Chaotic and Sensitive dependence of Gibbs measures at zero temperature

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1. Sensitive dependence of Gibbs measures at zero temperature.
2. Phenomenon of non-convergence of Gibbs measures as temperature tends to zero.

Chaotic dependence at zero temperature

An interaction (or potential) Φ is *chaotic at zero temperature*, if there is a sequence of inverse temperatures $(\beta_\ell)_{\ell\in\mathbb{N}}$ such that $\beta_\ell \to +\infty$ as $\ell \to +\infty$, and such that the following property holds: If for each ℓ in $\mathbb{N}$ we choose an arbitrary Gibbs measure ρ_ℓ for the interaction $\beta_\ell \cdot \Phi$, then the sequence $(\rho_\ell)_{\ell\in\mathbb{N}}$ does not converge.

XY models

Denote the circle by $\mathbb{T} := \mathbb{R}/\mathbb{Z}$, endowed with the (additive) group structure inherited from $\mathbb{R}$. Given a function $U\colon \mathbb{T} \to \mathbb{R}$, consider the nearest-neighbor interaction Φ_U on $\mathbb{T}^{\mathbb{Z}}$ defined by

$$\Phi_U(\{k, k+1\})\left((\theta_n)_{n\in\mathbb{Z}}\right) := -U(\theta_k - \theta_{k+1}).$$

When U is continuous there is a unique Gibbs measure for the interaction Φ_U, and this measure is translation invariant. Denote this measure by ρ_U.

Chaotic dependence at zero temperature in XY models

Theorem (van Enter, Ruszel '07)

There is a function U such that the interaction Φ_U is chaotic at zero temperature.

Remark. Valid in $\mathbb{Z}^d$.

Chaotic dependence at zero temperature in XY models

Denote by $\pi\colon \mathbb{T}^{\mathbb{Z}} \to \mathbb{T}$ the projection defined by

$$\pi\left((\theta_n)_{n\in\mathbb{Z}}\right) := \theta_0 - \theta_1 .$$

We have

$$\pi_* \rho_{\beta \cdot U} = \left(\frac{\exp(\beta \cdot U)}{\int_{\mathbb{T}} \exp(\beta \cdot U(\theta)) d\theta} \right) \text{Leb} .$$

Classical lattice systems with finite-state space or symbolic spaces

Let $d \geq 1$ be an integer. Given a finite set F containing at least 2 elements, consider the space $\Sigma := F^{\mathbb{Z}^d}$ endowed with the distance dist defined for distinct elements $(\theta_n)_{n\in\mathbb{Z}^d}$ and $(\theta'_n)_{n\in\mathbb{Z}^d}$ of Σ, by

$$\mathrm{dist}\left((\theta_n)_{n\in\mathbb{Z}^d}, (\theta'_n)_{n\in\mathbb{Z}^d}\right) := 2^{-\min\{\|n\|:\theta_n\neq\theta'_n\}},$$

where $\|\cdot\|$ is the sup-norm. Denote by σ the action of $\mathbb{Z}^d$ on Σ by translations.

The *topological pressure of a continuous function* $\varphi\colon \Sigma \to \mathbb{R}$, is

$$P(\varphi) := \sup \left\{ h_\nu + \int \varphi \mathrm{d}\nu : \nu \in \mathscr{M}_\sigma \right\}.$$

A *equilibrium state for the potential* φ is a measure ν at which the supremum above is attained.
If the dimension d is 1, then there is a unique equilibrium state.

Chaotic dependence at zero temperature in symbolic spaces

Theorem (Chazottes, Hochman '10)

For $d = 1$. There is a minimal closed invariant subset X of Σ such that the potential $\varphi(x) := -dist(x, X)$, for every x in Σ, is chaotic at zero temperature.

Remarks.

1. X is not uniquely ergodic.
2. φ is Lipschitz.

Chaotic dependence at zero temperature in symbolic spaces

Theorem (Chazottes, Hochman '10)

For $d \geq 3$. There is a locally constant potential which is chaotic at zero temperature for translation invariant Gibbs measures.

Remark.
The translation invariant Gibbs measures accumulate on measures supported on a subshift of finite type.

Sensitive dependence at zero temperature

An interaction (or potential) Φ is *sensitive at zero temperature*, if for every sequence of inverse temperatures $(\beta_\ell)_{\ell\in\mathbb{N}}$ such that $\beta_\ell \to +\infty$ as $\ell \to +\infty$ there is an arbitrarily small perturbation $\widetilde{\Phi}$ of Φ such that the following property holds: If for each ℓ in $\mathbb{N}$ we choose an arbitrary Gibbs measure ρ_ℓ for the interaction $\beta_\ell \cdot \widetilde{\Phi}$, then the sequence $(\rho_\ell)_{\ell\in\mathbb{N}}$ does not converge.

Theorem (Sensitive dependence for some quadratic-like maps)

There is a continuous family of real quadratic-like maps $\left(f_{\underline{\sigma}}\right)_{\underline{\sigma}\in\{+,-\}^{\mathbb{N}}}$*, a continuous families of probability measures*

$$\left(\rho_{\infty}^{+}(\underline{\sigma})\right)_{\underline{\sigma}\in\{+,-\}^{\mathbb{N}}} \quad \textit{and} \quad \left(\rho_{\infty}^{-}(\underline{\sigma})\right)_{\underline{\sigma}\in\{+,-\}^{\mathbb{N}}},$$

and a continuous function $A : \{+,-\}^{\mathbb{N}} \to (0,+\infty)$*, such that the following properties hold.*

1. *For each* $\underline{\sigma}$ *in* $\{+,-\}^{\mathbb{N}}$ *the measures* $\rho_{\infty}^{+}(\underline{\sigma})$ *and* $\rho_{\infty}^{-}(\underline{\sigma})$ *are distinct, and each of them is invariant by* $f_{\underline{\sigma}}$ *and supported on a real periodic orbit of* $f_{\underline{\sigma}}$*.*
2. *For each* $\underline{\sigma}$ *in* $\{+,-\}^{\mathbb{N}}$ *the map* $f_{\underline{\sigma}}$ *is essentially topologically exact. Moreover, for each* $t > 0$ *there is a unique equilibrium state* $\rho_t^{\mathbb{R}}(\underline{\sigma})$ *(resp.* $\rho_t(\underline{\sigma})$*) of* $f_{\underline{\sigma}}|_{I(f_{\underline{\sigma}})}$ *(resp.* $f_{\underline{\sigma}}|_{J(f_{\underline{\sigma}})}$*) for the potential* $-t\log|Df_{\underline{\sigma}}|$*.*

[3.] There are constants $C > 0$ and $\varkappa > 0$ such that for every sequence $\underline{\sigma} = (\sigma(m))_{m\in\mathbb{N}}$ in $\{+,-\}^{\mathbb{N}}$, the following properties hold. Let m and $\widehat{m}$ be integers such that

$$\widehat{m} \geq m \geq 1 \quad \text{and} \quad \sigma(m) = \cdots = \sigma(\widehat{m})$$

and let t be in $[A(\underline{\sigma})m, A(\underline{\sigma})\widehat{m}]$. Then the equilibrium state $\rho_t^{\mathbb{R}}(\underline{\sigma})$ (resp. $\rho_t(\underline{\sigma})$) of $f_{\underline{\sigma}}|_{I(f_{\underline{\sigma}})}$ (resp. $f_{\underline{\sigma}}|_{J(f_{\underline{\sigma}})}$) is super-exponentially close to $\rho_\infty^{\sigma(m)}(\underline{\sigma})$:

$$\rho_t^{\mathbb{R}}(\underline{\sigma})\left(B\left(\mathrm{supp}(\rho_\infty^{\sigma(m)}(\underline{\sigma})), \exp(-\varkappa t^2)\right)\right) \geq 1 - C\exp(-\varkappa t^2)$$

$$\left(\text{resp. } \rho_t(\underline{\sigma})\left(B\left(\mathrm{supp}(\rho_\infty^{\sigma(m)}(\underline{\sigma})), \exp(-\varkappa t^2)\right)\right) \geq 1 - C\exp(-\varkappa t^2)\right).$$

A configuration $(\theta_n)_{n\in\mathbb{Z}}$ in $\mathbb{T}^{\mathbb{Z}}$ is *ferromagnetic* (resp. *antiferromagnetic*), if for every n we have $\theta_{n+1} = \theta_n$ (resp. $\theta_{n+1} = \theta_n + \frac{1}{2}$). The *ferromagnetic* (resp. *antiferromagnetic*) *phase* is the measure on $\mathbb{T}^{\mathbb{Z}}$ that is evenly distributed on ferromagnetic (resp. antiferromagnetic) configurations.

Theorem (Sensitive dependence of Gibbs measures on the interaction)

There is a smooth function $U_0 \colon \mathbb{T} \to \mathbb{R}$ such that for every sequence of positive numbers $(\widehat{\beta}_\ell)_{\ell\in\mathbb{N}}$ satisfying $\widehat{\beta}_\ell \to +\infty$ as $\ell \to +\infty$, the following property holds: There is an arbitrarily small smooth perturbation U of U_0 such that the sequence of Gibbs measures $\left(\rho_{\widehat{\beta}_\ell \cdot U}\right)_{\ell\in\mathbb{N}}$ accumulates at the same time on the ferromagnetic and the antiferromagnetic phases.

Theorem (Sensitive dependence of Gibbs measures on the potential)

There is a Lipschitz continuous potential $\varphi_0 \colon \Sigma \to \mathbb{R}$ *and complementary open subsets* U^+ *and* U^- *of* Σ*, such that for every sequence of positive numbers* $(\widehat{\beta}_\ell)_{\ell \in \mathbb{N}}$ *satisfying* $\widehat{\beta}_\ell \to +\infty$ *as* $\ell \to +\infty$*, the following property holds: There is an arbitrarily small Lipschitz continuous perturbation* φ *of* φ_0 *such that if for each* ℓ *we choose an arbitrary translation invariant Gibbs measure* ρ_ℓ *for the potential* $\widehat{\beta}(\ell) \cdot \varphi$*, then the sequence* $(\rho_\ell)_{\ell \in \mathbb{N}}$ *accumulates at the same time on a measure supported on* U^+ *and on a measure supported on* U^-*.*

Corollary

Assume that the dimension d is 1*. Let* μ^+ *and* μ^- *be ergodic measures defined on a Lebesgue space having the same finite entropy. Then, provided the finite set F is sufficiently large, there is a Lipschitz continuous potential* $\varphi\colon \Sigma \to \mathbb{R}$ *such that the one-parameter family of Gibbs measures* $(\rho_{\beta\cdot\varphi})_{\beta>0}$ *accumulates at the same time on a measure isomorphic to* μ^+ *and on a measure isomorphic to* μ^- *as* $\beta \to +\infty$*.*

About the proof in the symbolic case

Lemma

Let X and X' be disjoint compact subsets of Σ, each of which is invariant by σ, and such that $h_{top}(\sigma|_X) > h_{top}(\sigma|_{X'})$. Moreover, let $\varphi\colon \Sigma \to \mathbb{R}$ be a Lipschitz continuous function attaining its maximum precisely on $X \cup X'$. Then for every δ in $(0, 1)$ and every neighborhood U of X there is $\beta_0 > 0$ such that for every $\beta \geq \beta_0$ and every translation invariant Gibbs measure ρ for the potential $\beta \cdot \varphi$ we have

$$\rho(U) \geq 1 - \delta.$$

About the proof in the symbolic case

Lemma

Let $\varphi_0 \colon \Sigma \to \mathbb{R}$ *be a Lipschitz continuous function and let* $\beta_0 \geq 0$ *be given. Then for every* $\delta > 0$ *and every continuous function* $\psi \colon \Sigma \to \mathbb{R}$ *there is* $\varepsilon > 0$ *such that for every Lipschitz continuous function* $\varphi \colon \Sigma \to \mathbb{R}$ *satisfying* $\|\varphi - \varphi_0\|_{Lip} \leq \varepsilon$ *and every translation invariant Gibbs measure* ρ *for the potential* $\beta_0 \cdot \varphi$ *there is a translation invariant Gibbs measure* ν *for the potential* $\beta_0 \cdot \varphi_0$ *such that*

$$\left| \int \psi d\rho - \int \psi d\nu \right| < \delta.$$