1} = f_{\alpha+1} \circ r$, since $f_{\alpha+1}$ is the kernel of $g_{\alpha+1}$. On the other hand, the function $\mathrm{Hom}(X_{\alpha+1}, Y) \to \mathrm{Hom}(X_\alpha, Y)$ is onto since $\mathrm{Ext}^1(X_{\alpha+1}/X_\alpha, Y) = 0$, so there exists a morphism $r' : X_{\alpha+1} \to Y$ such that $r = r' \circ X_\alpha^{\alpha+1}$. Set $s_{\alpha+1} := f_{\alpha+1} \circ r' + t$. We have:

$$\begin{aligned} s_{\alpha+1} \circ X_\alpha^{\alpha+1} &= f_{\alpha+1} \circ r' \circ X_\alpha^{\alpha+1} + t \circ X_\alpha^{\alpha+1} = f_{\alpha+1} \circ r + t \circ X_\alpha^{\alpha+1} \\ &= l_\alpha^{\alpha+1} \circ s_\alpha - t \circ X_\alpha^{\alpha+1} + t \circ X_\alpha^{\alpha+1} = l_\alpha^{\alpha+1} \circ s_\alpha. \end{aligned}$$

For every $\gamma \leq \alpha$, we can write $X_\gamma^{\alpha+1} = X_\alpha^{\alpha+1} \circ X_\gamma^\alpha$. We have:

$$\begin{aligned} (\rho_{\alpha+1} \circ s_{\alpha+1}) \circ X_\gamma^{\alpha+1} &= \rho_{\alpha+1} \circ (s_{\alpha+1} \circ X_\alpha^{\alpha+1}) \circ X_\gamma^\alpha = (\rho_{\alpha+1} \circ l_\alpha^{\alpha+1}) \circ s_\alpha \circ X_\gamma^\alpha \\ &= \rho_\alpha \circ s_\alpha \circ X_\gamma^\alpha = \rho_\gamma \circ s_\gamma. \end{aligned}$$

So we have constructed $s_{\alpha+1}$ compatible with every s_γ with $\gamma \leq \alpha$. Hence the family $(\rho_\gamma \circ s_\gamma : X_\gamma \to Z)$ defines a cocone of vertex Z. So by the universal property of the colimit $X_\beta = \mathrm{CoLim}_{\alpha<\beta}$, there exists a unique morphism $s_\beta : X_\beta \to Z$ such that the following diagram commutes:

$$\begin{tikzcd} X_\gamma \arrow[rr, "X_\gamma^\alpha"] \arrow[dr, "X_\gamma^\beta"'] \arrow[ddr, bend right, "\rho_\gamma \circ s_\gamma"'] & & X_\alpha \arrow[dl, "X_\alpha^\beta"] \arrow[ddl, bend left, "\rho_\alpha \circ s_\alpha"] \\ & X_\beta \arrow[d, dotted, "\exists!\ s_\beta"] & \\ & Z & \end{tikzcd}$$

Moreover, $(g \circ s_\beta) \circ X_\gamma^\beta = g \circ \rho_\gamma \circ s_\gamma = X_\gamma^\beta \circ g_\gamma \circ s_\gamma = X_\gamma^\beta$. Applying the universal property of X_β again, we conclude that $g \circ s_\beta = \mathrm{id}_{X_\beta}$ since the following diagram commutes:

$$\begin{tikzcd}
X_\gamma \arrow[rr, "X_\gamma^\alpha"] \arrow[dr, "X_\gamma^\beta"'] \arrow[ddr, bend right, "X_\gamma^\beta"'] & & X_\alpha \arrow[dl, "X_\alpha^\beta"] \arrow[ddl, bend left, "X_\gamma^\beta"] \\
& X_\beta \arrow[d, shift right, "\mathrm{id}_{X_\beta}"'] \arrow[d, shift left, "g \circ s_\beta"] & \\
& X_\beta &
\end{tikzcd}$$

$\square$

Lemma 2.2.4 (see (21, Lemma 7.3.1))

Given a set S, there exists a limit ordinal λ such that if $(\alpha_s)_{s\in S}$ is a family of ordinals such that $\alpha_s < \lambda$ for all $s \in S$, then there exists an ordinal $\lambda' < \lambda$ such that $\alpha_s \leq \lambda'$ for all $s \in S$.

Corollary 2.2.5

(1) [See (21, Corollary 7.3.2)] If S and λ are as in the lemma above and if $(Y_\alpha)_{\alpha<\lambda}$ is a family of subsets of a set Y such that $Y_\alpha \subseteq Y_{\alpha'}$ when $\alpha \leq \alpha' < \lambda$ and such that $Y = \bigcup_{\alpha<\lambda} Y_\alpha$, then for any function $f : S \to Y$ there is an $\alpha < \lambda$ such that $f(S) \subseteq Y_\alpha$.

(2) [Categorical version of (21, Corollary 7.3.2)] Let X be an object of a Grothendieck category $\mathcal{C}$. If $S := \mathrm{Card}(X)$ and λ are as in the lemma above, and if $Y = \mathrm{CoLim}_{\alpha<\lambda} Y_\alpha$, where the family of objects $(Y_\alpha)_{\alpha<\lambda}$ is a transfinite composition, then for any morphism $h : X \to Y$ there is an $\alpha < \lambda$ such that $h(X)$ is a subobject of Y_α.

In order to give a proof for **(2)**, we need to recall the notion of locally presentable categories.

Definition 2.2.7 (see (1, Definition 1.17)). Let λ be a regular cardinal , i.e. λ is infinite and cannot be expressed as $\lambda = \sum_{i<\alpha} \lambda_i$ where $\lambda_i < \lambda$ and $\alpha < \lambda$.

(1) A partially ordered set is called λ-directed provided that every subset of cardinality smaller than λ has an upper bound.

(2) A diagram $F : \Sigma \to \mathcal{C}$ such that $\mathrm{Ob}(\Sigma)$ is a λ-directed poset is called a λ-directed diagram.

(3) A colimit of a λ-directed diagram is called λ-directed.

(4) An object K of a category is called λ-presentable provided that its hom functor $\mathrm{Hom}_{\mathcal{C}}(K, -)$ preserves λ-directed colimits. An object is called presentable if it is λ-presentable for some λ.

(5) A category is called locally λ-presentable provided that it is cocomplete, and has a set $\mathcal{A}$ of λ-presentable objects such that every object if a λ-directed colimit of objects of from $\mathcal{A}$.

Example 2.2.5. Every Grothendieck category $\mathcal{C}$ is locally presentable. Moreover, every object of $\mathcal{C}$ is presentable.

Example 2.2.6.

(1) The successor of of any infinite cardinal, such as $\aleph_1$ (successor of $\aleph_0$), is a regular cardinal.

(2) In the proof of the previous lemma, the ordinal λ is defined as the least ordinal such that $\mathrm{Card}(\lambda) = \aleph_{\beta+1}$, where $\mathrm{Card}(S) \leq \aleph_\beta$. So $\mathrm{Card}(\lambda)$ is a regular cardinal.

Proof of Corollary 2.2.5 (2).

First, $\mathcal{C}$ is locally γ-presentable, for some regular cardinal γ. Note that the ordinal λ in the previous lemma can be taken such that $\mathrm{Card}(\lambda) \geq \gamma$. Since every object in $\mathcal{C}$ is γ-presentable, in particular we have that $\mathrm{Hom}_{\mathcal{C}}(X,-)$ preserves γ-directed colimits. On the other hand, $\mathrm{Card}(\lambda) \geq \gamma$ implies that $\mathrm{Hom}_{\mathcal{C}}(X,-)$ preserves colimits indexed by $\{\alpha : \alpha < \lambda\}$. This condition means that every morphism $X \to Y = \mathrm{CoLim}_{\alpha<\lambda} Y_\alpha$ factors through some α. Therefore, the result follows. $\square$

Theorem 2.2.6 (Eklof and Trlifaj's Theorem)

Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in a Grothendieck category with (functorially) enough projective objects. If $(\mathcal{A}, \mathcal{B})$ is cogenerated by a set $\mathcal{S} \subseteq \mathcal{A}$, then $(\mathcal{A}, \mathcal{B})$ is (functorially) right complete.

Proof.

First, notice that $(\mathcal{A}, \mathcal{B})$ is cogenerated by a single object, namely the direct sum $A = \bigoplus\{S : S \in \mathcal{S}\}$. Since $\mathcal{C}$ has enough projective objects, there is a short exact sequence $K \hookrightarrow P \twoheadrightarrow A$, where P is a projective object. For $X \in \mathrm{Ob}(\mathcal{C})$, we have a short exact sequence $K^{(\mathrm{Hom}(K,X))} \hookrightarrow P^{(\mathrm{Hom}(K,X))} \twoheadrightarrow A^{(\mathrm{Hom}(K,X))}$, since the direct sum of short exact sequences is exact. By the universal property of biproducts, there is a unique morphism $K^{(\mathrm{Hom}(K,X))} \xrightarrow{g_0} X$ such that the triangle

$$\begin{tikzcd} K \arrow[r, "i_f"] \arrow[dr, "f"'] & K^{(\mathrm{Hom}(K,X))} \arrow[d, dotted, "\exists!\ g_0"] \\ & X \end{tikzcd}$$

commutes for every $f \in \mathrm{Hom}(K,X)$. Taking the pushout of the maps g_0 and $K^{(\mathrm{Hom}(K,X))} \to P^{(\mathrm{Hom}(K,X))}$ we get the following diagram with exact rows:

$$\begin{tikzcd}
K^{(\mathrm{Hom}(K,X))} \arrow[r, hook] \arrow[d, "g_0"'] & P^{(\mathrm{Hom}(K,X))} \arrow[r, two heads] \arrow[d] & A^{(\mathrm{Hom}(K,X))} \arrow[d, equal] \\
X \arrow[r, hook] & X_1 \arrow[r, two heads] & A^{(\mathrm{Hom}(K,X))}
\end{tikzcd}$$

Setting $X_0 = X$, we have $X_1/X_0 \cong A^{(\mathrm{Hom}(K,X))}$. Let $\mathcal{D}$ denote the class of direct sums of copies of A. Using transfinite induction, one can compute a transfinite extension of X by $\mathcal{D}$. We have already constructed X_0 and X_1. Assuming X_α is constructed, $X_{\alpha+1}$ is given by the pushout diagram

$$\begin{tikzcd}
K^{(\mathrm{Hom}(K,X_\alpha))} \arrow[r, hook] \arrow[d, "g_\alpha"'] & P^{(\mathrm{Hom}(K,X_\alpha))} \arrow[r, two heads] \arrow[d] & A^{(\mathrm{Hom}(K,X_\alpha))} \arrow[d, equal] \\
X_\alpha \arrow[r, hook] & X_{\alpha+1} \arrow[r, two heads] & A^{(\mathrm{Hom}(K,X_\alpha))}
\end{tikzcd}$$

So $X_{\alpha+1}/X_\alpha \in \mathcal{D}$. If β is a limit ordinal and X_α is constructed for every $\alpha < \beta$, set $X_\beta = \mathrm{CoLim}_{\alpha<\beta} X_\alpha$. By the principle of transfinite induction, we obtain a transfinite extension $B = \mathrm{CoLim}_{\alpha<\lambda} X_\alpha$ of X by $\mathcal{D}$[6], for some ordinal λ. Consider the sequence $X \hookrightarrow B \twoheadrightarrow B/X$. We check $B \in \mathcal{B}$ and $B/X \in \mathcal{A}$. Note $B \in \mathcal{B}$ if, and only if, $\mathrm{Ext}^1(A, B) = 0$. Since the sequence $K \hookrightarrow P \twoheadrightarrow A$ derives in an exact sequence $\mathrm{Hom}(A, B) \hookrightarrow \mathrm{Hom}(P, B) \to \mathrm{Hom}(K, B) \to \mathrm{Ext}^1(A, B)$, we have that $\mathrm{Ext}^1(A, B) = 0$ if, and only if, $\mathrm{Hom}(P, B) \to \mathrm{Hom}(K, B)$ is surjective. Consider a morphism $K \xrightarrow{f} B$. By Corollary 2.2.5, the ordinal λ can be chosen as a limit ordinal such that f can be factored as a composition $K \xrightarrow{f'} X_\alpha \to B$, for some $\alpha < \lambda$. Since λ is a limit ordinal, $\alpha + 1 < \lambda$. So from the previous construction there exists a morphism $P \xrightarrow{g} X_{\alpha+1}$ satisfying $g \circ (K \to P) = (X_\alpha \to X_{\alpha+1}) \circ f'$. We get $f = (X_{\alpha+1} \to B) \circ (X_\alpha \to X_{\alpha+1}) \circ f' = [(X_{\alpha+1} \to B) \circ g] \circ (K \to P)$ and hence the map $\mathrm{Hom}(P, B) \to \mathrm{Hom}(K, B)$ is surjective.

---

6. It is possible to replace $\mathcal{D}$ by $\{A\}$, constructing a refinement of $(X_\alpha : \alpha < \lambda)$ such that $X_{\alpha+1}/X_\alpha \cong A$ for every $\alpha < \lambda$.

It is only left to show $B/X \in \mathcal{A}$. Let $Y \in \mathcal{B}$. Note that $B/X = \mathrm{CoLim}_{\alpha<\lambda}(X_\alpha/X)$. By construction, $\mathrm{Ext}^1(X_0/X, Y) \cong \mathrm{Ext}^1(0, Y) = 0$ and $\mathrm{Ext}^1(\frac{(X_{\alpha+1}/X)}{(X_\alpha/X)}, Y) \cong \mathrm{Ext}^1(X_{\alpha+1}/X_\alpha, Y) = 0$. By Eklof's Lemma, $\mathrm{Ext}^1(\mathrm{CoLim}_{\alpha<\lambda}(X_\alpha/X), Y) = 0$ and hence $B/X \in \mathcal{A}$.

<u>Functoriality</u>: Now we show that the object B above is functorial in X, provided that $\mathcal{C}$ has functorially enough projective objects and that $(\mathcal{A}, \mathcal{B})$ is functorially right complete. We have a natural epimorphism $P \twoheadrightarrow \mathrm{id}_{\mathcal{C}}$ with that $P(A)$ is projective. Consider a morphism $X \xrightarrow{f} Y$. First, we have a triangle:

$$\begin{tikzcd}
P(A) \arrow[r, "i_h"] \arrow[dr, "j_{f\circ h}"'] & P(A)^{(\mathrm{Hom}(K,X))} \arrow[d, dotted, "\exists!\ \varphi"] \\
 & P(A)^{(\mathrm{Hom}(K,Y))}
\end{tikzcd}$$

On the other hand, we have two commutative pullback squares

$$\begin{tikzcd}
K^{(\mathrm{Hom}(K,X))} \arrow[r, "\gamma"] \arrow[d, "s"'] & P^{(\mathrm{Hom}(K,X))} \arrow[d, "r"] \\
X \arrow[r, "X_0^1"'] & X_1
\end{tikzcd}
\qquad
\begin{tikzcd}
K^{(\mathrm{Hom}(K,Y))} \arrow[r, "\delta"] \arrow[d, "t"'] & P^{(\mathrm{Hom}(K,Y))} \arrow[d, "l"] \\
Y \arrow[r, "Y_0^1"'] & Y_1
\end{tikzcd}$$

Using the universal property of biproducts, one can show that $l \circ \varphi \circ \gamma = Y_0^1 \circ f \circ s$. It follows by the universal property of pushouts that there exists a unique morphism $X_1 \xrightarrow{f_1} Y_1$ such that the following diagram commutes:

$$\begin{tikzcd}
K^{(\mathrm{Hom}(K,X))} \arrow[r, "\gamma"] \arrow[d, "s"'] & P(A)^{(\mathrm{Hom}(K,X))} \arrow[d, "r"] \arrow[ddr, bend left, "l\circ\varphi"] & \\
X \arrow[r, "X_0^1"'] \arrow[drr, bend right, "Y_0^1\circ f"'] & X_1 \arrow[dr, dotted, "f_1"] & \\
 & & Y_1
\end{tikzcd}$$

By transfinite induction, we can construct morphisms $X_\alpha \xrightarrow{f_\alpha} Y_\alpha$ with $\alpha < \lambda$ (notice λ only depends on K) satisfying similar conditions. Using the universal property of colimits, we obtain:

$$\begin{tikzcd}
X_\alpha \arrow[rr, "X_\alpha^{\alpha'}"] \arrow[dr, "\rho_\alpha"'] \arrow[ddr, bend right, "\pi_\alpha \circ f_\alpha"'] & & X_{\alpha'} \arrow[dl, "\rho_{\alpha'}"] \arrow[ddl, bend left, "\pi_{\alpha'} \circ f_{\alpha'}"] \\
& \mathrm{CoLim}_{\alpha<\lambda} X_\alpha \arrow[d, dotted, "\exists!"', "\mathrm{CoLim}_{\alpha<\lambda} f_\alpha"] & \\
& \mathrm{CoLim}_{\alpha<\lambda} Y_\alpha &
\end{tikzcd}$$

It is not hard to show that $f \mapsto \mathrm{CoLim}_{\alpha<\lambda}(f_\alpha)$ is functorial. $\square$

Example 2.2.7.

(1) In any Grothendieck category with a generator G, note that by the Baer Criterion the injective cotorsion pair $(\mathrm{Ob}(\mathcal{C}), \mathcal{I}_0(\mathcal{C}))$ is cogenerated by the set of quotients G/I, with I running over the set of subobjects of G.

(2) Given an associative ring R, recall that a left R-module is projective if, and only if, it is a direct summand of a free module. It follows that $(\mathcal{P}_0, \mathrm{Ob}({}_R\mathbf{Mod}))$ is cogenerated by $\{R\}$.

2.3 Hovey's correspondence

We shall show that from two compatible and functorially complete cotorsion pairs in an Abelian category $\mathcal{C}$, we can construct an Abelian model structure. We start describing how to obtain a weak factorization system from a complete cotorsion pair. As a first approach to this construction, we know that if $(\mathcal{A}, \mathcal{B})$ is a complete cotorsion pair, then for every $X \in \mathrm{Ob}(\mathcal{C})$ there is an exact sequence $B \hookrightarrow A \twoheadrightarrow X$, with $A \in \mathcal{A}$ and $B \in \mathcal{B}$. This gives us a factorization of the morphism $B \xrightarrow{0} X$, where one of the factors is an epimorphism with kernel in $\mathcal{B}$. The class of all the epimorphisms having this property will turn out to be the right class of a weak factorization system. Probably the reader has already guessed what will be the left class of that system. Yes!, all the monomorphisms with cokernel in $\mathcal{A}$.

Definition 2.3.1. Given a cotorsion pair $(\mathcal{A}, \mathcal{B})$ and a weak factorization system $(\mathcal{L}, \mathcal{R})$ on an Abelian category $\mathcal{C}$ (with enough projective or injective objects), we shall say that $(\mathcal{L}, \mathcal{R})$ is a <u>cotorsion factorization system</u> with respect to $(\mathcal{A}, \mathcal{B})$ if the following two conditions are satisfied:

(1) $l : X \to Y$ is in $\mathcal{L}$ if, and only if, l is a monomorphism and $\mathrm{CoKer}(l) \in \mathcal{A}$.

(2) $r : W \to Z$ is in $\mathcal{R}$ if, and only if, r is an epimorphism and $\mathrm{Ker}(r) \in \mathcal{B}$.

Lemma 2.3.1

If $(\mathcal{L}, \mathcal{R})$ is a cotorsion factorization system on $\mathcal{C}$ with respect to $(\mathcal{A}, \mathcal{B})$, then $(\mathcal{A}, \mathcal{B})$ is complete.

Moreover, if $(\mathcal{L}, \mathcal{R})$ is a functorial cotorsion factorization system, then $(\mathcal{A}, \mathcal{B})$ is functorially complete.

Proof.

For if $X \in \mathrm{Ob}(\mathcal{C})$, then we can factor the map $0 \to X$ as follows:

$$\begin{tikzcd} & A \arrow[dr, "r"] & \\ 0 \arrow[ur, hook, "l"] \arrow[rr] & & X \end{tikzcd}$$

with $l \in \mathcal{L}$ and $r \in \mathcal{B}$. Since $r \in \mathcal{B}$, it is an epimorphism with $\mathrm{Ker}(r) \in \mathcal{B}$. On the other hand, $0 \to A$ is a monomorphism with $A = \mathrm{CoKer}(0 \to A) \in \mathcal{B}$. So we get an exact sequence $0 \to \mathrm{Ker}(r) \to A \to X \to 0$ with $A \in \mathcal{A}$ and $\mathrm{Ker}(r) \in \mathcal{B}$, i.e. $(\mathcal{A}, \mathcal{B})$ is left complete. Factoring the map $X \to 0$ as above, one can show that $(\mathcal{A}, \mathcal{B})$ is right complete. Hence, $(\mathcal{A}, \mathcal{B})$ is complete.

<u>Functoriality</u>: We show that $(\mathcal{A}, \mathcal{B})$ is functorially complete if $(\mathcal{L}, \mathcal{R})$ is equipped with a functorial factorization (L, R). For every $X \in \mathrm{Ob}(\mathcal{C})$, we have a commutative diagram

$$\begin{tikzcd}
B_X \arrow[dr, hook, "L(0 \to X)"'] & & \\
& A_X \arrow[dr, "R(0 \to X)"] & \\
0 \arrow[ur] \arrow[rr] & & X
\end{tikzcd}$$

where the sequence $B_X \hookrightarrow A_X \twoheadrightarrow X$ is exact. We prove A_X is functorial in X. Given a map $X \xrightarrow{f} Y$ in $\mathcal{C}$, derive $\mathrm{Hom}(A_X, -)$ from the sequence $B_Y \hookrightarrow A_Y \twoheadrightarrow Y$ to get an exact sequence $\mathrm{Hom}(A_X, A_Y) \to \mathrm{Hom}(A_X, Y) \to \mathrm{Ext}^1(A_X, B_Y) = 0$. So there exists a morphism $A_X \to A_Y$ such that the following square commutes:

$$\begin{tikzcd}
A_X \arrow[r] \arrow[d] & A_Y \arrow[d] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Define $\mathcal{C} \xrightarrow{F} \mathcal{C}$ as follows:

- For every $X \in \mathrm{Ob}(\mathcal{C})$, $F(X) = A_X$.
- For every morphism $X \xrightarrow{f} Y$, $F(f)$ is the filler $A_X \to A_Y$ in the above square.

It is not hard to see that F defines a functor. Moreover, note that for every morphism $X \xrightarrow{f} Y$, the morphism $R(0 \to X) \xrightarrow{R(0,f)} R(0 \to Y)$ in $\mathrm{Map}(\mathcal{C})$ is given by the pair $(F(f), f)$:

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "R(0 \to X)"'] & F(Y) \arrow[d, "R(0 \to Y)"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

So R defines a natural epimorphism $F \to \mathrm{id}_{\mathcal{C}}$. Hence $(\mathcal{A}, \mathcal{B})$ is functorially left complete. The rest follows in a similar way. $\square$

Lemma 2.3.2

Let $\mathcal{A}$ be a class closed under extensions.

The class $\mathbf{Mono}_{\mathcal{A}}(\mathcal{C})$ of monomorphisms with cokernel in $\mathcal{A}$ is closed under compositions.	The class $\mathbf{Epi}_{\mathcal{A}}(\mathcal{C})$ of epimorphisms with kernel in $\mathcal{A}$ is closed under compositions.

Proof.

We only prove the left statement. Let $f : X \to Y$ and $g : Y \to Z$ be in $\mathrm{Momo}_{\mathcal{A}}(\mathcal{C})$. It is clear that $g \circ f$ is a monomorphism. By Snake's Lemma, we have a short exact sequence $0 \to \mathrm{CoKer}(f) \to \mathrm{CoKer}(g \circ f) \to \mathrm{CoKer}(g) \to 0$, with the two ends in $\mathcal{A}$. Since $\mathcal{A}$ is closed under extensions, we have $\mathrm{CoKer}(g \circ f) \in \mathcal{A}$. $\square$

Theorem 2.3.3

If $(\mathcal{A},\mathcal{B})$ is a complete cotorsion pair in an Abelian category $\mathcal{C}$, then the two classes

$$\mathcal{L} := \{l \in \mathbf{Mono}(\mathcal{C}) : \mathrm{CoKer}(l) \in \mathcal{A}\} \text{ and } \mathcal{R} := \{r \in \mathbf{Epi}(\mathcal{C}) : \mathrm{Ker}(r) \in \mathcal{B}\}$$

form a cotorsion factorization system $(\mathcal{L},\mathcal{R})$ on $\mathcal{C}$. Moreover, if $(\mathcal{A},\mathcal{B})$ is functorially complete, then $(\mathcal{L},\mathcal{R})$ is a functorial weak factorization system. Conversely, if $(\mathcal{L},\mathcal{R})$ is a weak factorization system on $\mathcal{C}$ such that $\mathcal{L} \subseteq \mathbf{Mono}(\mathcal{C})$, $\mathcal{R} \subseteq \mathbf{Epi}(\mathcal{C})$ and $\mathrm{Ext}^1_{\mathcal{C}}(A,X) = 0$ for every $A \in \mathrm{CoKer}(\mathcal{L})$ and $X \in \mathrm{Ker}(\mathcal{R})$, then $(\mathrm{CoKer}(\mathcal{L}),\mathrm{Ker}(\mathcal{R}))$ is a cotorsion pair and $(\mathcal{L},\mathcal{R})$ is a cotorsion factorization system with respect to $(\mathrm{CoKer}(\mathcal{L}),\mathrm{Ker}(\mathcal{R}))$.

Proof.

$(\Longrightarrow)$ For the first part, suppose that $(\mathcal{A},\mathcal{B})$ is a complete cotorsion pair in $\mathcal{C}$. We prove conditions **(1)**, **(2)**, and **(3)** of Theorem 2.1.1.

Condition (1): $l \pitchfork r$ for every $l \in \mathcal{L}$ and $r \in \mathcal{R}$: This follows by Lemma 2.1.4.

Condition (2): $\mathcal{L}$ is closed under codomain retracts and $\mathcal{R}$ is closed under domain retracts: We only prove the statement concerning $\mathcal{L}$, since the other one is dual. Suppose we are given a commutative diagram

$$\begin{tikzcd} & Z \arrow[dl, "u"'] \arrow[d, "l"] \arrow[dr, "u"] & \\ X \arrow[r, "f"'] & Y \arrow[r, "g"'] & X \end{tikzcd}$$

where $l \in \mathcal{L}$ and $g \circ f = \mathrm{id}_X$. Note u is monic since $l = f \circ u$ and l is monic. It is only left to show that $\mathrm{CoKer}(u) \in \mathcal{A}$. By the universal property of cokernels, there exist maps $f' : \mathrm{CoKer}(u) \to \mathrm{CoKer}(l)$ and $g' : \mathrm{CoKer}(l) \to \mathrm{CoKer}(u)$ such that the following diagram commutes:

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, two heads, "u'"] & Y \arrow[r, "g"] \arrow[d, two heads, "l'"] & X \arrow[d, two heads, "u'"] \\
\mathrm{CoKer}(u) \arrow[r, "f'"'] & \mathrm{CoKer}({\color{red}l}) \arrow[r, "g'"'] & \mathrm{CoKer}(u)
\end{tikzcd}
$$

We have $(g' \circ f') \circ u' = g' \circ l' \circ f = u' \circ g \circ f = u'$. Since u' is epic, we have $g' \circ f' = \mathrm{id}_{\mathrm{CoKer}(u)}$, i.e. $\mathrm{CoKer}(u)$ is a retract of $\mathrm{CoKer}({\color{red}l}) \in {\color{red}\mathcal{A}}$. Hence $\mathrm{CoKer}(u) \in {\color{red}\mathcal{A}}$ since ${\color{red}\mathcal{A}}$ is closed under retracts.

Condition (3): Every morphism $X \xrightarrow{f} Y$ can be factored as $f = {\color{blue}r} \circ {\color{red}l}$ where ${\color{red}l} \in {\color{red}\mathcal{L}}$ and ${\color{blue}r} \in {\color{blue}\mathcal{R}}$:

<u>Case 1</u>: f is a monomorphism. Then we have an exact sequence $X \overset{f}{\hookrightarrow} Y \twoheadrightarrow C$ where $C = \mathrm{CoKer}(f)$. Since $({\color{red}\mathcal{A}}, {\color{blue}\mathcal{B}})$ is complete, there exists a short exact sequence ${\color{blue}B} \hookrightarrow {\color{red}A} \twoheadrightarrow C$, where ${\color{red}A} \in {\color{red}\mathcal{A}}$, ${\color{blue}B} \in {\color{blue}\mathcal{B}}$. Taking the pullback of $Y \twoheadrightarrow C$ and ${\color{red}A} \twoheadrightarrow C$, we get the following commutative diagram:

$$
\begin{tikzcd}
 & {\color{blue}B} \arrow[r, equal] \arrow[d, hook] & {\color{blue}B} \arrow[d, hook] \\
X \arrow[r, hook] \arrow[d, equal] & Y \times_C {\color{red}A} \arrow[r, two heads] \arrow[d, two heads] & {\color{red}A} \arrow[d, two heads] \\
X \arrow[r, hook, "f"'] & Y \arrow[r, two heads] & C
\end{tikzcd}
$$

Note that in the left bottom commutative square, $(X \hookrightarrow Y \times_C {\color{red}A}) \in {\color{red}\mathcal{L}}$ and $(Y \times_C {\color{red}A} \to Y) \in {\color{blue}\mathcal{R}}$. So condition **(3)** holds in the class of monic maps.

<u>Case 2</u>: f is a epimorphism. This case is dual.

<u>Case 3</u>: General case. Given any map $X \xrightarrow{f} Y$, write it as the composition $X \xrightarrow{(\mathrm{id}_X, f)} X \oplus Y \xrightarrow{\rho_Y} Y$, where $X \oplus Y$ is the biproduct of X and Y, ρ_Y is the projection of $X \oplus Y$ onto Y, and (id_X, f) is the only map such that $\rho_X \circ (\mathrm{id}_X, f) = \mathrm{id}_X$ and $\rho_Y \circ (\mathrm{id}_X, f) = f$. It is not hard to see that (id_X, f) is monic and ρ_Y is an epic. So by Case 1 we can write $(\mathrm{id}_X, f) = r' \circ l'$ with $l' \in \mathcal{L}$ and $r' \in \mathcal{R}$. On the other hand, since $\rho_Y \circ r'$ is epic, by Case 2 we can write $\rho_Y \circ r' = r \circ l''$ with $l'' \in \mathcal{L}$ and $r \in \mathcal{R}$. So we obtain $f = r \circ (l'' \circ l')$. It suffices to check that $l'' \circ l' \in \mathcal{L}$. This is a consequence of the fact that $\mathcal{L}$ is closed under compositions. For if l'' and l' are maps in $\mathcal{L}$ such that the composition $l'' \circ l'$ makes sense, then it is clear that $l'' \circ l'$ is monic. By Snake's Lemma, there is an exact sequence $0 \to \mathrm{CoKer}(l') \to \mathrm{CoKer}(l'' \circ l') \to \mathrm{CoKer}(l'') \to 0$ with $\mathrm{CoKer}(l'), \mathrm{CoKer}(l'') \in \mathcal{A}$. It follows $\mathrm{CoKer}(l'' \circ l') \in \mathcal{A}$ since $\mathcal{A}$ is closed under extensions. Dually, the class $\mathcal{R}$ is also closed under compositions.

$$
\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "{(\mathrm{id}_X, f)}"'] \arrow[dd, "l'"'] & & Y \\
& X \oplus Y \arrow[ur, "\rho_Y"'] & \\
X' \arrow[ur, "r'"'] \arrow[rr, "l''"'] & & Y' \arrow[uu, "r"']
\end{tikzcd}
$$

Figure 2.5: Factorizations in cotorsion factorization systems.

Condition (3*): $(\mathcal{L}, \mathcal{R})$ is equipped with a functorial factorization if $(\mathcal{A}, \mathcal{B})$ is functorially complete: Let $X \xrightarrow{f} Y$ be a morphism in $\mathcal{C}$.

<u>**Case 1**</u>**:** f is a monomorphism: Let $C = \mathrm{CoKer}(f)$. Since $(\mathcal{A}, \mathcal{B})$ is functorially left complete, there exists a functor $F : \mathcal{C} \to \mathcal{C}$ along with a natural epimorphism $F \to \mathrm{id}_{\mathcal{C}}$ such that $F(Z) \in \mathcal{A}$ and $B_Z := \mathrm{Ker}(F(Z) \twoheadrightarrow Z) \in \mathcal{B}$ for every $Z \in \mathrm{Ob}(\mathcal{C})$. We have the following commutative diagram

$$
\begin{tikzcd}
 & & B_C \arrow[rr, equal] \arrow[d, hook] & & B_C \arrow[d, hook] \\
X \arrow[rr, hook, "L^{\mathrm{monic}}(f)"] \arrow[d, equal] & & Y \times_C F(C) \arrow[rr, two heads, "\varphi_f"] \arrow[d, two heads, "R^{\mathrm{monic}}(f)"] & & F(C) \arrow[d, two heads, "p_f"] \\
X \arrow[rr, hook, "f"'] & & Y \arrow[rr, two heads, "f'"'] & & C
\end{tikzcd}
$$

where $L^{\mathrm{monic}}(f) \in \mathcal{L}$ and $R^{\mathrm{monic}}(f) \in \mathcal{R}$. We prove that L^{monic} and R^{monic} give rise to functors $\mathbf{Mono}(\mathcal{C}) \to \mathbf{Mono}(\mathcal{C})$, where the class $\mathbf{Mono}(\mathcal{C})$ of monic maps of $\mathcal{C}$ is considered as a full subcategory of $\mathbf{Map}(\mathcal{C})$. Let $(u, v) : f \to g$ be a morphism in $\mathbf{Mono}(\mathcal{C})$. We want to construct a map $L^{\mathrm{monic}}(u, v) : L^{\mathrm{monic}}(f) \to L^{\mathrm{monic}}(g)$, i.e. a commutative square

$$
\begin{tikzcd}
X \arrow[r, hook, "L^{\mathrm{monic}}(f)"] \arrow[d, "u"'] & Y \times_C F(C) \arrow[d, "L^{\mathrm{monic}}(u,v)"] \\
W \arrow[r, hook, "L^{\mathrm{monic}}(g)"'] & Z \times_D F(D)
\end{tikzcd}
$$

Note that we have the following commutative diagram with exact rows, for which we want to find a filler $Y \times_C F(C) \to Z \times_D F(D)$:

$$
\begin{tikzcd}
0 \arrow[r] & X \arrow[rr, "L^{\mathrm{monic}}(f)"] \arrow[dr, "u"] \arrow[dd, equal] & & Y \times_C F(C) \arrow[rr, "\varphi_f"] \arrow[dr, dotted] \arrow[dd] & & F(C) \arrow[rr] \arrow[dr, "F(w)"] \arrow[dd] & & 0 & \\
 & 0 \arrow[r] & W \arrow[rr, "L^{\mathrm{monic}}(g)"] \arrow[dd, equal] & & Z \times_D F(D) \arrow[rr, "\varphi_g"] \arrow[dd] & & F(D) \arrow[rr] \arrow[dd] & & 0 \\
0 \arrow[r] & X \arrow[rr, "f"] \arrow[dr, "u"] & & Y \arrow[rr, "f'"] \arrow[dr, "v"] & & C \arrow[rr] \arrow[dr, "w"] & & 0 & \\
 & 0 \arrow[r] & W \arrow[rr, "g"'] & & Z \arrow[rr, "g'"'] & & D \arrow[rr] & & 0
\end{tikzcd}
$$

where w is the only morphism satisfying $w \circ f' = g' \circ v$ (by the universal property of cokernels). Note that $g' \circ (v \circ R^{\text{monic}}(f)) = w \circ f' \circ R^{\text{monic}}(f) = w \circ p_f \circ \varphi_f = p_g \circ (F(w) \circ \varphi_f)$, and by the universal property of pullbacks there exists a unique morphism $Y \times_C F(C) \overset{L^{\text{monic}}}{\rightarrow}{}_{u,v}\, Z \times_D F(D)$ such that the following diagram commutes:

$$
\begin{tikzcd}
Y \times_C F(C) \arrow[dr, dotted, "\exists!\ {L^{\text{monic}}}_{u,v}"] \arrow[drr, bend left, "F(w) \circ \varphi_f"] \arrow[ddr, bend right, "v \circ R^{\text{monic}}(f)"'] & & \\
& Z \times_D F(D) \arrow[r, "\varphi_g"] \arrow[d, "R^{\text{monic}}(g)"'] & F(D) \arrow[d, "p_g"] \\
& Z \arrow[r, "g'"'] & D
\end{tikzcd}
$$

We check that ${L^{\text{monic}}}_{u,v} \circ L^{\text{monic}}(f) = L^{\text{monic}}(g) \circ u$. This is a consequence of the following equalities and the universal property of pullbacks:

$$
\begin{aligned}
\varphi_g \circ ({L^{\text{monic}}}_{u,v} \circ L^{\text{monic}}(f)) &= F(w) \circ \varphi_f \circ L^{\text{monic}}(f) = F(w) \circ 0 = 0, \\
R^{\text{monic}}(g) \circ ({L^{\text{monic}}}_{u,v} \circ L^{\text{monic}}(f)) &= v \circ R^{\text{monic}}(f) \circ L^{\text{monic}}(f) = v \circ f = g \circ u, \\
\varphi_g \circ (L^{\text{monic}}(g) \circ u) &= 0 \circ u = 0, \\
R^{\text{monic}}(g) \circ (L^{\text{monic}}(g) \circ u) &= g \circ u.
\end{aligned}
$$

$$
\begin{tikzcd}
X \arrow[dr, bend left=10, "{L^{\text{monic}}}_{u,v} \circ L^{\text{monic}}(f)"] \arrow[dr, bend right=10, "L^{\text{monic}}(g) \circ u"'] \arrow[drr, bend left, "0"] \arrow[ddr, bend right, "g \circ u"'] & & \\
& Y \times_D F(D) \arrow[r, "\varphi_g"] \arrow[d, "R^{\text{monic}}(g)"'] & F(D) \arrow[d, "p_g"] \\
& Z \arrow[r, "g'"'] & D
\end{tikzcd}
$$

Define a correspondence $(u, v) \mapsto L^{\text{monic}}(u, v) := (u, {L^{\text{monic}}}_{u,v})$. The fact that L^{monic} is a functor can be proven by using the universal property of pullbacks and the functoriallity of F. Similarly, R^{monic} is also a functor of the subcategory of $\text{Map}(\mathcal{C})$ formed by the monomorphisms of $\mathcal{C}$.

<u>Case 2</u>: The case where f is an epimorphism follows similarly, taking pushouts of monomorphisms instead of pullbacks. We get a functorial factorization $(L^{\text{epi}}, R^{\text{epi}}) : \mathbf{Epi}(\mathcal{C}) \to \mathbf{Epi}(\mathcal{C})$, along with its respective commutative pushout diagrams.

<u>Case 3</u>: For the general case, for every map $X \xrightarrow{f} Y$ we have he following commutative diagram where $C = \text{CoKer}((\text{id}_X, f))$ and $K = \text{Ker}(R^{\text{epi}}(\rho_Y \circ R^{\text{monic}}(\text{id}_X, f)))$:

$$\begin{tikzcd}
 & X \arrow[rr, "f"] \arrow[dl, "L^{\text{monic}}((\text{id}_X, f))"'] \arrow[dr, "(\text{id}_X, f)"] & & Y \\
(X \oplus Y) \times_C F(C) \arrow[rr, "R^{\text{monic}}((\text{id}_X, f))"'] \arrow[drrr, bend right, "L^{\text{epi}}(\rho_Y \circ R^{\text{monic}}(\text{id}_X, f))"'] & & X \oplus Y \arrow[ur, "\rho_Y"] & \\
 & & & {[(X \oplus Y) \times_C F(C)] \coprod_K G(K)} \arrow[uu, "R^{\text{epi}}(\rho_Y \circ R^{\text{monic}}(\text{id}_X, f))"']
\end{tikzcd}$$

Figure 2.6: Functorial factorizations in cotorsion factorization systems.

We construct functors $L, R : \text{Map}(\mathcal{C}) \to \text{Map}(\mathcal{C})$. Set

$$L(f) := L^{\text{epi}}(\rho_Y \circ R^{\text{monic}}((\text{id}_X, f))) \circ L^{\text{monic}}((\text{id}_X, f)) \text{ and}$$
$$R(f) := R^{\text{epi}}(\rho_Y \circ R^{\text{monic}}((\text{id}_X, f))).$$

Now suppose we are given a map $(X \xrightarrow{f} Y) \xrightarrow{(u,v)} (W \xrightarrow{g} Z)$ in $\text{Map}(\mathcal{C})$. Let $C = \text{CoKer}((\text{id}_X, f))$ and $D = \text{CoKer}((\text{id}_X, g))$. On the one hand, by the universal property of biproducts, there exists a unique map $X \oplus Y \xrightarrow{u'} W \oplus Z$ such that $u' \circ (\mathrm{id}_X, f) = (\mathrm{id}_W, g) \circ u$. On the other hand, we have a unique map $C \xrightarrow{u''} D$ in the following diagram:

$$\begin{tikzcd}
X \arrow[r, hook, "{(\mathrm{id}_X, f)}"] \arrow[d, "u"'] & X \oplus Y \arrow[r, two heads] \arrow[d, "u'"] & C \arrow[d, "u''"] \\
W \arrow[r, hook, "{(\mathrm{id}_W, g)}"'] & W \oplus Z \arrow[r, two heads] & D
\end{tikzcd}$$

Then there exists a unique morphism $(X \oplus Y) \times_C F(C) \to (W \oplus Z) \times_D F(D)$ such that the following diagram commutes:

$$\begin{tikzcd}
(X \oplus Y) \times_C F(C) \arrow[dr, dotted] \arrow[drr, bend left, "F(u'') \circ \pi_{F(C)}"] \arrow[ddr, bend right, "u' \circ \pi_{X \oplus Y}"'] & & \\
& (W \oplus Z) \times_D F(D) \arrow[r] \arrow[d] & F(D) \arrow[d] \\
& W \oplus Z \arrow[r] & D
\end{tikzcd}$$

Using again the universal property of pullbacks, we get the following commutative square:

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "L^{\mathrm{monic}}((\mathrm{id}_X, f))"'] & Y \arrow[d, "L^{\mathrm{monic}}((\mathrm{id}_W, g))"] \\
(X \oplus Y) \times_C F(C) \arrow[r] & (W \oplus Z) \times_D F(D)
\end{tikzcd}$$

Now let $K = \mathrm{Ker}(\rho_Y \circ R^{\mathrm{monic}}((\mathrm{id}_X, f)))$ and $Q = \mathrm{Ker}(\rho_Z \circ R^{\mathrm{monic}}((\mathrm{id}_W, g)))$. There exists a unique morphism $K \xrightarrow{v''} Q$ such that the following diagram commutes:

$$\begin{tikzcd}
K \arrow[r, hook, "j_K"] \arrow[d, "v''"] & (X \oplus Y) \times_C F(C) \arrow[r, two heads, "\rho_Y \circ R^{\text{monic}}((\mathrm{id}_X, f))"] \arrow[d, "v'"] & Y \arrow[d, "v"] \\
Q \arrow[r, hook, "j_Q"'] & (W \oplus Z) \times_D F(D) \arrow[r, two heads, "\rho_Z \circ R^{\text{monic}}((\mathrm{id}_W, g))"'] & Z
\end{tikzcd}$$

Recall we also have commutative diagrams:

$$\begin{tikzcd}
K \arrow[r, hook, "j_K"] \arrow[d] & (X \oplus Y) \times_C F(C) \arrow[r, two heads] \arrow[d] & Y \arrow[d, equal] \\
G(K) \arrow[r, hook, "i_K"'] & {[(X \oplus Y) \times_C F(C)] \coprod_K G(K)} \arrow[r, two heads] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
Q \arrow[r, hook, "j_Q"] \arrow[d] & (W \oplus Z) \times_D F(D) \arrow[r, two heads] \arrow[d] & Z \arrow[d, equal] \\
G(Q) \arrow[r, hook, "i_Q"'] & {[(W \oplus Z) \times_D F(D)] \coprod_Q G(Q)} \arrow[r, two heads] & Z
\end{tikzcd}$$

Then $(L^{\text{epi}}(\rho_Y \circ R^{\text{monic}}((\mathrm{id}_W, g))) \circ v') \circ j_K = i_Q \circ (G(v'') \circ (K \to G(K)))$. It follows there exists a unique morphism w such that the following diagram commutes:

$$\begin{tikzcd}
K \arrow[r] \arrow[d] & (X \oplus Y) \times_C F(C) \arrow[d] \arrow[ddr, bend left, "L^{\text{epi}}(\rho_Z \circ R^{\text{monic}}((\mathrm{id}_W, g))) \circ v'"] & \\
G(K) \arrow[r] \arrow[drr, bend right, "i_Q \circ G(v'')"'] & {[(X \oplus Y) \times_C F(C)] \coprod_K G(K)} \arrow[dr, dotted, "w"] & \\
 & & {[(W \oplus Z) \times_D F(D)] \coprod_Q G(Q)}
\end{tikzcd}$$

Using the universal property again,

$$R^{\text{epi}}(\rho_Z \circ R^{\text{monic}}((\mathrm{id}_W, g))) \circ w = v \circ R^{\text{epi}}(\rho_Y \circ R^{\text{monic}}((\mathrm{id}_X, f))).$$

We have the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "L^{\mathrm{monic}}((\mathrm{id}_X, f))"'] & W \arrow[d, "L^{\mathrm{monic}}((\mathrm{id}_W, g))"] \\
(X \oplus Y) \times_C F(C) \arrow[r, "v'"] \arrow[d, "L^{\mathrm{epi}}(\rho_Y \circ R^{\mathrm{monic}}((\mathrm{id}_X, f)))"'] & (W \oplus Z) \times_D F(D) \arrow[d, "L^{\mathrm{epi}}(\rho_Z \circ R^{\mathrm{monic}}((\mathrm{id}_W, g)))"] \\
{[(X \oplus Y) \times_C F(C)] \coprod_K G(K)} \arrow[r, "w"] \arrow[d, "R^{\mathrm{epi}}(\rho_Y \circ R^{\mathrm{monic}}((\mathrm{id}_X, f)))"'] & {[(W \oplus Z) \times_D F(D)] \coprod_Q G(Q)} \arrow[d, "R^{\mathrm{epi}}(\rho_Z \circ R^{\mathrm{monic}}((\mathrm{id}_W, g)))"] \\
Y \arrow[r, "v"'] & Z
\end{tikzcd}$$

We set $L((u,v)) := (u,w)$ and $R((u,v)) := (w,v)$. It is not hard to show that (L, R) defines a functorial factorization on $\mathcal{C}$.

($\Longleftarrow$) Assume $(\mathcal{L}, \mathcal{R})$ is a weak factorization system such that $\mathcal{L} \subseteq \mathbf{Mono}(\mathcal{C})$, $\mathcal{R} \subseteq \mathbf{Epi}(\mathcal{C})$ and $\mathrm{Ext}^1_{\mathcal{C}}(A, X) = 0$ for every $A \in \mathrm{CoKer}(\mathcal{L})$ and $X \in \mathrm{Ker}(\mathcal{R})$. First, we show the classes $\mathrm{CoKer}(\mathcal{L})$ and $\mathrm{Ker}(\mathcal{R})$ are given by $\{X \in \mathrm{Ob}(\mathcal{C}) : (0 \to X) \in \mathcal{L}\}$ and $\{Y \in \mathrm{Ob}(\mathcal{C}) : (Y \to 0) \in \mathcal{R}\}$, respectively. Let $X \in \mathrm{CoKer}(\mathcal{L})$. Then there is a map $Y \xrightarrow{l} Z$ in $\mathcal{L}$ such that $X = \mathrm{CoKer}(l)$. Since the sequence $Y \overset{l}{\hookrightarrow} Z \twoheadrightarrow X$ is exact, we have $0 \to X$ is the cobase change of l along $Y \to 0$. Since $\mathcal{L}$ is closed under cobase changes, we get $(0 \to X) \in \mathcal{L}$. The other inclusion follows by the definition of $\mathrm{CoKer}(\mathcal{L})$. Hence $\mathrm{CoKer}(\mathcal{L}) = \{X \in \mathrm{Ob}(\mathcal{C}) : (0 \to X) \in \mathcal{L}\}$. Similarly, $\mathrm{Ker}(\mathcal{R}) = \{Y \in \mathrm{Ob}(\mathcal{C}) : (Y \to 0) \in \mathcal{R}\}$.

Since $\mathrm{Ext}^1_{\mathcal{C}}(A, X) = 0$ for every $A \in \mathrm{CoKer}(\mathcal{L})$ and $X \in \mathrm{Ker}(\mathcal{R})$, we only need to show ${}^{\perp}(\mathrm{Ker}(\mathcal{R})) \subseteq \mathrm{CoKer}(\mathcal{L})$ and $(\mathrm{CoKer}(\mathcal{L}))^{\perp} \subseteq \mathrm{Ker}(\mathcal{R})$ in order to prove $(\mathrm{CoKer}(\mathcal{L}), \mathrm{Ker}(\mathcal{R}))$ is a cotorsion pair. Let $X \in {}^{\perp}(\mathrm{Ker}(\mathcal{R}))$. Suppose $r \in \mathcal{R}$. Notice $\mathrm{Ker}(r) \in \mathrm{Ker}(\mathcal{R})$. Since $0 \to X$ is monic, r is epic, and $\mathrm{Ext}^1_{\mathcal{C}}(X, \mathrm{Ker}(r)) =$

0, we have $(0 \to X) \pitchfork r$ for every $r \in \mathcal{R}$, by Lemma 2.1.4. We have $X \in \{Z \in \mathrm{Ob}(\mathcal{C}) : (0 \to Z) \in \mathcal{L}\} = \mathrm{CoKer}(\mathcal{L})$. The inclusion $(\mathrm{CoKer}(\mathcal{L}))^{\perp} \subseteq \mathrm{Ker}(\mathcal{R})$ follows similarly. $\square$

As an interesting property for cotorsion factorization systems, we have that it is impossible to obtain unique factorizations.

Proposition 2.3.4

If $\mathcal{C}$ is a nonzero Abelian category, then there are no cotorsion factorization systems on $\mathcal{C}$ which are factorization systems.

Proof.

Suppose $(\mathcal{L}, \mathcal{R})$ is a cotorsion factorization system with respect to a cotorsion pair $(\mathcal{A}, \mathcal{B})$, which is a factorization system. Then the class $\mathcal{R}$ is closed under diagonals. Let $r \in \mathcal{R}$. Then we have a short exact sequence $B \hookrightarrow W \overset{r}{\twoheadrightarrow} Z$ where $B \in \mathcal{B}$. On the one hand, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
B \arrow[r, hook] \arrow[d, equal] & W \times_Z W \arrow[r, two heads, "\pi_1"] \arrow[d, "\pi_2"'] & W \arrow[d, two heads, "r"] \\
B \arrow[r, hook] & W \arrow[r, two heads, "r"'] & Z
\end{tikzcd}$$

Since $\pi_1 \circ \delta(r) = \mathrm{id}_W$, we have that $\delta(r)$ is monic. On the other hand, $r \in \mathcal{R}$ implies that $\delta(r) \in \mathcal{R}$, and so $\delta(r)$ is epic. Then, $\delta(r)$ is an isomorphism, and hence $B = 0$. It follows r is an isomorphism. Therefore, $\mathcal{R} = \mathbf{Iso}(\mathcal{C})$. Similarly, $\mathcal{L} = \mathbf{Iso}(\mathcal{C})$. We have a weak factorization system $(\mathbf{Iso}(\mathcal{C}), \mathbf{Iso}(\mathcal{C}))$ on $\mathcal{C}$, so every morphism in $\mathcal{C}$ is an isomorphism. Therefore, every object in $\mathcal{C}$ is isomorphic to the zero object 0, getting a contradiction since $\mathcal{C} \neq \{0\}$. $\square$

Definition 2.3.2. A model structure $(\mathcal{C}_{of}, \mathcal{F}_{ib}, \mathcal{W}_{eak})$ on an Abelian category $\mathcal{C}$ is called Abelian if the following two conditions are satisfied:

(1) $f \in \mathcal{C}_{of}$ (resp. $f \in \mathcal{C}_{of} \cap \mathcal{W}_{eak}$) if, and only if, f is monic with cofibrant (resp. trivially cofibrant) cokernel.

(2) $g \in \mathcal{F}_{ib}$ (resp. $g \in \mathcal{F}_{ib} \cap \mathcal{W}_{eak}$) if, and only if, g is epi with fibrant (resp. trivially fibrant) kernel.

Example 2.3.1. The following are examples of Abelian model categories. All of these model structures were discovered before the notion of Abelian model structures.

(1) Recall that a ring R is called a quasi-Frobenius ring if the projective and injective left (or right) R-modules coincide. Two homomorphisms $f, g : M \to N$ are said to be stably equivalent (denoted $f \sim g$) if $f - g$ factors through a projective module. A homomorphism $f : M \to N$ is said to be a stable equivalence if there exists a homomorphism $h : N \to M$ such that $h \circ f \sim \mathrm{id}_M$ and $f \circ h \sim \mathrm{id}_N$. On ${}_R\mathbf{Mod}$, with R a quasi-Frobenius ring, there is the following model structure:

- The class of weak equivalences is the class of stable equivalences.
- The fibrations are given by the epimorphisms.
- The cofibrations are given by the monomorphisms.

Details can by found in (36, Section 2.2). The Hovey pair in this case is given by $(\mathrm{Ob}({}_R\mathbf{Mod}), \mathcal{P}_0)$ and $(\mathcal{P}_0, \mathrm{Ob}({}_R\mathbf{Mod}))$. We shall recall this model structure in Chapter 4.

Most of the model structures we present in this thesis are on the category $\mathbf{Ch}({}_R\mathbf{Mod})$ of complexes over modules. We start giving two well known examples on this category.

(2) The projective model structure on $\mathbf{Ch}({}_R\mathbf{Mod})$: on the category $\mathbf{Ch}({}_R\mathbf{Mod})$ of chain complexes over a ring R is given by the following three classes:

- The weak equivalences are given by the quasi-isomorphisms, i.e. chain maps $f : X \to Y$ such that each induced group homomorphism $H_n(f) : H_n(X) \to H_n(Y)$ is an isomorphism.
- The class of fibrations is given by the class of epimorphisms, i.e. chain maps $f : X \to Y$ such that $f_n : X_n \to Y_n$ is an epimorphism of modules, for every $n \in \mathbb{Z}$. A chain map is a trivial cofibration if an only if it has the right lifting property with respect to every chain map of the form $S^{n-1}(R) \to D^n(R)$.
- At this moment, we are only going to say that the cofibrations are given by the class of maps which have the left lifting property with respect to trivial fibrations. More details can be found in (36, Section 2.3). Hovey's correspondence will provide a nice characterization for cofibrations and trivial cofibrations.

(3) The injective model structure on $\mathbf{Ch}({}_R\mathbf{Mod})$: given by the following classes of morphisms.

- The weak equivalences are, as above, the quasi-isomorphisms.
- The trivial fibrations are given by the epimorphisms with injective kernel.
- The cofibrations are all the maps having the left lifting property with respect to the trivial fibrations.

More details on this structure are given (36, Section 2.3). This model structure was probably first discovered by A. Joyal in (40) on the category of complexes over a Grothendieck category.

Example 2.3.2. We give an example of a **NON**-Abelian model structure on $\mathbf{Ch}(\mathcal{C})$. The absolute model structure is defined by the following classes of maps:

- A chain map if a weak equivalence if it is a chain homotopy equivalence.

  Recall that two chain maps $f, g : X \to Y$ are said to be chain homotopic if there exists a family of maps $(D^n : X_n \to Y_{n+1})_{n \in \mathbb{Z}}$ such that $f_n - g_n = \partial^Y_{n+1} \circ D_n + D_{n-1} \circ \partial^X_n$. A chain map $f : X \to Y$ is said to be a chain homotopy equivalence if there exists a chain map $g : Y \to X$ such that $g \circ f$ and $f \circ g$ are homotopic to id_X and id_Y, respectively.

- A chain map is a cofibration if it is a degreewise split monomorphism.

- A chain map is a fibration if it is a degreewise split epimorphism.

This model structure can be found in (14, Example 3.4). If this model structure were Abelian, then every monomorphism with cofibrant cokernel would be a degreewise split monomorphism. Note that every object X in $\mathbf{Ch}(\mathcal{C})$ is cofibrant, since $0 \to X$ is clearly a degreewise split monomorphism. The map $S^0(2\mathbb{Z}) \to S^0(\mathbb{Z})$ is a monomorphism with cofibrant cokernel which is not a degreewise split monomorphism. Hence, the absolute model structure on $\mathbf{Ch}({}_{\mathbb{Z}}\mathbf{Mod})$ is not Abelian.

Definition 2.3.3. Two cotorsion pairs $(\mathcal{A}, \mathcal{B}')$ and $(\mathcal{A}', \mathcal{B})$ are said to be compatible if $\mathcal{A}' = \mathcal{A} \cap \mathcal{W}$ and $\mathcal{B} = \mathcal{B}' \cap \mathcal{W}$, for some class $\mathcal{W}$. If in addition the class $\mathcal{W}$ is thick, we shall say that $(\mathcal{A} \cap \mathcal{W}, \mathcal{B})$ and $(\mathcal{A}, \mathcal{B} \cap \mathcal{W})$ form a Hovey pair.

Definition 2.3.4. Let $(\mathcal{C}_{of}, \mathcal{F}_{ib}, \mathcal{W}_{eak})$ be a model structure on a bicomplete Abelian category $\mathcal{C}$, and $(\mathcal{A} \cap \mathcal{W}, \mathcal{B})$ and $(\mathcal{A}, \mathcal{B} \cap \mathcal{W})$ be a Hovey pair in $\mathcal{C}$. We shall say that $(\mathcal{C}_{of}, \mathcal{F}_{ib}, \mathcal{W}_{eak})$ is a cotorsion model structure with respect to $(\mathcal{A} \cap \mathcal{W}, \mathcal{B})$ and $(\mathcal{A}, \mathcal{B} \cap \mathcal{W})$ if:

(1) $(\mathcal{C}_{of} \cap \mathcal{W}_{eak}, \mathcal{F}_{ib})$ is a cotorsion factorization system with respect to $(\mathcal{A} \cap \mathcal{W}, \mathcal{B})$.

(2) $(\mathcal{C}_{of}, \mathcal{F}_{ib} \cap \mathcal{W}_{eak})$ is a cotorsion factorization system with respect to $(\mathcal{A}, \mathcal{B} \cap \mathcal{W})$.

By Lemma 2.3.1, the pairs $(\mathcal{A} \cap \mathcal{W}, \mathcal{B})$ and $(\mathcal{A}, \mathcal{B} \cap \mathcal{W})$ in the previous definition are complete.

Remark 2.3.1. Note that if $(\mathcal{C}_{of}, \mathcal{F}_{ib}, \mathcal{W}_{eak})$ is a cotorsion model structure with respect to a Hovey pair $(\mathcal{A} \cap \mathcal{W}, \mathcal{B})$ and $(\mathcal{A}, \mathcal{B} \cap \mathcal{W})$, then $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{W}$ are the classes of cofibrant, fibrant and trivial objects, respectively. This is clear for $\mathcal{A}$ and $\mathcal{B}$. If $W \in \mathcal{W}$, then we can write $0 \to W$ as a cofibration followed by a trivial fibration. So we have a commutative diagram

$$\begin{tikzcd} & A \arrow[dr, two heads, "r"] & \\ 0 \arrow[ur, hook, "l"] \arrow[rr] & & W \end{tikzcd}$$

with $A \in \mathcal{A}$ and $\mathrm{Ker}(r) \in \mathcal{B} \cap \mathcal{W}$. We get an exact sequence $\mathrm{Ker}(r) \hookrightarrow A \twoheadrightarrow W$ where W, $\mathrm{Ker}(r) \in \mathcal{W}$. Since $\mathcal{W}$ is thick, we have $A \in \mathcal{A} \cap \mathcal{W}$. So l is a trivial cofibration, and $0 \to W$ is a weak equivalence. The converse follows similarly.

Some of the previous results in this chapter allows us to state and prove Hovey's correspondence using our terminology of cotorsion factorization systems. The reader can check the original statement in (35, Theorem 2.2).

Theorem 2.3.5 (Hovey's correspondence)

Let $\mathcal{C}$ be a bicomplete Abelian category.

(1) Every Abelian model structure $(\mathcal{C}_{of}, \mathcal{F}_{ib}, \mathcal{W}_{eak})$ on $\mathcal{C}$ is a cotorsion model structure with respect to $(\mathrm{CoKer}(\mathcal{C}_{of}) \cap \mathcal{W}, \mathrm{Ker}(\mathcal{F}_{ib}))$ and $(\mathrm{CoKer}(\mathcal{C}_{of}), \mathrm{Ker}(\mathcal{F}_{ib}) \cap \mathcal{W})$, where $\mathcal{W}$ is the class of trivial objects.

(2) If $(\mathcal{A} \cap \mathcal{W}, \mathcal{B})$ and $(\mathcal{A}, \mathcal{B} \cap \mathcal{W})$ are functorially complete cotorsion pairs in $\mathcal{C}$ such that the class $\mathcal{W}$ is thick, then there exists a unique cotorsion model structure on $\mathcal{C}$ with respect to $(\mathcal{A} \cap \mathcal{W}, \mathcal{B})$ and $(\mathcal{A}, \mathcal{B} \cap \mathcal{W})$.

$\boxed{\textbf{Proof}}$.

Part (1): Let $(\mathcal{C}_{of}, \mathcal{F}_{ib}, \mathcal{W}_{eak})$ be an Abelian model structure on $\mathcal{C}$. By Theorem 2.3.3, we have two cotorsion factorization systems $(\mathcal{C}_{of} \cap \mathcal{W}_{eak}, \mathcal{F}_{ib})$ and $(\mathcal{C}_{of}, \mathcal{F}_{ib} \cap \mathcal{W}_{eak})$ with respect to the cotorsion pairs $(\mathrm{CoKer}(\mathcal{C}_{of} \cap \mathcal{W}_{eak}), \mathrm{Ker}(\mathcal{F}_{ib}))$ and $(\mathrm{CoKer}(\mathcal{C}_{of}), \mathrm{Ker}(\mathcal{F}_{ib} \cap \mathcal{W}_{eak}))$, respectively. On the other hand, it is clear that $\mathrm{CoKer}(\mathcal{C}_{of} \cap \mathcal{W}_{eak}) = \mathrm{CoKer}(\mathcal{C}_{of}) \cap \mathcal{W}$ and $\mathrm{Ker}(\mathcal{F}_{ib} \cap \mathcal{W}_{eak}) = \mathrm{Ker}(\mathcal{F}_{ib}) \cap \mathcal{W}$. It is only left to show that $\mathcal{W}$ is thick. Since $\mathcal{W}_{eak}$ is closed under retracts of maps, we have that $\mathcal{W}$ is closed under retracts of objects in $\mathcal{W}$. Now let $0 \to W' \xrightarrow{f} W \xrightarrow{g} W'' \to 0$ be a short exact sequence where two out of three of the objects W, W' and W'' are in $\mathcal{W}$.

(i) The case $W', W'' \in \mathcal{W}$ is proven in (35, Lemma 4.3).

(ii) Suppose $W, W'' \in \mathcal{W}$. Factor f as $f = p \circ i$, where $i : W' \overset{\sim}{\hookrightarrow} Q$ is a trivial cofibration and $p : Q \twoheadrightarrow W$ is a fibration. By the universal property of cokernels, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
W' \arrow[r, hook, "i", "\sim"'] \arrow[d, equal] & Q \arrow[r, two heads] \arrow[d, two heads, "p"] & \mathrm{CoKer}(i) \arrow[d, "q"] \\
W' \arrow[r, hook, "f"'] & W \arrow[r, two heads, "g"'] & W''
\end{tikzcd}$$

First, note that q is epic. Using Snake's Lemma, we have $\mathrm{Ker}(q) \cong \mathrm{Ker}(p)$, and so $\mathrm{Ker}(q)$ is fibrant. Then, q is a fibration. On the other hand, we have a commutative triangle

$$\begin{tikzcd}
& 0 \arrow[dl] \arrow[dr] & \\
\mathrm{CoKer}(i) \arrow[rr, two heads, "q"'] & & W''
\end{tikzcd}$$

where the maps $0 \to \mathrm{CoKer}(i)$ and $0 \to W''$ are weak equivalences, since $\mathrm{CoKer}(i)$ is trivially cofibrant and $W'' \in \mathcal{W}$. By the $3{\times}2$ axiom, we have

q is a weak equivalence, and so a trivial fibration. It follows $\mathrm{Ker}(p)$ is trivially fibrant. We have that p is a trivial fibration. Consider the following commutative triangle:

$$\begin{tikzcd} & 0 \arrow[dl] \arrow[dr] & \\ Q \arrow[rr, two heads, "p"'] & & W \end{tikzcd}$$

Since $0 \to W$ and p are weak equivalences, so is $0 \to Q$. Using a similar triangle with base i, we conclude that $0 \to W$ is a weak equivalence.

(iii) The case $W, W' \in \mathcal{W}$ is similar to **(ii)**.

Part (2): Define the following classes of morphisms:

- $\mathbf{Mono}_{\mathcal{A}}(\mathcal{C})$ = monomorphisms with cokernel in $\mathcal{A}$.
- $\mathbf{Mono}_{\mathcal{A}\cap\mathcal{W}}(\mathcal{C})$ = monomorphisms with cokernel in $\mathcal{A} \cap \mathcal{W}$.
- $\mathbf{Epi}_{\mathcal{B}}(\mathcal{C})$ = epimorphisms with kernel in $\mathcal{B}$.
- $\mathbf{Epi}_{\mathcal{B}\cap\mathcal{W}}(\mathcal{C})$ = epimorphisms with kernel in $\mathcal{B} \cap \mathcal{W}$.
- $\mathcal{W}_e$ = maps of the form $e \circ m$ with $m \in \mathbf{Mono}_{\mathcal{A}\cap\mathcal{W}}(\mathcal{C})$ and $e \in \mathbf{Epi}_{\mathcal{B}\cap\mathcal{W}}(\mathcal{C})$.

By Lemma 2.3.2, the first four classes are closed under compositions. First, we check $\mathbf{Mono}_{\mathcal{A}\cap\mathcal{W}}(\mathcal{C}) = \mathbf{Mono}_{\mathcal{A}}(\mathcal{C}) \cap \mathcal{W}_e$. The equality $\mathbf{Epi}_{\mathcal{B}\cap\mathcal{W}}(\mathcal{C}) = \mathbf{Epi}_{\mathcal{B}}(\mathcal{C}) \cap \mathcal{W}_e$ follows similarly. The inclusion $\subseteq$ is clear. Now let $f \in \mathbf{Mono}_{\mathcal{A}}(\mathcal{C}) \cap \mathcal{W}_e$. Write $f = e \circ m$, where $m \in \mathbf{Mono}_{\mathcal{A}\cap\mathcal{W}}(\mathcal{C})$ and $e \in \mathbf{Epi}_{\mathcal{B}\cap\mathcal{W}}(\mathcal{C})$. By Snake's Lemma, we have a short exact sequence $0 \to \mathrm{Ker}(e) \to \mathrm{CoKer}(m) \to \mathrm{CoKer}(f) \to 0$. Moreover, $\mathrm{Ker}(e), \mathrm{CoKer}(m) \in \mathcal{W}$. Since $\mathcal{W}$ is thick and $f \in \mathbf{Mono}_{\mathcal{A}}(\mathcal{C})$, we have $\mathrm{CoKer}(f) \in \mathcal{A} \cap \mathcal{W}$. Hence $f \in \mathbf{Mono}_{\mathcal{A}\cap\mathcal{W}}(\mathcal{C})$.

By the previous two equalities and Theorem 2.3.3, $(\mathbf{Mono}_{\mathcal{A}}(\mathcal{C}) \cap \mathcal{W}_e, \mathbf{Epi}_{\mathcal{B}}(\mathcal{C}))$ and $(\mathbf{Mono}_{\mathcal{A}}(\mathcal{C}), \mathbf{Epi}_{\mathcal{B}}(\mathcal{C}) \cap \mathcal{W}_e)$ are functorial cotorsion factorization systems. It is only left to show that $\mathcal{W}_e$ satisfies the 3×2 axiom.

Suppose we are given a composite morphism $X \xrightarrow{f} Y \xrightarrow{g} Z$.

(i) f and g are weak equivalences: By the definition of weak equivalence and using the factorizations in $(\mathbf{Mono}_{\mathcal{A}}(\mathcal{C}), \mathbf{Epi}_{\mathcal{B}}(\mathcal{C}) \cap \mathcal{W}_e)$, we can obtain a commutative diagram

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, hook, "m"'] & & Y \arrow[rr, "g"] \arrow[dr, hook, "m'"'] & & Z \\
& W \arrow[ur, two heads, "e"'] \arrow[dr, hook, "m''"'] & & W' \arrow[ur, two heads, "e'"'] & \\
& & W'' \arrow[ur, two heads, "e''"'] & &
\end{tikzcd}$$

where $m, m' \in \mathbf{Mono}_{\mathcal{A}\cap\mathcal{W}}(\mathcal{C})$, $m'' \in \mathbf{Mono}_{\mathcal{A}}(\mathcal{C})$, $e, e', e'' \in \mathbf{Epi}_{\mathcal{B}\cap\mathcal{W}}(\mathcal{C})$.

We may assume $g \circ f = e \circ m$ with $f \in \mathbf{Epi}_{\mathcal{B}\cap\mathcal{W}}(\mathcal{C})$, $g \in \mathbf{Mono}_{\mathcal{A}\cap\mathcal{W}}(\mathcal{C})$, $e \in \mathbf{Epi}_{\mathcal{B}\cap\mathcal{W}}(\mathcal{C})$ and $m \in \mathbf{Mono}_{\mathcal{A}}(\mathcal{C})$. We show $m \in \mathbf{Mono}_{\mathcal{A}\cap\mathcal{W}}(\mathcal{C})$, i.e. $\mathrm{Ker}(m) \in \mathcal{A} \cap \mathcal{W}$. By the universal property of cokernels, there exists a morphism $\mathrm{CoKer}(m) \xrightarrow{q} \mathrm{CoKer}(g)$ such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d, two heads, "f"] & W \arrow[r, two heads] \arrow[d, two heads, "e"] & \mathrm{CoKer}(m) \arrow[d, "q"] \\
Y \arrow[r, hook] & Z \arrow[r, two heads] & \mathrm{CoKer}(g)
\end{tikzcd}$$

We have a short exact sequence $0 \to \mathrm{Ker}(f) \to \mathrm{Ker}(e) \to \mathrm{Ker}(q) \to 0$ by Snake's Lemma, where the first two terms belong to $\mathcal{W}$. Since $\mathcal{W}$ is thick, we conclude that $\mathrm{Ker}(q) \in \mathcal{W}$. Note that q is an epimorphism. Then we have a short exact sequence $\mathrm{Ker}(q) \hookrightarrow \mathrm{CoKer}(m) \twoheadrightarrow \mathrm{CoKer}(g)$ where the end terms are in $\mathcal{W}$. We use again the fact that $\mathcal{W}$ is thick to conclude that $\mathrm{CoKer}(m) \in \mathcal{W}$. The result follows.

(ii) g and $g \circ f$ are weak equivalences: Using the definition of weak equivalence and factoring f, we can get a commutative diagram

$$
\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, hook, "m"'] & & Y \arrow[rr, "g"] \arrow[dr, hook, "m'"'] & & Z \\
& W \arrow[ur, two heads, "e"'] \arrow[dr, hook, "m''"'] & & W' \arrow[ur, two heads, "e'"'] & \\
& & W'' \arrow[ur, two heads, "e''"'] & &
\end{tikzcd}
$$

where $m \in \mathbf{Mono}_{\mathcal{A}}(\mathcal{C})$, $m', m'' \in \mathbf{Mono}_{\mathcal{A}\cap\mathcal{W}}(\mathcal{C})$, $e, e', e'' \in \mathbf{Epi}_{\mathcal{B}\cap\mathcal{W}}(\mathcal{C})$. Then we may assume $f \in \mathbf{Mono}_{\mathcal{A}}(\mathcal{C})$. In this simpler situation, we have the following commutative diagram

$$
\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[rr, "g"] \arrow[dr, "m"'] & & Z \\
& & W \arrow[ur, "e"'] &
\end{tikzcd}
$$

where $m \in \mathbf{Mono}_{\mathcal{A}\cap\mathcal{W}}(\mathcal{C})$ and $e \in \mathbf{Epi}_{\mathcal{B}\cap\mathcal{W}}(\mathcal{C})$. We rewrite the previous diagram as follows:

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "m \circ f"'] & Y \arrow[r, "g"] \arrow[d, "m"] & Z \arrow[d, equal] \\
W \arrow[r, equal] & W \arrow[r, "e"'] & Z
\end{tikzcd}
$$

Note that $m \circ f \in \mathbf{Mono}_{\mathcal{A}}(\mathcal{C})$. Since $g \circ f \in \mathcal{W}_e$, we can write $g \circ f$ as $X \xrightarrow{m'} W' \xrightarrow{e'} Y$, where $m' \in \mathbf{Mono}_{\mathcal{A}\cap\mathcal{W}}(\mathcal{C})$ and $e' \in \mathbf{Epi}_{\mathcal{B}\cap\mathcal{W}}(\mathcal{C})$. Since $m \circ f \in \mathbf{Mono}_{\mathcal{A}}(\mathcal{C})$ and $e' \in \mathbf{Epi}_{\mathcal{B}\cap\mathcal{W}}(\mathcal{C})$, there exists a filler $d : W \to W'$ such that the following commutes:

$$
\begin{tikzcd}
X \arrow[r, "m \circ f"] \arrow[d, equal] & W \arrow[r, "e"] \arrow[d, "d"] & Z \arrow[d, equal] \\
X \arrow[r, "m'"'] & W' \arrow[r, "e'"'] & Z
\end{tikzcd}
$$

Write $d = W \overset{m''}{\to} W'' \overset{e''}{\to} W'$, where $m'' \in \mathbf{Mono}_{\mathcal{A}}(\mathcal{C})$ and $\mathbf{Epi}_{\mathcal{B}\cap\mathcal{W}}(\mathcal{C})$. Then there is an exact sequence $\mathrm{Ker}(e) \hookrightarrow \mathrm{Ker}(e' \circ e'') \twoheadrightarrow \mathrm{CoKer}(m'')$, where $\mathrm{Ker}(e)$ and $\mathrm{Ker}(e' \circ e'')$ are in $\mathcal{W}$. It follows $\mathrm{CoKer}(m'') \in \mathcal{A} \cap \mathcal{W}$. Consider the composition $m' = e'' \circ (m'' \circ m \circ f)$. We use Snake's Lemma again to obtain a short exact sequence $\mathrm{Ker}(e'') \hookrightarrow \mathrm{CoKer}(m'' \circ m \circ f) \twoheadrightarrow \mathrm{CoKer}(m')$, with the end terms in $\mathcal{W}$. We get $\mathrm{CoKer}(m'' \circ m \circ f) \in \mathcal{W}$. It follows that $m'' \circ m \circ f \in \mathbf{Mono}_{\mathcal{A}\cap\mathcal{W}}(\mathcal{C})$. Applying the same reasoning, we can show that $m \circ f \in \mathbf{Mono}_{\mathcal{A}\cap\mathcal{W}}(\mathcal{C})$. Similarly, we conclude that $f \in \mathbf{Mono}_{\mathcal{A}\cap\mathcal{W}}(\mathcal{C})$.

(iii) f and $g \circ f$ are weak equivalences: Similar to **(ii)**.

By the previous remark, we conclude that this cotorsion model structure just obtained is Abelian. $\square$

We finish the section by giving some conditions under which we can deduce the completeness of compatible cotorsion pairs.

Proposition 2.3.6

Let $(\mathcal{A}\cap\mathcal{W},\mathcal{B})$ and $(\mathcal{A},\mathcal{B}\cap\mathcal{W})$ be two compatible cotorsion pairs in an Abelian category $\mathcal{C}$.

(1) Suppose $({}^{\perp}\mathcal{W},\mathcal{W})$ is a cotorsion pair cogenerated by a set $\mathcal{S}_{\mathcal{W}}$. If $(\mathcal{A}\cap\mathcal{W},\mathcal{B})$ is also cogenerated by a set $\mathcal{S}_{\mathcal{A}\cap\mathcal{W}}$, then $(\mathcal{A},\mathcal{B}\cap\mathcal{W})$ is cogenerated by $\mathcal{S}=\mathcal{S}_{\mathcal{A}\cap\mathcal{W}}\cup\mathcal{S}_{\mathcal{W}}$.

(2) Suppose $\mathcal{C}$ has enough projective and injective objects. If $({}^{\perp}\mathcal{W},\mathcal{W})$ and $(\mathcal{W},\mathcal{W}^{\perp})$ are complete cotorsion pairs, then $(\mathcal{A},\mathcal{B}\cap\mathcal{W})$ is complete if, and only if, $(\mathcal{A}\cap\mathcal{W},\mathcal{B})$ is.

(2*) Moreover, if $\mathcal{C}$ has functorially enough projective and injective objects, and if $({}^{\perp}\mathcal{W},\mathcal{W})$ and $(\mathcal{W},\mathcal{W}^{\perp})$ are functorially complete, then $(\mathcal{A},\mathcal{B}\cap\mathcal{W})$ is functorially complete if, and only if, $(\mathcal{A}\cap\mathcal{W},\mathcal{B})$ is.

Proof.

Part **(1)** follows from the equality $\mathcal{B}\cap\mathcal{W}=(\mathcal{S}_{\mathcal{A}\cap\mathcal{W}})^{\perp}\cap(\mathcal{S}_{\mathcal{W}})^{\perp}=(\mathcal{S}_{\mathcal{A}\cap\mathcal{W}}\cup\mathcal{S}_{\mathcal{W}})^{\perp}$.

Part **(2)**: We only prove the implication ($\Longrightarrow$), since the other is dual. So suppose $(\mathcal{A},\mathcal{B}\cap\mathcal{W})$ is complete and let X be an object in $\mathcal{C}$. Since $(\mathcal{W},\mathcal{W}^{\perp})$ is complete, there exists a short exact sequence $X\hookrightarrow C\twoheadrightarrow W$, where $C\in\mathcal{W}^{\perp}$ and $W\in\mathcal{W}$. Since $(\mathcal{A},\mathcal{B}\cap\mathcal{W})$ is complete, there exists a short exact sequence $B\hookrightarrow A\twoheadrightarrow W$, where $A\in\mathcal{A}$ and $B\in\mathcal{B}\cap\mathcal{W}$. Taking the pullback of the morphisms $C\twoheadrightarrow W$ and $A\twoheadrightarrow W$, we get the following commutative diagram:

$$\begin{tikzcd}
& B \arrow[r, equal] \arrow[d, hook] & B \arrow[d, hook] \\
X \arrow[r, hook] \arrow[d, equal] & C\times_W A \arrow[r, two heads] \arrow[d, two heads] & A \arrow[d, two heads] \\
X \arrow[r, hook] & C \arrow[r, two heads] & W
\end{tikzcd}$$

Since $\mathcal{W}$ is closed under extensions, we have $A \in \mathcal{A} \cap \mathcal{W}$. It suffices to show $C \times_W A \in \mathcal{B}$. Note that $\mathcal{A} \cap \mathcal{W} \subseteq \mathcal{W}$ implies $\mathcal{W}^{\perp} \subseteq (\mathcal{A} \cap \mathcal{W})^{\perp} = \mathcal{B}$. Then $C \in \mathcal{B}$. It follows $C \times_W A \in \mathcal{B}$ since $\mathcal{B}$ is closed under extensions. We have obtained a short exact sequence $X \hookrightarrow C \times_W A \twoheadrightarrow A$ with $C \times_W A \in \mathcal{B}$ and $A \in \mathcal{A} \cap \mathcal{W}$. By the Salce's Lemma, $(\mathcal{A} \cap \mathcal{W}, \mathcal{B})$ is complete. $\square$

2.4 Induced cotorsion pairs of chain complexes

In this last section we study some methods developed by Jim Gillespie to construct certain cotorsion pairs in the category of complexes over an Abelian category $\mathbf{Ch}(\mathcal{C})$, from a complete cotorsion pair in $\mathcal{C}$. This methods appear in (27). We shall present their proofs based on the arguments given by Gillespie, but we introduced some modifications and remarks according to our needs. The importance of these results lie in a better understanding of homological dimensions in $\mathbf{Ch}(\mathcal{C})$.

Definition 2.4.1. Let $\mathcal{D}$ be a class of objects of an Abelian category $\mathcal{C}$. A complex D over $\mathcal{C}$ is:

(1) An $\underline{\mathcal{D}\text{-complex}}$ if D is exact (i.e. $D \in \mathcal{E}$) and $Z_m(D) \in \mathcal{D}$ for every $m \in \mathbb{Z}$. We denote this class of complexes by $\widetilde{\mathcal{D}}$.

(2) A $\underline{\text{degreewise } \mathcal{D}\text{-complex}}$ if $D_m \in \mathcal{D}$ for every $m \in \mathbb{Z}$. We denote this class of complexes by $\mathrm{dw}\widetilde{\mathcal{D}}$.

(3) An $\underline{\text{exact degreewise } \mathcal{D}\text{-complex}}$ if $X \in \mathrm{dw}\widetilde{\mathcal{D}} \cap \mathcal{E}$. We denote this class of complexes by $\mathrm{ex}\widetilde{\mathcal{D}}$.

Given two chain complexes X and Y in $\mathbf{Ch}(\mathcal{C})$, the complex $\mathrm{Hom}'(X, Y)$ is defined by at each $n \in \mathbb{Z}$ by

$$\mathrm{Hom}'(X, Y)_n := \textstyle\prod_{k \in \mathbb{Z}} \mathrm{Hom}_R(X_k, Y_{n+k}).$$

Then an element $f \in \mathrm{Hom}(X,Y)_n$ is a set of maps $f = (f_k : X_k \to X_{k+n})_{k\in\mathbb{Z}}$. The boundary maps $\mathrm{Hom}'(X,Y)_n \to \mathrm{Hom}'(X,Y)_{n-1}$ are defined at every $f = (f_k)_{k\in\mathbb{Z}} \in \mathrm{Hom}'(X,Y)_n$ by

$$\partial_n^{\mathrm{Hom}'(X,Y)}(f) := (\partial_{k+n}^Y \circ f_k - (-1)^n f_{k-1} \circ \partial_k^X)_{k\in\mathbb{Z}}.$$

Definition 2.4.2. Let $(\mathcal{A},\mathcal{B})$ be a cotorsion pair in an Abelian category $\mathcal{C}$. A complex X over $\mathcal{C}$ is:

A <u>differential graded $\mathcal{A}$-complex</u> if $X_m \in \mathcal{A}$ for every $m \in \mathbb{Z}$, and if every chain map $X \to Y$ is null homotopic (or equivalently, $\mathrm{Hom}'(X,Y)$ is exact) whenever Y is a $\mathcal{B}$-complex. Denote by $\mathrm{dg}\widetilde{\mathcal{A}}$ this class of complexes.	A <u>differential graded $\mathcal{B}$-complex</u> if $X_m \in \mathcal{B}$ for every $m \in \mathbb{Z}$, and if every chain map $Y \to X$ is null homotopic (or equivalently, $\mathrm{Hom}'(Y,X)$ is exact) whenever Y is an $\mathcal{A}$-complex. Denote by $\mathrm{dg}\widetilde{\mathcal{B}}$ this class of complexes.

Example 2.4.1.

(1) Given a class $\mathcal{D} \subseteq \mathrm{Ob}(\mathcal{C})$, $D^m(D)$ is a $\mathcal{D}$-complex for every $m \in \mathbb{Z}$ and every $X \in \mathcal{X}$.

(2) If $(\mathcal{A},\mathcal{B})$ is a cotorsion pair in $\mathcal{C}$, then $S^m(A)$ is a differential graded $\mathcal{A}$-complex for every $m \in \mathbb{Z}$ whenever $A \in \mathcal{A}$. Dually, $S^m(B)$ is a dg $\mathcal{B}$-complex for every $m \in \mathbb{Z}$ whenever $B \in \mathcal{B}$.

Proposition 2.4.1

Let $\mathcal{D}$ be a class of objects in $\mathcal{C}$, and $(\mathcal{A},\mathcal{B})$ be a cotorsion pair in $\mathcal{C}$.

(1) The classes $\widetilde{\mathcal{D}}$, $\mathrm{dw}\widetilde{\mathcal{D}}$, $\mathrm{ex}\widetilde{\mathcal{D}}$, $\mathrm{dg}\widetilde{\mathcal{A}}$ and $\mathrm{dg}\widetilde{\mathcal{B}}$ are closed under suspensions.

(2) If $\mathcal{D}$ is a resolving or a coresolving class, then so is $\widetilde{\mathcal{D}}$.

(3) If $\mathcal{A}$ is resolving then so is $\mathrm{dg}\widetilde{\mathcal{A}}$. Dually, if $\mathcal{B}$ is coresolving then so is $\mathrm{dg}\widetilde{\mathcal{B}}$.

$\boxed{\textbf{Proof}}$.

(1) It is clear that $\widetilde{\mathcal{D}}$, dw$\widetilde{\mathcal{D}}$ and ex$\widetilde{\mathcal{D}}$ are closed under suspensions. It is only left to show the result for dg$\widetilde{\mathcal{A}}$, since for dg$\widetilde{\mathcal{B}}$ is dual. Let $X \in$ dg$\widetilde{\mathcal{A}}$ and $k \in \mathbb{Z}$. It is clear that $(\Sigma^k(X))_n \in \mathcal{A}$ for every $n \in \mathbb{Z}$. Now if $\Sigma^k(X) \to Y$ is a chain map with $Y \in \widetilde{\mathcal{B}}$. Then we get a chain map $X \to \Sigma^{-k}(Y)$, where $\Sigma^{-k}(Y) \in \widetilde{\mathcal{B}}$ since $\widetilde{\mathcal{B}}$ is closed under suspensions. It follows $X \to \Sigma^{-k}(Y)$ is null homotopic. Applying Σ^k to this map, we obtain that $\Sigma^k(X) \to Y$ is null homotopic.

(2) We only prove the case where $\mathcal{D}$ is a coresolving class. If P is a projective chain complex then $Z_m(P)$ is a projective object of $\mathcal{C}$, for every $m \in \mathbb{Z}$. So $Z_m(P) \in \mathcal{D}$ for every $m \in \mathbb{Z}$. On the other hand, every projective chain complex is exact and so $P \in \widetilde{\mathcal{D}}$.

Consider a short exact sequence of complexes $D' \hookrightarrow D \twoheadrightarrow D''$. If $D', D'' \in \widetilde{\mathcal{D}}$, then D is exact since the class of exact complexes is thick. On the other hand, for every $m \in \mathbb{Z}$, we have an exact sequence $Z_m(D') \hookrightarrow Z_m(D) \twoheadrightarrow Z_m(D'')$, by Lemma 1.8.3. Since $Z_m(D'), Z_m(D'') \in \mathcal{D}$ and $\mathcal{D}$ is closed under extensions, we have $Z_m(D) \in \mathcal{D}$.

(3) Suppose $\mathcal{A}$ is a resolving class. We first show that dg$\widetilde{\mathcal{A}}$ is closed under extensions and under taking kernels of epimorphisms. Suppose we are given an exact sequence $X' \hookrightarrow X \twoheadrightarrow X''$ of complexes. Then for every $m \in \mathbb{Z}$ we have ant exact sequence $X'_m \hookrightarrow X_m \twoheadrightarrow X''_m$ in $\mathcal{C}$. If $X', X'' \in$ dg$\widetilde{\mathcal{A}}$ then $X'_m, X''_m \in \mathcal{A}$. It follows $X_m \in \mathcal{A}$ since $\mathcal{A}$ is closed under extensions. Now we show that $\mathrm{Hom}'(X, Y)$ is an exact complex whenever $Y \in \widetilde{\mathcal{B}}$. For every $k \in \mathbb{Z}$, the short exact sequence $X'_k \hookrightarrow X_k \twoheadrightarrow X''_k$ derives in an exact sequence

$$\mathrm{Hom}_{\mathcal{C}}(X''_k, Y_{m+k}) \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(X_k, Y_{m+k}) \to \mathrm{Hom}_{\mathcal{C}}(X'_k, Y_{m+k}) \to \cancelto{0}{\mathrm{Ext}^1_{\mathcal{C}}(X''_k, Y_{m+k})} \to \cdots$$

Since chain homology commutes with direct products and so the product of exact sequences is exact, we obtain the following diagram with exact rows:

$$\begin{tikzcd}
\prod_{k\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(X''_k, Y_{m+k}) \arrow[r, hook] \arrow[d, equal] & \prod_{k\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(X_k, Y_{m+k}) \arrow[r, two heads] \arrow[d, equal] & \prod_{k\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(X'_k, Y_{m+k}) \arrow[d, equal] \\
\mathrm{Hom}'(X'', Y) \arrow[r, hook] & \mathrm{Hom}'(X, Y) \arrow[r, two heads] & \mathrm{Hom}'(X', Y)
\end{tikzcd}$$

where $\mathrm{Hom}'(X', Y)$ and $\mathrm{Hom}'(X'', Y)$ are exact. It follows $\mathrm{Hom}'(X, Y)$ is exact. Hence $X \in \mathrm{dg}\widetilde{\mathcal{A}}$. If $X, X'' \in \mathrm{dg}\widetilde{\mathcal{A}}$, then similarly it follows that $X' \in \mathrm{dg}\widetilde{\mathcal{A}}$. It is only left to show that $\mathrm{dg}\widetilde{\mathcal{A}}$ contains the class of projective chain complexes. Given a projective chain complex P, it is clear that $P_m \in \mathcal{P}_0(\mathcal{C})$ for every $m \in \mathbb{Z}$. Now we show that every chain map $P \to Y$ is null homotopic whenever $Y \in \widetilde{\mathcal{B}}$. We know P can be written as a direct sum $P = \bigoplus_{m\in\mathbb{Z}} D(Z_m(P))$. For each m we have an inclusion $D^m(Z_m(P)) \overset{i^m}{\hookrightarrow} P$. It is clear that every composition $f \circ i^m$ is null homotopic. On the other hand, $f = \bigoplus_{m\in\mathbb{Z}} f \circ i^m$. It follows f is null homotopic. $\square$

Definition 2.4.3. Let X and Y be two chain complexes over $\mathcal{C}$. Let $\mathrm{Ext}^1_{dw}(X, Y)$ denote the subgroup of $\mathrm{Ext}^1_{\mathbf{Ch}(\mathcal{C})}(X, Y)$ given by all the classes of short exact sequences $Y \hookrightarrow Z \twoheadrightarrow X$ which are degreewise split, i.e. $0 \to Y_k \to Z_k \to X_k \to 0$ is split for every $k \in \mathbb{Z}$.

Due to several results by J. Gillespie, we shall see how to construct from $(\mathcal{A}, \mathcal{B})$ several cotorsion pairs in $\mathbf{Ch}(\mathcal{C})$ involving the classes given in Definitions 2.4.1 and 2.4.2. Before stating these results, we need the following lemma:

Lemma 2.4.2 (see (27, Lemma 2.1))

$\mathrm{Ext}^1_{\mathrm{dw}}(X, Y) \cong H_{-1}(\mathrm{Hom}'(X, Y))$, for every pair of chain complexes X and Y over $\mathcal{C}$.

For every $m \in \mathbb{Z}$, using the help of the suspension functor $\Sigma^{-m-1}(-)$, the previous equality can be "shifted" to $\mathrm{Ext}^1_{\mathrm{dw}}(X, \Sigma^{-m-1}(Y)) \cong H_m(\mathrm{Hom}'(X,Y))$.

Theorem 2.4.3 (see (27, Proposition 3.6))

Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in $\mathcal{C}$.

If $\mathcal{C}$ has enough injective objects, then $(\widetilde{\mathcal{A}}, \mathrm{dg}\widetilde{\mathcal{B}})$ is a cotorsion pair in $\mathbf{Ch}(\mathcal{C})$.	If $\mathcal{C}$ has enough projective objects, then $(\mathrm{dg}\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}})$ is a cotorsion pair in $\mathbf{Ch}(\mathcal{C})$.

In the original statement of the previous theorem, the hypothesis assumed is that $\mathcal{C}$ has enough $\mathcal{A}$-objects and enough $\mathcal{B}$-objects, that is for every object $X \in \mathrm{Ob}(\mathcal{C})$ there exists an epimorphism $A \twoheadrightarrow X$ and a monomorphism $X \hookrightarrow B$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$. In (27, Corollary 3.8), the hypothesis that $\mathcal{C}$ has enough projective or injective objects is also considered.

Proof.

We prove the left statement, based in the arguments given in (27, Proposition 3.6).

(i) $\widetilde{\mathcal{A}} \subseteq {}^{\perp}(\mathrm{dg}\widetilde{\mathcal{B}})$ and $\mathrm{dg}\widetilde{\mathcal{B}} \subseteq (\widetilde{\mathcal{A}})^{\perp}$: Let $X \in \widetilde{\mathcal{A}}$ and $Y \in \mathrm{dg}\widetilde{\mathcal{B}}$. Note that for every $m \in \mathbb{Z}$ we have an exact sequence $Z_m(X) \hookrightarrow X_m \twoheadrightarrow Z_{m-1}(X)$ since X is exact. On the other hand, $Z_m(X), Z_{m-1}(X) \in \mathcal{A}$, and so $X_m \in \mathcal{A}$ since $\mathcal{A}$ is closed under extensions. Now consider a class $S = [Y \hookrightarrow Z \twoheadrightarrow X]$ in $\mathrm{Ext}^1_{\mathbf{Ch}(\mathcal{C})}(X,Y)$. We have $H_{-1}(\mathrm{Hom}'(X,Y)) = 0$ since $\mathrm{Hom}'(X,Y)$ is exact. So by the previous lemma, $\mathrm{Ext}^1_{\mathrm{dw}}(X,Y) = 0$. Each short exact sequence $Y_m \hookrightarrow Z_m \twoheadrightarrow X_m$ splits since $X_m \in \mathcal{A}$ and $Y_m \in \mathcal{B}$. It follows $[Y \hookrightarrow Z \twoheadrightarrow X]$ is in $\mathrm{Ext}^1_{\mathrm{dw}}(X,Y) = 0$. Hence $\mathrm{Ext}^1_{\mathbf{Ch}(\mathcal{C})}(X,Y) = 0$.

(ii) $\widetilde{\mathcal{A}} \supseteq {}^{\perp}(\mathrm{dg}\widetilde{\mathcal{B}})$: Let $X \in {}^{\perp}(\mathrm{dg}\widetilde{\mathcal{B}})$. First we show X is exact. Fix $m \in \mathbb{Z}$. Since $\mathcal{C}$ has enough injective objects, there exists a monic $X_m/B_m(X) \xrightarrow{f} I$ for some injective $I \in \mathrm{Ob}(\mathcal{C})$. Consider the composition $f \circ \pi^X_m$, where π^X_m is the canonical projection $X_m \to \frac{X_m}{B_m(X)}$. We have a chain map $X \xrightarrow{S^m(f\circ\pi^X_m)} S^m(I)$ given by

$$(S^m(f \circ \pi^X_m))_k = \begin{cases} f \circ \pi^X_m & \text{if } k = m \\ 0 & \text{if } k \leq m. \end{cases}$$

On the other hand,

$$H_n(\mathrm{Hom}'(X, S^m(I))) \cong \mathrm{Ext}^1_{\mathrm{dw}}\left(X, \sum^{-n-1}(S^m(I))\right) \subseteq \cancelto{0}{\mathrm{Ext}^1_{\mathbf{Ch}(\mathcal{C})}(X, \Sigma^{-n-1}(S^m(I)))}.$$

since $\mathrm{dg}\widetilde{\mathcal{B}}$ is closed under suspensions. So $\mathrm{Hom}'(X, S^m(I))$ is exact and the map $S^m(f \circ \pi^X_m)$ is null homotopic, i.e. there exists a map $X_{m-1} \xrightarrow{D_m} I$ such that $D_m \circ \partial^X_m = f \circ \pi^X_m$. On the one hand, $B_m(X) \subseteq Z_m(X)$ for every $m \in \mathbb{Z}$. On the other hand, $Z_m(X) \subseteq B_m(X)$ by the previous equality. Hence X is exact.

Now we check $\frac{X_m}{B_m(X)} \in \mathcal{A}$ for every $m \in \mathbb{Z}$. Let $B \in \mathcal{B}$. Since X is exact, by Proposition 1.6.3 we have $\mathrm{Ext}^1_{\mathcal{C}}(X_m/B_m(X), B) \cong \mathrm{Ext}^1_{\mathbf{Ch}(\mathcal{C})}(X, S^m(B))$. Recall that $S^m(B) \in \mathrm{dg}\widetilde{\mathcal{B}}$ by Example 2.4.1 **(2)**. Since $X \in {}^{\perp}(\mathrm{dg}\widetilde{\mathcal{B}})$, we have $\mathrm{Ext}^1_{\mathbf{Ch}(\mathcal{C})}(X, S^m(B)) = 0$. So $\mathrm{Ext}^1_{\mathcal{C}}(X_m/B_m(X), B) = 0$ and $\frac{X_m}{B_m(X)} \in \mathcal{A}$ for every $m \in \mathbb{Z}$. By exactness of X, we finally get $Z_{m-1}(X) \cong X_m/Z_m(X) \cong X_m/B_m(X) \in \mathcal{A}$, for every $m \in \mathbb{Z}$.

(iii) $\mathrm{dg}\widetilde{\mathcal{B}} \supseteq (\widetilde{\mathcal{A}})^{\perp}$: Let $Y \in (\widetilde{\mathcal{A}})^{\perp}$. For every $A \in \mathcal{A}$, $\mathrm{Ext}^1_{\mathbf{Ch}(\mathcal{C})}(D^m(A), Y) = 0$ since $D^m(A) \in \widetilde{\mathcal{A}}$, by Example 2.4.1 **(1)**, and $Y \in (\widetilde{\mathcal{A}})^{\perp}$. On the other hand, $\mathrm{Ext}^1_{\mathcal{C}}(A, Y_m)$ and $\mathrm{Ext}^1_{\mathbf{Ch}(\mathcal{C})}(D^m(A), Y)$ are isomorphic by Proposition 1.6.2. Then $\mathrm{Ext}^1_{\mathcal{C}}(A, Y_m) = 0$, and so $Y_m \in \mathcal{B}$.

Now let $X \in \widetilde{\mathcal{A}}$, we check $\mathrm{Hom}'(X, Y)$ is exact. By Lemma 2.4.2, we only need to check that $\mathrm{Ext}^1_{\mathrm{dw}}(X, Y) = 0$. But this follows from the fact that

$\mathrm{Ext}^1_{\mathrm{dw}}(X,Y)$ is a subgroup of $\mathrm{Ext}^1_{\mathbf{Ch}(\mathcal{C})}(X,Y)$ and that $\mathrm{Ext}^1_{\mathbf{Ch}(\mathcal{C})}(X,Y) = 0$, since $X \in \widetilde{\mathcal{A}}$ and $Y \in (\widetilde{\mathcal{A}})^{\perp}$. □

Lemma 2.4.4 (see (27, Lemma 3.9))

If $X \in \widetilde{\mathcal{A}}$ and $Y \in \widetilde{\mathcal{B}}$, then every chain map $X \to Y$ is null homotopic.

Theorem 2.4.5 (see (27, Lemma 3.14))

Let $(\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion pair in $\mathcal{C}$, and $\mathcal{E}$ be the class of exact chain complexes.

If $\mathcal{C}$ has enough projective objects, then $\widetilde{\mathcal{B}} = \mathrm{dg}\widetilde{\mathcal{B}} \cap \mathcal{E}$.

If $\mathcal{C}$ has enough injective objects, then $\widetilde{\mathcal{A}} = \mathrm{dg}\widetilde{\mathcal{A}} \cap \mathcal{E}$.

This result appears in the given reference under the hypothesis that $(\mathrm{dg}\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}})$ and $(\widetilde{\mathcal{A}}, \mathrm{dg}\widetilde{\mathcal{B}})$ are right and left complete, respectively. The next proof of the previous result is based on the arguments given by Gillespie in (27, Lemma 3.10 & Theorem 3.12).

Proof.

We first show that $\widetilde{\mathcal{A}} \subseteq \mathrm{dg}\widetilde{\mathcal{A}} \cap \mathcal{E}$. Let $X \in \widetilde{\mathcal{A}}$. Then it is clear that $X \in \mathcal{E}$, and so for every $m \in \mathbb{Z}$ we have an exact sequence $Z_m(X) \hookrightarrow X_m \twoheadrightarrow Z_{m-1}(X)$, where $Z_m(X), Z_{m-1}(X) \in \mathcal{A}$. Since $\mathcal{A}$ is closed under extensions, we get $X_m \in \mathcal{A}$. The rest follows by the previous lemma.

Given $X \in \mathrm{dg}\widetilde{\mathcal{A}} \cap \mathcal{E}$, we only need to show that $Z_m(X) \in \mathcal{A}$ for every $m \in \mathbb{Z}$. Let $B \in \mathcal{B}$. Since $\mathcal{C}$ has enough injective objects, there is an exact right injective resolution of B, say

$$\mathbf{I} = B \overset{\partial^{\mathbf{I}}_{-1}}{\hookrightarrow} I^0 \overset{\partial^{\mathbf{I}}_{0}}{\to} I^1 \overset{\partial^{\mathbf{I}}_{1}}{\to} \cdots \to I^n \to \cdots .$$

Note that we have an exact sequence $B \hookrightarrow I^0 \twoheadrightarrow Z_1(I)$. Since $\mathcal{B}$ is coresolving by Proposition 2.2.1, we have $Z_1(I) \in \mathcal{B}$. We can repeat this argument recursively and conclude that $Z_m(I) \in \mathcal{B}$ for every $m \in \mathbb{Z}$. Hence $I \in \widetilde{\mathcal{B}}$ and so $\mathrm{Ext}^1_{\mathbf{Ch}(\mathcal{C})}(X, I) = 0$ since $X \in \mathrm{dg}\widetilde{\mathcal{A}}$ (Theorem 2.4.3). On the other hand, there is an exact sequence

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(Z_{m-1}(X), B) \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(X_m, B) \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(Z_m(X), B) \arrow[dll, out=-10, in=170] \\
& \mathrm{Ext}^1_{\mathcal{C}}(Z_{m-1}(X), B) \arrow[r] & \cancelto{0}{\mathrm{Ext}^1_{\mathcal{C}}(X_m, B)} \arrow[r] & \cdots
\end{tikzcd}
$$

where the last term is zero since $X_m \in \mathcal{A}$. It follows $\mathrm{Ext}^1_{\mathcal{C}}(Z_{m-1}(X), B) = 0$ if, and only if, $\mathrm{Hom}_{\mathcal{C}}(X_m, B) \to \mathrm{Hom}_{\mathcal{C}}(Z_m(X), B)$ is surjective. Consider a map $Z_m(X) \xrightarrow{f} B$. We construct a chain map $X \to \Sigma^{-m}(\mathbf{I})$. Set $f_k = 0$ for every $k > m+1$. Let $f_{m+1} := f \circ \widehat{\partial^X_{m+1}} : X_{m+1} \to B$, where $\widehat{\partial^X_{m+1}}$ is the restriction of the map $X_{m+1} \xrightarrow{\partial^X_{m+1}} X_m$ onto its image. Since I^0 is injective, there exists a morphism $X_m \xrightarrow{f_m} I^0$ such that the following diagram commutes:

$$
\begin{tikzcd}
Z_m(X) \arrow[r, hook, "j^X_m"] \arrow[d, "f"'] & X_m \arrow[r, two heads, "\rho^X_m"] \arrow[ddl, "f_m"] & Z_{m-1}(X) \\
B \arrow[d, "\partial^{\mathbf{I}}_1"'] & & \\
I^0 & &
\end{tikzcd}
$$

Then $(\partial^{\mathbf{I}}_0 \circ f_m) \circ j^X_m = (\partial^{\mathbf{I}}_0 \circ \partial^{\mathbf{I}}_{-1}) \circ f = 0$, and so there is a unique morphism $Z_{m-1}(X) \xrightarrow{f'_{m-1}} I^1$ such that $f'_{m-1} \circ \rho^X_m = \partial^I_0 \circ f_n$. Since the sequence

$$\mathrm{Hom}_{\mathcal{C}}(Z_{m-2}(X), I^1) \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(X_{m-1}, I^1) \twoheadrightarrow \mathrm{Hom}_{\mathcal{C}}(Z_{m-1}(X), I^1)$$

is exact, there exists a morphism $X_{m-1} \xrightarrow{f_{m-1}} I^1$ such that $f_{m-1} \circ j^X_{m-1} = f'_{m-1}$. Then we have $f_{m-1} \circ \partial^X_m = f_{m-1} \circ j^X_{m-1} \circ \widehat{\partial^X_m} = f'_{m-1} \circ \widehat{\partial^X_m} = \partial^I_0 \circ f_m$. Recursively, we construct a chain map

$$X \downarrow \Sigma^{-m}(\mathbf{I}) \quad = \quad \begin{tikzcd} \cdots \arrow[r] & X_{m+2} \arrow[r] \arrow[d] & X_{m+1} \arrow[r] \arrow[d, "f_{m+1}"] & X_m \arrow[r] \arrow[d, "f_m"] & X_{m-1} \arrow[r] \arrow[d, "f_{m-1}"] & \cdots \\ \cdots \arrow[r] & 0 \arrow[r] & B \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & \cdots \end{tikzcd}$$

which is null homotopic, since $X \in \mathrm{dg}\widetilde{\mathcal{A}}$ and $\Sigma^{-m}(\mathbf{I}) \in \widetilde{\mathcal{B}}$. So there exists a map $X_m \xrightarrow{D} B$ such that $D \circ \partial^X_{m+1} = f_{m+1}$, i.e. $D \circ j^X_m \circ \widehat{\partial^X_{m+1}} = f \circ \widehat{\partial^X_{m+1}}$. Since $\widehat{\partial^X_{m+1}}$ is epic, we get $D \circ j^X_m = f$. Hence $\mathrm{Hom}_{\mathcal{C}}(X_m, B) \to \mathrm{Hom}_{\mathcal{C}}(Z_m(X), B)$ is surjective and so $\mathrm{Ext}^1_{\mathcal{C}}(Z_{m-1}(X), B) = 0$ for every $m \in \mathbb{Z}$, i.e. $Z_{m-1}(X) \in \mathcal{A}$. $\square$

Remark 2.4.1. Consider the trivial cotorsion pair $(\mathcal{P}_0(\mathcal{C}), \mathrm{Ob}(\mathcal{C}))$, which is complete provided $\mathcal{C}$ has enough projective objects. Then we get two induced cotorsion pairs $(\widetilde{\mathcal{P}_0(\mathcal{C})}, \mathrm{Ob}(\mathbf{Ch}(\mathcal{C})))$ and $(\mathrm{dg}\widetilde{\mathcal{P}_0(\mathcal{C})}, \mathrm{dg}\widetilde{\mathrm{Ob}(\mathcal{C})} \cap \mathcal{E}) = (\mathrm{dg}\widetilde{\mathcal{P}_0(\mathcal{C})}, \mathcal{E})$. So a complex X is dg-projective if, and only if, $\mathrm{Ext}^1_{\mathbf{Ch}(\mathcal{C})}(X, E) = 0$ for every exact complex E. Similarly, a chain complex Y is dg-injective if, and only if, $\mathrm{Ext}^1_{\mathbf{Ch}(\mathcal{C})}(E, Y) = 0$ for every exact complex X, provided $\mathcal{C}$ has enough injective objects.

The following two results, proven by J. Gillespie in (25, Propositions 3.2 & 3.3), describe more methods to get cotorsion pairs in $\mathbf{Ch}(\mathcal{C})$ from a cotorsion pair in $\mathcal{C}$.

Theorem 2.4.6 (see (25, Proposition 3.2))

If $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair in $\mathcal{C}$, then so are $(\mathrm{dw}\widetilde{\mathcal{A}}, (\mathrm{dw}\widetilde{\mathcal{A}})^{\perp})$ and $({}^{\perp}(\mathrm{dw}\widetilde{\mathcal{B}}), \mathrm{dw}\widetilde{\mathcal{B}})$ in $\mathbf{Ch}(\mathcal{C})$. The class $(\mathrm{dw}\widetilde{\mathcal{A}})^{\perp}$ is given by all complexe Y for which $Y_m \in \mathcal{B}$ for every $m \in \mathbb{Z}$ and for which each map $X \to Y$ is homotopic to 0 whenever $X \in \mathrm{dw}\widetilde{\mathcal{A}}$. Dually, the class ${}^{\perp}(\mathrm{dw}\widetilde{\mathcal{B}})$ is given by all the complexes X for which $X_m \in \mathcal{A}$ for every $m \in \mathbb{Z}$ and for which each map $X \to Y$ is homotopic to 0 whenever $Y \in \mathrm{dw}\widetilde{\mathcal{B}}$.

$\boxed{\textbf{Proof}}$.

We only present the proof (appearing in the given reference) for $({}^{\perp}(\mathrm{dw}\widetilde{\mathcal{B}}), \mathrm{dw}\widetilde{\mathcal{B}})$. Let $\mathcal{X}$ be the class of all chain complexes X such that $X_m \in \mathcal{A}$ for every $m \in \mathbb{Z}$, and such that every chain map $X \to Y$ is null homotopic whenever $Y \in \mathrm{dw}\widetilde{\mathcal{B}}$.

(i) Let $X \in \mathcal{X}$ and $Y \in \mathrm{dw}\widetilde{\mathcal{B}}$. Consider $[Y \hookrightarrow Z \twoheadrightarrow X] \in \mathrm{Ext}^1_{\mathbf{Ch}(\mathcal{C})}(X, Y)$. Since every chain map $X \to Y$ is null homotopic, we have $\mathrm{Hom}'(X, Y)$ is exact and so $\mathrm{Ext}^1_{\mathrm{dw}}(X, Y) = 0$. On the other hand, for every $m \in \mathbb{Z}$ we have an exact sequence $Y_m \hookrightarrow Z_m \twoheadrightarrow X_m$, which is split exact since $X_m \in \mathcal{A}$ and $Y_m \in \mathcal{B}$. It follows $[Y \hookrightarrow Z \twoheadrightarrow X] \in \mathrm{Ext}^1_{\mathrm{dw}}(X, Y) = 0$. Hence every $[Y \hookrightarrow Z \twoheadrightarrow X]$ is zero and $\mathrm{Ext}^1_{\mathbf{Ch}(\mathcal{C})}(X, Y) = 0$. We have $\mathrm{dw}\widetilde{\mathcal{B}} \subseteq \mathcal{X}^{\perp}$ and $\mathcal{X} \subseteq {}^{\perp}(\mathrm{dw}\widetilde{\mathcal{B}})$.

(ii) Let $Y \in \mathcal{X}^{\perp}$. Consider $A \in \mathcal{A}$ and the disk complex $D^m(A)$. It is clear that $(D^m(A))_k \in \mathcal{A}$ for every $k \in \mathbb{Z}$. If $D^m(A) \xrightarrow{f} Z$ is a chain map with $Z \in \mathrm{dw}\widetilde{\mathcal{B}}$, then set $D_k = 0$ for every $k \neq m$, and $D_m = f_m$. It is easy to verify that $(D_k)_{k \in \mathbb{Z}}$ is a chain homotopy from f to 0. Then $D^m(A) \in \mathcal{X}$. So $\mathrm{Ext}^1_{\mathcal{C}}(A, Y_m) \cong \mathrm{Ext}^1_{\mathbf{Ch}(\mathcal{C})}(D^m(A), Y) = 0$ for every $A \in \mathcal{A}$, i.e. $Y_m \in \mathcal{B}$ for every $m \in \mathbb{Z}$. Hence $Y \in \mathrm{dw}\widetilde{\mathcal{B}}$ and $\mathcal{X}^{\perp} \subseteq \mathrm{dw}\widetilde{\mathcal{B}}$.

(iii) Let $X \in {}^{\perp}(\mathrm{dw}\widetilde{\mathcal{B}})$. For every $B \in \mathcal{B}$, $\mathrm{Ext}^1_{\mathcal{C}}(X_m, B) \cong \mathrm{Ext}^1_{\mathbf{Ch}(\mathcal{C})}(X, D^{m+1}(B)) = 0$, i.e. $X_m \in \mathcal{A}$ for every $m \in \mathbb{Z}$. Let $Y \in \mathrm{dw}\widetilde{\mathcal{B}}$. Since $\mathrm{Ext}^1_{\mathbf{Ch}(\mathcal{C})}(X, Y) = 0$ and $\mathrm{dw}\widetilde{\mathcal{B}}$ is closed under suspensions, $\mathrm{Ext}^1_{\mathbf{Ch}(\mathcal{C})}(X, \Sigma^{-m-1}(Y)) = 0$ and so $\mathrm{Ext}^1_{\mathrm{dw}}(X, \Sigma^{-m-1}(Y)) = 0$. It follows $\mathrm{Hom}'_{\mathbf{Ch}(\mathcal{C})}(X, Y)$ is exact by Lemma 2.4.2, i.e. every chain map $X \to Y$ is null homotopic. Therefore $X \in \mathcal{X}$. $\square$

Theorem 2.4.7 (see (25, Proposition 3.3))

Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in $\mathcal{C}$.

If either $\mathcal{C}$ has enough injective objects, or if $\mathcal{B}$ contains a cogenerator of finite injective dimension, then $(\mathrm{ex}\widetilde{\mathcal{A}}, (\mathrm{ex}\widetilde{\mathcal{A}})^{\perp})$ is a cotorsion pair in $\mathbf{Ch}(\mathcal{C})$, where $(\mathrm{ex}\widetilde{\mathcal{A}})^{\perp}$ coincides with the class of all complexes Y for which $Y_m \in \mathcal{B}$ for every $m \in \mathbb{Z}$, and for which every chain map $X \to Y$ is null homotopic whenever $X \in \mathrm{ex}\widetilde{\mathcal{A}}$.

If either $\mathcal{C}$ has enough projective objects, or if $\mathcal{A}$ contains a generator of finite projective dimension, then $({}^{\perp}(\mathrm{ex}\widetilde{\mathcal{B}}), \mathrm{ex}\widetilde{\mathcal{B}})$ is a cotorsion pair in $\mathbf{Ch}(\mathcal{C})$, where ${}^{\perp}(\mathrm{ex}\widetilde{\mathcal{B}})$ coincides with the class of all complexes X for which $X_m \in \mathcal{A}$ for every $m \in \mathbb{Z}$, and for which every chain map $X \to Y$ is null homotopic whenever $Y \in \mathrm{ex}\widetilde{\mathcal{B}}$.

Proof.

We prove the left statement. Suppose $\mathcal{C}$ has enough injective objects. This case is not included in the original statement appearing in (25, Proposition 3.3). Using this hypothesis we can give an easier proof of the result. Let $\mathcal{Y}$ be the class of all chain complexes Y such that $Y_m \in \mathcal{B}$ for every $m \in \mathbb{Z}$, and such that every chain map $X \to Y$ is null homotopic whenever $X \in \mathrm{ex}\widetilde{\mathcal{A}}$.

(i) The inclusions $\mathrm{ex}\widetilde{\mathcal{A}} \subseteq {}^{\perp}\mathcal{Y}$ and $\mathcal{Y} \subseteq (\mathrm{ex}\widetilde{\mathcal{A}})^{\perp}$ are proven as in the previous theorem.

(ii) Let $X \in {}^{\perp}\mathcal{Y}$. Consider a differential graded injective complex I. We show $I \in \mathcal{Y}$. Note first that $I_m \in \mathcal{B}$ since I_m is injective. Moreover, every chain map $Z \to I$ with $Z \in \mathrm{ex}\widetilde{\mathcal{A}}$ is null homotopic since Z is exact and I is DG injective. Then $I \in \mathcal{Y}$ and $\mathrm{Ext}^1_{\mathbf{Ch}(\mathcal{C})}(X, I) = 0$. Hence X is exact since $(\mathcal{E}, \mathrm{dg}\widetilde{\mathcal{I}_0(\mathcal{C})})$ is a cotorsion pair. On the other hand, for every $B \in \mathcal{B}$ we have $\mathrm{Ext}^1_{\mathcal{C}}(X_m, B) \cong \mathrm{Ext}^1_{\mathbf{Ch}(\mathcal{C})}(X, D^{m+1}(B)) = 0$, since $D^{m+1}(B) \in \mathcal{Y}$. Therefore $X_m \in \mathcal{A}$ for every $m \in \mathbb{Z}$. We have ${}^{\perp}\mathcal{Y} \subseteq \mathrm{ex}\widetilde{\mathcal{A}}$.

(iii) Let $Y \in (\text{ex}\widetilde{\mathcal{A}})^\perp$. For every $A \in \mathcal{A}$, $\text{Ext}^1_{\mathcal{C}}(A, Y_m) \cong \text{Ext}^1_{\mathbf{Ch}(\mathcal{C})}(D^m(A), Y) = 0$ since $D^m(A) \in \text{ex}\widetilde{\mathcal{A}}$. So $Y_m \in \mathcal{B}$ for every $m \in \mathbb{Z}$. On the other hand, $\text{Ext}^1_{\text{dw}}(X, Y) \subseteq \text{Ext}^1_{\mathbf{Ch}(\mathcal{C})}(X, Y) = 0$, for every $X \in \text{ex}\widetilde{\mathcal{A}}$, implies that the complex $\text{Hom}'(X, Y)$ is exact, i.e. every chain map $X \to Y$ is null homotopic whenever $X \in \text{ex}\widetilde{\mathcal{A}}$. Therefore, $\mathcal{Y} \supseteq (\text{ex}\widetilde{\mathcal{A}})^\perp$. □

Remark 2.4.2 (A. Joyal)**.** It is possible to get other descriptions of the classes $(\text{dw}\widetilde{\mathcal{A}})^\perp$ and ${}^\perp(\text{dw}\widetilde{\mathcal{B}})$, using the orthogonality defined by the $\text{Ext}^1_{\text{dw}}(-,-)$. In this sense, define $(\text{dw}\widetilde{\mathcal{A}})^{\perp_{\text{dw}}}$ as the class of all complexes Y such that $\text{Ext}^1_{\text{dw}}(X, Y)$ is 0 for every $X \in \text{dw}\widetilde{\mathcal{A}}$. The class ${}^{\perp_{\text{dw}}}(\text{dw}\widetilde{\mathcal{B}})$ is defined similarly. We have the following characterizations of $(\text{dw}\widetilde{\mathcal{A}})^\perp$ and ${}^\perp(\text{dw}\widetilde{\mathcal{B}})$ in terms of the dw-orthogonal classes just defined.

(1) $(\text{dw}\widetilde{\mathcal{A}})^\perp = (\text{dw}\widetilde{\mathcal{A}})^{\perp_{\text{dw}}} \cap \text{dw}\widetilde{\mathcal{B}}$: Let $Y \in (\text{dw}\widetilde{\mathcal{A}})^\perp$ and $X \in \text{dw}\widetilde{\mathcal{A}}$. By the previous theorem, we have $Y \in \text{dw}\widetilde{\mathcal{B}}$. Then $\text{Ext}^1_{\mathcal{C}}(X_m, Y_m) = 0$ for every $m \in \mathbb{Z}$. This implies $\text{Ext}^1_{\text{dw}}(X, Y) = \text{Ext}^1_{\mathbf{Ch}(\mathcal{C})}(X, Y)$. Since $Y \in (\text{dw}\widetilde{\mathcal{A}})^\perp$ and $X \in \text{dw}\widetilde{\mathcal{A}}$, we obtain $\text{Ext}^1_{\mathbf{Ch}(\mathcal{C})}(X, Y) = 0$ and therefore $\text{Ext}^1_{\text{dw}}(X, Y) = 0$. The other inclusion follows similarly.

(2) ${}^\perp(\text{dw}\widetilde{\mathcal{B}}) = {}^{\perp_{\text{dw}}}(\text{dw}\widetilde{\mathcal{B}}) \cap \text{dw}\widetilde{\mathcal{A}}$: The proof is similar to **(1)**.

We know that the cotorsion pairs $(\text{dg}\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}})$ and $(\widetilde{\mathcal{A}}, \text{dg}\widetilde{\mathcal{B}})$ are compatible if the inducing pair $(\mathcal{A}, \mathcal{B})$ is hereditary. The author does not know if the same holds for the cotorsion pairs $(\text{dw}\widetilde{\mathcal{A}}, (\text{dw}\widetilde{\mathcal{A}})^\perp)$ and $(\text{ex}\widetilde{\mathcal{A}}, (\text{ex}\widetilde{\mathcal{A}})^\perp)$ (or for $({}^\perp(\text{dw}\widetilde{\mathcal{B}}), \text{dw}\widetilde{\mathcal{B}})$ and $({}^\perp(\text{ex}\widetilde{\mathcal{B}}), \text{ex}\widetilde{\mathcal{B}})$). Consider the case $\mathcal{C} = {}_R\mathbf{Mod}$. Since $\text{dg}\widetilde{\mathcal{P}_0(\mathcal{C})} \subseteq \text{dw}\widetilde{\mathcal{A}}$ and $\text{ex}\widetilde{\mathcal{A}} \subseteq \text{dw}\widetilde{\mathcal{A}}$, we have $(\text{dw}\widetilde{\mathcal{A}})^\perp \subseteq (\text{dg}\widetilde{\mathcal{P}_0(\mathcal{C})})^\perp = \mathcal{E}$ and $(\text{dw}\widetilde{\mathcal{A}})^\perp \subseteq (\text{ex}\widetilde{\mathcal{A}})^\perp$, and so $(\text{dw}\widetilde{\mathcal{A}})^\perp \subseteq (\text{ex}\widetilde{\mathcal{A}})^\perp \cap \mathcal{E}$. Similarly, ${}^\perp(\text{dw}\widetilde{\mathcal{B}}) \subseteq {}^\perp(\text{ex}\widetilde{\mathcal{B}}) \cap \mathcal{E}$. The next result provides a relationship between completeness of these degreewise cotorsion pairs and the remaining inclusions.

Proposition 2.4.8

If the pair $(\mathrm{dw}\widetilde{\mathcal{A}}, (\mathrm{dw}\widetilde{\mathcal{A}})^{\perp})$ is complete, then $(\mathrm{dw}\widetilde{\mathcal{A}})^{\perp} = (\mathrm{ex}\widetilde{\mathcal{A}})^{\perp} \cap \mathcal{E}$.	If the pair $({}^{\perp}(\mathrm{dw}\widetilde{\mathcal{B}}), \mathrm{dw}\widetilde{\mathcal{B}})$ is complete, then ${}^{\perp}(\mathrm{dw}\widetilde{\mathcal{B}}) = {}^{\perp}(\mathrm{ex}\widetilde{\mathcal{B}}) \cap \mathcal{E}$.

Proof.

We only prove the left statement. It suffices to show the inclusion $(\mathrm{ex}\widetilde{\mathcal{A}})^{\perp} \cap \mathcal{E} \subseteq (\mathrm{dw}\widetilde{\mathcal{A}})^{\perp}$. Let $Y \in (\mathrm{ex}\widetilde{\mathcal{A}})^{\perp} \cap \mathcal{E}$. Since $(\mathrm{dw}\widetilde{\mathcal{A}}, (\mathrm{dw}\widetilde{\mathcal{A}})^{\perp})$ is complete, there is a short exact sequence $Y \hookrightarrow X \twoheadrightarrow A$, with $X \in (\mathrm{dw}\widetilde{\mathcal{A}})^{\perp}$ and $A \in \mathrm{dw}\widetilde{\mathcal{A}}$. Note that $A \in \mathrm{dw}\widetilde{\mathcal{A}} \cap \mathcal{E}$ since $X \in (\mathrm{dw}\widetilde{\mathcal{A}})^{\perp} \subseteq \mathcal{E}$, $Y \in \mathcal{E}$ and $\mathcal{E}$ is thick. It follows $\mathrm{Ext}^1(A, Y) = 0$ and that the previous sequence splits. We have that Y is a direct summand of $X \in (\mathrm{dw}\widetilde{\mathcal{A}})^{\perp}$, and hence $Y \in (\mathrm{dw}\widetilde{\mathcal{A}})^{\perp}$ since $(\mathrm{dw}\widetilde{\mathcal{A}})^{\perp}$ is closed under direct summands. $\square$

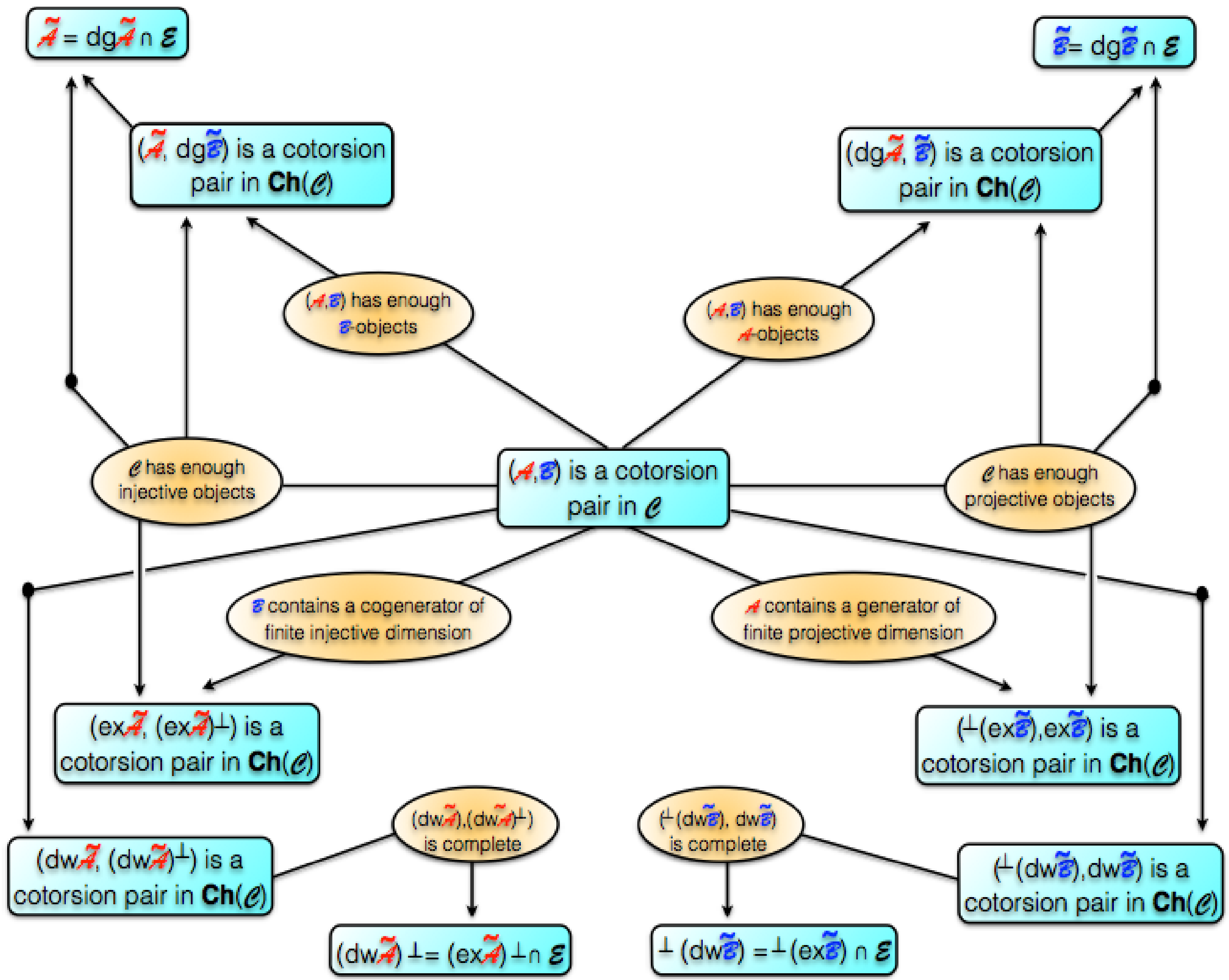

Figure 2.7: Inducing cotorsion pairs in chain complexes: This chart summarizes the different methods to obtain the induced, from a cotorsion pair in $\mathcal{C}$, the cotorsion pairs in $\mathbf{Ch}(\mathcal{C})$ involving the corresponding classes of differential graded and degreewise complexes.

CHAPTER III

ABELIAN MODEL STRUCTURES AND HOMOLOGICAL DIMENSIONS OF CHAIN COMPLEXES

"Science, my lad, is made up of mistakes, but they are mistakes which it is useful to make, because they lead little by little to the truth."

Jules Verne.

Some of the first model structures that appeared in homological algebra are related to the notions of projective, injective and flat modules. For example, if R is a Frobenius ring (i.e. the classes of projective and injective modules coincide) then there is an Abelian model structure on ${}_R\mathbf{Mod}$ where ${}_R\mathbf{Mod}$ itself is the class of cofibrant and fibrant objects, and $\mathcal{P}_0({}_R\mathbf{Mod}) = \mathcal{I}_0({}_R\mathbf{Mod})$ is the class of trivial objects. The reader can find the details in (36, Section 2.2).

On the category of chain complexes, one of the first model structures ever constructed was the projective model structure. In this case, the class of cofibrant objects is given by the differential graded chain complexes, the class $\mathcal{E}$ of exact chain complexes represents the trivial objects, and the whole category $\mathbf{Ch}({}_R\mathbf{Mod})$ forms the class of fibrant objects. This structure was discovered before Hovey's Correspondence was established. Cofibrations and fibrations are defined by using orthogonality with respect to certain classes of chain maps involving sphere and disk complexes centred at R. The injective model structure is constructed in a similar way, with the same trivial objects and the differential graded injective complexes forming the class of fibrant objects (The reader can see the details in (36)). James Gillespie was able to generalize, to the category of chain complexes, the argument given by E. Enochs that shows that $(\mathcal{F}_0, (\mathcal{F}_0)^\perp)$ is cogenerated by a set. This allowed him to construct a model structure on $\mathbf{Ch}({}_R\mathbf{Mod})$ where $\text{dg}\widetilde{\mathcal{F}_0}$ is the class of cofibrant objects, $(\widetilde{\mathcal{F}_0})^\perp$ the class of fibrant objects, and $\mathcal{E}$ the class of trivial objects.

The goal of this chapter is to generalize these model structures for every homological dimension. We mean that the classes of differential graded n-projective, n-injective and n-flat chain complexes appear as the classes of cofibrant and fibrant objects of certain Abelian model structures on $\mathbf{Ch}({}_R\mathbf{Mod})$. We shall do the same with the classes of degreewise n-projective, n-injective and n-flat complexes. In the end we present some of the model structures obtained in a categorical setting, using a functor known as Mitchell's equivalence.

3.1 Cotorsion pairs and homological dimensions of modules

In this section, we study some relatively recent results concerning cotorsion pairs and homological dimensions in the category of left R-modules. Namely, we shall see that the classes of n-projective and n-flat modules constitute the left halves of two complete cotorsion pairs. The idea in each case is to construct for every n-projective (res. n-flat) module, a transfinite extension of "small" n-projective (resp. "small" n-flat) modules. The construction of these transfinite extension is based on a method, probably first introduced by Edgar Enochs and coauthors, known as the *zig-zag procedure*. We shall explain how this method works in both the projective and flat cases, in order to study several generalizations to the category of chain complexes in the following sections. In the case of injective objects, the fact that $\mathcal{I}_n(\mathcal{C})$ is the right half of a cotorsion pair in a Grothendieck category $\mathcal{C}$ is a consequence of Baer's Criterion.

We start with the projective case. Given an Abelian category $\mathcal{C}$, recall that $(\mathcal{P}_0(\mathcal{C}), \mathrm{Ob}(\mathcal{C}))$ is a cotorsion pair. We also know that $\mathcal{C}$ has enough projective objects if, and only if, $(\mathcal{P}_0(\mathcal{C}), \mathrm{Ob}(\mathcal{C}))$ is complete. The category of left R-modules ${}_R\mathbf{Mod}$ is an example where this occurs. We use the notation $\mathcal{P}_0(\mathcal{C}) = \mathcal{P}_0$ whenever $\mathcal{C} = {}_R\mathbf{Mod}$ or $\mathbf{Mod}_R$. So $(\mathcal{P}_0, \mathrm{Ob}({}_R\mathbf{Mod}))$ is complete. This result is also a consequence of Eklof and Trlifaj's Theorem, since $(\mathcal{P}_0, \mathrm{Ob}({}_R\mathbf{Mod}))$ is cogenerated by the set $\{R\}$ (Recall that a module is projective if, and only if, it is a direct summand of a free module).

For any projective dimension, denote $\mathcal{P}_n = \mathcal{P}_n(\mathcal{C})$ whenever $\mathcal{C}$ is ${}_R\mathbf{Mod}$ or $\mathbf{Mod}_R$. We shall show that $(\mathcal{P}_n, (\mathcal{P}_n)^\perp)$ is a complete cotorsion pair. In 2001, S. T. Aldrich, E. E. Enochs, O. M. G. Jenda and L. Oyonarte proved this fact by giving a generating set, which is deduced from the zig-zag procedure. We shall see later that this result is valid in the category of modules over a ringoid (See Section 10 of Chapter 1).

As we mentioned above, $(\mathcal{P}_0, \mathrm{Ob}({}_R\mathbf{Mod}))$ is cogenerated by $\{R\}$. Can we assert a similar result in Abelian categories? The answer is yes, but imposing some conditions on $\mathcal{C}$. First, recall that R is a generator of ${}_R\mathbf{Mod}$. So we ask our category $\mathcal{C}$ to have a generator G. In general, G needs not to be projective. For instance, the category $\mathcal{C}$ of sheaves of Abelian groups on a topological space X is a Grothendieck category, and so it has a generator, but it needs not have enough projective sheaves. The following result is easy to prove.

Proposition 3.1.1

If P is a projective object of an Abelian category $\mathcal{C}$ with generator G, then P is a direct summand of a direct sum $G^{(I)}$ (for some index set I) of copies of G. If G is projective, the converse holds.

The following result is proven in (31, Lemma 3.1) for the category of left R-modules, but the same argument works for any Abelian category.

Proposition 3.1.2 (Eilenberg's Trick. See (31, Lemma 3.1))

Let $\mathcal{S}$ be a set of objects of a cocomplete Abelian category $\mathcal{C}$, and let $\mathrm{Add}(\mathcal{S})$ denote the class of direct sums of copies of objects from $\mathcal{S}$. If D is a direct summand of an object of $\mathrm{Add}(\mathcal{S})$, then there exists an object $C \in \mathrm{Add}(\mathcal{S})$ such that $D \oplus C = C$.

Proof.

Note there exists an object Q such that $E = D \oplus Q$ belongs to $\mathrm{Add}(\mathcal{S})$. Then the module

$$C = D \oplus Q \oplus D \oplus Q \oplus D \oplus Q \oplus \cdots = E \oplus E \oplus E \oplus \cdots$$

belongs to $\mathrm{Add}(\mathcal{S})$. The result follows since

$$C = D \oplus (Q \oplus D \oplus Q \oplus D \oplus Q \oplus \cdots) = D \oplus (D \oplus Q \oplus D \oplus Q \oplus D \oplus Q \oplus \cdots) = D \oplus C.$$

$\square$

Corollary 3.1.3

For every left (or right) projective R-module P there exists a free module F such that $P \oplus F \cong F$.

For a moment, we work in the category of left R-modules ${}_R\mathbf{Mod}$. One of the applications of the previous corollary is the construction free resolutions of finite length for every n-projective module. Suppose $X \in \mathcal{P}_n$ and let

$$\mathbf{P} = (0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to X \to 0)$$

be an exact and finite left projective resolution of length n. By Eilenberg's Trick, there is a free module L_0 such that $P_0 \oplus L_0 \cong L_0$. If we take the direct sum of $\mathbf{P}$ and $D^1(L_0)$, we get the exact sequence

$$\begin{aligned}\mathbf{P^1} &= (0 \to P_n \to P_{n-1} \to \cdots \to P_1 \oplus L_0 \to P_0 \oplus L_0 \to X \to 0)\\ &= (0 \to P_n \to P_{n-1} \to \cdots \to P_1 \oplus L_0 \to L_0 \to X \to 0).\end{aligned}$$

Since $P_1 \oplus L_0$ is projective, there exists a free module L_1 such that $(P_1 \oplus L_0) \oplus L_1 \cong L_1$. Taking the direct sum of $\mathbf{P^1}$ and $D^2(L_1)$, we get the exact sequence

$$\begin{aligned}\mathbf{P^2} &= (0 \to P_n \to P_{n-1} \to \cdots \to (P_1 \oplus L_0) \oplus L_1 \to L_0 \to X \to 0)\\ &= (0 \to P_n \to P_{n-1} \to \cdots \to L_1 \to L_0 \to X \to 0).\end{aligned}$$

At the $n-1$th step, we have obtained an exact sequence

$$\mathbf{P^{n-1}} = (0 \to P_n \oplus L_{n-1} \to L_{n-1} \to \cdots \to L_1 \to L_0 \to X \to 0).$$

Finally, let L_n be a free module such that $(P_n \oplus L_{n-1}) \oplus L_n \cong L_n$. We obtain a left free resolution of length n:

$$\begin{aligned}\mathbf{P^n} &= (0 \to (P_n \oplus L_{n-1}) \oplus L_n \to L_{n-1} \oplus L_n \to \cdots \to L_1 \to L_0 \to X \to 0)\\ &= (0 \to L_n \to L_{n-1} \oplus L_n \to \cdots \to L_1 \to L_0 \to X \to 0).\end{aligned}$$

Hence, every n-projective module has an exact and finite left free resolution of length n.

Corollary 3.1.4

Every n-projective left (or right) n-projective module has a free left resolution of length n.

By a map between left (or right) resolutions (by some class of objects) we shall mean a chain map between the given resolutions. In this sense, the sequence $\mathbf{F}' = (0 \to F'_n \to F'_{n-1} \to \cdots \to F'_1 \to F'_0 \to X' \to 0)$ is said to be a subresolution of $\mathbf{F} = (0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to X \to 0)$ if there is a monic map from $\mathbf{F}'$ to $\mathbf{F}$. We shall prove that given an $x \in X$, with $X \in \mathcal{P}_n$, we can construct a "small" left free subresolution of $\mathbf{P}$ of length n with $x \in X'$.

Definition 3.1.1. Let $\kappa \geq \mathrm{Card}(R)$ be a fixed infinite cardinal number. A set S is said to be κ-small (or simply small) if $\mathrm{Card}(S) \leq \kappa$. For a given class of modules $\mathcal{F}$, we denote by $\mathcal{F}^{\leq \kappa}$ the set of κ-small modules which are in $\mathcal{F}$.

The following result is due to Aldrich et al. (5, Proposition 4.1). It is a tool used to prove that $(\mathcal{P}_n, (\mathcal{P}_n)^{\perp})$ is a complete cotorsion pair. For the reader convenience, we present their proof to introduce the zig-zag procedure.

Lemma 3.1.5 (Aldrich, Enochs, Jenda & Oyonarte.)

If X is an n-projective module and $x \in X$, then there exists a κ-small n-projective submodule $X' \subseteq X$ containing x such that X/X' is also n-projective, where $\kappa \geq \mathrm{Card}(R)$ is a given infinite cardinal.

Proof.

Consider a free resolution $0 \to F_n \xrightarrow{\partial_n} F_{n-1} \to \cdots \to F_0 \xrightarrow{\partial_0} X \to 0$. Let $\mathcal{B}_i$ be a basis of F_i. We shall find small linearly independent sets $B_i \subseteq \mathcal{B}_i$ and construct an exact sequence of the form $0 \to \langle B_n \rangle \to \cdots \to \langle B_1 \rangle \to \langle B_0 \rangle$, and then take $X' = \mathrm{Im}(\partial_0|_{\langle B_0 \rangle})$. For this, we use a technique called the zig-zag procedure. Since ∂_0 is surjective, we can find a finite subset $Z_0 \subseteq \mathcal{B}_0$ such that $x \in \partial_0(\langle Z_0 \rangle)$. Now we choose a small subset $Z_1 \subseteq \mathcal{B}_1$ such that $\partial_1(\langle Z_1 \rangle) \supseteq \mathrm{Ker}(\partial_0|_{\langle Z_0 \rangle})$. Then we choose a small subset $Z_2 \subseteq \mathcal{B}_2$ such that $\partial_2(\langle Z_2 \rangle) \supseteq \mathrm{Ker}(\partial_1|_{\langle Z_1 \rangle})$. We continue this procedure until we get a set a small subset $Z_n \subseteq \mathcal{B}_n$ satisfying $\partial_n(\langle Z_n \rangle) \supseteq \mathrm{Ker}(\partial_{n-1}|_{\langle Z_{n-1} \rangle})$. Now choose a small subset $Z^{(1)}_{n-1} \subseteq \mathcal{B}_{n-1}$ containing Z_{n-1} such that $\partial_n(\langle Z_n \rangle) \subseteq < Z^{(1)}_{n-1} >$. Then choose a small subset $\mathcal{B}_{n-2} \supseteq Z^{(1)}_{n-2} \supseteq Z_{n-2}$ such that $\partial_{n-1}(< Z^{(1)}_{n-1} >) \subseteq < Z^{(1)}_{n-2} >$. Continue this procedure to construct small sets $Z^{(1)}_{n-3}, \ldots, Z^{(1)}_0$ with $\mathcal{B}_{n-i} \supseteq Z^{(1)}_{n-i} \supseteq Z_{n-i}$, and $\partial_{n-i+1}(< Z^{(1)}_{n-i+1} >) \subseteq < Z^{(1)}_{n-i} >$, where $3 \leq i \leq n$. Now choose a small set $Z^{(1)}_1 \subseteq Z^{(2)}_1 \subseteq \mathcal{B}_1$ such that $\partial_1(< Z^{(2)}_1 >) \supseteq \mathrm{Ker}(\partial_0|_{< Z^{(1)}_0 >})$. Then enlarge $Z^{(1)}_2$ and so on. Continue this zig-zag procedure indefinitely, and set $B_i = \bigcup_{k=0}^{\infty} Z^{(k)}_i$, where $Z^{(0)}_i = Z_i$. Note that each B_i is small. Let $\widetilde{\partial_i} = \partial_i|_{\langle B_i \rangle}$. By construction, we get an exact sequence $0 \to \langle B_n \rangle \xrightarrow{\widetilde{\partial_n}} \cdots \xrightarrow{\widetilde{\partial_2}} \langle B_1 \rangle \xrightarrow{\widetilde{\partial_1}} \langle B_0 \rangle$. Let $X' = \partial_0(\langle B_0 \rangle)$. We have a free (and hence projective) resolution of X' given by the sequence

$$0 \to \langle B_n \rangle \xrightarrow{\widetilde{\partial_n}} \cdots \xrightarrow{\widetilde{\partial_2}} \langle B_1 \rangle \xrightarrow{\widetilde{\partial_1}} \langle B_0 \rangle \xrightarrow{\widetilde{\partial_0}} X' \to 0.$$

Hence X' is n-projective. Note also that X' is small and that $x \in X'$. The quotient of the resolutions of X and X' gives rise to a projective resolution of X/X' of length n, and so X/X' is also n-projective. $\square$

Before generalizing the previous proposition to the category $\mathbf{Mod}(\mathfrak{R})$, we need to introduce some notation and recall the notion of a free module. It is important that the reader checks Section 10 of Chapter 1 before continuing through the rest of the current section, if he/she does not recall in detail the corresponding terminology of modules over a ringoid.

Definition 3.1.2. If M is a $\mathfrak{R}$-module we shall say that a submodule $N \subseteq M$ is generated by a family $\{x_i\}_{i\in I}$ of homogenous elements if N is the smallest submodule of M which contains all the elements x_i. Let $N \subseteq M$ be a submodule generated by a family $\{x_i\}_{i\in I}$ of homogenous elements of M. Then an element $x \in M(a)$ belongs to $N(a)$ if and only if it is an admissible linear combination $x = \sum_{i\in I} r_i \cdot x_i$ (where $r_i = 0$ for all but finitely many $i \in I$).

Definition 3.1.3. We shall say that a family $\{x_i\}_{i\in I}$ of homogenous elements of M is a basis of M if every homogenous element $x \in M$ can be written uniquely as an admissible linear combination $x = \sum_{i\in I} r_i \cdot x_i$. We shall say that M is free if it admits a basis.

We recall Yoneda's Lemma.

Lemma 3.1.6 (Yoneda)

If $F : \mathcal{C} \to \mathbf{Set}$ is a functor from a locally small category $\mathcal{C}$ to $\mathbf{Set}$, then for every $X \in \mathrm{Ob}(\mathcal{C})$ there is a bijection $\mathrm{Hom}_{[\mathcal{C},\mathbf{Set}]}(\mathrm{Hom}_{\mathcal{C}}(X,-),F) \to F(X)$ which sends each natural transformation $\alpha : \mathrm{Hom}_{\mathcal{C}}(X,-) \to F$ to the element $\alpha_X(\mathrm{id}_X) \in F(X)$.

If $\mathfrak{R}$ is a ringoid and M is a left $\mathfrak{R}$-module, then it follows from Yoneda's Lemma that for every $a \in \mathrm{Ob}(\mathfrak{R})$ and every $x \in M(a)$ there is a unique map of $\mathfrak{R}$-modules $\alpha : \mathrm{Hom}_{\mathfrak{R}}(a,-) \to M$ such that $x = \alpha_a(\mathrm{id}_a)$. More generally, if $\{a_i\}_{i\in I}$ be a family of objects of $\mathfrak{R}$, let us put

$$\langle a_i : i \in I\rangle := \bigoplus_{i\in I} \mathrm{Hom}_{\mathfrak{R}}(a_i,-)$$

and $[i] := (u_i)_{a_i}(\mathrm{id}_{a_i})$, where $u_i : \mathrm{Hom}_{\mathfrak{R}}(a_i,-) \to \langle a_i : i \in I\rangle$ is the inclusion. Consider a family of elements $\{x_i\}_{i\in I}$ in $\prod_{i\in I} M(a_i)$. Then for each $i \in I$ we can write $x_i = (\alpha^i)_{a_i}(\mathrm{id}_{a_i})$, where $\alpha^i : \mathrm{Hom}_{\mathfrak{R}}(a_i,-) \to M$ is a map of left $\mathfrak{R}$-modules.

There exists a unique map $f : \langle a_i : i \in I \rangle \to M$ such that the following triangle commutes:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{R}}(a_i, -) \arrow[r, "u_i"] \arrow[dr, "\alpha^i"'] & \langle a_i : i \in I \rangle \arrow[d, dotted, "\exists! \ \alpha"] \\
& M
\end{tikzcd}$$

Note that $\alpha_{a_i}([i]) = \alpha_{a_i} \circ (u_i)_{a_i}(\mathrm{id}_{a_i}) = (\alpha^i)_{a_i}(\mathrm{id}_{a_i}) = x_i$ for every $i \in I$. It follows that the $\mathfrak{R}$-module $\langle a_i : i \in I \rangle$ is freely generated by the elements $[i]$ of grade a_i for $i \in I$.

Definition 3.1.4. The family $\{a_i\}_{i \in I}$ is defining a map $| - | : I \to \mathrm{Ob}(\mathfrak{R})$ if we put $|i| = a_i$ for $i \in I$. We shall say that the set I equipped with the map $| - | : I \to \mathrm{Ob}(\mathfrak{R})$ is $\underline{\mathfrak{R}\text{-graded}}$. If I is an $\mathfrak{R}$-graded set, then the $\mathfrak{R}$-module $\langle I \rangle = \bigoplus_{i \in I} \mathrm{Hom}_{\mathfrak{R}}(|i|, -)$ is freely generated by elements $[i]$ of grade $|i|$ for $i \in I$.

Proposition 3.1.7

An $\mathfrak{R}$-module M is free if, and only if, it is isomorphic to a coproduct of $\mathfrak{R}$-modules $\mathrm{Hom}_{\mathfrak{R}}(a_i, -)$ for a family $\{a_i\}_{i \in I}$ of objects of $\mathfrak{R}$, $M \cong \bigoplus_{i \in I} \mathrm{Hom}_{\mathfrak{R}}(a_i, -)$.

Proof.

($\Longleftarrow$) Follows by the comments above.

($\Longrightarrow$) Suppose M is a free left module over $\mathfrak{R}$ admitting a basis $\{x_i\}_{i \in I}$. Consider the natural transformation $\alpha : \langle a_i : i \in I \rangle \to M$ given above, where $x_i \in M(a_i)$. We check that α is a natural isomorphism, i.e. $\alpha_b : \bigoplus_{i \in I} \mathrm{Hom}_{\mathfrak{R}}(a_i, b) \to M(b)$ is an isomorphism for every $b \in \mathrm{Ob}(\mathfrak{R})$. Let $x \in M(b)$. We can write x as a unique admissible linear combination $x = \sum_{i \in I} r_i \cdot x_i$, where $r_i \in \mathrm{Hom}_{\mathfrak{R}}(a_i, b)$ for every $i \in I$. Since α is a natural transformation, we have that $M(r_i) \circ \alpha_{a_i} =$

$\alpha_b \circ \langle a_i : i \in I\rangle (r_i)$. Then

$$x = \sum_{i\in I} r_i \cdot x_i = \sum_{i\in I} M(r_i)(x_i) = \sum_{i\in I} M(r_i)(\alpha_{a_i}([i])) = \sum_{i\in I} \alpha_b \circ \langle a_i : i \in I\rangle (r_i)([i])$$
$$= \sum_{i\in I} \alpha_b(r_i \cdot [i]) = \alpha_b\left(\sum_{i\in I} r_i \cdot [i]\right), \text{ where } \textstyle\sum_{i\in I} r_i \cdot [i] \text{ is unique} \qquad \square$$

Corollary 3.1.8

Every free (left) $\mathfrak{R}$-module is projective.

Proof.

Since the direct sum of projective objects is projective, by the previous proposition it suffices to show that $\mathrm{Hom}_{\mathfrak{R}}(a, -)$ is projective in $\mathbf{Mod}(\mathfrak{R})$. We prove that the functor $\mathrm{Hom}_{\mathbf{Mod}(\mathfrak{R})}(\mathrm{Hom}_{\mathfrak{R}}(a, -), -) : \mathbf{Mod}(\mathfrak{R}) \to \mathbf{Ab}$ is exact. Suppose we are given a short exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathbf{Mod}(\mathfrak{R})$. The sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathbf{Mod}(\mathfrak{R})}(\mathrm{Hom}_{\mathfrak{R}}(a, -), M') \arrow[r] & \mathrm{Hom}_{\mathbf{Mod}(\mathfrak{R})}(\mathrm{Hom}_{\mathfrak{R}}(a, -), M) \arrow[dl] \\
& \mathrm{Hom}_{\mathbf{Mod}(\mathfrak{R})}(\mathrm{Hom}_{\mathfrak{R}}(a, -), M'') \arrow[r] & 0
\end{tikzcd}$$

is exact since it is isomorphic (in $\mathbf{Ab}$) to $0 \to M'(a) \to M(a) \to M''(a) \to 0$ by Yoneda's Lemma. $\square$

Proposition 3.1.9 (Eilenberg's Trick for modules over a ringoid)

For every projective $\mathfrak{R}$-module P, there exists a free $\mathfrak{R}$-module F together with an isomorphism $P \oplus F \cong F$.

Proof.

This is a consequence of Proposition 3.1.2 with $\mathcal{S}$ the set of representable functors $\mathrm{Hom}_{\mathfrak{R}}(a, -)$. $\square$

Corollary 3.1.10

Every n-projective (left) $\mathfrak{R}$-module has a free left resolution of length n.

Definition 3.1.5. Let κ be a regular cardinal strictly greater than the cardianlity of $\mathrm{Hom}_{\mathfrak{R}}(a, b)$ for every $a, b \in \mathrm{Ob}(\mathfrak{R})$. We shall say that a $\mathfrak{R}$-module M is κ-small if $\mathrm{Card}(M(a)) \leq \kappa$, for every $a \in \mathrm{Ob}(\mathfrak{R})$. As we did in ${}_R\mathbf{Mod}$, if $\mathcal{X}$ is a class of modules over a ringoid $\mathfrak{R}$, then $\mathcal{X}^{\leq\kappa}$ shall denote the set of $X \in \mathcal{X}$ such that X is κ-small.

Lemma 3.1.11 (Generalization of Lemma 3.1.5)

Let M be a n-projective $\mathfrak{R}$-module. Then for every homogeneous element $x \in M(a)$ there exists a κ-small submodule $N \hookrightarrow M$ such that:

(1) $x \in N(a)$.

(2) The $\mathfrak{R}$-modules N and M/N are n-projective.

Proof.

By the previous corollary, we start with a free resolution in $\mathbf{Mod}(\mathfrak{R})$:

$$0 \to \langle I_n \rangle \overset{\partial_n}{\to} \langle I_{n-1} \rangle \to \cdots \to \langle I_1 \rangle \overset{\partial_1}{\to} \langle I_0 \rangle \overset{\partial_0}{\to} M \to 0,$$

where I_k is an $\mathfrak{R}$-graded set for every $0 \leq k \leq n$.

The map $(\partial_0)_a : \langle I_0 \rangle (a) = \bigoplus_{i \in I_0} \mathrm{Hom}_{\mathfrak{R}}([i], a) \to M(a)$ is surjective, so we can find a finite number of maps $r_{i_1} : [i_1] \to a, \ldots, r_{i_k} : [i_k] \to a$ such that $x =$

$(\partial_0)_a(r_{i_1} + \cdots + r_{i_k})$. Then $Z_0 = \{i_1, \ldots, i_k\}$ is a finite subset of I_0 such that $x \in \partial_0(\langle Z_0 \rangle)$ (by abuse of notation, this shall mean that $x \in (\partial_0)_a \circ \langle Z_0 \rangle (a)$). Consider the natural transformation $\partial_0|_{\langle Z_0 \rangle} : \langle Z_0 \rangle \to M$ and let $y \in \mathrm{Ker}(\partial_0|_{\langle Z_0 \rangle})$ of degree b. Since $\langle I_1 \rangle (b) \to \langle I_0 \rangle (b) \to M(b)$ is exact, there exists $y' \in \langle I_1 \rangle (b)$ such that $y = (\partial_1)_b(y')$. We can write $y' = \sum_{i \in Z_1^y} r_i \cdot [i]$, where Z_1^y is a finite subset of I_1. Let $Z_1 = \bigsqcup\{Z_1^y : y \in \mathrm{Ker}(\partial_0|_{\langle Z_0 \rangle})\}$. In order to estimate the number of elements of Z_1^y, note that for each $y \in \mathrm{Ker}(\partial_0|_{\langle Z_0 \rangle})$ we have a unique tuple $(\rho_i : i \in Z_0)$, with $\rho_i \in \mathrm{Hom}_{\mathfrak{R}}([i], b)$. Then we have

$$\begin{aligned}
\mathrm{Card}(Z_1) &= \sum \left\{ \mathrm{Card}(Z_1^y) : y \in \mathrm{Ker}(\partial_0|_{\langle Z_0 \rangle}) \right\} \\
&\leq \mathrm{Card}(\mathrm{Ker}(\partial_0|_{\langle Z_0 \rangle})) \text{ since each } Z_1^y \text{ is finite,} \\
&\leq \mathrm{Card}(\{(r_i : i \in Z_0) : y \in \mathrm{Ker}(\partial_0|_{\langle Z_0 \rangle})\}) \\
&\leq \prod_{i \in Z_0} \mathrm{Card}(\mathrm{Hom}_{\mathfrak{R}}([i], b)) \leq \kappa
\end{aligned}$$

Then we have Z_1 is a κ-small subset of I_1 such that $\partial_1(\langle Z_1 \rangle) \supseteq \mathrm{Ker}(\partial_0|_{\langle Z_0 \rangle})$, i.e. $(\partial_1)_b(\langle Z_1 \rangle (b)) \supseteq \mathrm{Ker}((\partial_0)_b|_{\langle Z_0 \rangle (b)})$ for every $b \in \mathrm{Ob}(\mathfrak{R})$. In a similar way, we can find a κ-small subset $Z_2 \subseteq I_2$ such that $\partial_2(\langle Z_2 \rangle) \supseteq \mathrm{Ker}(\partial_1|_{\langle Z_1 \rangle})$. We keep repeating this procedure until we get a κ-small subset $Z_n \subseteq I_n$ such that $\partial(\langle Z_n \rangle) \supseteq \mathrm{Ker}(\partial_{n-1}|_{\langle Z_{n-1} \rangle})$.

The next step in the zig-zag procedure consists in choosing a κ-small subset $Z_{n-1}^{(1)} \subseteq I_{n-1}$, containing Z_{n-1}, such that $\partial_n(\langle Z_n \rangle) \subseteq < Z_{n-1}^{(1)} >$. Let $y \in \partial_n(\langle Z_n \rangle)$ of degree b. Then $y = (\partial_n)_b(z)$, where $z = \sum_{i \in Z_n} r_i \cdot [i]$. We have $y = \sum_{i \in Z_n} (\partial_n)_b(r_i \cdot [i])$. On the other hand, $(\partial_n)_b(r_i \cdot [i]) = \sum_{j \in Z_{n-1}^{(1),y,i}} q_j \cdot [j]$ for a finite subset $Z_{n-1}^{(1),y,i} \subseteq I_{n-1}$. Thus $y = \sum_{i \in Z_n} \sum_{j \in Z_{n-1}^{(1),y,i}} q_j \cdot [j] = \sum_{j \in Z_{n-1}^{(1)}} q_j \cdot [j]$, where $Z_{n-1}^{(1)}$ is the disjoint union $\bigsqcup\{Z_{n-1}^{(1),y,i} : y \in \partial_n(\langle Z_n \rangle) \text{ and } i \in Z_n\}$. We have

$$\begin{aligned}
\mathrm{Card}(Z_{n-1}^{(1)}) &= \sum \left\{ \mathrm{Card}(Z_{n-1}^{(1),y,i}) : y \in \partial_n(\langle Z_n \rangle) \text{ and } i \in Z_n \right\} \\
&\leq \mathrm{Card}(\{(r_i : i \in Z_n) : y \in \partial_n(\langle Z_n \rangle)\}) \leq \kappa.
\end{aligned}$$

Then $Z^{(1)}_{n-1}$ is a κ-small subset of I_{n-1} such that $\partial_n(\langle Z_n \rangle) \subseteq < Z^{(1)}_{n-1} >$. Note that we can construct $Z^{(1)}_{n-1}$ containing Z_{n-1}. Similarly, there is a κ-small subset $Z^{(1)}_{n-2} \subseteq I_{n-2}$ containing Z_{n-2} such that $\partial_{n-1}(< Z^{(1)}_{n-1} >) \subseteq < Z^{(1)}_{n-2} >$.

At this point, we just need to mimic the argument given in the proof of Lemma 3.1.5, with the corresponding considerations for $\mathbf{Mod}(\mathfrak{R})$. Set $J_k := Z_k \cup Z^{(1)}_k \cup \cdots$ for every $0 \leq k \leq n$. It is clear that $\langle J_k \rangle := \bigoplus_{i \in J_k} \mathrm{Hom}_{\mathfrak{R}}([i], -)$ is a κ-small submodule of $\langle I_k \rangle$. By construction, we have an exact sequence

$$0 \to \langle J_n \rangle \xrightarrow{\partial_n} \langle J_{n-1} \rangle \to \cdots \to \langle J_1 \rangle \xrightarrow{\partial_1} \langle J_0 \rangle \xrightarrow{\partial_0} N \to 0,$$

where $N = \mathrm{CoKer}(\langle J_1 \rangle \xrightarrow{\partial_1} \langle J_0 \rangle)$. Note that $x \in N$ and that each $\langle J_k \rangle$ is projective by Corollary 3.1.8. It is only left to show that M/N is n-projective. It suffices to take the quotient of the resolution of M by the resolution of N, to get an exact sequence

$$0 \to \langle I_n \rangle / \langle J_n \rangle \to \langle I_{n-1} \rangle / \langle J_{n-1} \rangle \to \cdots \to \langle I_1 \rangle / \langle J_1 \rangle \to \langle I_0 \rangle / \langle J_0 \rangle \to M/N \to 0.$$

It is not hard to check that $\langle I_k \rangle / \langle J_k \rangle \cong \langle I_k - J_k \rangle$. So the previous sequence is a projective resolution of length n of M/N. $\square$

We are almost ready to show that the cotorsion pair $(\mathcal{P}_n(\mathbf{Mod}(\mathfrak{R})), (\mathcal{P}_n(\mathbf{Mod}(\mathfrak{R})))^{\perp})$ is cogenerated by a set. This turns out to be a consequence of Proposition 3.1.14.

Lemma 3.1.12

Let $(\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion pair in an Abelian category $\mathcal{C}$.

The class of left n-$\mathcal{A}$-objects is closed under extensions.	The class of right n-$\mathcal{B}$-objects is closed under extensions.

Proof.

We only prove the left assertion. First, note that X is a left n-$\mathcal{A}$-object if, and only if, $\mathrm{Ext}^i_{\mathcal{C}}(X, B) = 0$ for every $i > n$ and for every $B \in \mathcal{B}$. We verify this for $n = 1$, the rest follows by using derived long exact sequences of the functor $\mathrm{Ext}(-,-)$ and syzygies of X. So suppose we are given a $B \in \mathcal{B}$ and a short exact sequence $A_1 \hookrightarrow A_0 \twoheadrightarrow X$, where $A_0, A_1 \in \mathcal{A}$. Then we obtain a long exact sequence $\cdots \to \cancelto{0}{\mathrm{Ext}^1_{\mathcal{C}}(A_1, B)} \to \mathrm{Ext}^2_{\mathcal{C}}(X, B) \to \cancelto{0}{\mathrm{Ext}^2_{\mathcal{C}}(A_0, B)} \to \cdots$, where $\mathrm{Ext}^2_{\mathcal{C}}(A_0, B) = 0$ by Proposition 2.2.1. Then $\mathrm{Ext}^2_{\mathcal{C}}(X, B) = 0$. Now suppose the last equality holds for every $B \in \mathcal{B}$. There is an exact sequence $K \hookrightarrow P \twoheadrightarrow X$ where P is a projective object (and so it is in $\mathcal{A}$). Let $B \in \mathcal{B}$, we have a long exact sequence $\cdots \to \cancelto{0}{\mathrm{Ext}^1_{\mathcal{C}}(P, B)} \to \mathrm{Ext}^1_{\mathcal{C}}(K, B) \to \cancelto{0}{\mathrm{Ext}^2_{\mathcal{C}}(X, B)} \to \cdots$ where $\mathrm{Ext}^1_{\mathcal{C}}(P, B) = 0$ and $\mathrm{Ext}^2_{\mathcal{C}}(X, B) = 0$. So $K \in \mathcal{A}$ and hence X is a left 1-$\mathcal{A}$-object.

Now suppose we are given an exact sequence $X' \hookrightarrow X \twoheadrightarrow X''$, where X' and X'' are left n-$\mathcal{A}$-objects. Let $B \in \mathcal{B}$. We have a derived long exact sequence $\cdots \to \mathrm{Ext}^i_{\mathcal{C}}(X'', B) \to \mathrm{Ext}^i_{\mathcal{C}}(X, B) \to \mathrm{Ext}^i_{\mathcal{C}}(X', B) \to \cdots$, where for every $i > n$ we know $\mathrm{Ext}^i_{\mathcal{C}}(X', B) = 0$ and $\mathrm{Ext}^i_{\mathcal{C}}(X'', B) = 0$. It follows $\mathrm{Ext}^i_{\mathcal{C}}(X, B) = 0$ for every $B \in \mathcal{B}$ and every $i > n$, i.e. X is a left n-$\mathcal{A}$-object. $\square$

Proposition 3.1.13

Let $(\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion pair in $\mathbf{Mod}(\mathfrak{R})$. Suppose for each n-$\mathcal{A}$-module X and each $x \in X$, there exists a κ-small left n-$\mathcal{A}$-module $X_x \subseteq X$ (with κ as in the previous definition) such that $x \in X_x$ and that X/X_x is also a left n-$\mathcal{A}$-module. Then every left n-$\mathcal{A}$-module is a transfinite extension of the set of κ-small left n-$\mathcal{A}$-modules.

$\boxed{\textbf{Proof}}$.

Let X be a left $\mathfrak{R}$-module as described in the statement. Choose any $x_0 \in X$. Then there exists a small left n-$\mathcal{A}$-module X_0 such that $x_0 \in X_0$ and such that X/X_0 is also a left n-$\mathcal{A}$-module. Now choose a class $x_1 + X_0 \neq 0 + X_0$. Then there exists a small left n-$\mathcal{A}$-module X_1/X_0 such that $x_1 + X_0 \in X_1/X_0$ and such that $X/X_1 \cong (X/X_0)/(X_1/X_0)$ is left n-$\mathcal{A}$. Note that $X_0 \subseteq X_1$ and that X_1 is small since $\mathrm{Card}(X_1) = \mathrm{Card}(X_1/X_0) \cdot \mathrm{Card}(X_0)$. Since we have a short exact sequence $X_0 \hookrightarrow X_1 \twoheadrightarrow X_1/X_0$, where X_0 and X_1/X_0 are left n-$\mathcal{A}$-modules, by the previous lemma X_1 is also a left n-$\mathcal{A}$-module. We used transfinite induction to construct the desired transfinite extension. Suppose X_α is constructed, for a given (non limit) ordinal $\alpha > 1$. Then we can obtain $X_{\alpha+1}$ from X_α as we with X_1 from X_0. Finally, if $\beta > 0$ is a limit ordinal, we set $X_\beta := \bigcup_{\alpha<\beta} X_\alpha$. Then X is the transfinite extension $X = \mathrm{CoLim}_{\alpha<\lambda} X_\alpha$, for some ordinal $\lambda > 0$. $\square$

Proposition 3.1.14

Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in an Abelian category $\mathcal{C}$. If $\mathcal{S} \subseteq \mathcal{A}$ is a set objects in $\mathcal{C}$ such that every object $A \in \mathcal{A}$ is a transfinite extension of $\mathcal{S}$, then $(\mathcal{A}, \mathcal{B})$ is cogenerated by $\mathcal{S}$.

$\boxed{\textbf{Proof}}$.

We need to show $\mathcal{B} = \mathcal{S}^\perp$. The inclusion $\mathcal{B} \subseteq \mathcal{S}^\perp$ is clear. So suppose $Y \in \mathcal{S}^\perp$ and let $A \in \mathcal{A}$. By hypothesis, A is a transfinite extension of $\mathcal{S}$, say $A = \mathrm{CoLim}_{\alpha<\lambda}(S_\alpha)$ for some ordinal λ. Since $S_0, S_{\alpha+1}/S_\alpha \in \mathcal{S}$ for every non limit ordinal α, we have $\mathrm{Ext}^1_{\mathcal{C}}(S_0, Y) = 0$ and $\mathrm{Ext}^1_{\mathcal{C}}(S_{\alpha+1}/S_\alpha, Y) = 0$, it follows by Eklof's Lemma that $\mathrm{Ext}^1_{\mathcal{C}}(A, Y) = 0$, i.e. $Y \in \mathcal{B}$. $\square$

Theorem 3.1.15 (Generalization of (5, Theorem 4.2))

$(\mathcal{P}_n(\mathbf{Mod}(\mathfrak{R})), (\mathcal{P}_n(\mathbf{Mod}(\mathfrak{R})))^{\perp})$ is a hereditary complete cotorsion pair.

Proof.

We only need to show that $\mathcal{P}_n(\mathbf{Mod}(\mathfrak{R})) = {}^{\perp}((\mathcal{P}_n(\mathbf{Mod}(\mathfrak{R})))^{\perp})$. The inclusion $\mathcal{P}_n(\mathbf{Mod}(\mathfrak{R})) \subseteq {}^{\perp}((\mathcal{P}_n(\mathbf{Mod}(\mathfrak{R})))^{\perp})$ is clear. For the other inclusion, consider the set

$$(\mathcal{P}_n(\mathbf{Mod}(\mathfrak{R})))^{\leq\kappa} := \{S \in \mathcal{P}_n(\mathbf{Mod}(\mathfrak{R})) : S \text{ is } \kappa\text{-small}\},$$

and let $B = \bigoplus\{S : S \in (\mathcal{P}_n(\mathbf{Mod}(\mathfrak{R})))^{\leq\kappa}\}$. Since every n-projective left $\mathfrak{R}$-module is a transfinite extension of $(\mathcal{P}_n(\mathbf{Mod}(\mathfrak{R})))^{\leq\kappa}$ by Lemma 3.1.11 and Proposition 3.1.13, we have that $Y \in (\mathcal{P}_n(\mathbf{Mod}(\mathfrak{R})))^{\perp}$ if, and only if, $\mathrm{Ext}^1_{\mathfrak{R}}(B, Y) = 0$ [1]. Let $X \in {}^{\perp}((\mathcal{P}_n(\mathbf{Mod}(\mathfrak{R})))^{\perp})$ and consider a short exact sequence $K \hookrightarrow P \twoheadrightarrow X$, where P is a projective left $\mathfrak{R}$-module. Applying the same procedure from the proof of Eklof and Trlijaf's Theorem [2], we can construct a short exact sequence $K \hookrightarrow A \twoheadrightarrow L$, where A is the union of a continuous chain $(M_\alpha : \alpha < \lambda)$ such that $M_0 = K$, $M_{\alpha+1}/M_\alpha \cong B$ and $\mathrm{Ext}^1_{\mathfrak{R}}(B, A) = 0$. By the previous comments, $A \in (\mathcal{P}_n(\mathbf{Mod}(\mathfrak{R})))^{\perp}$. On the other hand, $L \cong A/K$, and so L is the union of the continuous chain $(L_\alpha : \alpha < \lambda)$ where $L_\alpha = M_\alpha/K$ for every $\alpha < \lambda$. Then $L_0 = 0$ and $L_{\alpha+1}/L_\alpha \cong B \in \mathcal{P}_n(\mathbf{Mod}(\mathfrak{R}))$. We show $L \in \mathcal{P}_n(\mathbf{Mod}(\mathfrak{R}))$. Let N be any module. Since L_0 and $L_{\alpha+1}/L_\alpha$ are both n-projective, for every $i > n$ we have $0 = \mathrm{Ext}^i_{\mathfrak{R}}(L_0, N) = \mathrm{Ext}^1_{\mathfrak{R}}(L_0, N')$ and $0 = \mathrm{Ext}^i_{\mathfrak{R}}(L_{\alpha+1}/L_\alpha, N) = \mathrm{Ext}^1_{\mathfrak{R}}(L_{\alpha+1}/L_\alpha, N')$, where $N' \in \Omega^{i-1}(N)$. By Eklof's Lemma, we get $0 = \mathrm{Ext}^1_{\mathfrak{R}}(L, N') = \mathrm{Ext}^i_{\mathfrak{R}}(L, N)$ for every $i > n$ and for every $N \in \mathrm{Ob}(\mathbf{Mod}(\mathfrak{R}))$. Hence, $L \in \mathcal{P}_n(\mathbf{Mod}(\mathfrak{R}))$. Taking the pushout of $K \hookrightarrow P$

1. We are replacing $\mathrm{Ext}^1_{\mathbf{Mod}(\mathfrak{R})}(-,-)$ by $\mathrm{Ext}^i_{\mathfrak{R}}(-,-)$ for simplicity.

2. Notice $\mathbf{Mod}(\mathfrak{R})$ is a Grothendieck category since $\mathbf{Ab}$ is and $\mathfrak{R}$ is small

and $K \hookrightarrow A$, we get a commutative diagram:

$$\begin{tikzcd}
K \arrow[r, hook] \arrow[d, hook] & P \arrow[r, two heads] \arrow[d, hook] & X \arrow[d, equal] \\
A \arrow[r, hook] \arrow[d, two heads] & A \coprod_K P \arrow[r, two heads] \arrow[d, two heads] & X \\
L \arrow[r, equal] & L &
\end{tikzcd}$$

Note that $\mathrm{Ext}^1_R(X, A) = 0$, the central row splits and so X is a direct summand of $A \coprod_K P$. Since $P, L \in \mathcal{P}_n(\mathbf{Mod}(\mathfrak{R}))$ and the central column is exact, we have that $A \coprod_K P \in \mathcal{P}_n(\mathbf{Mod}(\mathfrak{R}))$ and hence $X \in \mathcal{P}_n(\mathbf{Mod}(\mathfrak{R}))$. By the previous proposition, $(\mathcal{P}_n(\mathbf{Mod}(\mathfrak{R})), (\mathcal{P}_n(\mathbf{Mod}(\mathfrak{R})))^\perp)$ is a complete cotorsion pair.

To show $(\mathcal{P}_n(\mathbf{Mod}(\mathfrak{R})), (\mathcal{P}_n(\mathbf{Mod}(\mathfrak{R})))^\perp)$ is hereditary, we only need to check that $\mathcal{P}_n(\mathbf{Mod}(\mathfrak{R}))$ is closed under taking kernels of epimorphisms. So suppose we are given an exact sequence $X' \hookrightarrow X \twoheadrightarrow X''$ with $X, X'' \in \mathcal{P}_n(\mathbf{Mod}(\mathfrak{R}))$. For any module Y, we get a derived long exact sequence

$$\cdots \to \mathrm{Ext}^i_{\mathfrak{R}}(X, Y) \to \mathrm{Ext}^i_{\mathfrak{R}}(X', Y) \to \mathrm{Ext}^{i+1}_{\mathfrak{R}}(X'', Y) \to \cdots .$$

If $i > n$, then $\mathrm{Ext}^i_{\mathfrak{R}}(X, Y) = 0$ and $\mathrm{Ext}^{i+1}_{\mathfrak{R}}(X'', Y) = 0$. It follows $\mathrm{Ext}^i_{\mathfrak{R}}(X', Y) = 0$ for every $i > n$ and every $Y \in \mathrm{Ob}(\mathbf{Mod}(\mathfrak{R}))$. Therefore, $\mathcal{P}_n(\mathbf{Mod}(\mathfrak{R}))$ is resolving. □

Now it is time to go back to the category ${}_R\mathbf{Mod}$ and study the relationship between cotorsion pairs and flat dimensions of modules. We recall the arguments given by Enochs to show that $(\mathcal{F}_0, (\mathcal{F}_0)^\perp)$ is a complete cotorsion pair, and how to adapt these arguments, via de zig-zag procedure, to obtain a similar result for the class of n-flat modules.

Definition 3.1.6. Let $- \otimes_R - : \mathbf{Mod}_R \times {}_R\mathbf{Mod} \to \mathbf{Ab}$ be the usual tensor product functor. A right R-module F is <u>flat</u> if the covariant functor $- \otimes_R F$:

${}_R\mathbf{Mod} \to \mathbf{Ab}$ is left exact. We shall denote by $\mathcal{F}_0$ the class of flat right R-modules. Similarly, a left R-module F is said to be flat if the functor $F \otimes_R - : \mathbf{Mod}_R \to \mathbf{Ab}$ is exact.

We can obtain left derived functors of $- \otimes_R -$. Let M be a right R-module, and consider a projective resolution of M, say $\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$. Applying the functor $- \otimes_R N$, for a left R-module N, we get a complex

$$\cdots \to P_2 \otimes_R N \to P_1 \otimes_R N \to P_0 \otimes_R N \to M \otimes_R N \to 0.$$

After deleting $M \otimes_R N$ we get $\cdots \to P_2 \otimes_R N \to P_1 \otimes_R N \to P_0 \otimes_R N \to 0$.

Definition 3.1.7. The $\underline{n\text{-th torsion group}}$ $\mathrm{Tor}_n^R(M, N)$ is defined as the n-th homology group of the previous complex:

$$\mathrm{Tor}_n^R(M,N) := \begin{cases} \frac{\mathrm{Ker}(P_n \otimes_R N \to P_{n-1} \otimes_R N)}{\mathrm{Im}(P_{n+1} \otimes_R N \to P_n \otimes_R N)} & \text{for } n > 0, \\ \frac{P_0 \otimes_R N}{\mathrm{Im}(P_1 \otimes_R N \to P_0 \otimes_R N)} & \text{for } n = 0. \end{cases}$$

We summarize the most useful properties of $\mathrm{Tor}_n^R(-,-)$. The reader can see the proof, for instance, in (46) or (52). Recall that a module M is (left) n-flat if there exists an exact left flat resolution of M of length n (Definition 1.7.1).

Proposition 3.1.16 (Properties of torsion functors)

(1) (46, Proposition 3.2 a)) $\mathrm{Tor}_0^R(M,N) = M \otimes_R N$, for every $M \in \mathrm{Ob}(\mathbf{Mod}_R)$ and $N \in \mathrm{Ob}({}_R\mathbf{Mod})$.

(2) (46, Theorem 3.4 a)) If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence in $\mathbf{Mod}_R$, then there exists the following long exact sequence in $\mathbf{Ab}$:

$$\cdots \longrightarrow \mathrm{Tor}_2^R(M'',N) \longrightarrow \mathrm{Tor}_1^R(M',N) \longrightarrow \mathrm{Tor}_1^R(M,N) \longrightarrow \mathrm{Tor}_1^R(M'',N) \longrightarrow M' \otimes_R N \longrightarrow M \otimes_R N \longrightarrow M'' \otimes_R N \longrightarrow 0$$

(3) (46, Proposition 4.5) The following conditions are equivalent for a left R-module N.

(a) N is left n-flat.

(b) $\mathrm{Tor}_i^R(M,N) = 0$ for every $M \in \mathrm{Ob}(\mathbf{Mod}_R)$ and every $i > n$.

Before constructing cotorsion pairs concerning the class of flat modules, we need to recall the following isomorphism that relates extension and torsion.

Theorem 3.1.17 (see (21, Theorem 3.2.1))

Let R and S be rings, M a left R-module, and N an (S,R)-bimodule. If I is an injective left S-module, then $\mathrm{Ext}_R^i(M, \mathrm{Hom}_S(N,I)) \cong \mathrm{Hom}_S(\mathrm{Tor}_i^R(N,M), I)$ for all $i \geq 0$.

This theorem is very useful to check that the class of flat modules $\mathcal{F}_0$ is the left half of a cotorsion pair. Instead of showing this, we prove the more general result that the class $\mathcal{F}_n$ of n-flat modules is the left half of a cotorsion pair.

Definition 3.1.8. Given a left R-module M, the Pontryagin or character module of M is the right R-module $M^+ := \mathrm{Hom}_R(M, \mathbb{Q}/\mathbb{Z})$.

Proposition 3.1.18

The class of left n-flat modules $\mathcal{F}_n$ is the left half a cotorsion pair $(\mathcal{F}_n, (\mathcal{F}_n)^\perp)$.

Proof.

Note that it suffices to show that ${}^\perp({\mathcal{F}_n}^\perp) \subseteq \mathcal{F}_n$. So let $M \in {}^\perp({\mathcal{F}_n}^\perp)$ and N be a right R-module. Consider the left R-module $\mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z})$. For every n-flat module L and every n-cosyzygy N' of $\mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z})$, we have by the previous theorem and dimension shifting that $\mathrm{Ext}^1_R(L, N') \cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Tor}^R_{n+1}(N, L), \mathbb{Q}/\mathbb{Z})$, since $\mathbb{Q}/\mathbb{Z}$ is an injective left $\mathbb{Z}$-module. On the other hand, $\mathrm{Tor}^R_{n+1}(N, L) = 0$, since L is n-flat (Proposition 3.1.16). So $N' \in {\mathcal{F}_n}^\perp$. Then we obtain $0 = \mathrm{Ext}^1_R(M, N') \cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Tor}^R_{n+1}(N, M), \mathbb{Q}/\mathbb{Z})$. The fact that $\mathbb{Q}/\mathbb{Z}$ is a cogenerator of **Ab** implies $\mathrm{Tor}^R_{n+1}(N, M) = 0$ [3] for every right R-module N, i.e. M is n-flat. $\square$

In the paper (8), Enochs proved that $(\mathcal{F}_0, (\mathcal{F}_0)^\perp)$ is a complete cotorsion pair, by specifying a set of generators for it. We shall recall Enochs' proof, which mostly consists in constructing transfinite extensions for flat modules F of pure small submodules of F. This shall allow us to generalize part of his arguments to show that $(\mathcal{F}_n, (\mathcal{F}_n)^\perp)$ is also complete. In the proof of (31, Theorem 4.1.3), the reader can find another argument for this, but we think the proof presented later is simpler.

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3. More generally, if G is a cogenerator of an Abelian category $\mathcal{C}$ and if X is an object of $\mathcal{C}$ such that every morphism $X \to G$ is zero, then X is the zero object. Using the dual of Proposition 1.9.1, there exists a monomorphism $X \hookrightarrow G^{[I]}$, where $G^{[I]}$ is the product of copies of G over the index set $I = \mathrm{Hom}_{\mathcal{C}}(X, G)$. Since $\mathrm{Hom}_{\mathcal{C}}(X, G) = \{0\}$, we have an monomorphism $X \hookrightarrow G$, which is zero and hence X has to be the zero object.

Definition 3.1.9. A submodule S of a left R-module N is a pure submodule of M if the sequence $0 \to M \otimes_R S \to M \otimes_R N$ is exact for every right R-module M.

Pure subcomules are important in the sense that they will give rise to a set of generator for the cotorsion pair $(\mathcal{F}_0, (\mathcal{F}_0)^{\perp})$. We shall see that every flat module F can be written as a transfinite extension of small pure submodules of F. Before showing this, we present the following property of pure submodules.

Proposition 3.1.19 (see (23, Lemma 9.1))

If S is a pure submodule of a flat left R-module F, then both S and F/S are also flat.

Proof.

The proof we give here is different from that presented in the cited reference. We need to show that $- \otimes_R S$ and $- \otimes_R F/S$ are left exact functors. Consider a short exact sequence $0 \to M' \to M \to M'' \to 0$. Since S is a pure submodule of F and F is flat, we have the following diagram with exact rows and columns:

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
0 \arrow[r] & M' \otimes_R S \arrow[r, "\alpha'"] \arrow[d, "f_S"] & M' \otimes_R F \arrow[r, "\beta'"] \arrow[d, "f_F"] & M' \otimes_R F/S \arrow[r] \arrow[d, "f_{F/S}"] & 0 \\
0 \arrow[r] & M \otimes_R S \arrow[r, "\alpha"] \arrow[d, "g_S"] & M \otimes_R F \arrow[r, "\beta"] \arrow[d, "g_F"] & M \otimes_R F/S \arrow[r] \arrow[d, "g_{F/S}"] & 0 \\
0 \arrow[r] & M'' \otimes_R S \arrow[r, "\alpha''"] \arrow[d] & M'' \otimes_R F \arrow[r, "\beta''"] \arrow[d] & M'' \otimes_R F/S \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Note that f_S is injective since the upper left square commutes and the morphisms $f_F \circ \alpha'$ and α are monic. It is only left to show that $f_{F/S}$ is also monic. We show this by a diagram chasing argument. Let $x \in \mathrm{Ker}(f_{F/S})$. Since β' is epic, there exists an element $y \in M' \otimes_R F$ such that $x = \beta'(y)$. Then $0 = f_{F/S}(x) = f_{F/S} \circ \beta'(y) = \beta \circ f_F(y)$. Since the second row is exact, there exists an element $z' \in M \otimes_R S$ such that $f_F(y) = \alpha(z')$. We have $\alpha'' \circ g_S(z') = g_F \circ \alpha(z') = g_F \circ f_F(y) = 0$, and so $g_S(z') = 0$ (α'' is monic). Since the left column is exact, there exists $y' \in M' \otimes_R S$ such that $z' = f_S(y')$. One can verify that $y = \alpha'(y')$, using the fact that g_F is monic. It follows $x = \beta'(y) = \beta' \circ \alpha'(y') = 0$. Therefore, $f_{F/S}$ is monic. $\square$

From now on, consider a fixed infinite cardinal $\kappa > \mathrm{Card}(R)$.

Lemma 3.1.20 (see (21, Lemma 5.3.12 & Proposition 7.4.3))

Let F be a flat module. For each $x \in F$ there exists a κ-small pure submodule $S \subseteq F$ such that $x \in S$.

Using this lemma along with the previous proposition, it is possible to construct a transfinite extension of κ-small flat modules for every flat module F. It follows by Proposition 3.1.14 and Eklof and Trlifaj's Theorem that the Enochs cotorsion pair $(\mathcal{F}_0, (\mathcal{F}_0)^{\perp})$ is complete. It is time to show that the same is true for $(\mathcal{F}_n, (\mathcal{F}_n)^{\perp})$. We start with a generalization of the previous lemma.

Lemma 3.1.21

Let $M \in \mathcal{F}_n$ with a flat resolution

$$(1) = (0 \to F_n \xrightarrow{f_n} F_{n-1} \to \cdots \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \to 0)$$

and N be a small submodule of M. Then there exists a flat subresolution (i.e. a subcomplex)

$$0 \to S'_n \to \cdots \to S'_1 \to S'_0 \to N' \to 0$$

of (1) such that S'_k is a κ-small and pure submodule of F_k, for every $0 \leq k \leq n$, and such that $N \subseteq N'$. In this case, we shall say that N' is a n-pure submodule of M. Moreover, if N is part of a subresolution of (1)

$$0 \to S_n \to \cdots \to S_1 \to S_0 \to N \to 0$$

where S_k is a small and pure submodule of F_k, for every $0 \leq k \leq n$, then the above resolution of N' can be constructed in such a way that it contains the resolution of N.

Proof.

For every $x \in N$ there exists $y_x \in F_0$ such that $x = f_0(y_x)$. Consider the set $Y := \{y_x : x \in N \text{ and } f_0(y_x) = x\}$ and the submodule $\langle Y \rangle \subseteq F_0$. Since $\langle Y \rangle$ is small, there exists a small pure submodule $S_0(1) \subseteq F_0$ such that $\langle Y \rangle \subseteq S_0(1)$. Note that $f_0(S_0(1)) \supseteq N$. Now consider $\mathrm{Ker}(f_0|_{S_0(1)})$ and let A be a set of preimages of $\mathrm{Ker}(f_0|_{S_0(1)})$ such that $f_1(\langle A \rangle) \supseteq \mathrm{Ker}(f_0|_{S_0(1)})$. It is easy to see that $\langle A \rangle$ is a small submodule of F_1, so we can embed it into a small pure submodule $S_1(1) \subseteq F_1$. Hence we have $f_1(S_1(1)) \supseteq \mathrm{Ker}(f_0|_{S_0(1)})$. Now consider $\mathrm{Ker}(f_1|_{S_1(1)})$ and repeat the same process above in order to find a small pure submodule $S_2(1) \subseteq F_2$ such that $f_2(S_2(1)) \supseteq \mathrm{Ker}(f_1|_{S_1(1)})$. Keep doing this until find a small pure submodule $S_n(1) \subseteq F_n$ such that $f_n(S_n(1)) \supseteq \mathrm{Ker}(f_{n-1}|_{S_{n-1}(1)})$. Now $f_n(S_n(1))$ is a small submodule of F_{n-1}, so there is a small pure submodule $S_{n-1}(2) \subseteq F_{n-1}$ such that $f_n(S_n(1)) \subseteq S_{n-1}(2)$. Repeat this process until find a small pure submodule $S_0(2) \subseteq F_0$ such that $f_1(S_1(2)) \subseteq S_0(2)$. If we now consider $\mathrm{Ker}(f_0|_{S_0(2)}) \subseteq F_0$, we repeat the same argument above to find a small pure submodule $S_1(3) \subseteq F_1$ such that $f_1(S_1(3)) \supseteq \mathrm{Ker}(f_0|_{S_0(2)})$. Keep repeating this zig-zag procedure infinitely many times and set $S_k = \bigcup_{i \geq 1} S_k(i)$, for every $0 \leq k \leq n$. Note that each S_k is a pure submodule of F_k. By construction, we obtain an exact sequence of the form

$$(2) = (0 \to S_n \to S_{n-1} \to \cdots \to S_1 \to S_0 \to Q \to 0),$$

where $Q = \mathrm{CoKer}(f_1|_{S_1}) \subseteq M$. If we take the quotient of (1) by (2), we get an exact sequence $0 \to F_n/S_n \to \cdots \to F_1/S_1 \to F_0/S_0 \to M/Q \to 0$. Since each S_k is a pure submodule of F_k, we know from the previous proposition that S_k and F_k/S_k are flat modules, for every $0 \leq k \leq n$. Therefore, Q is a small n-flat submodule with $N \subseteq Q$ such that M/Q is also n-flat. The rest of the statement can be proven in a similar way. $\square$

Corollary 3.1.22

Every left n-flat module M is a transfinite extension of a set of κ-small left n-flat modules. It follows $(\mathcal{F}_n, (\mathcal{F}_n)^{\perp})$ is cogenerated by the set

$$(\mathcal{F}_n)^{\leq \kappa} := \{S \in \mathcal{F}_n : \mathrm{Card}(S) \leq \kappa\}.$$

There is more to say on cotorsion pairs involving the classes $\mathcal{F}_0$ and $\mathcal{F}_n$. Since $(\mathcal{F}_0, (\mathcal{F}_0)^{\perp})$ is complete, for every module M there exists an epimorphism $F \twoheadrightarrow M$ with $F \in \mathcal{F}_0$ and its kernel in $(\mathcal{F}_0)^{\perp}$. This epimorphism is not necessarily a flat cover. But recall that E. Enochs proved that for every M there is a flat cover. This is a consequence of the fact that $(\mathcal{F}_0, (\mathcal{F}_0)^{\perp})$ is perfect.

Definition 3.1.10. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is said to be <u>perfect</u> if $\mathcal{A}$ is a covering class and $\mathcal{B}$ is an enveloping class.

Theorem 3.1.23 (See (31, Corollary 5.32))

If $(\mathcal{A}, \mathcal{B})$ is a complete cotorsion pair in ${}_R\mathbf{Mod}$ and $\mathcal{A}$ is closed under direct limits, then $(\mathcal{A}, \mathcal{B})$ is perfect.

The arguments appearing in the proof from the given reference carry over to any Grothendieck category. Since the torsion functor $\mathrm{Tor}_i^R(-,-)$ preserves arbitrary colimits, we have that $\mathcal{F}_0$ is closed under direct limits. It follows that $(\mathcal{F}_0, (\mathcal{F}_0)^\perp)$ is a perfect cotorsion pair and so the Flat Cover Conjecture is settled. By the same argument, the pair $(\mathcal{F}_n, (\mathcal{F}_n)^\perp)$ is also perfect, i.e. every left R-module has an n-flat cover. This result also appears in (31, Theorem 4.1.3).

We close this section by showing that the class $\mathcal{I}_n(\mathcal{C})$ of n-injective objects in a Grothendieck category $\mathcal{C}$ is the right half of a hereditary complete cotorsion pair. Recall that if G is a generator of $\mathcal{C}$, then an object Y is n-injective if, and only if, $\mathrm{Ext}_{\mathcal{C}}^{n+1}(G/J, Y) = 0$ for every subobject $J \subseteq G$ (see Section 9 of Chapter 1).

Proposition 3.1.24

If $\mathcal{C}$ is a Grothendieck category with generator G, then $({}^\perp(\mathcal{I}_n(\mathcal{C})), \mathcal{I}_n(\mathcal{C}))$ is a cotorsion pair cogenerated by a set.

Proof.

It suffices to show $({}^\perp(\mathcal{I}_n(\mathcal{C})))^\perp \subseteq \mathcal{I}_n(\mathcal{C})$. Let $Y \in ({}^\perp(\mathcal{I}_n(\mathcal{C})))^\perp$ and J be a subobject of G. We have $\mathrm{Ext}_{\mathcal{C}}^{n+1}(G/J, Y) \cong \mathrm{Ext}_{\mathcal{C}}^1(S, Y)$, where $S \in \Omega^n(G/J)$. Note that $S \in {}^\perp(\mathcal{I}_n(\mathcal{C}))$. For if X is an n-injective object of $\mathcal{C}$, then $\mathrm{Ext}_{\mathcal{C}}^1(S, X) \cong \mathrm{Ext}_{\mathcal{C}}^{n+1}(G/J, X) = 0$. It follows $\mathrm{Ext}_{\mathcal{C}}^{n+1}(G/J, Y) = 0$. By the Corollary 1.9.6, $({}^\perp(\mathcal{I}_n(\mathcal{C})), \mathcal{I}_n(\mathcal{C}))$ is cogenerated by the set of all $S \in \Omega^n(G/J)$ with J running over the set of all subobjects of G. $\square$

It is not hard to see that the class $\mathcal{I}_n(\mathcal{C})$ is coresolving, and hence $({}^{\perp}(\mathcal{I}_n(\mathcal{C})), \mathcal{I}_n(\mathcal{C}))$ is a hereditary cotorsion pair by Proposition 2.2.1.

3.2 n-Projective model structures

The goal in this section is to construct a new Abelian model structure on $\mathbf{Ch}({}_R\mathbf{Mod})$ where the class of cofibrant objects is given by the differential graded n-projective complexes (with $n > 0$), and the trivial objects by the exact chain complexes. The motivation of this problem comes from the case $n = 0$, for which we know the existence of Hovey's projective model structure (See Example 2.3.1).

Theorem 3.2.1 (**Projective model structure**)
There is a unique Abelian model structure on $\mathbf{Ch}({}_R\mathbf{Mod})$ where the (trivial) cofibrations are the monomorphisms with differential graded projective (resp. projective) cokernel, the (trivial) fibrations are the epimorphisms (resp. with exact kernel), and the trivial objects are the exact chain complexes.

For nonzero projective dimension, we want to prove the following result.

Theorem 3.2.2 (**n-Projective model structure**)
There is a unique Abelian model structure on $\mathbf{Ch}({}_R\mathbf{Mod})$ where the (trivial) cofibrations are the monomorphisms with cokernel in $\mathrm{dg}\widetilde{\mathcal{P}_n}$ (resp. $\widetilde{\mathcal{P}_n}$), the (trivial) fibrations are the epimorphisms with kernel in $(\widetilde{\mathcal{P}_n})^{\perp}$ (resp. $(\mathrm{dg}\widetilde{\mathcal{P}_n})^{\perp}$), and the trivial objects are the exact chain complexes.

Remark 3.2.1. If $(\mathcal{C}_{of}, \mathcal{F}_{ib}, \mathcal{W}_{eak})$ is an Abelian model structure such that $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{E}$ are the classes of cofibrant, fibrant and trivial objects, respectively, then $\mathcal{W}_{eak}$ is the class of quasi-isomorphisms. For if $h \in \mathcal{W}_{eak}$, we can write $h = X \xrightarrow{f} Y \xrightarrow{g} Z$ where $f \in \mathcal{C}_{of} \cap \mathcal{W}_{eak}$ and $g \in \mathcal{F}_{ib} \cap \mathcal{W}_{eak}$. We have a short exact sequence $X \overset{f}{\hookrightarrow} Y \twoheadrightarrow A$ with A exact. It is known we can get a long homology exact sequence $\cdots \to \cancel{H_{n+1}(A)} \xrightarrow{0} H_n(X) \xrightarrow{H_n(f)} H_n(Y) \to \cancel{H_n(A)} \xrightarrow{0} H_{n-1}(X) \to \cdots$. Then $H_n(f)$ is an isomorphism, for every $n \in \mathbb{Z}$. Similarly, one can show that g is a quasi-isomorphism. Hence, h is also a quasi-isomorphism. Conversely, if $(\mathcal{C}_{of}, \mathcal{F}_{ib}, \mathcal{W}_{eak})$ is an Abelian model structure such that $\mathcal{W}_{eak}$ is the class of quasi-isomorphisms, then $\mathcal{E}$ is the class of trivial objects. For if X is a trivial object, then $0 \to X$ is a quasi-isomorphism. This means $0 = H_n(0) \cong H_n(X)$ for every $n \in \mathbb{Z}$, i.e. X is an exact complex.

In the previous section we showed that $(\mathcal{P}_n, (\mathcal{P}_n)^\perp)$ is a complete and hereditary cotorsion pair. Then by Theorems 2.4.3 and 2.4.5 we obtain two cotorsion pairs $(\widetilde{\mathcal{P}_n}, \mathrm{dg}\widetilde{(\mathcal{P}_n)^\perp})$ and $(\mathrm{dg}\widetilde{\mathcal{P}_n}, \widetilde{(\mathcal{P}_n)^\perp})$, $\widetilde{\mathcal{P}_n} = \mathrm{dg}\widetilde{\mathcal{P}_n} \cap \mathcal{E}$, $\widetilde{(\mathcal{P}_n)^\perp} = \mathrm{dg}\widetilde{(\mathcal{P}_n)^\perp} \cap \mathcal{E}$ and $\mathcal{E}$ is the class of exact chain complexes. By Proposition 1.8.4, the class $\widetilde{\mathcal{P}_n}$ represents the class of n-projective chain complexes. We can apply Lemma 3.1.11 to show that $(\widetilde{\mathcal{P}_n}, (\widetilde{\mathcal{P}_n})^\perp)$ is complete, after showing that $\mathbf{Ch}({}_R\mathbf{Mod})$ is equivalent to the category of left modules over certain ringoid.

Recall that the tensor product $\mathcal{C} \otimes \mathcal{D}$ of two pre-additive categories $\mathcal{C}$ and $\mathcal{D}$ is the pre-additive category defined by putting

(1) $\mathrm{Ob}(\mathcal{C} \otimes \mathcal{D}) = \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{D})$;

(2) $\mathrm{Hom}_{\mathcal{C}\otimes\mathcal{D}}((C, D), (C', D')) = \mathrm{Hom}_{\mathcal{C}}(C, C') \otimes \mathrm{Hom}_{\mathcal{D}}(D, D')$.

If $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ are pre-additive categories then we have a canonical isomorphism of pre-additive categories $[\mathcal{C} \otimes \mathcal{D}, \mathcal{E}] \simeq [\mathcal{C}, [\mathcal{D}, \mathcal{E}]]$. In particular, if $\mathcal{K}$ and $\mathfrak{R}$ are ringoids, then

$$[\mathcal{K}, \mathbf{Mod}(\mathfrak{R})] = [\mathcal{K}, [\mathfrak{R}, \mathbf{Ab}]] \simeq [\mathcal{K} \otimes \mathfrak{R}, \mathbf{Ab}] = \mathbf{Mod}(\mathcal{K} \otimes \mathfrak{R}).$$

If we consider a ring R and the ringoid $\mathfrak{S}$ defined in Example 1.10.1, then we have an isomorphism of additive categories $[\mathfrak{S}, \mathbf{Mod}(R)] \simeq \mathbf{Mod}(\mathfrak{S} \otimes R)$. This means that a chain complex of R-modules is a module over the ringoid $\mathfrak{S} \otimes R$, i.e. $\mathbf{Ch}({}_R\mathbf{Mod}) \cong \mathbf{Mod}(\mathfrak{S} \otimes R)$. Hence the following result follows.

Remark 3.2.2 (κ-small complexes, and some notations)**.** Let $\mathfrak{R}$ be a ringoid and κ be an infinite regular cardinal such that $\kappa > \mathrm{Card}(\mathrm{Hom}_{\mathfrak{R}}(a,b))$ for every $a, b \in \mathrm{Ob}(\mathfrak{R})$.

(1) If $\mathfrak{R}$ is the ringoid of Example 1.10.1 **(1)**, then we have that $\kappa > \mathrm{Card}(R)$. Note that a left module M over $\mathfrak{R}$ is κ-small if, and only of, it is κ-small as a left R-module (See Definition 3.1.1).

(2) We just saw above that $\mathbf{Ch}({}_R\mathbf{Mod})$ is equivalent to the category of left modules over the ringoid $\mathfrak{S} \otimes R$, with $\mathfrak{S}$ as in Example 1.10.1 **(2)**. It is not hard to see that $\kappa > \mathrm{Card}(R)$ and that the following conditions are equivalent for every chain complex X in $\mathbf{Ch}({}_R\mathbf{Mod})$:

(a) X is κ-small.

(b) $\mathrm{Card}(X_m) \leq \kappa$ for every $m \in \mathbb{Z}$.

(c) $\sum_{m \in \mathbb{Z}} \mathrm{Card}(X_m) \leq \kappa$.

Hence, we shall say for the rest of this work that a chain complex X in $\mathbf{Ch}({}_R\mathbf{Mod})$ is $\underline{\kappa\text{-small}}$ if each X_m is a κ-small module, where κ is an (infinite) regular cardinal satisfying $\kappa > \mathrm{Card}(R)$.

Theorem 3.2.3

The cotorsion pair $(\widetilde{\mathcal{P}_n}, (\widetilde{\mathcal{P}_n})^{\perp})$ is hereditary and complete.

Before continuing in our proof of Theorem 3.2.2, we want to comment some things on the theorem above. In (49, Theorem 4.7), the author proved Theorem 3.2.3

without using the theory of modules over ringoids. The proof given there is more complicated, and it consists in a sort of iteration of the zig-zag argument in a finite free[4] resolution of an n-projective complex to show Lemma 3.1.11 for the category $\mathbf{Ch}({}_R\mathbf{Mod})$.

In order to apply Hovey's correspondence to obtain the model structure described in Theorem 3.2.2, it is only left to show that $(\mathrm{dg}\widetilde{\mathcal{P}_n}, \mathrm{dg}\widetilde{(\mathcal{P}_n)^\perp} \cap \mathcal{E})$ is also complete. One way is to use Proposition 2.3.6, but after proving that $\mathcal{E}$ is the right and left half of two complete cotorsion pairs $({}^\perp\mathcal{E}, \mathcal{E})$ and $(\mathcal{E}, \mathcal{E}^\perp)$. This is and immediate consequence of Hovey's correspondence applied to the projective and injective model structures in Example 2.3.1 **(2)** and **(3)**.

Proposition 3.2.4 (See also (24, Section 2.3))

$({}^\perp\mathcal{E}, \mathcal{E})$ and $(\mathcal{E}, \mathcal{E}^\perp)$ are two complete cotorsion pairs in $\mathbf{Ch}({}_R\mathbf{Mod})$.

Although we already know that $(\mathrm{dg}\widetilde{\mathcal{P}_0}, \mathcal{E})$ is a complete cotorsion pair, we present a cogenerating set given by J. Rada and other authors in (12, Lemma 5.1), since we shall need this fact in the next chapter.

Lemma 3.2.5 (See (12, Lemma 5.1))

The cotorsion pair $(\mathrm{dg}\widetilde{\mathcal{P}_0}, \mathcal{E})$ is cogenerated by a set.

Proof.

Consider the disk complex $D^1(R) \in \widetilde{\mathcal{P}_0}$ and the sphere complex $S^0(R)$. Note that the quotient $D^1(R)/S^0(R)$ is isomorphic to the suspension $\Sigma^{-1}(S^0(R))$. Then we

4. In (49), a complex is defined to be free if it is exact and if every cycle group is a free left R-module.

have a partial projective resolution $0 \to S^0(R) \to D^1(R) \to \Sigma^{-1}(S^0(R)) \to 0$, from which we can compute the extension group $\mathrm{Ext}^1_{\mathbf{Ch}({}_R\mathbf{Mod})}(\Sigma^{-1}(S^0(R)), Y)$, for every chain complex $Y \in \mathrm{Ob}(\mathbf{Ch}({}_R\mathbf{Mod}))$. One can show $\mathrm{Hom}(S^0(R), Y) \cong Z_0(Y)$. Moreover, a map $S^0(R) \xrightarrow{f} Y$ has an extension $D^1(R) \xrightarrow{g} Y$ if, and only if, $x = f_0(1) \in Z_0(Y)$ is a boundary in Y. It follows $\mathrm{Ext}^1_{\mathbf{Ch}({}_R\mathbf{Mod})}(\Sigma^{-1}(S^0(R)), Y) \cong H_0(Y)$. One can generalize this isomorphism to $\mathrm{Ext}^1_{\mathbf{Ch}({}_R\mathbf{Mod})}(\Sigma^{k-1}(S^0(R)), Y) \cong H_k(Y)$, for every $k \in \mathbb{Z}$. So Y is exact if, and only if, $\mathrm{Ext}^1_{\mathbf{Ch}({}_R\mathbf{Mod})}(\Sigma^{k-1}(S^0(R)), Y)$ is 0 for every $k \in \mathbb{Z}$. Therefore, $\mathcal{E} = \mathcal{S}^\perp$, where $\mathcal{S} = \{\Sigma^{k-1}(S^0(R)) : k \in \mathbb{Z}\}$. $\square$

It follows $(\mathrm{dg}\widetilde{\mathcal{P}_n}, \mathrm{dg}\widetilde{(\mathcal{P}_n)^\perp} \cap \mathcal{E})$ is a complete cotorsion pair in $\mathbf{Ch}({}_R\mathbf{Mod})$. We have obtained two compatible and complete cotorsion pairs $(\mathrm{dg}\widetilde{\mathcal{P}_n} \cap \mathcal{E}, \mathrm{dg}\widetilde{(\mathcal{P}_n)^\perp})$ and $(\mathrm{dg}\widetilde{\mathcal{P}_n}, \mathrm{dg}\widetilde{(\mathcal{P}_n)^\perp} \cap \mathcal{E})$, so by Hovey's correspondence we obtain the n-projective model structure described in Theorem 3.2.2.

Another way to conclude that $(\mathrm{dg}\widetilde{\mathcal{P}_n}, \mathrm{dg}\widetilde{(\mathcal{P}_n)^\perp} \cap \mathcal{E})$ is complete is using (26, Proposition 3.8), where J. Gillespie shows that if $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair cogenerated by a set in a Grothendieck category $\mathcal{C}$, then so is $(\mathrm{dg}\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}})$. With respect to the class $\widetilde{\mathcal{A}}$, the author imposes an extra condition on the class $\mathcal{A}$ to prove that $(\widetilde{\mathcal{A}}, \mathrm{dg}\widetilde{\mathcal{B}})$ is also a cotorsion pair cogenerated by a set (See (26, Proposition 4.8 and 4.11)). One of the conditions needed is that $\mathcal{A}$ is closed under direct limits. However, concerning the particular case $\mathcal{F} = \mathcal{P}_n$, it is not true in general that $\mathcal{P}_n$ is closed under direct limits. This condition seems to be very related to the ring R. For example, H. Krause proved in (42, Lemma 5) that if R is a two-sided artinian ring, then $\mathcal{P}_n$ is closed under direct limits. Hence, in this case we can use Gillespie's results get the n-projective model structure on $\mathbf{Ch}({}_R\mathbf{Mod})$.

3.3 Degreewise n-projective model structures

We study the construction of another model structure on $\mathbf{Ch}({}_R\mathbf{Mod})$ from n-projective objects in ${}_R\mathbf{Mod}$. In (12), the authors proved the existence of the following model structure on $\mathbf{Ch}({}_R\mathbf{Mod})$.

Theorem 3.3.1 (**Degreewise projective model structure**)
There exists a unique Abelian model structure on $\mathbf{Ch}({}_R\mathbf{Mod})$ such that the (trivial) cofibrations are the monomorphisms with cokernel in $\mathrm{dw}\widetilde{\mathcal{P}_0}$ (resp. $\mathrm{ex}\widetilde{\mathcal{P}_0}$), the (trivial) fibrations are the epimorphisms with kernel in $(\mathrm{ex}\widetilde{\mathcal{P}_0})^{\perp}$ (resp. $(\mathrm{dw}\widetilde{\mathcal{P}_0})^{\perp}$), and the weak equivalences are the quasi-isomorphisms.

Our goal in this section is to show that the classes $\mathrm{dw}\widetilde{\mathcal{P}_n}$ and $\mathrm{ex}\widetilde{\mathcal{P}_n} = \mathrm{dw}\widetilde{\mathcal{P}_n} \cap \mathcal{E}$ of degreewise and exact degreewise n-projective chain complexes, respectively, form two compatible complete cotorsion pairs $(\mathrm{dw}\widetilde{\mathcal{P}_n}, (\mathrm{dw}\widetilde{\mathcal{P}_n})^{\perp})$ and $(\mathrm{ex}\widetilde{\mathcal{P}_n}, (\mathrm{ex}\widetilde{\mathcal{P}_n})^{\perp})$. A consequence of this result is the existence of the following new Abelian model structure on $\mathbf{Ch}({}_R\mathbf{Mod})$.

Theorem 3.3.2 (**Degreewise n-projective model structure**)
There exists a unique Abelian model structure on $\mathbf{Ch}({}_R\mathbf{Mod})$ where the (trivial) cofibrations are the monomorphisms with cokernel in $\mathrm{dw}\widetilde{\mathcal{P}_n}$ (resp. $\mathrm{ex}\widetilde{\mathcal{P}_n}$), the (trivial) fibrations are the epimorphisms with kernel in $(\mathrm{ex}\widetilde{\mathcal{P}_n})^{\perp}$ (resp. $(\mathrm{dw}\widetilde{\mathcal{P}_n})^{\perp}$), and the weak equivalences are the quasi-isomorphisms.

The completeness of the cotorsion pairs $(\mathrm{dw}\widetilde{\mathcal{P}_0}, (\mathrm{dw}\widetilde{\mathcal{P}_0})^{\perp})$ and $(\mathrm{ex}\widetilde{\mathcal{P}_0}, (\mathrm{ex}\widetilde{\mathcal{P}_0})^{\perp})$ is based on a theorem by I. Kaplansky, namely:

Theorem 3.3.3 (Kaplansky's Theorem. See (41))

Every projective module is a direct sum of countably generated projective modules.

So when one thinks of the completeness of the cotorsion pairs $(\mathrm{dw}\widetilde{\mathcal{P}_n}, (\mathrm{dw}\widetilde{\mathcal{P}_n})^{\perp})$ and $(\mathrm{ex}\widetilde{\mathcal{P}_n}, (\mathrm{ex}\widetilde{\mathcal{P}_n})^{\perp})$, a good question would be if it is possible to generalize Kaplansky's Theorem for n-projective modules. Let $M \in \mathcal{P}_n$ be an n-projective module along with a projective resolution of length n, say

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0.$$

We shall denote by $(\mathcal{P}_n)^{\aleph_0}$ the set of all modules M having a projective resolution as the previous one in which each P_i is countably generated.

By Kaplansky's Theorem we can write $P_k = \bigoplus_{i \in I_k} P^i_k$, where P^i_k is a countably generated projective module, for every $i \in I_k$ and every $0 \leq k \leq n$. Then we can rewrite the previous resolution as

$$0 \to \bigoplus_{i \in I_n} P^i_n \to \cdots \to \bigoplus_{i \in I_1} P^i_1 \to \bigoplus_{i \in I_0} P^i_0 \to M \to 0.$$

From now on we shall write any projective resolution of length n by using such direct sum decompositions. Notice that every module M having a finite projective resolution as the previous one belongs to $(\mathcal{P}_n)^{\aleph_0}$ if I_k is a countable set for every $0 \leq k \leq n$.

The fact that $(\mathrm{dw}\widetilde{\mathcal{P}_n}, (\mathrm{dw}\widetilde{\mathcal{P}_n})^{\perp})$ is a cotorsion pair in $\mathbf{Ch}({}_R\mathbf{Mod})$ is a consequence of Theorem 2.4.6. We shall prove that $(\mathrm{dw}\widetilde{\mathcal{P}_n}, (\mathrm{dw}\widetilde{\mathcal{P}_n})^{\perp})$ is cogenerated by the set $\mathrm{dw}\widetilde{(\mathcal{P}_n)^{\aleph_0}}$, by constructing a transfinite extension of $\mathrm{dw}\widetilde{(\mathcal{P}_n)^{\aleph_0}}$ for every degreewise n-projective complex. Then the completeness of $(\mathrm{dw}\widetilde{\mathcal{P}_n}, (\mathrm{dw}\widetilde{\mathcal{P}_n})^{\perp})$ shall be a consequence of Proposition 3.1.14. We need the following generalization of Kaplansky's Theorem:

Lemma 3.3.4 (Kaplansky's Theorem fon n-projective modules)
Let R be a Noetherian ring. Let $M \in \mathcal{P}_n$ and N be a countably generated submodule of M. Then there exists a transfinite extension of $(\mathcal{P}_n)^{\aleph_0}$ for M, say $(M_\alpha : \alpha < \lambda)$ with $\lambda > 1$, such that $M_1 \in (\mathcal{P}_n)^{\aleph_0}$ and $N \subseteq M_1$.

Proof.

Let $M \in \mathcal{P}_n$ and

$$0 \to \bigoplus_{i \in I_n} P_n^i \to \cdots \to \bigoplus_{i \in I_1} P_1^i \xrightarrow{f_1} \bigoplus_{i \in I_0} P_0^i \xrightarrow{f_0} M \to 0$$

be a projective resolution of M. We use transfinite induction. For $\alpha = 0$ set $M_0 = 0$. Now we construct M_1. Let $\mathcal{G}$ be a countable set of generators of N. Since f_0 is surjective, for every $g \in \mathcal{G}$ we can choose $y_g \in \bigoplus_{i \in I_0} P_0^i$ such that $g = f_0(y_g)$. Consider the set $Y = \{y_g : g \in \mathcal{G}\}$. Since Y is a countable subset of $\bigoplus_{i \in I_0} P_0^i$, we have that $\langle Y \rangle$ is a countably generated submodule of P_0. Choose a countable subset $I_0^{1,0} \subseteq I_0$ such that $\langle Y \rangle \subseteq \bigoplus_{i \in I_0^{1,0}} P_0^i$. Then $f_0(\langle Y \rangle) \subseteq N$. Consider $\mathrm{Ker}(f_0|_{\bigoplus_{i \in I_0^{1,0}} P_0^i})$. Since $\bigoplus_{i \in I_0^{1,0}} P_0^i$ is countably generated and $\mathrm{Ker}(f_0|_{\bigoplus_{i \in I_0^{1,0}} P_0^i})$ is a submodule of $\bigoplus_{i \in I_0^{1,0}} P_0^i$, we have that $\mathrm{Ker}(f_0|_{\bigoplus_{i \in I_0^{1,0}} P_0^i})$ is also countably generated, since R is Noetherian. Let $\mathcal{B}$ be a countable set of generators of $\mathrm{Ker}(f_0|_{\bigoplus_{i \in I_0^{1,0}} P_0^i})$. Let $b \in \mathcal{B}$, then $f(b) = 0$ and by exactness of the above sequence there exists $y_b \in \bigoplus_{i \in I_1} P_1^i$ such that $b = f_1(y_b)$. Let $Y' = \{y_b : b \in \mathcal{B}\}$. Note that Y' is a countable subset of $(f_1)^{-1}(\mathrm{Ker}(f_0|_{\bigoplus_{i \in I_0^{1,0}} P_0^i}))$. Then $\langle Y' \rangle$ is a countably generated submodule of $\bigoplus_{i \in I_1} P_1^i$. Hence there exists a countable subset $I_1^{1,0} \subseteq I_1$ such that $\bigoplus_{i \in I_1^{1,0}} P_1^i \supseteq \langle Y' \rangle$. Thus $f_1(\bigoplus_{i \in I_1^{1,0}} P_1^i) \supseteq f_1(\langle Y' \rangle)$. Now let $z \in \mathrm{Ker}(f_0|_{\bigoplus_{i \in I_0^{1,0}} P_0^i})$. Then $z = r_1 b_1 + \cdots + r_m b_m$, where each $b_j \in \mathcal{B}$. Since $b_j = f_1(y_{b_j})$ with $y_{b_j} \in Y'$, we get $z = f_1(r_1 y_{b_1} + \cdots + r_m y_{b_m}) \in f_1(\langle Y' \rangle)$. Hence, $\mathrm{Ker}(f_0|_{\bigoplus_{i \in I_0^{1,0}} P_0^i}) \subseteq f_1(\langle Y' \rangle) \subseteq f_1(\bigoplus_{i \in I_1^{1,0}} P_1^i)$. Use the same argument to find a countable subset $I_2^{1,0} \subseteq I_2$ such that $f_2(\bigoplus_{i \in I_2^{1,0}} P_2^i) \supseteq \mathrm{Ker}(f_1|_{\bigoplus_{i \in I_1^{1,0}} P_1^i})$. Repeat the same argument until find a countable subset $I_n^{1,0} \subseteq$

I_n such that $f_n(\bigoplus_{i\in I_n^{1,0}} P_n^i) \supseteq \mathrm{Ker}(f_{n-1}|_{\bigoplus_{i\in I_{n-1}^{1,0}} P_{n-1}^i})$. Now, $f_n(\bigoplus_{i\in I_n^{1,0}} P_n^i)$ is a countably generated submodule of $\bigoplus_{i\in I_{n-1}} P_{n-1}^i$. Then choose a countable subset $I_{n-1}^{1,0} \subseteq I_{n-1}^{1,1} \subseteq I_{n-1}$ such that $f_n(\bigoplus_{i\in I_n^{1,0}} P_n^i) \subseteq \bigoplus_{i\in I_{n-1}^{1,1}} P_{n-1}^i$. Repeat this process until find a countable subset $I_0^{1,0} \subseteq I_0^{1,1} \subseteq I_0$ satisfying $f_1(\bigoplus_{i\in I_1^{1,1}} P_1^i) \subseteq \bigoplus_{i\in I_0^{1,1}} P_0^i$. Now choose a countable subset $I_1^{1,1} \subseteq I_1^{1,2} \subseteq I_1$ such that $f_1(\bigoplus_{i\in I_2^{1,2}} P_1^i) \supseteq \mathrm{Ker}(f_0|_{\bigoplus_{i\in I_0^{1,1}} P_0^i})$. What we have been doing so far is called the zig-zag procedure. Keep repeating this procedure infinitely many times, and set $I_k^1 = \bigcup_{m\geq 0} I_k^{1,m}$, for every $0 \leq k \leq n$. By construction, we get the following exact sequence

$$0 \to \bigoplus_{i\in I_n^1} P_n^i \to \cdots \to \bigoplus_{i\in I_1^1} P_1^i \to \bigoplus_{i\in I_0^1} P_0^i \to M_1 \to 0$$

where $x \in M_1 := \mathrm{CoKer}(\bigoplus_{i\in I_1^1} P_1^i \to \bigoplus_{i\in I_0^1} P_0^i) \subseteq M$ and $N \subseteq M_1$. We take the quotient of the resolution of M by the resolution of M', and get the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \bigoplus_{i\in I_n^1} P_n^i \arrow[r] \arrow[d] & \cdots \arrow[r] & \bigoplus_{i\in I_1^1} P_1^i \arrow[r] \arrow[d] & \bigoplus_{i\in I_0^1} P_0^i \arrow[r] \arrow[d] & M_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bigoplus_{i\in I_n} P_n^i \arrow[r] \arrow[d] & \cdots \arrow[r] & \bigoplus_{i\in I_1} P_1^i \arrow[r] \arrow[d] & \bigoplus_{i\in I_0} P_0^i \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bigoplus_{i\in I_n - I_n^1} P_n^i \arrow[r] \arrow[d] & \cdots \arrow[r] & \bigoplus_{i\in I_1 - I_1^1} P_1^i \arrow[r] \arrow[d] & \bigoplus_{i\in I_0 - I_0^1} P_0^i \arrow[r] \arrow[d] & M/M_1 \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 & 0 & 0 &
\end{tikzcd}$$

where the third row is an exact sequence since the class of exact complexes is thick. Then we have a projective resolution of length n for M/M_1. Repeat the same procedure above for M/M_1, by choosing the class $x^1 + M_1 \in M/M_1 - \{0 + M_1\}$, in order to get an exact sequence

$$0 \to \bigoplus_{i \in I_n^2 - I_n^1} P_n^i \to \cdots \to \bigoplus_{i \in I_1^2 - I_1^1} P_1^i \to \bigoplus_{i \in I_0^2 - I_0^1} P_0^i \to M_2/M_1 \to 0,$$

for some $M_1 \subseteq M_2 \subseteq M$, with $I_k^2 - I_k^1$ is countable for every $0 \leq k \leq n$. Note that we have a projective resolution

$$0 \to \bigoplus_{i \in I_n^2} P_n^i \to \cdots \to \bigoplus_{i \in I_1^2} P_1^i \to \bigoplus_{i \in I_0^2} P_0^i \to M_2 \to 0$$

of M_2, since we have a commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \bigoplus_{i \in I_n^1} P_n^i \arrow[r] \arrow[d] & \cdots \arrow[r] & \bigoplus_{i \in I_1^1} P_1^i \arrow[r] \arrow[d] & \bigoplus_{i \in I_0^1} P_0^i \arrow[r] \arrow[d] & M_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bigoplus_{i \in I_n^2} P_n^i \arrow[r] \arrow[d] & \cdots \arrow[r] & \bigoplus_{i \in I_1^2} P_1^i \arrow[r] \arrow[d] & \bigoplus_{i \in I_0^2} P_0^i \arrow[r] \arrow[d] & M_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bigoplus_{i \in I_n^2 - I_n^1} P_n^i \arrow[r] \arrow[d] & \cdots \arrow[r] & \bigoplus_{i \in I_1^2 - I_1^1} P_1^i \arrow[r] \arrow[d] & \bigoplus_{i \in I_0^2 - I_0^1} P_0^i \arrow[r] \arrow[d] & M_2/M_1 \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 & 0 & 0 &
\end{tikzcd}$$

where the first and third rows are exact sequences, and then so is the second since the class of exact complexes is closed under extensions. We have that M_1 and M_2 are n-projective modules such that $M_1, M_2/M_1 \in (\mathcal{P}_n)^{\aleph_0}$. Now suppose that there is an ordinal β such that:

(1) M_α is an n-projective module, for every $\alpha < \beta$.

(2) $M_\alpha \subseteq M_{\alpha'}$ whenever $\alpha \leq \alpha' < \beta$.

(3) $M_{\alpha+1}/M_\alpha \in (\mathcal{P}_n)^{\aleph_0}$ whenever $\alpha + 1 < \beta$.

(4) $M_\gamma = \bigcup_{\alpha < \gamma} M_\alpha$ for every limit ordinal $\gamma < \beta$.

If β is a limit ordinal, then set $M_\beta = \bigcup_{\alpha < \beta} M_\alpha$. Otherwise there exists an ordinal

$\alpha < \beta$ such that $\beta = \alpha + 1$. In this case, construct $M_{\alpha+1} \in \mathcal{P}_n$ from M_α as we constructed M_2 from M_1, such that $M_{\alpha+1}/M_\alpha \in (\mathcal{P}_n)^{\aleph_0}$. By the Principle of Transfinite Induction, the result follows. $\square$

Theorem 3.3.5 (Generalization of (12, Theorem 4.4))

If R is a Noetherian ring, every n-projective chain complex is a transfinite extension of $\mathrm{dw}\widetilde{(\mathcal{P}_n)^{\aleph_0}}$.

Proof.

Let $X \in \mathrm{dw}\widetilde{\mathcal{P}_n}$ and write

$$X = \cdots \to X_{k+1} \xrightarrow{\partial_{k+1}} X_k \xrightarrow{\partial_k} X_{k-1} \to \cdots .$$

For each k one has a projective resolution of

$$0 \to \bigoplus_{i \in I_n(k)} P_n^i(k) \to \cdots \to \bigoplus_{i \in I_1(k)} P_1^i(k) \to \bigoplus_{i \in I_0(k)} P_0^i(k) \to X_k \to 0$$

of length k. We shall construct a transfinite extension of $\mathrm{dw}\widetilde{(\mathcal{P}_n)^{\aleph_0}}$ for X by using transfinite induction. For $\alpha = 0$ set $X^0 = 0$. For $\alpha = 1$, choose $m \in \mathbb{Z}$. Let S be a countably generated submodule of X_m. By the previous lemma, there exists a submodule $(\mathcal{P}_n)^{\aleph_0} \ni X_m^1 \subseteq X_m$ such that $S \subseteq X_m^1$. Note that X_m^1 is also countably generated. Then $\partial_m(X_m^1)$ is a countably generated submodule of X_{m-1}, and so there exists $(\mathcal{P}_n)^{\aleph_0} \ni X_{m-1}^1 \subseteq X_{m-1}$ such that $\partial_m(X_m^1) \subseteq X_{m-1}^1$. Repeat the same procedure infinitely many times in order to obtain a subcomplex

$$X^1 := \cdots \to X_{k+1}^1 \to X_k^1 \to X_{k-1}^1 \to \cdots$$

of X such that $X_k^1 \in (\mathcal{P}_n)^{\aleph_0}$ for every $k \in \mathbb{Z}$ (we are setting $X_k^1 = 0$ for every $k > m$). Hence $X^1 \in \mathrm{dw}\widetilde{(\mathcal{P}_n)^{\aleph_0}}$. Note from the proof of the previous lemma that the quotient X/X^1 is in dw$\widetilde{\mathcal{P}_n}$. We have

$$X/X^1 = \cdots \to X_{k+1}/X^1_{k+1} \to X_k/X^1_k \to X_{k-1}/X^1_{k-1} \to \cdots ,$$

where for every $k \leq m$ one has the following projective resolutions of length n for X^1_k and X_k/X^1_k:

$$0 \to \bigoplus_{i \in I^1_n(k)} P^i_n(k) \to \cdots \to \bigoplus_{i \in I^1_1(k)} P^i_1(k) \to \bigoplus_{i \in I^1_0(k)} P^i_0(k) \to X^1_k \to 0,$$

$$0 \to \bigoplus_{i \in I_n(k) - I^1_n(k)} P^i_n(k) \to \cdots \to \bigoplus_{i \in I_0(k) - I^1_0(k)} P^i_0(k) \to X_k/X^1_k \to 0.$$

Apply the same procedure above to the complex X/X^1, in order to obtain a chain subcomplex

$$X^2/X^1 = \cdots \to X^2_{k+1}/X^1_{k+1} \to X^2_k/X^1_k \to X^2_{k-1}/X^1_{k-1} \to \cdots$$

of X/X^1, such that for each $k \in \mathbb{Z}$ one has a projective resolution

$$0 \to \bigoplus_{i \in I^2_n - I^1_n} P^i_n(k) \to \cdots \to \bigoplus_{i \in I^2_0 - I^1_0} P^i_0(k) \to X^2_k/X^1_k \to 0$$

of length n for X^2_k/X^1_k, where each $I^2_j - I^1_j \subseteq I_j$ is countable. Now consider the chain complex

$$X^2 := \cdots \to X^2_{k+1} \to X^2_k \to X^2_{k-1} \to \cdots .$$

As we did in the proof of the previous lemma, we have that

$$0 \to \bigoplus_{i \in I^2_n(k)} P^i_n(k) \to \cdots \to \bigoplus_{i \in I^2_1(k)} P^i_1(k) \to \bigoplus_{i \in I^2_0(k)} P^i_0(k) \to X^2_k \to 0$$

is an exact sequence. So $X^2_k \in \mathcal{P}_n$ for every $k \in \mathbb{Z}$, and hence $X^2 \in$ dw$\widetilde{\mathcal{P}_n}$, with $X^2/X^1 \in$ dw$\widetilde{(\mathcal{P}_n)^{\aleph_0}}$. The rest of the proof follows by transfinite induction, as in the proof of the previous lemma. $\square$

Consider the class of exact degreewise n-projective complexes $\text{ex}\widetilde{\mathcal{P}_n} = \text{dw}\widetilde{\mathcal{P}_n} \cap \mathcal{E}$. By Theorem 2.4.7 we know $(\text{ex}\widetilde{\mathcal{P}_n}, (\text{ex}\widetilde{\mathcal{P}_n})^{\perp})$ is a cotorsion pair. We shall prove its completeness by constructing transfinite extension for $\text{ex}\widetilde{\mathcal{P}_n}$ of some subset of $\text{ex}\widetilde{\mathcal{P}_n}$.

Definition 3.3.1. Given a module $M \in \mathcal{P}_n$, consider a projective resolution

$$(*) = \left(0 \to \bigoplus_{i \in I_n} P_n^i \to \cdots \to \bigoplus_{i \in I_1} P_1^i \to \bigoplus_{i \in I_0} P_0^i \to M \to 0\right),$$

where each P_k^i is a countably generated projective module. We shall say that a projective resolution

$$(**) = \left(0 \to \bigoplus_{i \in I'_n} P_n^i \to \cdots \to \bigoplus_{i \in I'_1} P_1^i \to \bigoplus_{i \in I'_0} P_0^i \to N \to 0\right)$$

is a <u>nice subresolution</u> of $(*)$ if $I'_k \subseteq I_k$ for every $0 \leq k \leq n$ and $N \subseteq M$.

Let κ be an infinite cardinal with $\kappa \geq \text{Card}(R)$. We shall say that the resolution $(*)$ above is <u>κ-small</u> if $\text{Card}(I_k) \leq \kappa$ for every $0 \leq k \leq n$. Moreover, we say that $(**)$ is a <u>nice κ-small subresolution</u> of $(*)$ if each I'_k is a κ-small subset of I_k. We shall denote by $\mathcal{P}_n(\kappa)$ the set of n-projective modules with a κ-small projective resolution. Note that $\mathcal{P}_n(\kappa) \subseteq (\mathcal{P}_n)^{\leq \kappa}$.

We shall prove that every exact degreewise n-projective complex a transfinite extension of $\text{ex}\widetilde{\mathcal{P}_n(\kappa)} \subseteq (\text{ex}\widetilde{\mathcal{P}_n})^{\leq \kappa}$.

Lemma 3.3.6

Let $M \in \mathcal{P}_n$ with a projective resolution given by $(*)$. For every submodule $N \subseteq M$ with $\mathrm{Card}(N) \leq \kappa$, there exists a nice κ-small subresolution

$$(***) = \left(0 \to \bigoplus_{i\in I'_n} P^i_n \to \cdots \to \bigoplus_{i\in I'_1} P^i_1 \to \bigoplus_{i\in I'_0} P^i_0 \to N' \to 0\right)$$

of $(*)$ such that $N \subseteq N'$. Moreover, if N has an nice κ-small subresolution of M given by $(**)$, then $(***)$ can be constructed in such a way that $(**)$ is a nice subresolution of $(***)$.

$\boxed{\textbf{Proof}}$.

Since f_0 is surjective, for every $x \in N$ choose $y_x \in \bigoplus_{i\in I_0} P^i_0$ with $x = f_0(y_x)$. Let $Y = \{y_x : x \in N\}$. Note that $\langle Y\rangle$ is a κ-small submodule of $\bigoplus_{i\in I_0} P^i_0$. So there exists a κ-small subset $I^0_0 \subseteq I_0$ such that $\langle Y\rangle \subseteq \bigoplus_{i\in I^0_0} P^i_0$. We have $f_0(\bigoplus_{i\in I^0_0} P^i_0) \supseteq N$. Now consider the submodule $\mathrm{Ker}(f_0|_{\bigoplus_{i\in I^0_0}} P^i_0)$ of $f_0(\bigoplus_{i\in I^0_0} P^i_0)$, which is κ-small since $f_0(\bigoplus_{i\in I^0_0} P^i_0)$ is. Then we can choose a κ-small subset $I^0_1 \subseteq I_1$ such that $f_1(\bigoplus_{i\in I^0_1} P^i_1) \supseteq \mathrm{Ker}(f_0|_{\bigoplus_{i\in I^0_0} P^i_0})$. Repeat the same argument until find a κ-small subset $I^0_n \subseteq I_n$ such that $f_n(\bigoplus_{i\in I^0_n} P^i_n) \supseteq \mathrm{Ker}(f_{n-1}|_{\bigoplus_{i\in I^0_{n-1}} P^i_{n-1}})$. Since $f_n(\bigoplus_{i\in I^0_n} P^i_n)$ is a κ-small submodule of $\bigoplus_{i\in I_{n-1}} P^i_{n-1}$, we can choose a κ-small subset $I^0_{n-1} \subseteq I^1_{n-1} \subseteq I_{n-1}$ such that $f_n(\bigoplus_{i\in I^0_n} P^i_n) \subseteq \bigoplus_{i\in I^1_{n-1}} P^i_{n-1}$. From this point just use the zig-zag procedure to get κ-small subsets $I'_k = \bigcup_{j\geq 0} I^j_k \subseteq I_k$ and an exact sequence

$$0 \to \bigoplus_{i\in I'_n} P^i_n \to \cdots \to \bigoplus_{i\in I'_1} P^i_1 \to \bigoplus_{i\in I'_0} P^i_0 \to N' \to 0$$

where $N' := \mathrm{CoKer}(\bigoplus_{i\in I'_1} P^i_1 \to \bigoplus_{i\in I'_0} P^i_0)$ and $N \subseteq N' \subseteq M$.

Now suppose

$$0 \to \bigoplus_{i \in I'^N_n} P^i_n \to \cdots \to \bigoplus_{i \in I^N_1} P^i_1 \to \bigoplus_{i \in I^N_0} P^i_0 \to N \to 0$$

is a nice κ-small subresolution of $(*)$. Take the quotient of $(*)$ by this resolution and get the following resolution of M/N:

$$0 \to \bigoplus_{i \in I_n - I^N_n} P^i_n \to \cdots \to \bigoplus_{i \in I_1 - I^N_1} P^i_1 \to \bigoplus_{i \in I_0 - I^N_0} P^i_0 \to \frac{M}{N} \to 0.$$

Repeat the argument above using this sequence and the κ-small submodule $\langle z + N \rangle$, where $z \notin N$. Then we get a projective subresolution

$$0 \to \bigoplus_{i \in I'_n - I^N_n} P^i_n \to \cdots \to \bigoplus_{i \in I'_1 - I^N_1} P^i_1 \to \bigoplus_{i \in I'_0 - I^N_0} P^i_0 \to \frac{N'}{N} \to 0$$

of the previous one, where each set $I'_k - I^N_k$ is a κ-small set. As we did in the proof of Lemma 3.3.4, we have that

$$(***) = \left(0 \to \bigoplus_{i \in I'_n} P^i_n \to \cdots \to \bigoplus_{i \in I'_1} P^i_1 \to \bigoplus_{i \in I'_0} P^i_0 \to N' \to 0 \right)$$

is the desired nice κ-small subresolution of $(*)$. □

Lemma 3.3.7

Let $X \in \mathrm{dw}\widetilde{\mathcal{P}_n}$ and Y be a bounded above κ-small subcomplex of X. Then there exists a (bounded above) subcomplex Y' of X such that $Y \subseteq Y'$ and $Y' \in \mathrm{dw}\widetilde{\mathcal{P}_n(\kappa)}$.

Proof.

We are given the following commutative diagram

$$\begin{array}{ccccccccc}
Y = & \cdots & \longrightarrow & 0 & \longrightarrow & Y_m & \xrightarrow{\partial_m} & Y_{m-1} & \longrightarrow \cdots \\
 & & & \downarrow & & \downarrow & & \downarrow & \\
X = & \cdots & \longrightarrow & X_{m+1} & \xrightarrow{\partial_{m+1}} & X_m & \xrightarrow{\partial_m} & X_{m-1} & \longrightarrow \cdots
\end{array}$$

Since X_m is an n-projective module, we can consider a projective resolution for each m of the form

$$0 \to \bigoplus_{i \in I_n(m)} P_n^i(m) \to \cdots \to \bigoplus_{i \in I_1(m)} P_1^i(m) \to \bigoplus_{i \in I_0(m)} P_0^i(m) \to X_m \to 0.$$

By the previous lemma, there exists a submodule Y'_m of X_m containing Y_m, along with a nice κ-small subresolution

$$0 \to \bigoplus_{i \in I'_n(m)} P_n^i(m) \to \cdots \to \bigoplus_{i \in I'_1(m)} P_1^i(m) \to \bigoplus_{i \in I'_0(m)} P_0^i(m) \to Y'_m \to 0.$$

Note that $\mathrm{Card}(\partial_m(Y'_m) + Y_{m-1}) \leq \kappa$ and $Y_{m-1} \subseteq \partial_m(Y'_m) + Y_{m-1} \subseteq X_{m-1}$. Now choose a submodule $Y'_{m-1} \subseteq X_{m-1}$ such that $\partial_m(Y'_m) + Y_{m-1} \subseteq Y'_{m-1}$ and Y'_{m-1} has a nice κ-small subresolution of a fixed resolution of X_{m-1}. Repeat this process infinitely many times to obtain a complex $Y' := \cdots \to 0 \to Y'_m \to Y'_{m-1} \to \cdots$ such that $Y \subseteq Y' \subseteq X$ and $Y' \in \mathrm{dw}\widetilde{\mathcal{P}_n(\kappa)}$. □

In (12, Theorem 4.6), the authors proved that every exact degreewise projective chain complex is a transfinite extension of $(\mathrm{ex}\widetilde{\mathcal{P}_0})^{\leq \kappa}$. The previous lemmas allow us to prove that this is also valid for every projective dimension.

Theorem 3.3.8 (Generalization of (12, Theorem 4.6))

Every exact n-projective chain complex is a transfinite extension of $\mathrm{ex}\widetilde{\mathcal{P}_n(\kappa)}$.

Proof.

Let $X \in \mathrm{ex}\widetilde{\mathcal{P}_n}$. We construct a transfinite extension of $\mathrm{ex}\widetilde{\mathcal{P}_n(\kappa)}$ for X using transfinite induction. For $\alpha = 0$ set $X^0 = 0$. For the case $\alpha = 1$ let $m \in \mathbb{Z}$ be arbitrary and let $T_1 \subseteq X_m$ be a κ-small submodule of X_m. Then there exists a κ-small submodule Y_m^1 of X_m such that $T_1 \subseteq Y_m^1$ and that Y_m^1 has a nice κ-small projective subresolution of a given resolution of X_m. Note that $\partial_m(Y_m^1)$ is a submodule of

X_{m-1} with cardinality $\leq \kappa$, so there exists a submodule Y^1_{m-1} of X_{m-1} such that $\partial_m(Y^1_m) \subseteq Y^1_{m-1}$ and that Y^1_{m-1} has a nice κ-small projective subresolution of a given resolution of X_{m-1}. Keep repeating this argument infinitely many times. We obtain a chain complex Y^1 of the form $\cdots \to 0 \to Y^1_m \to Y^1_{m-1} \to \cdots$ which is a subcomplex of X and $Y^1 \in \mathrm{dw}\widetilde{\mathcal{P}_n(\kappa)}$. Note that Y^1 is not necessarily exact. We shall construct a complex X^1 from Y^1 such that $X^1 \subseteq X$ and $X^1 \in \mathrm{ex}\widetilde{\mathcal{P}_n(\kappa)}$. The rest of this proof uses an argument similar to the one used in (12, Theorem 4.6). Fix any $p \in \mathbb{Z}$. Then $\mathrm{Card}(Y^1_p) \leq \kappa$ and so $\mathrm{Card}(Z_p(Y^1)) \leq \kappa$. Since X is exact and $\mathrm{Card}(Z_p(Y^1)) \leq \kappa$, there exists a submodule $U \subseteq X_{p+1}$ with $\mathrm{Card}(U) \leq \kappa$ such that $Z_p(Y^1) \subseteq \partial_{p+1}(U)$. Let C^1 be a κ-small subcomplex of X such that $U \subseteq C_{p+1}$, $C_j = 0$ for every $j > p+1$, and that each C_j with $j \leq p$ has a nice κ-small projective subresolution of a given resolution of X_j. Since $Y^1 + C$ is a bounded above subcomplex of X, by the previous lemma there exists a κ-small subcomplex Y^2 of X such that $Y^1 + C \subseteq Y^2$ and that each Y^2_j has a nice κ-small projective subresolution of a given resolution of X_j. Note that $Z_p(Y^1) \subseteq \partial_{p+1}(Y^2_{p+1})$. Construct Y^3 from Y^2 as we just constructed Y^2 from Y^1, and so on, making sure to use the same $p \in \mathbb{Z}$ at each step. Set $X^1 := \bigcup_{j=1}^{\infty} Y^j \subseteq X$. Note that X^1 is exact at p. Repeat this argument to get exactness at any level. So we may assume that X^1 is an exact complex. Every X^1_k has a nice κ-small projective subresolution of the given resolution of X_k. For every j one has a projective subresolution of the form

$$0 \to \bigoplus_{i \in I^j_n(k)} P^i_n(k) \to \cdots \to \bigoplus_{i \in I^j_1(k)} P^i_1(k) \to \bigoplus_{i \in I^j_0(k)} P^i_0(k) \to Y^j_k \to 0,$$

where $I^1_l(k) \subseteq I^2_l(k) \subseteq \cdots$ for every $0 \leq l \leq n$, by Lemma 3.3.6. If we take the union of all of the previous sequences, then $\bigcup_{j \geq 1} I^j_l(k) \subseteq I_l(k)$ for every $0 \leq l \leq n$, and so we obtain the exact sequence

$$0 \to \bigoplus_{i \in \bigcup_{j \geq i} I^j_n(k)} P^i_n(k) \to \cdots \to \bigoplus_{i \in \bigcup_{j \geq 1} I^j_0(k)} P^i_0(k) \to \bigcup_{j \geq 1} Y^j_k = X^1_j \to 0.$$

Hence, $X^1 \in \mathrm{ex}\widetilde{\mathcal{P}_n(\kappa)}$. Now consider the quotient

$$\frac{X}{X^1} = \cdots \to \frac{X_{k+1}}{X^1_{k+1}} \to \frac{X_k}{X^1_k} \to \frac{X_{k-1}}{X^1_{k-1}} \to \cdots .$$

Note that each X_k/X^1_k is n-projective and that X/X^1 is exact. We apply the same procedure above to the complex X/X^1 in order to get a complex $X^2/X^1 \subseteq X/X^1$ such that $X^2/X^1 \in \mathrm{ex}\widetilde{\mathcal{P}_n(\kappa)}$. Note that X^2 is an exact complex since the class of exact complexes is closed under extensions, and so $X^2 \in \mathrm{ex}\widetilde{\mathcal{P}_n}$. The rest of the proof follows by using transfinite induction. □

The previous theorem implies that $(\mathrm{ex}\widetilde{\mathcal{P}_n}, (\mathrm{ex}\widetilde{\mathcal{P}_n})^\perp)$ is a complete cotorsion pair, since it is cogenerated by $\mathrm{ex}\widetilde{\mathcal{P}_n(\kappa)}$. Note that for this result we did not need R to be Noetherian.

In the case where R is a Noetherian ring, we have two complete cotorsion pairs $(\mathrm{dw}\widetilde{\mathcal{P}_n}, (\mathrm{dw}\widetilde{\mathcal{P}_n})^\perp)$ and $(\mathrm{ex}\widetilde{\mathcal{P}_n}, (\mathrm{ex}\widetilde{\mathcal{P}_n})^\perp)$. By Proposition 2.4.8, $(\mathrm{ex}\widetilde{\mathcal{P}_n})^\perp \cap \mathcal{E} = (\mathrm{dw}\widetilde{\mathcal{P}_n})^\perp$, since $(\mathrm{dw}\widetilde{\mathcal{P}_n}, (\mathrm{dw}\widetilde{\mathcal{P}_n})^\perp)$ is complete. It follows these two complete cotorsion pairs are compatible, and thus we obtain a proof of Theorem 3.3.2 in the case where R is a Noetherian ring. We shall see later a proof where R is not necessarily Noetherian. Note that we could have used Proposition 2.3.6 and 2.4.8 to show that the pair $(\mathrm{ex}\widetilde{\mathcal{P}_n}, (\mathrm{ex}\widetilde{\mathcal{P}_n})^\perp)$ is complete. However, we consider of more interest to generalize the arguments given in (12) on the completeness of $(\mathrm{ex}\widetilde{\mathcal{P}_0}, (\mathrm{ex}\widetilde{\mathcal{P}_0})^\perp)$.

3.4 n-Injective and degreewise n-injective model structures

The objective in this section is to get the dual of the n-projective and degreewise n-projective model structures on $\mathbf{Ch}({}_R\mathbf{Mod})$. The results are summarized in the following two theorems.

Theorem 3.4.1 (**n-Injective model structure**)

There exists a unique Abelian model structure on $\mathbf{Ch}({}_R\mathbf{Mod})$, where the (trivial) fibrations are the epimorphisms with kernel in $\mathrm{dg}\widetilde{\mathcal{I}_n}$ (resp. $\widetilde{\mathcal{I}_n}$), the (trivial) cofibrations are the monomorphisms with cokernel in ${}^{\perp}(\widetilde{\mathcal{I}_n})$ (resp. ${}^{\perp}(\mathrm{dg}\widetilde{\mathcal{I}_n})$), and the weak equivalences are the quasi-isomorphisms.

Theorem 3.4.2 (**Degreewise n-injective model structure**)

There exists a unique Abelian model structure on $\mathbf{Ch}({}_R\mathbf{Mod})$, where the (trivial) fibrations are the epimorphisms with kernel in $\mathrm{dw}\widetilde{\mathcal{I}_n}$ (resp. $\mathrm{ex}\widetilde{\mathcal{I}_n}$), the (trivial) cofibrations are the monomorphisms with cokernel in ${}^{\perp}(\mathrm{ex}\widetilde{\mathcal{I}_n})$ (resp. ${}^{\perp}(\mathrm{dw}\widetilde{\mathcal{I}_n})$), and the weak equivalences are the quasi-isomorphisms.

In a Grothendieck category $\mathcal{C}$ with enough projective objects, Proposition 3.1.24 and Theorem 2.4.3 imply that $(\mathrm{dg}{}^{\perp}(\widetilde{\mathcal{I}_n(\mathcal{C})}), \widetilde{\mathcal{I}_n(\mathcal{C})})$ and $({}^{\perp}(\widetilde{\mathcal{I}_n(\mathcal{C})}), \mathrm{dg}\widetilde{\mathcal{I}_n(\mathcal{C})})$ are cotorsion pairs in $\mathbf{Ch}(\mathcal{C})$. These pairs are also compatible since $({}^{\perp}(\mathcal{I}_n(\mathcal{C})), \mathcal{I}_n(\mathcal{C}))$ is hereditary (see Theorem 2.4.5). By Proposition 1.8.4, we know $\widetilde{\mathcal{I}_n(\mathcal{C})} = \mathcal{I}_n(\mathbf{Ch}(\mathcal{C}))$. It follows $(\mathrm{dg}{}^{\perp}(\widetilde{\mathcal{I}_n(\mathcal{C})}), \widetilde{\mathcal{I}_n(\mathcal{C})})$ is complete. In the case $\mathcal{C} = {}_R\mathbf{Mod}$ or $\mathbf{Mod}_R$, we have two complete cotorsion pairs $(\mathcal{E}, \mathcal{E}^{\perp})$ and $({}^{\perp}\mathcal{E}, \mathcal{E})$. For simplicity, write $\mathcal{I}_n({}_R\mathbf{Mod}) = \mathcal{I}_n$. Hence, by Proposition 2.3.6 **(2)**, $(\widetilde{{}^{\perp}(\mathcal{I}_n)}, \mathrm{dg}\widetilde{\mathcal{I}_n})$ is a complete cotorsion pair. Then Theorem 3.4.1 follows by Hovey's correspondence. Theorem 3.4.2 follows in a similar way, we only need to prove the following proposition.

Proposition 3.4.3

Let $\mathcal{C}$ be a Grothedieck category with a generator G. Then the cotorsion pair $({}^{\perp}(\mathrm{dw}\widetilde{\mathcal{I}_n(\mathcal{C})}), \mathrm{dw}\widetilde{\mathcal{I}_n}(\mathcal{C}))$ is cogenerated by a set.

$\boxed{\textbf{Proof}}$.

We show that $\mathrm{dw}\widetilde{\mathcal{I}_n(\mathcal{C})} = \mathcal{S}^\perp$, where $\mathcal{S}$ is the set of all disk complexes $D^m(C)$ with $C \in \Omega^n(G/J)$ and J running over the set of subobjects of G. First, we see that $\mathcal{S} \subseteq {}^\perp(\mathrm{dw}\widetilde{\mathcal{I}_n(\mathcal{C})})$. Let $D^m(C) \in \mathcal{S}$ and $Y \in \mathrm{dw}\widetilde{\mathcal{I}_n(\mathcal{C})}$. Using Proposition 1.6.2, we have $\mathrm{Ext}^1_{\mathbf{Ch}(\mathcal{C})}(D^m(C), Y) \cong \mathrm{Ext}^1_{\mathcal{C}}(C, Y_m) \cong \mathrm{Ext}^{n+1}_{\mathcal{C}}(G/J, Y_m) = 0$, since Y_m is n-injective. Then, $D^m(C) \in {}^\perp(\mathrm{dw}\widetilde{\mathcal{I}_n(\mathcal{C})})$. This implies $\mathrm{dw}\widetilde{\mathcal{I}_n(\mathcal{C})} = ({}^\perp(\mathrm{dw}\widetilde{\mathcal{I}_n(\mathcal{C})}))^\perp \subseteq \mathcal{S}^\perp$. The other inclusion follows in the same way. $\square$

There is not much to say regarding these two model structures. As the reader can appreciate, the proof of the first theorem is just a consequence of more general results, while the arguments given for the previous proposition are mainly applications of basic homological algebra. Actually, the existence of the degreewise n-injective model structure is a consequence of a more general result proven by J. Gillespie in (25, Theorem 4.7). We decided to include this section due to some applications in the next chapter.

3.5 Degreewise n-flat model structures

So far in this chapter we have obtain Abelian model structures on $\mathbf{Ch}({}_R\mathbf{Mod})$ from projective and injective dimensions in ${}_R\mathbf{Mod}$. So, it is only left to analyze the flat case. This part of this thesis is motivated by early investigations due to S. T. Aldrich, E. E. Enochs, J. R. García Rozas and L. Oyonarte, on the completeness of the cotorsion pairs $(\mathrm{dw}\widetilde{\mathcal{F}_0}, (\mathrm{dw}\widetilde{\mathcal{F}_0})^\perp)$ and $(\mathrm{ex}\widetilde{\mathcal{F}_0}, (\mathrm{ex}\widetilde{\mathcal{F}_0})^\perp)$, where $\mathrm{dw}\widetilde{\mathcal{F}_0}$ and $\mathrm{ex}\widetilde{\mathcal{F}_0}$ are the classes of degreewise flat and exact degreewise flat chain complexes, respectively. Since these two cotorsion pairs turn out to be compatible and complete, we have the following model structure on $\mathbf{Ch}({}_R\mathbf{Mod})$.

Theorem 3.5.1 (**Degreewise flat model structure**)

There exists a unique Abelian model structure on $\mathbf{Ch}({}_R\mathbf{Mod})$ where the (trivial) cofibrations are the monomorphisms with cokernel in $\mathrm{dw}\widetilde{\mathcal{F}_0}$ (resp. $\mathrm{ex}\widetilde{\mathcal{F}_0}$), the (trivial) fibrations are the epimorphisms with kernel in $(\mathrm{ex}\widetilde{\mathcal{F}_0})^\perp$ (resp. $(\mathrm{dw}\widetilde{\mathcal{F}_0})^\perp$), and the weak equivalences are the quasi-isomorphisms.

In this section, we prove that the induced cotorsion pairs $(\mathrm{dw}\widetilde{\mathcal{F}_n}, (\mathrm{dw}\widetilde{\mathcal{F}_n})^\perp)$ and $(\mathrm{ex}\widetilde{\mathcal{F}_n}, (\mathrm{ex}\widetilde{\mathcal{F}_n})^\perp)$ are complete, by using a method that we call the stairway zig-zag. This modified zig-zag procedure also works to show the completeness of the pair $(\mathrm{dw}\widetilde{\mathcal{P}_n}, (\mathrm{dw}\widetilde{\mathcal{P}_n})^\perp)$ and $(\mathrm{ex}\widetilde{\mathcal{P}_n}, (\mathrm{ex}\widetilde{\mathcal{P}_n})^\perp)$ (without assuming R Noetherian, as in Section 3.3). After showing that $(\mathrm{dw}\widetilde{\mathcal{F}_n}, (\mathrm{dw}\widetilde{\mathcal{F}_n})^\perp)$ and $(\mathrm{ex}\widetilde{\mathcal{F}_n}, (\mathrm{ex}\widetilde{\mathcal{F}_n})^\perp)$ are also compatible, the following result shall follow.

Theorem 3.5.2 (**Degreewise n-flat model structure**)

There exists a unique Abelian model structure on $\mathbf{Ch}({}_R\mathbf{Mod})$ where the (trivial) cofibrations are the monomorphisms with cokernel in $\mathrm{dw}\widetilde{\mathcal{F}_n}$ (resp. $\mathrm{ex}\widetilde{\mathcal{F}_n}$), the (trivial) fibrations are the epimorphisms with kernel in $(\mathrm{ex}\widetilde{\mathcal{F}_n})^\perp$ (resp. $(\mathrm{dw}\widetilde{\mathcal{F}_n})^\perp$), and the weak equivalences are the quasi-isomorphisms.

For the rest of this section, let $\mathcal{A}$ denote either the class of projective modules or flat modules, and $\mathcal{A}_n$ denote the class of left n-$\mathcal{A}$-modules, where $\mathcal{A}_0 = \mathcal{A}$.

Remark 3.5.1. From Lemmas 3.3.6 and 3.1.21, we have that for every $A \in \mathcal{A}_n$ and for every small submodule $0 \neq N \subseteq A$, there exists a small submodule $A' \subseteq A$ in $\mathcal{A}_n$ such that $N \subseteq A'$ and $A/A' \in \mathcal{A}_n$.

Theorem 3.5.3

Let $X \in \mathrm{ex}\widetilde{\mathcal{A}_n}$ and $x \in X$ (i.e. $x \in X_m$ for some $m \in \mathbb{Z}$). Then there exists a complex $X \supseteq Y \in (\mathrm{ex}\widetilde{\mathcal{A}_n})^{\leq \kappa}$ such that $x \in Y$ and $X/Y \in \mathrm{ex}\widetilde{\mathcal{A}_n}$.

The following proof is based on an argument given in (4, Proposition 4.1), where the authors prove that $(\mathrm{dw}\widetilde{\mathcal{F}_0}, (\mathrm{dw}\widetilde{\mathcal{F}_0})^{\perp})$ and $(\mathrm{ex}\widetilde{\mathcal{F}_0}, (\mathrm{ex}\widetilde{\mathcal{F}_0})^{\perp})$ are complete cotorsion pairs.

Proof of Theorem 3.5.3.

Assume without loss of generality that $x \in X_0$. Consider the submodule $\langle x \rangle \subseteq X_0$. Since $X_0 \in \mathcal{A}_n$ and $\langle x \rangle$ is small, we can embed $\langle x \rangle$ into a submodule $(\mathcal{A}_n)^{\leq \kappa} \ni Y_0^1 \subseteq X_0$ such that $X_0/Y_0^1 \in \mathcal{A}_n$. We can construct a κ-small and exact subcomplex

$$L^1 := (\cdots \to L_2^1 \to L_1^1 \to Y_0^1 \to \partial_0(Y_0^1) \to 0 \to \cdots),$$

since X is exact. The fact that $\partial_0(Y_0^1)$ is κ-small implies that it is contained in a submodule $(\mathcal{A}_n)^{\leq \kappa} \ni Y_{-1}^2 \subseteq X_{-1}$ such that $X_{-1}/Y_{-1}^2 \in \mathcal{A}_n$. As above, we can construct a κ-small and exact subcomplex of the form

$$L^2 := (\cdots \to L_2^2 \to L_1^2 \to L_0^2 \to Y_{-1}^2 \to \partial_{-1}(Y_{-1}^2) \to 0 \to \cdots).$$

Note that it is possible to construct L^2 containing L^1. Now embed L_0^2 into a submodule $(\mathcal{A}_n)^{\leq \kappa} \ni Y_0^3 \subseteq X_0$ such that $X_0/Y_0^3 \in \mathcal{A}_n$. Again, construct a subcomplex

$$L^3 := (\cdots \to L_2^3 \to L_1^3 \to Y_0^3 \to Y_{-1}^2 + \partial_0(Y_0^3) \to \partial_{-1}(Y_{-1}^2) \to 0 \to \cdots)$$

containing L^2, which is κ-small and exact. Now let $Y_1^4 \in (\mathcal{A}_n)^{\leq \kappa}$ be a submodule of X_1 containing L_1^3 such that $X_1/Y_1^4 \in \mathcal{A}_n$, and construct an exact and κ-small complex L^4 containing L^3 of the form

$$L^4 := (\cdots \to L_2^4 \to Y_1^4 \to Y_0^3 + \partial_1(Y_1^4) \to Y_{-1}^2 + \partial_0(Y_0^3) \to \partial_{-1}(Y_{-1}^2) \to 0 \to \cdots).$$

Embed $Y_0^3 + \partial_1(Y_1^4)$ into a submodule $(\mathcal{A}_n)^{\leq\kappa} \ni Y_0^5 \subseteq X_0$ such that $X_0/Y_0^5 \in \mathcal{A}_n$. Construct an exact and κ-small subcomplex

$$L^5 := (\cdots \to L_2^5 \to L_1^5 \to Y_0^5 \to Y_{-1}^2 + \partial_0(Y_0^5) \to \partial_{-1}(Y_{-1}^2) \to 0 \to \cdots)$$

containing L^4. In a similar way, construct κ-small and exact complexes

$$L^6 := (\cdots \to L_1^6 \to L_0^6 \to Y_{-1}^6 \to \partial_{-1}(Y_{-1}^6) \to 0 \to \cdots)$$

and

$$L^7 := (\cdots \to L_1^7 \to L_0^7 \to L_{-1}^7 \to Y_{-2}^7 \to \partial_{-2}(Y_{-2}^7) \to 0 \to \cdots),$$

such that $Y_{-1}^6 \in \mathcal{A}_n$ is a κ-small submodule of X_{-1} containing $Y_{-1}^2 + \partial_0(Y_0^5)$, and $Y_{-2}^7 \in \mathcal{A}_n$ is a small submodule of X_{-2} containing $\partial_{-1}(Y_{-1}^6)$. We have the following commutative diagram of subcomplexes of X where the $k+1$-th complex can be constructed in such a way that it contains the k-th complex:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
L_2^1 \arrow[r] \arrow[d] & L_1^1 \arrow[r] \arrow[d] & Y_0^1 \arrow[r] \arrow[d] & \partial_0(Y_0^1) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
L_2^2 \arrow[r] \arrow[d] & L_1^2 \arrow[r] \arrow[d] & L_0^2 \arrow[r] \arrow[d] & Y_{-1}^2 \arrow[r] \arrow[d] & \partial_{-1}(Y_{-1}^2) \arrow[r] \arrow[d] & 0 \arrow[d] \\
L_2^3 \arrow[r] \arrow[d] & L_1^3 \arrow[r] \arrow[d] & Y_0^3 \arrow[r] \arrow[d] & \partial_0(Y_0^3) + Y_{-1}^2 \arrow[r] \arrow[d] & \partial_{-1}(Y_{-1}^2) \arrow[r] \arrow[d] & 0 \arrow[d] \\
L_2^4 \arrow[r] \arrow[d] & Y_1^4 \arrow[r] \arrow[d] & Y_0^3 + \partial_1(Y_1^4) \arrow[r] \arrow[d] & \partial_0(Y_0^3) + Y_{-1}^2 \arrow[r] \arrow[d] & \partial_{-1}(Y_{-1}^2) \arrow[r] \arrow[d] & 0 \arrow[d] \\
L_2^5 \arrow[r] \arrow[d] & L_1^5 \arrow[r] \arrow[d] & Y_0^5 \arrow[r] \arrow[d] & Y_{-1}^2 + \partial_0(Y_0^5) \arrow[r] \arrow[d] & \partial_{-1}(Y_{-1}^2) \arrow[r] \arrow[d] & 0 \arrow[d] \\
L_2^6 \arrow[r] \arrow[d] & L_1^6 \arrow[r] \arrow[d] & L_0^6 \arrow[r] \arrow[d] & Y_{-1}^6 \arrow[r] \arrow[d] & \partial_{-1}(Y_{-1}^6) \arrow[r] \arrow[d] & 0 \arrow[d] \\
L_2^7 \arrow[r] \arrow[d] & L_1^7 \arrow[r] \arrow[d] & L_0^7 \arrow[r] \arrow[d] & L_{-1}^7 \arrow[r] \arrow[d] & Y_{-2}^7 \arrow[r] \arrow[d] & \partial_{-2}(Y_{-2}^7) \arrow[d] \\
L_2^8 \arrow[r] \arrow[d] & L_1^8 \arrow[r] \arrow[d] & L_0^8 \arrow[r] \arrow[d] & Y_{-1}^8 \arrow[r] \arrow[d] & \partial_{-1}(Y_{-1}^8) + Y_{-2}^7 \arrow[r] \arrow[d] & \partial_{-2}(Y_{-2}^7) \arrow[d] \\
L_2^9 \arrow[r] \arrow[d] & L_1^9 \arrow[r] \arrow[d] & Y_0^9 \arrow[r] \arrow[d] & \partial_0(Y_0^9) + Y_{-1}^8 \arrow[r] \arrow[d] & \partial_{-1}(Y_{-1}^8) + Y_{-2}^7 \arrow[r] \arrow[d] & \partial_{-2}(Y_{-2}^7) \arrow[d] \\
L_2^{10} \arrow[r] \arrow[d] & Y_1^{10} \arrow[r] \arrow[d] & \partial_1(Y_1^{10}) + Y_0^9 \arrow[r] \arrow[d] & \partial_0(Y_0^9) + Y_{-1}^8 \arrow[r] \arrow[d] & \partial_{-1}(Y_{-1}^8) + Y_{-2}^7 \arrow[r] \arrow[d] & \partial_{-2}(Y_{-2}^7) \arrow[d] \\
Y_2^{11} \arrow[r] \arrow[d] & \partial_2(Y_2^{11}) + Y_1^{10} \arrow[r] \arrow[d] & \partial_1(Y_1^{10}) + Y_0^9 \arrow[r] \arrow[d] & \partial_0(Y_0^9) + Y_{-1}^8 \arrow[r] \arrow[d] & \partial_{-1}(Y_{-1}^8) + Y_{-2}^7 \arrow[r] \arrow[d] & \partial_{-2}(Y_{-2}^7) \arrow[d] \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{tikzcd}$$

Note that the submodules Y_i^k appear according to the following "stairway-like" pattern:

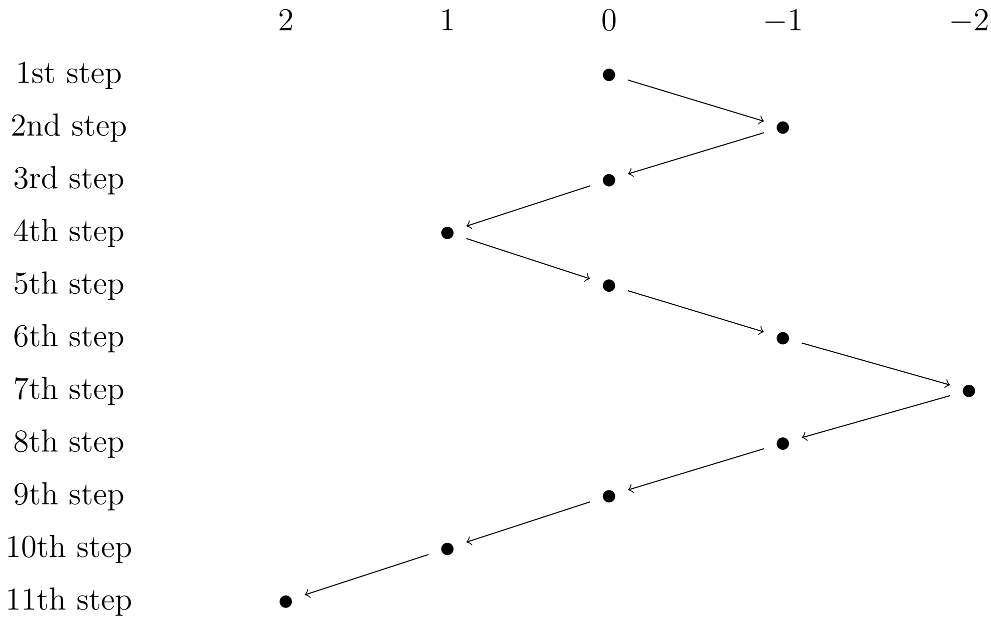

Figure 3.1: Stairway zig-zag.

The previous diagram is sort of inspired in the following painting.

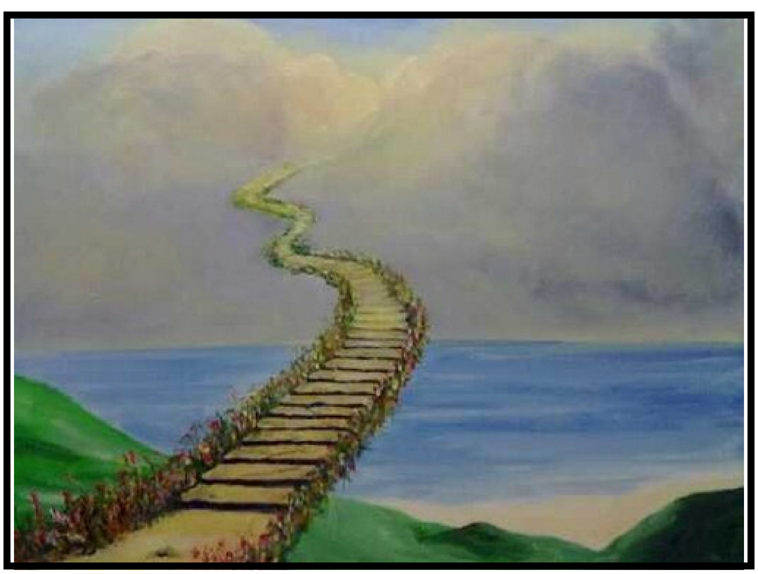

Figure 3.2: *Stairway to heaven.* © Paul Taylor.

Let $Y := \bigcup_{n\geq 1} L^n$, where $Y_i := \bigcup_{n\geq 1}(L^n)_i$. It is clear that Y is an exact complex. We check that Y is also in $(\mathrm{dw}\widetilde{\mathcal{A}_n})^{\leq\kappa}$. For example,

$$Y_0 = Y_0^1 \cup L_0^2 \cup Y_0^3 \cup (Y_0^3 + \partial_1(Y_1^4)) \cup Y_0^5 \cup \cdots = Y_0^1 \cup Y_0^3 \cup Y_0^5 \cup \cdots$$

is κ-small.

At this point, we split the proof into two cases:

(1) $\mathcal{A} = \mathcal{P}_0$: Consider

$$(1) = \left(0 \to \bigoplus_{i\in I_n} P_n^i \to \cdots \to \bigoplus_{i\in I_1} P_1^i \to \bigoplus_{i\in I_0} P_0^i \to X_0 \to 0\right)$$

a projective resolution of X_0 of length n, where each direct sum consists of countably generated projective modules. By Lemma 3.3.6, we can construct Y_0^1 containing $\langle x\rangle$ with a subresolution of the form

$$(2) = \left(0 \to \bigoplus_{i\in I_n^1} P_n^i \to \cdots \to \bigoplus_{i\in I_1^1} P_1^i \to \bigoplus_{i\in I_0^1} P_0^i \to Y_0^1 \to 0\right),$$

where each $I_k^1 \subset I_k$ is κ-small. Note that the quotient of (1) by (2) yields a projective resolution of X_0/Y_0^1 of length n, so $X_0/Y_0^1 \in \mathcal{P}_n$. Using Lemma 3.3.6 again, we can construct a subresolution containing (2), say

$$(3) = \left(0 \to \bigoplus_{i\in I_n^3} P_n^i \to \cdots \to \bigoplus_{i\in I_1^3} P_1^i \to \bigoplus_{i\in I_0^3} P_0^i \to Y_0^3 \to 0\right)$$

such that $X_0/Y_0^3 \in \mathcal{P}_n$. We keep applying Lemma 3.3.6 to get an ascending chain of subresolutions of (1):

$$0 \to \bigoplus_{i\in I_n^1} P_n^i \to \cdots \to \bigoplus_{i\in I_1^1} P_1^i \to \bigoplus_{i\in I_0^1} P_0^i \to Y_0^1 \to 0,$$

$$0 \to \bigoplus_{i\in I_n^3} P_n^i \to \cdots \to \bigoplus_{i\in I_1^3} P_1^i \to \bigoplus_{i\in I_0^3} P_0^i \to Y_0^3 \to 0,$$

$$0 \to \bigoplus_{i\in I_n^5} P_n^i \to \cdots \to \bigoplus_{i\in I_1^5} P_1^i \to \bigoplus_{i\in I_0^5} P_0^i \to Y_0^5 \to 0, \ \ldots$$

Now we take the union of this ascending chain and get the exact complex

$$(4) = \left(0 \to \bigcup_j \bigoplus_{i \in I_n^j} P_n^i \to \cdots \to \bigcup_j \bigoplus_{i \in I_1^j} P_1^i \to \bigcup_j \bigoplus_{i \in I_0^j} P_0^i \to \bigcup_j Y_0^j \to 0\right)$$
$$= \left(0 \to \bigoplus_{i \in \bigcup_j I_n^j} P_n^i \to \cdots \to \bigoplus_{i \in \bigcup_j I_1^j} P_1^i \to \bigoplus_{i \in \bigcup_j I_0^j} P_0^i \to Y_0 \to 0\right)$$

Since each $\bigcup_j I_k^j$ is a small subset of I_k, we have that the previous sequence is a $(\mathcal{P}_0)^{\leq \kappa}$-subresolution of (1). Note also that the quotient of (1) by (4) yields a projective resolution of X_0/Y_0 of length n. Then $Y_0 \in (\mathcal{P}_n)^{\leq \kappa}$. In a similar way, we can show that $Y_m \in (\mathcal{P}_n)^{\leq \kappa}$ and $X_m/Y_m \in \mathcal{P}_n$, for every $m \in \mathbb{Z}$. Hence, $Y \in (\mathrm{ex}\widetilde{\mathcal{P}_n})^{\leq \kappa}$. It follows $X/Y \in \mathrm{ex}\widetilde{\mathcal{P}_n}$ since the class of exact complexes is thick.

(2) $\mathcal{A} = \mathcal{F}_0$: Consider a left flat resolution of length n,

$$(1) = (0 \to F_n \to \cdots \to F_1 \to F_0 \to X_0 \to 0).$$

By Lemma 3.1.21, we can construct a subresolution

$$0 \to S_n^1 \to \cdots \to S_1^1 \to S_0^1 \to Y_0^1 \to 0,$$

where $\langle x \rangle \subseteq Y_0^1$, and each S_k^1 is a small and pure submodule of F_k. As we did in the previous case, applying Lemma 3.1.21 infinitely many times, we can get an ascending chain of subresolutions

$$0 \to S_n^1 \to \cdots \to S_1^1 \to S_0^1 \to Y_0^1 \to 0,$$
$$0 \to S_n^3 \to \cdots \to S_1^3 \to S_0^3 \to Y_0^3 \to 0,$$
$$0 \to S_n^5 \to \cdots \to S_1^5 \to S_0^5 \to Y_0^5 \to 0,$$
$$\vdots$$

Taking the union of these subresolutions yields an exact sequence

$$(2) = \left(0 \to \bigcup_j S_n^j \to \cdots \to \bigcup_j S_1^j \to \bigcup_j S_0^j \to Y_0 \to 0\right),$$

where each $\bigcup_j S_k^j$ is a small and pure submodule of F_k and the quotient $F_k / \bigcup_j S_k^j$ is flat. Then we have that $Y_0 \in (\mathcal{F}_n)^{\leq \kappa}$ and $X_0/Y_0 \in \mathcal{F}_n$ (take the quotient of (1) by (2)). In a similar way, we have $Y_m \in (\mathcal{F}_n)^{\leq \kappa}$ and $X_m/Y_m \in \mathcal{F}_n$, for every $m \in \mathbb{Z}$. It follows $Y \in (\mathrm{ex}\widetilde{\mathcal{F}_n})^{\leq \kappa}$ and $X/Y \in \mathrm{ex}\widetilde{\mathcal{F}_n}$. $\square$

It follows from the previous result that $(\mathrm{ex}\widetilde{\mathcal{P}_n}, (\mathrm{ex}\widetilde{\mathcal{P}_n})^{\perp})$ and $(\mathrm{ex}\widetilde{\mathcal{F}_n}, (\mathrm{ex}\widetilde{\mathcal{F}_n})^{\perp})$ are complete cotorsion pairs. Using a similar argument one can show that the pairs $(\mathrm{dw}\widetilde{\mathcal{P}_n}, (\mathrm{dw}\widetilde{\mathcal{P}_n})^{\perp})$ and $(\mathrm{dw}\widetilde{\mathcal{F}_n}, (\mathrm{dw}\widetilde{\mathcal{F}_n})^{\perp})$ are also complete. By Proposition 2.4.8, the pairs $(\mathrm{dw}\widetilde{\mathcal{F}_n}, (\mathrm{dw}\widetilde{\mathcal{F}_n})^{\perp})$ and $(\mathrm{ex}\widetilde{\mathcal{F}_n}, (\mathrm{ex}\widetilde{\mathcal{F}_n})^{\perp})$ are compatible. Therefore, Theorem 3.5.2 follows. Similarly, we can get another proof of Theorem 3.3.2.

3.6 n-Flat model structures

This section is devoted to study the completeness of the two cotorsion pairs $(\widetilde{\mathcal{F}_n}, \mathrm{dg}\widetilde{(\mathcal{F}_n)^{\perp}})$ and $(\mathrm{dg}\widetilde{\mathcal{F}_n}, \widetilde{(\mathcal{F}_n)^{\perp}})$, induced by $(\mathcal{F}_n, (\mathcal{F}_n)^{\perp})$. Specifically, we show that $(\widetilde{\mathcal{F}_n}, \mathrm{dg}\widetilde{(\mathcal{F}_n)^{\perp}})$ is cogenerated by a set, based on an argument given by S. T. Aldrich, E. E. Enochs, J. R. García Rozas and L. Oyonarte in (4), where authors proved that every flat complex is a transfinite extension of a set of κ-small flat complexes. For the case $n = 0$, the completeness of $(\widetilde{\mathcal{F}_0}, \mathrm{dg}\widetilde{(\mathcal{F}_0)^{\perp}})$ and $(\mathrm{dg}\widetilde{\mathcal{F}_0}, \widetilde{(\mathcal{F}_0)^{\perp}})$ was proven by J. Gillespie in (27), using the notions of pure and dg-pure subcomplexes. The compatibility of these two pairs and Hovey's correspondence allowed Gillespie to find the following model structure[5] on $\mathbf{Ch}({}_R\mathbf{Mod})$.

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5. The same author also constructed the flat model structure on the category of complexes of $\mathcal{O}$-modules on a topological space T, where $\mathcal{O}$ is a sheaf of rings on T. See (28, Corollary 4.12).

Theorem 3.6.1 (**Flat model structure**)
There exists a unique Abelian model structure on $\mathbf{Ch}({}_R\mathbf{Mod})$ where the (trivial) cofibrations are the monomorphisms with cokernel in $\mathrm{dg}\widetilde{\mathcal{F}_0}$ (resp. $\widetilde{\mathcal{F}_0}$), the (trivial) fibrations are the epimorphisms with kernel in $(\widetilde{\mathcal{F}_0})^\perp$ (resp. $(\mathrm{dg}\widetilde{\mathcal{F}_0})^\perp$), and the weak equivalences are the quasi-isomorphisms.

Once we settle the completeness and compatibility of the pairs $(\widetilde{\mathcal{F}_n}, \mathrm{dg}\widetilde{(\mathcal{F}_n)^\perp})$ and $(\mathrm{dg}\widetilde{\mathcal{F}_n}, \widetilde{(\mathcal{F}_n)^\perp})$, we shall deduce the following theorem.

Theorem 3.6.2 (**n-Flat model structure**)
There exists a unique Abelian model structure on $\mathbf{Ch}({}_R\mathbf{Mod})$ where the (trivial) cofibrations are the monomorphisms with cokernel in $\mathrm{dg}\widetilde{\mathcal{F}_n}$ (resp. $\widetilde{\mathcal{F}_n}$), the (trivial) fibrations are the epimorphisms with kernel in $(\widetilde{\mathcal{F}_n})^\perp$ (resp. $(\mathrm{dg}\widetilde{\mathcal{F}_n})^\perp$), and the weak equivalences are the quasi-isomorphisms.

First, we recall the definition of the standard tensor product on $\mathbf{Ch}({}_R\mathbf{Mod})$, from which one can define another tensor product on $\mathbf{Ch}({}_R\mathbf{Mod})$ more appropriate to define flat complexes. Given two chain complexes $X \in \mathbf{Ch}(\mathbf{Mod}_R)$ and $Y \in \mathbf{Ch}({}_R\mathbf{Mod})$, the tensor product complex $X \otimes Y$ is the chain complex (of Abelian groups groups) given by

$$(X \otimes Y)_n := \textstyle\bigoplus_{k \in \mathbb{Z}} X_k \otimes_R Y_{n-k},$$

where the boundary maps $(X \otimes Y)_n \to (X \otimes Y)_{n-1}$ are defined by

$$\partial_n^{X \otimes Y}(x \otimes y) := \partial^X(x) \otimes y + (-1)^{|x|} x \otimes \partial^Y(y).$$

This construction defines a functor $- \otimes -$, from which one constructs the left derived functors $\mathrm{Tor}_i(-,-)$, with $i \geq 0$.

The bar tensor product of X and Y is the complex of groups $X\overline{\otimes}Y$ given by

$$(X\overline{\otimes}Y)_n := (X \otimes Y)_n / B_n(X \otimes Y),$$

for every $n \in \mathbb{Z}$, whose boundary maps $(X\overline{\otimes}Y)_n \to (X\overline{\otimes}Y)_{n-1}$ are given by

$$\partial_n^{X\overline{\otimes}Y}(\overline{x \otimes y}) := \overline{\partial^X(x) \otimes y}.$$

As far as the author knows, the definition of this tensor product appeared first in (24). The left derived functors of $-\overline{\otimes}-$ shall be denoted by $\overline{\mathrm{Tor}}_i(-,-)$.

Definition 3.6.1. A chain complex X in $\mathbf{Ch}({}_R\mathbf{Mod})$ is said to be flat if the functor $-\overline{\otimes}X : \mathbf{Ch}(\mathbf{Mod}_R) \to \mathbf{Ab}$ is left exact.

Why does one consider $\overline{\otimes}$ instead of $\otimes$ to define flat complexes? The answer is given by the following proposition.

Proposition 3.6.3 (see (24, Proposition 5.1.2))

A chain complex X in $\mathbf{Ch}({}_R\mathbf{Mod})$ is flat if, and only if, $X \in \widetilde{\mathcal{F}_0}$.

As we did in Proposition 1.8.4, we have the following result.

Proposition 3.6.4

A chain complex X is n-flat if, and only if, it is exact and $Z_m(X) \in \mathcal{F}_n$, for every $m \in \mathbb{Z}$.

In (4, Proposition 3.1), the authors prove that each element of a flat chain complex F is contained in a κ-small flat subcomplex $L \subseteq F$ such that the quotient F/L is also flat. The following theorem is a generalization of this assertion, for n-flat complexes. As in previous sections, fix an infinite regular cardinal κ satisfying $\kappa \geq \mathrm{Card}(R)$.

Theorem 3.6.5

For any left n-flat complex $X \in \widetilde{\mathcal{F}_n}$ and any element $x \in X$ (i.e. $x \in X_k$ for some $k \in \mathbb{Z}$), there exists a κ-small n-flat subcomplex $L \subseteq X$ such that $x \in L$ and $X/L \in \widetilde{\mathcal{F}_n}$.

Proof.

Without loss of generality, assume $k = 0$. Write

$$X = (\cdots \to X_1 \to X_0 \to X_{-1} \to \cdots).$$

Consider the submodule $\langle x \rangle \subseteq X_0$. We show there exists a κ-small submodule $L_1^1 \subseteq X_1$ such that $\partial_1(L_1^1) = \mathrm{Ker}(\partial_0|_{\langle x \rangle})$. Let $rx \in \mathrm{Ker}(\partial_0|_{\langle x \rangle})$. Since X is exact, there exists $b_r \in X_1$ such that $rx = \partial_1(b_r)$. Let

$$B = \{b_r : rx \in \mathrm{Ker}(\partial_0|_{\langle x \rangle}) \text{ and } rx = \partial_1(b_r)\},$$

and set $L_1^1 = \langle B \rangle$. It is clear that $\partial_1(L_1^1) = \mathrm{Ker}(\partial_0|_{\langle x \rangle})$. Also, $\mathrm{Card}(L_1^1) \leq \aleph_0 \cdot \mathrm{Card}(R) \leq \aleph_0 \cdot \kappa = \kappa$. Now consider $\mathrm{Ker}(\partial_1|_{L_1^1})$. As we did before, we can construct a submodule $L_2^1 \subseteq X_2$ such that $\partial_2(L_2^1) = \mathrm{Ker}(\partial_1|_{L_1^1})$ and $\mathrm{Card}(L_2^1) \leq \kappa$. Keep repeating this procedure infinitely many times in order to get a κ-small and exact complex

$$L^1 := (\cdots \to L_3^1 \xrightarrow{\partial_3} L_2^1 \xrightarrow{\partial_2} L_1^1 \xrightarrow{\partial_1} \langle x \rangle \xrightarrow{\partial_0} \partial_0(\langle x \rangle) \to 0 \to \cdots).$$

Notice that $x \in L^1$.

Now consider $\mathrm{Ker}(\partial_0|_{\langle x \rangle})$ as a submodule of the left n-flat module $Z_0(X)$. Then, by Lemma 3.1.21, $\mathrm{Ker}(\partial_0|_{\langle x \rangle})$ can be embedded into a κ-small and n-pure submodule S_0^2 of $Z_0(X)$. Repeat the procedure above to get a κ-small complex

$$L^2 := (\cdots \to L_3^2 \xrightarrow{\partial_3} L_2^2 \xrightarrow{\partial_2} L_1^2 \xrightarrow{\partial_1} \langle x \rangle + S_0^2 \xrightarrow{\partial_0} \partial_0(\langle x \rangle) \to 0 \to \cdots),$$

which is exact at L_i^2 for every i. Moreover, L_1^2 is constructed in such a way that $\partial_1(L_1^2) = \operatorname{Ker}(\partial_0|_{\langle x\rangle + S_0^2})$. Note also that $\partial_0(\langle x\rangle + S_0^2) = \partial_0(\langle x\rangle)$. Finally, $S_0^2 \subseteq \operatorname{Ker}(\partial_0|_{\langle x\rangle}) \subseteq \operatorname{Ker}(\partial_0|_{\langle x\rangle + S_0^2})$. Now if $rx + s \in \operatorname{Ker}(\partial_0|_{\langle x\rangle + S_0^2})$, then we have $0 = \partial_0(rx) + \partial_0(s) = \partial_0(rx)$ and so $rx \in \operatorname{Ker}(\partial_0|_{\langle x\rangle}) \subseteq S_0^2$. It follows that $\operatorname{Ker}(\partial_0|_{\langle x\rangle + S_0^2}) = S_0^2$. Hence L^2 is an exact complex. Note also that L^2 can be constructed in such a way that $L_1^1 \subseteq L_1^2$. For if we have the sequence $\langle x\rangle + S_0^2 \to \partial_0(\langle x\rangle) \to 0$, then for any $rx+s \in \operatorname{Ker}(\partial_0|_{\langle x\rangle + S_0^2})$ there exists $b_{r,s}$ such that $rx+s = \partial_1(b_{r,s})$. For every $rx+s$ with $s=0$, choose $b_{r,0}$ as the b_r chosen in the construction of L^1. Thus, we get $L_1^2 \supseteq L_1^1$. Repeating this procedure, we may assume $L^1 \subseteq L^2$.

Using the Lemma 3.1.21 again, we can embed $\partial_0(\langle x\rangle)$ into a κ-small and n-pure submodule $S_{-1}^3 \subseteq Z_{-1}(X)$. Then construct an exact complex

$$L^3 := (\cdots \to L_2^3 \xrightarrow{\partial_2} L_1^3 \xrightarrow{\partial_1} L_0^3 \xrightarrow{\partial_0} S_{-1}^3 \to 0 \to \cdots) \supseteq L^2.$$

Now embed $\operatorname{Ker}(\partial_0|_{L_0^3})$ into a κ-small and n-pure submodule $S_0^4 \subseteq Z_0(X)$ and construct an κ-small and exact complex

$$L^4 := (\cdots \to L_2^4 \xrightarrow{\partial_2} L_1^4 \xrightarrow{\partial_1} L_0^3 + S_0^4 \xrightarrow{\partial_0} S_{-1}^3 \to \cdots) \supseteq L^3.$$

Notice $\operatorname{Ker}(\partial_0|_{L_0^3 + S_0^4}) = S_0^4$.

Consider $\operatorname{Ker}(\partial_1|_{L_1^4}) \subseteq Z_1(X)$ and let S_1^5 be a n-pure and κ-small submodule of $Z_1(X)$ such that $\operatorname{Ker}(\partial_1|_{L_1^4}) \subseteq S_1^5$. Then construct a small and exact complex L^5 containing L^4 having the form

$$L^5 := (\cdots \to L_2^5 \xrightarrow{\partial_2} L_1^4 + S_1^5 \xrightarrow{\partial_1} L_0^3 + S_0^4 \xrightarrow{\partial_0} S_1^3 \to 0 \to \cdots)$$

with $\operatorname{Ker}(\partial_1|_{L_1^4 + S_1^5}) = S_1^5$. Now find a κ-small n-pure submodule $S_2^6 \subseteq Z_2(X)$ such that $\operatorname{Ker}(\partial_2|_{L_2^5}) \subseteq S_2^6$. Construct a small and exact complex

$$L^6 := (\cdots \to L_3^6 \xrightarrow{\partial_3} L_2^5 + S_2^6 \xrightarrow{\partial_2} L_1^4 + S_1^5 \xrightarrow{\partial_1} L_0^3 + S_0^4 \xrightarrow{\partial_0} S_{-1}^3 \to 0 \to \cdots) \supseteq L^5,$$

where $\mathrm{Ker}(\partial_2|_{L_2^5+S_2^6}) = S_2^6$.

Keep repeating this process, an we have that for any $n \geq 4$ there exists a κ-small an exact complex

$$L^n := (\cdots \to L_{n-3}^n \xrightarrow{\partial_{n-3}} L_{n-4}^{n-1}+S_{n-4}^n \to \cdots \to L_1^4+S_1^5 \xrightarrow{\partial_1} L_0^3+S_0^4 \xrightarrow{\partial_0} S_{-1}^3 \to 0 \to \cdots)$$

such that $\mathrm{Ker}(\partial_{n-j}|_{L_{n-j}^{n-j+3}+S_{n-j}^{n-j+4}}) = S_{n-j}^{n-j-4}$ is a n-pure (and so n-flat) submodule of $Z_{n-j}(X)$, for every $3 < j \leq n$. Now consider the complex $L = \bigcup_{n\geq 4} L^n$. It is easy to see that L is κ-small and exact, and that it has the form

$$L := (\cdots \to L_3^6 + S_3^7 \xrightarrow{\partial_3} L_2^5 + S_2^6 \xrightarrow{\partial_2} L_1^4 + S_1^5 \xrightarrow{\partial_1} L_0^3 + S_0^4 \xrightarrow{\partial_0} S_{-1}^3 \to 0 \to \cdots).$$

Also, $Z_n(L)$ is an n-pure submodule of $Z_n(X)$, so it is n-flat. Hence, $L \in \widetilde{\mathcal{F}_n}$.

We obtained a short exact sequence of chain complexes $L \hookrightarrow X \twoheadrightarrow X/L$. Note that X/L is exact, since L and X are. By Lemma 1.8.3, we have an exact sequence $Z_n(L) \hookrightarrow Z_n(X) \twoheadrightarrow Z_n(X/L)$. It follows that $Z_n(X/L) \cong Z_n(X)/Z_n(L)$, which is n-flat since $Z_n(X)$ is n-flat and $Z_n(L)$ is an n-pure submodule of $Z_n(X)$. Therefore, $X/L \in \widetilde{\mathcal{F}_n}$. □

The previous theorem provides a cogenerating set for the pair $(\widetilde{\mathcal{F}_n}, (\widetilde{\mathcal{F}_n})^\perp)$, namely the set of κ-small n-flat complexes $(\widetilde{\mathcal{F}_n})^{\leq\kappa}$, by Propositions 3.1.13 and 3.1.14. So we have a complete cotorsion pair $(\widetilde{\mathcal{F}_n}, (\widetilde{\mathcal{F}_n})^\perp)$. For every fixed chain complex $X \in \mathrm{Ob}(\mathbf{Ch}(\mathbf{Mod}_R))$, the functor $\overline{\mathrm{Tor}_1}(X, -)$ preserves direct limits (We shall see later this fact in the proof of Proposition 4.7.18). It follows the class $\widetilde{\mathcal{F}_n}$ is closed under direct limits, and hence the pair $(\widetilde{\mathcal{F}_n}, (\widetilde{\mathcal{F}_n})^\perp)$ is perfect by Theorem 3.1.23. It follows that every chain complex in $\mathbf{Ch}({}_R\mathbf{Mod})$ has an n-flat cover. The first approaches to this result are given in (24) in the case $n = 0$, for complexes over a commutative Noetherian ring with finite Krull dimension.

We conclude this section by giving a description for exact chain complexes whose cycles are pure submodules.

Definition 3.6.2 (see (27, Definition 4.3))**.** We shall say that $S \in \mathrm{Ob}(\mathbf{Ch}({}_R\mathbf{Mod}))$ is a pure subcomplex of a complex X if the sequence $0 \to Y\overline{\otimes}S \to Y\overline{\otimes}X$ is exact, for every $Y \in \mathrm{Ob}(\mathbf{Ch}(\mathbf{Mod}_R))$

Proposition 3.6.6

S is a pure subcomplex of a flat complex F if, and only if, S is exact and $Z_m(S)$ is a pure submodule of $Z_m(F)$, for every $m \in \mathbb{Z}$.

Proof.

Suppose S is a pure subcomplex of F. Then S is flat (See (27, Lemma 4.7)). Since flat complexes are exact, it follows S is exact. It suffices to show $Z_m(S)$ is a pure submodule of $Z_m(F)$. Let M be a right R-module. Consider the sphere complex $S^0(M)$. Since S is a pure subcomplex of F, we have an exact sequence $S^0(M)\overline{\otimes}S \hookrightarrow S^0(M)\overline{\otimes}F$. At each $n \in \mathbb{Z}$, we have $(S^0(M) \otimes X)_m = M \otimes_R X_m$, for every complex $X \in \mathrm{Ob}(\mathbf{Ch}({}_R\mathbf{Mod}))$. Recall the boundary map $\partial_{m+1}^{S^0(M)\otimes X}$ is given by $y \otimes x \mapsto y \otimes \partial_{m+1}^X(x)$ on generators. It is easy to see the equality $B_m(S^0(M) \otimes X) = M \otimes_R B_m(X)$. It follows $M \otimes_R B_m(X) \to M \otimes_R X_m$ is injective and so $(S^0(M)\overline{\otimes}X)_m \cong \frac{(S^0(M)\otimes X)_m}{B_m(S^0(M)\otimes X)} = \frac{M\otimes_R X_m}{M\otimes_R B_m(X)} \cong M \otimes_R \frac{X_m}{B_m(X)}$. Since S and F are exact, we get $(S^0(M)\overline{\otimes}S)_m \cong M \otimes_R Z_{m-1}(S)$ and $(S^0(M)\overline{\otimes}F)_m \cong M \otimes_R Z_{m-1}(F)$. For every $m \in \mathbb{Z}$, the sequence $(S^0(M)\overline{\otimes}S)_m \hookrightarrow (S^0(M)\overline{\otimes}F)_m$ is exact, so $M \otimes_R Z_m(S) \hookrightarrow M \otimes_R Z_m(F)$ is an exact sequence by the previous isomorphism. Hence, $Z_m(S)$ is a pure submodule of $Z_m(F)$.

Now suppose S is an exact subcomplex of F such that $Z_m(S)$ is a pure submodule of $Z_m(F)$. Let A be a complex in $\mathbf{Ch}(\mathbf{Mod}_R)$. We want to show the sequence $A\overline{\otimes}S \hookrightarrow A\overline{\otimes}F$ is exact. Since $Z_{n-k}(S)$ is a pure submodule of $Z_{n-1}(F)$, we have $A_k \otimes_R Z_{n-k}(S) \hookrightarrow A_k \otimes_R Z_{n-k}(F)$ is exact. Since S and F are exact complex, we obtain the following commutative diagram where the top and the bottom row are exact (Recall Lemma 1.8.3):

$$\begin{array}{ccccc}
 & & 0 & & 0 \\
 & & \downarrow & & \downarrow \\
0 & \longrightarrow & A_k \otimes_R Z_{n-k}(S) & \longrightarrow & A_k \otimes_R Z_{n-k}(F) \\
 & & \downarrow & & \downarrow \\
0 & \longrightarrow & A_k \otimes_R S_{n-k} & \longrightarrow & A_k \otimes_R F_{n-k} \\
 & & \downarrow & & \downarrow \\
0 & \longrightarrow & A_k \otimes_R Z_{n-k-1}(S) & \longrightarrow & A_k \otimes_R Z_{n-k-1}(F) \\
 & & \downarrow & & \downarrow \\
 & & 0 & & 0
\end{array}$$

The columns of this diagram are also exact since $Z_{n-k}(S)$, $Z_{n-k}(F)$ and $Z_{n-k}(F/S)$ are flat modules. Since the class of short exact sequences is closed under extensions, we have that the sequence $A_k \otimes_R S_{n-k} \hookrightarrow A_k \otimes_R F_{n-k}$ is exact. It follows $(A \otimes S)_n \hookrightarrow (A \otimes F)_n$ is exact. Then $A \otimes S \hookrightarrow A \otimes F$ is an exact sequence of complexes, and so $B_n(A \otimes S) \hookrightarrow B_n(A \otimes F)$ is exact. Since the class of short exact sequences is thick, we have that $\frac{(A\otimes S)_n}{B_n(A\otimes S)} \hookrightarrow \frac{(A\otimes F)_n}{B_n(A\otimes F)}$ is exact. Hence, $A\overline{\otimes}S \hookrightarrow A\overline{\otimes}F$ is exact. $\square$

Using this characterization, we may say that given an n-flat complex X, for each $x \in X$ there exists a small n-pure subcomplex $L \subseteq X$ such that $x \in L$[6]. From Section 3.1 we know that $(\mathcal{F}_n, (\mathcal{F}_n)^{\perp})$ is a cotorsion pair. This, along with the fact that ${}_R\mathbf{Mod}$ is a category with enough projective and injective modules, implies

6. n-pure subcomplexes are defined in the same way as n-pure submodules.

that $(\mathrm{dg}\widetilde{\mathcal{F}_n} \cap \mathcal{E}, (\widetilde{\mathcal{F}_n})^{\perp})$ and $(\mathrm{dg}\widetilde{\mathcal{F}_n}, (\widetilde{\mathcal{F}_n})^{\perp} \cap \mathcal{E})$ are compatible cotorsion pairs. The former pair is complete and so by Proposition 2.3.6, the later is also complete. Therefore, Theorem 3.6.2 follows.

3.7 Model structures on complexes over Abelian categories with a progenerator

In this chapter, we have constructed Abelian model structures on $\mathbf{Ch}({}_R\mathbf{Mod})$ from the notion of homological dimensions of modules. Our way to proceed has consisted in generalizing results in the zero dimensional case (i.e. results on projective, injective and flat modules or complexes) to the nonzero dimensional case, or from the module case to the chain complex case. In this sense, we have applied several zig-zag-like techniques in order to remark the construction of transfinite extensions of κ-small (- exact or not - degreewise) n-projective and (- exact or not - degreewise) n-flat complexes. Very recently in (15), the authors have proven that $(\widetilde{\mathcal{A}}, \mathrm{dg}\widetilde{\mathcal{B}})$ is a cotorsion pair in $\mathbf{Ch}({}_R\mathbf{Mod})$ cogenerated by a set if $(\mathcal{A}, \mathcal{B})$ is also cogenerated by a set in ${}_R\mathbf{Mod}$. The author is not aware if such a result still holds in the context of modules over ringoids. The techniques applied in the given reference are different and more complicated from the ones we have shown throughout this chapter. We have preferred to keep our particular approach to this problem from the notion of homological dimensions, due to the applications (Chapter 4) of some of the notions and procedures given so far.

We devote this last section to present some categorical versions of the model structures concerning the projective and injective homological dimensions, by using Mitchell's and Freyd's equivalences (See Sections 1.9 and 1.10).

Note that every equivalence of Abelian categories $\mathcal{C} \to \mathcal{D}$ induces an equivalence of the corresponding categories of chain complexes $\mathbf{Ch}(\mathcal{C}) \to \mathbf{Ch}(\mathcal{D})$.

Let $\mathcal{C} \overset{F}{\to} \mathcal{D} \overset{G}{\leftarrow} \mathcal{C}$ be an equivalence of categories. By abuse of notation, given a class of objects $\mathcal{X}$ in $\mathcal{C}$, $F(\mathcal{X})$ shall denote the class of objects $Y \in \mathrm{Ob}(\mathcal{D})$ such that $Y \cong F(X)$ for some $X \in \mathcal{X}$. The class $G(\mathcal{Y})$ is defined similarly.

Proposition 3.7.1 (Some properties of equivalence of categories)

Let $\mathcal{C} \overset{F}{\to} \mathcal{D} \overset{G}{\leftarrow} \mathcal{C}$ be an equivalence of Abelian categories categories.

(1) F and G preserves exact sequences in $\mathcal{C}$ and $\mathcal{D}$, respectively.

(2) For every $X, Y \in \mathrm{Ob}(\mathcal{C})$ and $Z, W \in \mathrm{Ob}(\mathcal{D})$, there exist group monomorphisms:

$\mathrm{Ext}^1_{\mathcal{C}}(X,Y) \hookrightarrow \mathrm{Ext}^1_{\mathcal{D}}(F(X),F(Y))$ and $\mathrm{Ext}^1_{\mathcal{D}}(Z,W) \hookrightarrow \mathrm{Ext}^1_{\mathcal{C}}(G(Z),G(W))$.

(3) $(\mathcal{A},\mathcal{B})$ is a cotorsion pair in $\mathcal{C}$ if, and only if, $(F(\mathcal{A}),F(\mathcal{B}))$ is a cotorsion pair in $\mathcal{D}$.

(4) $(\mathcal{A},\mathcal{B})$ is cogenerated by a set in $\mathrm{Ob}(\mathcal{C})$ if, and only if, $(F(\mathcal{A}),F(\mathcal{B}))$ is cogenerated by a set in $\mathrm{Ob}(\mathcal{D})$.

(5) $(\mathcal{A},\mathcal{B})$ is complete in $\mathcal{C}$ if, and only if, $(F(\mathcal{A}),F(\mathcal{B}))$ is complete in $\mathcal{D}$.

(6) $(\mathcal{A},\mathcal{B})$ is hereditary in $\mathcal{C}$ if, and only if, $(F(\mathcal{A}),F(\mathcal{B}))$ is hereditary in $\mathcal{D}$.

We do not give a proof of this properties since **(1)** is well known and the rest are straightforward. Concerning equivalences of categories and induced cotorsion pairs, we have the following properties.

Proposition 3.7.2

Let $\mathcal{C} \overset{F}{\to} \mathcal{D} \overset{G}{\leftarrow} \mathcal{C}$ be an equivalence of categories, $\mathbf{Ch}(\mathcal{C}) \overset{\widetilde{F}}{\to} \mathbf{Ch}(\mathcal{D}) \overset{\widetilde{G}}{\leftarrow} \mathbf{Ch}(\mathcal{C})$, and $(\mathcal{A},\mathcal{B})$ be a cotorsion pair in $\mathcal{C}$. Then $\widetilde{F(\mathcal{A})} = \widetilde{F}(\widetilde{\mathcal{A}})$, $\widetilde{F}(\mathrm{dg}\widetilde{\mathcal{A}}) = \mathrm{dg}\widetilde{F(\mathcal{A})}$, $\widetilde{F}(\mathrm{dw}\widetilde{\mathcal{A}}) = \mathrm{dw}\widetilde{F(\mathcal{A})}$, and $\widetilde{F}(\mathrm{ex}\widetilde{\mathcal{A}}) = \mathrm{ex}\widetilde{F(\mathcal{A})}$. The same equalities also hold for the class $\mathcal{B}$.

Corollary 3.7.3

Let $\mathcal{C}$ be an Abelian category with a progenerator P and arbitrary coproducts of copies of P.

(1) There is a unique Abelian model structure on $\mathbf{Ch}(\mathcal{C})$ such that $\mathrm{dg}\widetilde{\mathcal{P}_n(\mathcal{C})}$ is the class of cofibrant objects, $(\widetilde{\mathcal{P}_n(\mathcal{C})})^{\perp}$ the class of fibrant objects, and $\mathcal{E}$ the class of trivial objects.

(2) There is a unique Abelian model structure on $\mathbf{Ch}(\mathcal{C})$ such that $\mathrm{dw}\widetilde{\mathcal{P}_n(\mathcal{C})}$ is the class of cofibrant objects, $(\mathrm{ex}\widetilde{\mathcal{P}_n(\mathcal{C})})^{\perp}$ the class of fibrant objects, and $\mathcal{E}$ the class of trivial objects.

(1') There is a unique Abelian model structure on $\mathbf{Ch}(\mathcal{C})$ such that $\mathrm{dg}\widetilde{\mathcal{I}_n(\mathcal{C})}$ is the class of fibrant objects, ${}^{\perp}(\widetilde{\mathcal{I}_n(\mathcal{C})})$ the class of cofibrant objects, and $\mathcal{E}$ the class of trivial objects.

(2') There is a unique Abelian model structure on $\mathbf{Ch}(\mathcal{C})$ such that $\mathrm{dw}\widetilde{\mathcal{I}_n(\mathcal{C})}$ is the class of fibrant objects, ${}^{\perp}(\mathrm{ex}\widetilde{\mathcal{I}_n(\mathcal{C})})$ the class of cofibrant objects, and $\mathcal{E}$ the class of trivial objects.

$\boxed{\textbf{Proof}}$.

We only prove the left half. First, we need to note that since P is a projective generator of $\mathcal{C}$, then $\bigoplus_{m\in\mathbb{Z}} D^{m+1}(P)$ is a projective generator of $\mathbf{Ch}(\mathcal{C})$, but not necessarily a progenerator. However, $\mathcal{G} = \{D^{m+1}(P) : m \in \mathbb{Z}\}$ is a set of finite projective generators of $\mathbf{Ch}(\mathcal{C})$, and so $\mathbf{Ch}(\mathcal{C})$ is equivalent to the category of right modules over the ringoid $\mathfrak{R} = \mathrm{End}_{\mathcal{C}}(\mathcal{G})$. It follows some of the results presented in this chapter and the previous one, for Grothendieck categories, are valid in $\mathbf{Ch}(\mathcal{C})$ for our particular choice of $\mathcal{C}$.

(1) Consider the n-projective model structure on $\mathbf{Ch}(\mathbf{Mod}_R)$, where $R = \mathrm{Hom}_{\mathcal{C}}(P, P)$. In this occasion, let $\mathcal{E}_R$ and $\mathcal{E}$ the classes of exact chain complexes in $\mathbf{Ch}(\mathbf{Mod}_R)$

and $\mathbf{Ch}(\mathcal{C})$, respectively. We have two compatible and complete cotorsion pairs $(\mathrm{dg}\widetilde{\mathcal{P}_n} \cap \mathcal{E}_R, (\widetilde{\mathcal{P}_n})^\perp)$ and $(\mathrm{dg}\widetilde{\mathcal{P}_n}, (\widetilde{\mathcal{P}_n})^\perp \cap \mathcal{E}_R)$. Consider Mitchell's equivalence (Theorem 1.9.3) $F : \mathcal{C} \to \mathbf{Mod}_R$ and let $G : \mathbf{Mod}_R \to \mathcal{C}$ be its natural inverse, i.e. GF and FG are naturally isomorphic to $\mathrm{id}_{\mathcal{C}}$ and $\mathrm{id}_{\mathbf{Mod}_R}$, respectively. By Proposition 3.7.1, we have two complete cotorsion pairs $(\widetilde{G}(\mathrm{dg}\widetilde{\mathcal{P}_n} \cap \mathcal{E}), \widetilde{G}((\widetilde{\mathcal{P}_n})^\perp))$ and $(\widetilde{G}(\mathrm{dg}\widetilde{\mathcal{P}_n}), \widetilde{G}((\widetilde{\mathcal{P}_n})^\perp \cap \mathcal{E}))$. It is not hard to see that $G(\mathcal{P}_n) = \mathcal{P}_n(\mathcal{C})$. Using this and Proposition 3.7.1 **(2)**, we can also show that $G((\mathcal{P}_n)^\perp) = (\mathcal{P}_n(\mathcal{C}))^\perp$. On the one hand, note that $\widetilde{G}(\mathrm{dg}\widetilde{\mathcal{P}_n} \cap \mathcal{E}_R) = \widetilde{G}(\mathrm{dg}\widetilde{\mathcal{P}_n}) \cap \mathcal{E}$ and $\widetilde{G}((\widetilde{\mathcal{P}_n})^\perp \cap \mathcal{E}) = \widetilde{G}((\widetilde{\mathcal{P}_n})^\perp) \cap \mathcal{E}$. On the other hand, by Proposition 3.7.2 **(2)** we have $\widetilde{G}(\mathrm{dg}\widetilde{\mathcal{P}_n}) = \mathrm{dg}\widetilde{G(\mathcal{P}_n)} = \mathrm{dg}\widetilde{\mathcal{P}_n(\mathcal{C})}$ and $\widetilde{G}((\widetilde{\mathcal{P}_n})^\perp) = \widetilde{G}(\mathrm{dg}\widetilde{(\mathcal{P}_n)^\perp}) = \mathrm{dg}G\widetilde{((\mathcal{P}_n)^\perp)} = \mathrm{dg}(\mathcal{P}_n(\mathcal{C}))^\perp = (\widetilde{\mathcal{P}_n(\mathcal{C})})^\perp$. It follows we have two compatible and complete cotorsion pairs of the form $(\mathrm{dg}\widetilde{\mathcal{P}_n(\mathcal{C})} \cap \mathcal{C}, (\widetilde{\mathcal{P}_n(\mathcal{C})})^\perp)$ and $(\mathrm{dg}\widetilde{\mathcal{P}_n(\mathcal{C})}, (\widetilde{\mathcal{P}_n(\mathcal{C})})^\perp \cap \mathcal{E})$. Finally, the result follows by Hovey's correspondence.

(2) Starting from the degreewise n-projective model structure and using some arguments similar to **(1)**, we have two compatible and complete cotorsion pairs $(\mathrm{dw}\widetilde{\mathcal{P}_n(\mathcal{C})} \cap \mathcal{E}, \widetilde{G}((\mathrm{ex}\widetilde{\mathcal{P}_n})^\perp))$ and $(\mathrm{dw}\widetilde{\mathcal{P}_n(\mathcal{C})}, \widetilde{G}((\mathrm{ex}\widetilde{\mathcal{P}_n})^\perp) \cap \mathcal{E})$. It suffices to show $\widetilde{G}((\mathrm{ex}\widetilde{\mathcal{P}_n})^\perp) = (\mathrm{ex}\widetilde{\mathcal{P}_n(\mathcal{C})})^\perp$. Let $X \in \widetilde{G}((\mathrm{ex}\widetilde{\mathcal{P}_n})^\perp)$. Write $X \cong \widetilde{G}(X')$ for some complex $X' \in (\mathrm{ex}\widetilde{\mathcal{P}_n})^\perp$. For every complex $Z \in \mathrm{ex}\widetilde{\mathcal{P}_n(\mathcal{C})}$, we have $\mathrm{Ext}^1_{\mathbf{Ch}(\mathcal{C})}(Z, X)$ embedded into the group $\mathrm{Ext}^1(F(Z), FG(X'))$, where $F(Z) \in \mathrm{ex}\widetilde{\mathcal{P}_n}$. Then $\mathrm{Ext}^1(F(Z), FG(X')) \cong \mathrm{Ext}^1(F(Z), X') = 0$, and hence $\mathrm{Ext}^1(Z, X) = 0$, i.e. $X \in (\mathrm{ex}\widetilde{\mathcal{P}_n(\mathcal{C})})^\perp$. The other inclusion follows in a similar way. $\square$

MODEL STRUCTURE	COFIBRANT OBJECTS	FIBRANT OBJECTS	TRIVIAL OBJECTS	TRIVIALLY COFIBRANT OBJECTS	TRIVIALLY FIBRANT OBJECTS
On chain complexes over a ring					
n-projective	$\mathrm{dg}\widetilde{\mathcal{P}_n}$	$(\widetilde{\mathcal{P}_n})^{\perp}$	$\mathcal{E}$	$\widetilde{\mathcal{P}_n}$	$(\mathrm{dg}\widetilde{\mathcal{P}_n})^{\perp}$
degreewise n-projective	$\mathrm{dw}\widetilde{\mathcal{P}_n}$	$(\mathrm{ex}\widetilde{\mathcal{P}_n})^{\perp}$	$\mathcal{E}$	$\mathrm{ex}\widetilde{\mathcal{P}_n}$	$(\mathrm{dw}\widetilde{\mathcal{P}_n})^{\perp}$
n-injective	${}^{\perp}(\widetilde{\mathcal{I}_n})$	$\mathrm{dg}\widetilde{\mathcal{I}_n}$	$\mathcal{E}$	${}^{\perp}(\mathrm{dg}\widetilde{\mathcal{I}_n})$	$\widetilde{\mathcal{I}_n}$
degreewise n-injective	${}^{\perp}(\mathrm{ex}\widetilde{\mathcal{I}_n})$	$\mathrm{dw}\widetilde{\mathcal{I}_n}$	$\mathcal{E}$	${}^{\perp}(\mathrm{dw}\widetilde{\mathcal{I}_n})$	$\mathrm{ex}\widetilde{\mathcal{I}_n}$
n-flat	$\mathrm{dg}\widetilde{\mathcal{F}_n}$	$(\widetilde{\mathcal{F}_n})^{\perp}$	$\mathcal{E}$	$\widetilde{\mathcal{F}_n}$	$(\mathrm{dg}\widetilde{\mathcal{F}_n})^{\perp}$
degreewise n-flat	$\mathrm{dw}\widetilde{\mathcal{F}_n}$	$(\mathrm{ex}\widetilde{\mathcal{F}_n})^{\perp}$	$\mathcal{E}$	$\mathrm{ex}\widetilde{\mathcal{F}_n}$	$(\mathrm{dw}\widetilde{\mathcal{F}_n})^{\perp}$

Table 3.1: **SUMMARY OF MODEL STRUCTURES**

CHAPTER IV

ABELIAN MODEL STRUCTURES AND GORENSTEIN HOMOLOGICAL DIMENSIONS

"I have noticed even people who claim everything is predestined, and that we can do nothing to change it, look before they cross the road."

Stephen Hawking.

In the previous chapter, we studied how model structures and homological dimensions are related. In most of the results, we did not need to impose special conditions on the ring R. However, if R is a Gorenstein ring, we have the chance to work with another type of homological algebra, described in terms of Gorenstein-projective, Gorenstein-injective and Gorenstein-flat modules. Gorenstein homological algebra can also be done in certain categories known as Gorenstein categories, which are basically Grothendieck categories with a generator of finite projective dimension, with some axioms concerning projective and injective dimensions of its objects. This chapter describes some connections between the theory of model categories and Gorenstein homological algebra, mainly the construction of Abelian model structures by proving completeness of certain cotorsion pairs obtained from classes of objects with finite Gorenstein homological dimensions. Several results are presented in a categorical setting, but there are some cases where it is more convenient to give the definitions and statements for modules and chain complexes (for instance, during our study of the Gorenstein-flat dimension).

If R is a Gorenstein ring, one of the interesting things that occurs is that a left R-module has finite projective dimension if, and only if, it has finite injective dimension. The author is not aware where the notion of Gorenstein category comes from, but one possible guess could be that Gorenstein rings are motivated in the concept of quasi-Frobenius rings. Recall that in the category of modules over a quasi-Frobenius ring, the classes of projective and injective modules coincide. So Gorenstein rings seem to be a generalization of quasi-Frobenius rings. In this sense, does the model structure presented in Example 2.3.1 has a generalization in ${}_R\mathbf{Mod}$, with R a Gorenstein ring? Settling this question shall be the starting point to study how to construct new Abelian model structures from Gorenstein homological dimensions.

4.1 Gorenstein categories

Throughout this section, $\mathcal{C}$ shall be a Gorenstein category. The author knows two definitions of this type of category. The first one (in chronological order) was given by J. R. García Rozas and appears in (24, Definition 3.1.1). For our purposes, it is better to drop this definition and use the one given by E. E. Enochs, S. Estrada and the same García Rozas in (20, Definition 2.18). The author is not aware if these two definitions are equivalent or not.

Definition 4.1.1. A Grothendieck category $\mathcal{C}$ is said to be a Gorenstein category if the following conditions are satisfied:

(1) For any object X of $\mathcal{C}$, $\mathrm{pd}(X) < \infty$ if, and only if, $\mathrm{id}(X) < \infty$.

(2) The supremum $FDP(\mathcal{C}) := \sup\{\text{pd}(X) : \text{pd}(X) < \infty\}$ and $FDI(\mathcal{C}) := \sup\{\text{id}(X) : \text{id}(X) < \infty\}$ are both finite.

(3) $\mathcal{C}$ has a generator of finite projective dimension.

Remark 4.1.1.

(1) If $\mathcal{C} =_R \mathbf{Mod}$, then $FDP(\mathcal{C})$ is called the big finitistic dimension of R.

(2) Every Gorenstein category has enough projective object. For let $X \in \text{Ob}(\mathcal{C})$ and G be a generator of $\mathcal{C}$ of finite projective dimension, say n. Then by Proposition 1.9.1 there exists an epimorphism $G^{(I)} \twoheadrightarrow X$, where $I = \text{Hom}_{\mathcal{C}}(G, X)$. On the other hand, there exists a projective object P and an epimorphism $P \twoheadrightarrow G$, since G has finite projective dimension. Then we get an epimorphism $P^{(I)} \twoheadrightarrow G^{(I)}$. Hence, we have obtain an epimorphism $P^{(I)} \twoheadrightarrow X$, where $P^{(I)}$ is a projective object. It follows by Proposition 1.9.1 again, that P is a projective generator of $\mathcal{C}$, so condition **(3)** in the previous definition can be replaced by: **(3')** $\mathcal{C}$ has a projective generator.

Example 4.1.1.

(1) A ring R is an Iwanaga-Gorenstein ring if it is left and right Noetherian, and it has finite injective dimension as a left and right R-module. It can be show that both injective dimensions are equal to some nonnegative integer n. Then R is called an n-Iwanaga-Gorenstein ring. In (21, Section 9.1) it is proven that if $\mathcal{P}$, $\mathcal{I}$, and $\mathcal{F}$ are the classes of left R-modules with finite projective, finite injective, and finite flat dimension, respectively, then $\mathcal{P} = \mathcal{I} = \mathcal{F} = \mathcal{P}_n = \mathcal{I}_n = \mathcal{F}_n$.

(2) Every quasi-Frobenius ring R is a 0-Gorenstein ring, and so ${}_R\mathbf{Mod}$ is a Gorenstein category. Note first that since R is projective (as a left or right R-module), then it is injective (i.e., R is a self-injective ring). On the other hand, in the definition of quasi-Frobenius rings given Example 2.3.1 **(1)**, it is not mentioned that R is left and right Noetherian, but it is known that a ring if quasi-Frobenius if, and only if, it is left (or right) Noetherian and left (resp. right) self-injective.

(3) Consider the category ${}_R\mathbf{Mod}$ where R is a field $\mathbb{K}$. So ${}_R\mathbf{Mod} = \mathbf{Vect}_{\mathbb{K}}$ is the category of $\mathbb{K}$-vector spaces. Then every vector space is projective and injective. It follows that $\mathbb{K}$ is a quasi-Frobenius ring and that $\mathbf{Vect}_{\mathbb{K}}$ is a Gorenstein category.

(4) If $\mathcal{C}$ is a Gorenstein category, then so is $\mathbf{Ch}(\mathcal{C})$. For conditions **(1)** and **(2)** of the previous definition, it suffices to notice that if X is a complex with finite projective dimension, then $X \in \widetilde{\mathcal{P}_n(\mathcal{C})}$, for $n = FDP(\mathcal{C})$, by Proposition 1.8.4. To check condition **(3')**, if G is a projective generator of $\mathcal{C}$, then $\bigoplus_{m\in\mathbb{Z}} D^m(G)$ is a generator of $\mathbf{Ch}(\mathcal{C})$ (see Example 1.9.1 **(5)**), which is projective since each $D^m(G)$ is. In particular, $\mathrm{Ch}({}_R\mathbf{Mod})$ is a Gorenstein category if R is a Gorenstein ring.

4.2 Gorenstein-projective and Gorenstein-injective objects and model structures

This section consists in giving the Gorenstein-like version of Examples 2.3.1 **(2)** and **(3)**. In (35, Theorem 8.6), M. Hovey constructs a unique Abelian model structure on ${}_R\mathbf{Mod}$ (with R a Gorenstein ring) where the cofibrant objects are the Gorenstein-projective modules and the trivial objects are the modules with finite projective dimension. There is also another model structure with the same trivial objects such that the Gorenstein-injective modules form the class of fibrant objects. Hovey's method consists in proving that the classes of Gorenstein-projective and Gorenstein-injective modules are the left and right halves, respectively, of two cotorsion pairs cogenerated by a set. We shall present Hovey's results in the context of Gorenstein categories. Let $\mathcal{W}(\mathcal{C})$ denote the class of objects in a Gorenstein category with finite projective dimension (eq. finite injective dimension). It is easy to see that this class is thick and closed under direct summands.

Definition 4.2.1. An object X in a Gorenstein category $\mathcal{C}$ is said to be Gorenstein-projective if there exists an exact sequence of projective objects

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

such that $X = \mathrm{Ker}(P^0 \to P^1)$, which is also $\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{P}_0(\mathcal{C}))$-exact (See Definition 1.4.2).

Theorem 4.2.1 (M. Hovey. **Gorenstein-projective model structure**)

If $\mathcal{C}$ is a Gorenstein category, then there exists a unique Abelian model structure on $\mathcal{C}$, where the (trivial) cofibrations are the monomorphisms with Gorenstein-projective (resp. projective) cokernel, the (trivial) fibrations are the epimorphisms (resp. whose kernel has finite projective dimension), and $\mathcal{W}(\mathcal{C})$ is the class of trivial objects.

Definition 4.2.2. An object X in a Gorenstein category $\mathcal{C}$ is said to be Gorenstein-injective if there exists an exact sequence of injective objects

$$\cdots \to I_1 \to I_0 \to I^0 \to I^1 \to \cdots$$

such that $X = \mathrm{Ker}(I^0 \to I^1)$, which is also $\mathrm{Hom}_{\mathcal{C}}(\mathcal{I}_0(\mathcal{C}), -)$-exact.

Theorem 4.2.2 (M. Hovey. **Gorenstein-injective model structure**)

If $\mathcal{C}$ is a locally Noetherian Gorenstein category, then there exists a unique Abelian model structure on $\mathcal{C}$, where the (trivial) fibrations are the epimorphisms with Gorenstein-injective (resp. injective) kernel, the (trivial) cofibrations are the monomorphisms (resp. whose cokernel has finite injective dimension), and $\mathcal{W}(\mathcal{C})$ is the class of trivial objects.

We denote by $\mathcal{GP}_0(\mathcal{C})$ and $\mathcal{GI}_0(\mathcal{C})$ the classes of Gorenstein-projective and Gorenstein-injective objects of $\mathcal{C}$, respectively.

We first focus on Theorem 4.2.1. The first step is to show that the classes $\mathcal{GP}_0(\mathcal{C})$ and $\mathcal{W}(\mathcal{C})$ form a complete cotorsion pair $(\mathcal{GP}_0(\mathcal{C}), \mathcal{W}(\mathcal{C}))$. Since $FDI(\mathcal{C}) < \infty$, there exists a nonnegative integer $N > 0$ such that $\mathcal{W}(\mathcal{C}) = \mathcal{I}_N(\mathcal{C})$. On the other hand, from the previous chapter we know that $({}^{\perp}(\mathcal{I}_N(\mathcal{C})), \mathcal{I}_N(\mathcal{C}))$ is a complete cotorsion pair. It follows $({}^{\perp}(\mathcal{W}(\mathcal{C})), \mathcal{W}(\mathcal{C}))$ is a complete cotorsion pair in any Gorenstein category $\mathcal{C}$. So it suffices to show that the classes $\mathcal{GP}_0(\mathcal{C})$ and ${}^{\perp}(\mathcal{W}(\mathcal{C}))$ coincide.

Proposition 4.2.3 (see (20, Theorem 2.25))

If $\mathcal{C}$ is a Gorenstein category, then $\mathcal{GP}_0(\mathcal{C}) = {}^{\perp}(\mathcal{W}(\mathcal{C}))$.

Proof.

If X is a Gorenstein-projective object, then there is an exact and $\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{P}_0(\mathcal{C}))$-exact sequence $\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ such that $X = \mathrm{Ker}(P_0 \to P^0)$. Let $W \in \mathcal{W}(\mathcal{C})$. Using the exact left projective resolution

$$\cdots \to P_2 \to P_1 \to X \to 0,$$

we can compute $\mathrm{Ext}^1_{\mathcal{C}}(X, W)$. Since W has finite projective dimension, we have $K \in \Omega^k(W)$ is projective, for some $k \geq 0$. Since the sequence

$$0 \to \mathrm{Hom}_{\mathcal{C}}(X, K) \to \mathrm{Hom}_{\mathcal{C}}(P_1, K) \to \mathrm{Hom}_{\mathcal{C}}(P_2, K) \to \cdots$$

is exact, it follows $\mathrm{Ext}^i_{\mathcal{C}}(X, K) = 0$, for every $i > 0$. Then for $k = 0$, the result follows immediately. In the case $k = 1$, we are given a short exact sequence $0 \to Q_1 \to Q_0 \to W \to 0$, with Q_0 and Q_1 projective. Then we have a long exact sequence $\cdots \to \cancelto{0}{\mathrm{Ext}^1_{\mathcal{C}}(X, Q_0)} \to \mathrm{Ext}^1_{\mathcal{C}}(X, W) \to \cancelto{0}{\mathrm{Ext}^2_{\mathcal{C}}(X, Q_1)} \to \cdots$. Hence, $\mathrm{Ext}^i_{\mathcal{C}}(X, W) = 0$ for every $i > 0$. The general result follows similarly.

The other implication is based on the arguments given in (20, Theorem 2.25). Suppose $X \in {}^{\perp}(\mathcal{W}(\mathcal{C}))$. Since $\mathcal{C}$ has enough projective objects by Remark 4.1.1, we have an exact left projective resolution $\cdots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} X \to 0$. Since $({}^{\perp}(\mathcal{W}(\mathcal{C})), \mathcal{W}(\mathcal{C}))$ is complete, there exists an embedding $X \overset{g^0}{\hookrightarrow} W^0$, for some $W^0 \in \mathcal{W}(\mathcal{C})$ (such an embedding is a $\mathcal{W}(\mathcal{C})$-pre-enveloping). Consider a short exact sequence $K^0 \hookrightarrow P^0 \twoheadrightarrow W^0$, where P^0 is projective. Since the class $\mathcal{W}(\mathcal{C})$ is thick, we have $K^0 \in \mathcal{W}(\mathcal{C})$. Since $\mathrm{Ext}^1_{\mathcal{C}}(X, K^0) = 0$, there exists a morphism $X \xrightarrow{f^0} P^0$ such that the following diagram commutes:

$$\begin{tikzcd}
 & & X \arrow[dl, "f^0"'] \arrow[d, hook, "g^0"] \\
K^0 \arrow[r, hook] & P^0 \arrow[r, two heads] & W^0
\end{tikzcd}$$

It is not hard to show that f^0 is also a $\mathcal{W}(\mathcal{C})$-pre-envelope. In a similar way, construct a $\mathcal{W}(\mathcal{C})$-pre-envelope $\mathrm{CoKer}(f^0) \xrightarrow{\rho^1} P^1$ and set $f^1 := \rho^1 \circ j^1$, where j^1 is the cokernel $P^0 \to \mathrm{CoKer}(f^0)$ of f^0. Repeating this procedure, we get an exact sequence $0 \to X \to P^0 \to P^1 \to \cdots$. Then we obtain a long exact sequence

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

of projective objects with $X = \mathrm{Ker}(P^0 \to P^1)$. It is only left to show that this sequence is $\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{P}_0(\mathcal{C}))$-exact. For P projective, we have a sequence

$$0 \to \mathrm{Hom}_{\mathcal{C}}(X, P) \xrightarrow{f_0^*} \mathrm{Hom}_{\mathcal{C}}(P_0, P) \xrightarrow{f_1^*} \mathrm{Hom}_{\mathcal{C}}(P_1, P) \to \cdots.$$

Since $X \in {}^{\perp}(\mathcal{W}(\mathcal{C}))$ and $\mathcal{W}(\mathcal{C})$ is coresolving, we have $\mathrm{Ext}^i_{\mathcal{C}}(X, P) = 0$ for every $i > 0$. Hence, the previous sequence is exact. Now consider

$$\cdots \to \mathrm{Hom}_{\mathcal{C}}(P^1, P) \xrightarrow{(f^1)^*} \mathrm{Hom}_{\mathcal{C}}(P^0, P) \xrightarrow{(f^0)^*} \mathrm{Hom}_{\mathcal{C}}(X, P) \to 0.$$

First, we show $(f^0)^*$ is onto. For a morphism $X \xrightarrow{h} P$, there exists a morphism $P^0 \xrightarrow{h'} P$ such that $h' \circ f^0 = h$, i.e. $h = (f^0)^*(h')$, since f^0 is a $\mathcal{W}(\mathcal{C})$-pre-envelope.

Finally, we show $\mathrm{Im}(f^1)^* = \mathrm{Ker}(f^0)^*$ (the rest of the equalities follow in the same way). It suffices to show $\mathrm{Ker}(f^0)^* \subseteq \mathrm{Im}(f^1)^*$. Let $P^0 \xrightarrow{h} P$ be a morphism such that $h \circ f^0 = 0$. Then there exists a morphism $\mathrm{CoKer}(f^0) \xrightarrow{h'} P$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, hook] & P^0 \arrow[r, two heads, "j^1"] \arrow[dr, "h"'] & \mathrm{CoKer}(f^0) \arrow[d, "h'"] \\ & & P \end{tikzcd}$$

Since $\mathrm{CoKer}(f^0) \xrightarrow{\rho^1} P^1$ is a $\mathcal{W}(\mathcal{C})$-pre-envelope, there exists a morphism $P^1 \xrightarrow{h''} P$ such that $h'' \circ \rho^1 = h'$. Then we have $h = h' \circ j^1 = h'' \circ \rho^1 \circ j^1 = h'' \circ f^1 = (f^1)^*(h'')$. $\square$

From the previous result, we have that $(\mathcal{GP}_0(\mathcal{C}), \mathcal{W}(\mathcal{C}))$ is a complete cotorsion pair. Recall also that $(\mathcal{P}_0(\mathcal{C}), \mathrm{Ob}(\mathcal{C}))$ is complete in any Abelian category with enough projectives. We shall see that these two pairs are also compatible. In (21, Proposition 10.2.3), it is proven that the projective dimension of a Gorenstein-projective left R-module is either zero of infinite. So if we denote by $\mathcal{GP}_0$ the class of Gorenstein-projective modules in ${}_R\mathbf{Mod}$, and by $\mathcal{W}$ the class of modules (over an Iwanaga-Gorenstein ring R) with finite projective dimension, then we obtain the equality $\mathcal{P}_0 = \mathcal{GP}_0 \cap \mathcal{W}$. The arguments to show this equality easily carry over to any Gorenstein category.

Proposition 4.2.4 (see (35, Corollary 8.5) for the case $\mathcal{C} = {}_R\mathbf{Mod}$)

Let $\mathcal{C}$ be a Gorenstein category. Then $\mathcal{P}_0(\mathcal{C}) = \mathcal{GP}_0(\mathcal{C}) \cap \mathcal{W}(\mathcal{C})$.

Proof.

The inclusion $\mathcal{P}_0(\mathcal{C}) \subseteq \mathcal{GP}_0(\mathcal{C}) \cap \mathcal{W}(\mathcal{C})$ is clear. Now suppose X is a Gorenstein-projective object with finite projective dimension. Then there is an exact sequence $W \hookrightarrow P \twoheadrightarrow X$, where P is projective and W has finite projective dimension. Since $\mathrm{Ext}^1_{\mathcal{C}}(X, W) = 0$, the previous sequence splits and so $P \cong W \oplus X$. It follows X is projective, since $\mathcal{P}_0(\mathcal{C})$ is closed under direct summands. $\square$

We have obtained two compatible and complete cotorsion pairs $(\mathcal{GP}_0(\mathcal{C}), \mathcal{W}(\mathcal{C}))$ and $(\mathcal{P}_0(\mathcal{C}), \mathrm{Ob}(\mathcal{C}))$ in any Gorenstein category. Therefore, Theorem 4.2.1 follows by Hovey's correspondence.

The rest of this subsection shall be devoted to prove that the class $\mathcal{GI}_0(\mathcal{C})$ of Gorenstein-injective objects generates a complete cotorsion pair $(\mathcal{W}(\mathcal{C}), \mathcal{GI}_0(\mathcal{C}))$, provided $\mathcal{C}$ is a locally Noetherian Gorenstein category.

Definition 4.2.3. Let $\mathcal{C}$ be a Grothendieck category with a generator G. Then the class $\mathrm{sub}(G)$ of subobjects of G is a set (See (47, Lemma 1, page 111)). We say that $\mathcal{C}$ is locally Noetherian if each nonempty subset $\mathcal{U}$ of $\mathrm{sub}(G)$ contains a maximal subobject, where a subobject $Y \in \mathcal{U}$ is said to be maximal in $\mathcal{U}$ if $Y' \in \mathcal{U}$ and $Y \subseteq Y'$ always imply $Y = Y'$.

Remark 4.2.1. The notion of locally Noetherian Grothendieck category depends on the choice of a generator.

In (35, Theorem 8.4), Hovey proves that $(\mathcal{W}, \mathcal{GI}_0)$ is a cotorsion pair in ${}_R\mathbf{Mod}$ cogenerated by the set of all ith syzygies of indecomposable injective objects, provided R is a Gorenstein ring. The proof uses the fact that every injective left R-module can be decomposed into a direct sum of indecomposable injective modules, provided R is left Noetherian. The generalization of this property to any locally Noetherian category is due to E. Matlis.

Definition 4.2.4. An injective object X in a Grothendieck category $\mathcal{C}$ is said to be indecomposable if $X \neq 0$ and if, for each decomposition $X = X_1 \oplus X_2$ into a direct sum of injective objects, either $X = X_1$ or $X = X_2$. If X is not indecomposable, then it is said to be decomposable.

Theorem 4.2.5 (E. Matlis (47, Theorem 4, page 208))

Let $\mathcal{C}$ be a locally Noetherian category. Each injective object I in $\mathcal{C}$ may be decomposed into a coproduct of indecomposable injective objects $I = \bigoplus_{\alpha \in A} I_\alpha$.

Hovey also uses a corollary of Eklof and Trlijaf's Theorem given in (31, Corollary 3.2.3). Doing some slight modifications, we get the following generalization to any Grothendieck category.

Lemma 4.2.6 ((31, Corollary 3.2.3) for Grothendieck categories)

Let X be an object of a Grothendieck category with enough projective objects. Let $\mathcal{Z}_X$ denote the class of all objects Z such that there is an exact sequence $P \hookrightarrow Z \twoheadrightarrow W$, where P is projective and W is a transfinite extension of $\{X\}$. Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in $\mathcal{C}$. The following conditions are equivalent:

(1) $(\mathcal{A}, \mathcal{B})$ is cogenerated by $\{X\}$.

(2) $\mathcal{A}$ consists of all direct summands of elements of $\mathcal{Z}_X$.

Proof.

First, we show **(1)** $\Longrightarrow$ **(2)**. Suppose $\mathcal{B} = X^\perp$ and let $A \in \mathcal{A}$. Since $\mathcal{C}$ has enough projectives, there exists an exact sequence $K \hookrightarrow P \twoheadrightarrow A$, where P is projective. By the proof of Eklof and Trlijaf's Theorem, we can construct for K an exact sequence $K \hookrightarrow B \twoheadrightarrow W$, where $B \in \mathcal{B}$ and W is a transfinite extension $(W_\alpha : \alpha < \lambda)$ of the class of direct sums of copies of X. Taking the pushout of $K \hookrightarrow P$ and $K \hookrightarrow B$, we get the following diagram with exact rows and columns:

$$
\begin{tikzcd}
K \arrow[r, hook] \arrow[d, hook] & P \arrow[r, two heads] \arrow[d, hook] & A \arrow[d, equal] \\
B \arrow[r, hook] \arrow[d, two heads] & B \coprod_K P \arrow[r, two heads] \arrow[d, two heads] & A \\
W \arrow[r, equal] & W &
\end{tikzcd}
$$

Since $A \in \mathcal{A}$ and $B \in \mathcal{B}$, the second row splits and so A is a direct summand of $B \coprod_K P \in \mathcal{Z}_X$.

Let Y be a direct summand of an element in $Z \in \mathcal{Z}_X$. Then there is a short exact sequence $P \hookrightarrow Z \twoheadrightarrow W$, where P is projective and W is a transfinite extension $(W_\alpha : \alpha < \lambda)$ of the class of direct sums of copies of X. Then there are index sets I_0 and I_α such that $W_0 \cong X^{(I_0)}$ and $W_{\alpha+1}/W_\alpha \cong X^{(I_\alpha)}$. For $B \in \mathcal{B}$, we have $\operatorname{Ext}^1_{\mathcal{C}}(W_0, B) \cong \operatorname{Ext}^1_{\mathcal{C}}(X^{(I_0)}, B) \cong \prod_{I_0} \operatorname{Ext}^1_{\mathcal{C}}(X, B) = 0$ and $\operatorname{Ext}^1_{\mathcal{C}}(W_{\alpha+1}/W_\alpha, B) \cong \operatorname{Ext}^1_{\mathcal{C}}(X^{(I_\alpha)}, B) \cong \prod_{I_\alpha} \operatorname{Ext}^1_{\mathcal{C}}(X, B) = 0$. By Eklof's Lemma, $\operatorname{Ext}^1_{\mathcal{C}}(W, B) = 0$. It follows $\operatorname{Ext}^1_{\mathcal{C}}(Z, B) = 0$, for every $B \in \mathcal{B}$, i.e. $Z \in \mathcal{A}$. We have that Y is a direct summand of an element in $\mathcal{A}$, and hence $Y \in \mathcal{A}$ since $\mathcal{A}$ is closed under direct summands.

Now we show **(1)** $\Longleftarrow$ **(2)**. In the previous part, we proved that $\mathcal{Z}_X \subseteq \mathcal{A}$. Note that $X \in \mathcal{Z}_X$. It follows $\mathcal{B} \subseteq X^\perp$. Now let $Y \in X^\perp$ and $A \in \mathcal{A}$. Let $Z \in \mathcal{Z}_X$ such that A is a direct summand of Z. We are given an exact sequence $P \hookrightarrow Z \twoheadrightarrow W$ as in the definition of $\mathcal{Z}_X$. Using Eklof's Lemma, one can show $\operatorname{Ext}^1_{\mathcal{C}}(W, Y) = 0$. On the other hand, $\operatorname{Ext}^1_{\mathcal{C}}(P, Y) = 0$. It follows $\operatorname{Ext}^1_{\mathcal{C}}(Z, Y) = 0$, i.e. $Z \in {}^\perp Y$. Since ${}^\perp Y$ is closed under direct summands, we get $A \in {}^\perp Y$. Hence $Y \in \mathcal{B}$. $\square$

Proposition 4.2.7 ((35, Theorem 8.4) for Gorenstein categories)
Let $\mathcal{C}$ be a locally Noetherian Gorenstein category with a projective generator G. Let $\mathcal{S}$ denote the set of all objects $S \in \Omega^i(J)$, where $i \geq 0$ and J runs over the set of indecomposable injective objects of $\mathcal{C}$. Then $\mathcal{S} \cup \{G\}$ cogenerates a cotorsion pair $(\mathcal{W}(\mathcal{C}), (\mathcal{W}(\mathcal{C}))^{\perp})$.

Proof.

First, we show the inclusion $\mathcal{W}(\mathcal{C}) \supseteq {}^{\perp}((\mathcal{S} \cup \{G\})^{\perp})$. Let $Y \in {}^{\perp}((\mathcal{S} \cup \{G\})^{\perp})$. By the previous lemma, Y is a direct summand of a element $Z \in \mathcal{Z}_X$, where $X = \bigoplus\{S : S \in \mathcal{S} \cup \{G\}\}$. Thus, there is an exact sequence $P \hookrightarrow Z \twoheadrightarrow W$, where P is projective and W is a transfinite extension $(W_\alpha : \alpha < \lambda)$ of the class of direct sums of copies of X. For each $S \in \mathcal{S}$, we have an exact sequence $0 \to S \to P_{i-1} \to \cdots \to P_1 \to P_0 \to J \to 0$, for some $i \geq 0$ and some indecomposable injective object J, where each P_k is projective. Since $J, P_k \in \mathcal{W}$, for every $0 \leq k \leq i-1$, and $\mathcal{W}$ is thick, we have S is also in $\mathcal{W}$. On the other hand, $G \in \mathcal{W}$. It follows $X \in \mathcal{W}$. Then, $W_0, W_{\alpha+1}/W_\alpha \in \mathcal{W}$ for every $\alpha < \lambda$, since $\mathcal{W}$ is closed under direct sums. Let C be an object of $\mathcal{C}$. We can find a nonnegative integer n such that $\mathrm{Ext}^1_{\mathcal{C}}(W_0, C') \cong \mathrm{Ext}^{n+1}_{\mathcal{C}}(W_0, C) = 0$ and $\mathrm{Ext}^1_{\mathcal{C}}(W_{\alpha+1}/W_\alpha, C') \cong \mathrm{Ext}^{n+1}_{\mathcal{C}}(W_{\alpha+1}/W_\alpha, C) = 0$ where $C' \in \Omega^{-n}(C)$. By Eklof's Lemma, we have $\mathrm{Ext}^{n+1}_{\mathcal{C}}(W, C) \cong \mathrm{Ext}^1_{\mathcal{C}}(W, C') = 0$, i.e. $W \in \mathcal{W}$. Since $\mathcal{W}$ is closed under extensions, we conclude $Z \in \mathcal{W}$. Hence, $Y \in \mathcal{W}$ since $\mathcal{W}$ is closed under direct summands.

Finally, we prove the inclusion $\mathcal{W} \subseteq {}^{\perp}((\mathcal{S} \cup \{G\})^{\perp})$. Let $W \in \mathcal{W}$ and $Y \in (\mathcal{S} \cup \{G\})^{\perp}$. We need to show $\mathrm{Ext}^1_{\mathcal{C}}(W, Y) = 0$. Let I be an injective object. By Matlis Theorem, we have a decomposition $I = \bigoplus_{\alpha \in A} I_\alpha$, where each I_α is an indecomposable injective object. For every $i \geq 0$, we have $\mathrm{Ext}^{i+1}_{\mathcal{C}}(I_\alpha, Y) \cong \mathrm{Ext}^1_{\mathcal{C}}({I_\alpha}', Y) = 0$, where ${I_\alpha}' \in \Omega^i(I_\alpha)$. It follows $\mathrm{Ext}^{i+1}_{\mathcal{C}}(I, Y) = 0$ for every $i \geq 0$

and for every injective object I. Suppose W has injective dimension at most 1 (the general case follows by induction). Then we have a short exact sequence $W \hookrightarrow I_0 \twoheadrightarrow I_1$. Using the long exact sequence

$$\cdots \to \mathrm{Ext}^1_{\mathcal{C}}(I_0, Y) \to \mathrm{Ext}^1_{\mathcal{C}}(W, Y) \to \mathrm{Ext}^2_{\mathcal{C}}(I_1, Y) \to \cdots,$$

we get $\mathrm{Ext}^1_{\mathcal{C}}(W, Y) = 0$ since $\mathrm{Ext}^1_{\mathcal{C}}(I_0, Y) = 0$ and $\mathrm{Ext}^2_{\mathcal{C}}(I_1, Y) = 0$. Hence, $\mathcal{W} \subseteq {}^{\perp}((\mathcal{S} \cup \{G\})^{\perp})$. □

As we did in the Gorenstein-projective case, one can show that $(\mathcal{W}(\mathcal{C}))^{\perp} = \mathcal{GI}_0(\mathcal{C})$. By the previous proposition, $(\mathcal{W}(\mathcal{C}), \mathcal{GI}_0(\mathcal{C}))$ is a complete cotorsion pair. We can also show that $\mathcal{I}_0(\mathcal{C}) = \mathcal{GI}_0(\mathcal{C}) \cap \mathcal{W}(\mathcal{C})$. Hence, Theorem 4.2.2 follows.

Remark 4.2.2. The completeness of the pairs $(\mathcal{GP}_0(\mathcal{C}), \mathcal{W}(\mathcal{C}))$ and $(\mathcal{W}(\mathcal{C}), \mathcal{GI}_0(\mathcal{C}))$ is also proven in (20, Theorems 2.24 and 2.25) for every Gorenstein category $\mathcal{C}$ (without assuming that $\mathcal{C}$ is locally Noetherian for the latter pair). The proofs given there consist in constructing for every object X in $\mathcal{C}$ a short exact sequence $0 \longrightarrow W \longrightarrow C \longrightarrow X \longrightarrow 0$ with $C \in \mathcal{GP}_0(\mathcal{C})$ and $W \in \mathcal{W}(\mathcal{C})$, and in showing that $(\mathcal{W}(\mathcal{C}), \mathcal{GI}_0(\mathcal{C}))$ is a small cotorsion pair. At some points in the rest of this chapter we shall assume that $\mathcal{C}$ is locally Noetherian, in order to use the given cogenerating set $\mathcal{S} \cup \{G\}$ of $(\mathcal{W}(\mathcal{C}), \mathcal{GI}_0(\mathcal{C}))$.

4.3 Cotorsion pairs from Gorenstein-projective and Gorenstein-injective dimensions

In this section we are going to study the notions and some properties of Gorenstein-projective and Gorenstein-injective dimensions, before constructing two cotorsion pairs involving the classes of objects with bounded Gorenstein-projective and Gorenstein-injective dimension.

Let $\mathcal{C}$ be a Gorenstein category. Since $(\mathcal{GP}_0(\mathcal{C}), \mathcal{W}(\mathcal{C}))$ is a complete cotorsion pair, for every object X of $\mathcal{C}$ there exists an epimorphism $C_0 \twoheadrightarrow X$ where C_0 is a Gorenstein-projective object. This allows us to construct an exact sequence

$$\begin{tikzcd}
\cdots \arrow[rr] & & C_1 \arrow[rr, "f_0"] \arrow[dr, two heads, "p_1"] & & C_0 \arrow[r, "f_0"] & X \arrow[r] & 0 \\
& W_1 \arrow[ur, hook, "i_1"'] & & W_0 \arrow[ur, hook, "i_0"'] & & &
\end{tikzcd}$$

where C_k is a Gorenstein-projective object for every $k \geq 0$, $W_0 \overset{i_0}{\hookrightarrow} C_0$ is the kernel of f_0, and $W_k \overset{i_k}{\hookrightarrow} C_k$ is the kernel of p_{k-1} for every $k > 0$. Dually, we can construct a long exact sequence $0 \to X \to D^0 \to D^1 \to \cdots$, where D^k is a Gorenstein-injective object for every $k \geq 0$.

Proposition 4.3.1

Let $\mathcal{C}$ be a Gorenstein category.

Every object has an exact left $\mathcal{GP}_0(\mathcal{C})$-resolution.	Every object has an exact right $\mathcal{GI}_0(\mathcal{C})$-resolution.

Proof.

We only prove the left statement. Let C be a Gorenstein-projective object. We show that the sequence $\cdots \to \mathrm{Hom}_{\mathcal{C}}(C, C_1) \to \mathrm{Hom}_{\mathcal{C}}(C, C_0) \to \mathrm{Hom}_{\mathcal{C}}(C, X) \to 0$ is exact. Consider the short exact sequence $W_0 \overset{i_0}{\hookrightarrow} C_0 \overset{f_0}{\twoheadrightarrow} X$. Since $\mathrm{Ext}^1_{\mathcal{C}}(C, W_0) = 0$, the morphism $\mathrm{Hom}_{\mathcal{C}}(C, C_0) \to \mathrm{Hom}_{\mathcal{C}}(C, X)$ is onto. It is only left to show that $\mathrm{Ker}((f_k)_*) \subseteq \mathrm{Im}((f_{k+1})_*)$. Let $C \xrightarrow{h} C_k$ be a morphism such that $0 = f_k \circ h = i_k \circ p_k \circ h$. Note $p_k \circ h = 0$ since i_k is monic. The exactness of $W_k \overset{i_k}{\hookrightarrow} C_k \overset{p_{k-1}}{\twoheadrightarrow} W_{k-1}$ implies there exists a morphism $C \xrightarrow{h'} W_k$ such that $i_k \circ h' = h$. On the other hand, $\mathrm{Ext}^1_{\mathcal{C}}(C, W_{k+1}) = 0$ implies there exists a morphism $C \xrightarrow{h''} C_{k+1}$ such that the following diagram commutes:

$$\begin{tikzcd} & & & C \arrow[dl, "h''"'] \arrow[d, "h'"] \\ W_{k+1} \arrow[r, hook, "i_{k+1}"'] & C_{k+1} \arrow[rr, two heads, "p_k"'] & & W_k \end{tikzcd}$$

We have $h = i_k \circ h' = i_k \circ p_k \circ h'' = f_k \circ h'' = (f_k)_*(h'')$. $\square$

Now we know that every object X in a Gorenstein category $\mathcal{C}$ with enough projectives has an exact left $\mathcal{GP}_0(\mathcal{C})$-resolution. The class of left r-$\mathcal{GP}_0(\mathcal{C})$-objects (with $r \geq 0$) shall be denoted by $\mathcal{GP}_r(\mathcal{C})$. We shall refer to an object in $\mathcal{GP}_r(\mathcal{C})$ as a Gorenstein-r-projective object (See Definition 1.7.1). We call the left $\mathcal{GP}_0(\mathcal{C})$-dimension the Gorenstein-projective dimension, and it is denoted by Gpd. Dually, the class of right r-$\mathcal{GI}_0(\mathcal{C})$-objects (with $r \geq 0$) shall be denoted by $\mathcal{GI}_r(\mathcal{C})$. We shall refer to an object in $\mathcal{GI}_r(\mathcal{C})$ as a Gorenstein-r-injective object. We call the right $\mathcal{GI}_0(\mathcal{C})$-dimension the Gorenstein-injective dimension, and it is denoted by Gid.

Proposition 4.3.2 ((21, Proposition 11.5.7) and (21, Proposition 11.2.5))

The following are equivalent in a Gorenstein category $\mathcal{C}$.

- **(1)** $X \in \mathcal{GP}_r(\mathcal{C})$.
- **(2)** $\mathrm{Ext}^i_{\mathcal{C}}(X, W) = 0$ for every $i > r$ and every $W \in \mathcal{W}(\mathcal{C})$.
- **(3)** $\mathrm{Ext}^{r+1}_{\mathcal{C}}(X, W) = 0$ for every $W \in \mathcal{W}(\mathcal{C})$.
- **(4)** $\Omega^r_{\mathcal{GP}_0(\mathcal{C})}(X) \subseteq \mathcal{GP}_0(\mathcal{C})$.
- **(5)** $\Omega^r(X) \subseteq \mathcal{GP}_0(\mathcal{C})$.

- **(1')** $Y \in \mathcal{GI}_r(\mathcal{C})$.
- **(2')** $\mathrm{Ext}^i_{\mathcal{C}}(W, Y) = 0$ for every $i > r$ and every $W \in \mathcal{W}(\mathcal{C})$.
- **(3')** $\mathrm{Ext}^{r+1}_{\mathcal{C}}(W, Y) = 0$ for every $W \in \mathcal{W}(\mathcal{C})$.
- **(4')** $\Omega^{-r}_{\mathcal{GI}_0(\mathcal{C})}(Y) \subseteq \mathcal{GI}_0(\mathcal{C})$.
- **(5')** $\Omega^{-r}(Y) \subseteq \mathcal{GI}_0(\mathcal{C})$.

Corollary 4.3.3

If $\mathcal{C}$ is a Gorenstein category, then:

(1) $\sup\{\mathrm{Gpd}(X) : X \in \mathrm{Ob}(\mathcal{C})\} \leq FDI(\mathcal{C})$.

(2) $\mathcal{P}_r(\mathcal{C}) = \mathcal{GP}_r(\mathcal{C}) \cap \mathcal{W}(\mathcal{C})$ for every $0 \leq r \leq FDI(\mathcal{C})$.

(1') $\sup\{\mathrm{Gid}(Y) : Y \in \mathrm{Ob}(\mathcal{C})\} \leq FDP(\mathcal{C})$.

(2') $\mathcal{I}_r(\mathcal{C}) = \mathcal{GI}_r(\mathcal{C}) \cap \mathcal{W}(\mathcal{C})$ for every $0 \leq r \leq FDP(\mathcal{C})$.

Proof.

We only prove the left statement. Let $X \in \mathrm{Ob}(\mathcal{C})$ and $W \in \mathcal{W}(\mathcal{C})$. Then $\mathrm{id}(W) \leq FDI(\mathcal{C})$, and so $\mathrm{Ext}_{\mathcal{C}}^{FDI(\mathcal{C})+1}(X, W) = 0$. We have that every X is Gorenstein-n-projective. Hence **(1)** follows.

The inclusion $\mathcal{P}_r(\mathcal{C}) \subseteq \mathcal{GP}_r(\mathcal{C}) \cap \mathcal{W}(\mathcal{C})$ is clear. Now let $X \in \mathcal{GP}_r(\mathcal{C}) \cap \mathcal{W}(\mathcal{C})$. Then every $K \in \Omega^r(X)$ is in $\mathcal{GP}_0(\mathcal{C})$. Since $\mathcal{W}(\mathcal{C})$ is thick, we also have $K \in \mathcal{W}(\mathcal{C})$. Then $K \in \mathcal{GP}_0 \cap \mathcal{W}(\mathcal{C}) = \mathcal{P}_0(\mathcal{C})$. It follows $X \in \mathcal{P}_r(\mathcal{C})$. Therefore, **(2)** holds. $\square$

We are ready to show that the classes $\mathcal{GP}_r(\mathcal{C})$ and $\mathcal{GI}_r(\mathcal{C})$ are the left and right halves, respectively, of two cotorsion pairs.

Proposition 4.3.4

Let $\mathcal{C}$ be a Gorenstein category.

$(\mathcal{GP}_r(\mathcal{C}), (\mathcal{GP}_r(\mathcal{C}))^{\perp})$ is a cotorsion pair for every $0 \leq r \leq FDI(\mathcal{C})$.

$({}^{\perp}(\mathcal{GI}_r(\mathcal{C})), \mathcal{GI}_r(\mathcal{C}))$ is a cotorsion pair in for every $0 \leq r \leq FDP(\mathcal{C})$.

Proof.

We only prove the left statement. It suffices to show that ${}^{\perp}((\mathcal{GP}_r(\mathcal{C}))^{\perp}) \subseteq \mathcal{GP}_r(\mathcal{C})$. Let $X \in {}^{\perp}((\mathcal{GP}_r(\mathcal{C}))^{\perp})$ and $0 \to K \to P_{r-1} \to \cdots \to P_1 \to P_0 \to X \to 0$ be a partial left projective resolution of X. By Proposition 4.3.2, we only need to show that K is Gorenstein-projective.

We use induction on r. Suppose $r = 1$ and let $W \in \mathcal{W}(\mathcal{C})$. We have the long exact sequence $\cdots \to \cancelto{0}{\mathrm{Ext}^1_{\mathcal{C}}(P_0, W)} \to \mathrm{Ext}^1_{\mathcal{C}}(K, W) \to \mathrm{Ext}^2_{\mathcal{C}}(X, W) \to \cdots$. On the other hand, $\mathrm{Ext}^2_{\mathcal{C}}(X, W) \cong \mathrm{Ext}^1_{\mathcal{C}}(X, L)$, where $L \in \Omega^{-1}(W)$. We show $L \in (\mathcal{GP}_1(\mathcal{C}))^{\perp}$. Let $Y \in \mathcal{GP}_1(\mathcal{C})$ and consider a short exact sequence $W \hookrightarrow I \twoheadrightarrow L$ where I is injective. Then $\cdots \to \cancelto{0}{\mathrm{Ext}^1_{\mathcal{C}}(Y, I)} \to \mathrm{Ext}^1_{\mathcal{C}}(Y, L) \to \cancelto{0}{\mathrm{Ext}^2_{\mathcal{C}}(Y, W)} \to \cdots$ is the derived long exact sequence, where $\mathrm{Ext}^2_{\mathcal{C}}(Y, W) = 0$ since $Y \in \mathcal{GP}_1(\mathcal{C})$ and $W \in \mathcal{W}(\mathcal{C})$. Then $\mathrm{Ext}^1_{\mathcal{C}}(Y, L) = 0$ for every $Y \in \mathcal{GP}_1(\mathcal{C})$, i.e. $L \in (\mathcal{GP}_1(\mathcal{C}))^{\perp}$. It follows $\mathrm{Ext}^2_{\mathcal{C}}(X, W) = 0$. Hence $\mathrm{Ext}^1_{\mathcal{C}}(K, W) = 0$ for every $W \in \mathcal{W}(\mathcal{C})$, i.e. $K \in \mathcal{GP}_0(\mathcal{C})$.

Suppose the result is true for every $1 \leq j \leq r - 1$. We have exact sequences $L \hookrightarrow P_0 \twoheadrightarrow X$ and $0 \to K \to P_{r-1} \to \cdots \to P_1 \to L \to 0$ for $L \in \Omega^1(X)$. If $Y \in (\mathcal{GP}_{r-1}(\mathcal{C}))^{\perp}$, we have $\mathrm{Ext}^1_{\mathcal{C}}(L, Y) \cong \mathrm{Ext}^1_{\mathcal{C}}(X, Y')$ where $Y' \in \Omega^{-1}(Y)$. Given $Z \in \mathcal{GP}_r(\mathcal{C})$, note $Z' \in \mathcal{GP}_{r-1}(\mathcal{C})$ for every $Z' \in \Omega^1(Z)$. We have $\mathrm{Ext}^1_{\mathcal{C}}(Z, Y') \cong \mathrm{Ext}^1_{\mathcal{C}}(Z', Y) = 0$. So $Y' \in (\mathcal{GP}_r(\mathcal{C}))^{\perp}$. It follows $\mathrm{Ext}^1_{\mathcal{C}}(L, Y) \cong \mathrm{Ext}^1_{\mathcal{C}}(X, Y') = 0$ for every $Y \in (\mathcal{GP}_{r-1}(\mathcal{C}))^{\perp}$. Hence $L \in {}^{\perp}((\mathcal{GP}_{r-1}(\mathcal{C}))^{\perp}) = \mathcal{GP}_{r-1}(\mathcal{C})$ and $X \in \mathcal{GP}_r(\mathcal{C})$. $\square$

Recall that in the third chapter we proved that $(\mathcal{P}_r, (\mathcal{P}_r)^{\perp})$ is a cotorsion pair, from the fact that every r-projective module is a transfinite extension of the set of κ-small r-projective modules, where $\kappa > \mathrm{Card}(R)$ is an infinite regular cardinal. Using the advantages provided by Gorenstein categories, we get the same result without constructing such transfinite extensions.

Corollary 4.3.5

If $\mathcal{C}$ is a Gorenstein category, then $(\mathcal{P}_r(\mathcal{C}), (\mathcal{P}_r(\mathcal{C}))^{\perp})$ is a cotorsion pair.

Proof.

We only need to show ${}^{\perp}((\mathcal{P}_r(\mathcal{C}))^{\perp}) \subseteq \mathcal{P}_r(\mathcal{C})$, which follows by the following implications:

$\mathcal{P}_r(\mathcal{C}) \subseteq \mathcal{GP}_r(\mathcal{C}) \Longrightarrow {}^{\perp}((\mathcal{P}_r(\mathcal{C}))^{\perp}) \subseteq {}^{\perp}((\mathcal{GP}_r(\mathcal{C}))^{\perp}) = \mathcal{GP}_r(\mathcal{C})$,

$\mathcal{P}_r(\mathcal{C}) \subseteq \mathcal{W}(\mathcal{C}) \Longrightarrow {}^{\perp}((\mathcal{P}_r(\mathcal{C}))^{\perp}) \subseteq {}^{\perp}((\mathcal{W}(\mathcal{C}))^{\perp}) = \mathcal{W}(\mathcal{C})$. $\square$

We know how to write $\mathcal{P}_r(\mathcal{C})$ and $\mathcal{I}_r(\mathcal{C})$ in terms of $\mathcal{GP}_r(\mathcal{C})$, $\mathcal{GI}_r(\mathcal{C})$ and $\mathcal{W}(\mathcal{C})$. Now we give similar equalities for $(\mathcal{GP}_r(\mathcal{C}))^{\perp}$ and ${}^{\perp}(\mathcal{GI}_r(\mathcal{C}))$.

Proposition 4.3.6

The following equalities hold in every Gorenstein category $\mathcal{C}$ and for every $0 \leq r \leq FDI(\mathcal{C})$:

$(\mathcal{GP}_r(\mathcal{C}))^{\perp} = (\mathcal{P}_r(\mathcal{C}))^{\perp} \cap \mathcal{W}(\mathcal{C})$.	${}^{\perp}(\mathcal{GI}_r(\mathcal{C})) = {}^{\perp}(\mathcal{I}_r(\mathcal{C})) \cap \mathcal{W}(\mathcal{C})$.

Proof.

The inclusion $(\mathcal{GP}_r(\mathcal{C}))^{\perp} \subseteq (\mathcal{P}_r(\mathcal{C}))^{\perp} \cap \mathcal{W}(\mathcal{C})$ follows as in the previous corollary. Now let $Y \in (\mathcal{P}_r(\mathcal{C}))^{\perp} \cap \mathcal{W}(\mathcal{C})$ and $X \in \mathcal{GP}_r(\mathcal{C})$. Since $(\mathcal{GP}_0(\mathcal{C}), \mathcal{W}(\mathcal{C}))$ is complete, there exists a short exact sequence $X \hookrightarrow W \twoheadrightarrow C$ with $C \in \mathcal{GP}_0(\mathcal{C})$ and $W \in \mathcal{W}(\mathcal{C})$. Note W also belongs to $\mathcal{GP}_r(\mathcal{C})$, and thus $W \in \mathcal{P}_r(\mathcal{C})$. We get an exact sequence $\cdots \to \cancelto{0}{\mathrm{Ext}^1_{\mathcal{C}}(W,Y)} \to \mathrm{Ext}^1_{\mathcal{C}}(X,Y) \to \cancelto{0}{\mathrm{Ext}^2_{\mathcal{C}}(C,Y)} \to \cdots$. It follows $Y \in (\mathcal{GP}_r(\mathcal{C}))^{\perp}$. $\square$

4.4 Gorenstein-r-projective model structures on modules over a Gorenstein ring

It is time to study the completeness of $(\mathcal{GP}_r(\mathcal{C}), (\mathcal{GP}_r(\mathcal{C}))^\perp)$ in the case $\mathcal{C} = {}_R\mathbf{Mod}$, where R is an n-Gorenstein ring. To simplify notation, we shall write $\mathcal{GP}_r({}_R\mathbf{Mod}) = \mathcal{GP}_r$. Notice that $({}^\perp\mathcal{W}, \mathcal{W})$ is a complete cotorsion pair, and that $(\mathcal{P}_r, (\mathcal{P}_r)^\perp)$ and $(\mathcal{GP}_r, (\mathcal{GP}_r)^\perp)$ are compatible. So by Proposition 2.3.6 we have that $(\mathcal{GP}_r, (\mathcal{GP}_r)^\perp)$ is cogenerated by a set, since $(\mathcal{P}_r, (\mathcal{P}_r)^\perp)$ is.

Proposition 4.4.1

If R is an n-Gorenstein ring, then $(\mathcal{GP}_r, (\mathcal{GP}_r)^\perp)$ is a complete cotorsion pair for every $0 \leq r \leq n$.

Therefore, the following result follows by Hovey's correspondence.

Theorem 4.4.2 (**Gorenstein-r-projective model structure**)

If R is an n-Gorenstein ring, then for each $0 \leq r \leq n$ there exists a unique Abelian model structure on ${}_R\mathbf{Mod}$ where the (trivial) cofibrations are the monomorphisms with cokernel in $\mathcal{GP}_r$ (resp. $\mathcal{P}_r$), the (trivial) fibrations are the epimorphisms with kernel in $(\mathcal{P}_r)^\perp$ (resp. $(\mathcal{GP}_r)^\perp$), and $\mathcal{W}$ is the class of trivial objects.

4.5 Gorenstein-r-injective model structures on Gorenstein categories

Unlike the previous section, we can obtain the dual of the previous theorem in every locally Noetherian Gorenstein category. In such a category $\mathcal{C}$, we know there are two compatible cotorsion pairs $({}^{\perp}(\mathcal{GI}_r(\mathcal{C})), \mathcal{GI}_r(\mathcal{C}))$ and $({}^{\perp}(\mathcal{I}_r(\mathcal{C})), \mathcal{I}_r(\mathcal{C}))$. Since the second pair is complete, we can use Proposition 2.3.6 to conclude that $({}^{\perp}(\mathcal{GI}_r(\mathcal{C})), \mathcal{GI}_r(\mathcal{C}))$ is complete. Then the theorem below follows. However, we take the opportunity to prove that $({}^{\perp}(\mathcal{GI}_r(\mathcal{C})), \mathcal{GI}_r(\mathcal{C}))$ is complete by providing a cogenerating set.

Theorem 4.5.1 (**Gorenstein-r-injective model structure** on $\mathcal{C}$)

Let $\mathcal{C}$ be a locally Noetherian Gorenstein category. Then for each $0 \leq r \leq FDP(\mathcal{C})$ there exists a unique Abelian model structure on $\mathcal{C}$, where the (trivial) fibrations are the epimorphisms with kernel in $\mathcal{GI}_r(\mathcal{C})$ (resp. $\mathcal{I}_r(\mathcal{C})$), the (trivial) cofibrations are the monomorphisms with cokernel in ${}^{\perp}(\mathcal{I}_r(\mathcal{C}))$ (resp. in ${}^{\perp}(\mathcal{GI}_r(\mathcal{C}))$), and $\mathcal{W}(\mathcal{C})$ is the class of trivial objects.

Theorem 4.5.2

If $\mathcal{C}$ is a locally Noetherian Gorenstein category, then $({}^{\perp}(\mathcal{GI}_r(\mathcal{C})), \mathcal{GI}_r(\mathcal{C}))$ is a cotorsion pair cogenerated by a set, for every $0 \leq r \leq FDP(\mathcal{C})$.

$\boxed{\textbf{Proof}}$.

Recall that $(\mathcal{W}(\mathcal{C}), \mathcal{GI}_0(\mathcal{C}))$ is cogenerated by the set $\mathcal{S}$ of all $S \in \Omega^i(J)$ where $i \geq 0$ and J runs over the set of all indecomposable injective objects of $\mathcal{C}$. Consider the set $\mathcal{S}(r)$ of all $S \in \Omega^i(J)$ where $i \geq r$ and J as above. We shall see that $({}^{\perp}(\mathcal{GI}_r(\mathcal{C})), \mathcal{GI}_r(\mathcal{C}))$ is cogenerated by $\mathcal{S}(r)$. First, we check that $\mathcal{S}(r) \subseteq {}^{\perp}(\mathcal{GI}_r(\mathcal{C}))$. Let $S \in \mathcal{S}(r)$ and consider $Y \in \mathcal{GI}_r(\mathcal{C})$. Then $S \in \Omega^i(J)$, for some $i \geq r$ and some indecomposable injective object J. We have $\mathrm{Ext}^1_{\mathcal{C}}(S, Y) \cong \mathrm{Ext}^{i+1}_{\mathcal{C}}(J, Y) = 0$, since $J \in \mathcal{W}(\mathcal{C})$, $Y \in \mathcal{GI}_r$ and $i + 1 \geq r + 1$.

Since $\mathcal{S}(r) \subseteq {}^{\perp}(\mathcal{GI}_r(\mathcal{C}))$ implies $\mathcal{GI}_r(\mathcal{C}) = ({}^{\perp}(\mathcal{GI}_r(\mathcal{C})))^{\perp} \subseteq (\mathcal{S}(r))^{\perp}$, it suffices to show that $(\mathcal{S}(r))^{\perp} \subseteq \mathcal{GI}_r(\mathcal{C})$. Let $Y \in (\mathcal{S}(r))^{\perp}$ and consider an exact partial right injective resolution of D, say $0 \to Y \to I^0 \to I^1 \to \cdots \to I^{r-1} \to D \to 0$. By Proposition 4.3.2, we only need to prove that D is Gorenstein-injective. Let $S \in \mathcal{S}$, i.e. $S \in \Omega^i(J)$ for some indecomposable injective object J and $i \geq 0$. Consider the short exact sequence $Y' \hookrightarrow I^{r-1} \twoheadrightarrow D$ where $Y' \in \Omega^{1-r}(Y)$. We have a long exact sequence $\cdots \to \cancelto{0}{\operatorname{Ext}^1_{\mathcal{C}}(S, I^{r-1})} \to \operatorname{Ext}^1_{\mathcal{C}}(S, D) \to \operatorname{Ext}^2_{\mathcal{C}}(S, Y') \to \cdots$, where $\operatorname{Ext}^2_{\mathcal{C}}(S, Y') \cong \operatorname{Ext}^{r+1}_{\mathcal{C}}(S, Y) \cong \operatorname{Ext}^1_{\mathcal{C}}(S', Y)$, where $S' \in \Omega^r(S)$. Since $S \in \Omega^i(J)$, we have $S' \in \Omega^r(\Omega^i(J)) = \Omega^{r+i}(J)$ with $r + i \geq r$, and so $S' \in \mathcal{S}(r)$. Hence $\operatorname{Ext}^2_{\mathcal{C}}(S, Y') \cong \operatorname{Ext}^1_{\mathcal{C}}(S', Y) = 0$. It follows $\operatorname{Ext}^1_{\mathcal{C}}(S, D) = 0$ for every $S \in \mathcal{S}$, i.e. D is Gorenstein-injective. $\square$

4.6 Gorenstein-flat dimension and model structures on modules

Throughout this section, R shall denote an n-Iwanaga-Gorenstein ring. We devote the next lines to study the Gorenstein-flat dimension and its relation to the notion of model structures. The idea, as for the other model structures we have obtained so far, is to construct to compatible complete cotorsion pairs concerning flat and Gorenstein-flat dimensions.

In (30), J. Gillespie and M. Hovey constructed the following new Abelian model structure on ${}_R\mathbf{Mod}$.

Definition 4.6.1. A left R-module $M \in {}_R\mathbf{Mod}$ is said to be <u>Gorenstein-flat</u> if there exists an exact sequence

$$\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

of flat modules with $M = \operatorname{Ker}(F^0 \to F^1)$, which is $(\mathcal{I}_0 \otimes_R -)$-exact sequence. We shall denote by $\mathcal{GF}_0$ the class of Gorentein flat modules.

Theorem 4.6.1 (**Gorenstein-flat model structure** on ${}_R\mathbf{Mod}$)

If R is an n-Gorenstein ring, then there exists a unique Abelian model structure on ${}_R\mathbf{Mod}$ where the (trivial) cofibrations are the monomorphisms with Gorenstein-flat (resp. flat) cokernel, the (trivial) fibrations are the epimorphisms with kernel in $(\mathcal{F}_0)^{\perp}$ (resp. in $(\mathcal{GF}_0)^{\perp}$), and $\mathcal{W}$ is the class of trivial objects.

In ${}_R\mathbf{Mod}$, with R a Gorenstein ring, we shall see it is possible to obtain exact left Gorenstein-flat resolutions for every left R-module. Then we can compute the Gorenstein-flat dimension of every module M, denoted by $\mathrm{Gfd}(M)$. We denote by $\mathcal{GF}_r$ the class of left r-$\mathcal{GF}_0$-modules. As it occurs with other homological dimensions, $\mathcal{GF}_r = \{M \in {}_R\mathbf{Mod} : \mathrm{Gfd}(M) \leq r\}$.

The goal of this section is to prove the following generalization of the previous theorem.

Theorem 4.6.2 (**Gorenstein-r-flat model structure** on ${}_R\mathbf{Mod}$)

If R is an n-Gorenstein ring, then for each $0 \leq r \leq n$ there exists a unique Abelian model structure on ${}_R\mathbf{Mod}$ where the (trivial) cofibrations are the monomorphisms with cokernel in $\mathcal{GF}_r$ (resp. in $\mathcal{F}_r$), the (trivial) fibrations are the epimorphisms with kernel in $(\mathcal{F}_r)^{\perp}$ (resp. $(\mathcal{GF}_r)^{\perp}$), and $\mathcal{W}$ is the class of trivial objects.

We recall the construction of the model structure described in Theorem 4.6.1, and later on we study the concept of the Gorenstein-flat dimension in order to obtain Theorem 4.6.2. Recall that the character module of $M \in_R \mathbf{Mod}$ is defined by the right R-module $M^+ := \mathrm{Hom}_R(M, \mathbb{Q}/\mathbb{Z})$ (Definition 3.1.8).

Lemma 4.6.3

(1) (21, Theorem 3.2.10) A left R-module F is flat if, and only if, its character module F^+ is an injective right R-module.

(2) [Corollary of **(1)**] A left R-module N is r-flat if, and only if, its character module N^+ is an r-injective right R-module.

Theorem 4.6.4 (see (21, Theorem 10.3.8))

Let R be an n-Gorenstein ring. Then the following conditions are equivalent:

(1) M is a Gorenstein-flat left R-module.

(2) M^+ is a Gorenstein-injective right R-module.

(3) $\operatorname{Tor}_i^R(I, M) = 0$ for all $i \geq 1$ and all injective right R-modules I.

(4) $\operatorname{Tor}_i^R(W, M) = 0$ for all $i \geq 1$ and all right R-modules $W \in \mathcal{W}$.

We shall call $(\mathcal{GF}_0)^\perp$ the class of Gorenstein-cotorsion modules. In (30), the authors mention that $(\mathcal{GF}_0, (\mathcal{GF}_0)^\perp)$ is a complete cotorsion pair. We shall prove this fact in this section, by giving a cogenerating set.

Proposition 4.6.5

$(\mathcal{GF}_0, (\mathcal{GF}_0)^\perp)$ is a cotorsion pair if R is an n-Gorenstein ring.

Proof.

It suffices to show ${}^\perp((\mathcal{GF}_0)^\perp) \subseteq \mathcal{GF}_0$. Let $M \in {}^\perp((\mathcal{GF}_0)^\perp)$. By Theorem 4.6.4, we only need to show $\operatorname{Tor}_1^R(W, M) = 0$ for every right R-module $W \in \mathcal{W}$. By Theorem 3.1.17, we have $\operatorname{Tor}_1^R(W, M)^+ = \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Tor}_1^R(W, M), \mathbb{Q}/\mathbb{Z}) \cong \operatorname{Ext}_R^1(M, \operatorname{Hom}_{\mathbb{Z}}(W, \mathbb{Q}/\mathbb{Z})) = \operatorname{Ext}_R^1(M, W^+)$ $(*)$. We also have $\operatorname{Ext}_R^1(E, W^+) \cong$

$\mathrm{Tor}_1^R(W, E)^+ = 0$ for every $E \in \mathcal{GF}_0$. It follows $W^+ \in (\mathcal{GF}_0)^\perp$ and hence $\mathrm{Ext}_R^1(M, W^+) = 0$. By the equality $(*)$, $\mathrm{Tor}_1^R(W, M) = 0$ for every $W \in \mathcal{W}$. $\square$

The next step is to construct for every Gorenstein-flat module a transfinite extension of the set of κ-small Gorenstein-flat modules, where κ is a infinite regular cardinal satisfying $\kappa \geq \mathrm{Card}(R)$. The procedure is very similar to that applied for the cotorsion pair $(\mathcal{F}_0, (\mathcal{F}_0)^\perp)$, where the main difference resides in a slight modification of the definition of pure submodules, specifically:

Definition 4.6.2. A submodule N of a left R-module M is said to be $\mathcal{W}$-pure if for every right R-module $W \in \mathcal{W}$, the sequence

$$0 \to W \otimes_R N \to W \otimes_R M \to W \otimes_R M/N \to 0$$

is exact. The short exact sequence $N \hookrightarrow M \twoheadrightarrow M/N$ is called a $\mathcal{W}$-pure exact sequence.

Proposition 4.6.6

Let S be a $\mathcal{W}$-pure submodule of a Gorenstein-flat left R-module E, with R an n-Gorenstein ring. Then S and E/S are also Gorenstein-flat.

Proof.

Consider the short exact sequence $S \hookrightarrow E \twoheadrightarrow E/S$. Then, given a right R-module $W \in \mathcal{W}$, we have the long exact sequence

$$\cdots \to \mathrm{Tor}_1^R(W, E/S) \to W \otimes_R S \to W \otimes_R E \to W \otimes_R E/S \to 0.$$

Since S is a $\mathcal{W}$-pure submodule of E, we have that the map $W \otimes_R S \to W \otimes_R E$ is injective. So $\mathrm{Tor}_1^R(W, E/S) = 0$ for every $W \in \mathcal{W}$. Hence E/S is Gorenstein-flat. Now we know $\mathrm{Tor}_1^R(W, E) = 0$ and $\mathrm{Tor}_1^R(W, E/S) = 0$. On the other hand, $\operatorname{Tor}_2^R(W, E/S) = \operatorname{Tor}_1^R(K, E/S)$ where $K \in \Omega^1(W)$. We are given an exact sequence $K \hookrightarrow P \twoheadrightarrow W$, where $K \in \mathcal{W}$ since $\mathcal{W}$ is thick, so $\operatorname{Tor}_1^R(K, E/S) = 0$. It follows $\operatorname{Tor}_2^R(W, E/S) = 0$. Since

$$\cdots \to \cancelto{0}{\operatorname{Tor}_2^R(W, E/S)} \to \operatorname{Tor}_1^R(W, S) \to \cancelto{0}{\operatorname{Tor}_1^R(W, E)} \to \cdots$$

is exact, we finally obtain $\operatorname{Tor}_1^R(W, S) = 0$ for every $W \in \mathcal{W}$. $\square$

Definition 4.6.3. We say a left R-module L is <u>$\mathcal{W}$-pure injective</u> if for every $\mathcal{W}$-pure exact sequence $N \hookrightarrow M \twoheadrightarrow M/N$, the sequence

$$0 \to \operatorname{Hom}_R(M/N, L) \to \operatorname{Hom}_R(M, L) \to \operatorname{Hom}_R(N, L) \to 0$$

is exact.

The proof of the following result is similar to Lemma 3.1.20.

Proposition 4.6.7

Given an infinite regular cardinal $\kappa \geq \operatorname{Card}(R)$, a Gorenstein-flat module E and an element $x \in E$, there exists a $\mathcal{W}$-pure submodule $S \subseteq E$ such that $x \in S$ and $\operatorname{Card}(S) \leq \kappa$.

Corollary 4.6.8

$(\mathcal{GF}_0, (\mathcal{GF}_0)^{\perp})$ is a cotorsion pair cogenerated by the set

$$(\mathcal{GF}_0)^{\leq \kappa} := \{S \in \mathcal{GF}_0 \, : \, \operatorname{Card}(S) \leq \kappa\}.$$

Remark 4.6.1. It is straightforward to show that Gorenstein-flat modules are closed under direct limits. Since $(\mathcal{GF}_0, (\mathcal{GF}_0)^{\perp})$ is a complete cotorsion pair with

$\mathcal{GF}_0$ closed under direct limits, it follows that $(\mathcal{GF}_0, (\mathcal{GF}_0)^{\perp})$ is a perfect cotorsion pair, i.e. every left R-module has a Gorenstein-flat cover and a Gorenstein-cotorsion cover (see (31) for details).

On the other hand, if we consider the class $(\mathcal{F}_0)^{\perp}$ of cotorsion modules, then $(\mathcal{F}_0, (\mathcal{F}_0)^{\perp})$ is a complete cotorsion pair cogenerated by $(\mathcal{F}_0)^{\leq\kappa}$, with κ as above (21, Proposition 7.4.3). The pairs $(\mathcal{GF}_0, (\mathcal{GF}_0)^{\perp})$ and $(\mathcal{F}_0, (\mathcal{F}_0)^{\perp})$ are compatible by the following result.

Proposition 4.6.9

Let R be an n-Gorenstein ring.

(1) (21, Corollary 10.3.4): The flat dimension of a Gorenstein-flat module is either zero or infinite. In other words, $\mathcal{F}_0 = \mathcal{GF}_0 \cap \mathcal{W}$.

(2) $(\mathcal{GF}_0)^{\perp} = (\mathcal{F}_0)^{\perp} \cap \mathcal{W}$.

Proof.

We only prove **(1)**, since **(2)** is similar to Proposition 4.3.6 (Also proven in (30, Proof of Theorem 3.12)). It is clear that $\mathcal{F}_0 \subseteq \mathcal{W}$. By Theorem 4.6.4, $\mathcal{F}_0 \subseteq \mathcal{GF}_0$. So $\mathcal{F}_0 \subseteq \mathcal{GF}_0 \cap \mathcal{W}$. Now let $E \in \mathcal{GF}_0 \cap \mathcal{W}$. By Theorem 4.6.4, E^+ is a Gorenstein-injective right R-module. On the other hand, we are given a left exact flat resolution $0 \to F_k \to F_{k-1} \to \cdots \to F_0 \to E \to 0$ for some $k \geq 0$, since $E \in \mathcal{W}$. Note that $\mathbb{Q}/\mathbb{Z}$ is an injective left $\mathbb{Z}$-module, so we get an exact sequence $0 \to E^+ \to {F_0}^+ \to \cdots \to {F_{k-1}}^+ \to {F_k}^+ \to 0$ after applying the exact functor $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$. By (21, Theorem 3.2.10), a left R-module is flat if, and only if, its character module is injective. Thus the previous sequence turns out to be an exact right injective resolution of E^+. It follows $E^+ \in \mathcal{W}$. So $E^+ \in \mathcal{GI}_0 \cap \mathcal{W} = \mathcal{I}_0$. Using again the theorem just cited, we conclude $E \in \mathcal{F}_0$. $\square$

We have obtained two compatible and complete cotorsion pairs $(\mathcal{GF}_0, (\mathcal{GF}_0)^\perp)$ and $(\mathcal{F}_0, (\mathcal{F}_0)^\perp)$. Hence, Theorem 4.6.1 follows by Hovey's correspondence.

Now we study the notion of Gorenstein-flat dimension. The completeness of $(\mathcal{GF}_0, (\mathcal{GF}_0)^\perp)$ allows us to construct exact left Gorenstein-flat resolutions for every left R-module. So the notion of Gorenstein-flat dimension makes sense in ${}_R\mathbf{Mod}$, with R a Gorenstein ring. The following proposition follows easily using the proposition above and basic homological algebra.

Theorem 4.6.10 (see (21, Proposition 11.7.5))

The following conditions are equivalent for every left R-module M over an n-Gorenstein ring R:

(1) $\mathrm{Gfd}(M) \leq r$.

(2) $\mathrm{Tor}_i^R(W, M) = 0$ for all $i \geq r+1$ and all $W \in \mathcal{W}$.

(3) $\mathrm{Tor}_i^R(I, M) = 0$ for all injective modules I and all $i \geq r+1$.

(4) Every rth Gorenstein-flat syzygy is Gorenstein-flat.

(5) Every rth flat syzygy of Gorenstein-flat.

(6) $\mathrm{Gid}(M^+) \leq r$.

Proposition 4.6.11

Let R be an n-Gorenstein ring. If $M \subseteq N$ is a $\mathcal{W}$-pure submodule of the left R-module N, then $\mathrm{Gfd}(M) \leq \mathrm{Gfd}(N)$.

Proof.

Suppose $\mathrm{Gfd}(N) = k < \infty$. We show $\mathrm{Tor}_{k+1}^R(W, M) = 0$ for every $W \in \mathcal{W}$. Let $W \in \mathcal{W}$ and $S \in \Omega^{k+1}(W)$. First, note that $S \in \mathcal{W}$, since $\mathcal{W}$ is thick. Consider a partial projective resolution of W, say $0 \to S \to P_k \to \cdots \to P_1 \to P_0 \to W \to 0$. We have the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & S \otimes_R M \arrow[r, "g"] \arrow[d, "l"'] & P_k \otimes_R M \arrow[r] \arrow[d, "r"] & \cdots \arrow[r] & P_0 \otimes_R M \arrow[r] \arrow[d] & W \otimes_R M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S \otimes_R N \arrow[r, "f"'] & P_k \otimes_R N \arrow[r] & \cdots \arrow[r] & P_0 \otimes_R N \arrow[r] & W \otimes_R N \arrow[r] & 0
\end{tikzcd}$$

Since $S, P_i \in \mathcal{W}$ for every $0 \leq i \leq k$ and M is a $\mathcal{W}$-pure submodule of N, we have that the columns are exact. Now consider the short exact sequence $S \hookrightarrow P_k \twoheadrightarrow S'$, where $S' \in \Omega^k(W)$. Then we have the derived long exact sequence $\cancel{\operatorname{Tor}_1^R(S', N)}^{\,0} \to S \otimes_R N \to P_k \otimes_R N \to S' \otimes_R N \to 0$, where $\operatorname{Tor}_1^R(S', N) = \operatorname{Tor}_{k+1}^R(W, N) = 0$. So f is monic. It follows that g is also monic, since $r \circ g = f \circ f$, where l and r are monic. Therefore, $\operatorname{Tor}_{k+1}^R(W, M) \cong \operatorname{Tor}_1^R(S', M) = 0$ and so $\operatorname{Gfd}(M) \leq k$. $\square$

As we did in Proposition 4.6.5, one can show that $(\mathcal{GF}_r, (\mathcal{GF}_r)^{\perp})$ is a cotorsion pair. We shall see that it is also complete. Recall that for every Gorenstein-flat module E and every $x \in E$, one can construct a $\mathcal{W}$-pure submodule $S \subseteq E$ with $\operatorname{Card}(S) \leq \kappa$, such that $x \in S$. One can apply the same reasoning to show that every submodule $E' \subseteq E$ with $\operatorname{Card}(E') \leq \kappa$ can be embedded into a $\mathcal{W}$-pure submodule $S \subseteq E$ with $\operatorname{Card}(S) \leq \kappa$. From this fact, one deduces the following result. The following lemma can be proven as Lemma 3.1.21. Then one can construct transfinite extensions of $(\mathcal{GF}_r)^{\leq \kappa}$ for every Gorenstein-r-flat module.

Lemma 4.6.12

Let R be an n-Gorenstein ring and M be in $\mathcal{GF}_r$ with a Gorenstein-flat resolution

$$(1) = (0 \to E_r \xrightarrow{f_r} E_{r-1} \to \cdots \to E_1 \xrightarrow{f_1} E_0 \xrightarrow{f_0} M \to 0)$$

and N be a submodule of M with $\mathrm{Card}(N) \leq \kappa$. Then there exists a Gorenstein-flat subresolution

$$0 \to S'_r \to S'_{r-1} \to \cdots \to S'_1 \to S'_0 \to N' \to 0$$

of (1) such that S'_k is a $\mathcal{W}$-pure submodule of E_k and $\mathrm{Card}(S'_k) \leq \kappa$, for every $0 \leq k \leq r$, and such that $N \subseteq N'$. Moreover, if N has a subresolution of (1)

$$0 \to S_r \to S_{r-1} \to \cdots \to S_1 \to S_0 \to N \to 0$$

where S_k is a $\mathcal{W}$-pure submodule of E_k with $\mathrm{Card}(S_k) \leq \kappa$, for every $0 \leq k \leq r$, then the above resolution of N' can be constructed in such a way that it contains the given resolution of N.

Theorem 4.6.13

If R is an n-Gorenstein ring, then $(\mathcal{GF}_r, (\mathcal{GF}_r)^{\perp})$ is a cotorsion pair cogenerated by $(\mathcal{GF}_r)^{\leq\kappa}$, for every $0 \leq r \leq n$.

As we did in the case $r = 0$, we can show that $\mathcal{F}_r = \mathcal{GF}_r \cap \mathcal{W}$ and $(\mathcal{GF}_r)^{\perp} = (\mathcal{F}_r)^{\perp} \cap \mathcal{W}$. Then we have two compatible and complete cotorsion pairs $(\mathcal{F}_r, (\mathcal{F}_r)^{\perp})$ and $(\mathcal{GF}_r, (\mathcal{GF}_r)^{\perp})$. Therefore, Theorem 4.6.2 follows.

4.7 Model structures on complexes over Gorenstein rings

In this section we present the analogues of the Gorenstein-r-projective and Gorenstein-r-flat model structures on the category of chain complexes over a Gorenstein ring.

The next result provides a characterization of the Gorenstein-projective and Gorenstein-injective chain complexes. This is also proven by García Rozas in (24, Theorem 3.3.5 & Corollary 3.2.3) for complexes over Iwanaga-Gorenstein rings. In the author's opinion, the proof given next is shorter and easier.

Proposition 4.7.1 (see (24, Theorem 3.3.5 & Corollary 3.2.3))

If $\mathcal{C}$ is a Gorenstein category, then:

$\mathcal{GP}_0(\mathbf{Ch}(\mathcal{C})) = \mathrm{dw}(\mathcal{GP}_0(\mathcal{C}))$.	$\mathcal{GI}_0(\mathbf{Ch}(\mathcal{C})) = \mathrm{dw}(\mathcal{GI}_0(\mathcal{C}))$.

Proof.

We only prove the left statement, since the right one is dual[1]. Suppose C is a Gorenstein-projective complex. For every $W \in \mathcal{W}(\mathcal{C})$, note that $D^{m+1}(W) \in \widetilde{\mathcal{W}(\mathcal{C})}$. So $0 = \mathrm{Ext}^1_{\mathbf{Ch}(\mathcal{C})}(C, D^{m+1}(W))$. By Proposition 1.6.2, we have the isomorphism $\mathrm{Ext}^1_{\mathbf{Ch}(\mathcal{C})}(C, D^{m+1}(W)) \cong \mathrm{Ext}^1_{\mathcal{C}}(C_m, W)$. Hence, C_m is Gorenstein-projective in $\mathcal{C}$, for every $m \in \mathbb{Z}$.

Now suppose X is a complex with $X_m \in \mathcal{GP}_0(\mathcal{C})$ for every $m \in \mathbb{Z}$. By know from the proof of Proposition 4.2.3 that it suffices to show that $\mathrm{Ext}^{i+1}_{\mathbf{Ch}(\mathcal{C})}(X, P) = 0$, for every $i \geq 0$ and for every projective complex P. Write $P = \bigoplus_{m \in \mathbb{Z}} D^{m+1}(Z_m(P))$. Note that in this case, $P = \prod_{m \in \mathbb{Z}} D^{m+1}(Z_m(P))$. We have $\mathrm{Ext}^{i+1}_{\mathbf{Ch}(\mathcal{C})}(X, P) \cong \mathrm{Ext}^{i+1}_{\mathbf{Ch}(\mathcal{C})}(X, \prod_{m \in \mathbb{Z}} D^{m+1}(Z_m(P))) \cong \prod_{m \in \mathbb{Z}} \mathrm{Ext}^{i+1}_{\mathbf{Ch}(\mathcal{C})}(X, D^{m+1}(Z_m(P)))$, where $0 = \mathrm{Ext}^{i+1}_{\mathcal{C}}(X_m, Z_m(P)) \cong \mathrm{Ext}^{i+1}_{\mathbf{Ch}(\mathcal{C})}(X, D^{m+1}(Z_m(P)))$ for every $m \in \mathbb{Z}$, by Proposition 1.6.2, since X_m is Gorenstein-projective in $\mathcal{C}$ and $Z_m(P)$ is projective. □

1. Note that we do not ask $\mathbf{Ch}(\mathcal{C})$ to be locally Noetherian, since indecomposable injective complexes are described in terms of indecomposable injective objects in $\mathcal{C}$.

If R is an n-Iwanaga-Gorenstein ring, we know by the previous example that $\mathbf{Ch}({}_R\mathbf{Mod})$ is a Gorenstein category. In this case, denote $\mathcal{GP}_0(\mathbf{Ch}({}_R\mathbf{Mod})) = \widehat{\mathcal{GP}_0}$ to simplify, and notice that $\mathcal{W}(\mathbf{Ch}({}_R\mathbf{Mod})) = \widetilde{\mathcal{W}}$. In this case, we can show that $(\widehat{\mathcal{GP}_0}, \widetilde{\mathcal{W}})$ is cogenerated by the union $\mathcal{X}$ of $\left\{\Sigma^k(S^0(R)) : k \in \mathbb{Z}\right\}$ and the of all nth syzygies $X \in \Omega^n(S^m(R/I))$ with $m \in \mathbb{Z}$ and I running over the set of left ideals of R. On the one hand, recall from the previous chapter that a complex W is exact if, and only if, it is right orthogonal to $\left\{\Sigma^k(S^0(R)) : k \in \mathbb{Z}\right\}$. On the other hand, $Z_m(W) \in \mathcal{W}$ if, and only if, $0 = \mathrm{Ext}^1_R(M, Z_m(W)) \cong \mathrm{Ext}^{n+1}_R(R/I, Z_m(W)) \cong \mathrm{Ext}^{n+1}(S^m(R/I), W) \cong \mathrm{Ext}^1(X, W)$, for some $M \in \Omega^n(R/I)$ and where $X \in \Omega^n(S^m(R/I))$ (recall Proposition 1.6.3).

The following lines are devoted to some comments on the Gorenstein-injective case. We can obtain decompositions as above of injective chain complexes in terms of indecomposable injective left R-modules. Let I be an injective chain complex. Then we can write $I \cong \bigoplus_{m\in\mathbb{Z}} D^{m+1}(Z_m(I))$. For every $m \in \mathbb{Z}$, $Z_m(I) \cong \bigoplus_{\alpha_m\in\Lambda_m} J_{\alpha_m}$, where J_{α_m} is an indecomposable injective module. Hence $D^{m+1}(Z_m(I)) \cong \bigoplus_{\alpha_m\in\Lambda_m} D^{m+1}(J_{\alpha_m})$, and so we have

$$I \cong \bigoplus_{m\in\mathbb{Z}}\left(\bigoplus_{\alpha_m\in\Lambda_m} D^{m+1}(J_{\alpha_m})\right) = \bigoplus\left\{D^{m+1}(J_{\alpha_m}) : (\alpha_{m+k})_{k\in\mathbb{Z}} \in \bigcup_{m\in\mathbb{Z}}\left(\prod_{k\in\mathbb{Z}}\Lambda_{m+k}\right)\right\},$$

where each $D^{m+1}(J_{\alpha_m})$ is an indecomposable injective complex.

A complex J is an indecomposable injective complex if, and only if, J is the disk complex of an indecomposable injective module. For if J is an indecomposable injective complex then write $J = \bigoplus_{m\in\mathbb{Z}} D^{m+1}(Z_m(J))$. Note that each $D^{m+1}(Z_m(J))$ is an injective complex, so it follows $D^{m_0+1}(Z_{m_0}(J)) = J$ for some $m_0 \in \mathbb{Z}$, and $D^{m+1}Z_m(J) = 0$ for every $m \neq m_0$. It is only left to show that $Z_{m_0}(J)$ is an indecomposable injective module. Suppose $Z_{m_0}(J) = A \oplus B$, where A and B are injective submodules of $Z_{m_0}(J)$. Then $J = D^{m_0+1}(A) \oplus D^{m_0+1}(B)$. Since $D^{m_0+1}(A)$ and $D^{m_0+1}(B)$ are injective complexes and J is indecomposable, we get $D^{m_0+1}(A) = J$ and $D^{m_0+1}(B) = 0$, or $D^{m_0+1}(B) = J$ and $D^{m_0+1}(A) = 0$. Then $A = Z_{m_0}(J)$ and $B = 0$, or $A = 0$ and $B = Z_{m_0}(J)$. Hence, $Z_{m_0}(J)$ is an indecomposable injective module. Now let $D^{m+1}(J)$ be a disk complex, where J is an indecomposable injective module. It is clear that $D^{m+1}(J)$ is an injective complex. Suppose $D^{m+1}(J) = X \oplus Y$, where X and Y are injective complexes. Then $J = A_m \oplus B_m$, $J = A_{m+1} \oplus B_{m+1}$, and $A_k, B_k = 0$ for every $k \neq m, m+1$. It follows $A_m = A_{m+1}$, $B_m = B_{m+1}$ and that A_m and B_m are injective modules. Since J is indecomposable, we get $A_m = J$ and $B_m = 0$, or $A_m = 0$ and $B_m = J$. Hence $A = D^{m+1}(J)$ and $B = 0$, or $A = 0$ and $B = D^{m+1}(J)$. In a similar way, one can show that these results hold for complexes over a locally Noetherian category.

With respect to Gorenstein-projective and Gorenstein-injective dimensions, we have the following characterization which follows from Proposition 4.7.1.

Corollary 4.7.2

If $\mathcal{C}$ is a Gorenstein category, then:

$\mathcal{GP}_r(\mathbf{Ch}(\mathcal{C})) = \mathrm{dw}(\mathcal{GP}_r(\mathcal{C}))$.	$\mathcal{GI}_r(\mathbf{Ch}(\mathcal{C})) = \mathrm{dw}(\mathcal{GI}_r(\mathcal{C}))$.

Proof.

We only prove the Gorenstein-projective case. Let X be a Gorenstein-r-projective chain complex. There exists an exact sequence

$$0 \to C_r \to C_{r-1} \to \cdots \to C_1 \to C_0 \to X \to 0$$

in $\mathbf{Ch}(\mathcal{C})$ such that C_i is Gorenstein-projective for every $0 \leq i \leq r$. For each $m \in \mathbb{Z}$, we have an exact sequence

$$0 \to (C_r)_m \to (C_{r-1})_m \to \cdots \to (C_1)_m \to (C_0)_m \to X_m \to 0$$

in $\mathcal{C}$. Since each C_i is Gorenstein-projective in $\mathbf{Ch}(\mathcal{C})$, we have $(C_i)_m$ is Gorenstein-projective in $\mathcal{C}$ by Proposition 4.7.1. It follows $X_m \in \mathcal{GP}_r(\mathcal{C})$.

Now suppose $X_m \in \mathcal{GP}_r(\mathcal{C})$ for every $m \in \mathbb{Z}$. Consider a partial exact left projective resolution

$$0 \to C_r \to P_{r-1} \to \cdots \to P_1 \to P_0 \to X \to 0.$$

It suffices to show that C_r is a Gorenstein-projective chain complex, by Proposition 4.3.2. For every integer m, we have an exact sequence

$$0 \to (C_r)_m \to (P_{r-1})_m \to \cdots \to (P_1)_m \to (P_0)_m \to X_m \to 0.$$

Note that each $(P_i)_m$ is projective in $\mathcal{C}$. Since $X_m \in \mathcal{GP}_r(\mathcal{C})$, we have $(C_r)_m \in \Omega^r(X_m) \subseteq \mathcal{GP}_0(\mathcal{C})$ by Proposition 4.3.2. Hence C_r is a Gorenstein-projective complex by Proposition 4.7.1. $\square$

Let's study the completeness of $(\mathcal{GP}_r(\mathcal{C}), (\mathcal{GP}_r(\mathcal{C}))^\perp)$ in the case $\mathcal{C} = \mathbf{Ch}({}_R\mathbf{Mod})$, with R an n-Gorenstein ring. We write $\mathcal{GP}_r(\mathbf{Ch}({}_R\mathbf{Mod})) = \widehat{\mathcal{GP}_r}$ to simplify the notation. As we did in Section 4.4, we can deduce that $(\widehat{\mathcal{GP}_r}, (\widehat{\mathcal{GP}_r})^\perp)$ is complete, since the pair $(\widetilde{\mathcal{P}_r}, (\widetilde{\mathcal{P}_r})^\perp)$ is also complete (See Theorem 3.2.3).

Proposition 4.7.3

If R is an n-Gorenstein ring, then for each $0 \leq r \leq n$ the pair $(\widehat{\mathcal{GP}_r}, (\widehat{\mathcal{GP}_r})^\perp)$ is complete.

Hence, the following result follows by Hovey's correspondence.

Theorem 4.7.4 (Gorenstein-r-projective model structure)

If R is an n-Gorenstein ring, then for each $0 \leq r \leq n$ there exists a unique Abelian model structure on $\mathbf{Ch}({}_R\mathbf{Mod})$ where the (trivial) cofibrations are the monomorphisms with cokernel in $\widehat{\mathcal{GP}_r}$ (resp. $\widetilde{\mathcal{P}_r}$), the (trivial) fibrations are the epimorphisms with kernel in $(\widetilde{\mathcal{P}_r})^{\perp}$ (resp. $(\widehat{\mathcal{GP}_r})^{\perp}$), and $\widetilde{\mathcal{W}}$ is the class of trivial objects.

We devote the rest of the section to study the Gorenstein-flat dimension and its relationship with to the notion of model structures in the context of chain complexes. We shall need to work with a special extension functor $\overline{\mathrm{Ext}}(-,-)$ derived from $\overline{\mathrm{Hom}}(-,-)$. Cotorsion pairs defined by orthogonality in the sense of this extension functor shall be called <u>bar-cotorsion pairs</u>.

Definition 4.7.1. Let X and Y be two chain complexes in $\mathbf{Ch}({}_R\mathbf{Mod})$. From $\mathrm{Hom}'(X,Y)$ (see Definition 2.4.2) we construct the <u>bar-Hom functor</u> by setting the complex $\overline{\mathrm{Hom}}(X,Y)$ as

$$\overline{\mathrm{Hom}}(X,Y)_n := Z_n(\mathrm{Hom}'(X,Y)), \text{ for every } n \in \mathbb{Z}.$$

Let $f = (f_k)_{k\in\mathbb{Z}} \in \overline{\mathrm{Hom}}(X,Y)_n$. Note that for $n = 0$ we have $\partial_k^Y \circ f_k = f_{k-1} \circ \partial_k^X$, i.e. f is a chain map and $\overline{\mathrm{Hom}}(X,Y)_0 = \mathrm{Hom}(X,Y)$. For $n = 1$, we have $\partial_{k+1}^Y \circ f_k + f_{k-1} \circ \partial_k^X = 0$, i.e. f is a chain homotopy from 0 to 0. The set of homomorphisms $f = (f_k)_{k\in\mathbb{X}} \in \overline{\mathrm{Hom}}(X,Y)_n$ is known as a <u>map of degree n</u>. The boundary maps of the complex $\overline{\mathrm{Hom}}(X,Y)$ are given by

$$\partial_n^{\overline{\mathrm{Hom}}(X,Y)}(f) := (\partial_{k+n}^Y \circ f_k)_{k\in\mathbb{Z}}, \text{ for every } f \in \overline{\mathrm{Hom}}(X,Y)_n.$$

We shall denote the right derived functor obtained from $\overline{\mathrm{Hom}}(-,-)$ by $\overline{\mathrm{Ext}}^i(-,-)$.

Lemma 4.7.5 (Mentioned in (24, Page 87), no proof given)

For every pair of complexes $X, Y \in \mathbf{Ch}({}_R\mathbf{Mod})$ and for every $i \geq 0$, $\overline{\mathrm{Ext}}^i(X, Y)$ is a complex of the form

$$\cdots \to \mathrm{Ext}^i(X, \Sigma^{k+1}(Y)) \to \mathrm{Ext}^i(X, \Sigma^k(Y)) \to \mathrm{Ext}^i(X, \Sigma^{k-1}(Y)) \to \cdots .$$

Proof.

We use induction on i. It suffices to prove the case $i = 0$, i.e. we need to show that $\overline{\mathrm{Hom}}(X, Y)$ is the chain complex given by

$$\cdots \to \mathrm{Hom}(X, \Sigma^{k+1}(Y)) \to \mathrm{Hom}(X, \Sigma^k(Y)) \to \mathrm{Hom}(X, \Sigma^{k-1}(Y)) \to \cdots .$$

Every element $f = (f_k)_{k\in\mathbb{Z}} \in Z_m\mathrm{Hom}'(X, Y)$ satisfies the equality $\partial^Y_{k+m} \circ f_k = (-1)^m f_{k-1} \circ \partial^X_k$. On the other hand, $g \in \mathrm{Hom}(X, \Sigma^{-m}(Y))$ makes the following diagram commute for every $k \in \mathbb{Z}$:

$$\begin{tikzcd}
X_k \arrow[r, "\partial^X_k"] \arrow[d, "g_k"'] & X_{k-1} \arrow[d, "g_{k-1}"] \\
Y_{m+k} \arrow[r, "(-1)^m\partial^Y_{m+k}"'] & Y_{m+k-1}
\end{tikzcd}$$

Then $Z_m(\mathrm{Hom}'(X, Y)) = \mathrm{Hom}(X, \Sigma^{-m}(Y))$ for every $m \in \mathbb{Z}$. Moreover, the differential map $\delta_m : \mathrm{Hom}(X, \Sigma^{-m}(Y)) \to \mathrm{Hom}(X, \Sigma^{-m+1}(Y))$ is given by $\delta(g) = (\partial^Y_{m+k} \circ g_k)_{k\in\mathbb{Z}}$. The result follows.

Now consider a short exact sequence $K \hookrightarrow P \twoheadrightarrow X$, where P is a projective chain complex. We have a commutative diagram:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & \mathrm{Hom}(X,\Sigma^{-m-1}(Y)) \arrow[r] \arrow[d] & \mathrm{Hom}(P,\Sigma^{-m-1}(Y)) \arrow[r] \arrow[d] & \mathrm{Hom}(K,\Sigma^{-m-1}(Y)) \arrow[r] \arrow[d] & \mathrm{Ext}^1(X,\Sigma^{-m-1}(Y)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}(X,\Sigma^{-m}(Y)) \arrow[r] \arrow[d] & \mathrm{Hom}(P,\Sigma^{-m}(Y)) \arrow[r] \arrow[d] & \mathrm{Hom}(K,\Sigma^{-m}(Y)) \arrow[r] \arrow[d] & \mathrm{Ext}^1(X,\Sigma^{-m}(Y)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}(X,\Sigma^{-m+1}(Y)) \arrow[r] \arrow[d] & \mathrm{Hom}(P,\Sigma^{-m+1}(Y)) \arrow[r] \arrow[d] & \mathrm{Hom}(K,\Sigma^{-m+1}(Y)) \arrow[r] \arrow[d] & \mathrm{Ext}^1(X,\Sigma^{-m+1}(Y)) \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Also, there is an exact sequence

$$0 \to \overline{\mathrm{Hom}}(X,Y) \to \overline{\mathrm{Hom}}(P,Y) \to \overline{\mathrm{Hom}}(K,Y) \to \overline{\mathrm{Ext}}^1(X,Y) \to 0.$$

It follows $\overline{\mathrm{Ext}}^1(X,Y)$ is the complex given by the right column of the previous diagram. $\square$

Some interesting results arise when we replace Ext by $\overline{\mathrm{Ext}}$ in the definition of cotorsion pair.

Definition 4.7.2. Given two classes $\mathcal{A}$ and $\mathcal{B}$ of chain complexes in $\mathbf{Ch}({}_R\mathbf{Mod})$, we shall say that $\mathcal{A}$ and $\mathcal{B}$ form a <u>bar-cotorsion pair</u> $(\mathcal{A} \mid \mathcal{B})$ if:

(1) $\mathcal{A} = {}^{\overline{\perp}}\mathcal{B} = \{A \in \mathbf{Ch}({}_R\mathbf{Mod}) : \overline{\mathrm{Ext}}^1(A,B) = 0, \text{ for every } B \in \mathcal{B}\}$, and

(2) $\mathcal{B} = \mathcal{A}^{\overline{\perp}} = \{B \in \mathbf{Ch}({}_R\mathbf{Mod}) : \overline{\mathrm{Ext}}^1(A,B) = 0, \text{ for every } A \in \mathcal{A}\}$.

Definition 4.7.3. A class $\mathcal{D}$ of complexes is said to be <u>closed under suspensions</u> if $\Sigma^k(D) \in \mathcal{D}$ for every $D \in \mathcal{D}$ and every $k \in \mathbb{Z}$.

Lemma 4.7.6

Let $(\mathcal{A} \mid \mathcal{B})$ be a bar-cotorsion pair. Then $\mathcal{A}$ is closed under suspensions if, and only if, $\mathcal{B}$ is. The same result holds if $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair.

Proof.

We only prove $(\Longrightarrow)$. Suppose $(\mathcal{A} \mid \mathcal{B})$ is a bar-cotorsion pair and that $\mathcal{A}$ is closed under suspensions. First, note that $\mathrm{Ext}^i(X, \Sigma^k(Y))$ and $\mathrm{Ext}^i(\Sigma^{-k}(X), Y)$ are isomorphic, for every pair of complexes X and Y and every $k \in \mathbb{Z}$. Let $B \in \mathcal{B}$, $A \in \mathcal{A}$ and $m \in \mathbb{Z}$. The complexes

$$\cdots \to \mathrm{Ext}^1(A, \Sigma^{k+1}(\Sigma^m(B))) \to \mathrm{Ext}^1(A, \Sigma^k(\Sigma^m(B))) \to \mathrm{Ext}^1(A, \Sigma^{k-1}(\Sigma^m(B))) \to \cdots,$$

$$\cdots \to \mathrm{Ext}^1(\Sigma^{-m}(A), \Sigma^{k+1}(B)) \to \mathrm{Ext}^1(\Sigma^{-m}(A), \Sigma^k(B)) \to \mathrm{Ext}^1(\Sigma^{-m}(A), \Sigma^{k-1}(B)) \to \cdots$$

are isomorphic. Then by the previous lemma $\overline{\mathrm{Ext}}^1(A, \Sigma^m(B)) \cong \overline{\mathrm{Ext}}^1(\Sigma^{-m}(A), B) = 0$, for every $A \in \mathcal{A}$, since $B \in \mathcal{B}$ and $\mathcal{A}$ is closed under suspensions. Hence the result follows. $\square$

Theorem 4.7.7

Let $\mathcal{A}$ and $\mathcal{B}$ be two classes in $\mathbf{Ch}({}_R\mathbf{Mod})$ such that $\mathcal{A}$ is closed under suspensions. Then $(\mathcal{A} \mid \mathcal{B})$ is a bar-cotorsion pair if, and only if, $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair.

Proof.

Suppose $(\mathcal{A} \mid \mathcal{B})$ is a bar-cotorsion pair. Let $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Then $\overline{\mathrm{Ext}}^1(A, B) = 0$. By Lemma 4.7.5, we have $\mathrm{Ext}^1(A, B) = 0$ and so $\mathcal{A} \subseteq {}^\perp\mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{A}^\perp$. Now if $A \in {}^\perp\mathcal{B}$, we have $\mathrm{Ext}^1(A, B) = 0$ for every $B \in \mathcal{B}$. Since $\mathcal{B}$ is closed under suspensions by Lemma 4.7.6, we have $\Sigma^k(B) \in \mathcal{B}$ for every $k \in \mathbb{Z}$. Then $\mathrm{Ext}^1(A, \Sigma^k(B)) = 0$ for every $k \in \mathbb{Z}$. It follows $\overline{\mathrm{Ext}}^1(A, B) = 0$ for every

$B \in \mathcal{B}$ and so ${}^{\perp}\mathcal{B} \subseteq {}^{\overline{\perp}}\mathcal{B} = \mathcal{A}$. Now we show $\mathcal{B} \supseteq \mathcal{A}^{\perp}$. Let $B \in \mathcal{A}^{\perp}$. Then $\mathrm{Ext}^1(A, B) = 0$ for every $A \in \mathcal{A}$. Since $\mathcal{A}$ is closed under suspensions, we have $\mathrm{Ext}^1(\Sigma^{-k}(A), B) = 0$ for every $k \in \mathbb{Z}$. Then $\overline{\mathrm{Ext}}^1(A, B) \cong \mathrm{Ext}^1(A, \Sigma^k(B)) = 0$ for every $k \in \mathbb{Z}$, and so $B \in \mathcal{A}^{\overline{\perp}} = \mathcal{B}$. The converse can be proven in a similar way. $\square$

Definition 4.7.4. A bar-cotorsion pair $(\mathcal{A} \mid \mathcal{B})$ is <u>bar-cogenerated</u> by a set $\mathcal{S} \subseteq \mathcal{A}$ if $\mathcal{B} = \mathcal{S}^{\overline{\perp}}$.

Theorem 4.7.8

Let $\mathcal{A}$ and $\mathcal{B}$ be two classes in $\mathbf{Ch}({}_R\mathbf{Mod})$, and $\mathcal{S} \subseteq \mathcal{A}$ be a set closed under suspensions. If $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair cogenerated by $\mathcal{S}$, then $(\mathcal{A} \mid \mathcal{B})$ is a bar-cotorsion pair bar-cogenerated by $\mathcal{S}$. The converse is also true.

Proof.

Suppose $(\mathcal{A}, \mathcal{B})$ is cogenerated by $\mathcal{S}$. Then $\mathcal{B} = \mathcal{S}^{\perp}$, which is closed under suspensions by Lemma 4.7.6 (since $\mathcal{S}$ is closed under suspensions). By the same lemma, $\mathcal{A}$ is also closed under suspensions, and hence by Theorem 4.7.7 we have $(\mathcal{A} \mid \mathcal{B})$ is a bar-cotorsion pair. It is only left to show that $\mathcal{B} = \mathcal{S}^{\overline{\perp}}$. Let $B \in \mathcal{B}$. Then $\mathrm{Ext}^1(S, B) = 0$ for every $S \in \mathcal{S}$. Since $\mathcal{S}$ is closed under suspensions, we have $\Sigma^{-k}(S) \in \mathcal{S}$ for every $k \in \mathbb{Z}$ and every $S \in \mathcal{S}$. Then $\mathrm{Ext}^1(S, \Sigma^k(B)) \cong \mathrm{Ext}^1(\Sigma^{-k}(S), B) = 0$, for every $k \in \mathbb{Z}$ and every $S \in \mathcal{S}$. It follows $\overline{\mathrm{Ext}}^1(S, B) = 0$ for every $S \in \mathcal{S}$, i.e. $B \in \mathcal{S}^{\overline{\perp}}$. Now if $D \in \mathcal{S}^{\overline{\perp}}$ then $\mathrm{Ext}^1(S, D) = 0$ for every $S \in \mathcal{S}$ and so $D \in \mathcal{S}^{\perp} = \mathcal{B}$. The converse follows similarly. $\square$

Lemma 4.7.9 (Eklof's Lemma for $\overline{\mathrm{Ext}}^1(-,-)$)

Let Y be a chain complex in $\mathbf{Ch}({}_R\mathbf{Mod})$. If X is a transfinite extension $(X_\alpha)_{\alpha<\lambda}$ of the class ${}^{\perp}\{Y\}$, then $\overline{\mathrm{Ext}}^1(X, Y) = 0$.

Proof.

It suffices to note that $\overline{\mathrm{Ext}}^1(X_{\alpha+1}/X_\alpha, Y) = 0$ implies $\mathrm{Ext}^1(X_{\alpha+1}/X_\alpha, \Sigma^k(Y)) = 0$, for every $k \in \mathbb{Z}$. By the original Eklof's Lemma, $\mathrm{Ext}^1(X, \Sigma^k(Y)) = 0$ for every $k \in \mathbb{Z}$, and so $\overline{\mathrm{Ext}}^1(X, Y) = 0$. $\square$

Proposition 4.7.10 (Proposition 3.1.14 for bar-cotorsion pairs)

Let $(\mathcal{A} \mid \mathcal{B})$ be a bar-cotorsion pair in $\mathbf{Ch}({}_R\mathbf{Mod})$, and $\mathcal{S} \subseteq \mathcal{A}$ be a set. If every $X \in \mathcal{A}$ has an $\mathcal{S}$-filtration, then $(\mathcal{A} \mid \mathcal{B})$ is bar-cogenerated by $\mathcal{S}$.

Assume throughout the rest of this section that R is a commutative ring.

Definition 4.7.5. A chain complex $X \in \mathbf{Ch}({}_R\mathbf{Mod})$ is said to be Gorenstein-flat if there exists a $(\widetilde{\mathcal{I}_0}\overline{\otimes}-)$-exact sequence $\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$ of flat complexes such that $X = \mathrm{Ker}(F^0 \to F^1)$. We shall denote the class of all Gorenstein-flat complexes by $\widehat{\mathcal{GF}}_0$.

Definition 4.7.6. Given a complex $X = \cdots \to X_{m+1} \xrightarrow{\partial^X_{m+1}} X_m \xrightarrow{\partial^X_m} X_{m-1} \to \cdots$ in $\mathbf{Ch}({}_R\mathbf{Mod})$, the Pontryagin or character complex of X is the complex $X^+ \in \mathbf{Ch}(\mathbf{Mod}_R)$ given by

$$X^+ := (\cdots \to \mathrm{Hom}_{\mathbb{Z}}(X_{-m-1}, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\partial^{X^+}_m} \mathrm{Hom}_{\mathbb{Z}}(X_{-m}, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\partial^{X^+}_{m-1}} \mathrm{Hom}_{\mathbb{Z}}(X_{-m+1}, \mathbb{Q}/\mathbb{Z}) \to \cdots)$$

where the boundary maps are defined by $\partial^{X^+}_m := (-1)^{m-1}\mathrm{Hom}_{\mathbb{Z}}(\partial^X_{-m}, \mathbb{Q}/\mathbb{Z})$.

Proposition 4.7.11

$X^+ \cong \overline{\mathrm{Hom}}_{\mathbb{Z}}(X, D^0(\mathbb{Q}/\mathbb{Z}))$.

Proof.

Recall $\overline{\mathrm{Hom}}(X, D^0(\mathbb{Q}/\mathbb{Z}))_m$ is the kernel of the differential

$$\mathrm{Hom}'(X, D^0(\mathbb{Q}/\mathbb{Z}))_m \xrightarrow{\delta_m} \mathrm{Hom}'(X, D^0(\mathbb{Q}/\mathbb{Z})).$$

Moreover,

$$\begin{aligned}
\mathrm{Hom}'(X, D^0(\mathbb{Q}/\mathbb{Z}))_m &= \prod_{k\in\mathbb{Z}} \mathrm{Hom}_Z(X_k, D^0(\mathbb{Q}/\mathbb{Z})_{k+m}) \\
&= \mathrm{Hom}_{\mathbb{Z}}(X_{-m-1}, \mathbb{Q}/\mathbb{Z}) \times \mathrm{Hom}_{\mathbb{Z}}(X_{-m}, \mathbb{Q}/\mathbb{Z}), \\
\mathrm{Hom}'(X, D^0(\mathbb{Q}/\mathbb{Z}))_{m-1} &= \prod_{k\in\mathbb{Z}} \mathrm{Hom}_Z(X_k, D^0(\mathbb{Q}/\mathbb{Z})_{k+m-1}) \\
&= \mathrm{Hom}_{\mathbb{Z}}(X_{-m}, \mathbb{Q}/\mathbb{Z}) \times \mathrm{Hom}_{\mathbb{Z}}(X_{-m+1}, \mathbb{Q}/\mathbb{Z}).
\end{aligned}$$

Every $f = (f_k)_{k\in\mathbb{Z}} \in \mathrm{Hom}(X, D^0(\mathbb{Q}/\mathbb{Z}))_m$ has the form $f = (\cdots, 0, f_{-m-1}, f_{-m}, 0, \cdots)$. Now suppose $\delta_m(f) = 0$. Then we have $\partial_{k+m}^{D^0(\mathbb{Q}/\mathbb{Z})} \circ f_k - (-1)^m f_{k-1} \circ \partial_k^X = 0$ for every $k \in \mathbb{Z}$. In particular:

- For $k = -m$: $0 = \partial_0^{D^0(\mathbb{Q}/\mathbb{Z})} \circ f_{-m} - (-1)^m f_{-m-1} \circ \partial_{-m}^X$ and so we have $f_{-m} = (-1)^m f_{-m-1} \circ \partial_{-m}^X$.
- For $k = -m+1$: $0 = \partial_1^{D^0(\mathbb{Q}/\mathbb{Z})} \circ f_{-m+1} - (-1)^m f_{-m} \circ \partial_{-m+1}^X = f_{-m} \circ \partial_{-m+1}^X$.

We have the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & X_{-m+1} \arrow[r, "\partial_{-m+1}^X"] \arrow[d] & X_{-m} \arrow[r, "\partial_{-m}^X"] \arrow[d, "f_{-m}"] & X_{-m-1} \arrow[r] \arrow[d, "f_{-m-1}"] & X_{-m-2} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & \mathbb{Q}/\mathbb{Z} \arrow[r, equal] & \mathbb{Q}/\mathbb{Z} \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Define a map $\varphi_m : \overline{\mathrm{Hom}}(X, D^0(\mathbb{Q}/\mathbb{Z}))_m \longrightarrow (X^+)_m = \mathrm{Hom}_{\mathbb{Z}}(X_{-m-1}, \mathbb{Q}/\mathbb{Z})$ by setting $\varphi_m(f) = (-1)^m f_{-m-1}$ for every $m \in \mathbb{Z}$. It is clear that φ_m is an isomorphism. We show that $\varphi = (\varphi_m)_{m\in\mathbb{Z}}$ is a chain map, i.e. that the following diagram commutes for every $m \in \mathbb{Z}$:

$$\begin{tikzcd}
\overline{\mathrm{Hom}}(X, D^0(\mathbb{Q}/\mathbb{Z}))_m \arrow[r, "\partial_m^{\overline{\mathrm{Hom}}(X,D^0(\mathbb{Q}/\mathbb{Z}))}"] \arrow[d, "\varphi_m"'] & \overline{\mathrm{Hom}}(X, D^0(\mathbb{Q}/\mathbb{Z}))_{m-1} \arrow[d, "\varphi_{m-1}"] \\
(X^+)_m \arrow[r, "\partial_m^{X^+}"'] & (X^+)_{m-1}
\end{tikzcd}$$

For $f = (f_k)_{k\in\mathbb{Z}} \in \overline{\mathrm{Hom}}(X, D^0(\mathbb{Q}/\mathbb{Z}))_m$ we have:

$$\begin{aligned}
\partial_m^{X^+} \circ \varphi_m(f) &= (-1)^{m-1} \cdot (-1)^m \mathrm{Hom}_{\mathbb{Z}}(\partial_{-m}^X, \mathbb{Q}/\mathbb{Z})(f_{-m-1}) \\
&= (-1)^{m-1} \cdot \left[(-1)^m f_{-m-1} \circ \partial_{-m}^X\right] \\
&= (-1)^{m-1} f_{-m} = \varphi_{m-1}(\cdots, 0, f_{-m}, 0, \cdots) \\
&= \varphi_{m-1}((\partial_{m+k}^{D^0(\mathbb{Q}/\mathbb{Z})} \circ f_k)_{k\in\mathbb{Z}}) \\
&= \varphi_{m-1} \circ \partial_m^{\overline{\mathrm{Hom}}(X,D^0(\mathbb{Q}/\mathbb{Z}))}(f).
\end{aligned}$$
□

Proposition 4.7.12

Given three chain complexes X, Y and Z, we have the following isomorphisms of complexes:

(1) $\overline{\mathrm{Hom}}(X \overline{\otimes} Y, Z) \cong \overline{\mathrm{Hom}}(X, \overline{\mathrm{Hom}}(Y, Z))$.

(2) If R is commutative, then $X \overline{\otimes} Y \cong Y \overline{\otimes} X$.

(3) $X \overline{\otimes} (Y \overline{\otimes} Z) \cong (X \overline{\otimes} Y) \overline{\otimes} Z$.

Moreover, for every $i > 0$:

(4) $\overline{\mathrm{Ext}}^i(X, Y^+) \cong \overline{\mathrm{Tor}}_i(X, Y)^+$.

(5) If R is commutative, then $\overline{\mathrm{Tor}}_i(X, Y) \cong \overline{\mathrm{Tor}}_i(X, Y)$.

$\boxed{\textbf{Proof}}$.

Parts **(1)**, **(2)** and **(3)** are proven in (24, Proposition 4.2.1). We only prove **(4)**, since **(5)** follows similarly. The case $i = 1$ is stated in (24, Lemma 5.4.2). We use induction on $i > 0$. Suppose $i = 1$ and consider an exact sequence $K \hookrightarrow P \twoheadrightarrow X$ with P projective. Derive $\overline{\mathrm{Hom}}(-, Y^+)$ and $-\overline{\otimes}Y$ to obtain long exact sequences

$$0 \to \overline{\mathrm{Hom}}(X, Y^+) \to \overline{\mathrm{Hom}}(P, Y^+) \to \overline{\mathrm{Hom}}(K, Y^+) \to \overline{\mathrm{Ext}}^1(X, Y^+) \to \cdots$$

and

$$\cdots \to \overline{\mathrm{Tor}}_1(X, Y) \to K\overline{\otimes}Y \to P\overline{\otimes}Y \to X\overline{\otimes}Y \to 0.$$

Then apply $\overline{\mathrm{Hom}}(-, D^0(\mathbb{Q}/\mathbb{Z}))$ to the first sequence, since $D^0(\mathbb{Q}/\mathbb{Z})$ is an injective chain complex, we get the long exact sequence

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & \overline{\mathrm{Hom}}(X\overline{\otimes}Y, D^0(\mathbb{Q}/\mathbb{Z})) = (X\overline{\otimes}Y)^+ \arrow[r] & \overline{\mathrm{Hom}}(P\overline{\otimes}Y, D^0(\mathbb{Q}/\mathbb{Z})) = (P\overline{\otimes}Y)^+ \arrow[dl, out=-10, in=170] \\
& \overline{\mathrm{Hom}}(K\overline{\otimes}Y, D^0(\mathbb{Q}/\mathbb{Z})) = (K\overline{\otimes}Y)^+ \arrow[r] & \overline{\mathrm{Hom}}(\overline{\mathrm{Tor}}_1(X, Y), D^0(\mathbb{Q}/\mathbb{Z})) = \overline{\mathrm{Tor}}_1(X, Y)^+ \arrow[r] & \cdots
\end{tikzcd}$$

Using the first isomorphism of Proposition 4.7.12, we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & (X\overline{\otimes}Y)^+ \arrow[r] \arrow[d, "\cong"] & (P\overline{\otimes}Y)^+ \arrow[r] \arrow[d, "\cong"] & (K\overline{\otimes}Y)^+ \arrow[r] \arrow[d, "\cong"] & \overline{\mathrm{Tor}}_1(X, Y)^+ \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \overline{\mathrm{Hom}}(X, Y^+) \arrow[r] & \overline{\mathrm{Hom}}(P, Y^+) \arrow[r] & \overline{\mathrm{Hom}}(K, Y^+) \arrow[r] & \overline{\mathrm{Ext}}^1(X, Y^+) \arrow[r] & \cdots
\end{tikzcd}$$

By the Five Lemma, $\overline{\mathrm{Tor}}_1(X, Y)^+ \to \overline{\mathrm{Ext}}^1(X, Y^+)$ is an isomorphism. Now suppose $\overline{\mathrm{Tor}}_{i-1}(X, Y)^+ \cong \overline{\mathrm{Ext}}^{i-1}(X, Y^+)$ for $i > 1$. We have the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \overline{\mathrm{Tor}}_{i-1}(P, Y)^+ \arrow[r] \arrow[d] & \overline{\mathrm{Tor}}_{i-1}(K, Y)^+ \arrow[r] \arrow[d, "\cong"] & \overline{\mathrm{Tor}}_i(X, Y)^+ \arrow[r] \arrow[d] & \overline{\mathrm{Tor}}_i(P, Y)^+ \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \overline{\mathrm{Ext}}^{i-1}(P, Y^+) \arrow[r] & \overline{\mathrm{Ext}}^{i-1}(K, Y^+) \arrow[r] & \overline{\mathrm{Ext}}^i(X, Y^+) \arrow[r] & \overline{\mathrm{Ext}}^i(P, Y^+) \arrow[r] & \cdots
\end{tikzcd}$$

Since P is flat, we have $\overline{\mathrm{Tor}}_j(P,Y)^+ = 0$ and $\overline{\mathrm{Ext}}^j(P,Y^+) = 0$, for every $j \geq 1$. It follows that $\overline{\mathrm{Tor}}_i(X,Y)^+ \longrightarrow \overline{\mathrm{Ext}}^i(X,Y^+)$ is an isomorphism. $\square$

Theorem 4.7.13 (see (24, Theorem 5.4.3))

Let $X \in \mathbf{Ch}({}_R\mathbf{Mod})$ be a chain complex. The following conditions are equivalent if R is an commutative n-Gorenstein ring:

(1) X is a Gorenstein-flat complex.

(2) X^+ is a Gorenstein-injective complex in $\mathbf{Ch}(\mathbf{Mod}_R)$.

(3) X_m is a Gorenstein-flat module, for every $m \in \mathbb{Z}$.

(4) $\overline{\mathrm{Tor}}_1(W,X) = 0$ for all $W \in \widetilde{\mathcal{W}} \subseteq \mathbf{Ch}(\mathbf{Mod}_R)$.

Definition 4.7.7. A chain complex $Y \in \mathbf{Ch}({}_R\mathbf{Mod})$ is <u>Gorenstein-cotorsion</u> if $\overline{\mathrm{Ext}}^1(X,Y) = 0$ for every $X \in \widehat{\mathcal{GF}}_0$. We denote by $(\widehat{\mathcal{GF}}_0)^\perp$ the class of Gorenstein-cotorsion complexes.

Proposition 4.7.14

Let R be a commutative n-Gorenstein ring. The classes $\widehat{\mathcal{GF}}_0$ and $(\widehat{\mathcal{GF}}_0)^\perp$ form a bar-cotorsion pair $(\widehat{\mathcal{GF}}_0 \mid (\widehat{\mathcal{GF}}_0)^\perp)$.

$\boxed{\textbf{Proof}}$.

We only have to show that ${}^\perp((\widehat{\mathcal{GF}}_0)^\perp) \subseteq \widehat{\mathcal{GF}}_0$. Let $X \in {}^\perp((\widehat{\mathcal{GF}}_0)^\perp)$. So let $W \in \widetilde{\mathcal{W}}$. By Proposition 4.7.12, we have $\overline{\mathrm{Tor}}_1(W,X)^+ \cong \overline{\mathrm{Tor}}_1(X,W)^+ \cong \overline{\mathrm{Ext}}^1(X,W^+)$. Now let E be a Gorenstein-flat complex. Then $\overline{\mathrm{Ext}}^1(E,W^+) \cong \overline{\mathrm{Tor}}_1(W,E)^+ = 0$, i.e. $W^+ \in (\widehat{\mathcal{GF}}_0)^\perp$ and so $\overline{\mathrm{Ext}}^1(X^+,W) \cong \overline{\mathrm{Ext}}^1(X,W^+) = 0$, i.e. X^+ is a Gorenstein-injective complex. Hence X is Gorenstein-flat by the previous theorem. $\square$

By Theorem 4.7.13, a complex X is Gorenstein-flat if, and only if, X_m is a Gorenstein-flat module, for every $m \in \mathbb{Z}$. Using this equivalence, it follows that $\widehat{\mathcal{GF}_0}$ is closed under suspensions. So by the previous proposition, Lemma 4.7.6 and Theorem 4.7.7, $(\widehat{\mathcal{GF}_0}, (\widehat{\mathcal{GF}_0})^{\perp})$ is a cotorsion pair. Recall that in (4, Proposition 4.1) it is proven that for every chain complex $X \in \mathrm{dw}\widetilde{\mathcal{F}_0}$ and every $x \in X$, there exists a subcomplex $S \subseteq X$ in $(\mathrm{dw}\widetilde{\mathcal{F}_0})^{\leq \kappa}$ (where κ is a fix regular cardinal number with $\mathrm{Card}(R) < \kappa$) such that $x \in S$ and $X/S \in \mathrm{dw}\widetilde{\mathcal{F}_0}$. The same arguments can be applied to $\widehat{\mathcal{GF}_0}$. So using this result, one can show that every Gorenstein-flat complex is a transfinite extension of $(\widehat{\mathcal{GF}_0})^{\leq \kappa}$, and hence $(\widehat{\mathcal{GF}_0}, (\widehat{\mathcal{GF}_0})^{\perp})$ is a cotorsion pair cogenerated by $(\widehat{\mathcal{GF}_0})^{\leq \kappa}$.

Proposition 4.7.15

Given a commutative n-Gorenstein ring R. Let E be a Gorenstein-flat complex and $x \in E$. Then there exists a Gorenstein-flat subcomplex $E' \subseteq E$ with $\mathrm{Card}(E') \leq \kappa$, such that $x \in E'$ and E/E' is also Gorenstein-flat.

Proposition 4.7.16

If R is a commutative n-Gorenstein ring, then

(1) $\widetilde{\mathcal{F}_0} = \widehat{\mathcal{GF}_0} \cap \widetilde{\mathcal{W}}$.

(2) $(\widehat{\mathcal{GF}_0})^{\perp} = (\widetilde{\mathcal{F}_0})^{\perp} \cap \widetilde{\mathcal{W}}$.

Proof.

---

(1) Let F be a flat complex. Then $-\overline{\otimes} F$ is an exact functor, and so $\overline{\mathrm{Tor}}_1(W, F) = 0$ for every $W \in \widetilde{\mathcal{W}}$. Hence F is Gorenstein-flat. On the other hand, it is clear that $F \in \widetilde{\mathcal{W}}$.

Now let $E \in \widehat{\mathcal{GF}_0} \cap \widetilde{\mathcal{W}}$. Then $E^+ \in \widehat{\mathcal{GI}_0}$. On the other hand, $\mathrm{fd}(E) = k < \infty$, so there exists an exact sequence $0 \to F_k \to F_{k-1} \to \cdots \to F_1 \to F_0 \to E \to 0$

where F_i is flat for every $0 \leq i \leq k$. Then

$$0 \to E^+ = \overline{\mathrm{Hom}}(X, D^0(\mathbb{Q}/\mathbb{Z})) \to {F_0}^+ \to {F_1}^+ \to \cdots \to {F_{k-1}}^+ \to {F_k}^+ \to 0$$

is exact since $D^0(\mathbb{Q}/\mathbb{Z})$ is an injective complex (notice that $\mathrm{Ext}^1(Y, D^0(\mathbb{Q}/\mathbb{Z})) = 0$ implies $\overline{\mathrm{Ext}}^1(Y, D^0(\mathbb{Q}/\mathbb{Z})) = 0$, for every complex Y), and ${F_i}^+$ is an injective complex for every $0 \leq i \leq k$ (the version of (21, Theorem 3.2.10) for chain complexes is a direct consequence of Proposition 4.7.12 **(4)**). So $\mathrm{id}(E^+) \leq k < \infty$ and $E^+ \in \widetilde{\mathcal{W}}$. We have $E^+ \in \widehat{\mathcal{GI}_0} \cap \widetilde{\mathcal{W}} = \widetilde{\mathcal{I}_0}$. It follows E is flat.

(2) Similar to Proposition 4.6.9 **(2)**. □

From this result, we have $(\widehat{\mathcal{GF}_0}, (\widehat{\mathcal{GF}_0})^{\perp})$ and $(\widetilde{\mathcal{F}_0}, (\widetilde{\mathcal{F}_0})^{\perp})$ are compatible and complete cotorsion pairs. Hence the following theorem follows.

Theorem 4.7.17 (**Gorenstein-flat model structure**)

If R is an n-Gorenstein ring, then there exists a unique Abelian model structure on ${}_R\mathbf{Mod}$ where the (trivial) cofibrations are the monomorphisms with cokernel in $\widehat{\mathcal{GF}_0}$ (resp. in $\widetilde{\mathcal{F}_0}$), the (trivial) fibrations are the epimorphisms with kernel in $(\widetilde{\mathcal{F}_0})^{\perp}$ (resp. in $(\widehat{\mathcal{GF}_0})^{\perp}$), and $\widetilde{\mathcal{W}}$ is the class of trivial objects.

We say a few things more with respect to the class $\widehat{\mathcal{GF}_0}$.

Proposition 4.7.18

Let R be a commutative n-Gorenstein ring. The class $\widehat{\mathcal{GF}_0}$ is closed under direct limits.

Proof.

Let $X \in \mathbf{Ch}({}_R\mathbf{Mod})$ be a chain complex which is the direct limit of a direct system of Gorenstein-flat complexes, say $(E_i)_{i\in I}$. Let $W \in \widetilde{\mathcal{W}}$, we have $\overline{\mathrm{Tor}}_1(W, X) \cong \varinjlim_{i\in I} \overline{\mathrm{Tor}}_1(W, E_i) = 0$, since $\overline{\mathrm{Tor}}_1(W, -)$ preserves direct limits. We prove this last assertion. Let Y be any chain complex which is the direct limit of a direct system $(Y_i)_{i\in I}$. In (24, Proposition 4.2.1 5), it is proven that $(\varinjlim_{i\in I} Y_i)\overline{\otimes}W \cong \varinjlim_{i\in I} Y_i\overline{\otimes}W$. Since R is commutative, $W\overline{\otimes}(\varinjlim_{i\in I} Y_i) \cong \varinjlim_{i\in I} W\overline{\otimes}Y_i$ $(*)$. Let $K \hookrightarrow P \twoheadrightarrow X$ be a short exact sequence where P is a projective complex. Then for every $i \in I$, we have an exact sequence

$$\mathbf{S}_i = 0 \to \overline{\mathrm{Tor}}_1(X, Y_i) \to K\overline{\otimes}Y_i \to P\overline{\otimes}Y_i \to X\overline{\otimes}Y_i \to 0.$$

Direct limits commute with homology, so the direct limit of a direct system of exact sequences is an exact sequence. Hence

$$0 \to \varinjlim_{i\in I} \overline{\mathrm{Tor}}_1(X, Y_i) \to \varinjlim_{i\in I} K\overline{\otimes}Y_i \to \varinjlim_{i\in I} P\overline{\otimes}Y_i \to \varinjlim_{i\in I} X\overline{\otimes}Y_i \to 0$$

is an exact sequence since $(\mathbf{S}_i)_{i\in I}$ is a direct system of exact sequences. By $(*)$, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \varinjlim_{i\in I} \overline{\mathrm{Tor}}_1(X, Y_i) \arrow[r] \arrow[d] & \varinjlim_{i\in I} K\overline{\otimes}Y_i \arrow[r] \arrow[d, "\cong"] & \varinjlim_{i\in I} P\overline{\otimes}Y_i \arrow[r] \arrow[d, "\cong"] & \varinjlim_{i\in I} X\overline{\otimes}Y_i \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \overline{\mathrm{Tor}}_1(X, \varinjlim_{i\in I} Y_i) \arrow[r] & K\overline{\otimes}\varinjlim_{i\in I} Y_i \arrow[r] & P\overline{\otimes}\varinjlim_{i\in I} Y_i \arrow[r] & X\overline{\otimes}\varinjlim_{i\in I} Y_i \arrow[r] & 0
\end{tikzcd}$$

By the Five Lemma, $\varinjlim_{i\in I} \overline{\mathrm{Tor}}_1(X, Y_i) \longrightarrow \overline{\mathrm{Tor}}_1(X, \varinjlim_{i\in I} Y_i)$ is an isomorphism. Using induction, one can show that $\varinjlim_{i\in I} \overline{\mathrm{Tor}}_j(X, Y_i) \cong \overline{\mathrm{Tor}}_j(X, \varinjlim_{i\in I} Y_i)$ for every $j \in \mathbb{Z}_{\geq 0}$. $\square$

We have that $(\widehat{\mathcal{GF}}_0, (\widehat{\mathcal{GF}}_0)^{\perp})$ is a complete cotorsion pair where the class $\widehat{\mathcal{GF}}_0$ is closed under direct limits. So the pair $(\widehat{\mathcal{GF}}_0, (\widehat{\mathcal{GF}}_0)^{\perp})$ is perfect by Theorem 3.1.23. The following result follows.

Corollary 4.7.19 (see (24, Theorem 5.8.4))

Every chain complex $X \in \mathbf{Ch}({}_R\mathbf{Mod})$ has a Gorenstein-flat cover, provided that R is a commutative n-Iwanaga-Gorenstein ring.

In the author's point of view, the proof we just gave is simpler that the one appearing in the cited reference.

As it occurred in ${}_R\mathbf{Mod}$, we can consider in $\mathbf{Ch}({}_R\mathbf{Mod})$ the class $\widehat{\mathcal{GF}}_r$ of left r-$\widehat{\mathcal{GF}}_0$-complexes (or simply Gorenstein-r-flat complexes). It is easy to note that $\widehat{\mathcal{GF}}_r = \{X \in \mathbf{Ch}({}_R\mathbf{Mod}) : \mathrm{Gfd}(X) \leq r\}$. We prove that $\widehat{\mathcal{GF}}_r$ cogenerates a complete cotorsion pair in $\mathbf{Ch}({}_R\mathbf{Mod})$.

Proposition 4.7.20

Let $X \in \mathbf{Ch}({}_R\mathbf{Mod})$. Then $\mathrm{fd}(X) \leq r$ if, and only if, $\mathrm{id}(X^+) \leq r$, for every $r > 0$.

Proof.

Suppose X is a chain complex. If $\mathrm{fd}(X) \leq r$, for every chain complex $Y \in \mathbf{Ch}({}_R\mathbf{Mod})$ and every $i > r$, we have $\overline{\mathrm{Tor}}_i(Y, X)^+ = 0$. Hence $\overline{\mathrm{Ext}}^i(Y, X^+) = 0$, i.e. $\mathrm{id}(X) \leq r$. The other implication follows similarly. □

The following proposition is easy to prove.

Proposition 4.7.21

The following conditions are equivalent for any chain complex $X \in \mathbf{Ch}({}_R\mathbf{Mod})$ over a commutative n-Gorenstein ring R:

(1) X is a Gorenstein-r-flat complex.

(2) $\overline{\mathrm{Tor}}_i(W, X) = 0$ for all $i \geq r+1$ and all $W \in \widetilde{\mathcal{W}}$.

(3) Every Gorenstein-rth flat syzygy is Gorenstein-flat.

(4) Every rth flat syzygy is Gorenstein-flat.

(5) X_m is a Gorenstein-r-flat module for every $m \in \mathbb{Z}$.

(6) X^+ is a Gorenstein-r-injective complex.

The class $\widehat{\mathcal{GF}_r}$ of Gorenstein-r-flat complexes cogenerates a complete cotorsion pair. In fact, we have a bar-cotorsion pair $(\widehat{\mathcal{GF}_r} \mid (\widehat{\mathcal{GF}_r})^{\overline{\perp}})$. Since the class $\widehat{\mathcal{GF}_r}$ is closed under suspensions, it follows $(\widehat{\mathcal{GF}_r}, (\widehat{\mathcal{GF}_r})^{\overline{\perp}})$ is a cotorsion pair and $(\widehat{\mathcal{GF}_r})^{\perp} = (\widehat{\mathcal{GF}_r})^{\overline{\perp}}$. So $(\widehat{\mathcal{GF}_r}, (\widehat{\mathcal{GF}_r})^{\perp})$ is a cotorsion pair. By Lemma 4.6.12, along with some arguments used in the previous chapter for the class $\mathrm{dw}\widetilde{\mathcal{F}_r}$, we have that $(\widehat{\mathcal{GF}_r}, (\widehat{\mathcal{GF}_r})^{\perp})$ is a cotorsion pair cogenerated by $(\widehat{\mathcal{GF}_r})^{\leq \kappa}$. Moreover, it is easy to show that $\widetilde{\mathcal{F}_r} = \widehat{\mathcal{GF}_r} \cap \widetilde{\mathcal{W}}$ and $(\widehat{\mathcal{GF}_r})^{\perp} = (\widetilde{\mathcal{F}_r})^{\perp} \cap \widetilde{\mathcal{W}}$. Hence, we obtain the chain complex version of Theorem 4.6.2.

Theorem 4.7.22 (Gorenstein-r-flat model structure)

If R is an n-Gorenstein ring, then for each $0 \leq r \leq n$ there exists a unique Abelian model structure on ${}_R\mathbf{Mod}$ where the (trivial) cofibrations are the monomorphisms with cokernel in $\widehat{\mathcal{GF}_r}$ (resp. in $\widetilde{\mathcal{F}_r}$), the (trivial) fibrations are the epimorphisms with kernel in $(\widetilde{\mathcal{F}_r})^{\perp}$ (resp. in $(\widehat{\mathcal{GF}_r})^{\perp}$), and $\widetilde{\mathcal{W}}$ is the class of trivial objects.

It is also easy to see that the pair $(\widehat{\mathcal{GF}}_r, (\widehat{\mathcal{GF}}_r)^\perp)$ is perfect, since $\widehat{\mathcal{GF}}_r$ is closed under direct limits. Therefore, every chain complex over a commutative Gorenstein ring has a Gorenstein-r-flat cover, by Theorem 3.1.23.

4.8 Gorenstein homological dimensions and differential graded complexes

In this section, we shall consider an associative ring R with unit and the graded ring $A := R[x]/(x^2)$. We first show that ${}_A\mathbf{Mod}$ and $\mathbf{Ch}({}_R\mathbf{Mod})$ are isomorphic categories, in order to prove later that Gorenstein-r-projective A-modules and dg-r-projective chain complexes over R are in one-to-one correspondence, provided R satisfies certain conditions. The same holds for Gorenstein-r-injective and Gorenstein-r-flat A-modules. This was initially proven by M. Hovey and J. Gillespie for $r = 0$.

Definition 4.8.1. Recall that a <u>$\mathbb{Z}$-graded ring</u> A is a ring that has a direct sum decomposition into (Abelian) additive groups

$$A = \bigoplus_{n\in\mathbb{Z}} A_n = \cdots A_{-1} \oplus A_0 \oplus A_1 \oplus \cdots$$

such that the ring multiplication $\cdot$ satisfies $A_m \cdot A_n \subseteq A_{m+n}$, for every $m, n \in \mathbb{Z}$. A <u>graded module</u> is left module over a $\mathbb{Z}$-graded ring A with a direct sum decomposition $M = \bigoplus_{n\in\mathbb{Z}} M_n$ such that the product $\cdot : A \cdot M \to M$ satisfies $A_m \cdot M_n \subseteq M_{m+n}$, for every $m, n \in \mathbb{Z}$.

If we consider the ring of polynomials $R[x]$ and the ideal (x^2), the quotient $A := R[x]/(x^2)$ is a $\mathbb{Z}$-graded ring with a direct sum decomposition given by $R[x]/(x^2) = \cdots \oplus 0 \oplus (x) \oplus R \oplus 0 \oplus \cdots$, where the scalars $r \in R$ are the elements of degree 0, and the elements in the ideal (x) form the terms of degree -1. Every A-module can be viewed as a chain complex over R, and vice versa.

Let $\Phi : {}_A\mathbf{Mod} \to \mathbf{Ch}({}_R\mathbf{Mod})$ be the application defined as follows:

• Given a graded A-module $M = \bigoplus_{n\in\mathbb{Z}} M_n$, note that if $y \in M_n$ then $x \cdot y \in M_{n-1}$, since x has degree -1. Denote by $\Phi(M)_n$ the set M_n endowed with the structure of R-module provided by the graded multiplication. Let $\partial_n : \Phi(M)_n \to \Phi(M)_{n-1}$ be the map $y \mapsto x \cdot y$. It is clear that ∂_n is an R-homomorphism. Moreover, $\partial_{n-1} \circ \partial_n(y) = x \cdot (x \cdot y) = x^2 \cdot y = 0 \cdot y = 0$. Then, $\Phi(M) = (\Phi(M)_n, \partial_n)_{n\in\mathbb{Z}}$ is a chain complex over R.

• Let $M \xrightarrow{f} N$ be a homomorphism of graded A-modules. Then $f(M_n) \subseteq N_n$ and $f|_{M_n}$ is an R-homomorphism, for every $m \in \mathbb{Z}$. Let $\Phi(f)_n : \Phi(M)_n \to \Phi(N)_n$ be $f|_{M_n}$. We have $\Phi(f)_{n-1} \circ \partial_n^M(y) = f|_{M_{n-1}}(x \cdot y) = x \cdot f|_{M_n}(y) = \partial_n^N \circ \Phi(f)_n(y)$. So $\Phi(f) = (\Phi(f)_n)_{n\in\mathbb{Z}}$ is a chain map.

Note that $\Phi : {}_A\mathbf{Mod} \to \mathbf{Ch}({}_R\mathbf{Mod})$ defines a covariant functor. We show Φ is an isomorphism, by giving an inverse functor $\Psi : \mathbf{Ch}({}_R\mathbf{Mod}) \to {}_A\mathbf{Mod}$.

• Let $M = (M_n, \partial_n)_{n\in\mathbb{Z}}$ be a chain complex over R. Let $y \in M_n$ and define the product $r \cdot y = ry \in M_n$ for every $r \in R$, and $x \cdot y = \partial_n(y) \in M_{n-1}$. This gives rise to a graded A-module, that we denote by $\Psi(M) = (\Psi(M)_n)_{n\in\mathbb{Z}}$, where $\Psi(M)_n = M_n$ as sets.

• Given a chain map $f : M \to N$, we have $x \cdot f(y) = \partial \circ f(y) = f \circ \partial(y) = f(x \cdot y)$. Then f gives rise to a graded A-module homomorphism denoted by $\Phi(f)$.

Note that $\Psi : \mathbf{Ch}({}_R\mathbf{Mod}) \to {}_A\mathbf{Mod}$ is a functor. It is easy to show $\Psi \circ \Phi = \mathrm{Id}_{{}_A\mathbf{Mod}}$ and $\Phi \circ \Psi = \mathrm{Id}_{\mathbf{Ch}({}_R\mathbf{Mod})}$. It follows that Φ and Ψ map projective and injective objects into projective and injective objects, respectively. It is also easy to check that both Ψ and Φ preserves exact sequences. Concerning flat objects, it is important to recall how tensor products are defined for $\mathbb{Z}$-graded A-modules. Given two graded A-modules $M = (M_n)_{n\in\mathbb{Z}}$ and $N = (N_m)_{m\in\mathbb{Z}}$, the tensor product

$M \otimes_{\mathbb{Z}} N = (\bigoplus_{n+m=k} M_n \otimes_{\mathbb{Z}} N_m)_{k\in\mathbb{Z}}$ has also a $\mathbb{Z}$-graduation. Let Q be the sub-$\mathbb{Z}$-module generated by the elements $(a \cdot y) \otimes z - y \otimes (a \cdot z)$ where $a \in A$, $y \in M$ and $z \in N$. The tensor product of M and N over A is defined by $M \otimes_A N = (M \otimes_{\mathbb{Z}} N)/Q$. It is clear that $M \otimes_A N \cong \Phi(M) \overline{\otimes} \Phi(N)$ and $X \overline{\otimes} Y \cong \Psi(X) \otimes_A \Psi(Y)$ for every $M, N \in {}_A\mathbf{Mod}$ and $X, Y \in \mathbf{Ch}({}_R\mathbf{Mod})$. So it follows that M is flat in ${}_A\mathbf{Mod}$ if, and only if, $\Phi(M)$ is in $\mathbf{Ch}({}_R\mathbf{Mod})$. Similarly, X is flat in $\mathbf{Ch}({}_R\mathbf{Mod})$ if, and only if, $\Psi(X)$ is flat in ${}_A\mathbf{Mod}$.

The following lemma is straightforward.

Lemma 4.8.1

We have the following isomorphisms for every $i \geq 1$, $M, N \in {}_A\mathbf{Mod}$, and $Y, Z \in \mathbf{Ch}({}_R\mathbf{Mod})$:

(1) $\mathrm{Ext}^i_A(M, N) \cong \mathrm{Ext}^i(\Phi(M), \Phi(N))$.

(2) $\mathrm{Tor}^A_i(M, N) \cong \overline{\mathrm{Tor}}_i(\Phi(M), \Phi(N))$.

(3) $\mathrm{Ext}^i(Y, Z) \cong \mathrm{Ext}^i_A(\Psi(Y), \Psi(Z))$.

(4) $\overline{\mathrm{Tor}}_i(Y, Z) \cong \mathrm{Tor}^A_i(\Psi(Y), \Psi(Z))$.

Corollary 4.8.2

There is a one-to-one correspondence between the flat objects of ${}_A\mathbf{Mod}$ and the flat objects of $\mathbf{Ch}({}_R\mathbf{Mod})$, given by the functor Φ.

Theorem 4.8.3 (J. Gillespie and M. Hovey (30, Prop. 3.6, 3.8 and 3.10))

The functor $\Psi : \mathbf{Ch}({}_R\mathbf{Mod}) \to {}_A\mathbf{Mod}$ maps:

(1) dg-projective complexes into Gorenstein-projective A-modules,

(2) dg-injective complexes into Gorenstein-injective A-modules, and

(3) dg-flat complexes into Gorenstein-flat A-modules.

If R is a left and right Noetherian ring of finite global dimension, then the inverse functor $\Phi : {}_A\mathbf{Mod} \to \mathbf{Ch}({}_R\mathbf{Mod})$ maps:

(1') Gorenstein-projective A-modules into dg-projective complexes,

(2') Gorenstein-injective A-modules into dg-injective complexes, and

(3') Gorenstein-flat A-modules into dg-flat complexes.

Such a result can be extended to any homological dimension, but before stating and proving a generalization, it is important to note the following correspondence.

Corollary 4.8.4

Let R be an n-Iwanaga-Gorenstein ring. Then there is a one-to-one correspondence between the exact chain complexes over R and the A-modules in $\mathcal{W}$.

Proof.

Let E be an exact complex over R. Then $\operatorname{Ext}^1(X, E) = 0$ for every dg-projective complex X. Consider $\Psi(E)$ and let C be a Gorenstein-projective A-module. By the previous theorem, there exists a unique $X \in \mathrm{dg}\widetilde{\mathcal{P}_0}$ such that $C = \Psi(X)$. We have $\operatorname{Ext}^1_A(C, \Psi(E)) = \operatorname{Ext}^1_A(\Psi(X), \Psi(E)) \cong \operatorname{Ext}^1(X, E) = 0$. It follows $\Psi(E) \in \mathcal{W}$, since $(\mathcal{GP}_0, \mathcal{W})$ is a cotorsion pair. The mapping $E \mapsto \Psi(E)$ gives the desired correspondence. $\square$

Theorem 4.8.5

The functor $\Psi : \mathbf{Ch}({}_R\mathbf{Mod}) \to {}_A\mathbf{Mod}$ maps:

(1) dg-r-projective complexes into Gorenstein-r-projective A-modules,

(2) dg-r-injective complexes into Gorenstein-r-injective A-modules, and

(3) dg-r-flat complexes into Gorenstein-r-flat A-modules.

If R is a left and right Noetherian ring of finite global dimension, then the inverse functor $\Phi : {}_A\mathbf{Mod} \to \mathbf{Ch}({}_R\mathbf{Mod})$ maps:

(1') Gorenstein-r-projective A-modules into dg-r-projective complexes,

(2') Gorenstein-r-injective A-modules into dg-r-injective complexes, and

(3') Gorenstein-r-flat A-modules into dg-r-flat complexes.

Proof.

We only prove **(1)** and **(1')**, since the other assertions can be shown in a similar way. Let $X \in \mathrm{dg}\widetilde{\mathcal{P}_r}$. Consider a partial left projective resolution

$$0 \to C \to P_{r-1} \to \cdots \to P_1 \to P_0 \to \Psi(X) \to 0.$$

We show that C is a Gorenstein-projective A-module. Consider the complex $\Phi(C)$ and let E be an exact complex. We have $\mathrm{Ext}^1(\Phi(C), E) \cong \mathrm{Ext}^1(X, E')$, where $E' \in \Omega^{-r}(E)$. Note that $E' \in (\widetilde{\mathcal{P}_r})^{\perp}$. If fact, if $Z \in \widetilde{\mathcal{P}_r}$ then $\mathrm{Ext}^1(Z, E') \cong \mathrm{Ext}^{r+1}(Z, E) = 0$. Also, it is easy to check that $E' \in \mathcal{E}$. So $E' \in (\widetilde{\mathcal{P}_r})^{\perp} \cap \mathcal{E} = (\mathrm{dg}\widetilde{\mathcal{P}_r})^{\perp}$. It follows $\mathrm{Ext}^1(\Phi(C), E) \cong \mathrm{Ext}^1(X, E') = 0$, for every $E \in \mathcal{E}$. In other words, $\Phi(C)$ is dg-projective, where $C = \Psi(\Phi(C))$ is a Gorenstein-projective A-module.

Now suppose that R is a left and right Noetherian ring of finite global dimension. Note that Ψ and Φ define an one-to-one correspondence between r-projective complexes over R and r-projective A-modules. Let $X \in (\widetilde{\mathcal{P}_r})^{\perp}$ and consider $\Psi(X)$.

Let M be an r-projective A-module. Then $\Phi(M)$ is an r-projective complex. We have $\mathrm{Ext}^1_A(M, \Psi(X)) \cong \mathrm{Ext}^1(\Phi(M), X) = 0$. It follows $\Psi(X) \in (\mathcal{P}_r({}_A\mathbf{Mod}))^\perp$. Hence, Ψ and Φ give rise to a one-to-one correspondence between $(\widetilde{\mathcal{P}_r})^\perp$ and $(\mathcal{P}_r({}_A\mathbf{Mod}))^\perp$. Also, by the previous corollary, we have the same correspondence between $\mathcal{E}$ and $\mathcal{W}$. Since $(\mathrm{dg}\widetilde{\mathcal{P}_r})^\perp = (\widetilde{\mathcal{P}_r})^\perp \cap \mathcal{E}$ and $(\mathcal{P}_r)^\perp \cap \mathcal{W} = (\mathcal{GP}_r({}_A\mathbf{Mod}))^\perp$, we have that a complex Y is in $(\mathrm{dg}\widetilde{\mathcal{P}_r})^\perp$ if and only if $\Psi(Y)$ is in $(\mathcal{GP}_r({}_A\mathbf{Mod}))^\perp$. Since $\mathrm{dg}\widetilde{\mathcal{P}_r} = {}^\perp((\mathrm{dg}\widetilde{\mathcal{P}_r})^\perp)$ and $\mathcal{GP}_r({}_A\mathbf{Mod}) = {}^\perp((\mathcal{GP}_r({}_A\mathbf{Mod}))^\perp)$, we have that Φ maps Gorenstein-r-projective A-modules into dg-r-projective complexes. $\square$

MODEL STRUCTURE	COFIBRANT OBJECTS	FIBRANT OBJECTS	TRIVIAL OBJECTS	TRIVIALLY COFIBRANT OBJECTS	TRIVIALLY FIBRANT OBJECTS
On Gorenstein categories					
Gorenstein-projective	$\mathcal{GP}_0(\mathcal{C})$	$\mathcal{C}$	$\mathcal{W}(\mathcal{C})$	$\mathcal{P}_0(\mathcal{C})$	$\mathcal{W}(\mathcal{C})$
Gorenstein-injective	$\mathcal{C}$	$\mathcal{GI}_0(\mathcal{C})$	$\mathcal{W}(\mathcal{C})$	$\mathcal{W}(\mathcal{C})$	$\mathcal{I}_0(\mathcal{C})$
On locally Noetherian Gorenstein categories					
Gorenstein-r-injective	${}^\perp(\mathcal{I}_r(\mathcal{C}))$	$\mathcal{GI}_r(\mathcal{C})$	$\mathcal{W}(\mathcal{C})$	${}^\perp(\mathcal{GI}_r(\mathcal{C}))$	$\mathcal{I}_r(\mathcal{C})$
where $0 < r \leq FDP(\mathcal{C})$					
On modules over an n-Gorenstein ring					
Gorenstein-r-projective	$\mathcal{GP}_r$	$(\mathcal{P}_r)^\perp$	$\mathcal{W}$	$\mathcal{P}_r$	$(\mathcal{GP}_r)^\perp$
Gorenstein-flat	$\mathcal{GF}_0$	$(\mathcal{F}_0)^\perp$	$\mathcal{W}$	$\mathcal{F}_0$	$(\mathcal{GF}_0)^\perp$
Gorenstein-r-flat	$\mathcal{GF}_r$	$(\mathcal{F}_r)^\perp$	$\mathcal{W}$	$\mathcal{F}_r$	$(\mathcal{GF}_r)^\perp$
where $0 < r \leq n$					
On chain complexes over an n-Gorenstein ring					
Gorenstein-r-projective	$\widehat{\mathcal{GP}}_r$	$(\widetilde{\mathcal{P}}_r)^\perp$	$\widetilde{\mathcal{W}}$	$\widetilde{\mathcal{P}}_r$	$(\widehat{\mathcal{GP}}_r)^\perp$
where $0 < r \leq n$					
On chain complexes over a commutative n-Gorenstein ring					
Gorenstein-flat	$\widehat{\mathcal{GF}}_0$	$(\widetilde{\mathcal{F}}_0)^\perp$	$\widetilde{\mathcal{W}}$	$\widetilde{\mathcal{F}}_0$	$(\widehat{\mathcal{GF}}_0)^\perp$
Gorenstein-r-flat	$\widehat{\mathcal{GF}}_r$	$(\widetilde{\mathcal{F}}_r)^\perp$	$\widetilde{\mathcal{W}}$	$\widetilde{\mathcal{F}}_r$	$(\widehat{\mathcal{GF}}_r)^\perp$
where $0 < r \leq n$					

Table 4.1: SUMMARY OF MODEL STRUCTURES

CONCLUSION

We have established a connection between Abelian model structures and two theories of homological algebra. In some cases, such a connection provides an easy way to construct covers and envelopes of modules and complexes for certain (Gorenstein) homological dimensions. We emphasized the construction of n-projective and n-flat transfinite extensions of modules and complexes, since they give an interesting way to generalize techniques and results which hold in dimension 0,. Examples are the zig-zag arguments and Kaplansky's Theorem.

Most of our results are presented in a category theoretical setting. We also have rewritten some known results in the context of Abelian and Grothendieck categories (Chapter 2). For example, the author does not know a reference for Eklof and Trlifaj's Theorem besides the version given for modules. On the other hand, we wanted to take advantage of the definition of weak factorization systems, since model structures can be defined from them. The connection between complete cotorsion pairs and cotorsion factorization systems turns out to be an easier way to explain Hovey's Correspondence. From the author's point of view, this thesis provides, apart from the results we have gotten so far, a useful reference for the further study of the theory of Abelian model structures.

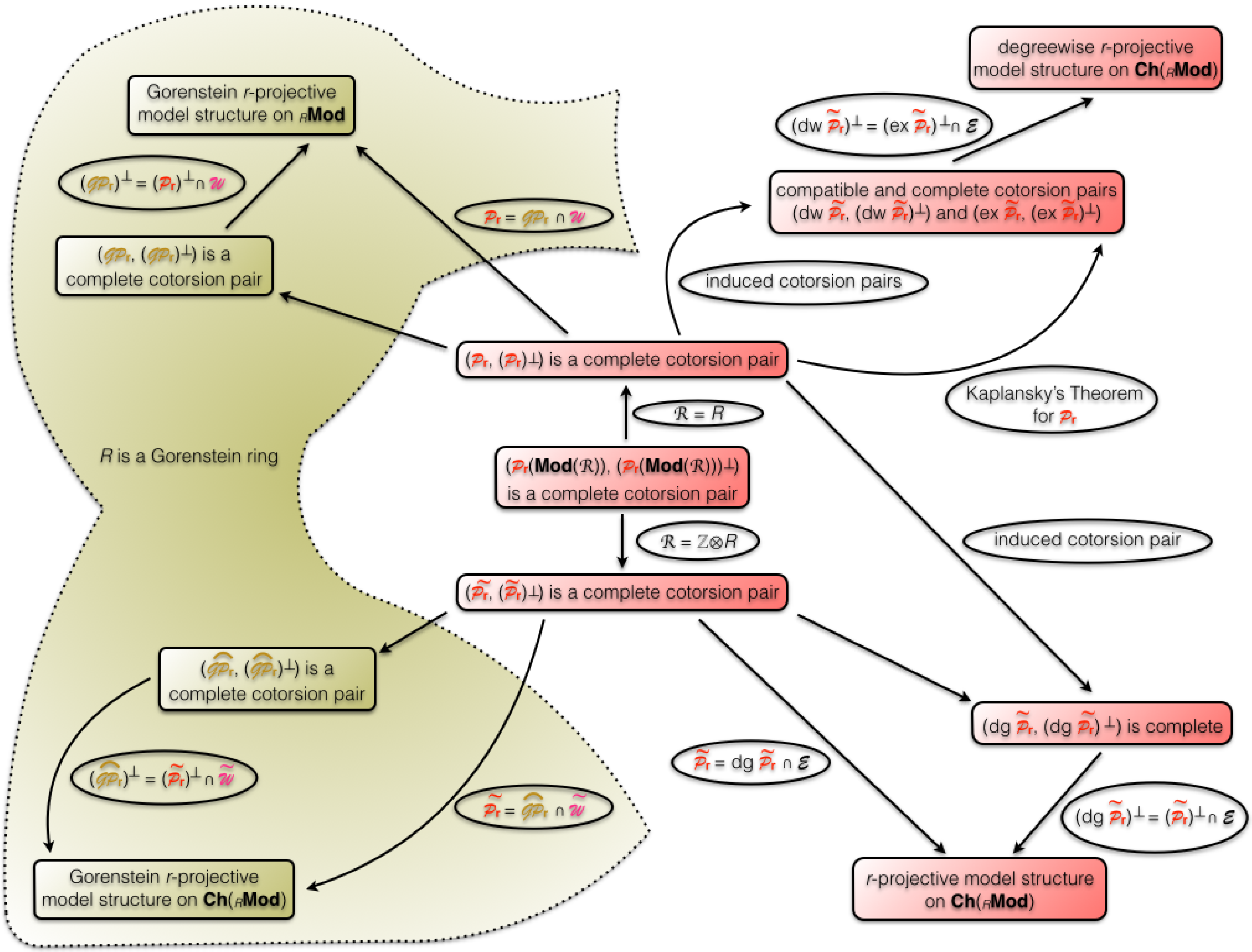

Figure C.1: Conclusions in the projective case.

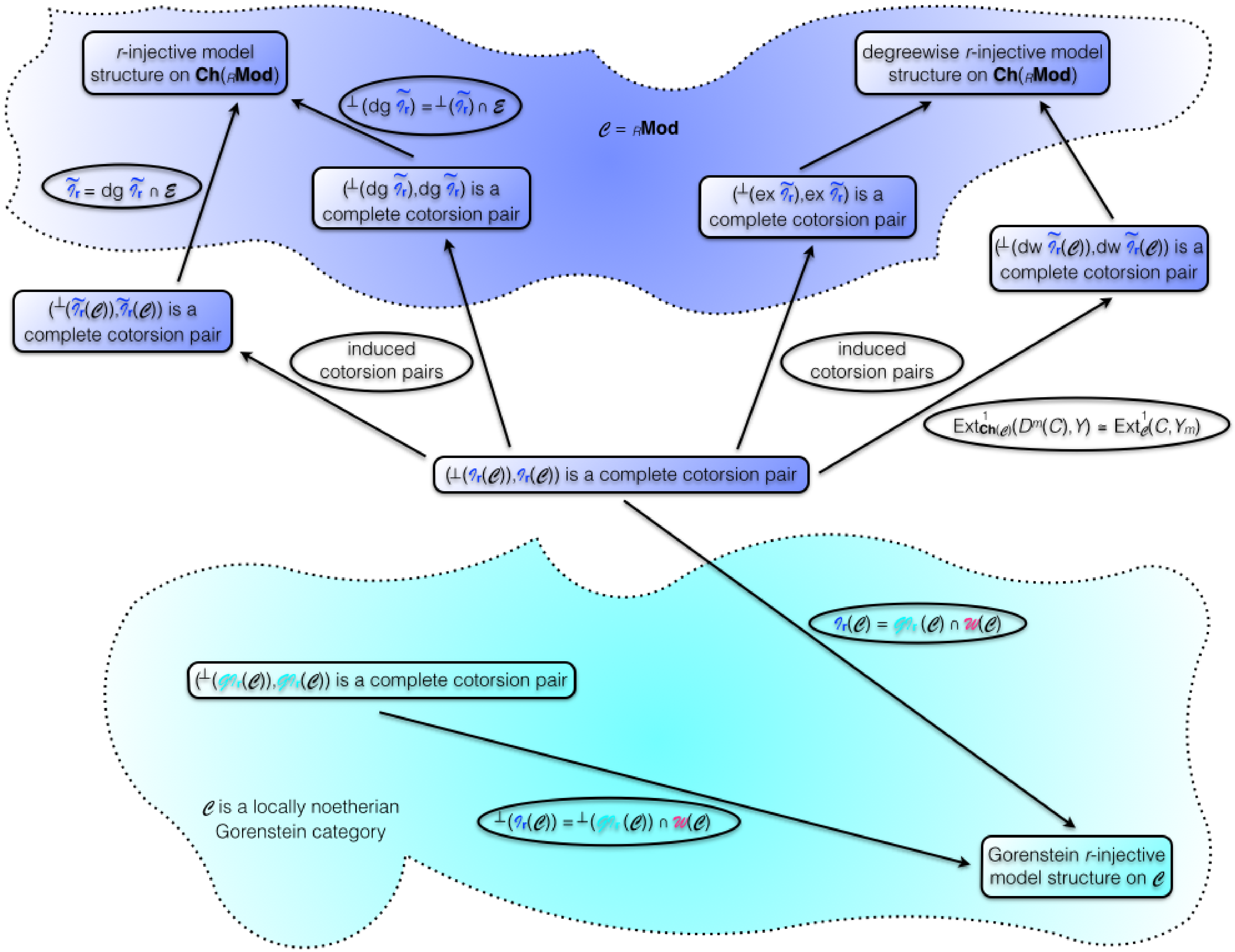

Figure C.2: Conclusions in the injective case.

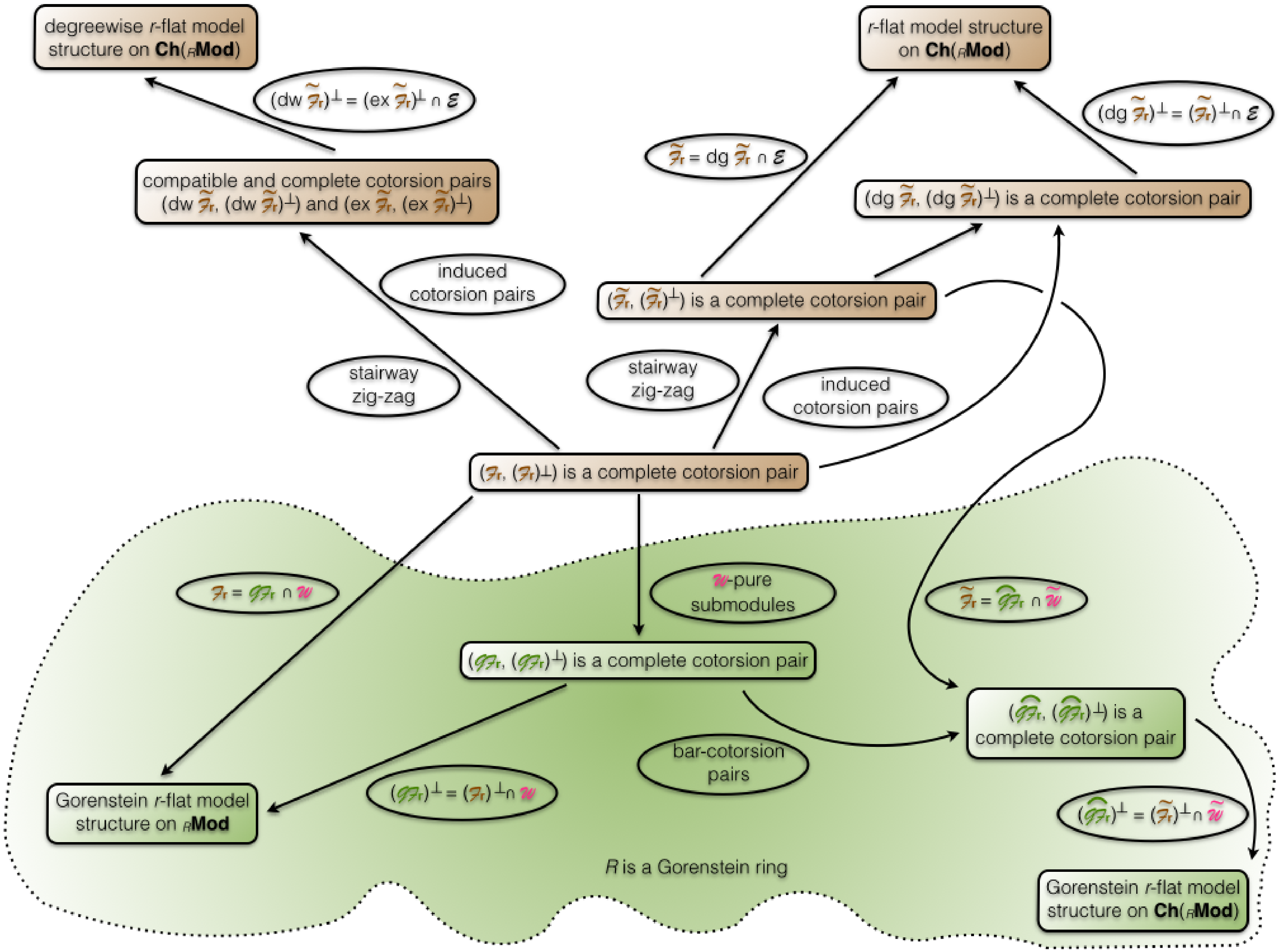

Figure C.3: Conclusions in the flat case.

The study of model structures on modules over specific rings does not only concern the case when R is a Gorenstein ring. There exists a certain class of rings that comprises the class of Gorenstein rings under a specific assumption. These rings are known nowadays as Ding-Chen rings (first introduced by N. Ding and J. Chen as n-FP rings). A ring R is said to be a Ding-Chen ring if it is both left and right coherent and $\text{FP-id}({}_RR) = \text{FP-id}(R_R) = n$, where $\text{FP-id}(-)$ denotes the FP-injective dimension, defined as

$$\text{FP-id}(M) := \min\{n \geq 0 : \text{Ext}^{n+1}_R(F, M) = 0 \ \forall \text{ finitely presented left } R\text{-module } F\}.$$

For Ding-Chen rings there is an equivalence involving FP-injective and flat dimensions, namely:

Theorem (N. Ding and J. Chen)

Let R be an n-FC ring and M be a right R-module. Then

$$\text{fd}(M) < \infty \iff \text{fd}(M) \leq n \iff \text{FP-id}(M) < \infty \iff \text{FP-id}(M) \leq n.$$

If $\mathcal{W}$ denotes the class of modules with finite FP-injective dimension, then $\mathcal{W}$ turns out to be the left and right halves of two complete cotorsion pairs. This result was proven by Ding and Lao in (17, Theorems 3.4) and (16, Theorem 3.8).

In (29), J. Gillespie proves that ${}^\perp\mathcal{W}$ and $\mathcal{W}^\perp$ coincide with the classes of Ding-projective and Ding-injective modules, respectively (see Corollaries 4.5 and 4.6 of the cited reference). Moreover, he also establishes the equalities ${}^\perp\mathcal{W} \cap \mathcal{W} = \mathcal{P}_0$ and $\mathcal{W}^\perp \cap \mathcal{W} = \mathcal{I}_0$. The following theorem follows.

Theorem (see (29, Theorem 4.7))

Let R be a Ding-Chen ring. Then there are two cofibrantly generated Abelian model structures on ${}_R\mathbf{Mod}$ each having $\mathcal{W}$ as the class of trivial objects. In the first model structure, each module is cofibrant while the (trivially) fibrant objects are the Ding-injective (resp. injective) modules. In the second model structure, each module is fibrant while the (trivially) cofibrant objects are the Ding-projective (resp. projective) modules.

It is known that Noetherian Ding-Chen rings are Gorenstein rings. So (35, Theorem 8.6) follows from the previous theorem in the particular case when R is Noetherian.

Question: Do there exist Ding analogues of the Gorenstein-r-projective and Gorenstein-r-injective modules structures in ${}_R\mathbf{Mod}$?

From the completeness of $({}^{\perp}\mathcal{W}, \mathcal{W})$ and $(\mathcal{W}, \mathcal{W}^{\perp})$ we can certainly do homological algebra in terms of Ding-projective and Ding-injective dimensions. Let $\mathcal{DP}_r$ (resp. $\mathcal{DI}_r$) denote the class of modules with Ding-projective (resp. Ding-injective) dimension at most r. As in the Gorenstein case, one can show that $(\mathcal{DP}_r, (\mathcal{DP}_r)^{\perp})$ and $({}^{\perp}(\mathcal{DI}_r), \mathcal{DI}_r)$ are cotorsion pairs.

We believe it is likely that some of the arguments in (16) and (17), along with results of this thesis, can be applied to show that the two cotorsion pairs above are complete. The corresponding compatibility equalities probably are not hard to prove. Hence, we are interested in giving a positive answer for the previous question in a near future, along with their respective generalizations to chain complexes.

Throughout this thesis we have only worked with model structures on the category of chain complexes $\mathbf{Ch}({}_R\mathbf{Mod})$ over a ring R with identity. The key element in the definition of chain complex is the relation $\partial^2 = 0$. But what if we replace this relation by $\partial^k = 0$, for some fixed $k > 2$? A sequence of modules and homomorphisms $X = (\cdots \to X_{n+1} \xrightarrow{\partial} X_n \xrightarrow{\partial} X_{n-1} \to \cdots)$ is called a <u>k-complex</u> if $\partial^k = 0$. This notion has already been considered by several authors. In (25, Page 688), the reader can find some recommended references for the study of the category $k\mathbf{Ch}({}_R\mathbf{Mod})$ of k-complexes. In (25, Theorem 4.5), M. Hovey and J. Gillespie construct the analogue of the projective model structure on the category of k-complexes. Moreover, they also show that the (trivial) cofibrations in $k\mathbf{Ch}({}_R\mathbf{Mod})$ are in one-to-one correspondence with the monomorphisms in ${}_{R[x]/(x^k)}\mathbf{Mod}$ with Gorenstein-projective (resp. projective) cokernel, provided R is left and right Noetherian and of finite global dimension.

The author is also interested in studying possible analogues of the (degreewise) r-projective, r-injective and r-flat model structures in the category of k-complexes, and their corresponding equivalence with the Gorenstein-r-projective, Gorenstein-r-injective and Gorenstein-r-flat model structures on ${}_{R[x]/(x^k)}\mathbf{Mod}$, with R as above.

Never a dull moment

Barry Karr

Skeptical Odysseys

APPENDIX I

COMMENTS ON MONOIDAL MODEL CATEGORIES AND HOMOLOGICAL DIMENSIONS

One question about the Abelian model structures we have obtained so far is that if they are monoidal with respect to a tensor product in the given category. In his paper (35, Theorem 7.2), M. Hovey gives necessary and sufficient conditions to check if an Abelian model category equipped with a closed symmetric monoidal structure is a monoidal model category. Those conditions concern certain closure and absorption properties for the classes of cofibrant objects with respect to the given tensor product. Namely, the product of cofibrant objects has to be cofibrant, and trivially cofibrant if one of them is trivial.

Definition A.1. A <u>symmetric monoidal structure</u> on a category $\mathcal{C}$ is given by a tensor product bifunctor $-\otimes- : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$, a <u>unit object</u> $S \in \mathrm{Ob}(\mathcal{C})$, and natural isomorphisms:

- <u>associativity</u>: $(- \otimes -) \otimes - \xrightarrow{a} - \otimes (- \otimes -)$ where

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \mapsto X \otimes (Y \otimes Z),$$

- <u>left unit</u>: $S \otimes - \xrightarrow{l} \mathrm{id}_{\mathcal{C}}$ where $l_Y : S \otimes Y \mapsto Y$,
- <u>right unit</u>: $- \otimes S \xrightarrow{r} \mathrm{id}_{\mathcal{C}}$ where $r_X : X \otimes S \mapsto X$,
- <u>braiding</u>: $-\otimes- \xrightarrow{b} -\otimes^{\mathrm{op}}-$ where $X\otimes^{\mathrm{op}}Y = Y\otimes X$ and $b_{X,Y} : X\otimes Y \to Y\otimes X$,

such that the following diagrams, called <u>coherence diagrams</u>, commute:

(1) <u>Pentagon identity</u>:

$$\begin{tikzcd}
 & (W \otimes X) \otimes (Y \otimes Z) \arrow[dr, "a_{W,X,Y\otimes Z}"] & \\
((W \otimes X) \otimes Y) \otimes Z \arrow[ur, "a_{W\otimes X,Y,Z}"] \arrow[d, "a_{W,X,Y} \otimes \mathrm{id}_Z"'] & & W \otimes (X \otimes (Y \otimes Z)) \\
(W \otimes (X \otimes Y)) \otimes Z \arrow[rr, "a_{W,X\otimes Y,Z}"'] & & W \otimes ((X \otimes Y) \otimes Z) \arrow[u, "\mathrm{id}_W \otimes a_{X,Y,Z}"']
\end{tikzcd}$$

(2) <u>Triangle identity</u>:

$$\begin{tikzcd}
(X \otimes S) \otimes Y \arrow[rr, "a_{X,S,Y}"] \arrow[dr, "r_X \otimes \mathrm{id}_Y"'] & & X \otimes (S \otimes Y) \arrow[dl, "\mathrm{id}_X \otimes l_Y"] \\
 & X \otimes Y &
\end{tikzcd}$$

(3) <u>Hexagon identity</u>:

$$\begin{tikzcd}
 & X \otimes (Y \otimes Z) \arrow[dr, "b_{X,Y\otimes Z}"] & \\
(X \otimes Y) \otimes Z \arrow[ur, "a_{X,Y,Z}"] \arrow[d, "b_{X,Y} \otimes \mathrm{id}_Z"'] & & (Y \otimes Z) \otimes X \arrow[d, "a_{Y,Z,X}"] \\
(Y \otimes X) \otimes Z \arrow[dr, "a_{Y,X,Z}"'] & & Y \otimes (Z \otimes X) \\
 & Y \otimes (X \otimes Z) \arrow[ur, "\mathrm{id}_Y \otimes b_{X,Z}"'] &
\end{tikzcd}$$

(4)

$$\begin{tikzcd} X \otimes Y \arrow[r, "b_{X,Y}"] \arrow[dr, equal, "id_{X\otimes Y}"'] & Y \otimes X \arrow[d, "b_{Y,X}"] \\ & X \otimes Y \end{tikzcd}$$

We denote the symmetric monoidal structure on $\mathcal{C}$ by the quintuple $(\otimes, a, b, r, l)$.

The following definitions appear in (36, Sections 4.1 and 4.2), which provides a detailed study of monoidal model categories.

Definition A.2. A symmetric monoidal structure $(\otimes, a, b, l, r)$ on $\mathcal{C}$ is said to be closed if for every object $Y \in \mathrm{Ob}(\mathcal{C})$ the functor $- \otimes Y : \mathcal{C} \to \mathcal{C}$ has a right adjoint functor $[Y, -] : \mathcal{C} \to \mathcal{C}$. This means that for all $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ we have a natural isomorphism $\mathrm{Hom}_{\mathcal{C}}(X \otimes Y, Z) \cong \mathrm{Hom}_{\mathcal{C}}(X, [Y, Z])$. Some authors call the right adjoint $[-, -]$ the internal hom.

Example A.1. Let R be a commutative ring. The following are examples of closed symmetric monoidal categories:

(1) $({}_R\mathbf{Mod}, \otimes_R)$, where $\otimes_R$ is the usual tensor product of modules, and R is the unit object.

(2) $(\mathbf{Ch}({}_R\mathbf{Mod}), \otimes)$, where $\otimes$ is the usual tensor product of complexes, and the unit object is given by $S^0(R)$. The internal hom is given by $\mathrm{Hom}'_{\mathbf{Ch}({}_R\mathbf{Mod})}(-, -)$ (see (36, Proposition 4.2.13) for details).

(3) $(\mathbf{Ch}({}_R\mathbf{Mod}), \overline{\otimes})$, where the unit is given by the 1-disk complex $D^1(R)$ (see (24, Proposition 4.2.1 4)). The internal hom is given by the bar-hom functor $\overline{\mathrm{Hom}}_{\mathbf{Ch}({}_R\mathbf{Mod})}(-, -)$.

Definition A.3. A monoidal model category is a model category $\mathcal{C}$ equipped with a symmetric monoidal structure $(\otimes, a, b, l, r)$ and an adjunction of two variables $(\mathrm{Hom}_l, \mathrm{Hom}_r, \varphi_l, \varphi_r) : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ such that the following conditions are satisfied:

(1) For every quadruple of objects $U, V, W, X \in \mathrm{Ob}(\mathcal{C})$, we have an induced morphism $f \square g : (V \otimes W) \coprod_{U \otimes W} (U \otimes X) \to V \otimes X$ making the following diagram commute:

$$\begin{tikzcd}
U \otimes W \arrow[r, "\mathrm{id}_U \otimes g"] \arrow[d, "f \otimes \mathrm{id}_W"'] & U \otimes X \arrow[d] \arrow[ddr, bend left] & \\
V \otimes W \arrow[r] \arrow[drr, bend right] & (V \otimes W) \coprod_{U \otimes W} (U \otimes X) \arrow[dr, dotted, "f \square g"] & \\
& & V \otimes X
\end{tikzcd}$$

If, given cofibrations $f : U \to V$ and $g : W \to X$ in $\mathcal{C}$, the induced map $f \square g$ is a cofibration, which is trivial if either f or g is.

(2) Using functorial factorizations, write $0 \to S$ as the composition $0 \hookrightarrow Q(S) \overset{\sim}{\twoheadrightarrow} S$ of a cofibration followed by a trivial fibration. The map $q : Q(S) \overset{\sim}{\twoheadrightarrow} S$ is called the cofibrant replacement for S. Then the maps $q \otimes X : Q(S) \otimes X \to S \otimes X$ and $X \otimes q : X \otimes Q(S) \to X \otimes S$ are weak equivalences for all cofibrant objects X.

Example A.2 (see (36, Prpopsotion 4.2.13))**.** The projective model structure is monoidal with respect to $\otimes$. However, the injective model structure is not monoiodal in general. For $R = \mathbb{Z}$, the cofibrations $S^0(\mathbb{Z}) \hookrightarrow S^0(\mathbb{Q})$ and $0 \to S^0(\mathbb{Z}_2)$ induce the map $(S^0(\mathbb{Z}) \hookrightarrow S^0(\mathbb{Q})) \square (0 \to S^0(\mathbb{Z}_2)) = S^0(\mathbb{Z}_2) \to 0$, which is not a cofibration.

The conditions given in the definition above could be difficult to check when we want to know if a given model structure is monoidal. In order to state Hovey's principal result on monoidal model categories, we recall the concept of proper exact sequences in the following definition taken from (44, Page 367), we adapt the notation to that used in this work.

Definition A.4. A class $\mathcal{P}$ of short exact sequences in an Abelian category $\mathcal{C}$ is said to be proper if the following conditions are satisfied:

(1) Any short exact sequence isomorphic to an element in $\mathcal{P}$ is also in $\mathcal{P}$, i.e. $\mathcal{P}$ is closed under isomorphisms.

(2) For any objects X and Y in $\mathcal{C}$, the sequence $X \hookrightarrow X \oplus Y \twoheadrightarrow Y$ is proper.

Consider a defined composition $X \xrightarrow{f} Y \xrightarrow{g} Z$:

(3) If f and g are monic and $X \overset{f}{\hookrightarrow} Y \twoheadrightarrow \mathrm{CoKer}(f)$ and $Y \overset{g}{\hookrightarrow} Z \twoheadrightarrow \mathrm{CoKer}(g)$ are proper, then so is $X \overset{g\circ f}{\hookrightarrow} Z \twoheadrightarrow \mathrm{CoKer}(g \circ f)$.

(4) If f and g are epic and $\mathrm{Ker}(f) \hookrightarrow X \overset{f}{\twoheadrightarrow} Y$ and $\mathrm{Ker}(g) \hookrightarrow Y \overset{g}{\twoheadrightarrow} Z$ are proper, then so is $\mathrm{Ker}(g \circ f) \hookrightarrow X \overset{g\circ f}{\twoheadrightarrow} Z$.

(5) If f and g are monic and $X \overset{g\circ f}{\hookrightarrow} Z \twoheadrightarrow \mathrm{CoKer}(g \circ f)$ proper, then so is the sequence $X \overset{f}{\hookrightarrow} Y \twoheadrightarrow \mathrm{CoKer}(f)$.

(6) If f and g are monic and $\mathrm{Ker}(g \circ f) \hookrightarrow X \overset{g\circ f}{\twoheadrightarrow} Z$ proper, then so is the sequence $\mathrm{Ker}(g) \hookrightarrow Y \overset{g}{\twoheadrightarrow} Z$.

Definition A.5. Given a proper class $\mathcal{P}$, an Abelian model structure on an Abelian category $\mathcal{C}$ is said to be compatible with $\mathcal{P}$ if the short exact sequences $X \overset{f}{\hookrightarrow} Y \twoheadrightarrow \mathrm{CoKer}(f)$ and $\mathrm{Ker}(g) \hookrightarrow W \overset{g}{\twoheadrightarrow} Z$ are proper for every cofibration f and every fibration g.

Theorem A.1 (M. Hovey. (35, Theorem 7.2))

Let $\mathcal{P}$ be a proper class of short exact sequences in a closed symmetric monoidal Abelian category $\mathcal{C}$, and that $\mathcal{C}$ is equipped with an Abelian model structure compatible with $\mathcal{P}$. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{W}$ denote the classes of cofibrant, fibrant and trivial objects, respectively. Then $\mathcal{C}$ is a monoidal model category if the following conditions are satisfied:

(1) The sequence $X \overset{f}{\hookrightarrow} Y \twoheadrightarrow \mathrm{CoKer}(f)$ is pure for every cofibration f.

(2) If $X, Y \in \mathcal{A}$, then $X \otimes Y \in \mathcal{A}$.

(3) If $X, Y \in \mathcal{A}$ and one of them is in $\mathcal{W}$, then $X \otimes Y \in \mathcal{A} \cap \mathcal{W}$.

(4) The unit S of the monoidal structure is in $\mathcal{A}$.

Conversely, if the model structure above is monoidal, then the above conditions hold.

Example A.3. J. Gillespie used the previous result to show that the flat model structure on $(\mathbf{Ch}({}_R\mathbf{Mod}), \otimes)$ is monoidal, with R a commutative ring. The class $\mathcal{P}$ in the theorem is the class of all short exact sequences (see (27, Corollary 5.1)). Gillespie also showed in (25, Subsection 5.2) that the degreewise flat model structure is not monoidal on $(\mathbf{Ch}({}_R\mathbf{Mod}), \otimes)$ in general. For the ring $\mathbb{Z}_4$, the complex $Y = \cdots \to \mathbb{Z}_4 \xrightarrow{\times 2} \mathbb{Z}_4 \xrightarrow{\times 2} \mathbb{Z}_4 \to \cdots$ is an exact degreewise flat complex, but $Y \otimes Y \notin \mathrm{ex}\widetilde{\mathcal{F}_0}$, since it is not even exact.

<u>When the degreewise flat model structure is monoidal?</u>:

We shall see that the degreewise flat model structure is monoidal for certain class of rings. It is clear that the unit $S^0(R)$ is cofibrant and that the tensor product of two degreewise flat chain complexes is degreewise flat (note that the tensor product of flat modules is flat). We study flatness with respect to the usual tensor product of complexes.

Proposition A.2

A chain complex $X \in \mathbf{Ch}({}_R\mathbf{Mod})$ over a commutative ring R is flat with respect to $\otimes$ if, and only if, it is degreewise flat.

Proof.

Let Y be a chain complex such that $- \otimes Y$ is exact. Consider an exact sequence of modules $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$. Then $0 \to S^0(A) \xrightarrow{S^0(\alpha)} S^0(B) \xrightarrow{S^0(\beta)} S^0(C) \to 0$ is exact in $\mathbf{Ch}({}_R\mathbf{Mod})$. It follows

$$0 \to S^0(A) \otimes Y \xrightarrow{S^0(\alpha)\otimes Y} S^0(B) \otimes Y \xrightarrow{S^0(\beta)\otimes Y} S^0(C) \otimes Y \to 0$$

is also exact. So for each $n \in \mathbb{Z}$ we have the exact sequence

$$0 \to (S^0(A) \otimes Y)_n = A \otimes_R Y_n \xrightarrow{\alpha\otimes_R Y_n} B \otimes_R Y_n \xrightarrow{\beta\otimes_R Y_n} C \otimes_R Y_n \to 0.$$

Now suppose $Y \in \mathrm{dw}\widetilde{\mathcal{F}_0}$. Consider a short exact sequence of chain complexes $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ and apply $- \otimes Y$. We need to check

$$0 \to (A \otimes Y)_n \xrightarrow{(\alpha\otimes Y)_n} (B \otimes Y)_n \xrightarrow{(\beta\otimes Y)_n} (C \otimes Y)_n \to 0$$

is exact, for every $n \in \mathbb{Z}$. In other words, we shall see

$$(*) = \left(0 \to \bigoplus_{k\in\mathbb{Z}} A_k \otimes_R Y_{n-k} \xrightarrow{\bigoplus_{k\in\mathbb{Z}} \alpha_k\otimes_R Y_{n-k}} \bigoplus_{k\in\mathbb{Z}} B_k \otimes_R Y_{n-k} \xrightarrow{\bigoplus_{k\in\mathbb{Z}} \beta_k\otimes_R Y_{n-k}} \bigoplus_{k\in\mathbb{Z}} C_k \otimes_R Y_{n-k} \to 0\right)$$

is exact. For every $k \in \mathbb{Z}$, the sequence

$$0 \to A_k \otimes_R Y_{n-k} \xrightarrow{\alpha_k\otimes_R Y_{n-k}} B_k \otimes_R Y_{n-k} \xrightarrow{\beta_k\otimes Y_{n-k}} C_k \otimes_R Y_{n-k} \to 0$$

is exact since Y_{n-k} is flat. It follows $(*)$ is exact since the direct sum of exact sequences is exact (homology commutes with direct sums). $\square$

Choose $\mathcal{P}$ the class of all short exact sequences. By the previous proposition, it follows that if $X \xrightarrow{f} Y$ is a cofibration in the degreewise flat model structure, then $0 \to X \xrightarrow{f} Y \to \mathrm{CoKer}(f) \to 0$ is a $\mathcal{P}$-pure sequence. Hence, the degreewise model structure satisfies conditions **(1)** and **(2)** and **(4)** of Theorem A.1. We already know that **(3)** is not true in general, but for certain rings, it happens that the tensor product of degreewise chain complexes turns out to be exact if one of them is exact.

Theorem A.3 (Künneth Exact Sequence, (46, Theorem 9.16)) Suppose R is a ring, with weak dimension at most 1. Suppose F and F' are degreewise flat chain complexes in $\mathbf{Ch}(\mathbf{Mod}_R)$ and $\mathbf{Ch}({}_R\mathbf{Mod})$, respectively. Then for every $n \in \mathbb{Z}$, there exists a exact sequence

$$0 \to \bigoplus_{i+j=n} H_i(F) \otimes H_j(F') \to H_n(F \otimes F') \to \bigoplus_{i+j=n-1} \mathrm{Tor}_1^R(H_i(F), H_j(F')) \to 0.$$

Recall that the <u>weak dimension</u> of a ring is defined as $\sup\{\mathrm{fd}(M) : M \in_R \mathbf{Mod}\}$ where $\mathrm{fd}(M) = \inf\{n \geq 0 : \mathrm{Tor}_{n+1}^R(-, M) \equiv 0\}$. The problem with the ring $\mathbb{Z}_4$ considered by Gillespie is that $\mathbb{Z}_2$ is a $\mathbb{Z}_4$-module with infinite flat dimension, since $\mathrm{Tor}_n^{\mathbb{Z}_4}(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2 \neq 0$ for every $n \geq 0$ (46, Chapter 3, Example 9).

If X and Y are degreewise flat chain complexes over a ring R with weak dimension at most 1, such that one of them is exact, then the previous theorem implies that $X \otimes Y$ has null homology, i.e. $X \otimes Y$ is exact. Therefore, the degreewise flat model structure on complexes over such a ring is monoidal. The same reasoning applies to the degreewise projective model structure (Recall the tensor product of projective modules is projective).

Gorenstein homological model structures are not monoidal:

We show the Gorenstein-projective model structure on modules over $\mathbb{Z}_4$ is not monoidal. First, we have to say that $\mathbb{Z}_4$ is a quasi-Frobenius ring, that is the classes of projective and injective $\mathbb{Z}_4$-modules coincide. Note that every module over such a ring is Gorenstien projective. For if M is a left R-module with R quasi-Frobinius, consider a left projective resolution and a right injective resolution of M, say $\cdots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$ and $0 \to M \xrightarrow{g^0} I^0 \xrightarrow{g^1} I^1 \to \cdots$. Taking the composition $g^0 \circ f_0$, we have an exact sequence $\cdots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{g^0 \circ f_0} I^0 \xrightarrow{g^1} I^1 \to \cdots$ such that $M = \mathrm{Ker}(g^1)$. It is clear that this sequence is $\mathrm{Hom}_R(-, \mathcal{P}_0) = \mathrm{Hom}_R(-, \mathcal{I}_0)$-exact. Hence M is Gorenstein-projective.

Back to the example $R = \mathbb{Z}_4$. There exists a left $\mathbb{Z}_4$-module with infinite projective dimension. Recall that the left global dimension of a ring R is defined as $\sup\{\mathrm{pd}(M) : M \in_R \mathbf{Mod}\}$. It is known that if R is a left Noetherian ring, then the left global dimension and the weak dimension of R coincide (46, Corollary 4.21). So $\mathbb{Z}_4$ has infinite left global dimension since it is left Noetherian with infinite weak dimension. It follows there exists a Gorenstein-projective $\mathbb{Z}_4$-module C with infinite projective dimension, and so M is not projective since $\mathcal{P}_0 = \mathcal{GP}_0 \cap \mathcal{W}$. Note also $\mathbb{Z}_4$ is a 0-Gorenstein ring, since $\mathbb{Z}_4$ is injective and left and right Noetherian.

The Gorenstein-projective model structure on modules over $\mathbb{Z}_4$ is not monoidal since condition **(3)** of Theorem A.1 does not hold: $\mathbb{Z}_4 \otimes_{\mathbb{Z}_4} C \cong C \notin \mathcal{GP}_0 \cap \mathcal{W}$ although $\mathbb{Z}_4 \in \mathcal{GP}_0 \cap \mathcal{W}$ and $C \in \mathcal{GP}_0$.

These arguments also work to show that the Gorenstein-projective model structure on $\mathbf{Ch}({}_R\mathbf{Mod})$ is not monoidal in general. It suffices to consider $D^1(\mathbb{Z}_4)$ and $S^0(C)$, where $D^1(\mathbb{Z}_4) \otimes S^0(C) \cong D^1(C)$, which is not projective but Gorenstein-projective.

In a similar way, one can show that the Gorenstein-injective and Gorenstein-flat model structures are not monoidal on modules or chain complexes over $\mathbb{Z}_4$.

<u>ARE THE n-PROJECTIVE AND n-FLAT MODEL STRUCTURES MONOIDAL?</u>: The answer is that they are not in general if $n \geq 1$. Consider the case $R = \mathbb{Z}$. Note that $\mathbb{Z}_2$ is a $\mathbb{Z}$-module in $\mathcal{P}_1$, since it is not projective and there exists a short exact sequence $0 \to \mathbb{Z} \stackrel{2\times}{\to} \mathbb{Z} \stackrel{\pi}{\to} \mathbb{Z}_2 \to 0$ where $2\times$ is the map $x \mapsto 2 \times x$ and π is the canonical projection $x \mapsto \overline{x} \in \{0,1\}$. Now let X be the complex given by the previous sequence, where $X_1 = \mathbb{Z}$, $X_0 = \mathbb{Z}$ and $X_{-1} = \mathbb{Z}_2$. We have $X \in \widetilde{\mathcal{P}_1} \subseteq \widetilde{\mathcal{P}_n}$. Consider also the complex $S^0(\mathbb{Z}_2)$, which is in $\mathrm{dg}\widetilde{\mathcal{P}_n}$ since $\mathrm{Ext}^1_{\mathbf{Ch}(\mathbb{Z})}(S^0(\mathbb{Z}_2), Y) \cong \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_2, Z_0Y) = 0$ for every $Y \in \widetilde{(\mathcal{P}_n)^{\perp}}$. Now we compute $S^0(\mathbb{Z}_2) \otimes X$:

$$(S^0(\mathbb{Z}_2) \otimes X)_m = \bigoplus_{k \in \mathbb{Z}} S^0(\mathbb{Z}_2)_k \otimes_{\mathbb{Z}} X_{m-k} = \mathbb{Z}_2 \otimes X_m = \begin{cases} \mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z} & \text{if } m = 1, \\ \mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z} & \text{if } m = 0, \\ \mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_2 & \text{if } m = -1, \\ 0 & \text{otherwise.} \end{cases}$$

It is not hard to see that $\partial_1^{S^0(\mathbb{Z}_2)\otimes X}$ is the zero map, so the sequence

$$\cdots \to 0 \to \mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z} \stackrel{\partial_1^{S^0(\mathbb{Z}_2)\otimes X}}{\to} \mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z} \stackrel{\partial_0^{S^0(\mathbb{Z}_2)\otimes X}}{\to} \mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_2 \to 0 \to \cdots$$

is not exact. We have that $0 \to S^0(\mathbb{Z}_2)$ is a cofibration, $0 \to X$ is a trivial cofibration, but the induced map $(0 \to S^0(\mathbb{Z}_2)) \square (0 \to X) = 0 \to S^0(\mathbb{Z}_2) \otimes X$ is cofibration but not a weak equivalence. Therefore, the n-projective model structure on $\mathbf{Ch}(\mathbb{Z})$ is not monoidal with respect to the tensor product $\otimes$.

We show also that the n-projective model structure on $\mathbf{Ch}(\mathbb{Z})$, with $n \geq 1$, is not monoidal with respect to $\overline{\otimes}$. Consider the complex X given above. We have

$$(S^0(\mathbb{Z}_2)\overline{\otimes}X)_m = \begin{cases} (S^0(\mathbb{Z}_2)\otimes X)_1/B_1(S^0(\mathbb{Z}_2)\otimes X) & \text{if } m=1, \\ (S^0(\mathbb{Z}_2)\otimes X)_0/B_0(S^0(\mathbb{Z}_2)\otimes X) & \text{if } m=0, \\ (S^0(\mathbb{Z}_2)\otimes X)_{-1}/B_{-1}(S^0(\mathbb{Z}_2)\otimes X) & \text{if } m=-1, \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z} & \text{if } m=1, \\ \mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z} & \text{if } m=0, \\ 0 & \text{otherwise.} \end{cases}$$

Then $S^0(\mathbb{Z}_2)\overline{\otimes}X = \cdots \to 0 \to \mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z} \xrightarrow{\partial_1^{S^0(\mathbb{Z}_2)\overline{\otimes}X}} \mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z} \to 0 \to \cdots$ where $\partial_1^{S^0(\mathbb{Z}_2)\overline{\otimes}X}$ is the zero map. Hence $S^0(\mathbb{Z}_2)\overline{\otimes}X$ is not an exact complex, and so the induced chain map $0 \to S^0(\mathbb{Z}_2)\overline{\otimes}X = (0 \to S^0(\mathbb{Z}_2))\square(0 \to X)$ is not a trivial cofibration.

These counterexamples also work for the n-flat model structure.

APPENDIX II

RELATIVE EXTENSIONS AND NATURAL TRANSFORMATIONS FROM DISK AND SPHERE CHAIN COMPLEXES

The extension functors $\mathrm{Ext}^i_{\mathcal{C}}(-,-)$ have their analogues in Gorenstein homological algebra. Suppose $\mathcal{C}$ is a Gorenstein category. As an application of the completeness of $(\mathcal{GP}_0(\mathcal{C}), \mathcal{W}(\mathcal{C}))$, for every object X we can construct an exact left Gorenstein-projective resolution of the form

$$\begin{tikzcd}
\cdots \arrow[rr] & & C_1 \arrow[rr, "f_0"] \arrow[dr, "p_1"] & & C_0 \arrow[r, "f_0"] & X \arrow[r] & 0 \\
& W_1 \arrow[ur, hook, "i_1"'] & & W_0 \arrow[ur, hook, "i_0"'] & & &
\end{tikzcd}$$

Let $\mathbf{C}_\bullet$ be the deleted complex $\cdots \to C_1 \to C_0 \to 0$. Given an object $Y \in \mathrm{Ob}(\mathcal{C})$, the mth cohomology of the complex $\mathrm{Hom}_{\mathcal{C}}(\mathbf{C}_\bullet, Y)$ is denoted by $\mathrm{GExt}^m_{\mathcal{C}}(X,Y)$.

Similarly, for every object $Y \in \mathrm{Ob}(\mathcal{C})$ we can construct a right Gorenstein-injective resolution $0 \to Y \to D^0 \to D^1 \to \cdots$. If $\mathbf{D}_\bullet$ denotes the deleted complex $0 \to D^0 \to D^1 \to \cdots$, then for every object $X \in \mathrm{Ob}(\mathcal{C})$, the mth cohomology group of the complex $\mathrm{Hom}_{\mathcal{C}}(X, \mathbf{D}_\bullet)$ coincides with $\mathrm{GExt}^m_{\mathcal{C}}(X,Y)$. We shall later give a proof of this fact.

We shall call the functors $\mathrm{GExt}^m_{\mathcal{C}}(X, -)$ and $\mathrm{GExt}^m_{\mathcal{C}}(-, Y)$ the Gorenstein-extension functors.

The objective of this appendix is to construct Gorenstein-like versions of Propositions 1.6.2 and 1.6.3. We do this by expressing $\mathrm{GExt}^1_{\mathcal{C}}(X, Y)$ as the subgroup of $E^{\mathcal{C}}(X, Y)$ composed by the classes of short exact sequences $Y \hookrightarrow Z \twoheadrightarrow X$ which are also $\mathrm{Hom}_{\mathcal{C}}(\mathcal{GP}_0(\mathcal{C}), -)$-exact.

Definition B.1. Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be Abelian categories and $T : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ be an additive functor contravariant in the first variable and covariant in the second. Let $\mathcal{F}$ and $\mathcal{G}$ be classes of objects of $\mathcal{C}$ and $\mathcal{D}$, respectively. Then T is said to be right balanced by $\mathcal{F} \times \mathcal{G}$ if:

(1) For every object X of $\mathcal{C}$, there is a $T(-, \mathcal{G})$-exact complex

$$\cdots \to F_1 \to F_0 \to X \to 0$$

with $F_i \in \mathcal{F}$ for every $i \geq 0$.

(2) For every object Y of $\mathcal{D}$, there is a $T(\mathcal{F}, -)$-exact complex

$$0 \to Y \to G^0 \to G^1 \to \cdots$$

with $G^i \in \mathcal{G}$ for every $i \geq 0$.

If, on the other hand, the complexes

$$\cdots \to F_1 \to F_0 \to X \to 0 \text{ and } 0 \to Y \to G^0 \to G^1 \to \cdots$$

are $T(\mathcal{G}, -)$-exact and $T(-, \mathcal{F})$-exact, respectively, then T is said to be left balanced by $\mathcal{G} \times \mathcal{F}$.

Example B.1.

(1) Let $\mathcal{C}$ be an Abelian category with enough projective and injective objects. The functor $\mathrm{Hom}_{\mathcal{C}}(-, -)$ is right balanced on $\mathcal{C} \times \mathcal{C}$ by $\mathcal{P}_0(\mathcal{C}) \times \mathcal{I}_0(\mathcal{C})$.

(2) If R is a left Noetherian ring, then $\mathrm{Hom}_R(-,-)$ is left balanced on ${}_R\mathbf{Mod} \times {}_R\mathbf{Mod}$ by $\mathcal{I}_0 \times \mathcal{I}_0$.

(3) Recall that a left (or right) R-module M is finitely presented if there is an exact sequence $0 \to K \to F \to M \to 0$ where F and K are finitely generated left R-modules and F is free. A ring R is left (resp. right) coherent if every finitely generated left (resp. right) ideal of R is finitely presented. For such an R, $\mathrm{Hom}_R(-,-)$ is left balanced on $\mathbf{Mod}_R \times \mathbf{Mod}_R$ by $\mathcal{F}_0 \times \mathcal{F}_0$.

The first example is a well know fact. For **(2)** and **(3)**, the reader can see the details in (21, Examples 8.3.4 & 8.3.6). The following result is proven in (21, Theorem 12.1.4) for the case $\mathcal{C} =_R \mathbf{Mod}$ with R an n-Iwanaga-Gorenstein ring.

Theorem B.1 (see (21, Theorem 8.2.14))

Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be Abelian categories and $T : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ be a functor contravariant in the first variable and covariant in the second. If $\mathcal{F}$ and $\mathcal{G}$ are classes of objects of $\mathcal{C}$ and $\mathcal{D}$, respectively, and T is right balanced on $\mathcal{C} \times \mathcal{D}$ by $\mathcal{F} \times \mathcal{G}$, then the complexes $T(\mathbf{F}_\bullet, Y)$ and $T(X, \mathbf{G}_\bullet)$ have isomorphic homology, for every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$.

We shall see in the next lemma that the Hom functor is right balanced by $\mathcal{GP}_0(\mathcal{C}) \times \mathcal{GI}_0(\mathcal{C})$. As a consequence, we have that the right derived functors $\mathrm{GExt}^i_{\mathcal{C}}(X, Y)$ of $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ can be computed by using left Gorenstein-projective resolutions of X or right Gorenstein-injective resolutions of Y.

Lemma B.2

Let $\mathcal{C}$ be a Gorenstein category. Then $\mathrm{Hom}_{\mathcal{C}}(-,-)$ is right balanced on $\mathcal{C} \times \mathcal{C}$ by $\mathcal{GP}_0(\mathcal{C}) \times \mathcal{GI}_0(\mathcal{C})$.

Proof.

We only prove that for every object $X \in \mathrm{Ob}(\mathcal{C})$ there exists an exact and $\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{GI}_0(\mathcal{C}))$-exact left Gorenstein-projective resolution

$$\cdots \to C_1 \to C_0 \to X \to 0.$$

A proof of the dual statement can be found in (21, Lemma 12.1.2) for the category ${}_R\mathbf{Mod}$. Let X be an object of $\mathcal{C}$. Let $\cdots \to C_1 \xrightarrow{f_1} C_0 \xrightarrow{f_0} X \to 0$ be an exact left Gorenstein-projective resolution of X such that:

(1) $\mathrm{Ker}(f_0) = W_0 \in \mathcal{W}(\mathcal{C})$.

(2) For every $k > 0$ there are short exact sequences $W_k \overset{i_k}{\hookrightarrow} C_k \overset{p_k}{\twoheadrightarrow} W_{k-1}$ with $f_k = i_{k-1} \circ p_k$.

Let $D \in \mathcal{GI}_0(\mathcal{C})$. We show

$$0 \to \mathrm{Hom}_{\mathcal{C}}(X, D) \to \mathrm{Hom}_{\mathcal{C}}(C_0, D) \to \mathrm{Hom}_{\mathcal{C}}(C_1, D) \to \cdots$$

is exact. Note that the sequence

$$0 \to \mathrm{Hom}_{\mathcal{C}}(W_{k-1}, D) \to \mathrm{Hom}_{\mathcal{C}}(C_k, D) \to \mathrm{Hom}_{\mathcal{C}}(W_k, D) \to 0$$

is exact for every $k > 0$, since $W_{k-1} \in \mathcal{W}(\mathcal{C})$ and $D \in \mathcal{GI}_0(\mathcal{C})$. So it suffices to show that the sequence

$$0 \to \mathrm{Hom}_{\mathcal{C}}(X, D) \to \mathrm{Hom}_{\mathcal{C}}(C_0, D) \to \mathrm{Hom}_{\mathcal{C}}(W_0, D) \to 0$$

is exact (Actually, we only need to show that $\mathrm{Hom}_{\mathcal{C}}(C_0, D) \to \mathrm{Hom}_{\mathcal{C}}(W_0, D)$ is surjective). Since D is Gorenstein-injective, there exists an exact sequence of injective objects $\cdots \to I_1 \overset{g_1}{\to} I_0 \overset{g_0}{\to} I^0 \overset{g^0}{\to} I^1 \overset{g^1}{\to} \cdots$ such that $D = \mathrm{Ker}(I^0 \to I^1)$. On the other hand, there exists $m \geq 0$ and an exact finite left projective resolution of W_0, say $0 \to P_m \overset{r_m}{\to} P_{m-1} \to \cdots \to P_1 \overset{r_1}{\to} P_0 \overset{r_0}{\to} W_0 \to 0$ for some nonnegative $m \in \mathbb{Z}$, since $W_0 \in \mathcal{W}(\mathcal{C})$. Consider a map $W_0 \overset{h}{\to} D$. Since $P_0, \ldots, P_k$ are projective, we can find fillers $P_j \overset{h_j}{\to} I_j$ for $0 \leq j < m$ and $P_m \overset{h_m}{\to} D' = \mathrm{Ker}(g_{m-1})$, to obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & P_m \arrow[r, "r_m"] \arrow[d, "h_m"'] & P_{m-1} \arrow[r] \arrow[d, "h_{m-1}"'] & \cdots \arrow[r] & P_1 \arrow[r, "r_1"] \arrow[d, "h_1"'] & P_0 \arrow[r, "r_0"] \arrow[d, "h_0"'] & W_0 \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & D' \arrow[r, "k"'] & I_{m-1} \arrow[r] & \cdots \arrow[r] & I_1 \arrow[r, "g_1"'] & I_0 \arrow[r, "g_0"'] & D \arrow[r] & 0
\end{tikzcd}$$

Consider the short exact sequence

$$0 \to P_m \overset{r_m}{\to} P_{m-1} \overset{d_m}{\to} \mathrm{Im}(r_{m-1}) \to 0.$$

We have $\mathrm{Ext}^1_{\mathcal{C}}(\mathrm{Im}(r_{m-1}), D') \cong \mathrm{Ext}^m_{\mathcal{C}}(W_0, D') = 0$ since $\mathrm{Im}(r_{m-1}) \in \Omega^{m-1}(W_0)$ and so D' is Gorenstein-injective. It follows there exists a morphism $P_{m-1} \overset{\varphi_{m-1}}{\to} D'$ such that $\varphi_{m-1} \circ r_m = h_m$. Working with the same exact sequence, we have $(h_{m-1} - g_m \circ \varphi_{m-1}) \circ r_m = 0$ and so there is a unique morphism $\mathrm{Im}(r_{m-1}) \overset{\widetilde{\varphi_{m-1}}}{\to} I_{m-1}$ such that $\widetilde{\varphi_{m-1}} \circ d_m = h_{m-1} - g_m \circ \varphi_{m-1}$. In a similar way, we can find a map $P_{m-2} \overset{\varphi_{m-1}}{\to} I_{m-1}$ such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(r_{m-2}) \arrow[r] \arrow[d, "\widetilde{\varphi_{m-1}}"'] & P_{k-2} \arrow[r] \arrow[dl, "\varphi_{m-2}"] & \mathrm{Im}(r_{m-2}) \arrow[r] & 0 \\
& I_{m-1} & & &
\end{tikzcd}$$

We keep on repeating this procedure until we find a map $W_0 \overset{\varphi_{-1}}{\to} I_0$ such that $\varphi_{-1} \circ r_0 = h_0 - g_1 \circ \varphi_0$. Note $(g_0 \circ \varphi_{-1} - h) \circ r_0 = g_0 \circ \varphi_{-1} \circ r_0 - h \circ r_0 = g_0 \circ (h_0 - g_1 \circ \varphi_0) - h \circ r_0 = g_0 \circ h_0 - h \circ r_0 = 0$. It follows $h = g_0 \circ \varphi_{-1}$ since r_0 is epic. On the other hand, since I_0 is injective, there exists a unique map $C_0 \overset{s_0}{\to} I_0$ such that the following diagram commutes:

$$
\begin{tikzcd}
0 \arrow[r] & W_0 \arrow[r, "i_0"] \arrow[d, "\varphi_{-1}"'] & C_0 \arrow[r, "f_0"] \arrow[dl, "s_0"] & X \arrow[r] & 0 \\
& I_0 & & &
\end{tikzcd}
$$

It follows $h = (g_0 \circ s_0) \circ i_0$, and hence $\mathrm{Hom}_{\mathcal{C}}(C_0, D) \to \mathrm{Hom}_{\mathcal{C}}(W_0, D)$ is onto. $\square$

Remark B.2.1 (see (21, Section 12.1)). We have the following properties:

(1) $\mathrm{GExt}^0_{\mathcal{C}}(X, Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, Y)$ for every pair $X, Y \in \mathrm{Ob}(\mathcal{C})$.

(2) X is a Gorenstein-projective object if, and only if, $\mathrm{GExt}^m_{\mathcal{C}}(X, Y) = 0$ for every $m > 0$ and every $Y \in \mathrm{Ob}(\mathcal{C})$.

(3) Y is a Gorenstein-injective object if, and only if, $\mathrm{GExt}^m_{\mathcal{C}}(X, Y) = 0$ for every $m > 0$ and every $X \in \mathrm{Ob}(\mathcal{C})$.

(4) Since $\mathcal{GP}_0(\mathcal{C})$ is a pre-covering class closed under finite direct sums, by Theorem 1.4.2 **(2)**, if $0 \to X' \to X \to X'' \to 0$ is a $\mathrm{Hom}_{\mathcal{C}}(\mathcal{GP}_0(\mathcal{C}), -)$-exact sequence, then there is a long exact sequence for every $Y \in \mathrm{Ob}(\mathcal{C})$:

$$\cdots \to \mathrm{GExt}^m_{\mathcal{C}}(X, Y) \to \mathrm{GExt}^m_{\mathcal{C}}(X', Y) \to \mathrm{GExt}^{m+1}_{\mathcal{C}}(X'', Y) \to \cdots$$

(5) Since $\mathcal{GI}_0(\mathcal{C})$ is a pre-enveloping class closed under finite direct sums, by the dual of Theorem 1.4.2 **(1)**, if $0 \to Y' \to Y \to Y'' \to 0$ is a $\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{GI}_0(\mathcal{C}))$-exact sequence, then there is a long exact sequence for every $X \in \mathrm{Ob}(\mathcal{C})$:

$$\cdots \to \mathrm{GExt}^m_{\mathcal{C}}(X, Y) \to \mathrm{GExt}^m_{\mathcal{C}}(X, Y'') \to \mathrm{GExt}^{m+1}_{\mathcal{C}}(X, Y') \to \cdots$$

Recall that in Chapter 1 we saw that for every pair of objects X and Y of an Abelian category $\mathcal{C}$ with enough projective and injective objects, the group $\mathrm{Ext}^1_{\mathcal{C}}(X, Y)$ is isomorphic to the group $E^{\mathcal{C}}(X, Y)$ of classes of short exact sequences $Y \hookrightarrow Z \twoheadrightarrow X$ under certain equivalence relation. Let $\mathcal{F}$ and $\mathcal{G}$ be classes of objects of $\mathcal{C}$ and define ${}_{\mathcal{F}}E(X, Y)$ (resp. $E_{\mathcal{G}}(X, Y)$) as the subset of $E^{\mathcal{C}}(X, Y)$ composed by the classes of short exact sequences $Y \hookrightarrow Z \twoheadrightarrow X$ which are $\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, -)$-exact (resp. $\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{G})$-exact). After showing that these subsets are actually subgroups of $E^{\mathcal{C}}(X, Y)$, we shall give a relationship between them and the right derived functors of $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ obtained by using left $\mathcal{F}$-resolutions of X and right $\mathcal{G}$-resolutions of Y.

Proposition B.3

Let $S_1 = Y \hookrightarrow Z_1 \twoheadrightarrow X$ and $S_2 = Y \hookrightarrow Z_2 \twoheadrightarrow X$ be two short exact sequences in an Abelian category $\mathcal{C}$, and $\mathcal{F}$ and $\mathcal{G}$ be classes of objects of $\mathcal{C}$.

If S_1 and S_2 are $\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, -)$-exact, then so is $S_1 +_B S_2$.	If S_1 and S_2 are $\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{G})$-exact, then so is $S_1 +_B S_2$.

Proof.

First we prove the left statement. Suppose S_1 and S_2 are $\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, -)$-exact. For the Baer sum $S_1 +_B S_2 = Y \overset{\alpha_1 +_B \alpha_2}{\hookrightarrow} Z_1 +_B Z_2 \overset{\beta_1 +_B \beta_2}{\twoheadrightarrow} X$, recall we have the pullback diagram

$$\begin{tikzcd}
 & Y \arrow[r, equal] \arrow[d, hook, "\widehat{\alpha_2}"'] & Y \arrow[d, hook, "\alpha_2"] \\
Y \arrow[r, hook, "\widehat{\alpha_1}"] \arrow[d, equal] & Z_1 \times_X Z_2 \arrow[r, two heads, "\rho_{Z_1}"] \arrow[d, two heads, "\rho_{Z_2}"] & Z_2 \arrow[d, two heads, "\beta_2"] \\
Y \arrow[r, hook, "\alpha_1"'] & Z_1 \arrow[r, two heads, "\beta_1"'] & X
\end{tikzcd}$$

Recall also that $\alpha_1 +_B \alpha_2 = \pi \circ \widehat{\alpha_1} = \pi \circ \widehat{\alpha_2}$, and that $\beta_1 +_B \beta_2$ is the filler satisfying $(\beta_1 +_B \beta_2) \circ \pi = \beta_1 \circ \rho_{Z_1} = \beta_2 \circ \rho_{Z_2}$. To show

$$0 \to \mathrm{Hom}_{\mathcal{C}}(F, Y) \to \mathrm{Hom}_{\mathcal{C}}(F, Z_1 +_B Z_2) \to \mathrm{Hom}_{\mathcal{C}}(F, X) \to 0$$

is exact for every $F \in \mathcal{F}$, it suffices to verify that $\mathrm{Hom}_{\mathcal{C}}(F, Z_1 +_B Z_2) \to \mathrm{Hom}_{\mathcal{C}}(F, X)$ is onto. Let $h \in \mathrm{Hom}_{\mathcal{C}}(F, X)$. Since the maps $\mathrm{Hom}_{\mathcal{C}}(F, Z_1) \to \mathrm{Hom}_{\mathcal{C}}(F, X)$ and

$\mathrm{Hom}_{\mathcal{C}}(F, Z_2) \to \mathrm{Hom}_{\mathcal{C}}(F, X)$ are surjective, there exist morphisms $F \xrightarrow{f_1} Z_1$ and $F \xrightarrow{f_2} Z_2$ such that $h = \beta_1 \circ f_1$ and $h = \beta_2 \circ f_2$. By the universal property of pullbacks, there exists a unique map $F \xrightarrow{f_{12}} Z_1 \times_X Z_2$ such that $\rho_{Z_1} \circ f_{12} = f_1$ and $\rho_{Z_2} \circ f_{12} = f_2$. Set $f := \pi \circ f_{12} : F \to Z_1 +_B Z_2$. We have $(\beta_1 +_B \beta_2) \circ f = h$.

For the right statement, there is no need to use the universal property of pullbacks. Let $G \in \mathcal{G}$. We show $\mathrm{Hom}_{\mathcal{C}}(Z_1 +_B Z_2, G) \to \mathrm{Hom}_{\mathcal{C}}(Y, G)$ is surjective. For $i = 1, 2$, let $Z_i \xrightarrow{l_i} G$ be maps such that $l_i \circ \alpha_i = h$. Then the sum $l_1 \circ \rho_{Z_1} + l_2 \circ \rho_{Z_2} : Z_1 \times_X Z_2 \to G$ satisfies $(l_1 \circ \rho_{Z_1} + l_2 \circ \rho_{Z_2}) \circ (\widehat{\alpha_1} - \widehat{\alpha_2}) = 0$. So by the universal property of cokernels, there is a unique map $Z_1 +_B Z_2 \xrightarrow{l} G$ such that $l \circ \pi = l_1 \circ \rho_{Z_1} + l_2 \circ \rho_{Z_2}$. It follows that $l \circ (\alpha_1 +_B \alpha_2) = h$. $\square$

Notice ${}_{\mathcal{F}}E(X, Y)$ and $E_{\mathcal{G}}(X, Y)$ are not empty since $Y \hookrightarrow Y \oplus X \twoheadrightarrow X$ is both $\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, -)$-exact and $\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{G})$-exact. After restricting $+_B$ on the products ${}_{\mathcal{F}}E(X, Y) \times {}_{\mathcal{F}}E(X, Y)$ and $E_{\mathcal{G}}(X, Y) \times E_{\mathcal{G}}(X, Y)$, we get binary operations

$${}_{\mathcal{F}}E(X, Y) \times {}_{\mathcal{F}}E(X, Y) \to {}_{\mathcal{F}}E(X, Y) \text{ and } E_{\mathcal{G}}(X, Y) \times E_{\mathcal{G}}(X, Y) \to E_{\mathcal{G}}(X, Y).$$

With these restrictions, both ${}_{\mathcal{F}}E(X, Y)$ and $E_{\mathcal{G}}(X, Y)$ are subgroups of $E^{\mathcal{C}}(X, Y)$.

Assume $\mathcal{F}$ is a pre-covering class of $\mathcal{C}$. Let $X \in \mathrm{Ob}(\mathcal{C})$ and $\mathbf{F}_\bullet$ be a deleted left $\mathcal{F}$-resolution of X. For every object $Y \in \mathrm{Ob}(\mathcal{C})$, denote the right nth derived functor of $\mathrm{Hom}_{\mathcal{C}}(-, Y)$ evaluated at X by

$$\mathcal{F}\text{-}\mathrm{Ext}^n_{\mathcal{C}}(X, Y) := R^n(\mathrm{Hom}_{\mathcal{C}}(-, Y))(X).$$

Dually, if $\mathcal{G}$ is a pre-enveloping class of $\mathcal{C}$, then for every object $Y \in \mathrm{Ob}(\mathcal{C})$ and every deleted right $\mathcal{G}$-resolution $\mathbf{G}_\bullet$ of Y, we denote the right nth derived functor of $\mathrm{Hom}_{\mathcal{C}}(X, -)$ evaluated at Y by

$$\mathrm{Ext}^n_{\mathcal{C}}\text{-}\mathcal{G}(X, Y) := R^n(\mathrm{Hom}_{\mathcal{C}}(X, -))(Y).$$

Theorem B.4

Let $\mathcal{F}$ and $\mathcal{G}$ be the left and right halves of two complete cotorsion pairs $(\mathcal{F}, \mathcal{F}^{\perp})$ and $({}^{\perp}\mathcal{G}, \mathcal{G})$ (so they are closed under finite biproducts, and $\mathcal{F}$ (resp. $\mathcal{G}$) is pre-covering (resp. pre-enveloping)). There are group isomorphisms:

$\mathcal{F}\text{-}\mathrm{Ext}^1_{\mathcal{C}}(X,Y) \cong {}_{\mathcal{F}}E(X,Y)$.	$\mathrm{Ext}^1_{\mathcal{C}}\text{-}\mathcal{G}(X,Y) \cong E_{\mathcal{G}}(X,Y)$.

Proof.

We only construct the left isomorphism. Consider a representative

$$S = Y \overset{\alpha}{\hookrightarrow} Z \overset{\beta}{\twoheadrightarrow} X$$

of a class in ${}_{\mathcal{F}}E(X,Y)$. Since $(\mathcal{F}, (\mathcal{F})^{\perp})$ is a complete cotorsion pair, we can obtain an exact left $\mathcal{F}$-resolution $\cdots \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} X \to 0$ (apply the reasoning given in Proposition 4.3.1). Recall $\mathcal{F}\text{-}\mathrm{Ext}^1_{\mathcal{F}}(X,Y) = \frac{\mathrm{Ker}(f_2^*)}{\mathrm{Im}(f_1^*)}$, where $f_1^* : g \in \mathrm{Hom}_{\mathcal{C}}(F_0, Y) \mapsto g \circ f_1 \in \mathrm{Hom}_{\mathcal{C}}(F_1, Y)$ and $f_2^* : h \in \mathrm{Hom}_{\mathcal{C}}(F_1, Y) \mapsto h \circ f_2 \in \mathrm{Hom}_{\mathcal{C}}(F_2, Y)$. Since S is $\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, -)$-exact, the sequence $\mathrm{Hom}_{\mathcal{C}}(F_0, S)$ is also exact. So there exists a morphism $g_0 : F_0 \to Z$ such that $f_0 = \beta \circ g_0$. Note that $\beta \circ (g_0 \circ f_1) = 0$, and since S is exact, there exists a unique homomorphism $g_S : F_1 \to Y$ such that $\alpha \circ g_S = g_0 \circ f_1$.

$$\begin{tikzcd}
 & & & & Y \arrow[d, hook, "\alpha"'] & \\
 & & & & Z \arrow[d, two heads, "\beta"'] & \\
\cdots \arrow[r] & F_2 \arrow[r, "f_2"'] & F_1 \arrow[r, "f_1"'] \arrow[rruu, "g_S"] & F_0 \arrow[r, "f_0"'] \arrow[ru, "g_0"] & X \arrow[r] & 0
\end{tikzcd}$$

On the other hand, $f_2^*(g_S) = g_S \circ f_2$, and $\alpha \circ (g_S \circ f_2) = (\alpha \circ g_S) \circ f_2 = (g_0 \circ f_1) \circ f_2 = g_0 \circ (f_1 \circ f_2) = 0$. Since α in a monomorphism, we have $g_S \circ f_2 = 0$. Then $g_S \in \mathrm{Ker}(f_2^*)$. One can check that the map $\Phi : {}_{\mathcal{F}}E(X,Y) \to \mathcal{F}\text{-}\mathrm{Ext}^1_{\mathcal{C}}(X,Y)$ defined by $\Phi([S]) := g_S + \mathrm{Im}(f_1^*)$, where $g_S + \mathrm{Im}(f_1^*)$ is the class of g_S in $\mathcal{F}\text{-}\mathrm{Ext}^1_{\mathcal{C}}(X,Y)$, is a well defined group homomorphism.

Now we show Φ is monic. Suppose $S = Y \overset{\alpha}{\hookrightarrow} Z \overset{\beta}{\twoheadrightarrow} X$ is a representative such that $g_S + \mathrm{Im}(f_1^*) = \Phi([S]) = 0 + \mathrm{Im}(f_1^*)$. Then $g_S = r \circ f_1$ for some morphism $F_0 \overset{r}{\to} Y$. It follows $(g_0 - \alpha \circ r) \circ f_1 = 0$ and $\beta \circ (g_0 - \alpha \circ r) = f_0$. Hence we may assume $g_s = 0$. Note that there is a unique morphism $X \overset{k_0}{\to} Z$ such that $k_0 \circ f_0 = g_0$, since $g_0 \circ f_1 = 0$ and the left $\mathcal{F}$-resolution of X is exact. It follows $(\beta \circ k_0) \circ f_0 = f_0$ and so $\beta \circ k_0 = \mathrm{id}_X$, since f_0 is epic.

To show that Φ is also epic, let $\overline{h} \in \mathcal{F}\text{-}\mathrm{Ext}^1_{\mathcal{C}}(X,Y)$. Then we have $h \circ f_2 = 0$, and so there exists a unique morphism $\mathrm{Ker}(f_0) \overset{h'}{\to} Y$ such that $h' \circ \widehat{f_1} = h$, where $f_1 = F_1 \overset{\widehat{f_1}}{\to} \mathrm{Im}(f_1) \overset{j_0}{\to} Y$.

$$\begin{tikzcd}
\mathrm{Im}(f_2) \arrow[r, hook, "j_1"] & F_1 \arrow[r, two heads, "\widehat{f_1}"] \arrow[dr, "h"'] & \mathrm{Ker}(f_0) \arrow[d, dotted, "\exists!\ h'"] \\
 & & Y
\end{tikzcd}$$

Taking the pushout of j_0 and h', we get the following commutative diagram with exact rows:

$$\begin{tikzcd}
\mathrm{Ker}(f_0) \arrow[r, hook, "j_0"] \arrow[d, "h'"'] & F_0 \arrow[r, two heads, "f_0"] \arrow[d, "i"'] & X \arrow[d, equal] \\
Y \arrow[r, hook, "\alpha"'] & Y \coprod_{\mathrm{Ker}(f_0)} F_0 \arrow[r, two heads, "\beta"'] & X
\end{tikzcd}$$

One can check that the following diagram commutes:

$$
\begin{tikzcd}
 & & & & & Y \arrow[d, hook, "\alpha"] & \\
 & & & & & Y \coprod_{\mathrm{Ker}(f_0)} F_0 \arrow[d, "\beta"] & \\
\cdots \arrow[r] & F_2 \arrow[r, "f_2"] & F_1 \arrow[r, "\widehat{f_1}"] \arrow[rr, bend right, "f_1"'] \arrow[uurrr, bend left, "h' \circ \widehat{f_1}"] & \mathrm{Ker}(f_0) \arrow[r, "j_0"] \arrow[uurr, "h'"] & F_0 \arrow[r, "f_0"] \arrow[ur, "i"] & X \arrow[r] & 0
\end{tikzcd}
$$

We have the following commutative diagram with exact rows:

$$
\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[r, "f_2"] & F_1 \arrow[r, "f_1"] \arrow[d, "h"'] & F_0 \arrow[r, "f_0"] \arrow[d, "i"'] & X \arrow[r] \arrow[d, equal] & 0 \\
 & 0 \arrow[r] & Y \arrow[r, "\alpha"'] & Y \coprod_{\mathrm{Ker}(f_0)} F_0 \arrow[r, "\beta"'] & X \arrow[r] & 0
\end{tikzcd}
$$

To show that the bottom row is $\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, -)$-exact, it suffices to verify that the homomorphism $\mathrm{Hom}_{\mathcal{C}}(F, Y \coprod_{\mathrm{Ker}(f_0)} F_0) \to \mathrm{Hom}_{\mathcal{C}}(F, X)$ is surjective, for every $F \in \mathcal{F}$. The diagram

$$
\begin{tikzcd}
\cdots \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(F, F_2) \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(F, F_1) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{C}}(F, F_0) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{C}}(F, X) \arrow[r] \arrow[d, equal] & 0 \\
 & 0 \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(F, Y) \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(F, Y \coprod_{\mathrm{Ker}(f_0)} F_0) \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(F, X) \arrow[r] & 0
\end{tikzcd}
$$

is commutative in **Ab**, where the top row is exact, so

$$
\mathrm{Hom}_{\mathcal{C}}\left(F, Y \coprod_{\mathrm{Ker}(f_0)} F_0\right) \to \mathrm{Hom}_{\mathcal{C}}(F, X)
$$

is onto. Then $h + \mathrm{Im}(f_1^*) = h' \circ \widehat{f_1} + \mathrm{Im}(f_1^*) = \Phi\left(\left[Y \hookrightarrow Y \coprod_{\mathrm{Ker}(f_0)} F_0 \twoheadrightarrow X\right]\right)$. $\square$

Corollary B.5

Let $\mathcal{C}$ be a Gorenstein category. There are group isomorphisms:

$\mathrm{GExt}^1_{\mathcal{C}}(X,Y) \cong {}_{\mathcal{GP}_0(\mathcal{C})}E(X,Y)$.	$\mathrm{GExt}^1_{\mathcal{C}}(X,Y) \cong E_{\mathcal{GI}_0(\mathcal{C})}(X,Y)$.

We shall present some natural isomorphisms for Gorenstein-extension functors involving disk and sphere chain complexes. Let $\mathcal{F}$ and $\mathcal{G}$ as in the previous theorem. In the case of disk complexes, we consider the subgroup ${}_{\widetilde{\mathcal{F}}}E(X, D^{n+1}(M))$ of $E^{\mathbf{Ch}(\mathcal{C})}(X, D^{n+1}(M))$, in order to construct an isomorphism to ${}_{\mathcal{F}}E(X_n, M)$. For example, if $\mathcal{F} = \mathcal{GP}_0(\mathcal{C})$, this situation seems to be inappropriate at a first glance, since we cannot replace the group $\mathrm{GExt}^1_{\mathbf{Ch}(\mathcal{C})}(X, D^{n+1}(M))$ by ${}_{\widetilde{\mathcal{GP}_0(\mathcal{C})}}E(X, D^{n+1}(M))$. However, we shall see ${}_{\widetilde{\mathcal{GP}_0(\mathcal{C})}}E(X, D^{n+1}(M))$ and ${}_{\mathcal{GP}_0(\mathbf{Ch}(\mathcal{C}))}E(X, D^{n+1}(M))$ turn out to be the same group if $\mathcal{C}$ is a Gorenstein category.

Proposition B.6

Let $\mathcal{C}$ be an Abelian category and $\mathcal{F}$ and $\mathcal{G}$ be classes of objects of $\mathcal{C}$. If $M \in \mathrm{Ob}(\mathcal{C})$ and $X, Y \in \mathbf{Ch}(\mathcal{C})$, then we have natural isomorphisms:

${}_{\mathcal{F}}E(X_n, M) \overset{\mathbf{(1)}}{\to} {}_{\mathrm{dw}\widetilde{\mathcal{F}}}E(X, D^{n+1}(M))$.	${}_{\mathcal{F}}E(M, Y_n) \overset{\mathbf{(1')}}{\to} {}_{\mathrm{dw}\widetilde{\mathcal{F}}}E(D^n(M), Y)$.
$E_{\mathcal{G}}(X_n, M) \overset{\mathbf{(2)}}{\to} E_{\mathrm{dw}\widetilde{\mathcal{G}}}(X, D^{n+1}(M))$.	$E_{\mathcal{G}}(M, Y_n) \overset{\mathbf{(2')}}{\to} E_{\mathrm{dw}\widetilde{\mathcal{G}}}(D^n(M), Y)$.

Proof.

The maps constructed in (27, Lemma 3.1) work to define **(1)**, **(2)**, **(1')** and **(2')**.

(1) Let $[S] = [0 \to D^{n+1}(M) \to Z \to X \to 0] \in {}_{\mathrm{dw}\widetilde{\mathcal{F}}}E(X, D^{n+1}(M))$. Since the sequence $0 \to D^{n+1}(M) \to Z \to X \to 0$ is exact in $\mathbf{Ch}(\mathcal{C})$, we have $0 \to M \to Z_n \to X_n \to 0$ is exact in $\mathcal{C}$. We show it is also $\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, -)$-exact. Let $F \in \mathcal{F}$. Then $D^n(F) \in \mathrm{dw}\widetilde{\mathcal{F}}$. We have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(F, M) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathcal{C}}(F, Z_n) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathcal{C}}(F, X_n) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_{\mathbf{Ch}(\mathcal{C})}(D^n(F), D^{n+1}(M)) \arrow[r] & \mathrm{Hom}_{\mathbf{Ch}(\mathcal{C})}(D^n(F), Z) \arrow[r] & \mathrm{Hom}_{\mathbf{Ch}(\mathcal{C})}(D^n(F), X) \arrow[r] & 0
\end{tikzcd}$$

Since the bottom row is exact and the vertical arrows are isomorphisms, we have that the top row is also exact. So $[0 \to M \to Z_n \to X_n \to 0] \in {}_{\mathcal{F}}E(X_n, M)$.

Define a map $\Phi : {}_{\mathrm{dw}\widetilde{\mathcal{F}}}E(X, D^{n+1}(M)) \to {}_{\mathcal{F}}E(X_n, M)$ by setting

$$\begin{tikzcd}
{[0} \arrow[r] & D^{n+1}(M) \arrow[r] & Z \arrow[r] \arrow[d, mapsto, "\Phi"] & X \arrow[r] & {0]} \\
{[0} \arrow[r] & M \arrow[r] & Z_n \arrow[r] & X_n \arrow[r] & {0]}
\end{tikzcd}$$

It is not hard to verify that Φ is a well defined group homomorphism.

Now we construct an inverse $\Psi : {}_{\mathcal{F}}E(X_n, M) \to {}_{\mathrm{dw}\widetilde{\mathcal{F}}}E(X, D^{n+1}(M))$ for Φ. Consider a class $[S] = [0 \to M \xrightarrow{\alpha} Z \xrightarrow{\beta} X_n \to 0] \in {}_{\mathcal{F}}E(X_n, M)$. Consider the pullback of β and ∂^X_{n+1}. We get the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\widetilde{\alpha}_{n+1}"] \arrow[d, "="] & Z \times_{X_n} X_{n+1} \arrow[r, "\widetilde{\beta}_{n+1}"] \arrow[d, "\partial^{\widetilde{Z}}_{n+1}"] & X_{n+1} \arrow[r] \arrow[d, "\partial^X_{n+1}"] & 0 \\
0 \arrow[r] & M \arrow[r, "\alpha"'] & Z \arrow[r, "\beta"'] & X_n \arrow[r] & 0
\end{tikzcd}$$

Let $\widetilde{Z}$ be the complex $\cdots \to X_{n+2} \xrightarrow{\partial^X_{n+2}} Z \times_{X_n} X_{n+1} \xrightarrow{\partial^{\widetilde{Z}}_{n+1}} Z \xrightarrow{\partial^{\widetilde{Z}}_{n}} \cdots$, where $\partial^{\widetilde{Z}}_n := \partial^X_n \circ \beta$, $\partial^{\widetilde{Z}}_{n+2}$ is the map induced by the universal property of pullbacks satisfying $\widetilde{\beta}_{n+1} \circ \partial^{\widetilde{Z}}_{n+2} = \partial^X_{n+2}$, and $\partial^{\widetilde{Z}}_k = \partial^X_k$ for every $k \neq n, n+1, n+2$. From this we get an exact sequence $D^{n+1}(M) \overset{\tilde{\alpha}}{\hookrightarrow} \tilde{Z} \overset{\tilde{\beta}}{\twoheadrightarrow} X$ in $\mathbf{Ch}(\mathcal{C})$, where $\tilde{\alpha}$ and $\tilde{\beta}$ are the chain maps given by:

$$\widetilde{\alpha}_k = \begin{cases} \alpha & \text{if } k = n, \\ \widetilde{\alpha}_{n+1} & \text{if } k = n+1, \\ 0 & \text{otherwise.} \end{cases} \quad \text{and} \quad \widetilde{\beta}_k = \begin{cases} \beta & \text{if } k = n, \\ \widetilde{\beta}_{n+1} & \text{if } k = n+1, \\ \mathrm{id}_{X_k} & \text{otherwise.} \end{cases}$$

We prove that the previous sequence is $\mathrm{Hom}(\mathrm{dw}\widetilde{\mathcal{F}}, -)$-exact. Let $F \in \mathrm{dw}\widetilde{\mathcal{F}}$ and suppose we are given a map $F \xrightarrow{f} X$. We want to find a chain map $g : F \longrightarrow \tilde{Z}$ such that $\tilde{\beta} \circ g = f$:

$$
\begin{tikzcd}
 & & \ddots \arrow[ddr] & & & & & \\
\ddots \arrow[ddr] & \ddots \arrow[ddr] & \ddots \arrow[ddr] & & & & & \\
 & & & F_{n+2} \arrow[dl, "g_{n+2}"'] \arrow[d, "f_{n+2}"] \arrow[ddr, "\partial^F_{n+2}"] & & & & \\
 & 0 \arrow[r] \arrow[ddr] & \tilde{Z}_{n+2} \arrow[r, equal] \arrow[ddr, "\partial^{\tilde{Z}}_{n+2}"'] & X_{n+2} \arrow[ddr, "\partial^X_{n+2}"] & & & & \\
 & & & & F_{n+1} \arrow[dl, "g_{n+1}"'] \arrow[d, "f_{n+1}"] \arrow[ddr, "\partial^F_{n+1}"] & & & \\
 & & M \arrow[r, "\alpha_{n+1}"'] \arrow[ddr, equal] & \tilde{Z}_{n+1} \arrow[r, "\beta_{n+1}"'] \arrow[ddr, "\partial^{\tilde{Z}}_{n+1}"'] & X_{n+1} \arrow[ddr, "\partial^X_{n+1}"] & & & \\
 & & & & & F_n \arrow[dl, "g_n"'] \arrow[d, "f_n"] \arrow[ddr, "\partial^F_n"] & & \\
 & & & M \arrow[r, "\alpha_n"'] \arrow[ddr] & \tilde{Z}_n \arrow[r, "\beta_n"'] \arrow[ddr, "\partial^{\tilde{Z}}_n"'] & X_n \arrow[ddr, "\partial^X_n"] & & \\
 & & & & & & F_{n-1} \arrow[dl, "g_{n-1}"'] \arrow[d, "f_{n-1}"] \arrow[ddr] & \\
 & & & & 0 \arrow[r] \arrow[ddr] & \tilde{Z}_{n-1} \arrow[r, equal] \arrow[ddr] & X_{n-1} \arrow[ddr] & \\
 & & & & & & & \ddots \\
 & & & & & \ddots & \ddots & \ddots
\end{tikzcd}
$$

We set $g_m = f_m$ if $m \geq n+2$ or $m \leq n-1$. Since the nth sequence is $\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, -)$-exact, there exists $g_n : F_n \to Z$ such that $\beta_n \circ g_n = f_n$. We have $\partial_n^{\widetilde{Z}} \circ g_n = \delta_n \circ g_n = \partial_n^X \circ \beta_n \circ g_n = \partial_n^X \circ f_n = f_{n-1} \circ \partial_n^F = g_{n-1} \circ \partial_n^F$. Now by the universal property of pullbacks, there exists a homomorphism $g_{n+1} : F_{n+1} \to Z \times_{X_n} X_{n+1}$ such that the following diagram commutes:

$$
\begin{tikzcd}
F_{n+1} \arrow[dr, "g_{n+1}"] \arrow[drr, bend left, "f_{n+1}"] \arrow[ddr, bend right, "g_n \circ \partial^F_{n+1}"'] & & \\
& Z \times_{X_n} X_{n+1} \arrow[r, "\widetilde{\beta}_{n+1}"] \arrow[d, "\partial^{\widetilde{Z}}_{n+1}"'] & X_{n+1} \arrow[d, "\partial^X_{n+1}"] \\
& Z \arrow[r, "\beta"'] & X_n
\end{tikzcd}
$$

In order to show that $g = (g_m)_{m \in \mathbb{Z}}$ is a chain map, it is only left to show that $g_{n+1} \circ \partial_{n+1}^F = \partial_{n+2}^{\widetilde{Z}} \circ g_{n+2} = \partial_{n+2}^{\widetilde{Z}} \circ f_{n+2}$. Note that the following diagram commutes:

$$
\begin{tikzcd}
F_{n+2} \arrow[dr, bend left=10, "g_{n+1} \circ \partial^F_{n+1}"] \arrow[dr, bend right=10, "\partial^{\widetilde{Z}}_{n+2} \circ f_{n+2}"'] \arrow[drr, bend left, "f_{n+1} \circ \partial^F_{n+2}"] \arrow[ddr, bend right, "0"'] & & \\
& Z \times_{X_n} X_{n+1} \arrow[r, "\widetilde{\beta}_{n+1}"] \arrow[d, "\partial^{\widetilde{Z}}_{n+1}"'] & X_{n+1} \arrow[d, "\partial^X_{n+1}"] \\
& Z \arrow[r, "\beta"'] & X_n
\end{tikzcd}
$$

By the universal property of pullbacks, we get $g_{n+1} \circ \partial_{n+2}^F = \partial_{n+2}^{\widetilde{Z}} \circ f_{n+2}$. Hence, we have a chain map $g : F \to \tilde{Z}$ such that $\tilde{\beta} \circ g = f$.

We set a map $\Psi : {}_{\mathcal{F}}E(X_n, M) \to {}_{\mathrm{dw}\widetilde{\mathcal{F}}}E(X, D^{n+1}(M))$ by

$$\begin{tikzcd}
{[0} \arrow[r] & M \arrow[r, "\alpha"] & Z \arrow[r, "\beta"] \arrow[d, mapsto, "\Psi"] & X_n \arrow[r] & 0] \\
{[0} \arrow[r] & D^{n+1}(M) \arrow[r, "\tilde{\alpha}"'] & \widetilde{Z} \arrow[r, "\tilde{\beta}"'] & X \arrow[r] & 0]
\end{tikzcd}$$

It is not hard to see that Ψ is a well defined group homomorphism such that $\mathrm{id}_{{}_{\mathrm{dw}\widetilde{F}}E(X, D^{n+1}(M))} = \Psi \circ \Phi$ and $\mathrm{id}_{\mathcal{F}E(X_n, M)} = \Phi \circ \Psi$.

(2) We use the same construction given in **(1)**. Given a class

$$[0 \to D^{n+1}(M) \to Z \to X \to 0]$$

in $E_{\mathrm{dw}\widetilde{\mathcal{G}}}(X, D^{n+1}(M))$, one can show as in **(1)** that the sequence

$$0 \to M \to Z_n \to X_n \to 0$$

is $\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{G})$-exact, by using Proposition 1.3.1 **(2)**.

Now if we are given an exact and $\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{G})$ sequence

$$0 \to M \xrightarrow{\alpha} Z \xrightarrow{\beta} X_n \to 0,$$

we show that the short exact sequence of complexes obtained by taking the pullback of β and ∂^X_{n+1} is $\mathrm{Hom}_{\mathbf{Ch}(\mathcal{C})}(-, \mathrm{dw}\widetilde{\mathcal{G}})$-exact. Let $G \in \mathrm{dw}\widetilde{\mathcal{G}}$. We construct a chain map $\widetilde{Z} \xrightarrow{h} G$ such that the following diagram commutes for a given chain map $D^{n+1}(M) \xrightarrow{f} G$:

$$
\begin{tikzcd}
\ddots \arrow[dr] & \ddots \arrow[dr] & \ddots \arrow[dr] & & & & & \\
\ddots \arrow[dr] & 0 \arrow[r] \arrow[d, "0"'] \arrow[dr] & X_{n+2} \arrow[r, equal] \arrow[dl, "h_{n+2}"'] \arrow[dr, "\partial^{\widetilde{Z}}_{n+2}"] & X_{n+2} \arrow[dr, "\partial^{X}_{n+2}"] & & & & \\
& G_{n+2} \arrow[dr, "\partial^{G}_{n+2}"'] & M \arrow[r, hook, "\widetilde{\alpha}_{n+1}"] \arrow[d, "f_{n+1}"'] \arrow[dr, equal] & \widetilde{Z}_{n+1} \arrow[r, two heads, "\widetilde{\beta}_{n+1}"] \arrow[dl, "h_{n+1}"'] \arrow[dr, "\partial^{\widetilde{Z}}_{n+1}"] & X_{n+1} \arrow[dr, "\partial^{X}_{n+1}"] & & & \\
& & G_{n+1} \arrow[dr, "\partial^{G}_{n+1}"'] & M \arrow[r, hook, "\alpha"] \arrow[d, "f_n"'] \arrow[dr] & Z \arrow[r, two heads, "\beta"] \arrow[dl, "h_n"'] \arrow[dr, "\partial^{\widetilde{Z}}_{n}"] & X_n \arrow[dr, "\partial^{X}_{n}"] & & \\
& & & G_n \arrow[dr, "\partial^{G}_{n}"'] & 0 \arrow[r] \arrow[d, "f_{n-1}"'] \arrow[dr] & X_{n-1} \arrow[r, equal] \arrow[dl, "h_{n-1}"'] \arrow[dr] & X_{n-1} \arrow[dr] & \\
& & & & G_{n-1} \arrow[dr] & \ddots & \ddots & \ddots \\
& & & & & \ddots & &
\end{tikzcd}
$$

For every $k \neq n, n+1$, we set $h_k = 0$. Since the sequence

$$0 \to M \to Z \to X_n \to 0$$

is $\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{G})$-exact, there exists a map $Z \xrightarrow{h'_{n+1}} G_{n+1}$ such that $f_{n+1} = h'_{n+1} \circ \alpha$. Set $h_{n+1} := h'_{n+1} \circ \partial^{\widetilde{Z}}_{n+1}$ and $h_n := \partial^G_{n+1} \circ h'_{n+1}$. We have:

$$
\begin{aligned}
h_{n+1} \circ \widetilde{\alpha}_{n+1} &= h'_{n+1} \circ \partial^{\widetilde{Z}}_{n+1} \circ \widehat{\alpha} = h'_{n+1} \circ \alpha = f_{n+1}, \\
h_n \circ \alpha &= \partial^G_{n+1} \circ h'_{n+1} \circ \alpha = \partial^G_{n+1} \circ f_{n+1} = f_n, \\
h_{n+1} \circ \partial^{\widetilde{Z}}_{n+2} &= h'_{n+1} \circ \partial^{\widetilde{Z}}_{n+1} \circ \partial^{\widetilde{Z}}_{n+2} = 0 = \partial^G_{n+1} \circ h_{n+2}, \\
h_n \circ \partial^{\widetilde{Z}}_{n+1} &= \partial^G_{n+1} \circ h'_{n+1} \circ \partial^{\widetilde{Z}}_{n+1} = \partial^G_{n+1} \circ h_{n+1}, \\
h_{n-1} \circ \partial^{\widetilde{Z}}_{n} &= 0 = \partial^G_n \circ \partial^G_{n+1} \circ h'_{n+1} = \partial^G_n \circ h_n.
\end{aligned}
$$

Hence, $h = (h_k : k \in \mathbb{Z})$ is a chain map satisfying $h \circ \widetilde{\alpha} = f$, and so $D^{n+1}(M) \overset{\widetilde{\alpha}}{\hookrightarrow} \widetilde{Z} \overset{\widetilde{\beta}}{\twoheadrightarrow} X$ is $\mathrm{Hom}_{\mathbf{Ch}(\mathcal{C})}(-, \mathrm{dw}\widetilde{G})$-exact. □

In the previous proof, note that when we chose $D^n(F)$ with $F \in \mathcal{F}$, we have that $D^n(F)$ is actually a complex in $\widetilde{\mathcal{F}}$. So we can restrict Φ and get a map ${}_{\widetilde{\mathcal{F}}}E(X, D^{n+1}(M)) \to {}_{\mathcal{F}}E(X_n, M)$. On the other hand, if we assume $\mathcal{F}$ is closed under extensions, we have that $F_n \in \mathcal{F}$ for every $F \in \widetilde{\mathcal{F}}$ (it suffices to consider the sequence $Z_n(F) \hookrightarrow F_n \twoheadrightarrow Z_{n-1}(F)$ for each $n \in \mathbb{Z}$). So under this additional hypothesis, we can obtain an inverse ${}_{\mathcal{F}}E(X_n, M) \to {}_{\widetilde{\mathcal{F}}}E(X, D^{n+1}(M))$ of the map ${}_{\widetilde{\mathcal{F}}}E(X, D^{n+1}(M)) \to {}_{\mathcal{F}}E(X_n, M)$. We have the following result:

Proposition B.7

Let $\mathcal{C}$ be an Abelian category and $\mathcal{F}$ and $\mathcal{G}$ be classes of objects of $\mathcal{C}$ which are closed under extensions. If $M \in \mathrm{Ob}(\mathcal{C})$ and $X, Y \in \mathbf{Ch}(\mathcal{C})$, then we have natural isomorphisms:

${}_{\mathcal{F}}E(X_n, M) \overset{(\mathbf{1})}{\to} {}_{\widetilde{\mathcal{F}}}E(X, D^{n+1}(M))$.	${}_{\mathcal{F}}E(M, Y_n) \overset{(\mathbf{1'})}{\to} {}_{\widetilde{F}}E(D^n(M), Y)$.
$E_{\mathcal{G}}(X_n, M) \overset{(\mathbf{2})}{\to} E_{\widetilde{\mathcal{G}}}(X, D^{n+1}(M))$.	$E_{\mathcal{G}}(M, Y_n) \overset{(\mathbf{2'})}{\to} E_{\widetilde{\mathcal{G}}}(D^n(M), Y)$.

It follows that when $\mathcal{F}$ and $\mathcal{G}$ are closed under extensions, ${}_{\widetilde{\mathcal{F}}}E(X, D^{n+1}(M)) \cong {}_{\mathrm{dw}\widetilde{\mathcal{F}}}E(X, D^{n+1}(M))$, $E_{\widetilde{\mathcal{G}}}(X, D^{n+1}(M)) \cong E_{\mathrm{dw}\widetilde{\mathcal{G}}}(X, D^{n+1}(M))$, ${}_{\widetilde{F}}E(D^n(M), Y) \cong {}_{\mathrm{dw}\widetilde{F}}E(D^n(M), Y)$, and $E_{\widetilde{\mathcal{G}}}(D^n(M), Y) \cong E_{\mathrm{dw}\widetilde{\mathcal{G}}}(D^n(M), Y)$. This seems to be a weird behaviour at a first glance, but this can be clarified in the following statement.

Proposition B.8

Let $\mathcal{C}$ be an Abelian category and $\mathcal{F}$ and $\mathcal{G}$ be classes of objects of $\mathcal{C}$ which are closed under extensions. Suppose we are given short exact sequences $S = 0 \to D^{n+1}(M) \to Z \to X \to 0$ and $S' = 0 \to Y \to Z \to D^n(M) \to 0$. Then:

(1) S is $\mathrm{Hom}(\mathrm{dw}\widetilde{\mathcal{F}}, -)$-exact iff it is $\mathrm{Hom}(\widetilde{\mathcal{F}}, -)$-exact.

(2) S is $\mathrm{Hom}(-, \mathrm{dw}\widetilde{\mathcal{G}})$-exact iff it is $\mathrm{Hom}(-, \widetilde{\mathcal{G}})$-exact.

(1') S' is $\mathrm{Hom}(\mathrm{dw}\widetilde{\mathcal{F}}, -)$-exact iff it is $\mathrm{Hom}(\widetilde{\mathcal{F}}, -)$-exact.

(2') S' is $\mathrm{Hom}(-, \mathrm{dw}\widetilde{\mathcal{G}})$-exact iff, it is $\mathrm{Hom}(-, \widetilde{\mathcal{G}})$-exact.

Proof.

We only prove **(1)**. The implication ($\Longrightarrow$) is clear, since $\widetilde{\mathcal{F}} \subseteq \mathrm{dw}\widetilde{\mathcal{F}}$ if $\mathcal{F}$ is closed under extensions. Now suppose S is $\mathrm{Hom}_{\mathbf{Ch}(\mathcal{C})}(\widetilde{\mathcal{F}}, -)$-exact. Note that S and the sequence $0 \to D^{n+1}(M) \to \widetilde{Z_n} \to X \to 0$ constructed in Proposition B.6 are equivalent, so the result will follow if we show that the latter sequence is $\mathrm{Hom}_{\mathbf{Ch}(\mathcal{C})}(\mathrm{dw}\widetilde{\mathcal{F}}, -)$-exact. Since $0 \to D^{n+1}(M) \to \widetilde{Z_n} \to X \to 0$ is $\mathrm{Hom}_{\mathbf{Ch}(\mathcal{C})}(\widetilde{\mathcal{F}}, -)$-exact, we know the sequence $0 \to M \to Z_n \to X_n \to 0$ is $\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, -)$-exact. Then as we did is the proof of Proposition B.6, we have that $0 \to D^{n+1}(M) \to \widetilde{Z_n} \to X \to 0$ is $\mathrm{Hom}_{\mathbf{Ch}(\mathcal{C})}(\mathrm{dw}\widetilde{\mathcal{F}}, -)$-exact. $\square$

Corollary B.9

Let $\mathcal{C}$ be an Abelian category. Let $M \in \mathrm{Ob}(\mathcal{C})$ and $X, Y \in \mathbf{Ch}(\mathcal{C})$. We have the following natural isomorphisms:

$${}_{\mathcal{GP}_0}E(X_n, M) \overset{(1)}{\to} {}_{\mathcal{GP}_0}E(X, D^{n+1}(M)).$$

$$E_{\mathcal{GI}_0}(X_n, M) \overset{(2)}{\to} E_{\mathcal{GI}_0}(X, D^{n+1}(M)).$$

$$E_{\mathcal{GI}_0}(M, Y_n) \overset{(1')}{\to} E_{\mathcal{GI}_0}(D^n(M), Y).$$

$${}_{\mathcal{GP}_0}E(M, Y_n) \overset{(2')}{\to} {}_{\mathcal{GP}_0}E(D^n(M), Y).$$

$\boxed{\textbf{Proof}}$.

It suffices to show that $\mathcal{GP}_0(\mathbf{Ch}(\mathcal{C})) \subseteq \mathrm{dw}\widetilde{\mathcal{GP}_0(\mathcal{C})}$ in any Abelian category (not necessarily Gorenstein), and apply the argument used in Proposition B.6. If C is a Gorenstein-projective complex, then there exists a $\mathrm{Hom}_{\mathbf{Ch}(\mathcal{C})}(-, \mathcal{P}_0(\mathbf{Ch}(\mathcal{C})))$-exact and exact sequence $\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ of projective complexes such that $C = \mathrm{Ker}(P^0 \to P^1)$. Let P be a projective object of $\mathcal{C}$. Then $D^{n+1}(P)$ is a projective complex. We have a commutative diagram

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & \mathrm{Hom}_{\mathbf{Ch}(\mathcal{C})}(P^0, D^{n+1}(P)) & \longrightarrow & \mathrm{Hom}_{\mathbf{Ch}(\mathcal{C})}(P_0, D^{n+1}(P)) & \longrightarrow & \cdots \\
 & & \downarrow \cong & & \downarrow \cong & & \\
\cdots & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}((P^0)_n, P) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}((P_0)_n, P) & \longrightarrow & \cdots
\end{array}$$

where the top row is exact. It follows

$$\cdots \to (P_1)_n \to (P_0)_n \to (P^0)_n \to (P^1)_n \to \cdots$$

is an exact and $\mathrm{Hom}_{\mathcal{C}}(-, P)$-exact sequence of projective objects of $\mathcal{C}$ such that $C_n = \mathrm{Ker}(P^0 \to P^1)_n = \mathrm{Ker}((P^0)_n \to (P^1)_n)$, i.e. C_n is Gorenstein-projective. $\square$

Remark B.1.

(1) If $\mathcal{C}$ is a Gorenstein category, we can give an easier proof of the previous corollary. Given a complex X, there is an exact sequence $W \hookrightarrow C \twoheadrightarrow X$ where $C \in \mathcal{GP}_0(\mathbf{Ch}(\mathcal{C}))$ and $W \in \widetilde{\mathcal{W}(\mathcal{C})}$. At the nth level we have an exact sequence $W_n \hookrightarrow C_n \twoheadrightarrow X_n$, with $C_n \in \mathcal{GP}_0(\mathcal{C})$ and $W_n \in \mathcal{W}(\mathcal{C})$. Since the former and latter sequences are $\mathrm{Hom}_{\mathbf{Ch}(\mathcal{C})}(\mathcal{GP}_0(\mathbf{Ch}(\mathcal{C})), -)$-exact and $\mathrm{Hom}_{\mathcal{C}}(\mathcal{GP}_0(\mathcal{C}), -)$-exact, respectively, by Theorem 1.4.2, Corollary B.5, and Proposition 1.3.1, we have a commutative diagram

$$
\begin{tikzcd}
\mathrm{Hom}(X, D^{n+1}(M)) \arrow[r, hook] \arrow[d, "\cong"] & \mathrm{Hom}(C, D^{n+1}(M)) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}(W, D^{n+1}(M)) \arrow[r, two heads] \arrow[d, "\cong"] & {}_{\mathcal{GP}_0(\mathbf{Ch}(\mathcal{C}))}E(X, D^{n+1}(M)) \arrow[d] \\
\mathrm{Hom}(X_n, M) \arrow[r, hook] & \mathrm{Hom}(C_n, M) \arrow[r] & \mathrm{Hom}(W_n, M) \arrow[r, two heads] & {}_{\mathcal{GP}_0(\mathcal{C})}E(X_n, M)
\end{tikzcd}
$$

The rightmost arrow is the only map induced by the universal property of cokernels making the right square commutative. It is not hard to see that this arrow is an isomorphism. It follows $\mathrm{GExt}^1_{\mathbf{Ch}(\mathcal{C})}(X, D^{n+1}(M)) \cong \mathrm{GExt}^1_{\mathcal{C}}(X_n, M)$, and similarly, $\mathrm{GExt}^1_{\mathbf{Ch}(\mathcal{C})}(D^n(M), Y) \cong \mathrm{GExt}^1_{\mathcal{C}}(M, Y_n)$.

(2) The same reasoning works to give a simpler proof of the isomorphisms given in Proposition 1.6.2, assuming $\mathcal{C}$ has enough projective and injective objects. Note that in the case where $\mathcal{F} = \mathcal{P}_0(\mathcal{C})$, we have $E^{\mathbf{Ch}(\mathcal{C})}(X, D^{n+1}(M)) = {}_{\widetilde{\mathcal{P}_0(\mathcal{C})}}E(X, D^{n+1}(M)) \cong {}_{\mathcal{P}_0(\mathcal{C})}E(X_n, M) = E^{\mathcal{C}}(X_n, M)$. The same applies for the class $\mathcal{G} = \mathcal{I}_0(\mathcal{C})$.

Proposition B.10

Let $\mathcal{C}$ be an Abelian category, and $\mathcal{F}$ and $\mathcal{G}$ be two classes of objects of $\mathcal{C}$ which are closed under extensions. Let $M \in \mathrm{Ob}(\mathcal{C})$ and $X, Y \in \mathrm{Ob}(\mathbf{Ch}(\mathcal{C}))$. There exist natural monomorphisms:

${}_{\mathcal{F}}E(\frac{X_n}{B_n(X)}, M) \overset{\mathbf{(1)}}{\hookrightarrow} {}_{\widetilde{\mathcal{F}}}E(X, S^n(M))$.

$E_{\mathcal{G}}(\frac{X_n}{B_n(X)}, M) \overset{\mathbf{(2)}}{\hookrightarrow} E_{\widetilde{\mathcal{G}}}(X, S^n(M))$.

${}_{\mathcal{F}}E(M, Z_n(Y)) \overset{\mathbf{(1')}}{\hookrightarrow} {}_{\widetilde{\mathcal{F}}}E(S^n(M), Y)$.

$E_{\mathcal{G}}(M, Z_n(Y)) \overset{\mathbf{(2')}}{\hookrightarrow} E_{\widetilde{\mathcal{G}}}(S^n(M), Y)$.

Proof.

(1) We consider the dual of the isomorphism given by J. Gillespie in (25, Lemma 4.2). Suppose $0 \to M \overset{\alpha}{\to} Z \overset{\beta}{\to} \frac{X_n}{B_n(X)} \to 0$ is an exact and $\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, -)$-exact sequence. Taking the pullback of β and $\pi^X_n : X_n \to \frac{X_n}{B_n(X)}$, we can construct the following commutative diagram with exact rows

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & X_{n+2} \arrow[r, equal] \arrow[d, "\partial^X_{n+2}"] & X_{n+2} \arrow[r] \arrow[d, "\partial^X_{n+2}"] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & X_{n+1} \arrow[r, equal] \arrow[d, "\partial^{\widetilde{Z}}_{n+1}"] & X_{n+1} \arrow[r] \arrow[d, "\partial^X_{n+1}"] & 0 \\
0 \arrow[r] & M \arrow[r, "\widetilde{\alpha}_n"] \arrow[d] & \widetilde{Z}_n \arrow[r, "\widetilde{\beta}_n"] \arrow[d, "\partial^{\widetilde{Z}}_n"] & X_n \arrow[r] \arrow[d, "\partial^X_n"] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & X_{n-1} \arrow[r, equal] \arrow[d, "\partial^X_{n-1}"] & X_{n-1} \arrow[r] \arrow[d, "\partial^X_{n-1}"] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & X_{n-2} \arrow[r, equal] \arrow[d] & X_{n-2} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

where $\widetilde{\alpha}_n$ and $\widetilde{\beta}_n$ are the morphisms appearing in the pullback diagram

$$
\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\widetilde{\alpha}_n"] \arrow[d, equal] & \widetilde{Z}_n \arrow[r, "\widetilde{\beta}_n"] \arrow[d, "\rho_Z"] & X_n \arrow[r] \arrow[d, "\pi^X_n"] & 0 \\
0 \arrow[r] & M \arrow[r, "\alpha"'] & Z \arrow[r, "\beta"'] & \frac{X_n}{B_n(X)} \arrow[r] & 0
\end{tikzcd}
$$

The arrow $\partial^{\widetilde{Z}}_{n+1}$ is the map induced by the universal property of pullbacks such that $\widetilde{\beta}_n \circ \partial^{\widetilde{Z}}_{n+1} = \partial^X_{n+1}$ and $\rho_Z \circ \partial^{\widetilde{Z}}_{n+1} = 0$, and $\partial^{\widetilde{Z}}_n := \partial^X_n \circ \widetilde{\beta}$. The central column is a complex and so we have an exact sequence $0 \to S^n(M) \xrightarrow{\widetilde{\alpha}} \widetilde{Z} \xrightarrow{\widetilde{\beta}} X \to 0$ in $\mathbf{Ch}(\mathcal{C})$. We show this sequence is also $\mathrm{Hom}_{\mathbf{Ch}(\mathcal{C})}(\widetilde{\mathcal{F}}, -)$-exact. Let $F \in \widetilde{\mathcal{F}}$ and consider a chain map $F \xrightarrow{f} X$. We construct a chain map $F \xrightarrow{h} \widetilde{Z}$ such that the following diagram commutes:

$$\begin{tikzcd}
 & \ddots \arrow[dr] & & & & \\
\ddots \arrow[dr] & \ddots \arrow[dr] & F_{n+1} \arrow[dl, "h_{n+1}"'] \arrow[d, "f_{n+1}"] \arrow[dr, "\partial^F_{n+1}"] & & & \\
 & X_{n+1} \arrow[r, equal] \arrow[dr, "\partial^{\widetilde{Z}}_{n+1}"'] & X_{n+1} \arrow[dr] & F_n \arrow[dl, "h_n"'] \arrow[d, "f_n"] \arrow[dr, "\partial^F_n"] & & \\
 & & \widetilde{Z}_n \arrow[r, "\widetilde{\beta}_n"] \arrow[dr, "\partial^{\widetilde{Z}}_n"'] & X_n \arrow[dr] & F_{n-1} \arrow[dl, "h_{n-1}"'] \arrow[d, "f_{n-1}"] \arrow[dr] & \\
 & & & X_{n-1} \arrow[r, equal] \arrow[dr] & X_{n-1} \arrow[dr] & \ddots \\
 & & & & \ddots & \ddots
\end{tikzcd}$$

Note that $\pi_n^X \circ f_n \circ \partial^F_{n+1} = 0$. Factoring ∂^F_{n+1} as $i_{B_n(F)} \circ \widehat{\partial^F_{n+1}}$, where $i_{B_n(F)} : B_n(F) \to F_n$ is the inclusion and $\widehat{\partial^F_{n+1}}$ is epic, we have $\pi_n^X \circ f_n \circ i_{B_n(F)} = 0$. By the universal property of cokernels, there is a unique map $\overline{f_n} : \frac{F_n}{B_n(F)} \to \frac{X_n}{B_n(X)}$ such that the following diagram commutes:

$$\begin{tikzcd}
B_n(F) \arrow[r, hook, "i_{B_n(F)}"] & F_n \arrow[r, two heads, "\pi_n^X"] \arrow[dr, "\pi_n^X \circ f_n"'] & \frac{F_n}{B_n(F)} \arrow[d, dotted, "\exists!\ \overline{f_n}"] \\
 & & \frac{X_n}{B_n(X)}
\end{tikzcd}$$

On the other hand, we have $\frac{F_n}{B_n(F)} \cong Z_{n-1}(F) \in \mathcal{F}$. Since the sequence $M \overset{\alpha}{\hookrightarrow} Z \overset{\beta}{\twoheadrightarrow} \frac{X_n}{B_n(X)}$ is $\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, -)$-exact, there exists a morphism $\frac{F_n}{B_n(F)} \xrightarrow{h'_n} Z$ such that the following diagram commutes:

$$\begin{tikzcd}
& & \frac{F_n}{B_n(F)} \arrow[dl, dotted, "\exists\, h'_n"'] \arrow[d, "\overline{f_n}"] \\
M \arrow[r, hook, "\alpha"'] & Z \arrow[r, two heads, "\beta"'] & \frac{X_n}{B_n(X)}
\end{tikzcd}$$

Since $\mathcal{F}$ is closed under extensions and F is exact, we have $F_n \in \mathcal{F}$. Using the universal property of pullbacks, we get the following commutative diagram

$$\begin{tikzcd}
F_n \arrow[drr, bend left, "f_n"] \arrow[dr, dotted, "\exists!\, h_n"] \arrow[ddr, bend right, "h'_n \circ \pi_n^F"'] & & \\
& Z \times_{\frac{X_n}{B_n(X)}} X_n \arrow[r, "\widetilde{\beta}_n"] \arrow[d, "\rho_Z"'] & X_n \arrow[d, "\pi_n^X"] \\
& Z \arrow[r, "\beta"'] & \frac{X_n}{B_n(X)}
\end{tikzcd}$$

Set $h_k = f_k$ for every $k \neq n$. We have $\widetilde{\beta}_k \circ h_k = f_k$ for every $k \in \mathbb{Z}$. We check $h = (h_k : k \in \mathbb{Z})$ is a chain map. The equality $h_n \circ \partial_{n+1}^F = \partial_{n+1}^{\widetilde{Z}} \circ f_{n+1}$ follows by the commutativity of the following diagram:

$$\begin{tikzcd}
F_{n+1} \arrow[drr, bend left, "f_n \circ \partial_{n+1}^F"] \arrow[dr, bend left, "h_n \circ \partial_{n+1}^F"] \arrow[dr, bend right, "\partial_{n+1}^{\widetilde{Z}} \circ f_{n+1}"'] \arrow[ddr, bend right, "0"'] & & \\
& Z \times_{\frac{X_n}{B_n(X)}} X_n \arrow[r, "\widetilde{\beta}_n"] \arrow[d, "\rho_Z"'] & X_n \arrow[d, "\pi_n^X"] \\
& Z \arrow[r, "\beta"'] & \frac{X_n}{B_n(X)}
\end{tikzcd}$$

On the other hand, $\partial_n^{\widetilde{Z}} \circ h_n = \partial_n^X \circ \widetilde{\beta}_n \circ h_n = \partial_n^X \circ f_n = f_{n-1} \circ \partial_n^F$. Therefore, h is a chain map satisfying $\widetilde{\beta} \circ h = f$, i.e. $0 \to S^n(M) \to \widetilde{Z} \to X \to 0$ is $\mathrm{Hom}_{\mathbf{Ch}(\mathcal{C})}(\widetilde{\mathcal{F}}, -)$-exact.

Since the map $E(\frac{X_n}{B_n(X)}, M) \to E(X, S^n(M))$ constructed by Gillespie is monic, the restriction ${}_{\mathcal{F}}E(\frac{X_n}{B_n(X)}, M) \to {}_{\widetilde{\mathcal{F}}}E(X, S^n(M))$ is also monic.

(2) Suppose $0 \to M \to \frac{Z_n}{B_n(Z)} \to \frac{X_n}{B_n(X)} \to 0$ is an exact and $\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{G})$-exact sequence. Let $G \in \widetilde{\mathcal{G}}$ and consider a chain map $S^n(M) \xrightarrow{f} G$. We construct a chain map $\widetilde{Z} \xrightarrow{h} G$ such that $h \circ \widetilde{\alpha} = f$, i.e. such that the diagram in the following page commutes.

Since $\partial_n^G \circ f_n = 0$, there exists a unique map $\overline{f_n}$ in completing the following commutative diagram

$$\begin{tikzcd}
Z_n(G) \arrow[r, hook, "i_{Z_n(G)}"] & G_n \arrow[r, two heads, "\widehat{\partial_n^G}"] & Z_{n-1}(G) \\
M \arrow[u, dotted, "\overline{f_n}"] \arrow[ur, "f_n"'] & &
\end{tikzcd}$$

On the other hand, $M \hookrightarrow Z \twoheadrightarrow \frac{X_n}{B_n(X)}$ is $\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{G})$-exact and $Z_n(G) \in \mathcal{G}$, so there is a morphism $Z \xrightarrow{h'_n} Z_n(G)$ such that $h'_n \circ \alpha = \overline{f_n}$. Set $h_k = 0$ for every $k \neq n$ and $h_n := i_{Z_n(G)} \circ h'_n \circ \rho_Z$.

$$\partial_n^G \circ h_n = 0 = h_{n-1} \circ \partial_n^{\widetilde{Z}},$$
$$h_n \circ \partial_{n+1}^{\widetilde{Z}} = i_{Z_n(G)} \circ h'_n \circ \rho_Z \circ \partial_{n+1}^{\widetilde{Z}} = 0 = \partial_{n+1}^G \circ h_{n+1},$$
$$h_n \circ \widetilde{\alpha}_n = i_{Z_n(G)} \circ h'_n \circ \rho_Z \circ \widetilde{\alpha}_n = i_{Z_n(G)} \circ h'_n \circ \alpha = i_{Z_n(G)} \circ \overline{f_n} = f_n.$$

Hence, h is a chain map satisfying $h \circ \widetilde{\alpha} = f$.

$$
\begin{tikzcd}
\ddots \arrow[dr] & \ddots \arrow[dr] & & & & & & \\
\ddots \arrow[dr] & 0 \arrow[r] \arrow[d, "f_{n+2}"'] \arrow[dr] & X_{n+2} \arrow[dl, "h_{n+2}"'] \arrow[dr, "\partial^X_{n+2}"] & & & & & \\
& G_{n+2} \arrow[dr, "\partial^G_{n+2}"'] & 0 \arrow[r] \arrow[d, "f_{n+1}"'] \arrow[dr] & X_{n+1} \arrow[dl, "h_{n+1}"'] \arrow[dr, "\partial^X_{n+1}"] & & & & \\
& & G_{n+1} \arrow[dr, "\partial^G_{n+1}"'] & M \arrow[r, "\widetilde{\alpha}_n"] \arrow[d, "f_n"'] \arrow[dr] & \widetilde{Z}_n \arrow[dl, "h_n"'] \arrow[dr, "\partial^X_n"] & & & \\
& & & G_n \arrow[dr, "\partial^G_n"'] & 0 \arrow[r] \arrow[d, "f_{n-1}"'] \arrow[dr] & X_{n-1} \arrow[dl, "h_{n-1}"'] \arrow[dr, "\partial^X_{n-1}"] & & \\
& & & & G_{n-1} \arrow[dr, "\partial^G_{n-1}"'] & 0 \arrow[r] \arrow[d, "f_{n-2}"'] \arrow[dr] & X_{n-2} \arrow[dl, "h_{n-2}"'] \arrow[dr] & \\
& & & & & G_{n-2} \arrow[dr] & \ddots & \ddots \\
& & & & & & \ddots &
\end{tikzcd}
$$

Similarly, the map $E_{\mathcal{G}}(\frac{X_n}{B_n(X)}, M) \overset{(\mathbf{2})}{\hookrightarrow} E_{\widetilde{\mathcal{G}}}(X, S^n(M))$ is also monic. $\square$

Remark B.2. If the complex X in the previous theorem is exact and $\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, -)$-exact, then the map ${}_{\mathcal{F}}E(\frac{X_n}{B_n(X)}, M) \hookrightarrow {}_{\widetilde{\mathcal{F}}}E(X, S^n(M))$ is invertible.

First, note that given a class of objects $\mathcal{F}$ in an Abelian category $\mathcal{C}$, if X is an exact and $\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, -)$-exact complex, then the sequence $Z_m(X) \hookrightarrow X_m \twoheadrightarrow Z_{m-1}(X)$ is also $\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, -)$-exact. Since the sequence

$$\cdots \to \mathrm{Hom}_{\mathcal{C}}(F, X_{m+1}) \to \mathrm{Hom}_{\mathcal{C}}(F, X_m) \to \mathrm{Hom}_{\mathcal{C}}(F, X_{m-1}) \to \cdots$$

is exact for every $F \in \mathcal{F}$, we obtain the exact sequence

$$\mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(F, \partial_m^X)) \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(F, X_m) \twoheadrightarrow \mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(F, \partial_{m-1}^X))$$

for every $m \in \mathbb{Z}$. It is not hard to see that $\mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(F, \partial_m^X)) \cong \mathrm{Hom}_{\mathcal{C}}(F, Z_m(X))$. Then the result follows.

In the case $\mathcal{F}$ is a class closed under extensions, the inverse of the homomorphism $E^{\mathcal{C}}(\frac{X_n}{B_n(X)}, M) \hookrightarrow E^{\mathbf{Ch}(\mathcal{C})}(X, S^n(M))$ restricted to the group ${}_{\widetilde{\mathcal{F}}}E(X, S^n(M))$ yields the inverse of ${}_{\mathcal{F}}E(\frac{X_n}{B_n(X)}, M) \hookrightarrow {}_{\widetilde{\mathcal{F}}}E(X, S^n(M))$ if the complex X is exact and $\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, -)$-exact. For if $0 \to S^n(M) \to Z \to X \to 0$ is exact and $\mathrm{Hom}(\widetilde{\mathcal{F}}, -)$-exact, then we can deduce that the sequence $0 \to M \to Z_n \to X_n \to 0$ is $\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, -)$-exact, by considering disk complexes $D^n(F)$ with $F \in \mathcal{F}$ and using Proposition 1.3.1. We show $0 \to M \to \frac{Z_n}{B_n(Z)} \xrightarrow{\overline{\beta_n}} \frac{X_n}{B_n(X)} \to 0$ is also $\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, -)$-exact. Consider a map $F \xrightarrow{f} \frac{X_n}{B_n(X)}$ with $F \in \mathcal{F}$. By the previous comments, there exists a map $F \xrightarrow{f'} X_n$ such that the following diagram commutes:

$$\begin{tikzcd}
 & & & F \arrow[dl, dotted, "\exists f'"'] \arrow[d, "f"] & \\
0 \arrow[r] & B_n(X) \arrow[r] & X_n \arrow[r, "\pi_n^X"'] & \frac{X_n}{B_n(X)} \arrow[r] & 0
\end{tikzcd}$$

It follows the existence of a map $F \xrightarrow{h'} Z_n$ making the following diagram commute:

$$\begin{tikzcd}
 & & & F \arrow[dl, dotted, "\exists h'"'] \arrow[d, "f'"] & \\
0 \arrow[r] & M \arrow[r] & Z_n \arrow[r, "\beta_n"'] & X_n \arrow[r] & 0
\end{tikzcd}$$

Set $h := \partial_n^Z \circ h'$. We have $\overline{\beta_n} \circ h = \overline{\beta_n} \circ \partial_n^Z \circ h' = \partial_n^X \circ \beta_n \circ h' = \partial_n^X \circ f' = f$, and hence $0 \to M \to \frac{Z_n}{B_n(Z)} \xrightarrow{\overline{\beta_n}} \frac{X_n}{B_n(X)} \to 0$ is $\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, -)$-exact.

Dually, if Y is exact and $\mathrm{Hom}_{\mathcal{C}}(-,\mathcal{G})$-exact, then $E_{\mathcal{G}}(M, Z_n(Y)) \to E_{\widetilde{\mathcal{G}}}(S^n(M), Y)$ is invertible.

Proposition B.11

Let $\mathcal{C}$ be an Abelian category. Let $M \in \mathrm{Ob}(\mathcal{C})$ and X and Y be exact chain complexes. There exist natural monomorphisms:

$E_{\mathrm{dw}\widetilde{\mathcal{G}}}(X, S^n(M)) \hookrightarrow E_{\mathcal{G}}(\frac{X_n}{B_n(X)}, M).$	${}_{\mathrm{dw}\widetilde{\mathcal{F}}}E(S^n(M), Y) \hookrightarrow {}_{\mathcal{F}}E(M, Z_n(Y)).$

Proof.

We only prove the left statement. It suffices to show that the restriction on $E_{\mathrm{dw}\widetilde{\mathcal{G}}}(X, S^n(M))$ of the isomorphism $E^{\mathbf{Ch}(\mathcal{C})}(X, S^n(M)) \hookrightarrow E^{\mathcal{C}}(\frac{X_n}{B_n(X)}, M)$ is well defined. So consider an exact and $\mathrm{Hom}_{\mathbf{Ch}(\mathcal{C})}(-, \mathrm{dw}\widetilde{\mathcal{G}})$-exact sequence

$$0 \to S^n(M) \to Z \to X \to 0.$$

If $G \in \mathcal{G}$, then $S^n(G) \in \mathrm{dw}\widetilde{G}$. By Proposition 1.3.1, we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathbf{Ch}(\mathcal{C})}(X, S^n(G)) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathbf{Ch}(\mathcal{C})}(Z, S^n(G)) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathbf{Ch}(\mathcal{C})}(S^n(M), S^n(G)) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(\frac{X_n}{B_n(X)}, G) \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(\frac{Z_n}{B_n(Z)}, G) \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(M, G) \arrow[r] & 0
\end{tikzcd}$$

where the top row is exact. It follows the bottom row is also exact. $\square$

The previous two propositions can be used to obtain an isomorphism between $\mathrm{GExt}^1_{\mathbf{Ch}(\mathcal{C})}(X, S^n(M))$ and $\mathrm{GExt}^1_{\mathcal{C}}(\frac{X_n}{B_n(X)}, M)$, but we also need the following lemma:

Lemma B.12

Let $\mathcal{C}$ be a Gorenstein category and $0 \to Y \xrightarrow{\alpha} Z \xrightarrow{\beta} X \to 0$ be an exact sequence in $\mathbf{Ch}(\mathcal{C})$.

If X is exact, then this sequence is $\mathrm{Hom}(\mathcal{GP}_0(\mathbf{Ch}(\mathcal{C})), -)$-exact if, and only if, it is $\mathrm{Hom}(\widetilde{\mathcal{GP}_0(\mathcal{C})}, -)$-exact.	If Y is exact, then this sequence is $\mathrm{Hom}(-, \mathcal{GI}_0(\mathbf{Ch}(\mathcal{C})))$-exact if, and only if, it is $\mathrm{Hom}(-, \widetilde{\mathcal{GI}_0(\mathcal{C})})$-exact.

$\boxed{\textbf{Proof}}$.

We prove the left statement. The implication $\Longrightarrow$ is clear. Now suppose the sequence $0 \to Y \to Z \to X \to 0$ is $\mathrm{Hom}_{\mathbf{Ch}(\mathcal{C})}(\widetilde{\mathcal{GP}_0(\mathcal{C})}, -)$-exact and that we are given a chain map $C \xrightarrow{f} X$ with $C \in \mathcal{GP}_0(\mathbf{Ch}(\mathcal{C}))$. Since $\mathcal{C}$ is Gorenstein, the pair $(\mathcal{GP}_0(\mathbf{Ch}(\mathcal{C})), \widetilde{\mathcal{W}(\mathcal{C})})$ is complete (see Proposition 4.2.3). So there exists a short exact sequence $0 \to W \to C' \xrightarrow{r} X \to 0$, where $W \in \widetilde{\mathcal{W}(\mathcal{C})}$ and $C' \in \mathcal{GP}_0(\mathbf{Ch}(\mathcal{C}))$. On the one hand, we have the induced cotorsion pairs $(\widetilde{\mathcal{GP}_0(\mathcal{C})}, \mathrm{dg}\widetilde{\mathcal{W}(\mathcal{C})})$ and $(\mathrm{dg}\widetilde{\mathcal{GP}_0(\mathcal{C})}, \widetilde{\mathcal{W}(\mathcal{C})})$, since $(\mathcal{GP}_0(\mathcal{C}), \mathcal{W}(\mathcal{C}))$ is complete. On the other hand, $\mathcal{GP}_0(\mathbf{Ch}(\mathcal{C})) = {}^{\perp}(\widetilde{\mathcal{W}(\mathcal{C})}) = \mathrm{dg}\widetilde{\mathcal{GP}_0(\mathcal{C})}$. It follows $\widetilde{\mathcal{GP}_0(\mathcal{C})} = \mathrm{dg}\widetilde{\mathcal{GP}_0(\mathcal{C})} \cap \mathcal{E} = \mathcal{GP}_0(\mathbf{Ch}(\mathcal{C})) \cap \mathcal{E}$. Since W and X are exact, we have $C' \in \mathcal{GP}_0(\mathbf{Ch}(\mathcal{C})) \cap \mathcal{E} = \widetilde{\mathcal{GP}_0(\mathcal{C})}$. Using the fact that $0 \to Y \to Z \to X \to 0$ is $\mathrm{Hom}_{\mathbf{Ch}(\mathcal{C})}(\widetilde{\mathcal{GP}_0(\mathcal{C})}, -)$-exact, we find a map $C' \xrightarrow{h} Z$ such that $\beta \circ h = r$. Furthermore, the sequence $0 \to W \to C' \xrightarrow{r} X \to 0$ is $\mathrm{Hom}_{\mathbf{Ch}(\mathcal{C})}(\mathcal{GP}_0(\mathbf{Ch}(\mathcal{C})), -)$-exact, so there is a map $C \xrightarrow{g} C'$ such that $r \circ g = f$. We have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[dr] & & & & \\
& W \arrow[dr] & & & \\
& & C' \arrow[d, "h"'] \arrow[dr, "r"] & C \arrow[l, "g"'] \arrow[d, "f"] & \\
0 \arrow[r] & Y \arrow[r] & Z \arrow[r, "\beta"'] & X \arrow[r] \arrow[dr] & 0 \\
& & & & 0
\end{tikzcd}$$

Therefore, $f = \beta \circ (h \circ g)$ and $0 \to Y \to Z \to X \to 0$ is $\mathrm{Hom}_{\mathbf{Ch}(\mathcal{C})}(\mathcal{GP}_0(\mathbf{Ch}(\mathcal{C})), -)$-exact. $\square$

Remark B.3.

(1) As we did in Remark B.1, we can prove that $\mathrm{GExt}^1_{\mathbf{Ch}(\mathcal{C})}(X, S^n(M))$ and $\mathrm{GExt}^1_{\mathcal{C}}(\frac{X_n}{B_n(X)}, M)$ are isomorphic assuming that X is an exact chain complex over a Gorenstein category $\mathcal{C}$. We start with a short exact sequence $W \hookrightarrow C \twoheadrightarrow X$ where $C \in \mathcal{GP}_0(\mathbf{Ch}(\mathcal{C}))$ and $W \in \widetilde{\mathcal{W}(\mathcal{C})}$. Notice that this sequence is $\mathrm{Hom}_{\mathcal{C}}(\mathcal{GP}_0(\mathbf{Ch}(\mathcal{C})), -)$-exact. By Lemma 1.8.2, we have the short exact sequence $\frac{W_n}{B_n(W)} \hookrightarrow \frac{C_n}{B_n(C)} \twoheadrightarrow \frac{X_n}{B_n(X)}$, with $\frac{C_n}{B_n(C)} \cong Z_{n-1}(C) \in \mathcal{GP}_0(\mathcal{C})$ (since $C \in \mathcal{GP}_0(\mathbf{Ch}(\mathcal{C})) \cap \mathcal{E} \cong \widetilde{\mathcal{GP}_0(\mathcal{C})})$ and $\frac{W_n}{B_n(W)} \cong Z_{n-1}(W) \in \mathcal{W}(\mathcal{C})$. The previous sequence is $\mathrm{Hom}_{\mathcal{C}}(\mathcal{GP}_0(\mathcal{C}), -)$-exact. Then the existence of the following commutative diagram follows:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{Ch}(\mathcal{C})}(X, S^n(M)) \arrow[r, hook] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathbf{Ch}(\mathcal{C})}(C, S^n(M)) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathbf{Ch}(\mathcal{C})}(W, S^n(M)) \arrow[r, two heads] \arrow[d, "\cong"] & E_{\mathcal{GP}_0(\mathbf{Ch}(\mathcal{C}))}(X, S^n(M)) \arrow[d, "\cong"] \\
\mathrm{Hom}_{\mathcal{C}}(\frac{X_n}{B_n(X)}, M) \arrow[r, hook] & \mathrm{Hom}_{\mathcal{C}}(\frac{C_n}{B_n(C)}, M) \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(\frac{W_n}{B_n(W)}, M) \arrow[r, two heads] & E_{\mathcal{GP}_0(\mathcal{C})}(\frac{X_n}{B_n(X)}, M)
\end{tikzcd}$$

The dual statement can be proven similarly.

(3) The same reasoning used in **(2)** works to give a simpler proof of the isomorphisms given in Proposition 1.6.3, assuming $\mathcal{C}$ has enough projective or injective objects.

BIBLIOGRAPHY

(1) Adámek, J., Herrlich, H., Rosický, J. and Tholen, W. (2002). *Injective hulls are not natural*. Algebra Universalis, Vol. 48, pp. 379-388.

(2) Adámek, J., Herrlich, H. and Strecker, G. E. (2004). *Abstract and Concrete Categories. The Joy of Cats.* Dover Publications, Inc. New York, New York, United States.

(3) Adámek, J. and Rosický, J. (1994). *Locally Presentable and Accesible Categories*. London Mathematical Society Lecture Note Series 189. Cambridge University Press. Cambridge, United Kingdom.

(4) Aldrich, S. T., Enochs, E. E., García Rozas, J. R. and Oyonarte, L. (2001). *Covers and Envelopes in Grothendieck Categories: Flat Covers of Complexes with Applications*. Journal of Algebra. Vol. 243. pp. 615-630.

(5) Aldrich, S. T., Enochs, E. E., Jenda, O. M. G. and Oyonarte, L. (2001). *Envelopes and Covers by Modules of Finite Injective and Projective Dimensions*. Journal of Algebra, Vol. 242, pp. 447-459.

(6) Angeleri-Hügel, L. and Mendoza, O. (2009). *Homological Dimensions in Cotorsion pairs*. Illinois Journal of Mathematics, Vol. 53, no. 1, pp. 251-263.

(7) Arndt, P. and Kapulkin, C. (2011). *Homotopy Theoretic Models of Type Theory*. Lecture Notes in Computer Science 6690. pp. 45-60.

(8) Bican, L., El Bashir, R. and Enochs, E. E. (2001). *All Modules Have Flat Covers*. Bulletin of the London Mathematical Society, Vol. 33, No. 4, pp. 385-390.

(9) Borceaux, F. (1994). *Handbook of Categorical Algebra 1. Basic Category Theory*. Encyclopedia of Mathematics and its Applications, Vol. 50. Cambridge University Press. Cambridge, United Kingdom.

(10) Borceaux, F. (1994). *Handbook of Categorical Algebra 2. Categories and Structures*. Encyclopedia of Mathematics and its Applications, Vol. 51. Cambridge University Press. Cambridge, United Kingdom.

(11) Bourbaki, N. (1970). *Elements of Mathematics. Algebra I. Chapters 1-3.* Springer Verlag. Berlin, Germany.

(12) Bravo, D., Enochs, E. E., Iacob, A. C., Jenda, O. M. G., and Rada, J. (2012). *Cotorsion pairs in C(R-Mod).* Rocky Mountain Journal of Mathematics. Vol. 42, No. 6, 1787-1802.

(13) Cartan, H. and Eilenberg, S. (1999). *Homological Algebra.* Princeton Landmarks in Mathematics. Princeton University Press. Princeton, New Jersey, United States.

(14) Christensen, D. and Hovey, M. (2002). *Quillen model structures for relative homological algebra.* Mathematical Proceedings of the Cambridge Mathematical Society. 133, Part 2, pp. 261-293.

(15) Cortés Izurdiaga, M., Estrada, S. and Guil Asensio, P. A. (2012). *A model structure approach to the finitistic dimension conjecture.* Mathematische Nachrichten. Vol. 285, Issue 7, pp. 821-833.

(16) Ding, N. and Mao, L. (2005). *Relative FP-projective modules.* Communications in Algebra. Vol. 33, No. 5, pp. 1587-1602.

(17) Ding, N. and Mao, L. (2007). *Envelopes and covers by modules of finite FP-injective and flat dimensions.* Communications in Algebra. Vol. 35, No. 13, pp. 833-849.

(18) Dugger, D. and Shipley, B. (2004). *K-Theory and Derived Equivalences.* Duke Mathematical Journal. Vol. 124, no. 3, pp. 587-617.

(19) Eklof, P. and Trlifaj, J. (2001). *How to make Ext vanish.* Bulletin of the London Mathematical Society, Vol. 33, No. 1, pp. 41-51.

(20) Enochs, E. E., Estrada, S. and García Rozas, J. R. (2008). *Gorenstein categories and Tate cohomology on projective schemes.* Mathematische Nachrichten. Vol. 281, no. 4., pp. 525-540.

(21) Enochs, E. E. and Jenda, O. M. G. (2011). *Relative Homological Algebra. Volume 1.* De Gruyter Expositions in Mathematics, Vol. 30. De Gruyter. Berlin, Germany.

(22) Enochs, E. E., Jenda, O. M. G. and Xu, J. (1996). *Orthogonality in the category of complexes.* Math. J. Okayama Univ. Vol. 38. pp. 25-46.

(23) Fuchs, L. and Salce, L. (2001). *Module over Non-Noetherian Domains*. Mathematical Surveys and Monographs. Vol. 84. American Mathematical Society. Providence, Rhode Island, United States.

(24) García Rozas, J. R. (1999). *Covers and Envelopes in the Category of Complexes of Modules*. Chapman & Hall/CRC Research Notes in Mathematics, Vol. 407. Boca Raton, Florida, United States.

(25) Gillespie, J. (2008). *Cotorsion Pairs and Degreewise Homological Model Structures*. Homology, Homotopy and Applications, Vol. 10, No. 1, pp. 283-304.

(26) Gillespie, J. (2007). *Kaplansky Classes and Derived Categories*. Mathematische Zeitschrift, Vol. 257, No. 4, pp. 811-843.

(27) Gillespie, J. (2004). *The Flat Model Structure on* $\mathrm{Ch}(R)$. Transactions of the American Mathematical Society, Vol. 356, No. 8, pp. 3369-3390.

(28) Gillespie, J. (2006). *The Flat Model Structure on Complexes of Sheaves*. Transactions of the American Mathematical Society, Vol. 358, N. 7, pp. 2855-2874.

(29) Gillespie, J. (2009). *Model Structures on Modules over Ding-Chen Rings*. Preprint. arXiv:0910.1942v1.

(30) Gillespie, J. and Hovey, M. (2010). *Gorenstein model structures and generalized derived categories*. Proceedings of the Edinburgh Mathematical Society. 53, no. 3, pp. 675-696.

(31) Göbel, R. and Trlifaj, J. (2006). *Approximations and Endomorphism Algebras of Modules*. De Gruyter Expositions in Mathematics. Vol. 41. Berlin, Germany.

(32) Grothendieck, A. (1957). *Sur quelques points d'algèbre homologique.* Tohoku Mathematical Journal. Vol. 9, No. 2, pp. 119-221.

(33) Holm, H. (2004). *Gorenstein Homological Dimensions*. Journal of Pure and Applied Algebra. Vol. 189, Issues 1-3, pp. 167-193.

(34) Hovey, M. (2006). *Cotorsion pairs and model categories*. American Mathematical Society.

(35) Hovey, M. (2002). *Cotorsion pairs, model category structures and representation theory*. Mathematical Zeitschrift. Vol. 241, pp. 553-592.

(36) Hovey, M. (2007). *Model Categories*. Mathematical Surveys and Monographs. Vol. 63. American Mathematical Society. Providence, Rhode Island, United States.

(37) Hovey, M. (2001). *Model category structures on chain complexes of sheaves*. Transactions of the American Mathematical Society. Vol. 353, no. 6, pp. 2441-2457.

(38) Igusa, K. *Ext*. people.brandeis.edu/∼igusa/Math101b/Ext.pdf

(39) Joyal, A. *Joyal's CatLab*. http://ncatlab.org/joyalscatlab/published/Home Page

(40) Joyal, A. (1984). *Letter to Alexander Grothendieck*.

(41) Kaplansky, I. (1958). *Projective modules*. Annals of Mathematics. Vol. 68 , no. 2, pp. 372-377.

(42) Krause, H. (1998). *Finitistic Dimension and Ziegler Spectrum*. Proceedings of the American Mathematical Society, Vol. 126, No. 4, pp. 983-987.

(43) Mac Lane, S. (1978). *Categories for the Working Mathematician*. Graduate Texts in Mathematics, Vol. 5. Springer Verlag. New York, New York, United States.

(44) Mac Lane, S. (1995). *Homology*. Classics in Mathematics. Springer Verlag. Berlin, Germany.

(45) Mantese, F. and Tonolo, A. (2008). *On classes defining a homological dimension*. (Invited paper). Models, Modules and Abelian Groups. De Gruyter, pp. 431-444.

(46) Osborne, M. S. (2000). *Basic Homological Algebra*. Graduate Texts in Mathematics, Vol. 196. Springer Verlag. New York, New York, United States.

(47) Pareigis, B. (1970). *Categories and Functors*. Pure and Applied Mathematics, Vol. 39. Academic Press. New York, New York, United States.

(48) Pérez, M. (2012). *Degreewise n-projective and n-flat model structures on chain complexes*. Version 2. Preprint. arXiv:1204.1358.

(49) Pérez, M. (2012). *Homological dimensions and abelian model structures on chain complexes*. Version 2 (Titled *n-Projective model structures on chain complexes*). Preprint. arXiv:1202.2481

(50) Pérez, M. (2012). *Gorenstein homological dimensions and abelian model structures*. Version 1. Preprint. arXiv:1212.1517.

(51) Quillen, D. G. (1967). *Homotopical Algebra*. Lecture Notes in Mathematics. Vol 43. Springer Verlag. Berlin, Germany.

(52) Rotman, J. J. (2009). *An Introduction to Homological Algebra*. Universitext. Second Edition. Springer Verlag. New York, New York, United States.

(53) Salce, L. (1979). *Cotorsion Theories for Abelian Groups*. Symposia Mathematica. Vol. XXIII. (Conf. Abelian Groups and Their Relationship to the Theory of Modules, INDAM, Rome, 1977.) pp. 11-32. Academic Press. London, United Kingdom.

(54) Saorín, M. *Teoría de Categorías*. Notes. Universidad de Murcia.

(55) Strom, A. (1972). *The homotopy category is a homotopy category*. Archiv der Mathematik. Vol 23, pp. 435–441.

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