Entropy-expansiveness and domination

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Abstract

Let $f : M \to M$ be a C^r-diffeomorphism, $r \geq 1$, defined on a compact boundary-less manifold M. We prove that if M is a surface and $K \subset M$ is a compact f-invariant set such that $T_K M = E \oplus F$ is dominated then f/K is entropy expansive. Moreover, if $\dim M > 2$ and $H(p)$, the f-homoclinic class of a hyperbolic periodic point p, has a dominated splitting and is isolated, then C^1-generically $f/H(p)$ is entropy expansive. Conversely, if there exists a C^1 neighborhood $\mathcal{U}$ of a diffeomorphism f defined on a compact surface and a homoclinic class $H(p)$ of an f- hyperbolic periodic point p, such that for every $g \in \mathcal{U}$ the continuation $H(p_g)$ of $H(p)$ is entropy-expansive then there is a dominated splitting for $H(p)$.

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1 Introduction

Since the seminal work of Smale [Sm] establishing the main goals to describe the long term evolution of a discrete or continuous time dynamical system, the strategy has been to prescribe some property at the infinitesimal level of the system that implies a definite behavior for the underlined dynamics. Examples include the concepts of hyperbolicity, partial hyperbolicity and dominated splitting. On the other hand one may ask what are the consequences at the infinitesimal level from a known behavior of the evolution system at the ambient manifold. But rarely a property displayed by a system solely implies an interesting behavior of the differential map acting at the tangent bundle. For instance, in [Ge, GK] it is proved that a (generalized) pseudo-Anosov map f is ergodic and even Bernoulli. For those maps there is at least one point p where the derivative $Df(p)$ is the identity map and so the dynamics at the tangent bundle cannot be characterized in terms of hyperbolicity or even dominance. Example 1 in this article is a generalized pseudo-Anosov map illustrating such a behavior. So, it is natural to ask which robust properties satisfied by the underlined systems has dynamical consequences at the tangent bundle level and vice versa. Several authors have worked in this line of ideas (see for instance [Ma2, Ma3, DPU, PPV, SV]).

Here by a robust property we mean a property shared by all system in a neighborhood of the original one. In this paper we study what are the consequences at the dynamical behavior of the tangent map Df of a diffeomorphism $f : M \to M$, assuming that f is robustly entropy expansive. In this direction we obtain that the tangent bundle has a Df-invariant dominated splitting. Reciprocally, we show, in the case of surfaces, that the existence of a dominated splitting for the tangent bundle implies robust entropy expansiveness for the diffeomorphism f. Thus robust entropy expansiveness is equivalent to the existence of a dominated spitting for surface diffeomorphisms.

We also give an example of a diffeomorphism that is not entropy expansive. This example is of class C^∞ and so it is asymptotically entropy expansive by a result of Buzzi [Bu]. The first example of a diffeomorphism that is not entropy expansive neither asymptotically entropy expansive was given by Misiurewicz in [Mi] answering a question posed by Bowen [Bo]. Nevertheless we add our example because of its nice properties: (1) it is defined on the sphere S^2, (2) it has no dominated splitting, (3) it is ergodic and even Bernoulli, (4) it admits analytic models. Moreover, a straightforward modification of this example shows that there are diffeomorphisms defined on manifolds of dimension greater than 2 that has a dominated splitting defined on a homoclinic class but that are not entropy expansive.

Let us now give precise definitions. Let M be a compact connected boundary-less Riemannian d-dimensional manifold and $f : M \to M$ a homeomorphism. Let K be a compact invariant subset of M and $\mathrm{dist} : M \times M \to I\!\!R^+$ a distance in M compatible with its Riemannian structure. For $E, F \subset K$, $n \in I\!\!N$ and $\delta > 0$ we say that E (n, δ)-spans F with respect to f if for each $y \in F$ there is $x \in E$ such that $\mathrm{dist}(f^j(x), f^j(y)) \le \delta$ for all $j = 0, \ldots, n-1$. Let $r_n(\delta, F)$ denote the minimum cardinality of a set that (n, δ)-spans F. Since K is compact $r_n(\delta, F) < \infty$. We define

$$h(f, F, \delta) \equiv \limsup_{n\to\infty} \frac{1}{n} \log(r_n(\delta, F))$$

and the topological entropy of f restricted to F as

$$h(f, F) \equiv \lim_{\delta\to 0} h(f, F, \delta)\,.$$

The last limit exists since $h(f, F, \delta)$ increases as δ decreases to zero.

For $x \in K$ let us define

$$\Gamma_\epsilon(x, f) \equiv \{y \in M \,/\, d(f^n(x), f^n(y)) \le \epsilon,\ n \in Z\!\!\!Z\}\,.$$

We will simply write $\Gamma_\epsilon(x)$ instead of $\Gamma_\epsilon(x, f)$ when it is understood which f we refer to.

Following Bowen (see [Bo]) we say that f/K is **entropy-expansive** or h**-expansive** for short, if and only if there exists $\epsilon > 0$ such that

$$h^*_f(\epsilon) \equiv \sup_{x\in K} h(f, \Gamma_\epsilon(x)) = 0\,.$$

The importance of f being h-expansive is that the topological entropy can be derived from its ϵ-estimate $h(f,K,\epsilon)$, as showed by [Bo, Theorem 2.4].

A similar notion to h-expansiveness, albeit weaker, is the notion of *asymptotically h-expansiveness* [Mi]: let K be a compact metric space and $f: K \to K$ an homeomorphism. We say that f is asymptotically h-expansive if and only if

$$\lim_{\epsilon \to 0} h_f^*(\epsilon) = 0\,.$$

Thus we do not require that for a certain $\epsilon > 0$ $h_f^*(\epsilon) = 0$ but that $h_f^*(\epsilon) \to 0$ when $\epsilon \to 0$. It has been proved by Buzzi that any C^∞ diffeomorphism defined on a compact manifold is asymptotically h-expansive.

Next we recall the notion of dominated splitting.

Definition 1.1. *We say that a compact f-invariant set $\Lambda \subset M$ admits a dominated splitting if the tangent bundle $T_\Lambda M$ has a continuous Df-invariant splitting $E \oplus F$ and there exist $C > 0$, $0 < \lambda < 1$, such that*

$$\|Df^n|E(x)\| \cdot \|Df^{-n}|F(f^n(x))\| \le C\lambda^n \;\forall x \in \Lambda,\, n \ge 0. \tag{1}$$

Our main results are the following:

Theorem A. *Let M be a compact boundaryless C^∞ surface and $f: M \to M$ be a C^r diffeomorphism such that $K \subset M$ is a compact f-invariant subset with a dominated splitting $E \oplus F$. Then f/K is h-expansive.*

Since the property of having a dominated splitting is open we may conclude that any g C^1 close to f is such that g/K_g is h-expansive where K_g is a continuation of $K = K_f$.

In case M is a d-dimensional manifold with $d \ge 3$ the existence of a dominated splitting is not enough to guarantee h-expansiveness as it is shown in the second example given below. Nevertheless a weaker result can be achieved:

Theorem B. *Let M be a compact boundaryless C^∞ d-dimensional manifold and $f: M \to M$ be a C^r diffeomorphism. Let $H(p)$ be an isolated f-homoclinic class associated to the f-hyperbolic periodic point p. Assume that $H(p)$ admits a dominated splitting. Then there is a C^1 neighborhood $\mathcal{U}$ of f such that for a residual subset $\mathcal{R} \subset \mathcal{U}$ any $g \in \mathcal{R}$ is h-expansive when restricted to $H(p_g)$.*

Observe that if the topological entropy of a map $f: M \to M$ vanishes, $h(f) = 0$, then f is h-expansive. For instance the identity map $id: M \to M$ is h-expansive. Nevertheless, robustness of h-expansiveness has a dynamical meaning as shows the following theorem.

Theorem C. *Let M be a compact boundaryless C^∞ surface and $f: M \to M$ be a C^r diffeomorphism. Let $H(p)$ be an f-homoclinic class associated to the f-hyperbolic periodic point p. Assume that there is a C^1 neighborhood $\mathcal{U}$ of f such that for any $g \in \mathcal{U}$ it holds that the continuation $H(p_g)$ of $H(p)$ is h-expansive. Then $H(p)$ has a dominated splitting.*

A natural question that arises is if Theorem C holds not only for surfaces but also for compact manifolds of any finite dimension. We believe that this is the case and it will be the subject of a forthcoming paper. This would imply that C^1 generically h-expansiveness of an isolated $H(p,f)$ is equivalent to the existence of a dominated splitting for $H(p,f)$.

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