Rigidity of Smooth Critical Circle Maps

Pablo Guarino
Universidade Federal Fluminense
Niterói, Rio de Janeiro, Brasil

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Definition

Critical circle map: orientation-preserving C^3 circle homeomorphism, with exactly one critical point of odd type.

We will focus on the case of **irrational** rotation number (no periodic orbits).

Topological Rigidity (Yoccoz 1984)

Any C^3 critical circle map f with $\rho(f) \in [0,1] \setminus \mathbb{Q}$ is **minimal**.

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Why are we interested on critical circle maps?

They belong to the "boundary of chaos": $\partial\{h_{top} > 0\}$.

What happens at the boundary of chaos?

- Simple topological dynamics: well-understood model, which is minimal.
- Simple ergodic behaviour: uniquely ergodic attractor supporting a global physical measure (full Lebesgue measure basin) with zero Lyapunov exponent.

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The **topology** of the system determines its **geometry**.

Theorem 1 (G.-de Melo 2012, available at arXiv: 1303.3470)

Any two C^3 critical circle maps with the same irrational rotation number of **bounded type** and the same odd criticality are conjugate to each other by a $C^{1+\alpha}$ circle diffeomorphism, for some universal $\alpha > 0$.

Recall that θ in $[0, 1]$ is of *bounded type* if $\exists\ \varepsilon > 0$:

$$\left|\theta - \frac{p}{q}\right| \geq \frac{\varepsilon}{q^2},$$

for any positive coprime integers p and q.

The set $\mathcal{BT} \subset [0, 1]$ of bounded type numbers has Hausdorff dimension equal to 1, but Lebesgue measure equal to zero.

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What about unbounded combinatorics?

Theorem 2 (G.-Martens-de Melo, work in progress)

Let f and g be two C^4 critical circle maps such that:

- $\rho(f) = \rho(g) \in [0,1]\setminus\mathbb{Q}$,
- the same odd criticality.

Let h be the conjugacy between f and g that maps the critical point of f to the critical point of g. Then:

- h is a C^1 diffeomorphism.
- For a full Lebesgue measure set of rotation numbers, h is a $C^{1+\alpha}$ diffeomorphism (for some universal $\alpha > 0$).

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Theorem 1 was known as the **Rigidity Conjecture**, formulated by Lanford, Feigenbaum, Kadanoff, Shenker and Rand among others, in the early eighties.

Both Theorem 1 and Theorem 2 were proved to be true in the **real-analytic** category by de Faria-de Melo 2000, Yampolsky 2003 and Khanin-Teplinsky 2007.

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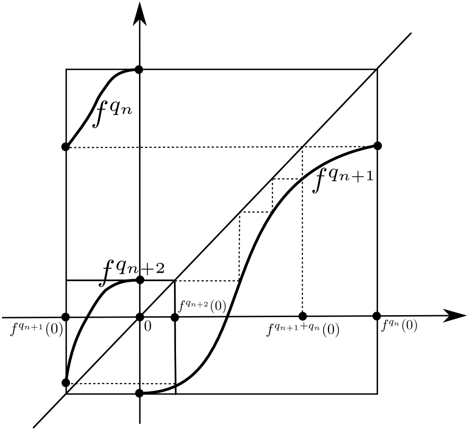
The renormalization operator:



In the figure $a_{n+1} = 4$.

Smooth conjugacy follows from **exponential contraction** of the **renormalization** operator.

Theorem (de Faria-de Melo 1999)

There exists $\mathbb{A} \subset [0,1]$ with:

- $\text{Leb}(\mathbb{A}) = 1$
- $\mathcal{BT} \subset \mathbb{A}$

such that for any two C^3 c.c.m. f and g with $\rho(f) = \rho(g) \in \mathbb{A}$ we have that if:

$$d_{C^0}\big(\mathcal{R}^n(f), \mathcal{R}^n(g)\big) \to 0 \quad \text{when} \quad n \to +\infty$$

exponentially fast, then f and g are $C^{1+\alpha}$ conjugate, for some universal $\alpha > 0$.

The remaining cases: exponential convergence in the C^2-metric implies C^1-rigidity (Khanin-Teplinsky 2007).

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Theorem A (G.-de Melo 2012)

There exist a C^ω-compact set $\mathcal{K}$ of real-analytic critical commuting pairs and $\lambda \in (0,1)$ such that:

Given a C^3 critical circle map f with **any** irrational rotation number there exist:

- $C > 0$, and
- $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{K}$,

such that for all $n \in \mathbb{N}$:

- $d_{C^2}\big(\mathcal{R}^n(f), f_n\big) \leq C\lambda^n$, and
- $\rho(f_n) = \rho\big(\mathcal{R}^n(f)\big)$.

To prove that Theorem A implies exponential contraction of $\mathcal{R}$ we use:

Hölder continuity (G.-Martens-de Melo, work in progress)

Let $\mathcal{K}$ be a C^3-compact set of critical commuting pairs.
There exist $C > 0$ and $\gamma \in (0, 1]$ such that if $f, g \in \mathcal{K}$ with $\rho(f) = \rho(g) \in [0, 1] \backslash \mathbb{Q} \Rightarrow$

$$d_{C^0}\big(\mathcal{R}(f), \mathcal{R}(g)\big) \leq C\big(d_{C^2}(f, g)\big)^{\gamma}.$$

Both C and γ are uniform in $\mathcal{K}$, they **do not** depend on the combinatorics.

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