Entropy-expansiveness and domination

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Abstract

Let $f : M \to M$ be a C^r-diffeomorphism defined in a compact boundaryless surface M, $d = 2, 3$. We prove that if K is a compact f-invariant subset of M with a dominated splitting then f/K is h-expansive. Reciprocally, if there exists a C^r neighborhood of f, $\mathcal{U}$, such that for $g \in \mathcal{U}$ there exists K_g compact invariant such that g/K_g is h-expansive then there is a dominated splitting for K_g.

1 Introduction

In order to obtain results about the complexity of the dynamics: recurrence, existence of periodic orbits, SRB measures, etc., one usually try to express dynamic properties at the infinitesimal level, ie: precise definitions are given prescribing the behaviour of the tangent map $Df : TM \to TM$ of a diffeomorphism $f : M \to M$. Examples of that are the concepts of hyperbolicity and the existence of dominated splittings. On the other hand a robust dynamic property (i.e. a property that holds for a system and all nearby ones) should leave its *impromptus* in the behaviour of the tangent map of those differentiable systems sharing that property. In [PPV], [SV] and [PPSV] it has been studied the influence of expansiveness when it holds in a homoclinic class H associated to a hyperbolic periodic point p such that H and the corresponding homoclinic classes H_g , for all diffeomorphism g nearby f, are expansive. It is proved there that in that case there exists dominated splitting and moreover that it is hyperbolic in the codimension one case ([PPV], [PPSV]). In the general codimension case we also obtain hyperbolicity adding an extra hypothesis called germ-expansiveness (see [SV]).
In this paper we relax expansiveness asking what should be the properties of the

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tangent map Df of a diffeomorphism f such that robustly exhibits h-expansiveness (entropy-expansiveness, see definitions below). We obtain that for such maps it exists a dominated splitting. On the other hand we prove that if K admits a dominated splitting then it is h-expansive. Thus robust h-expansiveness is equivalent to the existence of domination in the two dimensional case.

Moreover, we give here an example of a C^∞ diffeomorphism that is not h-expansive. By a result of Buzzi (see [Bu]) such an example is asymptotically h-expansive since it is C^∞. The first examples of diffeomorphisms that are not h-expansive and even not asymptotically h-expansive were given by Misiurewicz in [Mi] answering a question posed by Bowen. We give our example here because of its good properties from various points of view. First it is clear that it has not a dominated splitting. Second it is defined on S^2, is ergodic and even has Bernoulli property. Third it admitts analytic models a stronger property than being C^∞.

Let us now give precise definitions. Let M be a compact connected boundaryless Riemmanian d-dimensional manifold and $f : M \to M$ a homeomorphism. Let K be a compact invariant subset of M and $\mathrm{dist} : M \times M \to I\!R^+$ a distance in M compatible with its Riemannian structure. For $E, F \subset K$, $n \in I\!N$ and $\delta > 0$ we say that E (n, δ) spans F with respect to f if for each $y \in F$ there is $x \in E$ such that $\mathrm{dist}(f^j(x), f^j(y)) \le \delta$ for all $j+0, \dots, n-1$. Let $r_n(\delta, F)$ denote the minimum cardinality of a set that (n, δ) spans F. Since K is compact $r_n(\delta, F) < \infty$. We define

$$h(f, F, \delta) = \limsup_{n\to\infty} \frac{1}{n} \log(r_n(\delta, F))$$

and

$$h(f, F) = \lim_{\delta\to 0} h(f, F, \delta)\,.$$

The last limit exists since $h(f, F, \delta)$ increases as δ decreases to zero.

For $x \in K$ let us define

$$\Gamma_\epsilon(x) = \{y \in M \,/\, d(f^n(x), f^n(y)) \le \epsilon\}\,.$$

Following Bowen (see [Bo]) we say that f/K is **entropy-expansive** or **h-expansive** iff there exists $\epsilon > 0$ such that

$$h^*_f(\epsilon) = \sup_{x\in K} h(f, \Gamma_\epsilon(x)) \;=\; 0\,.$$

The importance of f being h-expansive is that the topological entropy of f restricted to K, $h(f/K)$, is equal to its estimate using ϵ: $h(f, K) = h(f, K, \epsilon)$. More precisely:

Theorem 1.1. *For all homeomorphism f defined in a compact invariant set K it holds*

$$h(f,K) \leq h(f,K,\epsilon) + h^*_f(\epsilon) \text{ in particular } h(f,K) = h(f,K,\epsilon) \text{ if } h^*_f(\epsilon) = 0\,.$$

Proof. See [Bo], Theorem 2.4. □

A weaker property of that of being h-expansive is that of being asymptotically h-expansive ([Mi]). Let K be a compact metric space and $f : K \to K$ an homeomorphism. We say that f is asymptotically h-expansive iff

$$\lim_{\epsilon \to 0} h^*_f(\epsilon) = 0\,.$$

Thus we do not require that for a certain $\epsilon > 0$ $h^*_f(\epsilon) = 0$ but that $h^*_f(\epsilon) \to 0$ when $\epsilon \to 0$. It has been proved by Buzzi that any C^∞ diffeomorphism defined on a compact manifold is asymptotically h-expansive. Hence our example although not h-expansive is asymptotically h-expansive.

Our main results are the following:

Definition 1.1. *We say that a compact f-invariant set Λ admits a dominated splitting if the tangent bundle $T_\Lambda M$ has a continuous f-invariant splitting $E \oplus F$ and there exist $C > 0$, $0 < \lambda < 1$ such that*

$$\|Df^n|E(x)\| \cdot \|Df^{-n}|F(f^n(x))\| \leq C\lambda^n \;\forall x \in \Lambda,\, n \geq 0.$$

Theorem A. *Let M be a compact boundary-less C^∞ surface and $f : M \to M$ be a C^r diffeomorphism such that $K \subset M$ is a compact f-invariant subset with a dominated splitting $E \oplus F$. Then f/K is h-expansive.*

Since the property of having a dominated splitting is open we may conclude that any g C^1 close to f is such that g/K_g is h-expansive.

In case M is a d-dimensional manifold with $d \geq 3$ dominance alone is not enough to guarantee h-expansiveness as it is shown in the examples.

Reciprocally one has

Theorem B. *Let M and $f : M \to M$ be as in Theorem A and $H(p)$ an f-homoclinic class associated to the f-hyperbolic periodic point p. Assume that there is a C^1 neighborhood $\mathcal{U}$ of f such that for any $g \in \mathcal{U}$ it holds that there is a continuation $H(p_g)$ of $H(p)$ such that $H(p_g)$ is h-expansive. Then $H(p)$ has a dominated splitting.*

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