EXPANSIVE DIFFEOMORPHISMS AND NON HYPERBOLIC PERIODIC POINTS

José L. Vieitez *
IMERL
Facultad de Ingeniería
Montevideo, URUGUAY

Abstract

Let M be a compact connected oriented three dimensional manifold and $f : M \to M$ an expansive diffeomorphism. Assume that there is a periodic point p for f such that $\Omega(f) \supset V(p)$ where $V(p)$ is a neighborhood of p. Then p is topologically hyperbolic provided at most one of its eigenvalues has modulus 1.

1 Introduction

Let M be a compact connected oriented three dimensional manifold and $f : M \to M$ an expansive diffeomorphism. Let $dist : M \times M \to \mathbb{R}$ be a metric defining its topology.

Definition 1.1 *We say that f is expansive if there exists a positive constant α such that if we have $x, y \in M$ and for every $n \in \mathbb{Z}$ it holds that $dist(f^n(x), f^n(y)) \leq \alpha$ then $x = y$. The number α is called an expansivity constant for f.*

Definition 1.2 *We say that p is a hyperbolic periodic point for f if there is $k \in \mathbb{N}$, $k > 0$ such that $f^k(p) = p$ and the associated tangent map $T_p f^k$ has all its*

*I wish to thank L'Université de Bourgogne for its hospitality during part of the preparation of this article

eigenvalues off the unit circle. The point is said to be a topologically hyperbolic periodic point for f if there is a homeomorphism h conjugating in a neighborhood of p, f^k with a hyperbolic linear map T, $h(p) = 0$, $h : V(p) \to V(0)$, $T : \mathbb{R}^3 \to \mathbb{R}^3$.

Let us assume now that $\Omega(f) \supset V(p)$ where p is a periodic point for f. Then p is topologically hyperbolic provided at most one of its eigenvalues has modulus 1.
As M is a compact manifold and f is expansive there are not Lyapunov stable points (see [Le2] lemma 2.7) thus there are no periodic attractors (repellers) for f. Therefore $T_p f^k$ cannot have the three eigenvalues of modulus less than one or of modulus greater than one. Thus in the hyperbolic case it has two eigenvalues of modulus greater than 1 and the other one with modulus less than 1 or it has two eigenvalues with modulus less than 1 and the other one with modulus greater than 1. Now assume that there is an eigenvalue of modulus one and only one. Then it must be real hence equal to 1 or -1. The other two eigenvalues can be of modulus less than 1 both, of modulus greater than 1 both, or one of them of modulus greater than 1 and the other of modulus less than 1. In this last case we have that the three eigenvalues, being of different modulus, are real. We will see that in the three cases p behaves as if it were hyperbolic, namely, it is topologically hyperbolic. It seems natural to ask what occurs in the case that p loose hyperbolicity in more than one eigenvalue. We have proved a similar result in the case that at least one eigenvalue lies outside the unit circle, but the methods used are different from those of this paper.
The following example, due to Lewowicz, shows that this is possible even in the case of a diffeomorphism conjugate to a linear Anosov isomorphism to loose hyperbolicity at all(see [Le1], section 6).
Let $F_c(x, y) = (2x - \frac{c}{2\pi}\sin(2\pi x) + y,\, x - \frac{c}{2\pi}\sin(2\pi x) + y)$. Then for $0 \leq c < 1$, F_c is Anosov. Moreover F_1 is conjugated to F_0, as every F_c, $c \in [0, 1]$, is. But F_1 has a non-hyperbolic fixed point at $(0, 0)$. See [Le1], in particular section 6 for details. Moreover, this example may be refined in order to have that the tangent map at $(0, 0)$ is the identity.

Let us state the main definitions.

Definition 1.3 *For $x \in M$ we define*

$$W^s_\epsilon(x, f) = \{y \in M / dist(f^k(x), f^k(y)) \leq \epsilon\, ; k \geq 0\}$$

as the local ϵ-stable set for the point x and the homeomorphism f.

Analogously we define the local ϵ-unstable set for x and f as $W^u_\epsilon(x,f) = W^s_\epsilon(x,f^{-1})$. If there is no ambiguity we shall usually omit any reference to ϵ and f and speak about local stable and unstable sets of the point x denoting them by $W^s_\epsilon(x)$ and $W^u_\epsilon(x)$ respectively. Observe that $W^s_\epsilon(x)$ and $W^u_\epsilon(x)$ verify that $f(W^s_\epsilon(x)) \subset W^s_\epsilon(f(x))$ and $f^{-1}(W^u_\epsilon(x)) \subset W^u_\epsilon(f^{-1}(x))$.

Definition 1.4 *We define*

$$W^s(x,f) = \{y \in M / \lim_{k \to +\infty} dist(f^k(x), f^k(y)) = 0\}$$

as the stable set for the point x and the homeomorphism f.

Analogously we define the unstable set for x, f as $W^u(x,f) = W^s(x,f^{-1})$. We usually will omit the reference to f. If α is a expansivity constant for f and $\epsilon \leq \alpha$, then $W^s_\epsilon(x) \subset W^s(x)$, $W^u_\epsilon(x) \subset W^u(x)$ (see [Vi1] lemma 1, for instance). The reader should notice that the stable and unstable sets *are not assumed* to be manifolds. Moreover, most of our work is devoted to the proof that these sets are true manifolds, at least in a topological sense.

Proposition 1.1 *Let $0 < \epsilon < \alpha$. There is r_0, $0 < r_0 \leq \epsilon$ such that for every $x \in M$ there exists a compact connected set $D(x) \subset W^s_\epsilon(x)$ ($C(x) \subset W^u_\epsilon(x)$) such that $x \in D(x)$ (resp.: $x \in C(x)$) and for all open set $A \subset B(x, r_0)$, $x \in A$, we have that $D(x) \cap \partial A \neq \emptyset$ (resp.: $C(x) \cap \partial A \neq \emptyset$).*

Proof. See [Le3], lemma 2.1, see also [Vi3] proposition 2.6.

■

Proposition 1.2 *Let $0 < \sigma < \epsilon$. There is $r > 0$ such that if $y \in W^s_\epsilon(x)$ and $dist(x,y) < r$ then $y \in W^s_\sigma(x)$.*

Proof. See [Le3], lemma 2.2.

■

We usually will take $\epsilon < \frac{\alpha}{2}$ and $r = r_1$ such that if $y \in W^s_{2\epsilon}(x)$ and $dist(x,y) < r_1$ then $y \in W^s_\epsilon(x)$. Moreover for such an ϵ we choose $r_1 \leq r_0$, where r_0 is that of 1.1. Let A, B be non-empty subsets of M and define $\text{dist}(A,B) = \inf\{\text{dist}(x,y)/x \in A,\ y \in B\}$.

Proposition 1.3 *Given r', $r_1 > r' > 0$, there exist $\lambda > 0$ and $\mu > 0$ such that if $dist(x,y) < \lambda$ then*

$$dist(W^s_\epsilon(x)\backslash B(x,r'), W^u_\epsilon(y)\backslash B(x,r')) > \mu$$

.

Proof. See [Vi3], proposition 2.8 and remark 2.9.

∎

Let $f : M \to M$ be an expansive homeomorphism, where M is a compact manifold.

Proposition 1.4 *(Existence of Lyapunov functions)*
There are functions U, V and W defined in a neighborhood N of the diagonal of $M \times M$, $U, V, W : N \to I\!R$ such that $V(x,y) = \Delta U(x,y) = U(f(x), f(y)) - U(x,y)$ and $W(x,y) = \Delta V(x,y) = V(f(x), f(y)) - V(x,y)$ $= \Delta(\Delta U) = U(f^2(x), f^2(y)) - 2U(f(x), f(y)) + U(x,y)$ with the properties that $U(x,y)$ and $W(x,y)$ vanish only at the diagonal of $M \times M$ and are positive elsewhere, and $V(x,y) > 0$ if $y \in W^u_\epsilon(x)$ and $V(x,y) < 0$ if $y \in W^s_\epsilon(x), y \neq x$, where $\epsilon > 0$ is less than some expansivity constant α.

Proof. See [Le3] §1, see also [Le1] §4.

∎

Definition 1.5 *Given a set A in a metric space (X,d) we define its δ-parallel body*

$$[A]_\delta = \{y \in X/ \inf_{x \in A} d(y,x) \leq \delta\}.$$

The Hausdorff distance $Hdist$ between two non empty compact sets $A, B \subset X$ is

$$Hdist(A,B) = \inf\{\delta \geq 0/A \subset [B]_\delta \text{ and } B \subset [A]_\delta\}.$$

With this distance the space $\mathcal{C} = \{C \subset M/C \neq \emptyset \text{ is compact}\}$ is a complete space (see [Fa]).

Definition 1.6 *We say that a sequence of compacta $\{C_n\}$ converges in the Hausdorff metric to C and write $Hlim\, C_n = C$ if given $\epsilon > 0$ there is $N \in I\!\!N$ such that for all $n \geq N$ we have $Hdist(C_n, C) < \epsilon$.*

As usual $Oxy = \{(x, y, z) \in I\!\!R^3/z = 0\}$ and $Oz = \{(x, y, z) \in I\!\!R^3/x = y = 0\}$, a horizontal square is a square located in a plane parallel to Oxy, a vertical line is a segment located in a straight line parallel to Oz.

Definition 1.7 *Given $f : M \to M$ and a point $x \in M$ we say that there is a local product structure in x (abbr.: x has an f-lps) if there is a neighborhood $V(x)$, and a homeomorphism $h : V(x) \to [-1, 1]^3$, the standard $3D$ cube, such that either h sends the local stable sets of points in $V(x)$ onto horizontal squares and the unstable sets onto vertical lines or, alternatively, sends the local stable sets of points in $V(x)$ onto vertical lines and the unstable sets onto horizontal squares.*

Observe that the local product structure is invariant by f, ie: if there is a local product structure for x then there is a local product structure for $f^k(x)$ for $k \in Z\!\!\!Z$. Our definition of local product structure differs from the usual one (see [Sh] chapter 8). We require the existence of local C^0 foliations given by the local stable and unstable sets.
We say that a set $A \subset M$ has an f-local product structure (abbr. A has an f-lps) if every point of A has an f-lps in M. By definition A is open.
The open ball of center x and radius $r > 0$ is the set $B(x, r) = \{y \in M/dist(x, y) < r\}$. As M is a compact $3D$-manifold there is $r_1 > 0$ such that for all $0 < r \leq r_1$ $B(x, r)$ is homeomorphic to B^3 the standard 3-cell of $I\!\!R^3$. We assume from now on that we take $r \leq r_1$. Throughout this paper we assume that M is a compact connected orientable smooth $3D$-manifold so we will usually avoid to state it.
We will call the eigenvalues of $T_p f^k$ of modulus different from 1 (here k is the period of p) by λ, μ.

2 Case of $|\lambda|<1$, $|\mu|>1$

2.1

We have that the three eigenvalues are real and taking a convinient n we will have that for f^{kn} p is a fixed point, $0<\lambda^n<1/4$, $\mu^n>4$ and the remaining eigenvalue is 1. Thus there is no loose of generality assuming that p is a fixed point for f and that the associated eigenvalues are $0<\lambda<1/4$, 1 and $\mu>4$. First of all we will prove that there is a 2-disk either in $W^s_\epsilon(p)$ or in $W^u_\epsilon(p)$. From this and topological properties of 2-disks in three dimensional spaces using the dynamical features of f we will obtain that p is topologically hyperbolic. We recall that any 2-disk locally separates $\mathbb{R}^3$, more exactly:

Theorem 2.1 *If D is a topological disk in $\mathbb{R}^3$, $x\in int(D)$ then there is $\epsilon>0$ such that for any neighborhood U of x there are two points y, $z\in U\backslash D$ such that any ϵ-arc from y to z crosses D an odd number of times. Moreover, there is a neighborhood V of x such that if x_1, x_2, x_3 are three points of $V\backslash D$, then some ϵ-arc in $\mathbb{R}^3\backslash D$ joins some two of them.*

Proof. See [Bi] Chapter IX.

■

Remark 2.2 *It is clear that we can replace $\mathbb{R}^3$ by M in theorem 2.1.*

The previous theorem shows that a topological 2-disk D locally has exactly two sides at each point of $\mathrm{int}(D)$.
Let $\epsilon>0$ be less than half the expansivity constant α and assume (as we may) that for all $q\in M$ for all $0<r\leq\alpha$, $B(q,r)\cong B^3=\{(x,y,z)\in\mathbb{R}^3\,/\,x^2+y^2+z^2<1\}$.

By the usual theory of invariant manifolds we have that there is a local strong unstable one dimensional manifold $W^{uu}_\epsilon(p)$, associated to μ, a one dimensional local strong stable manifold $W^{ss}_\epsilon(p)$, associated to λ, and local center stable, center unstable and center manifolds $W^{cs}_\epsilon(p)$, $W^{cu}_\epsilon(p)$, $W^c_\epsilon(p)$, respectively. $W^{cs}_\epsilon(p)$ is tangent to the invariant suspace $E^s\oplus E^c\subset T_pM$ generated by the eigenvectors associated to the eigenvalues λ and 1 and $W_\epsilon^{cu}(p)$ is tangent to the invariant suspace $E^u \oplus E^c \subset T_pM$ generated by the eigenvectors associated to the eigenvalues μ and 1. Thus $W_\epsilon^{cs}(p)$ and $W_\epsilon^{cu}(p)$ are transversal at p. There intersection gives $W_\epsilon^c(p)$ which is therefore a one dimensional manifold C tangent to the linear subspace spanned by the eigenvector associated to the eigenvalue 1. (See [Sh], chapter 5, principally Theorem III.7.)

In order to simplify notation we will call CS, CU, U, S to the center stable, center unstable, unstable and stable local manifolds, respectivelly. By transversality we have that $C \cap S = \{p\}$, and also $C \cap U = \{p\}$.

We have that C, U, S are C^r arcs of the same differentiability class of that of f, moreover, $U \subset CU$, $S \subset CS$. Let us remark the following property of C: there is a ball B around p such that $f(C) \cap B \subset C$. As p is a fixed point for f, there is a neighbourhood W of p in C such that if $x \in W$ then both $f(x)$ and $f^{-1}(x)$ are in C. We may assume that if a point x is in W then $\text{length}(x,p) < \epsilon \leq \frac{\alpha}{2}$, $\epsilon > 0$, here $\text{length}(x,p)$ stands for the length of the subarc of C joining p with x. Observe that $C \backslash \{p\}$ has two connected components C'^{+} and C'^{-}. We call $C^+ = C'^{+} \cup \{p\}$, $C^- = C'^{-} \cup \{p\}$ We have the following lemma.

Lemma 2.3 *For all $x \in C^+$, $x \neq p$, we have length$(f(x),p) <$ length(x,p) or length$(f(x),p) >$ length(x,p).*

Proof. As the eigenvalue associated to C is $1 > 0$ we have that if $\text{length}(f(x),p) = \text{length}(x,p)$, then we have that $f(x) = x$. As $\text{length}(x,p) < \alpha$ then $\text{dist}(x,p) < \alpha$. But there cannot be two fixed points p, x at a distance less than α, an expansivity constant for f. Thus $x = p$. On the other hand, if $\text{length}(f(x),p) < \text{length}(x,p)$ and there is another point $y \in C^+$ such that $\text{length}(f(y),p) > \text{length}(y,p)$, then we will have a partition of C'^{+} into two non void open subsets, namely $A = \{x \in C'^{+} / \text{length}(f(x),p) < \text{length}(x,p)\}$ and $B = \{x \in C'^{+} / \text{length}(f(x),p) > \text{length}(x,p)\}$. This contradicts connectedness of C'^{+}. Therefore if $A \neq \emptyset$ then $B = \emptyset$ and viceversa.

■

Remark: The same is true for C^-.

Corollary 2.4 *Assume, using 2.3, that for all $x \in C'^{+}$ it holds that length$(f(x),p) <$ length(x,p). Then $C^+ \subset W_\epsilon^s(p)$.*

Proof. If $x \in C^+$ then $\text{length}(f^n(x), p) < \text{length}(f^{n-1}(x), p)$ for all $n \geq 1$. As $C^+ \cong I\!R^3$ we have that that there is a limit $y = \lim_{n\to\infty} f^n(x)$. But $f(y) = f(\lim_{n\to\infty} f^n(x)) = \lim_{n\to\infty} f^{n+1}(x) = y$. Therefore, by 2.3, $y = p$.

∎

Corollary 2.5 *If $C^+ \subset X^s_\epsilon$, for all $x, y \in C^+$ there is $N > 0$ such that $dist(f^{-N}(x), f^{-N}(y)) > \alpha$*

Proof. It follows from the expansive properties of f.

∎

¿From now on we assume that $C^+ \subset X^s_\epsilon(p)$. By changing the inner product in T_pM, if it were necessary, we may assume that v_λ, v_1, v_μ, the eigenvectors corresponding to λ, 1, μ form an orthonormal basis which we identify with the canonical basis of $I\!R^3$, $\vec{i}$, $\vec{j}$, $\vec{k}$ via a coordinate map. Hence we assume that we are working in $I\!R^3$. With these identifications we have $p = \vec{0} = (0,0,0)$ the origin of $I\!R^3$.

Theorem 2.6 *There is a ball B around 0 and a C^r diffeomorphism h of the same class of differentiability of that of CS and U, which sends the connected component of $CS \cap B$ containing $\vec{0}$ onto a disk around 0 in the $x0y$ plane, S, C, U onto segments in the $0x$, $0y$, $0z$ respectively. We may assume that 0 is the center of these segments, and that C^+ is mapped into $0y^+$.*

Proof. We will construct h by steps. In the first we use a diffeomorphism h_1 which sends S onto $0x$, then, by another diffeomorphism h_2 we send $h_1(C)$ onto $0y$ without afecting $0x$, then h_3 sends $h_2 \circ h_1(U)$ onto $0z$ without afecting $0x$ and $0y$. Finally we push $h_3 \circ h_2 \circ h_1(CS)$ onto $x0y$ by a diffeomorphism h_4 without affecting $0z$, $0x$, $0y$. Let us parameterize S by a curve $\gamma : [-1,1] \to I\!R^3$, $\gamma(0) = \vec{0}$; $\gamma'(0) = \vec{i}$. Let us find $\delta > 0$ such that for all $s \in [-\delta, \delta]$ we have $(\gamma'(s), \vec{i}) > 0$. Hence we may write $\gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s))$ in the form $(x, \gamma_2 \circ \gamma_1^{-1}(x), \gamma_3 \circ \gamma_1^{-1}(x))$. We find a C^∞ bump function $b : [0, +\infty) \to [0, +\infty)$ such that $b(r) = 1$, for all $r \in [0, \delta]$, $b(r) = 0$, $\forall r \geq 1$, and $\gamma'(r) \leq 0\, \forall r \in [0, +\infty)$. Let $B(y,z) = b(\sqrt{y^2 + z^2})$ We define

$h_1(x,y,z) = (x, y - B(y,z)\gamma_2 \circ \gamma_1^{-1}(x), z - B(y,z)\gamma_3 \circ \gamma_1^{-1}(x))$. Then the Jacobian matrix of h_1 is

$$J_{h_1} =$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -B(y,z)\frac{\gamma_2'(\gamma_1^{-1}(x))}{\gamma_1'(\gamma_1^{-1}(x))} & 1 - B_y(y,z)\gamma_2 \circ \gamma_1^{-1}(x) & -B_z(y,z)\gamma_2 \circ \gamma_1^{-1}(x) \\ -B(y,z)\frac{\gamma_3'(\gamma_1^{-1}(x))}{\gamma_1'(\gamma_1^{-1}(x))} & -B_y(y,z)\gamma_3 \circ \gamma_1^{-1}(x) & 1 - B_z(y,z)\gamma_3 \circ \gamma_1^{-1}(x) \end{pmatrix}$$

The Jacobian determinant is $det(J_{h_1}) =$

$$= (1 - B_y(y,z)\gamma_2 \circ \gamma_1^{-1}(x))(1 - B_z(y,z)\gamma_3 \circ \gamma_1^{-1}(x)) -$$

$$- B_y(y,z)\gamma_3 \circ \gamma_1^{-1}(x) B_z(y,z)\gamma_2 \circ \gamma_1^{-1}(x)$$

We have that $B_y(y,z) = b'(\sqrt{y^2+z^2})\frac{y}{\sqrt{y^2+z^2}}$ which is clearly bounded. The same is true for $B_z(y,z)$. We have also that $\gamma_2 \circ \gamma_1^{-1}(x) =$

$$= \gamma_2(0) + \gamma_2'(0)\gamma_1^{-1}(x) + \gamma_2''(0)\frac{(\gamma_1^{-1}(x))^2}{2!} + o((\gamma_1^{-1}(x))^2) = O((\gamma_1^{-1}(x))^2)$$

But $x \sim s = \gamma_1^{-1}(x)$ for $s \to 0$. So $\gamma_2 \circ \gamma_1^{-1}(x) = O(x^2)$, and the same holds for $\gamma_3 \circ \gamma_1^{-1}(x)$. Hence there is $\delta > 0$ such that for all $x \in [0, \delta]$, $det(J_{h_1}) > 0$. Moreover, as $(x,y,z) \to (0,0,0)$ $J_{h_1}(x,y,z) \to Id$. This follows from the previous calculations and the fact that $B(y,z)$ is bounded and $\frac{\gamma_j'(\gamma_1^{-1}(x))}{\gamma_1'(\gamma_1^{-1}(x))} = O(x)$, $j = 2, 3$.
Moreover, assuming that γ is C^2, every component of J_{h_1} is C^1. . This is because $\gamma_j'(\gamma_1^{-1}(x))$, $j = 1,2,3$ are C^1 and $\gamma_1'(\gamma_1^{-1}(x)) \neq 0$, and the higher derivatives like $B_{yy}(y,z) = b''(\sqrt{y^2+z^2})\frac{y^2}{y^2+z^2} + b'(\sqrt{y^2+z^2})\frac{z^2}{(y^2+z^2)^{3/2}}$ has no problem due to the fact that terms like $\frac{z^2}{(y^2+z^2)^{3/2}} \to \infty$ when $(y,z) \to (0,0)$ for $b'(r) \equiv 0$, $\forall r \in [0, \delta]$.
We have also that $b''(r)$, $b'''(r) \ldots \equiv 0 \forall r \in [0, \delta]$, so these remarks can be extended to the higher derivatives, proving that h_1 is C^r whenever γ is.
Observe that $h_1(x, \gamma_2 \circ \gamma_1^{-1}(x), \gamma_3 \circ \gamma_1^{-1}(x)) = (x, 0, 0)$ and that every point of $y0z$ is fixed by h_1, in a suitable neighbourhood of $(0,0,0)$. Moreover, as $J_{h_1}(0,0,0) = Id$, the tangent vectors to $h_1(C)$ and $h_1(U)$ at $(0,0,0)$ are $\vec{j}$ and $\vec{k}$ respectively, i.e.: they rest the same as those of C and U. This allows us to repeat the previous construction finding h_2 and h_3 such that $h_3 \circ h_2 \circ h_1$ is a diffeomorphism sending S, C, U to the

$0x$, $0y$, $0z$ respectively.
Finally we take care of CS. The normal vector to $h_3 \circ h_2 \circ h_1(CS)$ is $\vec{k} = (0,0,1)$ therefore it may be writen, locally, as the graph of a C^r function $z = \phi(x,y)$ with $\phi : D \subset \mathbb{R}^2 \to \mathbb{R}$, $\phi(x,0) \equiv 0$, $\phi(0,y) \equiv 0$. Here D is a small disk around $(0,0,0)$. Again we define a C^∞ bump function b with similar properties as the previous one we have considered, and a diffeomorphism $(x,y,z) \mapsto (x,y,z-\phi(x,y)b(r))$. Here $r = \sqrt{x^2+y^2+z^2}$. Clearly this last diffeomorphism composed with the previous ones gives a diffeomorphism h which has the desired properties.

■

Locally, in a neighbourhood of $(0,0,0)$, we define a C^r diffeomomorphism $\hat{f} = h \circ f \circ h^{-1}$. As the jacobian matrix of h is the identity at the origin, $J_h(0,0,0) = Id$, we have that this map is well defined. Moreover, its tangent map at $\vec{0}$, $T\hat{f}_{\vec{0}}$ is the same as $Tf_{\vec{0}}$. Also $\hat{f}(x,0,0) = (\lambda x + r(x,0,0),0,0)$, $\hat{f}(0,y,0) = (0, y+s(0,y,0),0)$, $\hat{f}(0,0,z) = (0,0,\mu z + t(0,0,z))$ Finally we may write $\hat{f}(x,y,0) = (\lambda x + r(x,y,0), y + s(x,y,0),0)$. This express the local invariance of the $x0y$ plane. Moreover, $r(0,0,0) = s(0,0,0) = t(0,0,0) = 0$ and the same is true for the first partial derivatives at $(0,0,0)$. For instance, $r_x(0,0,0) = r_y(0,0,0) = r_z(0,0,0) = 0$. This allows us to write (using Taylor expansions for instance) $r(x,y,z)$ as a 'quadratic function':

$$r(x,y,z) = r_{11}(x,y,z)x^2 + 2r_{12}(x,y,z)xy + r_{22}y^2 + 2r_{13}xz + 2r_{23}yz + r_{33}z^2$$

Here $r_{ij}(x,y,z)$ are C^{r-2} functions defined in a neighbourhood of $\vec{0}$. The same is true for $s(x,y,z)$ and $t(x,y,z)$. As $C^+ \subset 0y^+$ and $\hat{f}$ restricted to C^+ is contractive we have that for $\hat{f}(0,y,0) = (0, y + s_{22}(0,y,0)y^2, 0)$ it holds that $0 < y + s_{22}(0,y,0)y^2 < y$. Hence $s_{22}(0,y,0) < 0$, $\forall y \in C'^+$

We want to show that there is a neighbourhood N of $\vec{0}$ such that $H^+ = N \cap \{(x,y,z)/x \in \mathbb{R},\ y \geq 0,\ z = 0\}$ is contained in $W^s_\epsilon(\vec{0})$. This will prove the existence of a disk in the local stable manifold of $\vec{0}$. The problem is that $\vec{0}$ is in the boundary of the disk, namely in the $0x$ axis. If we have that f contracts C^- too, then we may complete the picture glueing H^+ with the corresponding H^-, $H^- \subset \{(x,y,z)/x \in \mathbb{R},\ y \leq 0,\ z = 0\}$. But it may happen that f expands C^-. In this case we can construct a disk $G^- \subset \{(x,y,z)/x=0,\ y\leq 0,\ z\in I\!R\}$, contained in the local unstable set $W^u_\epsilon(\vec{0})$ with $\vec{0}$ in its boundary. We will show that this cannot occur; otherwise we arrive to a contradiction. Hence if the dynamics contracts C^+ it will contract C^- too, proving that there is a 2-disk in the local stable set. From this and the existence of an arc in the local unstable manifold we will prove that p is topologically hyperbolic.

Lemma 2.7 *There is a neighbourhood N_1 of $\vec{0}$ in $x0y$ such that $\hat{f}$ maps $H_1^+ = N_1 \cap \{(x,y,z)/x\in I\!R,\ y\geq 0,\ z=0\}$ into itself.*

Proof. Using 2.6 we know that there is a neighbourhood N_1 of $\vec{0}$ such that in it $\hat{f}(x,y,0) = (\lambda x + r(x,y,0), y + s(x,y,0), 0)$ where r, s are infinitesimals of order at least 2 with respect to the distance to $\vec{0}$. From this we have that $\hat{f}$ strongly contracts in the $0x$ direction and is nearby the identity in the $0y$ direction. Thus there is a neighbourhood N_2 of $\vec{0}$ such that $N_2 \cup \hat{f}(N_2) \cup \hat{f}^{-1}(N_2) \subset N_1$. Assume that there is a point $(x,y,0)$ in $N_2 \cap \{(x,y,z)/x\in I\!R,\ y\geq 0,\ z=0\}$ such that its image by $\hat{f}$ is not in $\{(x,y,z)/x\in I\!R,\ y\geq 0,\ z=0\}$. We may join $(x,y,0)$ with $(0,y,0)$ with an arc γ without intersections with $0x$. $\hat{f}(0,y,0) = (0, y+s(0,y,0), 0)$ with $y + s(0,y,0) > 0$. As $\hat{f}(x,y,0)$ belongs to $\{(x,y,z)/x\in I\!R,\ y\leq 0,\ z=0\}$, we have that there is a point of intersection of $\hat{f}(\gamma)$ with $0x$. This implies, by our choice of N_2, that γ also cuts $0x$, which contradicts our choice of N_2. To finish the proof we choose a $\delta > 0$ and define $N_1 = N_2 \times (-\delta, \delta)$

■

Lemma 2.8 *There is a C^{r-2} function $\hat{r}_{22}$ defined in $H_1 = N_1 \cap \{(x,y,z)/x\in I\!R,\ y\in I\!R,\ z=0\}$ such that in H_1, $r_{22} = x\hat{r}_{22}$*

Proof. As $0y$ is locally invariant in a neighbourhood of $\vec{0}$ we have that $r_{22}(0,y,0)=0$ for all y in a neighbourhood of 0.
Define $\hat{r}_{22}(x,y,z) = \frac{r_{22}(x,y,z)}{x}$. The previous remark implies that $\hat{r}_{22}$ is a well defined C^{r-2} function in H_1.

■

Remark 2.9 *Similarly we may write* $s_{11} = y\hat{s}_{11}$*, using that* $s_{11}(x,0,0) = 0$.

Let $(x_1, y_1, z_1) = \hat{f}(x,y,z)$ and, in general, $(x_n, y_n, z_n) = \hat{f}^n(x,y,z)$, $n \in \mathbb{Z}$, provided $\hat{f}^n(x,y,z)$ is defined.

Lemma 2.10 *There is* $\delta > 0$ *such that in* H_1^+ *and for* $-\delta \le x \le \delta$, $0 \le y \le \delta$ *we have that* $\mid \frac{x_1}{y_1} \mid \le \mid \frac{1}{2}\frac{x}{y} \mid$ *and* $\mid x_1 \mid \le \mid \frac{1}{2}x \mid$.

Proof.

$$\left|\frac{x_1}{y_1}\right| = \left|\frac{\lambda x + r(x,y,0)}{y + s(x,y,0)}\right| \le$$

$$\left|\frac{\lambda x}{y + s(x,y,0)}\right| + \left|\frac{r(x,y,0)}{y + s(x,y,0)}\right| \le$$

$$\left|\frac{\frac{x}{4y}}{1 - |\hat{s}_{11}x^2 y + 2s_{12}xy + s_{22}y^2|/|y|}\right| +$$

$$+ \left|\frac{r_{11}x^2 + 2r_{12}xy + \hat{r}_{22}xy^2}{y(1 - |\hat{s}_{11}x^2 + 2s_{12}x + s_{22}y|)}\right| =$$

$$= \left|\frac{x}{4y}\right|\left|\frac{1}{1 - |\hat{s}_{11}x^2 + 2s_{12}x + s_{22}y|}\right| +$$

$$+ \left|\frac{x}{4y}\right|\left|\frac{4(r_{11}x + 2r_{12}y + \hat{r}_{22}y^2)}{1 - |\hat{s}_{11}x^2 + 2s_{12}x + s_{22}y|}\right|$$

Taking into account that the functions s_{ij}, r_{ij}, $\hat{s}_{11}$ and $\hat{r}_{22}$ are bounded, it is clear from the above inequalities that $\mid \frac{x_1}{y_1} \mid \le \mid \frac{1}{2}\frac{x}{y} \mid$ for a a suitable choice of δ.
The proof of the second inequality follows from similar computations. $\mid x_1 \mid = \mid \lambda x + r(x,y,0) \mid = \mid \lambda x + r_{11}x^2 + 2r_{12}xy + \hat{r}_{22}xy^2 \mid = \mid x \mid\mid \lambda + r_{11}x + 2r_{12}y + \hat{r}_{22}y^2 \mid$. As $\lambda < \frac{1}{4}$ and all functions involved are bounded and are multiplied either by x or y, it is clear that we may choose $\delta > 0$ as desired.

■

Lemma 2.11 *There is* $\delta' > 0$ *such that the 2-disk given by* $|x| \le \delta'$, $0 \le y \le \delta$ *is contained in* $W^s_\epsilon(\vec{0})$ *(here* δ *is as in 2.10).*

Proof. We may choose $0 < \delta < \frac{1}{\sqrt{2}}\epsilon$ in 2.10. Let us find a $\delta \geq \delta'' > 0$ such that the segment $y = \delta$, $-\delta'' \leq x \leq \delta''$ is pushed down to the $x0y$ axis by $\hat{f}$. This is possible because $s_{22}(0, y, 0) < 0$ in $0 < y \leq \delta$ (see 2.6) so $y_1 = y + s(x, \delta, 0) < \delta$ for a suitable $\delta'' > 0$ if $| x | \leq \delta''$. Moreover, by 2.10, we have that $| x_1 | \leq \frac{1}{2} | x |$, and it is easy to see that the sign of x_1 is the same sign of x so the segment $y = \delta$, $-\delta'' \leq x \leq \delta''$ is mapped onto an arc which is topologically transversal to $0y$. Let δ' be the minimum between $| x_1(-\delta'', \delta, 0) |$ and $x_1(\delta'', \delta, 0)$. It is easy to see that the image β by $\hat{f}$ of $y = \delta$, $-\delta'' \leq x \leq \delta''$ separates $[-\delta', \delta'] \times [0, \delta]$ in the $x0y$ plane.
Consider a point $(x, y, 0) \in [-\delta', \delta'] \times [0, \delta] \times \{0\}$. We will prove that dist$(\hat{f}^n(x, y, 0), (0, 0, 0) leq\epsilon$ for all $n \geq 0$ therefore $[-\delta', \delta'] \times [0, \delta] \times \{0\}$ is contained in $W^s_\epsilon(\vec{0})$. By our choice of δ and δ' it is enough to prove that $[-\delta', \delta'] \times [0, \delta] \times \{0\}$ is mapped into itself by $\hat{f}$. This will guarantee at the same time that $\hat{f}^n(x, y, 0)$ makes sense and that the distance between $\hat{f}^n(x, y, 0)$ and $(0, 0, 0)$ is less than $\sqrt{\delta^2 + \delta'^2} \leq \sqrt{2}\delta \leq \epsilon$. So let us prove that $\hat{f}([-\delta', \delta'] \times [0, \delta] \times \{0\}) \subset [-\delta', \delta'] \times [0, \delta] \times \{0\}$. As $x0y$ is locally invariant it suffices to show that $| x_1 | \leq \delta'$ and $0 \leq y_1 \leq \delta$. $| x_1 | \leq \delta'$ follows from 2.10 and that $y_1 \geq 0$ is contained in 2.7. Thus the only thing that rest to prove is that $y_1 \leq \delta$. Assume it is not true. Then there is $(x, y, 0)$ such that for its image $(x_1, y_1, 0)$ it holds that $y_1 > \delta$. Suppose that $x > 0$ the case $x < 0$ is similar. We join $(x, y, 0)$ with the segment γ paralell to the $0x$ axis with $(0, y, 0)$. Every point $(t, y, 0)$, $0 \leq t \leq x$ of this segment is mapped into a point $(t_1, y_1, 0)$ with $0 \leq t_1 \leq \frac{1}{2}t \leq \frac{1}{2}\delta'$. Moreover $(0, y, 0)$ is mapped into a point $(0, y_1, 0)$ that is below β As β separates $[-\delta', \delta'] \times [0, \delta]$ we have that there is a point of intersection between it and $\hat{f}(\gamma)$. But this implies that their preimages have a point in common which is absurd.

■

Proposition 2.12 *There is a 2-disk D included in $W^s_\epsilon(p, f)$, where p is the periodic point associated to f, $p \in \partial D$.*

Proof. Follows from the previous lemmas which proves the existence of such a disk for $\vec{0}$ and $\hat{f}$. In fact we may choose $D = h^{-1}([-\delta', \delta'] \times [0, \delta] \times \{0\})$ where h is the diffeomorphism of 2.6 and δ' and δ are as in 2.10 and 2.11. There is $k > 0$ such that $\hat{f}$ and f^k are locally conjugated by the diffeomorphism h (see 2.6). As the tangent map

Th is nearby the identity in a neighbourhood of p, if we choose δ small enough in 2.10 we have that $\text{dist}(f^n(q), f^n(p)) \leq \epsilon$ for any point $q \in D = h([-\delta', \delta'] \times [0, \delta] \times \{0\})$ as desired.

■

As we have said above if C^- also behaves as a part of the local stable set of p then we are done glueing together the disk D with its analogous D' such that $h^{-1}(D')$ would be in the half of $x0y$ where $y \leq 0$. In this case p would be in the interior of the disk $D \cup D' \subset W^s_\epsilon(p)$.

On the other hand, it may happen that C^- behaves as a part of the local unstable set of p. The rest of this paragraph is devoted to prove that this latter possibility cannot occur. Otherwise we arrive to a contradiction. So we will assume that there is a 2-disk contained in the local unstable set of p, constructed in a similar way as we have constructed D, let us call it R.

First we assume that for all $0 < \sigma$ there is an open neighbourhood $A_\sigma \subset B(p, \sigma)$ such that a dense subset P_σ of A_σ has the property that every point x of P_σ has a non trivial sub-continuum $C(x) \subset W^u_\epsilon(x)$ such that $C(x) \cap \text{int}(D) \neq \emptyset$, $x \in C(x)$. We choose σ_0 such that $B(p, \sigma_0) \subset \Omega(f)$. This will allow us to find a point x in $B(p, \sigma_0)$ such that $W^s_\epsilon(x)$ will contain a 2-disk $D(x)$ such that will intersect R in more than one point hence violating expansivity.

Second we suppose that the previous assumption is false. This in turn will imply that every point z of D has nontrivial sub-continua $C^+(z)$ and $C^-(z)$ both contained in $W^u_\epsilon(z)$ and both reaching the boundary of the ball $B(z, r_0)$, $r_0 > 0$ fixed, with $C^+(z)$ and $C^-(z)$ locally separated by $\text{int}(D)$ if $z \in \text{int}(D)$. Moreover, every point z of $f^n(D)$ will have such subcontinua $C^+(z)$, $C^-(z)$ with analogous properties. But this will imply our first assumption (the existence of σ_0), thus reaching again a contradiction. This last step is accomplished by the device of taking limits in the Hausdorff metric of a sequence of 2-disks contained in the local stable set of a corresponding sequence of points in M. The simplest way we see to obtain such 2-disks is considering a point $z \in \text{int}(D)$ and taking limits of $f^{-n}(z)$ (this will give us the sequence of points). Considering small disks $D_n \subset \text{int}(D)$ around z, the named sequence of 2 disks will be given by $S_n = f^{-n}(D_n)$.

This 2-disks will accumulate. A point w between two of the accumulating disks, say S_n, S_m, will have the property that its local unstable set $W^u_\epsilon(w)$ will contain a subcontinuum $C(w)$ such that it intersects both S_n, S_m. This in turn will imply that σ_0 exists.

We need the following lemmas.

Lemma 2.13 *There is $r > 0$ such that for all $(x, y, z) \in B(\vec{0}, r)$ we have $|z_1| > 2|z|$, $z \neq 0$, where $(x_1, y_1, z_1) = \hat{f}(x, y, z)$.*

Proof. As in 2.11 we may prove that there is $r > 0$ such that if $z > 0$ (resp. $z < 0$) then $z_1 > 0$ (resp. $z_1 < 0$). This follows from the local invariance of the $0xy$ plane. Assume that $z > 0$. Let $z_1 = \mu z + t_{11}(x, y, z)x^2 + 2t_{12}(x, y, z)xy + t_{22}(x, y, z)y^2 + z(2t_{13}(x, y, z)x + 2t_{23}(x, y, z)y + t_{33}(x, y, z)z)$. Using the local invariance of $0xy$ we have that $t_{11}(x, y, 0)x^2 + 2t_{12}(x, y, 0)xy + t_{22}(x, y, 0)y^2 \equiv 0$. Therefore, using differentiability, we have

$$t_{11}(x, y, z)x^2 + 2t_{12}(x, y, z)xy + t_{22}(x, y, z)y^2 =$$

$$= t_{11}(x, y, 0)x^2 + 2t_{12}(x, y, 0)xy + t_{22}(x, y, 0)y^2 +$$

$$+(t'_{11z}(x, y, 0)x^2 + 2t'_{12z}(x, y, 0)xy + t'_{22z}(x, y, 0)y^2)z + \epsilon(x, y, z)z$$

with $\epsilon(x, y, z) \to 0$ when $z \to 0$. Thus $t_{11}(x, y, z)x^2 + 2t_{12}(x, y, z)xy + t_{22}(x, y, z)y^2 = (t'_{11z}(x, y, 0)x^2 + 2t'_{12z}(x, y, 0)xy + t'_{22z}(x, y, 0)y^2)z + \epsilon(x, y, z)z$ and using that t_{ij} and t'_{ij} are bounded functions we may find $r > 0$ such that in $B(\vec{0}, r)$ $|t_{11}(x, y, z)x^2 + 2t_{12}(x, y, z)xy + t_{22}(x, y, z)y^2 + z(2t_{13}(x, y, z)x + 2t_{23}(x, y, z)y + t_{33}(x, y, z)z)| \leq 2|z|$ Therefore, as $\mu > 4$, we have $|z_1| > 2|z|$. Analogously, if $z < 0$ then $z_1 < 2z$.

∎

Lemma 2.14 *Let $r > 0$ be as in 2.13 and $\epsilon < r$. Then $W^s_\epsilon(\vec{0})$ is contained in the $0xy$ plane.*

Proof. If $(x, y, z) \in B(\vec{0}, r)$, $z > 0$ then $z_1 > 2z$. If for $h = 1, 2, \ldots, n-1$ we have $(x_h, y_h, z_h) \in B(\vec{0}, r)$ then $z_n > 2z_{n-1} > 2^n z$, where $(x_h, y_h, z_h) = \hat{f}(x_{h-1}, y_{h-1}, z_{h-1})$, $(x_0, y_0, z_0) = (x, y, z)$. Hence there is $n \in I\!N$ such that $2^n z > r$ therefore $distn, y_n, z_n), (0, 0, 0)) > r$. This proves the lemma in the case $z > 0$. The case $z < 0$ is analogous.

■

Corollary 2.15 *There is $\epsilon > 0$ such that $W^s_\epsilon(p, f) \subset CS$.*

Proof. Follows from the fact that in a neighbourhood of p, $\hat{f} = h^{-1} \circ f \circ h$ and 2.14.

■

Assumption A. Assume now that for all $0 < \sigma$ there is an open neighbourhood $A_\sigma \subset B(p, \sigma)$ such that a dense subset P_σ of A_σ has the property that every point x of P_σ has a non trivial sub-continuu $C(x) \subset W^u_\epsilon(x)$ such that $C(x) \cap \text{int}(D) \neq \emptyset$, $x \in C(x)$. By our hypotheses in this article we may also assume that $B(p, \sigma) \subset \Omega(f)$. Observe that we may suppose that the diffeomorphism h of 2.6 sends CU onto a disk around $\vec{0}$ contained in the $0yz$ plane. That is, working with $\hat{f}$ instead of f, we will have $h(R) \subset CU \subset 0yz$. Anyway, in order to simplify notation we will continue writing R and D instead of $h(R)$ and $h(D)$ respectively, we also will identify $\hat{f}$ with f and $\vec{0}$ with p.
Assume that $q \in \text{int}(D) \subset W^s_\epsilon(p)$.

Lemma 2.16 *Given $\epsilon > 0$ there are $r_0 \geq r_1 > 0$, $\alpha > \epsilon \geq r_0$, with α a constant of expansivity of f such that*

- *For every $x \in M$ there are compact connected sets $D(x) \subset W^s_\epsilon(x)$, $C(x) \subset W^u_\epsilon(x)$, $x \in C(x) \cap D(x)$ and for all open set A such that $x \in A \subset B(x, r_0)$ it holds that $D(x) \cap \partial A \neq \emptyset$ and $C(x) \cap \partial A \neq \emptyset$*
- *There is $N \in I\!N$, depending on $q \in int(D)$, such that for $n \geq N$ the connected component S_n of $f^{-n}(int(D)) \cap B(f^{-n}(q), r_1)$ containing $f^{-n}(q)$ separates $B(f^{-n}(q), r_1)$ in two connected components and $S_n \subset W^s_\epsilon(f^{-n}(q))$*

Proof. The first part of the proposition is proved in [Le-To], Proposition 1.1. Moreover, this proposition is proved in a more general setting. Nevertheless, in order to do this paper more readable we give here a proof which adapts the above mentioned to the case of expansive homeomorphisms which simplifies the techniques. It suffices to prove that there is $D(x) \subset W^s_\epsilon(x)$ such that $D(x)$ is a continuum joining x with $\partial B(x, r_0)$. Take a point $x \in M$ and assume first that there is a positive $\rho \leq \epsilon$

such that for any $k \geq k_0$, with $k_0 \in I\!N$ such that $\frac{1}{k_0} \leq \epsilon$, there is $n_k > 0$ such that $f^{-\nu}(B(f^{n_k}(x), \frac{1}{k}))$ is not included in $B(f^{n_k-\nu}(x), \rho)$ for some ν, $0 \leq \nu \leq n_k$. Choose a point $y \in B(f^{n_k}(x), \frac{1}{k})$ such that $f^{-\nu}(y) \notin B(f^{n_k-\nu}(x), \rho)$ and an arc $\gamma_k(t)$, $t \in [0,1]$ such that $\gamma_k(0) = f^{n_k}(x)$, $\gamma_k(1) = y$ and $\gamma_k(t) \in B(f^{n_k}(x), \frac{1}{k})$ for all $t \in [0,1]$. Let t_k^* be the supremum of those $t \in [0,1]$ such that $f^{-\nu}(\gamma_k([0,t]) \subset B(f^{n_k-\nu}(x), \rho)$ for all $\nu = 0, 1, \ldots, n_k$. Therefore $f^{-\nu}(\gamma_k([0,t_k^*]) \subset \text{clos}(B(f^{n_k-\nu}(x), \rho))$ for all $\nu = 0, 1, \ldots, n_k$ and there is ν_k such that $f^{-\nu_k}(\gamma_k([0,t_k^*]) \cap \partial B(f^{n_k-\nu_k}(x), \rho) \neq \emptyset$. For each k we choose such a ν_k. The sequence ν_k cannot repeat a number an infinity of times. Otherwise we will have a sequence k_l diverging to $+\infty$ such that for a fixed $N = \nu_{k_l}$, $f^N(f^{-N}((\gamma_{k_l}([0,t_{k_l}^*])))) \subset B(f^{n_{k_l}}(x), \frac{1}{k_l})$. But $\text{diam}(f^{-N}((\gamma_{k_l}([0,t_{k_l}^*]))))$ is at least equal to ρ and therefore f^N cannot map it into arbitrarily small balls. Thus ν_k diverges to $+\infty$. On the other hand $n_k - \nu_k$ is a bounded sequence. For if it were not, we will have a subsequence $n_{k_l} - \nu_{k_l}$ diverging to $+\infty$. Assume without generality that $\lim_{k \to +\infty}(n_k - \nu_k) = +\infty$ in order to simplify notation. We may choose converging subsequences from $\{f^{n_k-\nu_k}(x)\}$ and $\{f^{-\nu_k}(\gamma_k(t_k^*)\}$ to points z and w respectively. Once again suppose that the sequences converge to simplify notation. We have that $\text{dist}(z, w) = \rho$ and that for any $n \in Z\!\!Z$ $\text{dist}(f^n(z), f^n(w)) \leq \rho$ which contradicts expansivity. For given $n \in Z\!\!Z$ we have that $f^n(f^{n_k-\nu_k}(x)) = f^{n+n_k-\nu_k}(x)$ converges to $f^n(x)$ and $f^{n-\nu_k}(\gamma_k(t_k^*))$ converges to $f^n(w)$. By our assumption that $n_k - \nu_k$ diverges and as ν_k diverges too, we may choose k such that $n + n_k - \nu_k$ is between 0 and n_k. But then $\text{dist}(f^{n+n_k-\nu_k}(x), f^{n-\nu_k}(\gamma_k(t_k^*))) \leq \rho$ which implies taking limits with $k \to +\infty$ that $\text{dist}(f^n(z), f^n(w)) \leq \rho$. As $n_k - \nu_k$ is bounded, the diameters of the arcs $f^{-n_k}(\gamma_k([0,t_k^*])$ are bounded away from zero, say, they all have diameter equal or greater than r_0 and moreover they are all contained in $B(x, \rho)$. Taking the Hausdorff limit of a converging subsequence of those arcs we obtain a continuum $D(x) \subset W_\epsilon^s(x)$ such that it intersects the boundary of $B(x, r_0)$ for a certain $r_0 > 0$. Up to this moment this number depends on x. Suppose now that the assumption made at the beginning does not hold. That is, for any ρ_m converging to zero with $m \to +\infty$ there is $k_m > 0$ such that for every $n \geq 0$, for all ν between 0 and n, $f^{-\nu}(B(f^n(x), \frac{1}{k_m})) \subset B(f^{n-\nu}(x), \rho_m)$. Therefore for any point y in the omega limit set of x, $\omega(x)$, we have that $f^{-\nu}(B(y, \frac{1}{k_m})) \subset B(f^{-\nu}(y), \rho_m)$. As $\rho_m \to 0$ we have that y is a Lyapunov stable point for f^{-1}. But this contradicts the fact that there are no Lyapunov stable points for an expansive homeomorphism (see [Le2], lemma 2.7). Compactness of M enables us to prove that we may choose $r_0 > 0$ independent of $x \in M$.

We give the idea of the proof of the second part of the lemma and refer the reader to [Vi3], lemmas 4.1 and 5.1 for details. As $\text{int}(D) \subset W^s_\epsilon(p)$, taking a point q in $\text{int}(D)$ and joining it by arcs $\gamma \subset D$ with ∂D we have that these arcs cannot increase their diameters for $n \to +\infty$. Hence, by expansivity, $\text{diam}(f^n(\gamma) \geq \alpha$ with $n < 0$. Moreover, as ∂D is compact there is a fixed $N < 0$ such that all the arcs joining q with ∂D have achieved a diameter at least α for a n such that $-1 \geq n \geq N$. A property of expansive homeomorphisms, (see [Vi1] section 2) ensures that once $\text{diam}(f^n(\gamma)) \geq \alpha$ then the same is true for any $n' < n$. Therefore, we have that for all arcs $\gamma \subset D$ joining q with ∂D we have that the diameters of $f^N(\gamma)$ is greater than α, and the same is true for every $n \leq N$. We may prove that this implies that $f^n(D)$ separates $B(f^n(q), \alpha)$. Moreover, the connected component of $f^n(D) \cap B(f^n(q), \alpha)$ containing $f^n(q)$ has the same property and separates in two components. Nevertheless we cannot expect it to be included in $W^s_\epsilon(f^n(q))$ The same property of expansive homeomorphisms mantioned above (see [Vi1] section 2) enables us to find r_1 such that for the smaller neighbourhood of $f^n(q)$, $B(f^n(q), r_1)$, the connected component of $f^n(D) \cap B(f^n(q), r_1)$ containing $f^n(q)$ still separates $B(f^n(q), r_1)$ and moreover is contained in $W^s_\epsilon(f^n(q))$ for a suitable choice of $r_1 > 0$ which may be done the same for all $q \in \text{int}(D)$. See [Vi3] for details.

■

Remark 2.17 *Let $y \in P_\sigma$. Then there is a neighbourhood $V(y)$ of y such that for all $x \in V(y)$ we have a continuum $C(x) \subset W^u_\epsilon(x)$ such that $C(x) \cap int(D) \neq \emptyset$.*

Lemma 2.18 *Given $\lambda > 0$ there is $\sigma > 0$ such that if dist) $< \sigma$, $x \in P_\sigma$ and $x \notin 0xy$ then there is a point $y \in V(x)$, $V(x)$ as in 2.17, such that $W^s_\epsilon(y) \supset D(y)$ a surface such that $D(y)$ separates $B(y, r_1)$ and the distance between any point of $D(y)$ and $0xy$ is less than λ.*

Proof. For a fixed $x \in P_\sigma$ consider a sequence $\{x_k\}$ converging to x when $k \to +\infty$ and such that there is a sequence $\{n_k\} \subset I\!N$ with $n_k \to +\infty$ when $k \to +\infty$ with the property that $f^{-n_k}(x_k) \to x$ too. As we suppose that there is a neighbourhood of p included in $\Omega(f)$ such sequences always exist if σ is small enough. For every x_k we have $C(x_k) \cap \mathrm{int}(D) \neq \emptyset$. Moreover expansivity implies that the intersection is a single point w_k and that $w_k \to w = C(x) \cap \mathrm{int}(D)$ when $k \to +\infty$. Using 2.16, we may find a sequence of surfaces $S_k \subset W^s_\epsilon(f^{-n_k}(w_k))$, $f^{-n_k}(w_k) \in S_k$ and S_k separates $B(f^{-n_k}(w_k), r_1)$ in two connected components. For, compactness of $\{w\} \cup \{w_k\}_{k \in I\!N}$ allows us to find a $K > 0$ such that the separation property of S_k holds independently of the point w_k if $k \geq K$ (see [Vi3] lemma 4.1 for more details). As $w_k \in W^u_\epsilon(x_k))$ by [Ma] lemma I, p. 315, we have that $f^{-n_k}(w_k) \to x$ as $k \to +\infty$. Thus if $dist) < \sigma$ the same is true for $f^{-n_k}(w_k)$ if we take k great enough. Choose k such that $dist(w_k), p) < \sigma$ and call u_k the corresponding point $f^{-n_k}(w_k)$ in order to simplify notation. Let us prove that we may find σ such that if $dist) < \sigma$ then for any point z of the chosen S_k we have that $distxy) < \lambda$. Otherwise we may construct a sequence of points u_k converging to p such that for a certain $\lambda_0 > 0$ fixed, there is a point $z_k \in S_k$ such that $dist, 0xy) \geq \lambda_0$. Taking a convergent subsequence from z_k we find a point $z \in W^s_\epsilon(p)$ such that $dist) \geq \lambda_0$. But this contradicts 2.14. To finish the proof we only have to pick up a point u_k, and the corresponding S_k such that if $z \in S_k$ then $distxy)\lambda$, and call them y and $D(y)$ respectively.

■

Lemma 2.19 Assumption A *is not compatible with the existence of the 2-disk* $R \subset W^u_\epsilon(p)$.

Proof. If we prove that there are more than one point of intersection between the local stable and unstable sets of two points then this fact contradicts expansivity and we are done. By arguments similar to those of Theorem B of [Vi3] we are able to prove that if y is close enough to p then $D(y)$ intersects $0z$. Moreover, the distance between any point of $D(y)$ and $0xy$ is as small as we wish for a suitable choice of y. Hence $D(y)$ will intersect R. Moreover, as $R \subset 0yz$ and a part of the boundary of R is included in $0z$ there will be points of R locally separated by $D(y)$. As both, $D(y)$, R are 2-disks, their intersection will contain more than one point, thus contradicting expansivity.

■

Assume now that **Assumption A** is false, then the following holds.

Assumption B. Assume that there are $\sigma_0 > 0$ and an open dense set $A \subset B(p, \sigma_0)$ such that if $y \in A$ then for all subcontinuum $C(y) \subset W^u_\epsilon(y)$ we have $C(y) \cap \text{int}(D) = \emptyset$. As in [Vi4] lemma 2.8 we may prove:

Lemma 2.20 *If* **Assumption B** *takes place then there is* $r_2 > 0$ *such that if* $x \in \cup_{n\in\mathbb{Z}} f^{-n}(int(D))$ *there is a continuum* $C(x) \subset W^u_\epsilon(x)$ *such that* $C(x) \setminus \{x\}$ *has at least two connected components each of which reaches* $\partial B(x, r_2)$.

Proof. It is more or less the same as that of [Vi4] lemma 2.8. The only difference is that we have to take care of the fact that the thesis holds only in $\cup_{n\in\mathbb{Z}} f^{-n}(\text{int}(D))$. Let $x \in W^s(p)$ and assume, without loss of generality, that p is fixed. Hence for all $0 < \sigma < \alpha$ $W^s(p) = \cup_{n\in\mathbb{N}} f^{-n}(W^s_\sigma(f^n(p)) = \cup_{n\in\mathbb{N}} f^{-n}(W^s_\sigma((p))$. Thus there is $N \in \mathbb{N}$ such that $x \in f^{-N}(W^s_\sigma(p))$ therefore, as $f^{-(n+1)}(W^s_\sigma(p) \supset f^{-n}(W^s_\sigma(p)$, $n \in \mathbb{Z}$, $x \in f^{-n}(W^s_\sigma(p)$ for all $n \geq N$. Let us choose $\sigma < \rho$ and let us assume first that $S(x)$ the connected component of $W^s_\epsilon(x) \cap B(x, r_1)$ containing x separates $B(x, r_1)$ in two connected components B^+_x, B^-_x. If there is no continuum $C(x) \subset W^u_\epsilon(x)$ such that $C(x) \cap \partial B(x, r_1) \cap B^-_x \neq \emptyset$ then, by 1.1 assuming that $r_1 \leq \frac{r_0}{2}$, we have that for all $y \in B(x, \lambda)$, $C(y) \cap \partial B(x, r_1) \cap B^+_x \neq \emptyset$. Take $y \in B(x, \lambda)$, $y \in B^-_x$. Hence $C(y) \cap S(x) \neq \emptyset$, and by expansivity it is a single point w. Moreover we may assume that y is such that $\text{dist}(y, w) < \lambda$, where $\lambda > 0$ has been chosen such that $w \in W^u_\epsilon(z)$ and $\text{dist}(z, w) < \lambda$ imply $w \in W^u_{\epsilon/k}(z)$. Thus $f^N(w) \in f^N(W^u_{\epsilon/k}(y)$. As N is fixed there is k_0 such that for all $k \geq k_0$ $f^N(W^u_{\epsilon/k}(y) \subset W^u_\epsilon(f^N(y))$ which contradicts assumption **B**. Thus $C(x) \cap \partial B(x, r_1) \cap B^-_x \neq \emptyset$ and also $C(x) \cap \partial B(x, r_1) \cap B^+_x \neq \emptyset$. Let now x be any point of $W^s(p)$. As in [Vi3] we may prove that there is n_1 such that the connected component $S(f^{-n_1}(x))$ of $W^s_\epsilon(f^{-n_1}(x)) \cap B(f^{-n_1}(x), r_1)$ containing x separates $B(f^{-n_1}(x), r_1)$. Hence the previous arguments apply to $f^{-n_1}(x)$ and we obtain a continuum $C(f^{-n_1}(x)) = C^+ \cup C^-$, $C^+ \subset B^+_{f^{-n_1}(x)}$, $C^- \subset B^-_{f^{-n_1}(x)}$, $C^+ \cap \partial B(f^{-n_1}(x), r_1) \neq \emptyset$, $C^+ \cap \partial B(f^{-n_1}(x), r_1) \neq \emptyset$. Let $\mathcal{U}$ be a Lyapunov function for f (see 1.4). We have that for all $y \in C(f^{-n_1}(x))$, $y \neq f^{-n_1}(x)$, $\Delta\mathcal{U}(f^{-n_1}(x), y) > 0$. Thus in the connected component $\mathcal{C}^+$ of $f^{n_1}(C^+_{f^{-n_1}(x)}) \cap clos(B(x, r_1))$ containing x the same is true. Using similar arguments as those in [Vi3], lemma 4.1, we may prove that there is $r_2 > 0$ such that $\mathcal{C}^+ \cap \partial B(x, r_2) \neq \emptyset$ and calling $\mathcal{C}^-$ to the connected component of $f^{n_1}(C^-_{f^{-n_1}(x)}) \cap clos(B(x, r_1))$ containing x, we also have that $\mathcal{C}^- \cap \partial B(x, r_2) \neq \emptyset$. As $\mathcal{C}^+$ and $\mathcal{C}^-$ are locally separated by $S(x)$, we obtain $C(x) = \mathcal{C}^+ \cup \mathcal{C}^-$ with the desired properties.

■

As in [Vi4] lemma 2.9 we may prove:

Lemma 2.21 **Assumption B** *implies* **Assumption A.**

Proof. Assume that **B** holds. Let $x \in W^s_\epsilon(p)$ and let us call $x_n = f^{-n}(x)$. As in [Vi3] and [Vi4], we may prove that there are a 2-disk S, $x \in \text{int}(S)$ and $N_0 \in I\!N$ such that for all $n \geq N_0$ S_n, the connected component of $f^{-n}(S) \cap B(x_n, r_1)$ containing x_n separates $B(x_n, r_1)$ and $S_n \subset W^s_\epsilon(x_n)$. Moreover, by **B**, for all x_n there is a continuum $C_n \subset W^u_\epsilon(x_n)$ such that it is separated by S_n in two connected components C_n^+ and C_n^- and both components reach $\partial B(x_n, r_2)$. We choose a subsequence $n_k \to +\infty$ such that $\{x_{n_k}\}$ converges to y in the metric of M and $\{S_{n_k}\}$ and $\{C_{n_k}\}$ converge to $D(y)$ and $C(y)$ respectively in the Hausdorff metric. In order to simplify notation we assume that $\{x_n\}$, $\{S_n\}$ and $\{C_n\}$ converge. By Cauchy's condition of convergence given $\sigma > 0$ there is $N_1 \geq N_0$ such that if n, $m \geq N_1$ then $\text{dist}(x_n, x_m) < \sigma$, and $\text{Hdist}(S_n, S_m) < \sigma$, $\text{Hdist}(C_n, C_m) < \sigma$. Let us choose $\sigma > 0$ so small that if z is in $B(y, \frac{2r_1}{3}) \backslash B(y, \frac{r_1}{3})$ and between S_n and S_m then a continuum $C(z) \subset W^u_\epsilon(z)$ intersects either S_n or S_m. The existence of $C(z)$ follows from 1.1, while 1.1 and 1.3 ensure that $C(z)$ intersects either S_n or S_m. Otherwise we will have a point w in S_n such that $\text{dist}(z, w) < \sigma$ and $\text{dist}(W^s_\epsilon(w) \cap \partial B(w, r_2/2), W^u_\epsilon(z) \cap \partial B(w, r_2/2)) < \sigma$ which contradicts 1.3. We also assume (see 1.2) that if $\text{dist}(w, z) < r_0$ and $w \in W^u_{2\epsilon}(z)$ ($w \in W^s_{2\epsilon}(z)$) then $w \in W^u_\epsilon(z)$ (resp. $w \in W^s_\epsilon(z)$). Choose z between S_n and S_m, with n, $m \geq N_1$ fixed. Assume $C(z) \cap S_n = \{w\}$. As $w \in W^s(p)$ there is a continuum $C(w) \subset W^u_\epsilon(w)$ such that $C(w) \backslash S_n = C^+ \cup C^-$ with C^+ and C^- continua locally separated by S_n and both reaching $\partial B(w, r_2)$. One of those pieces, C^+ or C^-, is in the same component with respect to S_n in which there are z and S_m. Let C^+ be that piece. Then $C^+ \cap S_m \neq \emptyset$ otherwise there will be points of $C(w)$ and S_n at a distance less than σ. But, by 1.2, $S_n \subset W^s_\epsilon(w)$ and also $C(w) \subset W^u_\epsilon(w)$ which contradicts

1.3 if $\sigma > 0$ is sufficiently small. Let us consider the closed set C given by the union of $C(z)$ with the closure of the subset of C^+ between S_n and S_m. We claim that $C \subset W_\epsilon^u(z)$. For let $v \in C$. The claim is clear if $v \in C(z)$. If $v \in C^+$ and is between S_n and S_m then for all $n \leq 0$ $\text{dist}(f^n(w), f^n(z)) \leq \epsilon$ and $\text{dist}(f^n(w), f^n(v)) \leq \epsilon$ so $\text{dist}(f^n(v), f^n(z)) \leq 2\epsilon$. But every point between S_n and S_m is at a distance less than r_0 from z if σ and r_2 are sufficiently small. Therefore $v \in W_\epsilon^u(z)$. Also we have a subcontinuum $C' \subset C$ such that C' joins z with $u \in S_m$. Thus we have obtained the following property

Every point z between S_n and S_m has a continuum $C' \subset W_\epsilon^u(z)$ which cuts both, S_n and S_m.

But with similar arguments to those used in [Vi4] we see that the above property implies a contradiction with assumption **B** for we may take z so close to S_n that applying f^n we have that $f^n(C')$ is in $W_\epsilon^u(f^n(z))$ and intersects $S(p)$. Moreover, we may take a small neighbourhood V of z such that if $z' \in V$ then the same holds for z'. But this implies assumption **A** thus contradicting **B**.

■

Lemma 2.22 *If C^+ is a part of $W_\epsilon^s(p)$ the same is true for C^-. Therefore there is $\epsilon > 0$ such that $W_\epsilon^s(p)$ is a 2-disk embedded in M such that $p \in int(W_\epsilon^s(p))$ and $W^s(p)$ is an immersed plane.*

Proof. That C^- is part of $W_\epsilon^s(p)$ follows from 2.19 and 2.21, taking into account that **Assumption B** is the negation of **Assumption A**. Therefore, as we have said above, there is a 2-disk $E \subset W_\epsilon^s(p)$, $E \subset CS$. By corollary 2.15 $W_\epsilon^s(p) \subset CS$ too. Expansivity implies that there is $N > 0$ such that $f^n(W_\epsilon^s(p)) \subset E$ if $n \geq N$ (see [Ma]). This suffices to complete the proof for we may write $W^s(p) = \cup_{n \in I\!N} f^{-n}(E)$, an increasing union of 2-disks.

■

Let us assume now that there is a local product structure invariant by f on a neighbourhood of p. This assumption will be justified after dealing with the other cases of values of the eigenvalues, namely, that case in which $|\lambda|$, $|\mu| < 1$ and that in which $|\lambda|$, $|\mu| > 1$.

Proposition 2.23 *Under the assumption 2.1 we have that the point p is a topologically hyperbolic periodic point.*

Proof. The proof of this is contained in proposition 2.16 of [Vi3].

■

3 Cases of $\mid \lambda \mid, \mid \mu \mid < 1$ or $\mid \lambda \mid, \mid \mu \mid > 1$

We will treat both cases simultaneously, as it is clear that taking f^{-1} we may reduce one to the other. Let us assume that $\mid \lambda \mid, \mid \mu \mid < 1$. In this case we have that the splitting of the tangent space of M at p is $T_pM = E^s \oplus E^c$ where E^s corresponds to λ, μ and E^c corresponds to the eigenvalue 1. By Theorem III.2 of [Sh], there are a local strong stable manifold S of dimension 2, and a local central unstable manifold C of dimension 1, C is tangent to E^c. As in section 2 we may write $C = C^+ \cup C^-$ where $C^+ \cap C^- = p$ and C^+ and C^- are line segments tangent to E^c. In the cited theorem, in spite of its name, C may contain points of the stable manifold. In our case we have the following dichotomie.

Lemma 3.1 *Either $C^+ \subset W^s_\epsilon(p)$ or $C^+ \subset W^u_\epsilon(p)$. Analogously for C^-.*

Proof. These are exactly 2.3 and 2.4.

■

We want to prove that C behaves as the local unstable manifold. This is the same to say that $C^+, C^- \subset W^u_\epsilon(p)$. So arguing once again by contradiction we suppose that C^+ is a part of $W^s_\epsilon(p)$. The idea of the proof is to prove that if this occurs then the local stable set of p will contain an open set of M. But this is not possible, for it implies the existence of a Lyapunov stable point for f therefore contradicting the expansive properties of f (see [Le2] lemma 2.7). As in section 2 we may identify via a diffeomorphism h the point p with $\vec{0}$ in $I\!R^3$, and f with $\hat{f} = h^{-1} \circ f \circ h$ in a ball around $\vec{0}$, in such a way, that S is sent by h onto a neighbourhood of $\vec{0}$ in the $0xy$ plane, and C is sent onto a neighbourhood of $\vec{0}$ in the $0z$ axis. Taking an iterate of f if it were necessary we may assume that $\mid \lambda \mid, \mid \mu \mid < \frac{1}{16}$. The diffeomorphism $\hat{f}$ has the following form in local coordinates around $\vec{0}$ depending on the following cases: λ and μ are real and $D\hat{f}_{\vec{0}}$ is diagonalizable. We do not exclude here that it may happens that $\lambda = \mu$.

$$\hat{f}(x, y, z) = (\lambda x + r(x, y, z), \mu y + s(x, y, z), z + t(x, y, z))$$

$\lambda = \mu$ is real and $D\hat{f}_{\vec{0}}$ is not diagonalizable

$$\hat{f}(x, y, z) = (\lambda x + y + r(x, y, z), \lambda y + s(x, y, z), z + t(x, y, z))$$

λ and μ are conjugate complex eigenvalues and therefore $D\hat{f}_{\vec{0}}$ is diagonalizable. In this cases there are real numbers $\rho < \frac{1}{16}$ and $\phi \in [0, 2\pi)$ such that $\lambda = \rho \exp^{\mathrm{i}\phi}$ and $\mu = \rho \exp^{-\mathrm{i}\phi}$ Then we may write $\hat{f}$ as

$$\hat{f}(x, y, z) = (x\rho \cos\phi - y\rho \sin\phi + r(x, y, z),$$

$$x\rho \sin\phi + y\rho \cos\phi + s(x, y, z), z + t(x, y, z))$$

We have supposed that the axes and the scalar product has been chosen in such a way that they correspond to the standard coordinates. Moreover the functions r, s, t can be expressed as in 2.6 as 'quadratic functions'. For instance we may express $r(x, y, z)$ as

$$r(x, y, z) = r_{11}(x, y, z)x^2 + 2r_{12}(x, y, z)xy + r_{22}(x, y, z)y^2 +$$

$$+ 2r_{13}(x, y, z)xz + 2r_{23}(x, y, z)yz + r_{33}(x, y, z)z^2$$

with the r_{ij} C^{r-2} functions if f is of class C^r. The same is true for $s(x, y, z)$ and $t(x, y, z)$. Observe that by our choice of λ we have that $4\lambda^2 + 4\lambda < \frac{17}{64} < \frac{1}{2}$. Let us write $(x_2, y_2, z_2) = \hat{f}^2(x, y, z)$. Clearly this make sense in a neighbourhood of $\vec{0}$.

Remark 3.2 *We identify C with a segment in $0z$. Therefore the function $\hat{f}$ is expressed as $\hat{f}(0,0,z) = (0, 0, z + t_{33}(0,0,z)z^2$. Thus if C^+ is a part of the $W^s_\epsilon(\vec{0})$ then$t_{33}(0,0,z)z^2 < 0$ and $z + t_{33}(0,0,z)z^2 > 0$ for $z > 0$. Therefore given $1 > \delta_0 > 0$ we also have that there is $\delta_1 > 0$, $\delta_1 < \delta_0$ such that if $\sqrt{x^2 + y^2} < \delta_1$ then $\delta_0 < \delta_0 + t_{33}(x, y, z)\delta_0^2$. That is to say that the plane parallel to $0xy$ for $(0, 0, \delta_0)$ is locally pushed down by $\hat{f}$. As in section 2 we have that $r_{33}(0, 0, z) = s_{33}(0, 0, z) \equiv 0$ and this allows us to write $r_{33}(x, y, z) = r_{33x}(0, 0, z)x + r_{33y}(0, 0, z)y + o(\sqrt{x^2 + y^2})$. Similarly for s_{33}.*

Lemma 3.3 *There is $\delta > 0$, $\delta < \delta_0$, such that if $\mid x \mid \leq \delta$, $\mid y \mid \leq \delta$ and $\mid z \mid \leq \delta_0$, δ_0 as in 3.2, then* $\max\{\mid x_2 \mid, \mid y_2 \mid\} < \frac{1}{2}\max\{\mid x \mid, \mid y \mid\}$

Proof. In fact, we don't need to take the double iteration except in the case in which $D\hat{f}_{\vec{0}}$ is not diagonalizable. As the three cases are similar and the former seems a bit more difficult we treate just this case leaving the others to the reader. Let us fix a compact neighbourhood around $\vec{0}$ and find a common bound K for $\mid r_{ij} \mid, \mid s_{ij} \mid, \mid t_{ij} \mid\mid r_{33x} \mid, \mid r_{33y} \mid, \mid s_{33x} \mid, \mid s_{33y} \mid$ in that neighbourhood. Let $\delta_0 > 0$ be such that $12K\delta_0(1+2\lambda) < \lambda$. In order to simplify notation we will omit writing (x, y, z) as part of the arguments of the functions involved. For instance we will write $r_{11}x^2$ instead of $r_{11}(x, y, z)x^2$. Nevertheless, we iterate two times, so we have to calculate the values of the functions not only in (x, y, z) but also in (x_1, y_1, z_1). There will be no confussion as we will write $r_{11}x_1^2$ when we calculate in (x_1, y_1, z_1). Assume $\max\{\mid x \mid, \mid y \mid\} = \mid x \mid$. Hence

$$\mid x_1 \mid = \mid \lambda x + y + r_{11}x^2 + 2r_{12}xy + r_{22}y^2 + 2r_{13}xz + 2r_{23}yz +$$

$$(r_{33x}x + r_{33y}y + o(\sqrt{x^2+y^2}))z^2 \mid \leq$$

$$\leq \lambda \mid x \mid + \mid y \mid + Kx^2 + 2K \mid xy \mid + \cdots +$$

$$+(K \mid x \mid + K \mid y \mid + \mid o(\sqrt{x^2+y^2}) \mid)z^2 \leq$$

$$\leq \lambda \mid x \mid + \mid x \mid + Kx^2 + 2Kx^2 + \cdots +$$

$$+(K \mid x \mid + K \mid x \mid + \mid o(\sqrt{x^2+y^2}) \mid)z^2$$

We may assume that $\mid o(\sqrt{x^2+y^2}) \leq K(\mid x \mid + \mid y \mid) \leq 2K \mid x \mid$. By the choice of δ_0 we then have that

$$\mid x_1 \mid \leq$$

$$\leq \mid x \mid (\lambda + 1 + 8K \mid x \mid + 4Kz^2) \leq$$

$$\leq \mid x \mid (\lambda + 1 + 12K\delta_0) \leq$$

$$\leq \mid x \mid (2\lambda + 1)$$

Analogously we find that

$$\mid y_1 \mid \leq 2\lambda \mid x \mid$$

Iterating again we have for $\mid x_2 \mid$

$$\mid x_2 \mid =$$
$$\mid \lambda^2 x + \lambda y + \lambda(r_{11}x^2 + \cdots) + \lambda y + (s_{11}x^2 + \cdots) +$$
$$+ r_{11}x_1^2 + 2r_{12}x_1y_1 + \cdots + (r_{33x}x_1 + r_{33y}y_1 + o(\sqrt{x_1^2 + y_1^2}))z_1^2 \mid$$

We have that z_1 is not necessarily less than δ_0, but still it is less than $\delta_0(1+2\lambda)$ as it is easy to see by straightforward computations similar as those used above. Similarly for x_1 we have $x_1 < \delta_0(1+2\lambda)$. Hence we may write, using our assumption that $\max\{\mid x \mid, \mid y \mid\} = \mid x \mid$, $\mid x_2 \mid \leq (\lambda^2 + 3\lambda) \mid x \mid + \lambda \max\{\mid x_1 \mid, \mid y_1 \mid\}$. A common bound is given by $\mid x_2 \mid \leq (4\lambda^2 + 4\lambda) \mid x \mid \leq \frac{1}{2} \mid x \mid$. Similarly we may write $\mid y_2 \mid \leq (4\lambda^2 + \lambda) \mid x \mid \leq \frac{1}{2} \mid x \mid$ Assuming that $\max\{\mid x \mid, \mid y \mid\} = \mid y \mid$ we achieve an analogous bound.

■

In the sequel we restrict ourselves to the neighbourhood N_0 of $\vec{0}$ given by $\mid x \mid \leq \delta$, $\mid y \mid \leq \delta$ and $\mid z \mid \leq \delta_0$.

Lemma 3.4 *There is a neighbourhood $N \subset N_0$ of $\vec{0}$ such that on it $\hat{f}$ maps $\{(x,y,z) \in N \,/\, z \geq 0\}$ into itself.*

Proof. The proof is similar to that of 2.7.

■

Definition 3.1 *We say that a point x is Lyapunov stable for $f : M \to M$ if given $\rho > 0$ there is $\sigma > 0$ such that for all $n \geq 0$ $f^n(B(x,\sigma) \subset B(f^n(x),\rho)$.*

Lemma 3.5 *If $C^+ \subset W^s_\epsilon(\vec{0})$ then any point of $C^+ \cap N \backslash \{\vec{0}\}$ is Lyapunov stable for $\hat{f}$.*

Proof. Let $q = (0,0,z_q)$ be any point in $C^+ \cap N$ and choose $\rho > 0$. Let $\sigma' > 0$ be such that $\sigma' < \rho$, $B(q,\sigma') \subset N$ and moreover that $\hat{f}$ push down any point of the disk D paralell to the $0xy$ plane centred in $(0,0,z_q+\sigma')$ and of radious σ'. Therefore its preimage $\hat{f}^{-1}(D)$ has the property that if $w \in \hat{f}^{-1}(D)$ then its height z is greater than the coordinate z of any point of D. By 3.3 we have that any point w of $B(q,\sigma')$ becomes nearer to the $0z$ axis under the positive iterations of $\hat{f}$. Let us see that no positive iterate $\hat{f}^n(w)$ of w has its z coordinate greater than $z_q+\sigma'$. Otherwise joining w with the $0z$ axis with a segment L paralell to the $0xy$ plane we will have that $\hat{f}^n(L)$ has a point, $\hat{f}^n(w)$, of height greater than $z_q+\sigma'$ and other, the iterate of the end point of L on $0z$, with height less than $z_q+\sigma'$. By 3.3 $\hat{f}^n(L)$ intersects D. Hence its preimage by $\hat{f}$ will have a point in $\hat{f}^{-1}(D)$. But choosing the first n such that $\hat{f}^n(L)$ has a point with height greater than $z_q+\sigma'$ we arrive to a contradiction. Therefore, by 3.4, for all $n \geq 0$ we have that $\hat{f}^n(B(q,\sigma')$ is contained in the solid cylinder Q given by $\sqrt{x^2+y^2} \leq \sigma'$, $0 \leq z \leq z_q+\sigma'$. Hence for σ' small enough $Q \subset W^s_\epsilon(\vec{0})$. By expansivity we have that $\lim_{n\to+\infty}(\mathrm{diam}\hat{f}^n(Q)) = 0$. Hence there is a $\sigma' \geq \sigma > 0$ such that $\hat{f}^n(B(q,\sigma) \subset B(\hat{f}^n(q),\rho)$ for all $n \geq 0$.

■

Proposition 3.6 *The local central unstable manifold C is the local unstable manifold, i.e.: $C \subset W^u_\epsilon(\vec{0})$.*

Proof. Lemma 3.3 proves that any point not in $C =$ a segment of $0z$ cannot tend to $\vec{0}$ under negative iterates of $\hat{f}$. This proves unicity. But 3.6 and 2.3 and 2.4 prove that $C \subset W^u_\epsilon(\vec{0})$.

■

Proposition 3.7 *Under the assumption 2.1 we have that the point p is a topologically hyperbolic periodic point.*

Proof. This is the same of [Vi3] proposition 2.16.

■

Remark 3.8 *We have to prove that $W^s_\epsilon(p)$ is unique, but if it were not iterating a 2-disk as in the following section 4, we obtain a separating $D(p)$ in $W^s_\epsilon(p)$ not in $0xy$ we then reason as follows: We take an arc joining the disk in $0xy$ included in $W^s_\epsilon(p)$ with a point in $D(p)$ in the region between both separating sets. There are points in* *the arc that have their local unstable sets intersecting the 2 disk in* $0xy$ *and other ones which intersect* $D(p)$*. Both sets of points in the arc are closed, by connectedness there is a point such that its local unstable set is in the region between the 2-disk in* $0xy$ *and* $D(p)$*. Taking a limit of these points and their local unstable sets when these points approach* p*, we obtain a* $C(p)$ *in the local unstable set of* p*. This contradicts that this local unstable set is contained in* $0z$*.*

4 Existence of local product structure

Perhaps it is worthwhile to remember that the definition of local product structure we usually use requires the existence of local C^0 foliatons, see definition 1.7. Thus we are going to prove here than in a neighbourhood of the periodic point p we have such a structure given by the local stable and unstable sets of the points of this neighbourhood. The idea of the proof is to try to produce a homoclinic intersection between $W^s(p)$ and $W^u(p)$. Once this is obtained we can show that there is a local product structure, at least in one side of $W^s(p)$. Using this structure and the fact that there is a neighbourhood of p in $\Omega(f)$ we cen extend the local product structure to a neighbourhood of p. Then we show that there is a homeomorphism h conjugating, in a neighbourhood of p, f with a linear hyperbolic map T; that is, p is topologically hyperbolic. As in [Vi4] we may prove that

Proposition 4.1 *If there is a neighbourhood* $V(p) \subset \Omega(f)$ *of* p *and* p *is a topologically hyperbolic periodic point for* f *then there is a point* x *of homoclinic intersection between* $W^s(p)$ *and* $W^u(p)$*. Moreover* $W^s(p)$ *is topologically transversal to* $W^u(p)$ *at* x*.*

Proof. It follows from the fact that **Assumption A** always holds that there are homoclinics points. That the intersection is topologically transversal is Theorem B of [Vi3].

■

Proposition 4.2 *There is a local product structure given by the stable and unstable sets, and therefore invariant by* f*, on a neighbourhood* $V' \subset V(p)$*.*

Proof. This is Theorem 5.3 of [Vi3].

■

Remark 4.3 *We remark that 4.2 does not say that $p \in V'$. If it were the case, we are done.*

By the previous remark 4.3 we have to extend the local product structure to a neighbourhood of p.

Proposition 4.4 *There is a neighbourhood $N(p)$ of p with an f-lps.*

Proof. Let $x \neq p$, $x \in W^s(p) \cap W^u(p)$. We may assume that $x \in 0xy \cap W^s_\epsilon(p)$. Let D be a 2-disk in $0xy \cap W^s_\epsilon(p)$ around x. As in the proof of 2.16 we may prove that $W^s_\epsilon(f^{-n}(x)$ contains a surface S_n which separates $B(f^{-n}(x), r_1)$ for a suitable $r_1 > 0$ and for all $n \geq N$. As $n \to +\infty$ we have that $f^{-n}(x) \to p$ and therefore S_n accumulates in $W^s_\epsilon(p)$. We prove that between to such S_n there is an f-lps as in [Vi3]. Moreover we may assume that between S_l and S_n there is another surface S_m. Therefore we have a local product structure in a neighbourhood of x in both sides of $0xy$ We simply iterate the local product structure obtained before by f^m to have $f^m(S_m)$ included in D. As we may assume that x is as close as desired to p, we may repeat the arguments of [Vi4] to obtain homoclinic intersections of $W^s(p)$ with $W^u(p)$ in both sides of $0xy$. Thus we have a neighbourhood with a local product structure in both halfspaces limited by $0xy$. Repeating the arguments of [Vi3], Theorem 5.3 carefully, we may prove that in both halfspaces H^+, H^- there are neighbourhoods N^+ and N^- respectively such that on them there are defined f-lps. These local product structures extend well to the $0xy$ plane and therefore the union $\mathrm{clos}(N^+) \cup \mathrm{clos}(N^-)$ gives a neighbourhood of p in which there is an f-lps. This finishes the proof.

■

References

[Bi] Bing R. H., *The Geometric Topology of 3-Manifolds,* AMS Colloquium Publications, **volume 40** 1983.

[Du-Hu] Duvall P. F. and Husch L. S., *Analysis on Topological Manifolds,* Fundamenta Mathematicae, **77** 1972.

[Fa] Falconer , *The Geometry of Fractal Sets,* Cambridge University Press, 1985.

[Fr-Ro] Franks J. and Robinson R. C., *A quasi-Anosov diffeomorphisms that is not Anosov,* Trans. Amer. Math. Soc., **233** (1976), p. 267-278.

[Ft] Fathi A., *Expansiveness, Hiperbolicity and Hausdorff Dimension,* Commun. Math. Phys., **126** (1989), p. 249-262.

[Fr] Franks, J., *Anosov diffeomorphisms,* Proceedings of the Symposium in Pure Mathematics, **14** (1970), p. 61-94.

[Le1] Lewowicz, J., *Lyapunov Functions and topological stability,* Journal of Differential Equations, **38 (2)** (1980), p. 192-209.

[Le2] Lewowicz, J., *Persistence in Expansive Systems,* Erg. Th. & Dyn. Sys., **3** (1983), p. 567-578.

[Le3] Lewowicz, J., *Expansive homeomorphisms of surfaces,* Bol. Soc. Bras. Mat, **20 (1)** (1989), p. 113-133.

[Le-To] Lewowicz, J. and Tolosa, J., *Genericity of Homeomorphisms with Connected Stable and Unstable Sets,* Contemporary Mathematics, **152** (1993), p. 203-213.

[Ma] Mañé R., *Expansive Homeomorphisms and Topological Dimension,* Trans. Amer. Math. Soc., **252** (1979), p. 313-319.

[Sh] Shub M., *Global Stability of Dynamical Systems,* Springer-Verlag, 1987.

[Sp] Spanier E. H., *Algebraic Topology,* Mc. Graw-Hill, 1966.

[Vi1] Vieitez, J. L., *Three Dimensional Expansive Homeomorphisms,* Pitman Research Notes in Mathematics, **285** (1993), p. 299-323.

[Vi2] VIEITEZ, J. L., *Expansive Homeomorphisms and Hyperbolic Diffeomorphisms on three manifolds,* Erg. Th. & Dyn. Sys., **16** (1996), p. 591-622.

[Vi3] VIEITEZ, J. L., *Three Dimensional Expansive Diffeomorphisms with Homoclinic Points,* Bol. Soc. Bras. Mat., **27** (1996), p. 55-90.

[Vi4] VIEITEZ, J. L., *Expansive Diffeomorphisms and perio points,* Preprint, .

[Wi, 1979] WILDER R. L., *Topology of Manifolds,* AMS Colloquium Publications, **Volume 32** 1979.