Ergod. Th. & Dynam. Sys. (1998), **18**, 567–608
Printed in the United Kingdom © *1998 Cambridge University Press*

A heteroclinic bifurcation of Anosov diffeomorphisms*

HEBER ENRICH†
IMERL Facultad de Ingeniería, C.C. 30 Montevideo, Uruguay
(e-mail: enrich@fing.edu.uy)

(*Received* 3 *August* 1995 *and accepted in revised form* 25 *February* 1997)

Abstract. We study some diffeomorphisms in the boundary of the set of Anosov diffeomorphisms mainly from the ergodic viewpoint. We prove that these diffeomorphisms, obtained by isotopy from an Anosov $f : M \mapsto M$ through a heteroclinic tangency, determine a manifold $\mathcal{M}$ of finite codimension in the set of C^r diffeomorphisms. We prove that any diffeomorphism F in $\mathcal{M}$ is conjugate to f; moreover, there exists a unique SRB measure for F, and F is Bernoulli with respect to this measure. In particular, if the dimension of M is two, and μ is a volume element, we prove that the isotopy can be taken such that the measure is preserved.

0. *Introduction*

0.1. The understanding of the process of loss of hyperbolicity is an old problem. One way it is lost, starting from an Anosov diffeomorphism, is by creating a map which is derived from Anosov, where one of the eigenvalues at a fixed point (or two if they are complex conjugate) is pushed to the boundary of the unit circle. They maintain some of the topological and ergodic properties (see, among others, [**L80**], [**C93**], and [**HY95**]). There are other ways to arrive at the boundary $\mathcal{B}$ of the Anosov diffeomorphisms with local bifurcations; for instance, as a consequence of several general ideas, in [**L80**] an example in which the stable and unstable manifolds of a fixed point are modified until they become tangent at the fixed point is shown. The result is a diffeomorphism conjugate to an Anosov map; in particular, there appear two invariant foliations conjugate to the stable and unstable foliations of the Anosov map. In the same article, an example of a Kupka–Smale map in $\mathcal{B}$ conjugate to an Anosov map is shown, which therefore inherits many interesting properties that can be considered as obtained from an Anosov map through a global bifurcation. In the same vein, we study here some dynamical properties of a kind of diffeomorphism in $\mathcal{B}$.

* To the memory of Professor R. Mañé, who has been my advisor during the preparation of this article, a part of which has been presented as a doctoral thesis.
† Also at IMPA/CNPq, Brazil.

Let us describe the diffeomorphisms we are interested in (see Figure 1). Let M^n be a C^∞ n-dimensional compact connected Riemannian manifold, $f : M^n \mapsto M^n$ a transitive C^r Anosov diffeomorphism with $r \geq 2$. Let P, Q be two different fixed points of f, and R an intersection point of the unstable manifold of P (denoted $W^u(P, f)$) with the stable manifold of Q ($W^s(Q, f)$). We modify f by isotopy so as to obtain a cubic tangency at the continuation of R between the unstable manifold of the continuation of P and the stable manifold of the continuation of Q. Then, we prove the following theorems.

THEOREM 1.
(a) *There exists a manifold $\mathcal{M}$ of codimension $n+1$ in $C^r(M^n)$ included in $\mathcal{B}$ such that for any $F \in \mathcal{M}$ there exists an isotopy F_t, $t \in [0,1]$, with $F_0 = f$, $F_1 = F$, F_t Anosov for $t \in [0,1)$; $R_{F_1} \in W^u(P_{F_1}, F_1) \cap W^s(Q_{F_1}, F_1)$ with $T_{R_{F_1}} W^u(P_{F_1}, F_1)$ and $T_{R_{F_1}} W^s(Q_{F_1}, F_1)$ intersecting in a one-dimensional fibre, where P_{F_1}, Q_{F_1}, and R_{F_1} are the continuations of P, Q and R, respectively.*
(b) *F_1 is conjugate to f (thus expansive and transitive).*
(c) *The Pesin region for F_1 has full probability. There exists a unique SRB measure for F_1 in M^n, and an ergodic attractor whose corresponding measure is the SRB measure. F_1 is Bernoulli with respect to this measure.*

THEOREM 2. *If f preserves a volume form μ, and we assume that there exist linearizing C^2 coordinates in a neighborhood of P, and also of Q, then:*
(a) *there exists an isotopy f_t such that $f_0 = f$, $f_1 \in \mathcal{B}$, f_t preserves the measure for $t \in [0,1]$, f_t is Anosov for $t \in [0,1)$; $R_{f_1} \in W^u(P_{f_1}, f_1) \cap W^s(Q_{f_1}, f_1)$ with $T_{R_{f_1}} W^u(P_{f_1}, f_1) = T_{R_{f_1}} W^s(Q_{f_1}, f_1)$;*
(b) *f_1 is conjugate to f (thus expansive and transitive);*
(c) *the Pesin region for f_1 has full probability and, with respect to the given measure, f_1 is ergodic. There exists an ergodic attractor whose corresponding measure is μ, such that the basin of attraction of the ergodic attractor has measure 1. Moreover, μ is the only SRB measure for f_1 and f_1 is Bernoulli with respect to this measure.*

The key fact in the theorems is that the invariant stable and unstable foliations persist in $\mathcal{B}$, determining the topological and ergodic properties.

We remark that the codimension of $\mathcal{M}$ seems rather excessive, the initial conjecture was that it could be two, because to obtain a cubic tangency as in the figure it is necessary to impose two conditions. The fact is that, according to our methods, we must ensure that the tangency of the foliations is obtained at the point R, to be able to use the properties of the points P and Q, and this implies more restrictions. If the tangency is obtained in a point near R, we could lose the hyperbolicity before $t = 1$.

Let us define some terms that appeared in our theorems. We denote as $C^r(M^n)$ the set of C^r diffeomorphisms in M^n. From now on the manifold will be denoted as M, omitting the dimension.

Definition 0.1. A homeomorphism $F : M \mapsto M$ is *expansive* if there exists a constant $\alpha > 0$ such that for $S, S^* \in M$, $d(F^i(S), F^i(S^*)) \leq \alpha$ for all $i \in \mathbb{Z}$ implies $S = S^*$.

Definition 0.2. A homeomorphism $F : M \mapsto M$ is *transitive* if there exists a point $S \in M$ with dense orbit.

Definition 0.3. Given a transformation $F : M \mapsto M$, we say that $A \subset M$ has *full probability* if, for every F-invariant probability μ, we have $\mu(A) = 1$.

Definition 0.4. We say that the point $S \in M$ is *regular for the diffeomorphism* F if there exist real numbers

$$\chi_1(S) > \chi_2(S) > \cdots > \chi_m(S)$$

(called Lyapounov exponents) and a decomposition

$$T_S M = E_1(S) \oplus E_2(S) \oplus \cdots \oplus E_m(S)$$

such that

$$\chi(S, v, F) = \lim_{i \to \pm\infty} \frac{1}{i} \log \|DF^i(S)v\|$$

exists and is equal to $\chi_j(S)$ for $0 \neq v \in E_j(S)$ and $1 \leq j \leq m$. We denote the set of regular points as $\tilde{\Lambda}$.

The *theorem of Oseledec* asserts that $\tilde{\Lambda}$ has full probability.

Definition 0.5. Given a diffeomorphism F, its *Pesin region* Λ is the set of regular points of M such that all the Lyapounov exponents are non-zero.

Definition 0.6. A *SRB probability* is an ergodic F-invariant probability which has absolutely continuous conditional measures on unstable manifolds (for more details, see §4.2).

Definition 0.7. Following [**PS89**], we say that an *ergodic attractor* for F is a F-invariant set $A \subset M$ with a F-invariant Borel probability μ on A (i.e. (A, ν) is a probability space) such that for some set $Y \subset M$ with positive Lebesgue measure:
(i) $\lim_{i\to\infty} d(F^i(S), A) = 0$ for $S \in Y$,
(ii) μ is F-ergodic,
(iii) the Lebesgue a.e. point $S \in Y$ is generic with respect to μ, that is, $\lim_{i\to\infty}(1/i)\sum_{j=0}^{i-1} \delta_{F^j(S)} = \mu$ with weak topology, where δ_T is the Dirac measure concentrated on T.

Definition 0.8. We say that a map $F : M \mapsto M$ is *Bernoulli* if it is equivalent to a Bernoulli shift, where $(M, \mathcal{A}, \mu)$ is a measure space with F-invariant measure.

Now we sketch the proof of the theorems. We first construct in §1 an isotopy f_t from f for $t = 0$ to a function $f_1 \in \mathcal{B}$ with f_1 satisfying the conditions of F_1 in the statement of Theorem 1. The condition we impose to define f_1 is that the angle near R between a local unstable invariant foliation defined near P (and extended to a neighborhood of R), and a local stable invariant foliation defined near Q (and extended to a neighborhood of R) must have, to a first approximation, a quadratic variation with the distance to R with large enough coefficient (see Lemma 1.4). We end the proof of Theorem 2(a) with the results of §2 which, with an argument of cones, states that f_t is Anosov for $t \in [0, 1)$; and with the results of §4.4, which shows that the isotopy can be taken such that the given measure is f_t-invariant.

We show that if a diffeomorphism F_1 near f_1 verifies a certain inequality (that of Lemma 1.4 with $t = 1$), then it is in $\mathcal{B}$ (§1.6). That inequality is obtained through

$n+1$ conditions (the tangency one plus n conditions due to the quadratic variation of the angle described above), determining a local chart of the manifold $\mathcal{M}$. This, together with the results of §2 (where we prove that we can arrive at $\mathcal{M}$ within the Anosov set), proves Theorem 1(a). In §3.1 we show the existence of two invariant fibre bundles (a 'stable' and an 'unstable' fibre bundle), through an appropriate set of cones. We integrate them (§3.2) to prove the expansivity of F_1 (Proposition 3.13) and the conjugation to f (Proposition 3.14) for the assertions of Theorems 1(b) and 2(b).

We study the ergodic properties in §4. First, we prove that the Pesin region has full measure observing that we can restrict ourselves to the points which have infinite forward and backward iterates through a neighborhood of R, and it is enough to study Lyapounov exponents of the return function to that neighborhood. In the case of Theorem 1, we construct the SRB measure adapting an idea of [**PS82**]. The other ergodic properties are proved following arguments of [**PS89**].

In the case of Theorem 1, we will make some additional hypotheses about the eigenvalues at P and Q. These hypotheses do not represent a restriction to our theorem because they can be obtained by isotopy without leaving the set of Anosov diffeomorphisms. In dimension two, we denote the respective eigenvalues at P and Q as λ_P, μ_P, and λ_Q, μ_Q and we will suppose that $|\mu_P| > 1 > |\lambda_P|$; $|\mu_Q| > 1 > |\lambda_Q|$ with $|\lambda_P\mu_P| > 1 > |\lambda_Q\mu_Q|$. For $n > 2$, we denote the eigenvalues at P and Q respectively as $\lambda_{P_1}, \dots, \lambda_{P_s}, \mu_{P_1}, \dots, \mu_{P_u}$ and $\lambda_{Q_1}, \dots, \lambda_{Q_s}, \mu_{Q_1}, \dots, \mu_{Q_u}$ with $u+s=n$, such that

$$|\mu_{P_1}| \geq \cdots \geq |\mu_{P_u}| > 1 > |\lambda_{P_1}| \geq \cdots \geq |\lambda_{P_s}|$$
$$|\mu_{Q_1}| \geq \cdots \geq |\mu_{Q_u}| > 1 > |\lambda_{Q_1}| \geq \cdots \geq |\lambda_{Q_s}|.$$

We suppose that

$$|\mu_{P_u}| > \max\{\exp\{\log|\lambda_{P_s}| \log|\mu_{Q_u}\mu_{Q_1}^{-2}|/\log|\mu_{Q_1}|\}, |\lambda_{P_1}\lambda_{P_s}^{-2}|, |\lambda_{P_1}\mu_{P_1}|\} \tag{1}$$
$$|\lambda_{Q_1}| < \min\{\exp\{\log|\mu_{Q_1}| \log|\lambda_{P_1}\lambda_{P_s}^{-2}|/\log|\lambda_{P_s}|\}, |\mu_{Q_u}\mu_{Q_1}^{-2}|, |\lambda_{Q_s}\mu_{Q_u}|\}. \tag{2}$$

These conditions are needed in order for F_t to be hyperbolic for $t < 1$. We also suppose that the non-resonance conditions for the set of eigenvalues with modulus larger than 1 and also for the set of eigenvalues with modulus smaller than 1 for the point P and also for Q are verified. Finally, we suppose (see Theorem 2 of [**S57**]) that

$$r > \max\{\log|\lambda_{P_s}|/\log|\lambda_{P_1}|, \log|\mu_{P_1}|/\log|\mu_{P_u}|, \log|\lambda_{Q_s}|/\log|\lambda_{Q_1}|, \log|\mu_{Q_1}|/\log|\mu_{Q_u}|\}.$$

We observe that these are open conditions: if they are verified for a diffeomorphism f, they are also verified for g in a C^r neighborhood of f.

We point out that Theorem 2 is asserted only for $n = 2$. It can be stated in higher-dimensional manifolds with additional hypotheses (n even, and $u = s$). In the bidimensional case, the conditions $|\lambda_P\mu_P| = 1 = |\lambda_Q\mu_Q|$ are close enough to the conditions $|\lambda_P\mu_P| > 1 > |\lambda_Q\mu_Q|$ so as to assert that the proof still works. However, in higher-dimensional manifolds, the conditions $\prod_{i=1}^{s}|\lambda_{P_i}|\prod_{i=1}^{u}|\mu_{P_i}| = 1 = \prod_{i=1}^{s}|\lambda_{Q_i}|\prod_{i=1}^{u}|\mu_{Q_i}|$ together with (1) and (2) are contradictory. If we write equality signs instead the inequality ones in (1) and (2) (as we did for $n = 2$) we have the restriction $s = u$. We do not insist on this.

There are some questions which remain to be answered. For instance, has the basin of attraction of the ergodic attractor Lebesgue measure one? What can we say about F_t beyond the boundary of the set of Anosov diffeomorphisms?

We remark that some of the topological results of our theorem in dimension two were obtained simultaneously and in an independent way in [**BDV94**].

1. *Definition of the manifold $\mathcal{M}$*

In this section we construct the isotopy. The idea is to deform f through a map $\theta_t : M \mapsto M$ which is different from the identity outside a neighborhood of R in the way shown in Figure 1 (see §1.1). It can be expected that the stable and unstable foliation will persist, and that outside a disconnected region near the iterates of R the foliations will not be very different from the initial ones, that near R the stable foliation will not vary too much and, roughly speaking, the unstable foliation will be mapped as θ_t. We want that for $t = 1$, near R, the angle between these two foliations has a quadratic variation with the distance to R. In fact, on account of P and Q, if we reach the wanted properties for a local unstable invariant foliation for P extended near R and a local stable invariant foliation for Q (also extended near R), the property works for any local invariant foliation and, indeed, for the global stable and unstable foliations. So, in §1.2 we define the local foliations with adequate smoothness conditions, which will be used in connection with two systems of coordinates (one at P, the other at Q). In this section, we construct θ_t verifying the mentioned quadratic variation, and show that to develop this construction we need to impose $n + 1$ conditions, thus determining the dimension of M.

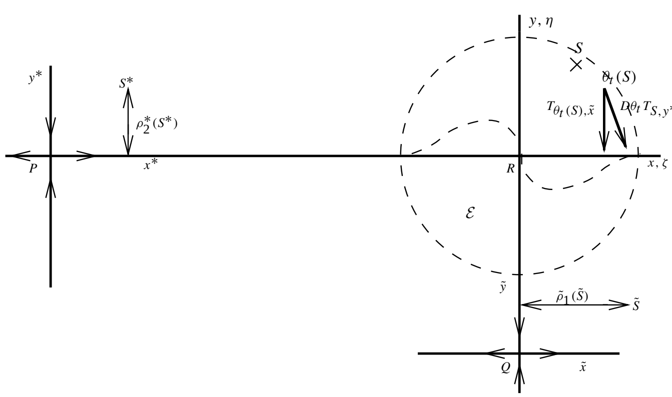

FIGURE 1. The bifurcation.

1.1. We begin by defining an isotopy $f_t = \theta_t \circ f$ with $t \in [0, 1]$, where θ_t is a C^r diffeomorphism, $\theta_t : M \mapsto M$, $\theta_t(R) = R$, $\theta_0 = \mathrm{id}$, $\theta_t = \mathrm{id}$ outside of a small neighborhood $\mathcal{E}$ of the point R. (Alternatively, we could have defined the isotopy as $f_t = f \circ \theta_t^*$, where $\theta_t^* = f^{-1} \circ \theta_t \circ f$.) We consider a C^r system of coordinates between a neighborhood $\mathcal{U}$ of R and an open set $\mathcal{U}^n$ of $\mathbb{R}^n$; we have to define θ_t for points in $\mathcal{U}$. We denote by $(\xi_1, \ldots, \xi_u, \eta_1, \ldots, \xi_s) = (\xi, \eta) \in \mathbb{R}^u \times \mathbb{R}^s$ the coordinates of a point, we suppose that R has coordinates $(0, 0)$ and that the connected component of $W^u(P, f) \cap \mathcal{U}$ that contains R has image $\eta = 0$; in the same way, we suppose that the connected component of $W^s(Q, f) \cap \mathcal{U}$ that contains R has image $\xi = 0$.

We have to impose some additional properties for the C^r diffeomorphism between $\mathcal{U}$ and $\mathcal{U}^n$ (see Lemma 1.1), so we consider new C^r coordinates $(x_1, \ldots, x_u, y_1, \ldots, y_s) = (x, y) \in \mathbb{R}^u \times \mathbb{R}^s$. We denote by $x = \Gamma(\xi, \eta)$, $y = \Delta(\xi, \eta)$ the equations of the change of coordinates, and assume the conditions

$$\Gamma(0, \eta) = 0 \tag{3}$$

$$\Delta(\xi, 0) = 0. \tag{4}$$

Indeed, $D_\xi \Gamma(0, 0)$ and $D_\eta \Delta(0, 0)$ are invertible. Our first step is to find a suitable $\theta_t \in C^r$:

$$\theta_t(x_1, \ldots, x_u, y_1, \ldots, y_s) = \begin{pmatrix} x_1 \cos t\gamma(\rho) + y_1 \sin t\gamma(\rho) \\ x_2 \\ \vdots \\ x_u \\ -x_1 \sin t\gamma(\rho) + y_1 \cos t\gamma(\rho) \\ y_2 \\ \vdots \\ y_s \end{pmatrix}$$

where $t \in [0, 1]$, $\rho^2 = \sum_{i=1}^u x_i^2 + \sum_{i=1}^s y_i^2$ and $\gamma : [0, +\infty) \mapsto \mathbb{R}$, $\gamma \in C^r$ is such that $\gamma(0) = \pi/2$; $0 \leq \gamma(x) < \pi/2$ for $x \in (0, \epsilon)$; $\gamma(x) = 0$ for $x \in [\epsilon, +\infty)$, where ϵ is a small enough positive number which we will determine later.

For $t = 1$, this transformation maps the tangent space to Ox_1 on O over the tangent space to Oy_1 on O. We observe that $\rho(S) = \rho(\theta_t(S))$ for $S \in \mathcal{E}$.

1.2. Before imposing more conditions on the change of coordinates (see §1.3) we consider two suitable linearizing systems of coordinates (see Figure 1): the first one at Q (whose image will be denoted $\tilde{\mathcal{U}} \subset \mathbb{R}^n$, and where the coordinates of a generic point will be denoted as $(\tilde{x}_1, \ldots, \tilde{x}_u, \tilde{y}_1, \ldots, \tilde{y}_s) = (\tilde{x}, \tilde{y}) \in \mathbb{R}^u \times \mathbb{R}^s = \mathbb{R}^n$), and the other one at P (with image $\mathcal{U}^*$ and coordinates denoted $(x_1^*, \ldots, x_u^*, y_1^*, \ldots, y_s^*) = (x^*, y^*) \in \mathbb{R}^u \times \mathbb{R}^s = \mathbb{R}^n$). In the case of Theorem 2, we take the systems of the hypothesis, in the case of Theorem 1, we now show how to construct them, adapting some ideas of [**PT93**]. We work at the point Q, similar considerations work for the point P. First we consider a linearizing C^1 system of coordinates for $f|_{W^s(Q,f)}$ and for $f|_{W^u(Q,f)}$ (see Theorem 2 of [**S57**]). Then we construct a C^1 invariant foliation $\mathcal{F}^u$ (by definition, this means that there exists C^1 local charts covering M such that at each one the foliation is 'flattened'; in particular, this implies that the leaves of the foliation are *at least* C^1). We construct the lift

$$\begin{array}{ccc} L(M) & \xrightarrow{Df} & L(M) \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & M \end{array}$$

where $L(M) = \{(S, L) : S \in M$ and L is an u-dimensional linear subspace of $T_S M\}$. Let $\mathcal{F}$ be a foliation of $\tilde{\mathcal{U}}$ (a neighborhood of Q), by u-dimensional manifolds. Then, $W_L = \{(S, L) : S \in \tilde{\mathcal{U}}$ and L is the tangent space of the $\mathcal{F}$-leaf through $S\}$ is a manifold in $L(M)$ of dimension n; and $\mathcal{F}$ is a f-invariant foliation if and only if W_L is a Df invariant manifold. If W_L is C^q then $\mathcal{F}$ is C^q, and we have even in this case that the tangent spaces $T_S\mathcal{F}(S)$ to the leaves of $\mathcal{F}$ depend C^q on S. We have that (Q, I_Q) (where I_Q is the unstable linear subspace corresponding to f at Q) is a fixed point of Df, and the $n + us$ eigenvalues of $D(Df)_{(Q,I_Q)}$ have absolute value:

$$|\lambda_{Q_i}| \text{ for } i = 1, \dots, s; \quad |\mu_{Q_j}| \text{ for } j = 1, \dots, u;$$
$$|\lambda_{Q_i}|/|\mu_{Q_j}| \text{ for } i = 1, \dots, s,\ j = 1, \dots, u$$

with $|\lambda_{Q_i}|/|\mu_{Q_j}| \le |\lambda_{Q_1}|/|\mu_{Q_u}| < |\lambda_{Q_s}| \le \dots \le |\lambda_{Q_1}| < 1 < |\mu_{Q_u}| \le \dots \le |\mu_{Q_1}|$ for $i = 1, \dots, s$, $j = 1, \dots, u$. Hence, there exists a n-dimensional center-unstable locally invariant C^1 manifold W_L containing (Q, I_Q) such that $T_{(Q,I_Q)}W_L$ is the eigenspace of $D(Df)_{(Q,I_Q)}$ corresponding to the eigenvalues

$$\lambda_{Q_s}, \dots, \lambda_{Q_1}, \mu_{Q_u}, \dots, \mu_{Q_1}.$$

W_L defines a f-invariant C^1 foliation $\mathcal{F}^u$ in a neighborhood $\tilde{\mathcal{U}}$ of Q, and $\mathcal{F}^u(Q) \subset W^u(Q, f)$ because $\mathcal{F}^u(Q)$ is an invariant submanifold tangent to I_Q, so $\mathcal{F}^u$ is an unstable foliation on a neighborhood of Q.

In the same way, we define a C^1 invariant foliation $\mathcal{F}^s$ on $\tilde{\mathcal{U}}$, working with the fixed point (Q, E_Q) of $Df : J(M) \mapsto J(M)$, where $J(M) = \{(S, J) : S \in M$ and J is a s-dimensional linear subspace of $T_S M\}$. The $n + su$ eigenvalues of $D(Df)_{(Q,E_Q)}$ are, in absolute value,

$$|\lambda_{Q_i}| \text{ for } i = 1, \dots, s; \quad |\mu_{Q_j}| \text{ for } j = 1, \dots, u;$$
$$|\mu_{Q_j}|/|\lambda_{Q_i}| \text{ for } i = 1, \dots, s,\ j = 1, \dots, u$$

with $|\lambda_{Q_s}| \le \dots \le |\lambda_{Q_1}| < 1 < |\mu_{Q_u}| \le \dots \le |\mu_{Q_1}| < |\mu_{Q_u}|/|\lambda_{Q_1}| \le |\mu_{Q_j}|/|\lambda_{Q_i}|$ for $i = 1, \dots, s$, $j = 1, \dots, u$. Now we consider the theory of normal hyperbolicity (see [**HPS77**]). W_J is q-normally hyperbolic for all q such that $|\mu_{Q_u}|/(|\lambda_{Q_1}||\mu_{Q_1}|^q) < 1$. As $\log|\mu_{Q_u}/\lambda_{Q_1}|/\log|\mu_{Q_1}| < 2$, then we can take $q = 2$, and this implies (see [**HPS77**, Theorem 6.1]) that *$\mathcal{F}^s$ is a C^2 foliation*.

We can define a C^1 system of coordinates with image in $\tilde{\mathcal{U}} \subset \mathbb{R}^n$, such that given a point $S \in \tilde{\mathcal{U}}$, we denote $(\tilde{x}_1, \dots, \tilde{x}_u)$ the coordinates of the point $\mathcal{F}^s(S) \cap W^u_{\text{loc}}(Q, f)$ in the linearizing coordinates just defined, and similarly $(\tilde{y}_1, \dots, \tilde{y}_s)$ are the coordinates of $\mathcal{F}^s(S) \cap W^s_{\text{loc}}(Q, f)$.

From now on, K stands for a constant number or vector whose value can vary in the different formulas, but the variation is independent of t or any other variable. For $n \ge 2$ we denote by $T_{S,\tilde{x}}$ the fibre of the tangent bundle of $\tilde{x} = K \in \mathbb{R}^u$ at the point S which (as $\mathcal{F}^s$ is C^2) is a C^1 function of S, and similarly by $T_{S,\tilde{y}}$ the fibre of the tangent bundle of $\tilde{y} = K \in \mathbb{R}^s$ at the point S (we only can assert that it varies continuously with S). We suppose that the coordinates at Q are extended so that $R \in \tilde{\mathcal{U}}$. Let $(0, \dots, 0, \tilde{r}_1, \dots, \tilde{r}_s) = (0, \tilde{r}) \in \mathbb{R}^u \times \mathbb{R}^s = \mathbb{R}^n$ be the coordinates of R. Let $\tilde{\rho}_1^2(S) = \sum_{i=1}^u \tilde{x}_i^2$; $\tilde{\rho}^2(S) = \sum_{i=1}^u \tilde{x}_i^2 + \sum_{j=1}^s (\tilde{y}_j - \tilde{r}_j)^2$. Let us suppose that the coordinates are extended so that

$$\tilde{U} = \{S = (0, \dots, 0, \tilde{y}_1, \dots, \tilde{y}_s) : \tilde{\rho}(S) \leq \tilde{\rho}(R)\} \subset \tilde{\mathcal{U}}$$

and we take $\tilde{d}$, $\tilde{e}$ small enough so that

$$\tilde{U} \subset \tilde{\mathcal{U}}_1 = \left\{ S = (\tilde{x}_1, \dots, \tilde{x}_u, \tilde{y}_1, \dots, \tilde{y}_s) : \sum_{i=1}^u \tilde{x}_i^2 \leq \tilde{d}; \sum_{j=1}^s \tilde{y}_j^2 \leq \tilde{e} \right\} \subset \tilde{\mathcal{U}}.$$

It is not a restriction to suppose that $\tilde{d} = 1$. We will denote by $W^u_{\text{loc}}(Q, f)$ the set

$$\left\{ S = (\tilde{x}_1, \dots, \tilde{x}_u, 0, \dots, 0) : \sum_{i=1}^u \tilde{x}_i^2 \leq 1 \right\}.$$

We suppose that $\mathcal{E}$ is small enough such that $\mathcal{E} \subset \tilde{\mathcal{U}}_1$.

We work in a similar way around P, we denote by T_{S,x^*} the fibre of the tangent bundle of $x^* = K \in \mathbb{R}^u$ at the point S, and by T_{S,y^*} the fibre of the tangent bundle of $y^* = K \in \mathbb{R}^s$ at the point S. We suppose that $R = (r_1^*, \dots, r_u^*, 0, \dots, 0) = (r^*, 0) \in \mathcal{U}^*$, where $\mathcal{U}^*$ is the domain of the linearizing coordinates. Let $\rho_2^{*2}(S) = \sum_{j=1}^s y_j^{*2}$; $\rho^{*2}(S) = \sum_{i=1}^s (x_i^* - r_i^*)^2 + \sum_{j=1}^s y_j^{*2}$. We suppose that the linearizing coordinates are extended so that

$$U^* = \{S = (x_1^*, \dots, x_u^*, 0, \dots, 0) : \rho^*(S) \leq \rho^*(R)\} \subset \mathcal{U}^*$$

and we take d^*, e^* small enough so that

$$U^* \subset \mathcal{U}_1^* = \left\{ S = (x_1^*, \dots, y_s^*) : \sum_{i=1}^u x_i^{*2} \leq d^*; \sum_{j=1}^s y_j^{*2} \leq e^* \right\} \subset \mathcal{U}^*.$$

We suppose that $e^* = 1$, and $\mathcal{E} \subset \mathcal{U}_1^*$. Let $W^s_{\text{loc}}(P, f) = \{S = (0, \dots, 0, y_1^*, \dots y_s^*) : \sum_{i=1}^s y_i^{*^2} \leq 1\}$.

If we assume $n > 2$, let $\zeta_{P_u}, \nu_{P_0}, \nu_{P_s}, \zeta_{Q_0}, \zeta_{Q_u}, \nu_{Q_0}$ be such that

$$|\mu_{P_u}| > \zeta_{P_u} > 1 > \nu_{P_0} > |\lambda_{P_1}| \geq |\lambda_{P_s}| > \nu_{P_s} > 0 \tag{5}$$

$$\zeta_{Q_0} > |\mu_{Q_1}| \geq |\mu_{Q_u}| > \zeta_{Q_u} > 1 > \nu_{Q_0} > |\lambda_{Q_1}| \tag{6}$$

$$\zeta_{P_u} > \max\{\exp\{\log \nu_{P_s} \log(\zeta_{Q_u} \zeta_{Q_0}^{-2}) / \log \zeta_{Q_0}\}, \nu_{P_0} \nu_{P_s}^{-2}\}$$

$$\nu_{Q_0} < \min\{\exp\{\log \zeta_{Q_0} \log(\nu_{P_0} \nu_{P_s}^{-2}) / \log \nu_{P_s}\}, \zeta_{Q_u} \zeta_{Q_0}^{-2}\}$$

(we recall our hypotheses). If $n = 2$, the reasoning that follows is valid just taking

$$\nu_{P_0} = \nu_{Ps} = |\lambda_P|; \quad \zeta_{P_u} = |\mu_P|; \quad \nu_{Q_0} = |\lambda_Q|; \quad \zeta_{Q_0} = \zeta_{Q_u} = |\mu_Q|.$$

Therefore, if $\mathcal{E}$ is small enough, we can iterate any point $S \in \mathcal{E}$ at least $m_1 = m_1(S)$ times by f without leaving $\tilde{\mathcal{U}}_1$, where m_1 is larger than

$$m(S) = \begin{cases} E(-\log \tilde{\rho}_1(S) / \log \zeta_{Q_0}) & \text{if } \tilde{\rho}_1(S) \neq 0 \\ \infty & \text{if } \tilde{\rho}_1(S) = 0, \end{cases}$$

and where $E(x)$ is the largest natural number smaller or equal to x. Similarly, we can iterate any point $S \in \mathcal{E}$ at least $p_1 = p_1(S)$ times by f^{-1} without leaving $\mathcal{U}_1^*$, where p_1 is larger than

$$p(S) = \begin{cases} E(\log \rho_2^*(S)/\log \nu_{P_s}) & \text{if } \rho_2^*(S) \neq 0 \\ \infty & \text{if } \rho_2^*(S) = 0. \end{cases}$$

1.3.

LEMMA 1.1. *There exists a C^∞ local chart (x, y) at R, and positive constants k and K, such that if $v(S) = (v_1(S), \dots, v_n(S)) \in T_{S,\tilde{x}}$ satisfies $\sum_{i=1}^n v_i^2(S) = 1$, then $|v_1(S)| \leq K\rho^2(S)$, $|v_i(S)| \leq K\rho(S)$ for $i = 2, \dots, u$, and $k < \sum_{i=u+1}^n |v_i(S)|$. Also, if $w(S) = (w_1(S), \dots, w_n(S)) \in T_{S,y^*}$ is a unit vector, then $k < \sum_{i=1}^u |w_i(S)|$, $|v_{u+1}(S)| \leq K\rho^2(S)$ and $|v_{u+i}(S)| \leq K\rho(S)$ for $i = 2, \dots, s$.*

Proof. Let $\xi = \tilde{\alpha}(\tilde{x}, \tilde{y})$ and $\eta = \tilde{\beta}(\tilde{x}, \tilde{y})$ be the equations of the change of coordinates between the $(\tilde{x}, \tilde{y})$ and (ξ, η) systems of coordinates; we know that $\tilde{\alpha}(0, \tilde{y}) = 0$, $\tilde{\beta}(0, \tilde{r}) = 0$ and that $D_{\tilde{x}}\tilde{\alpha}(0, \tilde{r})$ and $D_{\tilde{y}}\tilde{\beta}(0, \tilde{r})$ are invertible. The unit vector (with the usual inner product in $(\tilde{x}, \tilde{y})$) of $T_{S,\tilde{x}}$ with components $(0, \tilde{\delta}_b(S))$ in the $(\tilde{x}, \tilde{y})$ system has coordinates, in the (x, y) system,

$$v(S) = \begin{pmatrix} v_1(S) \\ \vdots \\ v_n(S) \end{pmatrix} = \begin{pmatrix} (D_\xi \Gamma D_{\tilde{y}}\tilde{\alpha} + D_\eta \Gamma D_{\tilde{y}}\tilde{\beta})\tilde{\delta}_b(S) \\ \\ (D_\xi \Delta D_{\tilde{y}}\tilde{\alpha} + D_\eta \Delta D_{\tilde{y}}\tilde{\beta})\tilde{\delta}_b(S) \end{pmatrix}.$$

At $S = R$ we obtain the vector $(0, D_\eta\Delta(0, 0)D_{\tilde{y}}\tilde{\beta}(0, \tilde{r})\tilde{\delta}_b(R))$ where $D_\eta\Delta(0, 0)D_{\tilde{y}}$ $\tilde{\beta}(0, \tilde{r})\tilde{\delta}_b(R)$ has a norm in $\mathbb{R}^s$ bounded away from zero when the vector $(0, \tilde{\delta}_b(R)) \in T_{R,\tilde{x}}$ varies, because the matrices $D_\eta\Delta(0, 0)$ and $D_{\tilde{y}}\tilde{\beta}(0, \tilde{r})$ are invertible. Therefore, the vector formed by the last s components of $v(S)$ when we take $(0, \tilde{\delta}_b(S)) \in T_{S,\tilde{x}}$, with S in a neighborhood of R, has the same property.

A generic term of the first row of $(D_\xi \Gamma D_{\tilde{y}}\tilde{\alpha} + D_\eta \Gamma D_{\tilde{y}}\tilde{\beta})\tilde{\delta}_b(S)$ is

$$\begin{aligned} a_{1,j}(\tilde{x}, \tilde{y}) &= \sum_{\tau=1}^u D_{\xi_\tau}\Gamma_1(\tilde{\alpha}(\tilde{x}, \tilde{y}), \tilde{\beta}(\tilde{x}, \tilde{y}))D_{\tilde{y}_j}\tilde{\alpha}_\tau(\tilde{x}, \tilde{y}) \\ &\quad + \sum_{l=1}^s D_{\eta_l}\Gamma_1(\tilde{\alpha}(\tilde{x}, \tilde{y}), \tilde{\beta}(\tilde{x}, \tilde{y}))D_{\tilde{y}_j}\tilde{\beta}_l(\tilde{x}, \tilde{y}), \end{aligned}$$

$j = 1, \dots, s$, where Γ_i stands for the i-component of $\Gamma(\xi, \eta)$, and a similar notation is used for the other functions. If we differentiate and evaluate in $(\tilde{x}, \tilde{y}) = (0, \tilde{r})$, we deduce that $D_{y_i}a_{1,j}(0, \tilde{r}) = 0$ for $i = 1, \dots, s$, $j = 1, \dots, s$, and also that if we impose derivative zero with respect to x_i, the function $\Gamma(\xi, \eta)$ has to satisfy, for $i = 1, \dots, u$, $j = 1, \dots, s$:

$$\sum_{\tau=1}^u \left[D_{\xi_\tau}\Gamma_1(0, 0)D^2_{\tilde{x}_i,\tilde{y}_j}\tilde{\alpha}_\tau(0, \tilde{r}) + D_{\tilde{x}_i}\tilde{\alpha}_\tau(0, \tilde{r}) \sum_{l=1}^s D^2_{\xi_\tau,\eta_l}\Gamma_1(0, 0)D_{\tilde{y}_j}\tilde{\beta}_l(0, \tilde{r}) \right] = 0. \quad (7)$$

We will see later that the above equations always have solutions. So we can write $|a_{1,j}(S)| \leq K\rho^2(S)$ for $j = 1, \dots, s$, and S near R.

As the terms (i, j), $i = 2, \dots, u$, $j = 1, \dots, s$, of $(D_\xi \Gamma D_{\tilde{y}} \alpha + D_\eta \Gamma D_{\tilde{y}} \tilde{\beta})(R)$ are zero, we have $|a_{i,j}(S)| \leq K\rho(S)$ for $i = 2, \dots, u$, $j = 1, \dots, s$.

These arguments prove the lemma if we take unitary vectors with the usual inner product in $(\tilde{x}, \tilde{y})$, and therefore with the usual inner product in (x, y). The same arguments prove the lemma for T_{S,y^*}, if we impose the conditions

$$\sum_{k=1}^{s} \left[D_{\eta_k} \Delta_1(0,0) D^2_{x_i^*, y_j^*} \beta_k^*(r^*, 0) + D_{y_j^*} \beta_k^*(r^*, 0) \sum_{l=1}^{u} D^2_{\xi_l, \eta_k} \Delta_1(0,0) D_{x_i^*} \alpha_l^*(r^*, 0) \right] = 0 \tag{8}$$

$i = 1, \dots, u$, $j = 1, \dots, s$, where $\xi = \alpha^*(x^*, y^*)$, $\eta = \beta^*(x^*, y^*)$ are the equations of the change of coordinates.

If we take

$$\begin{aligned} \Gamma_1(\xi, \eta) &= \xi_1 + \sum_{i=1}^{u} \sum_{j=1}^{s} s_{i,j} \xi_i \eta_j \\ \Gamma_i(\xi, \eta) &= \xi_i \quad \text{for } i = 1, \dots, u \\ \Delta_1(\xi, \eta) &= \eta_1 + \sum_{i=1}^{u} \sum_{j=1}^{s} t_{i,j} \xi_i \eta_j \\ \Delta_j(\xi, \eta) &= \eta_j \quad \text{for } j = 1, \dots, s \end{aligned}$$

it is easy to see that conditions (3) and (4) are verified, and that $s_{i,j}$ and $t_{i,j}$ can be taken so that (7) and (8) are verified. The transformation $x = \Gamma(\xi, \eta)$, $y = \Delta(\xi, \eta)$ is invertible in a neighborhood of $(0, 0)$. Then, it is enough to take $\mathcal{E}$ sufficiently small such that it falls in the region where the transformation is invertible. □

Remark 1.2. The admissible (x, y) local charts of Lemma 1.1 are independent of the $(\tilde{x}, \tilde{y})$ and (x^*, y^*) local charts.

Proof of the remark. Let us suppose that we start with a linearizing system of coordinates at Q denoted $(\hat{x}, \hat{y})$ which is different from the $(\tilde{x}, \tilde{y})$ system. Let $\hat{v}(S) = (\hat{\delta}_1(S), \dots, \hat{\delta}_n(S)) = (\hat{\delta}_a(S), \hat{\delta}_b(S)) \in \mathbb{R}^u \times \mathbb{R}^s$ be the components of a vector of $T_{S,\hat{x}}$ in the $(\tilde{x}, \tilde{y})$ system of coordinates with $\sum_{i=1}^n \hat{\delta}_i^2(S) = 1$.

We first claim that $|\hat{\delta}_i(S)| \leq K\tilde{\rho}^2(S)$ for $i = 1, \dots, u$. If $\tilde{\rho}_1(S)$ is small enough, after m iterates (see §1.2) the tangent of the angle (in the usual metric of the $(\tilde{x}, \tilde{y})$ system) between $\hat{v}(S)$ and $T_{S,\tilde{x}}$ is bounded by a constant $K^2 < \infty$. So, we can write

$$\zeta_{Q_u}^{2m} \sum_{i=1}^{u} \hat{\delta}_i^2(S) \Bigg/ \left(\nu_{Q_0}^{2m} \sum_{i=1}^{s} \hat{\delta}_{i+u}^2(S) \right) < K^2.$$

As $\sum_{i=1}^s \hat{\delta}_{i+u}^2(S) \leq 1$, then for $i = 1, \dots, u$, $\zeta_{Q_u}^m |\hat{\delta}_i(S)| / \nu_{Q_0}^m < K$, that is,

$$|\hat{\delta}_i(S)| < K(\nu_{Q_0}/\zeta_{Q_u})^m \leq K(\nu_{Q_0}/\zeta_{Q_u})^{-\log \tilde{\rho}_1(S)/\log \zeta_{Q_0}} \leq K\tilde{\rho}_1^2(S).$$

As $\tilde{\rho}_1^2(S) = \sum_{i=1}^u \tilde{x}_i^2 \leq \sum_{i=1}^u \tilde{x}_i^2 + \sum_{j=1}^s \tilde{y}_j^2 \leq K\rho^2(S)$ the claim is proved.

Now, the vector $\hat{v}(S)$ has components

$$\begin{pmatrix} (D_\xi \Gamma D_{\tilde{x}} \tilde{\alpha} + D_\eta \Gamma D_{\tilde{x}} \tilde{\beta}) \hat{\delta}_a(S) \\ (D_\xi \Delta D_{\tilde{x}} \tilde{\alpha} + D_\eta \Delta D_{\tilde{x}} \tilde{\beta}) \hat{\delta}_a(S) \end{pmatrix} + \begin{pmatrix} (D_\xi \Gamma D_{\tilde{y}} \tilde{\alpha} + D_\eta \Gamma D_{\tilde{y}} \tilde{\beta}) \hat{\delta}_b(S) \\ (D_\xi \Delta D_{\tilde{y}} \tilde{\alpha} + D_\eta \Delta D_{\tilde{y}} \tilde{\beta}) \hat{\delta}_b(S) \end{pmatrix}$$

in the (x, y) system of coordinates. We observe that the difference with the vector $v(S)$ obtained in Lemma 1.1 is only in the first of the two terms. The claim shows that $\|\hat{\delta}_a(S)\| \leq K\rho^2(S)$, and then it can be neglected in terms of our hypothesis. $\square$

1.4. We will show that the angle between $D\theta_t T_{S,y^*}$ and $T_{\theta_t(S),\tilde{x}}$ has a quadratic variation with the distance to R (see Lemma 1.4). Here we study two terms of the first component of $D\theta_t T_{S,y^*}$.

We work in the (x, y) system of coordinates. We start with the point

$$S = (X_1, \ldots, X_u, Y_1, \ldots, Y_s) = (X, Y).$$

Let us take a unit vector (with the metric of (x, y)) $w(S) = (w_1(S), \ldots, w_n(S)) \in T_{S,y^*}$. According to the definition of θ_t,

$$\begin{aligned} \theta_t(S) \;=\; & (X_1 \cos t\gamma(\rho) + Y_1 \sin t\gamma(\rho), X_2, \ldots, X_n, -X_1 \sin t\gamma(\rho) \\ & +Y_1 \cos t\gamma(\rho), Y_2, \ldots, Y_s) \end{aligned}$$

and so

$$\begin{aligned} & D\theta_t w(S) \\ & = \Bigg(\left[\cos t\gamma(\rho) + X_1(Y_1 \cos t\gamma(\rho) - X_1 \sin t\gamma(\rho)) \frac{t D_\rho \gamma(\rho)}{\rho}\right] w_1 \\ & \quad + \left[\sin t\gamma(\rho) + Y_1(Y_1 \cos t\gamma(\rho) - X_1 \sin t\gamma(\rho)) \frac{t D_\rho \gamma(\rho)}{\rho}\right] w_{u+1} \\ & \quad +(-X_1 \sin t\gamma(\rho) + Y_1 \cos t\gamma(\rho)) \left(\sum_{i=2}^{u} X_i w_i + \sum_{j=2}^{s} Y_j w_{u+j}\right) \frac{t D_\rho \gamma(\rho)}{\rho}, w_2, \ldots, w_u, \\ & \quad \left[-\sin t\gamma(\rho) - X_1(X_1 \cos t\gamma(\rho) + Y_1 \sin t\gamma(\rho)) \frac{t D_\rho \gamma(\rho)}{\rho}\right] w_1 \\ & \quad + \left[\cos t\gamma(\rho) - Y_1(X_1 \cos t\gamma(\rho) + Y_1 \sin t\gamma(\rho)) \frac{t D_\rho \gamma(\rho)}{\rho}\right] w_{u+1} \\ & \quad +(-X_1 \cos t\gamma(\rho) - Y_1 \sin t\gamma(\rho)) \left(\sum_{i=2}^{u} X_i w_i + \sum_{j=2}^{s} Y_j w_{u+j}\right) \frac{t D_\rho \gamma(\rho)}{\rho}, w_{u+2}, \ldots, w_n \Bigg) \end{aligned}$$

where everything is evaluated at S.

The strategy is to take the following term of the first component of $D\theta_t w(S)$ to be large:

$$\cos t\gamma(\rho) + X_1(Y_1 \cos t\gamma(\rho) - X_1 \sin t\gamma(\rho)) \frac{t D_\rho \gamma(\rho)}{\rho}.$$

This expression should be larger than $\hat{m}(1 - t + ta\rho^2)$, with $\hat{m}$ a constant, and a as large as required. Indeed, this cannot be true for $\gamma(\rho)$ fixed, so we consider γ depending also on a. We use the same letter γ to denote the new function depending not only on ρ, but also on a.

LEMMA 1.3. *For all a there exist $\gamma : [0, \infty) \mapsto \mathbb{R}, \gamma \in C^r$; $\hat{m} > 0$, $\hat{m}' > 0$ and $\epsilon = \epsilon(a)$ such that $\gamma(0) = \pi/2; 0 \leq \gamma(a\rho^2) < \pi/2$ for $0 < \rho < \epsilon(a)$; $\gamma(a\rho^2) = 0$ for*

$\rho \in [\epsilon(a), \infty)$; $\epsilon(a)$ goes to zero when a goes to infinity, and

$$\cos t\gamma(a\rho^2) + X_1(Y_1 \cos t\gamma(a\rho^2) - X_1 \sin t\gamma(a\rho^2)) \frac{t D_\rho \gamma(a\rho^2)}{\rho} \geq \hat{m}(1 - t + ta\rho^2)$$

$$\sin t\gamma(a\rho^2) + Y_1(Y_1 \cos t\gamma(a\rho^2) - X_1 \sin t\gamma(a\rho^2)) \frac{t D_\rho \gamma(a\rho^2)}{\rho} \leq \hat{m}'$$

for $t \in [0, 1]$, $\rho \in [0, \epsilon(a)]$.

Proof. We take a C^∞ function $b : \mathbb{R} \mapsto \mathbb{R}$ such that $b(x) = 0$ for $x \leq 0$, $b(x) = 1$ for $x \geq 1$, $0 \leq Db(x) \leq 4/3$. Next, we define

$$\gamma(x) = \begin{cases} [1 - b((2x-1)/(\pi^8 - 1))] & \\ \quad \times[(\pi/2) - x + b(2x)(x - 2^{-7/8}x^{1/8})] & \text{for } x \in [0, \pi^8/2] \\ 0 & \text{for } x \in (\pi^8/2, +\infty). \end{cases}$$

We observe that this implies $\epsilon(a) = \pi^4/\sqrt{2a}$ (this shows that a is related to the size of $\mathcal{E}$). After some computations, it is not difficult to verify that $\gamma(x)$ so defined is suitable. $\square$

1.5. Here d denotes the distance in the manifold and $A(G_1, G_2)$ the angle between the fibre bundles $G_1, G_2 \in T_{\theta_t(S)}M$. The parameter τ, defined in the next lemma, is essential in our reasoning.

LEMMA 1.4. *We fix t_0. Then, we can find $h > 0$ so that, given $\tau > 0$, there exists $a_0 = a_0(\tau)$ such that for any $a > a_0$, $S \in \mathcal{E}$, and $t \in [t_0, 1]$,*

$$\tan A(D\theta_t T_{S,y^*}, T_{\theta_t(S),\tilde{x}}) \geq h(1 - t) + \tau d^2(S, R).$$

Proof. We prove the lemma working in the (x, y) system of coordinates, and taking the usual inner product in $\mathbb{R}^n$; the equivalence between the metrics produces the lemma. For simplicity, we denote with the same letter A the angle between vectors with this metric, and use the same letters h and τ.

We first observe that for $S \in \mathcal{E}$, $H(S) = \|D\theta_t w(S)\|$ is bounded and bounded away from zero as S, a, t and $w(S)$ the unit vector in T_{S,y^*} vary, and this follows because $\|D\theta_t\|$ and $\|D\theta_t^{-1}\|$ are bounded. So, we can find b, B such that $0 < b \leq H(S) \leq B$.

By contradiction, suppose that given $h = \min\{1/(2Bn), \hat{m}/(2B)\}$ there exists $\tau = \tau(h) > 0$ such that for any a_0 we can find $a > a_0$, $S_a \in \mathcal{E}$, $t_a \in [t_0, 1]$ and

$$v(\theta_{t_a}(S_a)) = (v_1(\theta_{t_a}(S_a)), \ldots, v_n(\theta_{t_a}(S_a))) \in T_{\theta_{t_a}(S_a),\tilde{x}}; \quad \sum_{i=1}^n v_i^2(\theta_{t_a}(S_a)) = 1$$

$$w(S_a) = (w_1(S_a), \ldots, w_n(S_a)) \in T_{S_a,y^*}; \quad \sum_{i=1}^n w_i^2(S_a) = 1$$

with

$$\tan A(D\theta_{t_a} w(S_a), v(\theta_{t_a}(S_a))) < h(1 - t_a) + \tau(h)\rho^2(S_a).$$

Now,

$$\tan A(D\theta_{t_a} w(S_a), v(\theta_{t_a}(S_a))) \geq \left\| \frac{1}{H(S_a)} D\theta_{t_a} w(S_a) - v(\theta_{t_a}(S_a)) \right\|$$

and this last term is larger or equal to the absolute value of any component of

$$\frac{1}{H(S_a)} D\theta_{t_a} w(S_a) - v(\theta_{t_a}(S_a)).$$

Taking into account the contradiction hypothesis, there exists a sequence $\{a_l\}_{l\in\mathbb{Z}^+}$ going to infinity such that

$$\left|\frac{1}{H(S_{a_l})} w_i(S_{a_l}) - v_i(\theta_{t_{a_l}}(S_{a_l}))\right| < h(1-t_{a_l}) + \tau\rho^2(S_{a_l}), \quad i = 2, \dots, u$$

and then (we recall Lemma 1.1)

$$|w_i(S_{a_l})| < B[K\rho(S_{a_l}) + h(1-t_{a_l}) + \tau\rho^2(S_{a_l})], \quad i = 2, \dots, u.$$

As $B^2h^2(1-t_{a_l})^2(u-1) < 1/4$, then

$$\begin{aligned} w_1^2(S_{a_l}) &> 1-(u-1)B^2[K\rho(S_{a_l}) + h(1-t_{a_l}) + \tau\rho^2(S_{a_l})]^2 - sK\rho^2(S_{a_l}) \\ &> (3/4) - K\rho(S_{a_l}) > 1/4. \end{aligned}$$

These inequalities are valid taking a_l large enough because $\rho \leq \sqrt{\pi^8/(2a)}$. Then, $|w_1(S_{a_l})| > 1/2$. The absolute value of the first component of $(D\theta_{t_{a_l}} w(S_{a_l})/H(S_{a_l})) - v(\theta_{t_{a_l}}(S_{a_l}))$ is therefore larger than $\hat{m}(1-t_{a_l})/(2B) + (mt_{a_l}a_l/(2B) - \tilde{K})\rho^2(S_{a_l})$, where $\tilde{K}$ is a constant. Now we fix $a_l > 2(\tau + \tilde{K})B/(\hat{m}t_0)$. We deduce that the absolute value of the first component of $(D\theta_{t_{a_l}} w(S_{a_l})/H(S_{a_l}) - v(\theta_{t_{a_l}}(S_{a_l}))$ is larger than $h(1-t_{a_l}) + \tau\rho^2(S_{a_l})$ and this is a contradiction. □

We remark that if we increase τ, a_0 also increases, and therefore $\mathcal{E}$ decreases.

1.6. In this subsection we construct the manifold $\mathcal{M}$. We are interested in finding the conditions that we have to impose in order that we can write (for F such that $T_{R_F}W^s(Q_F, F)$ and $T_{R_F}W^u(P_F, F)$ intersect in a one-dimensional vector space)

$$\tan A(T_{S,y_F^*}, T_{S,\tilde{x}_F}) \geq \tau d^2(S, R_F) \tag{9}$$

for S near R_F and for τ (which will be fixed later) large enough (see Lemma 1.4). Let us suppose that inequality (9) is verified, then we can define an isotopy $F_t, t \in [0, 1]$ with $F_0 = f, F_1 = F$, and such that for $t < 1$, S near R_F, $\tan A(T_{S,y^*_{F_t}}, T_{S,\tilde{x}_{F_t}}) \geq h_{F_t} + \tau d^2(S, R_F)$. If F is near enough to f_1, then it will be seen that this condition is enough to prove that F_t is Anosov for $t < 1$ (see §2). Condition (9) will allow us to prove points (b) and (c) of our theorems, which will be done in §§2–4.

Now, we examine condition (9). First, let us take R_F as an intersection point of $W^u(P_F, F)$ and $W^s(Q_F, F)$ near to R, with $R_F \to R$ when $F \to f_1$, and let us impose that the intersection of $W^u(P_F, F)$ and $W^s(Q_F, F)$ is not transversal (in the differentiable sense). Therefore, we need to impose one condition, namely, that the volume of the parallelepiped spanned by a basis of $T_{R_F}W^u(P_F, F)$ and a basis of $T_{R_F}W^s(Q_F, F)$ is equal to zero. This implies that there exists a unit vector $e_{R_F,u+1} \in T_{R_F}W^u(P_F, F) \cap T_{R_F}W^s(Q_F, F)$. Therefore, there exists a unit vector $e_{R_F,1}$ which is not in $T_{R_F}W^u(P_F, F) \oplus T_{R_F}W^s(Q_F, F)$. We take $e_{R_F,1}$ such that it converges to the unit vector in $T_R M$ that defines the direction of Ox_1 at R when F goes to f_1. Let us take a local chart at R_F, $(x_{F,1}, \dots, x_{F,u}, y_{F,1}, \dots, y_{F,s}) = (x_F, y_F) \in \mathbb{R}^u \times \mathbb{R}^s$, with the coordinates of R_F equal to $(0,0)$, with $e_{R_F,1}$ in the direction corresponding to $Ox_{F,1}$ at O and $e_{R_F,u+1}$ in the direction corresponding to $Oy_{F,1}$ at O. Let us denote by $w_F(S) = (w_{F,1}(S), \dots, w_{F,n}(S))$ the components in the (x_F, y_F) system of coordinates of a generic unit vector (with the usual metric in $\mathbb{R}^n$) of T_{S,y_F^*}, and by $v_F(S) = (v_{F,1}(S), \dots, v_{F,n}(S))$ a generic unit vector of $T_{S,\tilde{x}_F}$. Let us denote by $p_1(S)$ the projection on the axis $Ox_{F,1}$ of the vector $w_F(S) - v_F(S)$ with minimum norm. We have $p_1(R_F) = 0$. Taking into account the fact that T_{S,y_F^*} and $T_{S,\tilde{x}_F}$ vary in a C^1 way with S, we can suppose that $p_1(S)$ is C^1 for S near R_F. We demand that

$$\left.\frac{\partial p_1(S)}{\partial x_{F,i}}\right|_{S=R_F} = 0 \quad \text{for } i = 1, \dots, u, \qquad \left.\frac{\partial p_1(S)}{\partial y_{F,j}}\right|_{S=R_F} = 0 \quad \text{for } j = 1, \dots, s.$$

These are n more conditions, which we will see do not depend on the choice of the local charts (so, we have $n+1$ conditions if we take into account the tangency one). In particular, they imply that the intersection between $W^u(P_F, F)$ and $W^s(Q_F, F)$ has to be at least cubic (i.e. there exists $v \neq 0$ such that $R^n = v \oplus T_{R_F} W^u(P_F, F) \oplus T_{R_F} W^s(Q_F, F)$ and the plane determined by R_F, v and $T_{R_F} W^u(P_F, F) \cap T_{R_F} W^s(Q_F, F)$ intersects $W^u(P_F, F)$ and $W^s(Q_F, F)$ in two curves with cubic intersection). It is cubic because the second-order terms are not zero for f, and because we can take the local charts varying C^r continuously at f_1. Therefore, there exists only one point of intersection of $W^u(P_F, F)$ and $W^u(Q_F, F)$, R_F, near R. For the diffeomorphisms verifying the n conditions the vector $e_{R_F,u+1}$ can be chosen such that it converges to the direction of Oy_1 at R when F goes to f_1, and so we can take the local chart at R_F varying continuously at $F = f_1$. In fact, we can take the local chart such that the manifold given by the equations $y_F = 0, x_{F,1} = 0$ is in $W^u(P_F, F)$. Reasoning as in Remark 1.2 shows that these conditions do not depend on the $(\tilde{x}_F, \tilde{y}_F)$ and (x_F^*, y_F^*) system of coordinates; a modification on these systems represents a variation of second order in the components of $w_F(S)$ and $v_F(S)$. They also do not depend on the (x_F, y_F) system of coordinates; we are demanding that $p_1(S)$ (equal to 0 for $S = R_F$) has an extreme near R_F. Taking into account the fact that the derivatives of first order are zero, we deduce that $\tan A(T_{S,y_F^*}, T_{S,\tilde{x}_F})$ varies, to a first approximation, in a quadratic way with S. As for $F = f_1$, this quadratic form is larger than $\tau d^2(S, R)$, and by the continuous variation of $\tan A(T_{S,y_F^*}, T_{S,\tilde{x}_F})$ with F at f_1 we deduce inequality (9). □

2. *F_t is Anosov for $t \in [0,1)$*

Now we begin to prove that the ideas developed at the beginning of §1 actually work. In this section we prove that, for θ_t as constructed in §1 and $t < 1$, F_t is Anosov by defining a suitable system of cones. In fact, we will prove in §3 that these cones persist for $t = 1$, so we are interested in the case $t = 1$ for some of the results of this section.

2.1. We take F_t near f_t such that in the same set $\mathcal{E}$ defined for f_1 condition (9) is verified (we will impose more conditions on the size of $\mathcal{E}$). We also suppose that the values ζ_{P_u}, ν_{P_0}, ν_{P_s}, ζ_{Q_0}, ζ_{Q_u}, ν_{Q_0} verify conditions (5) and (6) for the eigenvalues of

P_F and Q_F. Later we will impose more conditions on F_t. Here we find the stable and unstable vector bundle in $\mathcal{E}$ through two families of cones around T_{S,y^*} and $T_{S,\tilde{x}}$ (see Definition 2.1). In fact, if $t \in [0, t_0)$ for some $t_0 > 0$, there is nothing to prove because the set of Anosov diffeomorphims is open in the C^r topology. Then, we take $t \in [t_0, 1]$. We begin with some definitions.

Definition 2.1. Given $S \in M$ and a direct sum decomposition $T_S M = E_1 \oplus E_2$, we define the *ϵ-cone about E_1 by E_2* to be $C(S, \epsilon, E_1) = \{(v_1, v_2) \in E_1 \oplus E_2 : \|v_2\| \leq \epsilon \|v_1\|\}$.

Definition 2.2. An *almost hyperbolic splitting for $g : M \mapsto M$ on M* is a (not necessarily continuous) splitting $T_S M = E_{1,S} \oplus E_{2,S}$ for $S \in M$ such that $\dim E_{1,S}$ and $\dim E_{2,S}$ are constant and there exist bounded positive nowhere zero functions ϵ_1, ϵ_2 on M, a constant $\sigma > 1$ and $q \in \mathbb{Z}^+$ such that
(i)

$$Dg^q(C(S, \epsilon_2(S), E_{2,S})) \subset C(g^q(S), \epsilon_2(g^q(S)), E_{2,g^q(S)})$$
$$Dg^{-q}(C(S, \epsilon_1(S), E_{1,S})) \subset C(g^{-q}(S), \epsilon_1(g^{-q}(S)), E_{1,g^{-q}(S)})$$

(ii)

$$\|Dg^q(v)\| \geq \sigma \|v\| \text{ for } S \in M, v \in C(S, \epsilon_2(S), E_{2,S})$$
$$\|Dg^{-q}(v)\| \geq \sigma \|v\| \text{ for } S \in M, v \in C(S, \epsilon_1(S), E_{1,S}).$$

Then, the following theorem follows.

THEOREM 2.3. ([**NP73**, Theorem 3.1]) *g has an almost hyperbolic splitting if and only if g is Anosov.*

In fact, the theorem proved in [**NP73**] is stronger (it applies to prove hyperbolicity), but we reformulate it in the way we need here. (For the proof, see [**NP73**], also 15.1 of [**R89**], and Lemmas 3.1 and 4.3). Moreover, we can assert that there exists a suitable metric such that we can take $q = 1$. We suppose that we work with such a metric for the diffeomorphism f, and $\epsilon_1(S) = \epsilon_2(S) = 1/4$. We consider two continuous families of cones $C(S, \epsilon_1(S), (E_{1,S}))$ and $C(S, \epsilon_2(S), (E_{2,S}))$ for f about the stable by the unstable vector bundle and about the unstable by the stable vector bundle, respectively. We denote these families by $C_{1,S}$ and $C_{2,S}$, respectively, given by understanding the splitting and the functions $\epsilon_i(S)$. We prove the following.

PROPOSITION 2.4. *F_t has an almost hyperbolic splitting for $t \in [0, 1)$.*

We develop the proof in the remainder of the section. We only construct $C(S, \epsilon_2(S), E_{2,S})$ (depending also on t) for F_t, and prove the properties of Definition 2.2(i) and (ii) observing that q can be taken as large and σ as near to 1 as we want. The construction of $C(S, \epsilon_1(S), E_{1,S})$ and the proof of properties (i) and (ii) is similar.

2.2.

LEMMA 2.5. *There exist two families of cones (depending on t) which verify condition (i) of Definition 2.2 for $F_t, t \in [0,1)$ and $S \in M$; similarly for F_1 and $S \in M \setminus \{F_1^i(R) : i \in \mathbb{Z}\}$.*

Proof. We denote the systems of cones that we are going to construct as $C'_{1,S}$ and $C'_{2,S}$. First, we define $C'_{2,S}$ for points of $\mathcal{E}$. In the (x, y) system of coordinates, we take $E_{1,S} = T_{S,\tilde{x}_{F_t}}$, $E_{2,S} = T_{S,y^*_{F_t}}$, $\epsilon_1(S) = \epsilon_2(S) = 1/4$. This construction does not work for $S = R_{F_1}$ and $t = 1$ because $E_{1,R_{F_1}}$ and $E_{2,R_{F_1}}$ have a one-dimensional space in common. Next we work in the $(\tilde{x}, \tilde{y})$ system of coordinates (we recall the notation of §1.2). By the definition of $C'_{2,S}$ for $S \in \mathcal{E}$, the tangent of the angles between the vectors of $C'_{2,S}$ and the vectors of $T_{S,\tilde{x}_{F_t}}$ are larger than $\tilde{h} + \tilde{\tau}\tilde{\rho}^2(S)$, where $\tilde{h}$ depends on h_{F_t} and on the change of coordinates, and $\tilde{\tau}$, which depends on τ and on the change of coordinates, can be taken as large as wanted. We can take m_0 large enough depending on the Jordan form of $DF_t(Q)$ and on ζ_{Q_u}, ν_{Q_0} such that the angle of the vectors of $DF_t^m(C'_{2,S})$ $(S \in \mathcal{E})$ with $T_{F_t^m(S),\tilde{x}_{F_t}}$, for $m > m_0$, has tangent larger than

$$(\zeta_{Q_u}/\nu_{Q_0})^m(\tilde{h} + \tilde{\tau}\tilde{\rho}^2(S))$$

(while $F_t^m(S) \in \tilde{\mathcal{U}}_1$). Then, we increase τ so that for every $S \in \mathcal{E}$, we have $m(S) > m_0$ (we recall that if τ increases, then the diameter of $\mathcal{E}$ decreases). First, we suppose $\tilde{\rho}_1(S) \neq 0$ (i.e. $S \notin W^s(Q, F_t)$). Then,

$$(\zeta_{Q_u}/\nu_{Q_0})^m(\tilde{h} + \tilde{\tau}\tilde{\rho}^2(S)) \geq (\tilde{\rho}_1(S))^{(\log\nu_{Q_0} - \log\zeta_{Q_u})/\log\zeta_{Q_0}}(\tilde{h} + \tilde{\tau}\tilde{\rho}^2(S))\tilde{\tau}.$$

By the continuity of the family $C_{2,S}$, the fact that $T_{S,\tilde{y}_{F_t}}$ varies continuously with S, and because $T_{S,\tilde{y}_{F_t}}$ is equal to the unstable fibre if S belongs to $\tilde{y} = 0$, we deduce that $T_{S,\tilde{y}}$ is in the interior of $C_{2,S}$ if S is near to $\tilde{y} = 0$. We take $\mathcal{E}$ small enough (independent of F_t) such that the iterate that is leaving $\tilde{\mathcal{U}}_1$ of any point of $\mathcal{E}$ fulfills the former property. Then, we take $\tilde{\tau}$ large enough such that, denoting by i the iterate just leaving $\tilde{\mathcal{U}}_1$, $DF_t^i(C'_{2,S})$ is into $C_{2,F_t^i(S)}$. This is done independently of $F_1 \in \mathcal{M}$ if $\mathcal{M}$ is sufficiently small because the values of $\tilde{\tau}$ and the eigenvalues at P and Q vary continuously. Now, we define the new family of cones in the positive iterates of points of $\mathcal{E}$ in $\tilde{\mathcal{U}}_1$ such that $C'_{2,F_t^j(S)} = DF_t^j(C'_{2,S})$ for $S \in \mathcal{E}$, if $DF_t^j(C'_{2,S}) \not\subset C_{2,F_t^j(S)}$. If $DF_t^j(C'_{2,S}) \subset C_{2,F_t^j(S)}$, then we define $C'_{2,F_t^j(S)} = C_{2,F_t^j(S)}$ until $j = i$.

Analogously, we suppose now that $\tilde{\rho}_1(S) = 0$ (that is, $S \in W^s(Q, F_t)$). In such a case, the iterates by F_t of S also approach $\tilde{y} = 0$ (in fact, to Q), and, if either $t \neq 1$ or $\tilde{\rho}^2(S) \neq 0$ (or both), then

$$(\zeta_{Q_u}/\nu_{Q_0})^i(\tilde{h} + \tilde{\tau}\tilde{\rho}^2(S))$$

grows without bound as i grows. The conclusion is that, except for $t = 1$ and $\tilde{\rho}(S) = 0$ (that is, $S = R$), there exists $j_0 < \infty$ such that $DF_t^{j_0}(C'_{2,S}) \subset C_{2,F_t^{j_0}(S)}$ and $F_t^{j_0}(S) \in \tilde{\mathcal{U}}_1$. We define $C'_{2,F_t^j(S)} = DF_t^j(C'_{2,S})$ for $S \in \mathcal{E}$, if $DF_t^j(C'_{2,S}) \not\subset C_{2,F_t^j(S)}$; if $DF_t^j(C'_{2,S}) \subset C_{2,F_t^j(S)}$, then we define $C'_{2,F_t^j(S)} = C_{2,F_t^j(S)}$.

A similar reasoning for the negative iterates of $\mathcal{E}$ in $\mathcal{U}_1^*$ allows us to construct $C'_{2,S}$ for those iterates. To end the construction of $C'_{2,S}$, we take $C'_{2,S} = C_{2,S}$ for the points in which we have not yet defined $C'_{2,S}$. □

2.3. Now we begin the proof of condition (ii) of Definition 2.2. This proof ends in §2.4. In this subsection we continue considering t fixed in $[t_0, 1]$. Let us take the set of points $S \in M$ such that $S \notin \mathcal{U}_1^*$; $F_t(S) \in \mathcal{U}_1^*$; there exists $j = j(S)$ such that $F_t^j(S) \in \mathcal{E}$, and there exists $k^+ = k^+(S)$ such that $F_t^{k^+}(S) \in \tilde{\mathcal{U}}_1$; $F_t^{k^++1}(S) \notin \tilde{\mathcal{U}}_1$, with $F_t^i(S) \in \mathcal{U}_\infty^*$ for $i = 1, \dots, j$; $F_t^i(S) \in \tilde{\mathcal{U}}_\infty$ for $i = j, \dots, k^+$. For S as before, we denote by $\mathcal{D}$ the set of points $\{F_t^i(S) : i = 1, \dots, k^+(S)\}$, and $\mathcal{H} = M \setminus \mathcal{D}$.

LEMMA 2.6. *Let $T \in \mathcal{H}$, $F_t(T) \in \mathcal{D}$, $v \in C'_{2,T}$. Then, $\|DF_t^{k^+(T)}(T)v\| \geq 2\|v\|$.*

Proof. First we work in the $(\tilde{x}, \tilde{y})$ system of coordinates, with $t \in [t_0, 1]$. Let us take a vector $v \in C'_{2,S}$, $S \in \mathcal{E}$. We want to bound from below the expansion of this vector at leaving $\tilde{\mathcal{U}}_1$, we study this expansion with the usual inner product in (x, y). The norm of the component of v in the direction $T_{S,\tilde{x}_{F_t}}$ is larger than

$$\|v\| \sin\{\arctan[\tilde{h} + \tilde{\tau}\tilde{\rho}^2(S)]\} \geq \|v\| \frac{\tilde{\tau}\tilde{\rho}^2(S)}{\sqrt{1 + [\tilde{\tau}\tilde{\rho}^2(S)]^2}}.$$

We suppose that $\tilde{\rho}_1(S) \neq 0$. The number of iterates needed to leave $\tilde{\mathcal{U}}_1$ is larger than m, and in each iterate the former component of v increases with a larger than ζ_{Q_u} rate if m is large. Then, the expansion of any vector of $C'_{2,S}$ on leaving $\tilde{\mathcal{U}}_1$ is larger than

$$\zeta_{Q_u}^m \frac{\tilde{\tau}\tilde{\rho}^2(S)}{\sqrt{1 + [\tilde{\tau}\tilde{\rho}^2(S)]^2}} \geq (\tilde{\rho}_1(S))^{-\log\zeta_{Q_u}/\log\zeta_{Q_0}} \frac{\tilde{\tau}\tilde{\rho}^2(S)}{\sqrt{1 + [\tilde{\tau}\tilde{\rho}^2(S)]^2}}.$$

Always with $t \in [t_0, 1]$, we study the situation with the negative iterates of points of $\mathcal{E}$. Then, we work in the (x^*, y^*) system of coordinates. Let us take a point $S \in \mathcal{E}$, and let us consider the largest $i > 0$ such that $F_t^{-j}(S) \in \mathcal{U}_1^*$ for $j = 1, \dots, i$ (we suppose that $\rho_2^*(S) \neq 0$). We have

$$i \geq p = E(\log\rho_2^*(S)/\log\nu_{P_s}).$$

Let us consider the cone $C'_{2,F_t^{-i}(S)} = C_{2,F_t^{-i}(S)}$. The vectors in this cone determine an angle with $T_{F_t^{-i}(S),y^*_{F_t}}$ smaller than a number γ independent of the point. Then, a vector $v \in C'_{2,F_t^{-i}(S)}$ will expand until arriving at S with a factor larger than

$$\zeta_{P_u}^p \cos\gamma \geq K(\rho_2^*(S))^{\log\zeta_{P_u}/\log\nu_{P_s}}$$

where in a similar way as before we have considered only the component in $T_{F_t^{-i}(S),y^*_{F_t}}$.

We are ready to study the situation in which we iterate by F_t, $t \in [t_0, 1]$, a point $T \in \mathcal{H}$ that enters $\mathcal{D}$. We take a vector in $C'_{2,T}$. The norm of this vector (in the Riemannian metric in the manifold) will be increased, after leaving $\mathcal{D}$, by a factor larger than

$$K(\rho_2^*(S))^{\log\zeta_{P_u}/\log\nu_{P_s}}(\tilde{\rho}_1(S))^{-\log\zeta_{Q_u}/\log\zeta_{Q_0}} \frac{\tilde{\tau}\tilde{\rho}^2(S)}{\sqrt{1 + [\tilde{\tau}\tilde{\rho}^2(S)]^2}} \tag{10}$$

where S is the iterate of T that belongs to $\mathcal{E}$, and K takes into account the different changes of coordinates. We can write that

$$(\rho_2^*(S))^{\log\zeta_{P_u}/\log\nu_{P_s}}(\tilde{\rho}_1(S))^{-\log\zeta_{Q_u}/\log\zeta_{Q_0}} \geq K(\tilde{\rho}(S))^{(\log\zeta_{P_u}/\log\nu_{P_s})-(\log\zeta_{Q_u}/\log\zeta_{Q_0})} \geq K\tilde{\rho}^{-2}(S).$$

Thus, the norm of any vector in $C'_{2,T}$ is increased with a factor $e(T)$ larger than

$$K\tilde{\tau}/\sqrt{1+[\tilde{\tau}\tilde{\rho}^2(S)]^2}.$$

As $\tilde{\rho}(S)$ goes to zero when $\tilde{\tau}$ goes to infinity, increasing $\tilde{\tau}$ if necessary, we can write that the last term of (10) is larger than 2. Then, after going through $\mathcal{D}$, the vectors of $C'_{2,T}$ expand by a factor of at least 2. Analogously, we find that the vectors of $C'_{1,S}$ expand by a factor of at least two in each passage through $\mathcal{D}$ by F_t^{-1}. □

2.4.

LEMMA 2.7. *For each $F_t, t \in [t_0, 1)$, there exist $q_0 = q_0(F_t) \in \mathbb{N}$ and $\sigma_0 = \sigma_0(F_t) > 1$ such that $\|DF_t^q v\| \geq \sigma\|v\|$ for $v \in C'_{2,S}, S \in M, q \geq q_0, \sigma_0 \geq \sigma > 1$.*

Proof. We know that there exists $\sigma_1 > 1$ such that for any $S \in M$ and $v \in C_{2,S}$, $\|Dfv\| \geq \sigma_1\|v\|$ (we suppose that we are working with an adapted metric for f). First, we prove that given $F_t, t \in [t_0, 1)$ and $S \in \mathcal{E}$, there exist N_1 and N_2, with $N_1 = N_1(F_t, S) \leq N_2 = N_2(F_t)$ such that the vectors of the cone $C'_{2,S}$ contract by a factor larger than $c = c(F_t) < 1$ in each iterate, during at most N_1 iterates, and then, until leaving $\tilde{\mathcal{U}}$, the vectors of the cone grow by a rate of at least σ_1 in each iterate. We observe that we have contraction only after passing through $\mathcal{E}$; in other cases the expansion has a factor of at least σ_1 in each iterate. We work in the $(\tilde{x}, \tilde{y})$ system of coordinates. For any system of cones C_S, we write

$$P(C_S) = \inf_{0\neq v\in C_S}\sqrt{\sum_{i=1}^{u} v_i^2 \Bigg/ \sum_{j=1}^{s} v_{u+j}^2},$$

where $(v_1, \ldots, v_{u+s})$ are the components of v in the $(\tilde{x}, \tilde{y})$ system of coordinates. After N iterates, N large enough (depending on the Jordan form of $DF_1(Q)$ and on ζ_{Q_u}, ν_{Q_0}), if $S \in \mathcal{E}$ we have

$$P(DF_t^N(C'_{2,S})) \geq (\zeta_{Q_u}/\nu_{Q_0})^N[\tilde{h} + \tilde{\tau}\tilde{\rho}(S)] \geq (\zeta_{Q_u}/\nu_{Q_0})^N\tilde{h}.$$

Then, as $t < 1$, we need a finite number $N_1 = N_1(F_t, S)$ of iterates for $C'_{2,S}$ to be in the system of cones $C_{2,S}$: we can suppose that $\tilde{\mathcal{U}}_1$ is such that if a vector in $T_S M \in \tilde{\mathcal{U}}_1$ has a tangent of the angle with $T_{S,\tilde{x}}$ larger than a number $P_0 < \infty$ then it belongs to $C_{2,S}$. Therefore,

$$N_2(F_t) = \max\{E(\log(P_0/\tilde{h})/\log(\zeta_{Q_u}/\nu_{Q_0})) + 1, N_0\}$$
$$N_1(F_t, S) = \min\{N_2(F_t), m(S)\}$$

where we suppose that the number of iterates for any point of $\mathcal{E}$ leaving $\tilde{\mathcal{U}}_1$ is larger than N_0. This is possible taking $\mathcal{E}$ small enough. The existence of $c(F_t)$ is then clear.

Now, given a point T as in §2.1, let us define $\sigma_2(T)$, the average expansion in each iterate by the passage through $\mathcal{D}$, as the $1/k^+(T)$ power of $e(T)$. We are interested in

$$\tilde{\sigma} = \inf_T\{\sigma_2(T), \sigma_1\}.$$

Let us prove that $\tilde{\sigma} > 1$. We know that $\sigma_1 > 1$, and that $\sigma_2(T) \geq 1$ because $e(T) > 2$. By contradiction, let us suppose that there is a sequence of points T_i such that $\sigma_2(T_i)$ goes to 1 when i goes to infinity. As $e(T_i) > 2$, it is necessary that $k^+(T_i)$ goes to infinity when i goes to infinity. If $j = j(T_i)$ is such that $F_t^j(T_i) \in \mathcal{E}$, we denote $N_3(T_i) = N_1(F_t, F_t^j(T_i))$, and then

$$\sigma_2(T_i) = e(T_i)^{1/k^+(T_i)} \geq [c^{N_3(T_i)}\sigma_1^{k^+(T_i)-N_3(T_i)}]^{1/k^+(T_i)} = \sigma_1(c/\sigma_1)^{N_3(T_i)/k^+(T_i)} \underset{i\to\infty}{\longrightarrow} \sigma_1$$

because $N_3(T_i)$ is bounded for F_t fixed. But $\sigma_1 > 1$, and this is a contradiction.

For fixed F_t, let us fix

$$q_0 > 2(\log(\tilde{\sigma}/c)/\log\tilde{\sigma})N_2(F_t)$$

and

$$\sigma_0 = \tilde{\sigma}^{q_0}(c/\tilde{\sigma})^{2N_2(F_t)} > \tilde{\sigma}^{q_0}(c/\tilde{\sigma})^{q_0\log\tilde{\sigma}/\log(\tilde{\sigma}/c)} = 1.$$

We must check that σ_0 and q_0 verify the lemma. Let us take $v \in C'_{2,S}, S \in M, q \geq q_0, \sigma_0 \geq \sigma > 1$, and let us estimate $\|DF_t^q(v)\|$. In the most unfavourable case, we have an initial and a final contraction by the passage through $\mathcal{E}$. If we have more passages through $\mathcal{E}$, first we have passages near P and later near Q, and therefore, the average expansion in each iterate is not smaller than $\tilde{\sigma}$. For the iterates neither corresponding to the initial and final contractions, nor passages through $\mathcal{D}$, the expansion in each iterate is not smaller than $\tilde{\sigma}$. Then

$$\|DF_t^q v\| \geq c^{2N_2(F_t)}\tilde{\sigma}^{q-2N_2(F_t)}\|v\| \geq (c/\tilde{\sigma})^{2N_2(F_t)}\tilde{\sigma}^{q_0}\|v\| = \sigma_0\|v\|,$$

proving the lemma. □

3. *Topological properties of F_1*

In this section, we show that the two systems of cones defined in §2 persist for $t = 1$, defining two F_1-invariant fibre bundles E_S and I_S (see Lemma 3.3), which will we integrate to obtain two continuous foliations W^s and W^u whose leaves are C^1 (see Lemma 3.12). We denote W^s the leaf of the foliation through S with $W^s(S)$ (and we distinguish it from the stable manifold through S denoted $W^s(S, F_1)$ in the case that it exists, in fact they coincide). Then, we prove that F_1 is expansive (see Proposition 3.13) and conjugate to f (Proposition 3.14). We remark that the two F_1-invariant fibre bundles are not continuous: I_S is not continuous at Q_{F_1}, and in the case of the hypothesis of our Theorem 2, the continuity of I_S is asserted only at $M \setminus W^u(Q_{F_1}, F_1)$.

3.1. We begin by stating a lemma which will be used to construct the invariant fibre bundles.

LEMMA 3.1. *Let $S \in M$ such that there exist $\nu > 1$, $N_1(F_1^i(S)) > 0$ and $N_2(F_1^i(S)) > 0$ for $i \in \mathbb{Z}$ with*

$$\|DF_1^{-N_1(F_1^i(S))}v\| \geq \nu\|v\| \quad \textit{for } v \in C'_{1,F_1^{i+N_1(F_1^i(S))}(S)}$$

$$\|DF_1^{N_1(F_1^i(S))}v\| \geq \nu\|v\| \quad \textit{for } v \in C'_{2,F_1^i(S)}$$
$$\|DF_1^{-N_2(F_1^i(S))}v\| \geq \nu\|v\| \quad \textit{for } v \in C'_{1,F_1^i(S)}$$
$$\|DF_1^{N_2(F_1^i(S))}v\| \geq \nu\|v\| \quad \textit{for } v \in C'_{2,F_1^{i-N_2(F_1^i(S))}(S)}.$$

Then, for each $j \in \mathbb{Z}$, there exist in $T_{F_1^j(S)}M$ two linear spaces given by

$$E_{F_1^j(S)} = \cap_{i=1}^{\infty} DF_1^{-i}(C'_{1,F_1^{j+i}(S)}) \quad \textit{and} \quad I_{F_1^j(S)} = \cap_{i=1}^{\infty} DF_1^{i}(C'_{2,F_1^{j-i}(S)}).$$

The dimension of the spaces are the same as those of the stable and unstable vector bundles corresponding to f, respectively.

The proof can be easily obtained, i.e. following the arguments in 15.1 of [**R89**]. See also Lemma 4.3.

LEMMA 3.2. *For $S \notin \{F_1^i(R) : i \in \mathbb{Z}\}$, $\cap_{i=1}^{\infty} DF_1^{-i}(C'_{1,F_1^i(S)})$ and $\cap_{i=1}^{\infty} DF_1^{-i}(C'_{2,F_1^{-i}(S)})$ define two invariant linear spaces of $T_S M$, of dimension s and u, respectively, which we denote E_S and I_S, respectively.*

Proof. We only have to verify that we are in the hypothesis of the former lemma. We take as ν any number larger than 1. After the results of §2 we conclude that, although we can have an initial contraction, the vectors of $C'_{2,S}$ for $S \notin \{F_1^i(R) : i \in \mathbb{Z}\}$ expand after a number of iterates by DF_1 as much as we want, and the same thing happens with the vectors of $C'_{1,S}$, iterating by DF_1^{-1}. It is not difficult after this observation, to find suitable values of N_1 and N_2. □

LEMMA 3.3. *I_S can be extended to the whole manifold M such that if S_i converges to S with $S \in M \setminus W^u(Q_{F_1}, F_1)$, then I_{S_i} converges to I_S. Similarly, E_S can be defined in M with a similar property. Moreover, in the non-conservative case, I_S is continuous for $S \neq Q_{F_1}$ and E_S is continuous for $S \neq P_{F_1}$.*

Proof. We only develop the proof for I_S. First, in (a), we work with

$$S \in \mathcal{L} = M \setminus \overline{\{F_1^i(R_{F_1}) : i \in \mathbb{Z}\}} \cup \{W^u(Q, F_1)\}$$

where the bar indicates closure. Then, in (b), we define I_S in $\{F_1^i(R_{F_1}) : i \in \mathbb{Z}\}$ in a continuous way, and show the continuity at P. Finally, in (c), we prove the last assertion of the lemma.

(a) The system of cones that defines I_S can be taken continuous in $\mathcal{L}$: near the boundary of $\mathcal{E}$ we modify the cones in order that they vary continuously. We also take care of the positive iterates of $\mathcal{E}$ in the points in which we leave the definition of $C'_{2,F_1^i(S)}$ as $DF_1^i(C'_{2,S})$ to take $C_{2,F_1^i(S)}$. We cannot assure the continuity of $C'_{2,S}$ on the set of points with coordinates $(0, y^*)$ with $\rho_2^{*2}(0, y^*) \leq e^* = 1$ (see §2.2); on $W^u(Q_{F_1}, F_1)$, and on $\{F_1^i(R_{F_1}) : i \in \mathbb{Z}\}$ (in this last set $C'_{2,S}$ is not defined). It is not a restriction to suppose that S does not belong to the set of points with coordinates $(0, y^*)$ with $\rho_2^{*2}(0, y^*) \leq e^* = 1$; in other cases we iterate backwards taking into account the fact that I_S is invariant. Let us suppose that we have $S_i \in M$ converging to S such that I_{S_i} converges to $I_0 \neq I_S$. Then, for k large enough, I_0 does not belong to $\cap_{j=1}^{k} DF_1^j(C'_{2,F_1^{-j}(S)})$, and by the continuity of the system of cones, I_{S_i} does not belong to $\cap_{j=1}^{k} DF_1^j(C'_{2,F_1^{-j}(S_i)})$ for i large enough: a contradiction.

(b) Now we define I_S in $\{F_1^i(R_{F_1}) : i \in \mathbb{Z}\}$. Actually, we only define $I_{R_{F_1}}$; if we iterate, we obtain the result for $\{F_1^i(R_{F_1}) : i \in \mathbb{Z}\}$. Let us work in the (x^*, y^*) system of coordinates, we take $S = (x^*, y^*)$ near to R_{F_1}, and define $p_0 = p_0(S) = \max\{i \in \mathbb{Z}^+ : F_1^{-i}(S) \in \mathcal{U}_1^* \text{ for } i = 1, \dots, p_0\}$ (possibly $p_0(S) = \infty$). If $\rho_2^{*2}(S) \neq 0$, we know that

$$p_0 \geq E(\log \rho_2^*(S)/\log \nu_{P_s}).$$

The vectors of the cone $C'_{2,F_1^{-p_0}(S)} = C_{2,F_1^{-p_0}(S)}$ determine an angle with $y^* = K$ with tangent smaller than a constant k_0. Therefore, the tangent of the angle of the vectors of the cone $\cap_{i=1}^{p_0} DF_1^i(C'_{2,F_1^{-i}(S)})$ with $y^* = K$ is bounded from above by

$$(\nu_{P_0}/\zeta_{P_u})^{p_0} k_0 \leq (\rho_2^*(S))^{\log(\nu_{P_0}/\zeta_{P_u})/\log \nu_{P_s}} k_0 \leq (\rho_2^*(S))^2 k_0$$

which converges to zero when $\rho_2^*(S)$ converges to zero.

A similar argument works for $\rho_2^*(S) = 0$: let us take $S = (x^*, 0), S \neq R_{F_1}$. After a finite number i of negative iterates, $DF_1^{-i}(C'_{2,S}) \subset C_{2,F_1^{-i}(S)}$, and then the slope is bounded. Reasoning as before, the vectors of the cones $\cap_{i=1}^{k} DF_1^i(C'_{2,F_1^{-i}(S)})$ have slope smaller than $(\nu_{p_0}/\zeta_{P_u})^k$ tending to zero when k tends to infinity. Thus, the right way to define $I_{R_{F_1}}$ to obtain continuity is $I_{F_1^{-i}(R_{F_1})} = T_{F_1^{-i}(R_{F_1})} W^u(P_{F_1}, F_1), i \in \mathbb{Z}^+$, and similarly $I_{F_1^i(R_{F_1})} = T_{F_1^i(R_{F_1})} W^u(P_{F_1}, F_1)$, for $i \in \mathbb{Z}^+$. A similar argument works to verify the continuity at P_{F_1} of $I_{P_{F_1}} = T_{P_{F_1}}(P_{F_1}, F_1)$.

(c) Now we prove the last assertion of the lemma. We work in the $(\tilde{x}, \tilde{y})$ system of coordinates, and we prove the assertion for the points $(\tilde{x}, 0)$ with $\tilde{x} \neq 0, \tilde{\rho}(\tilde{x}, 0) \leq 1$. Let us denote by $\alpha(S)$ the angle between $C'_{2,S}$ and $T_{S,\tilde{x}}$. We study the worst situation, $S \in \mathcal{E}$. In such a case,

$$\tan \alpha(\tilde{S}) \geq \tan \tilde{\tau} \tilde{\rho}^2(\tilde{S}) \geq \tilde{\tau} \tilde{\rho}^2(\tilde{S}) \geq \tilde{\tau} \tilde{\rho}_1^2(\tilde{S}).$$

Iterating, for large enough i,

$$\tan \alpha(F_1^i(\tilde{S})) \geq \tilde{\tau} \tilde{\rho}_1^2(\tilde{S}) |\mu_{Q_u}/\lambda_{Q_1}|^i \geq \tilde{\tau} \tilde{\rho}_1^2(F_1^i(\tilde{S})) |\mu_{Q_u}/\lambda_{Q_1}|^i / |\mu_{Q_1}|^{2i}.$$

Let us consider a sequence $S_j \to S = (\tilde{x}, 0), \tilde{x} \neq 0$. If the sequence S_j consist only of iterates of points $\tilde{S}_j \in \mathcal{E}$ (obtained without leaving $\tilde{\mathcal{U}}_1$), then

$$\tan \alpha(S_j) = \tan \alpha(F_1^{i_j}(\tilde{S}_j)) \geq \tilde{\tau} \tilde{\rho}_1^2(S_j) |\mu_{Q_u}/\mu_{Q_1}^2 \lambda_{Q_1}|^{i_j} \underset{j \to \infty}{\longrightarrow} \infty$$

because $i_j \to_{j \to \infty} \infty$. If in the sequence S_j there are points which are not iterates of points of $\mathcal{E}$, the reasoning is easier. $\square$

From the previous arguments we have the following.

COROLLARY 3.4. (Of the proof of Lemma 3.3.) *For $S \in W^u(P_{F_1}, F_1)$, we can write $I_S = T_S W^u(P_{F_1}, F_1)$; if $S \in W^u(Q_{F_1}, F_1)$, then $I_S = T_S W^u(Q_{F_1}, F_1)$; if $S \in W^s(P_{F_1}, F_1)$, then $E_S = T_S W^s(P_{F_1}, F_1)$; and if $S \in W^s(Q_{F_1}, F_1)$, then $E_S = T_S W^s(Q_{F_1}, F_1)$.*

Proof. The first equality was obtained when we proved that $I_{F_1^i(R_{F_1})} = T_{F_1^i(R_{F_1})}W^u(P_{F_1}, F_1)$. The other ones are proved in a similar way. □

3.2. For S_1, S_2 in any manifold $\mathcal{N}$ immersed in M we denote by $d_{\mathcal{N}}(S_1, S_2)$ the Riemannian distance between S_1 and S_2 measured on $\mathcal{N}$. For $S \in \mathcal{N}$ we denote by $B_{\mathcal{N}}(S, t)$ the closed ball with centre at S and radius t on $\mathcal{N}$.

Definition 3.5. An u-dimensional manifold M^u is an *integral manifold* of I_S if the following conditions are verified: (i) $T_S M^u = I_S$ for $S \in M^u$, (ii) M^u is complete. We adopt a similar definition with respect to E_S.

PROPOSITION 3.6. *There exists a continuous invariant foliation W^u of M such that each leaf of this foliation is an integral manifold of I_S, and therefore is C^1. $W^u(Q_{F_1}, F_1)$ and $W^u(P_{F_1}, F_1)$ are leaves of this foliation. Moreover, if $W^u(S)$ is the leaf of W^u through any point S, and if S' belongs to $W^u(S)$, then*

$$\limsup_{i\to\infty} d_{W^u(F_1^i(S))}(F_1^{-i}(S), F_1^{-i}(S')) \text{ is bounded}$$

$$\liminf_{i\to\infty} d(F_1^{-i}(S), F_1^{-i}(S')) = 0$$

$$\lim_{i\to\infty} d_{W^u(F_1^i(S))}(F_1^i(S), F_1^i(S')) = \infty.$$

A similar assertion stands for the integral manifolds of E_S.

Proof. We divide the proof into several lemmas. We remark that the foliation is not C^1: at Q_{F_1} the fibre bundle I_S is not continuous. The following two lemmas are introductory to our proof of the proposition.

LEMMA 3.7. *Let M^u be any u-dimensional manifold immersed in M such that $T_S M^u \subset C'_{2,S}$ for every $S \in M^u$. Given τ_0, there exist $r(\tau_0) > 0$ and $s_0(x) \geq 0$ defined in $[0, r]$, $s_0(x) = 0$ only if $x = 0$, such that if $\tau > \tau_0$ and $d_{M^u}(S_1, S_2) = x$ then $d(S_1, S_2) \geq s_0(x)$.*

Proof. The more critical situation is that of R_{F_1} or its positive iterates, and for this reason we begin by regarding this case. We consider the $(\tilde{x}, \tilde{y})$ system of coordinates. If $S \neq F_1^i(R_{F_1}), i \geq 0$, then we have for any manifold M^u verifying the assumptions of the lemma that $T_S M^u \oplus T_{S,\tilde{x}} = T_S M$. We recall that $\alpha(S)$ denotes the angle between $C'_{2,S}$ and $T_{S,\tilde{x}}$. For S near R_{F_1}, and taking into account condition (9), we can write $\tan\alpha(S) \geq K\tilde{\tau}\tilde{\rho}_1^2(S)$. Iterating,

$$\tan\alpha(F_1^i(S)) \geq K\tilde{\tau}\tilde{\rho}_1^2(S)(\zeta_{Q_u}/\nu_{Q_0})^i \geq K\tilde{\tau}\tilde{\rho}_1^2(F_1^i(S))(\zeta_{Q_u}/\nu_{Q_0})^i/\zeta_{Q_0}^{2i} \geq K\tilde{\tau}\tilde{\rho}_1^2(F_1^i(S))$$

because $\zeta_{Q_u} \geq \nu_{Q_0}\zeta_{Q_0}^2$. This shows that near to each point $F_1^i(R_{F_1}), i \geq 0$, and for τ larger than a certain τ_0, the angle between $T_S M^u$ and $T_{S,\tilde{x}}$ grows faster than the square of the distance from S to $\tilde{y} = 0$ and with a coefficient independent of i; that is, it is the same one near to any of the points $F_1^i(R_{F_1}), i \in \mathbb{Z}^+$. Now, let us take a finite number of connected C^1 charts, (x', y'), with domain in totally normal neighborhoods covering M such that the coordinate lines coincide with those of $(\tilde{x}, \tilde{y})$ in $\mathcal{D}$. We also ask that for

$S \in M$, $T_{S,x'}$ (the fibre at S of the tangent vector bundle of $x' = K$) is into $C'_{1,S}$, and similarly $T_{S,y'} \subset C'_{2,S}$. Let r be smaller than the Lebesgue number of the covering, we take $0 < x < r$. We consider in any chart the set of points (S_1, S_2) such that S_1, S_2 belong to a u-dimensional immersed manifold with $T_S M^u \subset C'_{2,S}$ for $S \in M^u$, $d_{M^u}(S_1, S_2) = x$. A transversality argument (the transversality is at least topological) shows that the points $S_1 = (x'_{S_1}, y'_{S_1})$, $S_2 = (x'_{S_2}, y'_{S_2})$ must have different values of x'_{S_1} and x'_{S_2}. The set of numbers $d(\{x' = x'_{S_1}\}, \{x' = x'_{S_2}\})$ has a positive minimum that increases with τ. Then, taking the minimum of those values between the charts with $\tau = \tau_0$, we obtain a positive lower bound for $d(S_1, S_2)$ which we call $s_0(x)$. □

LEMMA 3.8. *Let M^s be an s-dimensional manifold such that $T_S M^s \subset C'_{1,S}$ for every $S \in M^s$. Then, given $S_1 \in M^s$, $\epsilon > 0$ with $\exp_{S_1}$ well defined for $v \in T_S M^u$, $\|v\| < \epsilon$ and $\rho > 0$, there exists $i_0 = i_0(S_1, \epsilon, \rho)$ (not depending on M^s) such that $\exp_{F^{-i}(S)}$ is well defined for $v \in T_S M^u$, $\|v\| < \epsilon$, for $i \geq i_0$.*

Proof. We have two cases to consider. If $S_1 \notin W^u(P_{F_1}, F_1)$ the result follows because F_1^{-1} actually expands. If $S_1 \in W^u(P_{F_1}, F_1)$, the lemma is a consequence of the topological transversality of M^s to $W^u(P, F_1)$. □

LEMMA 3.9. *$W^u(P_{F_1}, F_1)$ and $W^u(Q_{F_1}, F_1)$ are two integral manifolds of I_S. In the same way, $W^s(P_{F_1}, F_1)$ and $W^s(Q_{F_1}, F_1)$ are two integral manifolds of E_S.*

Proof. It is a consequence of Corollary 3.4 and the results of §2.3. □

LEMMA 3.10. *Let M^u be an integral manifold of I_S. Then, it is the only integral manifold through any of its points. Similarly if M^s is an integral manifold of E_S, then it is the only integral manifold through any of its points.*

Proof. We only prove the first assertion. We divide the proof into three steps: first we prove that the lemma is valid for $W^u(P_{F_1}, F_1)$, then for $W^u(Q_{F_1}, F_1)$, and finally for any integral manifold.

(i) Let $S \in W^u(P_{F_1}, F_1)$. Iterating backwards if necessary, we can suppose that $S \in \mathcal{U}_1^*$ with coordinates $(x^*, 0)$. Let $k(x^*, y^*)$ be the tangent of the largest angle of a vector of I_{S^*} with $y^* = K$, where $S = (x^*, y^*)$. Taking into account the hypothesis of our Theorems 1 and 2, it is easy to prove that $0 \leq k(S^*) \leq K\rho_2^*(S^*)$. The same reasoning of unicity of solutions of differential equations shows that $y^* = 0$ is the unique integral manifold through the points of $W^u_{\text{loc}}(P_{F_1}, F_1)$.

(ii) We prove now that $W^u(Q_{F_1}, F_1)$ is the only integral manifold of I_S through any of its points. Suppose that for some point $S \in W^u(Q_{F_1}, F_1)$ we have in any neighborhood of S another integral manifold M^u besides $W^u(Q_{F_1}, F_1)$. Iterating backwards if necessary, we can suppose that $\tilde{S} \in \tilde{\mathcal{U}}_1$. We have two cases to consider. In the first one, we can get $b \in \mathbb{R}$ and a neighborhood $\mathcal{A}$ of R_{F_1} such that if $S_1 \in B_{M^u}(S, b)$, for every $j \in \mathbb{Z}^+$ such that $F_1^{-i}(S_1) \in \tilde{\mathcal{U}}_1$, $0 \leq i \leq j$, then $F_1^{-j}(S_1) \notin \mathcal{A}$. In this case, we can apply the same reasoning as in (i) to get a contradiction. In the second case, we have a sequence of points $\{S_i\}_{i\in\mathbb{Z}^+}$, and $\{l_i\}_{i\in\mathbb{Z}^+}$, $l_i \in \mathbb{Z}^+$, l_i tending to infinity, such that $F_1^{-j}(S_i) \in \tilde{\mathcal{U}}_1$ for $0 \leq j \leq l_i$, and $F_1^{-l_i}(S_i)$ converges to R_{F_1} when i tends to infinity. We work in the

(x^*, y^*) system of coordinates. Taking into account the fact that $F_1^{-l_i}(M^u)$ is an integral manifold of I_S and the fact that for $S \in F_1^{-l_i}(M^u)$, $T_S F_1^{-l_i}(M^u) \subset C'_{2,S}$, we deduce that we can write the equation of $F_1^{-l_i}(M^u)$ in a neighborhood of $F_1^{-l_i}(S_i)$ as $y^* = \phi_i^*(x^*)$, with $(x^*, 0)$ in a neighborhood of R_{F_1}. Let $F_1^{-l_i}(S_i) = (x_i^*, y_i^*)$. We recall that in (i) we obtained that $k(S) \le K\rho_2^*(S)$, therefore we can write for x^* near of x_i^* that

$$\begin{aligned} &\rho_2^*((1-v)x_i^* + vx^*, \phi_i^*((1-v)x_i^* + vx^*)) \\ &\le \quad \rho_2^*(x_i^*, y_i^*) + K \int_0^v \rho_2^*((1-t)x_i^* + tx^*, \phi_i^*((1-t)x_i^* + tx^*))\, dt. \end{aligned}$$

By the Gronwall inequality,

$$\rho_2^*((1-v)x_i^* + vx^*, \phi_i^*((1-v)x_i^* + vx^*)) \le \rho_2^*(x_i^*, y_i^*)e^{Kv}$$

which shows that, near R_{F_1}, $F_1^{-l_i}(M^u)$ converges uniformly on compact neighborhoods near R_{F_1} to $W^u(P_{F_1}, F_1)$. But this is a contradiction, because on compact neighborhoods near R_{F_1} locally $F_1^{-l_i}(M^u)$ must converge uniformly to $W^s(Q_{F_1}, F_1)$.

(iii) We take $S_0 \in M \setminus \{W^u(P_{F_1}, F_1) \cup W^u(Q_{F_1}, F_1)\}$; we can suppose that $S_0 \in \mathcal{H}$. Suppose by contradiction that there exist two integral manifolds M_a^u, M_b^u, of I_S through S_0 which ramify at S_0. We take a C^1 manifold M^s with $T_S M^s \in C'_{1,S}$ for $S \in M^s$ such that $M^s \cap M_a^u$ is a point A and $M^s \cap M_b^u$ is a point B. We consider the iterates $F_1^{-i}(S_0) \in \mathcal{H}$ for $i > 0$. When i goes to infinity, $d(F_1^{-i}(A), F_1^{-i}(S_0))$ and $d(F_1^{-i}(B), F_1^{-i}(S_0))$ converge to zero with $F_1^{-i}(S_0) \in \mathcal{H}$ while $d_{F_1^{-i}(M^s)}(F_1^{-i}(A), F_1^{-i}(B))$ can be made as large as required for large enough i. Taking into account the fact that $T_S F_1^{-i}(M^s) \subset C'_{1,S}$ for $S \in F_1^{-i}(M^s)$, we arrive at a contradiction. □

LEMMA 3.11. *If F_1 is taken C^0 near of f, then $W^u(P_{F_1}, F_1)$, $W^s(P_{F_1}, F_1)$, $W^u(Q_{F_1}, F_1)$ and $W^s(Q_{F_1}, F_1)$ are dense in M.*

Proof. As f is Anosov, then it is topologically stable (see [**W70**] and [**L80**]), that is, there exists $\delta_0 > 0$ such that for every $0 < \delta^* < \delta_0$ there exists $\bar{\delta}$ with the property that for every $g \in \mathrm{Diff}(M)$ with $\rho_{C^0}(f, g) < \bar{\delta}$ there exists a unique $\varphi : M \mapsto M$ continuous and onto with $\varphi \circ g = f \circ \varphi$, and $\rho_{C^0}(\varphi, \mathrm{id}) < \delta^*$. We take $\epsilon_P > 0$ such that $\{S \in M : d(P, S) \le \epsilon_P\} \subset \mathcal{U}_1^*$ and $\epsilon_Q > 0$ such that $\{S \in M : d(Q, S) \le \epsilon_Q\} \subset \tilde{\mathcal{U}}_1$ (here $\tilde{\mathcal{U}}_1$ and $\mathcal{U}_1^*$ are the regions corresponding to the diffeomorphism f). Let us fix $0 < \delta^* < \frac{1}{2}\min\{\epsilon_P, \epsilon_Q\}$ and obtain $\bar{\delta}$. We also take F_1 such that $\rho_{C^0}(f, F_1) < \bar{\delta}$, and then there exists $\varphi : M \mapsto M$, φ continuous and onto, $\rho_{C^0}(\varphi, \mathrm{id}) < \delta^*$ such that $\varphi \circ F_1 = f \circ \varphi$. As $d(\varphi(P_{F_1}), P) < \delta^* < \epsilon_P$, we deduce that $\varphi(P_{F_1}) \subset \mathcal{U}_1^*$. As P is the only fixed point of f in $B(P, \epsilon_P)$, we conclude that $\varphi(P_{F_1}) = P$.

We claim that $\varphi(W^s(P_{F_1}, F_1)) = W^s(P, f)$ (similar assertions are valid for the other stable and unstable manifolds of P_{F_1}, Q_{F_1}). We first prove that $\varphi(W^s(P_{F_1}, F_1)) \subset W^s(P, f)$. We take $S \in W^s(P_{F_1}, F_1)$. As $F_1^i(S) \to_{i\to\infty} P_{F_1}$, and recalling the continuity of φ, we deduce that $f^i \circ \varphi(S) = \varphi \circ F_1^i(S) \to_{i\to\infty} P$, that is, $\varphi(S) \in W^s(P, f)$. Now we prove that $W^s(P, f) \subset \varphi(W^s(P_{F_1}, F_1))$. Let us take $S \in W^s(P, f)$. As φ is onto, there exists S_1 such that $\varphi(S_1) = S$. By contradiction, let us suppose that $S_1 \notin W^s(P_{F_1}, F_1)$. Then, there exists i such that $F_1^i(S_1) \in M \setminus \mathcal{U}_1^*$ while $d(f^i(S), P) < \delta^*$. Therefore,

$d(f^i(\varphi(S_1)), F_1^i(S_1)) = d(\varphi(F_1^i(S_1)), F_1^i(S_1)) < \delta^*$, but

$$d(f^i(\varphi(S_1)), F_1^i(S_1)) \geq d(F_1^i(S_1), P) - d(P, f^i(S)) > \epsilon_P - \delta^* > \delta^*,$$

a contradiction.

We have proved that $W^s(P_{F_1}, F_1)$ and $W^u(P_{F_1}, F_1)$ are δ^* dense, this follows from the claim and the fact that $W^s(P, f)$ and $W^u(P, f)$ are dense. We take an open set A. We have to prove that $W^s(P_{F_1}, F_1)$ intersects A. Let M^u be an u-dimensional manifold, $M^u \subset A$, $T_S M^u \subset C'_{2,S}$ for $S \in M^u$. Iterating forward if necessary, we can get a point $S_0 \in \mathcal{H}$ such that $B_{M^u}(S_0, r_1) \subset M^u$ for r_1 so large that for a transversality argument $W^s(P_{F_1}, F_1)$ must intersect M^u (and then A). We reason in a similar way for $W^u(P_{F_1}, F_1)$. □

LEMMA 3.12. *There exists a C^0 invariant foliation W^u such that each leaf of W^u is at least C^1; $T_S W^u(S) = I_S$. Moreover, if S' belongs to $W^u(S)$, then*

$$\limsup_{i\to\infty} d_{W^u(F_1^i(S))}(F_1^{-i}(S), F_1^{-i}(S')) \text{ is bounded}$$

$$\liminf_{i\to\infty} d(F_1^{-i}(S), F_1^{-i}(S')) = 0$$

$$\lim_{i\to\infty} d_{W^u(F_1^i(S))}(F_1^i(S), F_1^i(S')) = \infty.$$

Similar assertions are valid with respect to a foliation W^s tangent to E_S.

Proof. Let us take a point $S_0 \notin W^u(Q_{F_1}, F_1)$, we construct the leaf of the foliation through S (for $S_0 \in W^u(Q_{F_1}, F_1)$, such a leaf is $W^u(Q_{F_1}, F_1)$). We work with the coordinate charts of Lemma 3.7. We take a coordinate chart with S_0 in its domain, letting $\{(x^i, y^i)\}_{i\in\mathbb{Z}^+}$ be a sequence with (x^i, y^i) the coordinates of a point S_i in $W^u(P_{F_1}, F_1)$, such that $S_i \to_{i\to\infty} S_0$. A connected component of the intersection of $W^u(P_{F_1}, F_1)$ with the domain of the coordinate chart will be the graphic of a function $y = g(x)$. Let $y = g_i(x)$ be the equation of the connected component of $W^u(P_{F_1}, F_1)$ such that $y^i = g(x^i)$. Taking into account the Ascoli theorem, and the fact that in $M \setminus W^u(Q_{F_1}, F_1)$, I_S is continuous, we can suppose that $\{g_i(x)\}_{i\in\mathbb{Z}^+}$ is uniformly convergent on a compact set, defining in this way a local C^0 manifold W with $S_0 \in W$. To prove that W is a *local C^1 integral manifold* of I_S, we consider the local vector fields χ_j, $j = 1, \dots, u$, such that at the point S, χ_j belong to I_S, and have components $(0, \dots, 0, 1, 0, \dots, 0, a_{u+1}, \dots, a_n)$, with 1 in the j-term. Denoting $\tilde{S}$ a point with coordinates $(x_1, \dots, x_j, \dots, x_u, g(x_1, \dots, x_u))$, the curve with parameter $t \in \mathbb{R}$, t small enough, given by $\phi_t^j(\tilde{S}) = (x_1, \dots, x_j + t, \dots, x_u, g(x_1, \dots, x_j + t, \dots, x_u))$ is an integral curve of χ_j on $W^u(P_{F_1}, F_1)$. The sequence $\{\phi_t^j(S_i)\}_{i\in\mathbb{Z}^+}$ converges to a curve which we denote by $\phi_t^j(S_0)$. By the bounded convergence theorem of Lebesgue and Lemma 3.3, $\phi_t^j(S_0)$ is an integral curve of χ_j in W. Similarly, $\phi_s^i(\phi_t^j(S_0))$ for t fixed is an integral curve of χ_i. It follows immediately that $\phi_t^j(\phi_s^i(\tilde{S}_0)) = \phi_s^i(\phi_t^j(\tilde{S}_0))$ for $s, t \in \mathbb{R}$ small enough. This proves that W is described by a function $\Psi : B^u \mapsto M$, where B^u is a ball in $\mathbb{R}^u$ with centre zero such that if $\{e_i\}_{i=1,\dots,u}$ is the canonical basis of $\mathbb{R}^u$, then

$$\Psi\left(\sum_{i=1}^{u} t_i e_i\right) = \phi_{t_1}^1 \dots \phi_{t_u}^u(S_0).$$

As $\{\chi_j(S)\}_{j=1,\dots,u}$ is independent and generates $T_S M$, we deduce that W is a local integral manifold. Taking a finite number of subsequences of the prolongation of the connected components of $W^u(S_0, F_1)$ which define W, we deduce that any ball with centre in S_0 belongs to an extension of W unless we arrive at $W^u(Q_{F_1}, F_1)$. But reasoning as in Lemma 3.10(ii) we conclude that this is not possible, proving that the extension of W is complete. Therefore, we obtain for each point of M an unique integral manifold.

To end the proof of the proposition, the assertion about the boundness of

$$\limsup_{i\to\infty} d_{W^u(F_1^i(S))}(F_1^{-i}(S), F_1^{-i}(S'))$$

is immediate after the results of Lemma 2.6 (in fact, it will be seen in Lemma 4.18 that the limit is zero). We know that for S' in $W^u(S)$, if i_j is such that $F^{-i_j}(S) \in \mathcal{H}$, $i_j \to_{j\to\infty} \infty$

$$\lim_{j\to\infty} d(F_1^{-i_j}(S), F_1^{-i_j}(S')) = 0$$

and then follows the second assertion about the limit. The last limit of the lemma is clear since, no matter what the initial contraction is, any subsequent contractions (in $\mathcal{D}$) are compensated with expansions in $\mathcal{D}$. The distance grows without bound because of the expansions in $\mathcal{H}$. $\square$

3.3.

PROPOSITION 3.13. *F_1 is expansive with a constant independent of τ.*

Proof. For small enough $\epsilon > 0$ there exists $\delta_1(\epsilon) > 0$ (independent of τ) such that for S_1, S_2 with $d(S_1, S_2) < \delta_1(\epsilon)$, then $B_{W^u(S_1)}(S_1, \epsilon) \cap B_{W^s}(S_2, \epsilon)$ is a single point, which we denote $[S_1, S_2]$. Also, there exists $\delta_2 > 0$ and $\delta_3 > 0$ such that

$$F_1([B_{W^s(S)}(S, \delta_2), B_{W^u(S)}(S, \delta_2)]) \subset [B_{W^s(F_1(S))}(S, \delta_1(\epsilon)/2), B_{W^u(F_1(S))}(S, \delta_1(\epsilon)/2)]$$

and such that for any $S \in M$ and $S' \notin [B_{W^s(S)}(S, \delta_2), B_{W^u(S)}(S, \delta_2)]$ we have $d(S, S') > \delta_3$. We take $\alpha < \min\{\delta_1(\delta_2), \delta_3\}$, and let us assume that $d(F_1(S_1), F_2(S_2)) \le \alpha$ for $i \in \mathbb{Z}$, $S_1 \ne S_2$. If $S_1 \ne [S_1, S_2]$, after Lemma 3.12, there exists i such that $d_{W^u(F_1^i(S_1))}(F_1^i(S_1), F_1^i[S_1, S_2]) > \delta_2$, and therefore there exists, for $j > 0$, the first iterate such that $F_1^j(S_1) \notin [B_{W^s(S)}(S, \delta_2), B_{W^u(S)}(S, \delta_2)]$. Therefore, $d(F_1^j(S_1), F_1^j(S_2)) > \delta_3$, a contradiction. If $S_1 = [S_1, S_2]$, the reasoning is similar, but with F_1^{-1}. $\square$

3.4.

PROPOSITION 3.14. *F_1 is conjugate to f.*

Proof. Let us observe that the value of δ_1 is independent of the results obtained after Lemma 3.11. If in Lemma 3.11 we take $\delta^* < \alpha/2$, φ is one to one, and after the theorem of invariance of the domain, a homeomorphism. $\square$

4. *Ergodic properties of F_1*

We will show first that the Pesin region has full measure, by means of the family of cones. Then, we recall some ideas, mainly from [**P73**] and [**LY85**] which will be used to construct an SRB measure. The idea is to take a small open set on a leaf of the unstable foliation, to iterate it, and define in a suitable way a sequence of measures in the manifold, simply by assigning to any Borelian set the measure of the intersection of this Borelian set with the iterates of the open set on W^u. We can expect that an accumulation point of the sequence of measures will be an SRB measure. Other ergodic results are exposed in §4.3, from [**PS82**]. The conservative case is considered in §4.4.

4.1. We denote the Pesin region as Λ. In this subsection we will prove the following.

PROPOSITION 4.1. *Λ has full probability.*

We only have to study the points which have iterates in $\mathcal{E}$. Let us take μ a F_1-invariant probability, we suppose $\mu(\mathcal{E}) > 0$. Let us define first

$$\begin{aligned}\mathcal{E}^* \quad = \quad & \{S \in \mathcal{E} : F_1^n(S) \in \mathcal{E} \text{ for infinite values of } n \in \mathbb{Z}^+ \\ & \text{and also infinite values of } n \in \mathbb{Z}^-\}.\end{aligned}$$

We know that $\mu(\mathcal{E}^*) = \mu(\mathcal{E})$. Let $k_{\mathcal{E}^*} : \mathcal{E}^* \mapsto \mathbb{N}$ be defined as $k_{\mathcal{E}^*}(S) = \min\{i \in \mathbb{Z}^+ : F_1^i(S) \in \mathcal{E}\}$. It is known that $k_{\mathcal{E}^*} \in L^1(\mathcal{E}^*, \mu)$. Next, we define $\hat{F} : \mathcal{E}^* \mapsto \mathcal{E}^*$ as $\hat{F}(S) = F_1^{k_{\mathcal{E}^*}(S)}(S)$.

Let us define

$$\tilde{k}_{\mathcal{E}^*} = \lim_{m\to\infty} \frac{1}{m} \sum_{i=0}^{m-1} k_{\mathcal{E}^*} \circ \hat{F}^i.$$

After the ergodic theorem of Birkhoff, it is defined and finite a.e. It is clear that it is positive.

LEMMA 4.2. *Let $\mathcal{E}^* \subset M$, $\mu(\mathcal{E}^*) > 0$. Then, for a.e. point $S \in \mathcal{E}^*$, S a regular point for F_1, and for every $v \in T_S M$, $v \neq 0$, the following equality follows:*

$$\chi(S, v, \hat{F}) = \tilde{k}_{\mathcal{E}^*}(S)\chi(S, v, F_1).$$

Proof. Let us define $\tilde{k}_m = \sum_{i=0}^{m-1} k_{\mathcal{E}^*} \circ \hat{F}^i$. We have

$$\begin{aligned}\chi(S, v, \hat{F}) \quad &= \quad \lim_{m\to\infty} \left(\frac{1}{m}\right) \log \|D\hat{F}^m(S)v\| = \lim_{m\to\infty} \left(\frac{1}{m}\right) \log \|DF_1^{\tilde{k}_m(S)(S)} v\| \\ &= \quad \chi(S, v, F_1) \lim_{m\to\infty} \tilde{k}_m(S)/m. \qquad \square\end{aligned}$$

Therefore, in order to prove that almost every point of $\mathcal{E}^*$ is in the Pesin region for F_1, it is enough to prove that almost every point of $\mathcal{E}^*$ is in the Pesin region for $\hat{F}$. To do that, we use the following lemma.

LEMMA 4.3. *Let $C'_{1,S}$ and $C'_{2,S}$ for $S \in \mathcal{E}^*$ be as in §2.2, and suppose that there exist $N_1(S) > 0$, $N_2(S) > 0$ defined in $\mathcal{E}^*$ a.e. and $\nu > 1$ such that for a.e. $S \in \mathcal{E}^*$*

$$\|D\hat{F}^{-N_1(S)}v\| \geq \nu^{N_1(S)}\|v\| \quad \text{for } v \in C'_{1,\hat{F}^{N_1(S)}(S)}$$
$$\|D\hat{F}^{N_1(S)}v\| \geq \nu^{N_1(S)}\|v\| \quad \text{for } v \in C'_{2,S}$$
$$\|D\hat{F}^{-N_2(S)}v\| \geq \nu^{N_2(S)}\|v\| \quad \text{for } v \in C'_{1,S}$$
$$\|D\hat{F}^{N_2(S)}v\| \geq \nu^{N_2(S)}\|v\| \quad \text{for } v \in C'_{2,\hat{F}^{-N_2(S)}(S)}.$$

Then, a.e. the point of $\mathcal{E}^*$ is in the Pesin region for $\hat{F}$.

Proof. It is similar to that of Lemma 3.1. We observe that we are using the same notation for the linear spaces as in Lemma 3.1: they are the same except for the domain of definition. The two first inequalities do not guarantee the last two: it is possible that the Lebesgue measure of the set of points $\hat{F}^{N_1(S)}(S)$ be smaller than the Lebesgue measure of $\mathcal{E}^*$. Let $\mathcal{E}_1^* \subset \mathcal{E}^*$ be the set of points where the hypothesis is verified, $\mu(\mathcal{E}_1^*) = \mu(\mathcal{E}^*)$. We define $\mathcal{E}_2^* = \cap_{j\in\mathbb{Z}}\hat{F}^j(\mathcal{E}_1^*)$, $\mu(\mathcal{E}_2^*) = \mu(\mathcal{E}_1^*)$. We know by §2.2 that $D\hat{F}^{N_2(S)}(C'_{2,S}) \subset C'_{2,\hat{F}^{N_2(S)}(S)}$ and $D\hat{F}^{-N_2(S)}(C'_{1,\hat{F}^{N_2(S)}(S)}) \subset C'_{1,S}$ for every $S \in \mathcal{E}_2^*$. We fix $S \in \mathcal{E}_2^*$ and denote by induction $n_1 = -N_2(S), n_k = n_{k-1} - N_2(\hat{F}^{n_{k-1}}(S))$. Let $v_i \in C'_{2,\hat{F}^{n_k}(S)}$; $w_i = D\hat{F}^{-n_k}(v_i) \in C'_{2,S}, i = 1, 2$; $w_2 - w_1 \in T_{S,\tilde{x}} \in C'_{1,S}$. If we write $m_k = 2n_k - N_2(\hat{F}^{n_k}(S)) + N_2(S)$, we have

$$\begin{aligned}\|w_2 - w_1\| &\leq \nu^{n_k}\|v_2 - v_1\| \leq \nu^{n_k}(\|v_2 - v_1\|)\\ &\leq \nu^{m_k}(\|w_2\| + \|w_1\|) \leq \nu^{m_k}(2\|w_2\| + \|w_2 - w_1\|)\end{aligned}$$

so that

$$\|w_2 - w_1\| \leq 2\nu^{m_k}\|w_2\|/(1 - \nu^{m_k}) \underset{k\to\infty}{\longrightarrow} 0.$$

This shows that $D\hat{F}^{-n_k}(C'_{2,\hat{F}^{n_k}(S)})$, a decreasing sequence of cones, converges to a linear space $I_S \subset T_SM$ with $D\hat{F}^{N_2(S)}(I_S) = I_{\hat{F}^{N_2(S)}(S)}$. To find such space, we consider $\lim_{k\to\infty} D\hat{F}^{-n_k}(T_{\hat{F}^{n_k}(S),y^*})$. For any k, $D\hat{F}^{-n_k}T_{\hat{F}^{n_k}(S),y^*}$ is the graph of a continuous linear map $A^k : T_{S,y^*} \mapsto T_{S,\tilde{x}}$, whose graphic converges to I_S. This implies $I_S \oplus T_{S,\tilde{x}} = T_SM$. In the same way, we define $n'_1 = N_1(S), n'_k = n'_{k-1} + N_1(\hat{F}^{n'_{k-1}}(S))$, and show that $D\hat{F}^{-n'_k}C'_{1,\hat{F}^{n'_k}(S)}$ converges to a linear space E_S, the limit of $D\hat{F}^{-n'_k}T_{\hat{F}^{n'_k}(S)\tilde{x}}$. As $I_{\hat{F}^{n'_k}(S)} \oplus T_{\hat{F}^{n'_k}(S)\tilde{x}} = T_{\hat{F}^{n'_k}(S)}M$, we have $I_S \oplus E_S = T_SM$. It is clear that $\|D\hat{F}^{N_1(S)}v\| \geq \nu^{N_1(S)}\|v\|$ if $v \in I_S$ and $\|D\hat{F}^{-N_2(S)}v\| \leq \nu^{-N_2(S)}\|v\|$ if $v \in E_S$. Let $S \in \mathcal{E}_2^*$ be a regular point. Then, $\lim_{i\to\infty}\log\|D\hat{F}^i v\|/i \geq \log\nu > 0$ if $v \in T_SM \setminus E_S$ and $\lim_{i\to\infty}\log\|D\hat{F}^i v\|/i \leq -\log\nu < 0$ if $v \in E_S$, and, therefore, S belongs to the Pesin region. □

Next, we prove that the assumptions of Lemma 4.3 are verified. We take the systems of cones $C'_{1,S}$ and $C'_{2,S}$ restricted to $\mathcal{E}^*$. After the results of §2.2 and §2.3, for $S \in \mathcal{E}$, the norm of any vector $v \in C'_{2,S}$ (with the Riemannian metric) expands, after $k_{\mathcal{E}^*}(S)$ iterates, with a factor larger than

$$\begin{aligned}G(S) &= K\tilde{\tau}\rho^2(S)(\tilde{\rho}_1(S))^{-\log\zeta_{Q_u}/\log\zeta_{Q_0}}(\rho_2^*(\hat{F}(S)))^{\log\zeta_{P_u}/\log\nu_{P_s}}\\ &\geq K_1\tilde{\tau}(\rho(S)/\rho(\hat{F}(S)))^{-\log\zeta_{P_u}/\log\nu_{P_s}}.\end{aligned}$$

Let us take

$$\tilde{\tau} > 2^{1-(\log \zeta_{P_u} / \log \nu_{P_s})}/K_1.$$

In such a case, if $2\rho(S) \geq \rho(\hat{F}(S))$, then any vector v in $C'_{2,S}$ will expand, after one iterate by $\hat{F}$, with a factor larger than two. With a similar argument with large enough τ, if $\rho(S) \leq 2\rho(\hat{F}(S))$, then any vector $v \in C'_{1,\hat{F}(S)}$ will expand (with respect $\hat{F}^{-1}$) with a factor larger than two. Then, if $\rho(\hat{F}(S)) \in [\rho(S)/2, 2\rho(S)]$, we take $\nu = 2$ and $N_1(S) = 1$. Let us suppose that $\rho(\hat{F}(S)) \notin [\rho(S)/2, 2\rho(S)]$. After two iterates by $\hat{F}$ the norm of $v \in C'_{2,S}$ will be multiplied by a factor larger than

$$K_1^2 \tilde{\tau}^2 (\rho(S)/\rho(\hat{F}^2(S)))^{-\log \zeta_{P_u} / \log \nu_{P_s}}.$$

Therefore, if $2^2\rho(S) \geq \rho(\hat{F}^2(S))$, any vector in $C'_{2,S}$ will expand with a factor of at least four. In a similar way, if $\rho(S) \geq 2^2\rho(\hat{F}^2(S))$, any vector of $C'_{1,\hat{F}^2(S)}$ will expand by $\hat{F}^{-2}$ with a factor larger than four. Then, if $\rho(\hat{F}(S)) \notin [\rho(S)/2, 2\rho(S)]$, but $\rho(\hat{F}^2(S)) \in [\rho(S)/4, 4\rho(S)]$, we take $N_1(S) = 2$ and $\nu = 2$. If $\rho(\hat{F}(S)) \notin [\rho(S)/2, 2\rho(S)]$, $\rho(\hat{F}^2(S)) \notin [\rho(S)/4, 4\rho(S)]$, but $\rho(\hat{F}^3(S)) \in [\rho(S)/8, 8\rho(S)]$, we take $N_1(S) = 3$, and so on.

Given $a > 1$ and $t > 0$ small, we define the set

$$\Gamma_{a,t} = \mathcal{E}^* \cap (B(R_{F_1}, at) \setminus B(R_{F_1}, t/a)).$$

It is clear that $\lim_{i\to\infty} \mu(\Gamma_{(4/3)^i,t} \cap \mathcal{E}^*) = \mu(\mathcal{E}^*)$, and also that $\Gamma_{(4/3)^i,t} \subset [\rho(S)/2^i, 2^i\rho(S)]$ for $i \in \mathbb{Z}^+$, $S \in \Gamma_{4/3,t}$. Then, the set of points of $\Gamma_{4/3,t}$ such that $\rho(\hat{F}^i(S)) \notin \Gamma_{(4/3)^i,t}$ for every $i \in \mathbb{Z}^+$ has measure 0. As we can divide $\mathcal{E}^*$ into a countable set of sets $\Gamma_{4/3,t}$, it follows that for a.e. point in $\mathcal{E}^*$ we can define $N_1(S)$ with $\nu = 2$. A similar construction allows us to define $N_2(S)$ for a.e. $S \in \mathcal{E}^*$. Therefore, the hypotheses of Lemma 4.3 are verified, and the proposition is proved. □

4.2. In this subsection we work under the hypothesis of Theorem 1, and we aim to prove that F_1 has an SRB measure (Proposition 4.12). For simplicity, the reasonings will be developed only in dimension two. We begin by giving some definitions (see §1.3 of [**P73**]). Let $f : M \mapsto M$ be a diffeomorphism, we denote by $\Lambda^{l+}_{q,t}$ with $l, q, t \in \mathbb{Z}, q > t \geq 1$, the set of points $S \in M$ such that for $i \in \mathbb{Z}^+$, $j \in \mathbb{Z}$, and $\epsilon_q = (1/100)\log(1 + 2/q)$:

(i) there exist subspaces $E^{l+}_{1,S,q,t}$ and $E^{l+}_{2,S,q,t}$ (possibly one of them equal to 0_S), for which $T_S M = E^{l+}_{1,S,q,t} \oplus E^{l+}_{2,S,q,t}$.

(ii)

$$\|Df^i_{f^j(S)} v\| \leq l(q/(t+2))^i \exp\{\epsilon_q i + 4\epsilon_q |j|\}\|v\| \quad \text{and}$$
$$\|Df^{-i}_{f^j(S)} v\| \geq l^{-1}(q/(t+2))^{-i} \exp\{-\epsilon_q i - 4\epsilon_q |j|\}\|v\| \quad \text{for } v \in Df^j E^{l+}_{1,S,q,t};$$
$$\|Df^i_{f^j(S)} v\| \geq l^{-1}(q/t)^i \exp\{-\epsilon_q i - 4\epsilon_q |j|\}\|v\| \quad \text{and}$$
$$\|Df^{-i}_{f^j(S)} v\| \leq l(q/t)^{-i} \exp\{\epsilon_q i + 4\epsilon_q |j|\}\|v\| \quad \text{for } v \in Df^j E^{l+}_{2,S,q,t}.$$

(iii) Let $\gamma^{l+}_{q,t}(f^j(S))$ be the angle between $Df^j(E^{l+}_{1,S,q,t})$ and $Df^j(E^{l+}_{2,S,q,t})$. Then, $\gamma^{l+}_{q,t}(f^j(S)) \geq l^{-1}\exp\{-\epsilon_q |j|\}$.

Let us denote $\Lambda^+ = \cup_{q>t\geq 1; l\geq 1}\Lambda^{l^+}_{q,t}$. The theorem of Oseledec implies that Λ^+ has full probability. For $S \in \Lambda^+$, let us take q and t such that the dimension of $E^{l^+}_{2,S,q,t}$ is maximum, and denote $E^+_{2,S} = E^{l^+}_{2,S,q,t}$.

It is clear that for S a regular point, $E^+_{2,S}$ is equal to the direct sum of the spaces corresponding to positive Lyapounov exponent. We denote by $\Lambda^{l^+}_{u,q,t}$ the set of points $S \in \Lambda^{l^+}_{q,t}$ with $\dim E^+_{2,S} = u$.

Definition 4.4. For $S \in \Lambda^+$, the set

$$W^u(S, f) = \left\{ S^* \in M : \limsup_{i\to\infty} \log d(f^{-i}(S), f^{-i}(S^*))/i < 0 \right\}$$

is called *the unstable manifold of f at S*.

$W^u(S, f)$ is an immersed submanifold of M of class C^{r-1} tangent at S to $E^+_{2,S}$. If $\tilde{S} \in W^u(S, f)$, then $\tilde{S} \in \Lambda^+$ (see [**FHY83**]).

Definition 4.5. The set $\{W^u(S, f)\}_{S\in\Lambda^+}$ is called the *unstable foliation of f* and is denoted $W^u(f)$.

It is an invariant foliation. The following assertion follows from Theorems 2.2.1 and 2.3.1 of [**P73**] and of [**FHY83**].

THEOREM 4.6. *There exist $\tilde{\delta}^{l^+}_{q,t} > 0$, $0 < \chi^+_{q,t} < 1$, $L^+ > 0$ such that for $S \in \Lambda^{l^+}_{u,q,t}$ with $E^+_{2,S} = E^{l^+}_{2,S,q,t}$, there exists a local unstable manifold $V^+(S) \ni S$ such that:*
(i) $T_S V^+(S) = E^+_{2,S}$;
(ii) $V^+(S) = B_{V^+(S)}(S, \tilde{\delta}^{l^+}_{q,t})$ *(we recall the notation of §3.2);*
(iii) *for $\tilde{S} \in V^+(S)$ and $n \in \mathbb{Z}^+$ we have $d(f^{-n}(S), f^{-n}(\tilde{S})) \leq L(\chi^+_{q,t})^n d(S, \tilde{S})$;*
(iv) $W^u(S, f) = \cup_{n\geq 0} f^n(V^+(f^{-n}(S)))$.

We denote by $\Lambda^{l^-}_{q,t}$, $q > t \geq 1$, the set of points $S \in M$ such that for $i \in \mathbb{Z}^+$, $j \in \mathbb{Z}$, and $\epsilon_q = (1/100)\log(1 + 2/q)$:

(i) there exist subspaces $E^{l^-}_{1,S,q,t}$ and $E^{l^-}_{2,S,q,t}$ for which $T_S M = E^{l^-}_{1,S,q,t} \oplus E^{l^-}_{2,S,q,t}$.

(ii)

$$\begin{aligned}
&\|Df^i_{f^j(S)}v\| \leq l(t/q)^i \exp\{\epsilon_q i + 4\epsilon_q|j|\}\|v\| \quad \text{and}\\
&\|Df^{-i}_{f^j(S)}v\| \geq l^{-1}(t/q)^{-i} \exp\{-\epsilon_q i - 4\epsilon_q|j|\}\|v\| \quad \text{for } v \in Df^j E^{l^-}_{1,S,q,t};\\
&\|Df^i_{f^j(S)}v\| \geq l^{-1}((t+2)/q)^i \exp\{-\epsilon_q i - 4\epsilon_q|j|\}\|v\| \quad \text{and}\\
&\|Df^{-i}_{f^j(S)}v\| \leq l((t+2)/q)^{-i} \exp\{\epsilon_q i + 4\epsilon_q|j|\}\|v\| \quad \text{for } v \in Df^j E^{l^-}_{2,S,q,t}.
\end{aligned}$$

(iii) Let $\gamma^{l^-}_{q,t}(f^j(S))$ be the angle between $Df^j(E^{l^-}_{1,S,q,t})$ and $Df^j(E^{l^-}_{2,S,q,t})$. Then $\gamma^{l^-}_{q,t}(f^j(S)) \geq \exp\{\epsilon_q|j|\}/l$.

As before, $\Lambda^- = \cup_{q>t\geq 1; l\geq 1}\Lambda^{l^-}_{q,t}$ has full probability. For $S \in \Lambda^-$, let us take l, q, and t such that the dimension of $E^{l^-}_{1,S,q,t}$ is maximum, and denote $E^-_{1,S} = E^{l^-}_{1,S,q,t}$. It is well defined, and for S a regular point, $E^-_{1,S}$ is equal to the direct sum of the subspaces corresponding to the negative Lyapounov exponents.

The *stable manifold at* S, denoted $W^s(S, f)$, and the *stable foliation* $W^s(f)$ are defined as the unstable manifold by writing Λ^- and f^{-1} instead of Λ^+ and f. We denote the local stable manifold at S as $V^-(S)$.

If W is an immersed C^1 manifold of M, then it inherits a Riemannian structure from M, and we can define a corresponding Lebesgue measure. In particular, we are interested in the case in which W is an unstable manifold; the corresponding Lebesgue measure will be denoted by ν^u, given by the understood manifold. The Lebesgue measure on M will be denoted by ν.

Let $\mathcal{B}$ be the Borel σ-algebra on M completed with respect to some f-invariant Borel probability μ.

Definition 4.7. A *measurable partition* ξ *of* M is a partition of M such that, up to a set of measure zero, the quotient space M/ξ is separated by a countable number of measurable sets (see [**R62**]).

Associated with each measurable partition there is a *canonical system of conditional measures,* that is, for every S in a set of μ-invariant measure one, there is a probability measure μ_S^ξ defined on $\xi(S)$, the element of ξ containing S. These measures are characterized (up to sets of μ-measure zero) by the following properties: if $\mathcal{B}_\xi$ is the sub-σ-algebra of $\mathcal{B}$ whose elements are unions of elements of ξ, and $A \subset M$ is a $\mathcal{B}$-measurable set, then $S \mapsto \mu_S^\xi(A)$ is $\mathcal{B}$-measurable, and $\mu(A) = \int_M \mu_S^\xi(A)\, d\mu(S)$ (see [**LY85**]).

Definition 4.8. Let ξ be a measurable partition of M. We say that ξ is *subordinate to the* $W^u(f)$*-foliation* if for μ a.e. S, we have $\xi(S) \subset W^u(S, f)$ and $\xi(S)$ contains an open neighborhood of S in the submanifold topology of $W^u(S, f)$.

Definition 4.9. We say that μ has *absolutely continuous conditional measures on unstable manifolds* if for every measurable partition ξ subordinate to $W^u(f)$, μ_S^ξ is absolutely continuous with respect to ν^u for μ a.e. S.

THEOREM 4.10. ([**LY85**]) *μ has absolutely continuous conditional measures on unstable manifolds if and only if*

$$h_\mu(f) = \int_M \sum_{i:\chi_i(S)>0} \chi_i(S) \dim E_i(S)\, d\mu(S)$$

where $h_\mu(f)$ is the metric entropy of f.

For the proof, see [**LY85**]. In fact, it is shown there that μ_S^ξ is equivalent to ν^u for μ a.e. S if the formula of the theorem is verified.

Definition 4.11. We say that μ is an *SRB probability for* f if it is ergodic and has absolutely continuous conditional measures on unstable manifolds.

PROPOSITION 4.12. *In the hypothesis of Theorems 1(a) or 2, there exists a unique SRB probability for F_1. This measure has an associated ergodic attractor.*

Proof. We adapt a construction exposed in [**PS82**] (see also [**C93**]). The proof will be developed in this and in the following subsection. We begin with some preliminary results which we use later.

LEMMA 4.13. *For a.e.* $S_0 \in \Lambda$, *and* $S_1 \in W^u(S_0, F_1)$, *we have* $T_{S_1} W^u(S_0, F_1) = I_{S_1}$. *Analogously, for* $S_2 \in W^s(S_0, F_1)$, $T_{S_2} W^s(S_0, F_1) = E_{S_2}$.

Proof. We only prove the first of these two assertions. We know that the lemma is valid for $S_1 \in W^u(P_{F_1}, F_1)$ and that $\dim W^u(S_0, F_1) = u$ for a.e. S_0. We consider $S_0 \notin W^u(P_{F_1}, F_1)$ (and therefore S_1 is not an iterate of R_{F_1}). Then, there exists a sequence $\{l_i\}_{i \in \mathbb{Z}^+}$, $l_i = l_i(S_0) \in \mathbb{Z}^+$, l_i going to infinity when i goes to infinity, such that $F_1^{l_i}(S_0)$ has a neighborhood $\mathcal{K}(S_0, l_i) \subset W^u({F_1}^{l_i}(S_0), F_1)$ of uniform size such that if a point $\tilde{S}$ belongs to that neighborhood, then $C'_{1,\tilde{S}} = C_{1,\tilde{S}}$; $C'_{2,\tilde{S}} = C_{2,\tilde{S}}$. Let us suppose, by contradiction, that for some $S_1 \in W^u(S_0, F_1)$, $T_{S_1} W^u(S_0, F_1) \neq I_{S_1}$. Iterating backwards if necessary, we find l_i such that $F_1^{-l_i}(S_1) \in \mathcal{K}(S_0, l_i)$, with $T_{F_1^{l_i}} W^u(F_1^{-l_i}(S_0), F_1) \neq I_{F_1^{-l_i}(S_0)}$. For simplicity, let us suppose that $S_1 \in \mathcal{K}(S_0, l_i)$. Also iterating backwards if necessary, we can suppose that $T_{S_1} W^u(S_0, F_1) \in C'_{1,S}$, and then by continuity, $T_S W^u(S_0, F_1) \in C'_{1,S}$ for S in a neighborhood of S_1 on $W^u(S_0, F_1)$. Lemmas 3.7 and 3.8 assert that $d(F_1^{-i}(S_1), F_1^{-i}(S_0))$ cannot converge to zero when i goes to infinity, a contradiction. □

LEMMA 4.14. *For* $S_0 \in \Lambda$, *then* $W^u(S_0, F_1) \subset W^u(S_0)$ *and* $W^s(S_0, F_1) \subset W^s(S_0)$.

Proof. The contradiction hypothesis asserts that there are ramification points as like those in Part (iii) of the proof of Lemma 3.10. □

LEMMA 4.15. *There exists a set of full probability* $\Lambda' \subset \Lambda$ *such that for* $S_0 \in \Lambda'$, $W^s(S_0) \subset W^s(S_0, F_1)$.

Proof. (See Theorem 4.4 of [**P77**].) Let us take an invariant probability, $\bar{\mu}$, and l, q, t such that $\Lambda_{s,q,t}^{l-}$ has positive measure. We consider the corresponding diameter of the stable manifolds (see Theorem 4.6), which we denote by $\tilde{\delta}_{q,t}^{l-}$. Let S be a density point of $\Lambda_{s,q,t}^{l-}$. It is not a restriction to consider S in $\mathcal{H}$, at most we have to change the value of l. Let us take $A \subset \Lambda_{s,q,t}^{l-} \cap B(S, m)$, m smaller or equal to the value of r defined in Lemma 3.7 (we have fixed τ_0). For $\tilde{S} \in A$ we denote by $n_i(\tilde{S})$, $i = 1, 2, \ldots,$ the successive numbers for which $F_1^{n_i(\tilde{S})}(\tilde{S}) \in A$.

We claim that for $\bar{\mu}$ a.e. $\tilde{S} \in A$,

$$W^s(\tilde{S}, F_1) = \cup_{i=1}^{\infty} F_1^{-n_i(\tilde{S})}(V^-(F_1^{n_i(\tilde{S})}(\tilde{S}))).$$

To prove the claim, let us observe that if $\tilde{A}$ is the set of points $\tilde{S} \in A$ such that the sequence $\{n_i(\tilde{S})\}_{i \in \mathbb{Z}^+}$ is infinite, $\bar{\mu}(\tilde{A}) = \bar{\mu}(A)$. It is also clear that

$$W^s(\tilde{S}, F_1) \supset \cup_{i=1}^{\infty} F_1^{-n_i(\tilde{S})}(V^-(F_1^{n_i(\tilde{S})}(\tilde{S}))).$$

Let $\tilde{S} \in \tilde{A}$, $S^* \in W^s(\tilde{S}, F_1)$. As $\lim_{j \to \infty} d(F_1^j(S^*), F_1^j(\tilde{S})) = 0$, then for large enough i,

$$d_{W^s(F_1^{n_i(\tilde{S})}(\tilde{S}))}(F_1^{n_i(\tilde{S})}(\tilde{S}), F_1^{n_i(\tilde{S})}(S^*)) < \tilde{\delta}_{q,r}^{-l}/2$$

and so $F_1^{n_i(\tilde{S})}(S^*) \in V^-(F_1^{n_i(\tilde{S})}(\tilde{S}))$, proving the claim.

After the results of Lemma 3.12, we know that $\liminf_{j \to \infty} \operatorname{diam} F_1^j(B_{W^s(\tilde{S})}(\tilde{S}, m)) = 0$ (the limit is obtained by taking iterates in $\mathcal{H}$). So, there exists large enough i such that $F_1^{n_i(\tilde{S})}(B_{W^s(\tilde{S})}(\tilde{S}, m)) \subset V^-(F_1^{n_i(\tilde{S})}(\tilde{S}))$. □

LEMMA 4.16. $\varphi(W^s(S_1)) = W^s(\varphi(S_1), f)$. *Analogously,* $\varphi(W^u(S_1)) = W^u(\varphi(S_1), f)$.

Proof. The assertion is true for $S_1 = P_{F_1}$, and then, by density, reasoning as in Lemma 3.12, we conclude this lemma. □

We continue with the proof of Proposition 4.12. We suppose $S \in W^u(S^*, F_1)$ for some $S^* \in \Lambda$, let

$$\mathcal{J}_{F_1}(S) = |\operatorname{Jac} DF_1|_{T_S W^u(S^*, F_1)}|$$

with respect to the Riemannian volume ν^u on $W^u(S^*)$. Let us fix $S^* \in \Lambda'$. For simplicity, we take $S^* \notin W^u(Q_{F_1}, F_1)$; $S^* \notin W^u(P_{F_1}, F_1)$. For $m \in \mathbb{Z}^+$, $k \in \mathbb{Z}$, $S', \tilde{S} \in W^u(F_1^k(S^*), F_1)$, we denote

$$R_m(\tilde{S}, S') = \prod_{l=0}^{m-1} \mathcal{J}_{F_1}(F_1^{-l}(\tilde{S}))/\mathcal{J}_{F_1}(F_1^{-l}(S')).$$

LEMMA 4.17. *For $S', \tilde{S}$ as before, there exists* $R(\tilde{S}, S') = \lim_{m\to\infty} R_m(\tilde{S}, S')$.

Proof. We have to prove that the series of general terms

$$|(\mathcal{J}_{F_1}(F_1^{-l}(\tilde{S}))/\mathcal{J}_{F_1}(F_1^{-l}(S'))) - 1|$$

is convergent. Let us observe that in the compact product space whose points are (S, V^u), with $S \in M$, $V^u \in G_S(u, T_S M)$ (the Grassmanian of the subspaces of dimension u in $T_S M$), the function $|\det DF_1|_{V^u}(S)|$ is C^{r-1} and bounded from below by a positive number $\tilde{K}$. Therefore, $\mathcal{J}_{F_1}(F_1^{-m}(S')) \geq \tilde{K} > 0$. We know that $F_1^{-j}(S'), F_1^{-j}(\tilde{S}) \in V(F_1^{k-j}(S^*))$ for some j, and then, as on $V(F_1^{k-j}(S^*)), \mathcal{J}_{F_1}$ is C^{r-1}, so we conclude that for $l \geq j$

$$\begin{aligned} |\mathcal{J}_{F_1}(F_1^{-l}(\tilde{S})) - \mathcal{J}_{F_1}(F_1^{-l}(S'))| &\leq K d_{W^u(\tilde{S})}(F_1^{-l}(\tilde{S}), F_1^{-l}(S')) \\ &\leq K d(F_1^{-l}(\tilde{S}), F_1^{-l}(S')) \\ &\leq K(\chi_{F_1^{k-j}(S^*)})^{l-j}(2\delta_{F_1^{k-j}(S^*)}) \end{aligned}$$

with $0 < \chi_{F_1^{k-j}(S^*)} < 1$ (using Theorem 4.6). □

LEMMA 4.18. *There exist $\tilde{\epsilon} > 0$, $L > 0$ such that if $S_a, S_b \in W^u(S)$ for $W^u(S) \neq W^u(P_{F_1}, F_1)$, $W^u(S) \neq W^u(Q_{F_1}, F_1)$ with $d_{W^u(F_1^k(S), F_1)}(S_a, S_b) < \tilde{\epsilon}$, and such that the arc S_a, S_b on $W^u(S)$ is in $\mathcal{H}$, then $R(S_a, S_b) < L$.*

Proof. Let us consider first the case in which one of the points, say S_a, does not belong to $\tilde{\mathcal{U}}_1$, while $F_1^{-1}(S_a) \in \tilde{\mathcal{U}}_1$. We can suppose that the curve C with extreme points S_a, S_b on $W^u(S)$ is in $\tilde{\mathcal{U}}$ (the domain of the linearizing coordinates) just taking $\tilde{\epsilon}$ small. Let us suppose that $S_{1,a} = F_1^{-m_1}(S_a) \in \mathcal{E}$, and let $(\tilde{x}_{1,a}, \tilde{y}_{1,a})$ be the coordinates of $S_{1,a}$. It is not a restriction to suppose that $\tilde{\epsilon}$ is small enough such that $S_{1,b} = F_1^{-m_1}(S_b)$ is in $\tilde{\mathcal{U}}_1$, with coordinates $(\tilde{x}_{1,b}, \tilde{y}_{1,b})$. Let us fix $0 < \epsilon < \tilde{\epsilon}$, and take C with diameter ϵ. We project C over the axis $O\tilde{x}$ following the lines $\tilde{x} = K$; such a projection will have a smaller size than $K\epsilon$ with K a constant. We know that $m_1 = E(-\log|\tilde{x}_{1,a}|/\log|\mu_Q|)$; after m_1 iterates by F_1^{-1} that projection will have size bounded by $K\epsilon\mu_Q^{-m_1}$, where $\mu_Q^{-m_1}$ is of the order of $|\tilde{x}_{1,a}|$. As the angle determined by $\tilde{x} = K$ and $I_{S''}$ with $S'' \in \mathcal{E}$ varies as $\tilde{k}(\tilde{\rho}^2(S'')+$higher-order terms), we deduce that the difference $|\tilde{y}_{1,a} - \tilde{y}_{1,b}|$ is bounded by $K\epsilon^{1/3}\mu_Q^{-m_1/3}$ where K is a constant. Let us consider a vector in $I_{S''}$, it has the direction of

$$v(S'') = (\sin \tilde{k}(\tilde{\rho}^2(S'') + \text{higher-order terms}), \cos \tilde{k}(\tilde{\rho}^2(S'') + \text{higher-order terms})).$$

For S'' near to $S_{1,a}$, m_1 iterates, $v(S'')$ will take the components

$$(\mu_Q^{m_1} \sin \tilde{k}(\tilde{\rho}^2(S'') + \text{higher-order terms}), \lambda_Q^{m_1} \cos \tilde{k}(\tilde{\rho}^2(S'') + \text{higher-order terms})).$$

For $\mathcal{E}$ small, in any case the first component predominates, and therefore the rate of the moduli of the vectors corresponding to S_a, S_b will be given, in the $(\tilde{x}, \tilde{y})$ system of coordinates, by

$$\sin \tilde{k}(\tilde{\rho}^2(S_a) + \text{higher-order terms})/\sin \tilde{k}(\tilde{\rho}^2(S_b) + \text{higher-order terms}).$$

As the difference between the abscissas $\tilde{x}_{1,a}$ and $\tilde{x}_{1,b}$ is bounded by $K\epsilon|\tilde{x}_{1,a}|$, the difference between the ordinates is bounded by $K(\epsilon|\tilde{x}_{1,a}|)^{1/3}$; taking into account the fact that the arc of the unstable leaf between $S_{1,a}$ and $S_{1,b}$ has its curvature bounded, and that it is a C^r arc, we deduce that the rate will be smaller than $1 + K(\epsilon|\tilde{x}_{1,a}|)^{1/3}$. Now, as $\mathcal{E}$ is bounded, and therefore so is $|\tilde{x}_{1,a}|$, we can bound that rate (now taking the Riemannian metric on the manifold) by $1 + K\epsilon^{1/3}$.

Now we continue the reasoning in $\mathcal{U}_1^*$. We start with an interval in the unstable foliation of size bounded by $K\epsilon^{1/3}\mu_Q^{-m_1/3} < K_1\epsilon^{1/3}|y_{1,a}^*|$ where $S_{1,a} = (x_{1,a}^*, y_{1,a}^*)$, $S_{1,b} = (x_{1,b}^*, y_{1,b}^*)$, and K_1 is a constant. We need $p_1 = E(\log|y_{1,a}^*|/\log|\lambda_P|)$ iterates by F_1^{-1} to leave $\mathcal{U}_1^*$. Let $S_{2,a} = F_1^{-p_1}(S_{1,a})$, $S_{2,b} = F_1^{-p_1}(S_{1,b})$. As before, taking unit vectors belonging to $I_{S''}$ in the $\mathcal{U}_1^*$ system at $S_{2,a}$, $S_{2,b}$, after applying $F_1^{p_1}$, we obtain that the rate is again bounded by $1 + K\epsilon^{1/3}$.

We recall also (see Lemma 2.6) that the length of $F_1^{-m_1-p_1}(C)$ is smaller that $\epsilon/2$. Then, if we continue iterating by F_1^{-1}, until entering $\mathcal{D}$ again, we must change ϵ by a number smaller than $\epsilon/2$, and the rates in $\tilde{\mathcal{U}}_1$ and $\mathcal{U}_1^*$ will be bounded by $1+K\epsilon^{1/3}/2^{1/3}$, and so on.

If we consider the iterates in $\mathcal{H}$, the size of $F_1^{-j}(C)$ decreases in each iterate with a factor at most $\chi < 1$ and, reasoning as in Lemma 4.17, the rate in each iterate is bounded by $1 + K\chi^j\epsilon$ (*we consider only the iterates in which* $F_1^{-j}(C) \in \mathcal{H}$).

Taking S_a, S_b as in the conditions of the hypothesis, we consider the series of general term

$$|(\mathcal{J}_{F_1}(F_1^{-l}(S_a))/\mathcal{J}_{F_1}(F_1^{-l}(S_b))) - 1|.$$

Reordering it according to the position of the points (in $\mathcal{H}$, in $\mathcal{D} \cap \tilde{\mathcal{U}}_1$ or in $\mathcal{D} \cap \mathcal{U}_1^*$), we conclude that the series is uniformly convergent, and therefore, uniformly bounded. □

Remark 4.19. The lemma also follows if the points S_a, S_b are in a neighborhood of $W^s_{\text{loc}}(P_{F_1}, F_1)$ (the set of points with coordinates $(0, t)$, $0 \le |t| \le 1$, in the (x^*, y^*) system of coordinates). This is so because in a neighborhood of $W^s_{\text{loc}}(P_{F_1}, F_1)$, I_S is in $C_{2,S}$, and therefore the bound $1 + K\chi^j\epsilon$ is valid. After a finite number of iterates by F_1^{-1} they enter $\mathcal{H}$, and the proof ends as before.

Now we construct a measure, a component of which will be the SRB measure of our theorem. We begin with a construction of [**PS82**].

Let us fix $S^* \in \Lambda$ (the construction does not depend on S^*). We choose a small open neighborhood U_0 of S^* on $W^u(S^*)$ such that $\int_{U_0} R(S^*, S)\, d\nu^u(S) = 1$. Let us define $U_k = F_1^k(U_0)$, $c_0 = 1$, $c_k = [\prod_{l=0}^{k-1} \mathcal{J}_{F_1}(F_1^l(S^*))]^{-1}$, $k \geq 1$, and a measure $\tilde{\nu}_k$ on U_k: $d\tilde{\nu}_k(S) = c_k R(F_1^k(S^*), S)\, d\nu^u(S)$. The measures $\tilde{\nu}_k$ are probabilities on U_k. We define Borel measures ν_k on M so that for any Borel set $A \subset M$, $\nu_k(A) = \tilde{\nu}_k(A \cap U_k)$. Next, we define $\mu_i = (1/i) \sum_{k=0}^{i-1} \nu_k$. Any accumulation point μ' of the set $\{\mu_i\}$ is an F_1-invariant probability. Let W be a small manifold, which we take transversal to the foliation W^u (then, $\dim W = s$). We fix $t \leq s(r)$ (recall Lemma 3.7) and define the closed set $\mathcal{N} = \{S' \in M : \exists \tilde{S} \in W, S' \in B_{W^u(\tilde{S})}(\tilde{S}, t)\}$. We remark that we could take t depending on $\tilde{S}$. The definition of $\mathcal{N}$ allows us to define a function, $\pi : \mathcal{N} \mapsto W$, with $\pi(S') = \tilde{S}$. The sets $B_{W^u(\tilde{S})}(\tilde{S}, t)$ determine a measurable partition of $\mathcal{N}$ and then the conditional measures on the elements of the partition are well defined.

We define

$$A^{(k)} = \{\tilde{S} \in W : U_k \cap B_{W^u(\tilde{S})}(\tilde{S}, t) \neq \emptyset\}; \quad B^{(k)} = \{\tilde{S} \in W : \partial U_k \cap B_{W^u(\tilde{S})}(\tilde{S}, t) \neq \emptyset\};$$
$$C^{(k)} = A^{(k)} \setminus B^{(k)}; \quad D^{(k)} = \{S \in W^u(F_1^k(S^*)) : d_{W^u(F_1^k(S^*))}(S, \partial U_k) \leq 2t\}.$$

Let $h : M \mapsto \mathbb{R}^+$ be a continuous function with support in $\mathcal{N}$, we consider

$$g : W \cap \cup_{k\in\mathbb{Z}} W^u(f^k(S^*), F_1) \mapsto \mathbb{R}$$
$$g(\tilde{S}) = \int_{B_{W^u(\tilde{S})}(\tilde{S},t)} \frac{h(S')R(\tilde{S}, S')}{N_t(\tilde{S})}\, d\nu^u(S') \quad \text{where } N_t(\tilde{S}) = \int_{B_{W^u(\tilde{S})}(\tilde{S},t)} R(\tilde{S}, S')\, d\nu^u(S').$$

We can write

$$\begin{aligned} \int_{\mathcal{N}} h(S')\, d\nu_k(S') &= \sum_{\tilde{S}\in A^{(k)}} \int_{B_{W^u(\tilde{S})}(\tilde{S},t)\cap U_k} h(S')\, d\tilde{\nu}_k(S') \\ &= \sum_{\tilde{S}\in C^{(k)}} \int_{B_{W^u(\tilde{S})}(\tilde{S},t)\cap U_k} h(S')\, d\tilde{\nu}_k(S') \\ &\quad + \sum_{\tilde{S}\in B^{(k)}} \int_{B_{W^u(\tilde{S})}(\tilde{S},t)\cap U_k} h(S')\, d\tilde{\nu}_k(S') = I_1^{(k)} + I_2^{(k)}. \end{aligned}$$

The key fact is that $I_2^{(k)}$ goes to zero with k:

$$|I_2^{(k)}| \leq \max_{\mathcal{N}}\{h\} \tilde{\nu}_k(D^{(k)}) = K c_k \int_{D^{(k)}} R(F_1^k(S^*), \tilde{S})\, d\nu(\tilde{S}) \leq K \nu^u(F_1^{-k}(D^{(k)})).$$

Now, $F_1^{-k}(D^{(k)})$ is contained in an annulus based in the boundary of U_0 of measure going to 0 when k goes to infinity because the vectors on TW^u expand if they are iterated enough (except for one direction with base point on the iterates of R).

Let us consider

$$I_1^{(k)} = \sum_{\tilde{S}\in C^{(k)}} \int_{B_{W^u(\tilde{S})}(\tilde{S},t)\cap U_k} h(S')\, d\tilde{\nu}_k(S') = \int_W c_k N_t(\tilde{S}) R(F_1^k(S^*), \tilde{S}) g(\tilde{S})\, d\delta^{(k)}(\tilde{S}).$$

Here, $\delta^{(k)}$ is the measure in W, concentrated in $C^{(k)}$ such that for $J \subset C^{(k)}$, $\delta^{(k)} = \#J$. Let $i_j \to_{j\to\infty} \infty$ be such that $\lim_{j\to\infty}(1/i_j)\sum_{k=0}^{i_j-1}\nu_k = \mu'$ in the weak topology. Then

$$\begin{aligned}\int_{\mathcal{N}} h(S')\,d\mu' &= \lim_{j\to\infty}\frac{1}{i_j}\sum_{k=0}^{i_j-1}(I_1^{(k)}+I_2^{(k)}) = \lim_{j\to\infty}\frac{1}{i_j}\sum_{k=0}^{i_j-1} I_1^{(k)}\\ &= \lim_{j\to\infty}\int_W \frac{1}{i_j}\sum_{k=0}^{i_j-1} c_k N_t(\tilde S)R(F_1^k(S^*),\tilde S)g(\tilde S)\,d\delta^{(k)}(\tilde S).\end{aligned}$$

The sequence of measures $\{\sigma_{i_j}\}_{j\in\mathbb{Z}^+}$ with

$$d\sigma_{i_j} = \frac{1}{i_j}\sum_{k=0}^{i_j-1} c_k N_t(\tilde S)R(F_1^k(S^*),\tilde S)\,d\delta^{(k)}(\tilde S)$$

verifies $\sigma_{i_j}(W) \le 1$ for $j \in \mathbb{Z}^+$. We can write

$$\int_{\mathcal{N}} h(S')\,d\mu'(S') = \lim_{j\to\infty}\int_W d\sigma_{i_j}(\tilde S)\int_{B_{W^u(\tilde S)}(\tilde S,t)}\frac{R(\tilde S,S')h(S')}{N_t(\tilde S)}\,d\nu^u(S').$$

In the hypothesis of [**PS82**],

$$\int_{B_{W^u(\tilde S)}(\tilde S,t)}\frac{R(\tilde S,S')h(S')}{N_t(\tilde S)}\,d\nu^u(S')$$

is continuous as a function of S, and therefore the limit can be introduced under the integral sign, ending the construction, but this is not our case, and so we must modify this argument.

Definition 4.20. We define the *dynamical ball of p iterates and radius ϵ centered at S* as

$$B_p^s(S,\epsilon) = \{\tilde S \in M : d(F_1^i(\tilde S), F_1^i(S)) \le \epsilon, 0 \le i \le p\}.$$

LEMMA 4.21. *There exists a probability μ (a component of μ') such that for a.e. μ point S in M, and $0 < A < 1$, there exist a constant $C = C(S)$, $\epsilon_0 = \epsilon_0(S) > 0$, and an increasing sequence $\{p_i\}_{i\in\mathbb{Z}^+}$, $p_i = p_i(S,A)$, such that for $0 < \epsilon < \epsilon_0$*

$$\mu(B_{p_i}^s(S,\epsilon)) \le C\left[A^{p_i}\prod_{l=0}^{p_i-1}\mathcal{J}_{F_1}(F_1^l(S))\right]^{-1}.$$

Proof. First, we will prove that the assertion of the lemma is valid for the measure μ' and $S \in \Lambda$, $S \ne Q_{F_1}$. Later, we will prove that $\mu'(Q_{F_1}) \ne 1$. Then, the lemma follows taking μ such that $\mu(\cdot) = \mu'(\cdot)/\mu'(M \setminus Q_{F_1})$. For simplicity, we will assume that the metric we are working with is adapted to the cones in $\mathcal{H}$, that is, in Definition 2.2 we suppose $q = 1$. We denote the interior of B as int B. Given $\epsilon' > 0$, and for $S \in \mathcal{H} \cup W^s_{\mathrm{loc}}(P_{F_1}, F_1)$, we define $\mathcal{N}(S) = \mathrm{int}\{[\tilde S, S'] : \tilde S \in B_{W^u}(S,\epsilon'), S' \in B_{W^s}(S,\epsilon')\}$. We take $0 < \epsilon' < \tilde\epsilon$ ($\tilde\epsilon$ defined by Lemma 4.18), ϵ' independent of S such that for $S \in W^s_{\mathrm{loc}}(P_{F_1}, F_1)$, $\mathcal{N}(S)$ be in the neighborhood of Remark 4.19. Finally, let $H = \{S \in \mathcal{H} : \mathcal{N}(S) \in \mathcal{H}\}$.

(a) For $S \in \{H \setminus W^s(Q_{F_1}, F_1)\} \cup W^s_{\mathrm{loc}}(P_{F_1}, F_1)$ we take $\tilde\epsilon_0$ independent of S such that for all S as before, $B(S,\tilde\epsilon_0) \subset \mathcal{N}(S)$. We define $\epsilon_0(S) = \epsilon_0 < \tilde\epsilon_0$ for all S. For points in H, we take a sequence $\{p'_i\}_{i\in\mathbb{Z}^+}$, p'_i depending on S such that $p'_1 = 0$, and p'_{i+1} verifies $p'_{i+1} > p'_i$; $F_1^j(S) \notin \mathcal{H}$ for $j = p'_i + 1, \ldots, p'_{i+1} - 1$; $F_1^{p'_{i+1}}(S) \in \mathcal{H}$, or, if $F_1^{p'_i}(S) \in W^s_{\mathrm{loc}}(P_{F_1}, F_1)$, we define $p'_{i+k} = p'_i + k$ for $k \geq 0$. We define $W(S)$ as the connected component of $W^s(S, F_1) \cap \mathcal{N}(S)$ that contains S, and $X_S(\tilde{S})$ for $\tilde{S} \in W(S)$ the connected component of $W^u(S, F_1) \cap \mathcal{N}(S)$ that contains $\tilde{S}$.

(b) If $S \in M \setminus \{H \cup W^s_{\mathrm{loc}}(P_{F_1}, F_1) \cup W^s(Q_{F_1}, F_1)\}$, let $p'_1 = p'_1(S) > 0$ be such that $F_1^{p'_1}(S)$ is the first iterate in H. We define $\mathcal{N}(S)$ as $F_1^{-p'_1}(\mathcal{N}(F_1^{p'_1}(S)))$. We observe that if S_1, S_2 belong to a connected component of $F^i(\mathcal{N}(S)) \cap W^u$, then $\mathcal{J}_{F_1}(S_1)/\mathcal{J}_{F_1}(S_2) > J(S)$ for some $J(S)$ and for $i = 0, \ldots, p'_1 - 1$ because $\mathcal{J}_{F_1}$ is bounded and bounded away from zero.

Then, we define $\epsilon_0(S)$ smaller than the value ϵ_0 obtained in (a), such that $B(S, \epsilon_0(S)) \subset \mathcal{N}(S)$ and $F_1^{p'_1}(B(S, \epsilon_0(S))) \subset B(F_1^{p'_1}(S), \tilde{\epsilon}_0)$. We define p'_{i+1} depending on S so that $p'_{i+1} > p'_i$; $F_1^j(S) \notin H$ for $j = p'_i + 1, \ldots, p'_{i+1} - 1$; $F_1^{p'_{i+1}}(S) \in H$, or, if $F_1^{p'_i}(S) \in W^s_{\mathrm{loc}}(P_{F_1}, F_1)$, we define $p'_{i+k} = p'_i + k$ for $k \geq 0$.

(c) For $S \in W^s(Q_{F_1}, F_1) \setminus \{Q_{F_1}\} \cup \{F_1^j(R)\}_{j\in\mathbb{Z}}$, we take $j \geq 0$ such that $F_1^j(S)$ is the first positive iterate in $W^s_{\mathrm{loc}}(Q_{F_1}, F_1)$. For $k \geq j$, we define $\mathcal{N}_S(F_1^k(S)) = \mathrm{int}\{[\tilde{S}, S'] : \tilde{S} \in F_1^{k-j}(B_{W^s}(F_1^j(S), \epsilon'')$, $S' \in B_{W^u}(F_1^k(S), \epsilon'')\}$ taking $\epsilon'' = \epsilon''(S)$ such that $\mathcal{N}_S(F_1^k(S)) \subset \tilde{\mathcal{U}}_1$, and such that, for $k = j$, $\mathcal{N}_S(F_1^k(S))$ does not contain any iterate of R. We note the subindex S in the last definition: we can take that set independent of S, but we allow the dependence on S to simplify the arguments. It can be seen that, although $\mathcal{N}_S(F_1^j(S))$ can intersect $\mathcal{D}$, after at most a finite number of iterates, $\mathcal{N}_S(F_1^k(S))$ enters a zone where $C'_{2,S} \subset C_{2,S}$, obtaining the condition $\mathcal{J}_{F_1}(F_1^j(S_1))/\mathcal{J}_{F_1}(F_1^j(S_2)) > 1 + K\chi^j d_{W^u}(S_1, S_2)$ for S_1, S_2 in a connected component of $W^u \cap \mathcal{N}(F^k(S))$. The reasoning follows as in (b): we define $\mathcal{N}(S) = F_1^{-k}(\mathcal{N}_S(F_1^k(S)))$, and define $\epsilon_0(S, A)$ such that $B(S, \epsilon_0(S, A)) \subset \mathcal{N}(S)$; p'_1 is equal to the former k, and $p'_{i+1} = p'_i + 1$.

Now we define the sequence p_i. We observe that after a perturbation theorem (see Theorem 4.1 in [**R79**]†), for any $m \geq m_0 = m_0(S, A)$, m_0 independent of $\tilde{S}$ in $W(S)$,

$$\prod_{l=p'_1}^{m-1} \mathcal{J}_{F_1}(F_1^l(\tilde{S})) \Big/ \prod_{l=p'_1}^{m-1} \mathcal{J}_{F_1}(F_1^l(S)) \geq A^m.$$

We take the sequence $\{p_i\}_{i\in\mathbb{Z}^+}$, $p_i = p_i(S, A)$ as a subsequence of $\{p'_i\}_{i\in\mathbb{Z}^+}$ with $p_1 \geq m_0$. We take $p_0 = 0$. Therefore, we can write for $i \in \mathbb{Z}^+$ and $S' \in B^s_{p_i}(S, \epsilon_0(S, A)) \cap \cup_{k\in\mathbb{Z}^+} W^u(F_1^k(S^*))$, so that

$$\prod_{l=p'_1}^{p_i-1} \mathcal{J}_{F_1}(F_1^l(S')) = \frac{\prod_{l=p'_1}^{p_i-1} \mathcal{J}_{F_1}(F_1^l(S'))}{\prod_{l=p'_1}^{p_i-1} \mathcal{J}_{F_1}(F_1^l(\tilde{S}))} \prod_{l=p'_1}^{p_i-1} \mathcal{J}_{F_1}(F_1^l(\tilde{S})) \geq \frac{A^{p_i}}{L} \prod_{l=p'_1}^{p_i-1} \mathcal{J}_{F_1}(F_1^l(S)).$$

Now, defining $\mathcal{N}_S(F_1^k(S)) = \mathcal{N}(F_1^k(S))$ for $k \geq 1$, $F_1^k(S) \in H$, $\mathcal{N}_S(S) = \mathcal{N}(S)$ and $C(S, p_i)$ the connected component of $\mathcal{N}_S(S) \cap F_1^{-p_i}(\mathcal{N}_S(F_1^{p_i}(S)))$ that contains S, we

† The assertion follows from that theorem and former considerations about the associated filtrations in [**R79**] for the case $u = 1$. For $u > 1$, we apply the same theorem with $T^{\wedge u}$ instead of T (notation of [**R79**]). See also Theorem 7.2 of [**KS86**], although there the hypotheses are somewhat stronger.

see that for $\epsilon < \epsilon_0$

$$\mu'(B_{p_i}(S,\epsilon)) \leq \liminf_{j\to\infty} \tilde{\mu}_j(\cap_{l=0}^{i} F_1^{-p_l}(\mathcal{N}_S(F_1^{p_l}(S)))) = \liminf_{j\to\infty} \tilde{\mu}_j(C(S,p_i))$$

where $\tilde{\mu}_j$ is a measure in $\mathcal{N}(S)$ such that for $g(S)$ continuous supported in $\mathcal{N}(S)$ we have

$$\int_{\mathcal{N}(S)} g(S')\,d\tilde{\mu}_j(S') = \int_{W(S)} d\sigma_{i_j}(\tilde{S}) \int_{X(\tilde{S})} \frac{R(\tilde{S},S')g(S')}{N_t(\tilde{S})}\,d\nu^u(S').$$

So

$$\tilde{\mu}_j(\mathcal{N}(S)) = \int_{W(S)} d\sigma_{i_j}(\tilde{S}) \int_{X(\tilde{S})} \frac{R(\tilde{S},S')}{N_t(\tilde{S})}\,d\nu^u(S').$$

$R(\tilde{S},S')$ is bounded and bounded away from zero on $\mathcal{N}(S) \times \mathcal{N}(S)$ with a bound depending on S because of Lemma 4.18, Remark 4.19 and the former conditions in (a), (b) or (c) (at worst we have to take a finite number of forward or backward iterates to get out of $\mathcal{D}$, and then apply the lemma or the former conditions). Therefore, there exists $q = q(S)$ such that $q^{-1} < R(\tilde{S},S') < q$ and so we can write, examining the situation on each $X(\tilde{S})$, that

$$\tilde{\mu}_j(C(S,p_i)) \leq \int_{W(S)} d\sigma_{i_j}(\tilde{S}) \int_{X(F_1^{p_i}\tilde{S})} \frac{q^2}{J(S)^{p'_1}\epsilon''(S)A^{p_i}\prod_{l=p'_1}^{p_i-1}\mathcal{J}_{F_1}(F_1^l(S))}\,d\nu^u(S')$$

where ϵ'' is a lower bound of the lengths of $X(\tilde{S})$ for $\tilde{S} \in W(S)$. Observing that σ_{i_j} is a bounded measure we deduce the lemma (the difference with the lower limit of the productory is compensated with C).

As $\sum_{j\in\mathbb{Z}} \mu'(F_1^j(R)) = 0$, we are left to prove that $\mu'(Q_{F_1}) \neq 1$. This is true because the effects of the isotopy become more and more unimportant as we iterate the points in $\mathcal{D}$: let us assume that $\mu'(Q_{F_1}) = 1$. Let $\mathcal{N}(S) = \mathrm{int}\{[\tilde{S},S'] : \tilde{S} \in B_{W^s}(Q_{F_1},\epsilon'), S' \in B_{W^u}(Q_{F_1},\epsilon')\}$. In order that $\mu'(Q_{F_1}) = 1$, the only possibility is that σ_{i_j} (the measure on $B_{W^s}(Q_{F_1},\epsilon')$) is more and more concentrated with j near Q_{F_1}, and with the measure $\tilde{\mu}_j^u$ on $X(\tilde{S})$ given by

$$d\tilde{\mu}_j^u(S') = (R(\tilde{S},S')/N_t(\tilde{S}))\,d\nu^u(S')$$

also more and more concentrated S' near to Q_{F_1} for $\tilde{S}$ more and more near to Q_{F_1}. This last condition is not possible for points $\tilde{S} \in \mathcal{H}$, so we are left to study the case in which σ_{i_j} is more and more concentrated on $\mathcal{D}$ near to Q_{F_1}. However, after the results of §3 we observe that if $\tilde{S}$ in $B_{W^s}(Q_{F_1},\epsilon')$ is in $\mathcal{D}$, after a finite number of iterates it behaves as a point in $\mathcal{H}$: eventually I_S enters the cones corresponding to f, and after a finite number of iterates it behaves as a point leaving $\mathcal{D}$ and entering $\mathcal{H}$. So, we must have a sequence of points $\tilde{S}_i$ in $B_{W^s}(Q_{F_1},\epsilon)$ which approaches more and more to R. In this case, let us observe that, on account of the construction of the measure $\tilde{\nu}_k$ in U_k, the measure of sets around $\tilde{S}_i$ is the same as the measure of the iterate F_1^{-k} in U_0 for $\tilde{\nu}_0$. Therefore, after the expansion, the measure $\tilde{\nu}_k$ around $\tilde{S}_i$ can be written as $K(k,S)\nu^u(S)$ with $K(k,S')/K(k,S'')$ going to 1 with k going to infinity, for S', S'' near $\tilde{S}_i$. Let us assume the worst situation; let us suppose that $\tilde{S}_i$ coincides with R. In such a case, if we consider the points in $W^u(R, F_1)$ very near to R with abscissas (in the $(\tilde{x}, \tilde{y})$ system) in $[a/\mu_{Q_1}^{j+1}, a/\mu_{Q_1}^{j}]$, for j large and a in $[1, \mu_{Q_1}]$, the arc determined has measure $\tilde{\nu}_k$ of the order $K\mu_{Q_1}^{-j/3}$, with K a constant. That is, from the measure viewpoint, for high iterates of R, everything works as if the expanding eigenvalue were $\mu_{Q_1}^{1/3}$ instead μ_{Q_1}, and $W^u(R, F_1)$ were transversal. Therefore, we can see that the rate of the measure of the points near the iterates of R whose tangent space does not behave as if it were in $\mathcal{H}$ with respect the measure of the points in $\tilde{\mathcal{U}}_1$ in the same local unstable manifold remains bounded with bound smaller than 1. So, the measure of the set of points whose measure could accumulate in Q_{F_1} is smaller than 1, and therefore, $\mu(Q_{F_1}) \neq 1$. □

To end with the proof of the proposition, we adapt the proof of Proposition 5.1 in [**K88**]. Recalling the Brin–Katok definition of the entropy (see [**BK84**]) we have

$$\begin{aligned} h_\mu(F_1) &= \int_M \lim_{\epsilon\to 0}\left(\limsup_{p\to\infty}\frac{1}{p}\log[\mu(B_p^s(S,\epsilon))]^{-1}\right) d\mu(S) \\ &\geq \int_M \lim_{\epsilon\to 0}\left(\log A + \limsup_{i\to\infty}\frac{1}{p_i(S,A)}\log\prod_{l=0}^{p_i(S,A)-1}\mathcal{J}_{F_1}(F_1^l(S))\right) d\mu(S). \end{aligned}$$

As A can be taken arbitrarily near to 1, and as from the theorem of Oseledec it follows that for S in a set of full probability

$$\lim_{p\to\infty}\frac{1}{p}\log\prod_{l=0}^{p-1}\mathcal{J}_{F_1}(F_1^l(S)) = \sum_{i:\chi_i(S)>0}\chi_i(S)\dim E_i(S),$$

then

$$h_\mu(F_1) \geq \int_M \sum_{i:\chi_i(S)>0}\chi_i(S)\dim E_i(S)\, d\mu(S).$$

The inequality of Ruelle (see [**R78**]) asserts that

$$h_\mu(F_1) \leq \int_M \sum_{i:\chi_i(S)>0}\chi_i(S)\dim E_i(S)\, d\mu(S).$$

We conclude that

$$h_\mu(F_1) = \int_M \sum_{i:\chi_i(S)>0}\chi_i(S)\dim E_i(S)\, d\mu(S).$$

and this, after Theorem 4.10, implies that μ has absolutely continuous conditional measures on unstable manifolds. □

4.3. We now end the proof of Theorem 1(c), with some ideas from Theorem 3 in [**PS89**]. We know that Λ has measure one (§4.1), and that μ has absolutely continuous conditional measures on unstable manifolds.

Given a continuous function $g : M \mapsto \mathbb{R}$, let $\mathcal{G}_+$ and $\mathcal{G}_-$ be the sets of points $S \in M$ such that respectively the Birkhoff averages

$$\mathcal{B}_+(g,S) = \lim_{i\to+\infty}\frac{1}{i}\sum_{j=0}^{i-1}g(F_1^j(S)); \quad \mathcal{B}_-(g,S) = \lim_{i\to+\infty}\frac{1}{i}\sum_{j=0}^{i-1}g(F_1^{-j}(S))$$

exist. From the ergodic theorem of Birkhoff it follows that $\mathcal{G}_+ \cap \mathcal{G}_-$ is of full probability, and that the subset of points $\bar{\mathcal{G}}$ on which the two averages are equal is also of full probability and F_1-invariant.

We fix any density point S_0 of the measure, $W^u_\epsilon(S_0)$ and $W^s_\epsilon(S_0)$ with ϵ small such that the set $U_{S_0} = \{[S_1, S_2] : S_1 \in W^u_\epsilon(S_0), S_2 \in W^s_\epsilon(S_0)\}$ is well defined. We have that $\mu(U_{S_0}) > 0$. We consider $U_1 = U_{S_0} \cap \bar{\mathcal{G}} \cap \Lambda$; we know that $\mu(U_{S_0}) = \mu(U_1)$. As U_{S_0} is a measurable partition of U_{S_0} (see Definition 4.7 and subsequent comments), we can write for $U_2 \subset U_1$ that

$$\mu(U_2) = \int_{W^s_{\mathrm{loc}}(S_0)} \mu^u_{\tilde{S}}(U_2 \cap W^u_{\mathrm{loc}}(\tilde{S}))\, d\sigma(\tilde{S})$$

where $\mu^u_{\tilde{S}} \cong \nu^u_{\tilde{S}}$, with $\nu^u_{\tilde{S}}$ the Riemannian measure on $W^u(\tilde{S})$. We consider the set of points $\tilde{S} \in W^s_\epsilon(S_0)$ with $\mu^u_{\tilde{S}}(W^u_{\mathrm{loc}}(\tilde{S}, F_1)) > 0$. As $\mu(U_1) = \mu(U_{S_0}) > 0$, we know that for σ-almost every point $\tilde{S}_0$ in $W^s(S_0)$, $\mu^u_{\tilde{S}}(W^u_{\mathrm{loc}}(\tilde{S}_0, F_1) \cap U_1) = \mu^u_{\tilde{S}}(W^u_{\mathrm{loc}}(\tilde{S}_0, F_1)) > 0$, we fix $\tilde{S}_0$ with this property. We denote $B = W^u_{\mathrm{loc}}(\tilde{S}_0, F_1) \cap U_1$; we have that $\mu^u_{\tilde{S}_0}(W^u_{\mathrm{loc}}(\tilde{S}_0, F_1) \setminus B) = 0$. Let us consider $W^s_{\mathrm{loc}}(B, F_1)$. As $\nu^u_{\tilde{S}_0}(B) > 0$, and taking into account the absolute continuity of $W^s(B, F_1)$, we deduce that $W^s_{\mathrm{loc}}(B, F_1)$ has positive Lebesgue measure. The absolute continuity works when we have manifolds transverse to the stable foliation. In our case we have the transversality except at the iterates of the point R, where the transversality is only topological. So, the only manifold with which we may have some problem is $W^s(Q, F_1)$, but this manifold has a countable number of intersection points with any unstable manifold, and so this set has Lebesgue measure zero. Now, we saturate B: we consider the set $C = \cup_{i \in \mathbb{Z}} F_1^i(B)$, and let $Y_g = W^s(C, F_1)$.

LEMMA 4.22. *Let $\{g_i\}_{i \in \mathbb{Z}^+}$, $g_i : M \mapsto \mathbb{R}$ be a dense sequence in the unit ball of continuous functions $\{f \in C^0(M) : \|f\| \le 1\}$. Let $Y = \cap_{i \in \mathbb{Z}^+} Y_{g_i}$. Then $A = Y \cap \Lambda$ with the measure μ defined in §4.2 is an ergodic attractor.*

Proof. We first claim that $\mu(Y_g) = 1$. By contradiction, suppose $\mu(Y_g) < 1$; we consider a density point S_1 of the Borel probability defined by

$$\mu_1(D) = \mu((M \setminus Y_g) \cap D)/\mu(M \setminus Y_g).$$

There exists a sequence of sets $\{D_i\}_{i \in \mathbb{Z}^+}, D_i \subset \Lambda \cap (M \setminus Y_g), \mu_1(D_i) > 0$ with $D_i \in B(S_1, 1/i)$. We consider $W^s_t(S_1)$, so it follows from Lemma 4.16 that for large enough t it must intersect the interior of U, and so, taking account of Lemma 4.15, for large enough i and for almost every $S \in D_i$, $W^s_t(S, f_1) \cap U \neq \emptyset$. We observe that $W^s_t(D_i, f_1) \subset M \setminus Y_g$. As before, we construct a neighborhood $U_{S_1} \in B(S_1, 1/i)$ where we find a point $\tilde{S}_1$ with $\nu^u_{\tilde{S}_1}(W^u_{\epsilon_1}(\tilde{S}_1) \cap D_i) > 0$. Using the absolute continuity of the stable foliation, we conclude that there exists a set of positive Lebesgue measure in $W^u_{\mathrm{loc}}(\tilde{S}_0)$ which is not in B. This contradicts the fact that $\mu^u_{\tilde{S}_0}(W^u_{\mathrm{loc}}(\tilde{S}_0, f_1) \setminus B) = 0$ proving the claim. We have proved then that $\mu(A) = \mu(Y \cap \Lambda) = 1$.

Now considering $U_1 \cap Y$, we know that $\mu(U_1 \cap Y) = \mu(U_{S_0})$. Reasoning as before, but with $U_1 \cap Y$ instead U_1, we obtain a set $W^s_{\mathrm{loc}}(B, F_1) \subset Y$ with $\nu(W^s_{\mathrm{loc}}(B, F_1)) > 0$. Therefore, $\nu(Y) > 0$.

For $S \in U_1$ we know that the two Birkhoff averages $\mathcal{B}_+(g, S)$ and $\mathcal{B}_-(g, S)$ exist and are equal. By continuity, $\mathcal{G}^+$ consists of whole stable sets, and $\mathcal{G}^-$ consists of whole unstable sets. It follows that $\mathcal{B}_+(g, S)$ is a constant function on Y and therefore, for *every* continuous function $g : M \mapsto \mathbb{R}$, $\mathcal{B}_+(g, S)$ is a.e. constant. This constant must be $\int g\, d\mu$. Recalling Proposition 2.2 of [**M83**], we conclude that μ is ergodic. Moreover, we have proved that every point $S \in Y$ is generic with respect to μ. □

Now, we prove the unicity. Let us suppose there is another probability measure μ_1 absolutely continuous with respect to the unstable foliation; we take a density point of μ_1 and we reason with this point as we did with the point S_1. For fixed $g : M \mapsto \mathbb{R}$, we get a set $Y_{1,g}$ of positive Lebesgue measure which must intersect Y_g. Therefore, we can take a point S such that $\int g\, d\mu = \mathcal{B}_+(g, S) = \int g\, d\mu_1$ and, as this can be made for every continuous function, this implies $\mu = \mu_1$.

We are left to prove that F_1 is Bernoulli with respect to μ. In order to do this, we recall Theorem 5.10 of [**L83**], which asserts that if Λ has μ measure one, and if the system (M, F_1^j, μ) is ergodic for $j > 0$, then F_1 is Bernoulli with respect to μ. The same reasoning as before works to prove this last assertion, and this ends the proof of the Theorem 1.

4.4. Here we study the conservative case. First, we show that we can arrive at f_1 maintaining the given measure, and then we verify the ergodic properties. As we have stated before, we work with $n = 2$. We consider the (ξ, η) system of coordinates; by hypothesis there exists an element of volume given by $\omega(\xi, \eta)\, d\xi \wedge d\eta$ with $\omega \in C^r$. The equations $x = \Gamma(\xi, \eta)$, $y = \Delta(\xi, \eta)$ represent a change of coordinates. We want to impose that in the new system of coordinates the element of volume is given by $dx \wedge dy$, so we must impose the condition

$$D_\xi \Gamma(\xi, \eta) D_\eta \Delta(\xi, \eta) - D_\eta \Gamma(\xi, \eta) D_\xi \Delta(\xi, \eta) = \omega(\xi, \eta). \tag{11}$$

In §§1.1 and 1.3 we demanded conditions (3), (4), (7), and (8). Let us define

$$b = \frac{D_\xi \omega(0,0)}{2\omega(0,0)} + \frac{D_{x^*,y^*}\beta^*(r^*, 0)}{2D_{x^*}\alpha^*(r^*, 0) D_{y^*}\beta^*(r^*, 0)} \quad \text{and} \quad c = -\frac{D_{\tilde{x},\tilde{y}}\tilde{\alpha}(0, \tilde{r})}{D_{\tilde{x}}\tilde{\alpha}(0, \tilde{r}) D_{\tilde{y}}\tilde{\beta}(0, \tilde{r})}.$$

Clearly, $z = \xi \exp\{b\xi\}$ is an invertible function in a neighborhood of $\xi = 0$. We denote the inverse function with $\xi = \nu(z)$ (it also depends on b). Then,

$$\Gamma(\xi, \eta) = \xi \exp\{b\xi + c\eta\}$$

and

$$\Delta(\xi, \eta) = \int_0^\eta \frac{\omega(\nu(\xi \exp\{b\xi + c(\eta - t)\}), t) \exp\{-b\nu(\xi \exp\{b\xi + c(\eta - t)\}) - ct\}}{1 + b\nu(\xi \exp\{b\xi + c(\eta - t)\})}\, dt$$

verify conditions (3), (4), (7), (8), and (11). We only have to observe that $dx \wedge dy$ is θ_t invariant ($\det D\theta_t = 1$), and this proves that f_t preserves the original measure for $t \in [0, 1]$.

We have to prove Theorem 2(c). We know that Λ has measure one (§4.1), and that μ is an SRB measure by a result of Pesin (see Proposition 3.3.1 of [**P73**]). The reasoning of §4.3 is applicable, and this ends our theorem.

Acknowledgements. We thank Professor R. Mañé for his teachings, Professors J. Lewowicz and J. Palis for their constant support and Professors L. Diaz and M. Viana for many useful comments. We also thank IMPA for its hospitality and CNPq, Facultad de Ingeniería del Uruguay, and UNESCO for financial support.

REFERENCES

[BDV94] Ch. Bonatti, L. Diaz and F. Vuillemin. Topologically transverse nonhyperbolic diffeomorphisms at the boundary of the stable ones. *Bol. Soc. Bras. de Mat.* To appear.

[BK84] M. Brin and A. Katok. On local entropy. *Lecture Notes in Mathematics, 1007*. Springer, 1983, pp. 30–38.

[C93] M. Carvalho. Sinai–Ruelle–Bowen measures for N-dimensional derived from Anosov diffeomorphisms. *Ergod. Th. & Dynam. Sys.* **13** (1993), 21–44.

[FHY83] A. Fathi, M. R. Herman and J. C. Yoccoz. A proof of Pesin's stable manifold theorem. *Lecture Notes in Mathematics, 1007*. Springer, 1983, pp. 177–215.

[HPS77] M. Hirsch, C. Pugh and M. Shub. *Invariant Manifolds (Lecture Notes in Mathematics, 853)*. Springer, 1977.

[HY95] H. Hu and L. S. Young. Nonexistence of SBR measures for some diffeomorphisms that are 'Almost Anosov'. *Ergod. Th. & Dynam. Sys.* **15** (1995), 67–76.

[KS86] A. Katok and J. Strelcyn. *Invariant Manifolds, Entropy and Billiards; Smooth Maps with Singularities (Lecture Notes in Mathematics, 1222)*. Springer ,1986.

[K88] Yu. Kifer. *Random Perturbations of Dynamical Systems (Progress in Probabilities and Statistics)*. Boston, Birkhauser, 1988.

[L83] F. Ledrappier. Propiétés ergodiques des mesures de Sinai. *Publ. Math. I.H.E.S.* **59** (1983), 163–188.

[LY85] F. Ledrappier and L. S. Young. The metric entropy of diffeomorphisms. Part I: characterization of measures satisfying Pesin's entropy formula. *Ann. Math.* **122** (1985), 509–539.

[L80] J. Lewowicz. Lyapunov functions and topological stability. *J. Diff. Equations* **38** (1980), 192–209.

[M83] R. Mañé. *Ergodic Theory and Differentiable Dynamics*. Springer, 1983.

[NP73] S. Newhouse and J. Palis. Bifurcations of Morse Smale dynamical systems. *Dynamical Systems*. Ed. M. Peixoto. Academic, 1973, pp. 303–366.

[PM82] J. Palis and W. de Mello. *Geometric Theory of Dynamical Systems*. Springer, 1982.

[PT93] J. Palis and F. Takens. *Hyperbolicity and Sensitive Chaotic Dynamics at Homoclinic Bifurcations*. Cambridge University Press, Cambridge, 1993.

[P73] Ya. Pesin. Families of invariant manifolds corresponding to nonzero characteristic exponents. *Math. U.S.S.R. Izvestya* **10** (1973), 1261–1305.

[P77] Ya. Pesin. Characteristic Lyapounov exponents and smooth ergodic theory. *Russian Math. Surveys* **32**(4) (1977), 115–175.

[PS82] Ya. Pesin and Ya. Sinai. Gibbs measures for partially hyperbolic attractors. *Ergod. Th. & Dynam. Sys.* **2** (1982) 417–438.

[PS89] C. Pugh and M. Shub. Ergodic attractors. *Trans. Amer. Math. Soc.* **312** (1989), 1–54.

[R62] V. Rohlin. On the fundamental ideas of measure theory. *Amer. Math. Soc. Transl.* **1** (1962), 1–52.

[R78] D. Ruelle. An inequality for the entropy of differentiable maps. *Bol. Soc. Bras. de Mat.* **9** (1978), 83–87.

[R79] D. Ruelle. Ergodic theory of differentiable dynamical systems. *Publ. Math. I.H.E.S.* **50** (1979), 275–306.

[R89] D. Ruelle. *Elements of Differentiable Dynamics and Bifurcation Theory.* Academic, 1989.

[S57] S. Sternberg. Local contractions and a theorem of Poincaré. *Amer. J. Math.* **79** (1957), 809–824.

[S58] S. Sternberg. On the structure of local homoemorphisms of euclidian n-space, II. *Amer. J. Math.* **80** (1958), 623–631.

[W70] P. Walters. Anosov diffeomorphisms are topologically stable. *Topology* **9** (1970), 71–78.

[W85] M. Wojtkowski. Invariant families of cones and Lyapounov exponents. *Ergod. Th. & Dynam. Sys.* **5** (1985), 145–161.