Hydrodynamical stability of certain equilibrium flows of an ideal bidimensional fluid

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Abstract

We study here the dynamical stability of certain stationary flows of a bidimensional fluid confined in a circle. We apply, after adaptations, some results of Arnold ([A 1966]), which have been generalized by Holm el al. ([HMRW 1985]). This work is originated in a tridimensional helicoidal flow with high Reynolds theoretically described by Guarga ([G 1990]), and is an intent to establish a conjecture in connection with this phenomenon.

1 Introduction

We study the stability of a stationary Euler homogeneous incompressible flow moving in a circle. The analytic formulation of the velocity field of this flow is exposed in the section 4. A previous work on this theme has been developed by Luis Gustavo Sarasúa in [S 1996], there, some results about stability and unstability are obtained, and also interesting results are obtained about helicoidal vortex, supposed concentrated in a thread of radius 0. The problem has a hamiltonian formulation (see [EM 1970]); in the section 2 we resume from [HMRW 1985] (see also [P 1993]) some definitions of stability related with hamiltonian and Hamilton-Poisson mechanical systems in order to fix some ideas. The stability of hamiltonian systems is a hard problem, simple questions posed tenths of years ago are yet open. But some methods have succeeded in particular cases. One of these methods is the energy-Casimir method, created by Arnold in [A 1966] and systematized by Holm et al. in [HMRW 1985], particulary used in order to study the stability of several flows. For the ideal homogeneous incompressible tridimensional fluid flow, only trivial versions of stability are obtained: the dynamical stability of this kind of flows is an open problem. But for the ideal bidimensional flows case, the energy-Casimir method supplies the important Rayleigh-Arnold stability theorem.

In the section 3 we resume, from [G 1990], the physical problem which has originated this work, it consists in a fluid which moves in an infinite cylinder of radius δ_A, with a velocity field which has two analytical expressions, one in the nucleus $0 \leq \xi \leq 1$, with ξ the distance to the axis of the cylinder, and the other in the ring $1 \leq \xi \leq \delta_A$; and it is studied what happens with the velocity field, if under certain hypothesis, the radius δ_A increases to δ_B. The stationary solutions of [G 1990] of the Euler equations form a family of solutions depending on several parameters. In the section 4 is stated as a conjecture that at disappearing, or turning unstable some of these stationary solutions when the paramenters vary, in the surroundings there exist stable stationary solutions, and the velocity field varies next to some of these stationary solutions, in this way obtaining the helicoidal dynamics that show the experiments.

Taking into account this problem, we study a plane case obtained making void the axial velocity. The Rayleigh-Arnold theorem requires that the vorticity do not void, and this is not valid in the ring. So, before taking equal 0 the axial velocity, we have supposed, for $1 \leq \xi \leq \delta_A$, an azimuthal component of the velocity equal to $\frac{K}{2\xi} + \varepsilon\left(\xi - \frac{1}{\xi}\right)$, with small $|\varepsilon|$. We have supposed that the diameter of the cylinder increases to δ_B, and after an approximation, and considering the plane case, we determine the corresponding velocity field. An additional problem to apply the Rayleigh-Arnold theorem as in [HMRW 1985] is that the vorticity is not continuous. So, the thesis in the Rayleigh-Arnold theorem has been weakened: the theorem states that with smalll perturbations of the equilibrium solution, the velocity field and the vorticity "do not vary too much". In our case, we show that the velocity field has a "small variation": we prove that there exist values of the parameters such that the solution $\tilde{v}_e$ (bidimensional stationary velocity field in the circle D with vorticity $\tilde{\omega}_e$) of the section 4 has stable solutions as near as wanted (those corresponding to small $|\epsilon|$ different from 0).

More precisely, and using the notation $\|v_e - \tilde{v}_e\|_0 = \sup_{(x,y)\in D} \|v_e - \tilde{v}_e\|$; $\omega = \nabla \wedge v \cdot \vec{k}$; $|\omega - \tilde{\omega}_e|_0 = \sup_{(x,y)\in D} |\omega - \tilde{\omega}_e|$, we show

Theorem 1 *There exists an open set A in the set of parameters such that for any element of A, the corresponding stationary velocity field $\tilde{v}_e$ in the circle D verifies the following property: given $\varepsilon_0 > 0$, there exist stationary solutions v_e which are obtained from the tridimensional velocity field after canceling the axial component, and which verify*

1. $\|v_e - \tilde{v}_e\|_0 < \varepsilon_0$, $|\omega_e - \tilde{\omega}_e|_0 < \varepsilon_0$,

2. *Given $\varepsilon_1 > 0$, there exists $\delta > 0$ such that if v is a solenoidal vector field with $\|v - v_e\|_0 < \delta$, $|\omega - \omega_e|_0 < \delta$, then $\iint_D \|v(t) - v_e\|^2 \, dx\, dy < \epsilon_1$ for every $t \geq 0$ such that $v(t)$ has sense, where $v(t)$ denotes the velocity field of the solution of the Euler equation in D with $v(0) = v$.*

□

In fact, the proximity between v_e and $\tilde{v}_e$ can be obtained not only in the C^0 topology but C^r with any r. The Rayleigh-Arnold theorem can be applied in the nucleus, and, although it does not decide in the ring for the irrotational case, after the former theorem, it is possible to assert that for some values of the parameters there exist stationary stable solutions as near as wanted of those correspondig to the irrotational case in the ring.

The difficulty to prove this assertion in the ring for all the values of the parameters come frome the difficulties which arise from considering all the stationary solutions arbitrarely near of the irrotational one (it has been considered only a monoparametric family of such solutions: that obtained at varying ϵ). It is clear that in the bidimensional case, any velocity smooth field with 0 radial component, and azimuthal component depending on the radio is solution of the Euler equations. If in the hypothesis of the former theorem we remove the demand that the perturbation come from the tridimensional field of velocities, then the set of available solutions is larger. We emphasize that what we prove in the bidimensional case is not stability in a linear approximation, but Lyapounov stability (that is, taking into account small perturbations in a nonlinear problem). The difference between these kinds of stability is fundamental in the considered case, because the problem has hamiltonian character. The asymptotic stability in hamiltonian systems is not possible[1], for this reason the stability in linear approximation is always neutral, insufficient to argue about the stability of the equilibrium position in a nonlinear problem. We thank to Rafael Guarga for posing the phenomenon which originated this work, and to Raúl Tempone for many revealing talks.

2 Review of results on stability

Next we resume some ideas taken from [HMRW 1985] (see also [P 1993]); there the following notions are widely developed.

We start with a dynamical system determined by an equation $\dot{x} = X(x)$, where x is a variable which describes the state of the system we are studying. We will suppose that x belongs to a Banach space.

Definition 2.1 *An equilibrium state is a state x_e such that $\vec{0} = X(x_e)$.*

The study of the equilibrium states plays a central role in the theory on differential equations. Nevertheless, the equilibrium states must satisfy certain stability criteria for they be physically

[1] If we "slightly" perturb an equilibrium state, so that the perturbed system have a different energy from the initial one, and there is not energy exchange, the new solution will not approach the initial one.

noticeable. There are many possible stability definitions, in next we will emphasize four, relevant for hamiltonian systems, and we will set aside other ones which also are very important.

Definition 2.2 (Neutral stability) *An equilibrium point x_e is neutrally stable provided the spectrum of the linearized operator $DX(x_e)$ is purely imaginary.*

Definition 2.3 (Linearized stability) *An equilibrium point x_e is linearly stable relative to a norm $\|\cdot\|$ if given $\varepsilon > 0$, there exists $\delta > 0$ such that if δx verifies $\dot{(\delta x)} = DX(x_e) \cdot \delta x$ with $\|\delta x\| < \delta$ at $t = 0$, then $\|\delta x\| < \varepsilon$ for $t > 0$.*

Linearized stability implies neutral stability, the converse is not generally true, as shown by elementary examples.

Definition 2.4 (Formal stability) *An equilibrium state x_e is formally stable if there exists a real valued function which is constant on the trajectories of $\dot{x} = X(x)$ whose first differential vanishes at x_e, and whose second differential at x_e is definite.*

The second differential (or its opposite) provides a norm preserved by the linearized differential equation, so, with that norm, the formal stability implies linearized stability. The converse is not generally true.

Definition 2.5 (Lyapounov stability) *An equilibrium state x_e is Lyapounov stable if given $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|x(0) - x_0\| < \delta$, then x $\|x(t) - x_0\| < \varepsilon$, for any $t > 0$ where $x(t)$ is defined.*

It is a clasical result that *in finite dimension spaces* formal stability implies Lyapounov stability. Other implications are not true, in particular, there exist realistic examples of elasticity theory according to which, formally stable solutions have an infinite number of unstability directions. We are interested into Lyapounov stability.

The most known theorem regarding Lyapounov stability for hamiltonian mechanical system is the Lagrange-Dirichlet theorem.

Theorem 2.6 (Lagrange-Dirichlet) *Let us consider a hamiltonian mechanical system*

$$\dot{q}^i = \frac{\partial H}{\partial p_i}; \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}; \quad i = 1, \ldots, n,$$

Let (q_e, p_e) be an equilibrium state. If $D^2 H(q_e, p_e)$ is definite, then (q_e, p_e) is Lyapounov stable.

The proof is easy: in a neighborhood of (q_e, p_e) the level surfaces are diffeomorphic to concentric ellipsoids. As H is conserved, any point near to (q_e, p_e) remains in the asociated ellipsoid, and, therefore, in a neighborhood of (q_e, p_e).

$\square$

The energy-Casimir method applies to hamiltonian and Hamilton-Poisson systems, and consists in the following algorithm:

Step A Find a linear Poisson space P with a bracket $\{,\}$, and write the equations of the motion in the form $\dot{F} = \{F, H\}$

Step B Find a family of constants of the motion for the equation (that is, functions $C : P \mapsto R$ such that $\frac{dC}{dt}(x(t)) = 0$ for any C^1 solution of the equations). A way to find them is to look for Casimirs, that is, functions such that $\{C, G\} = 0$ for all G.

Step C Relate x_e to a constant C of the motion demanding that $H + C$ have a critical point at x_e.

Step D Let $\Delta x = x - x_e$ be a finite variation in P. Find Q_1 and Q_2 quadratic forms in P so that

$$Q_1(\Delta x) \leq H(x_e + \Delta x) - H(x_e) - DH(x_e)\Delta x$$

$$Q_2(\Delta x) \leq C(x_e + \Delta x) - C(x_e) - DC(x_e)\Delta x$$

for every $x \in P$. Demand that, for $\Delta x \neq 0$, $\quad Q_1(\Delta x) + Q_2(\Delta x) > 0$. Define the norm

$$||x||^2_{Q_1 Q_2} = Q_1(x) + Q_2(x)$$

Then, the following theorem follows:

Theorem 2.7 (Holm, Marsden, Raţiu, Weinstein) *If $H + C$ is continuous with the norm $|| \cdot ||_{Q_1 Q_2}$ at x_e, and if there exist solutions of the motion for any $t > 0$, then x_e is Lyapounov stable regarding the norm $|| \cdot ||_{Q_1 Q_2}$.*

For the proof, see [HMRW 1985] or [P 1993]. Let us observe that the starting space is a Banach space, and, therefore, there exists a norm with which the derivatives are computed, but the stability is determined regarding the norm $|| \cdot ||_{Q_1 Q_2}$, which can be different from the former one (in particular the space is not necessarily complete with this norm).

We will work in a bounded plane domain D (not necessarily simply connected), limited by differentiable curves $(\partial D)_i$ $i = 1, \ldots, g$. We consider in $P = \mathcal{H}_{div}(D)$, the space of solenoidal vector fields on D which are tangent to $(\partial D)_i$ $i = 1, \ldots, g$, the topology H^2 of functions whose derivatives up to order 2 in the distributional sense are square integrable. Let us define the Poisson bracket

$$\{F, G\}(v) = -\int_D v \cdot \left[\frac{\delta F}{\delta v}, \frac{\delta G}{\delta v}\right] dx\, dy$$

Where the functional derivative $\delta F/\delta G \in P$ is defined by

$$DF(v) \cdot \delta v = \langle \frac{\delta F}{\delta v}, \delta v \rangle = \int \frac{\delta F}{\delta v} \cdot \delta v\, dx\, dy$$

for any $\delta v \in P$, and

$$\left[\frac{\delta F}{\delta v}, \frac{\delta G}{\delta v}\right] = \left(\frac{\delta F}{\delta v} \cdot \nabla\right) \frac{\delta G}{\delta v} - \left(\frac{\delta G}{\delta v} \cdot \nabla\right) \frac{\delta F}{\delta v}$$

It is possible to prove that the Euler equations of motion of a fluid are given by

$$\dot{F} = \{F, H\},$$

where

$$H(v) = \frac{1}{2} \int_D |v|^2\, dx\, dy$$

(see [HMRW 1985] or [P 1993]).

The theorem 2.7, and the former considerations provide the following corollary:

Theorem 2.8 (Rayleigh-Arnold) *Let v belonging to the space $\mathcal{H}_{div}(D)$ of smooth solenoidal vector fields tangents to $(\partial D)_i$ be the velocity of a fluid in D, supposed incompressible, homogeneous and satisfying the Euler equations. Let v_e be a stationary solution of the equation in D with vorticity ω_e different form 0, let us suppose that the Bernoulli function depends on the vorticity ω_e. Last, let us suppose that there exist constants $0 < c \leq C < \infty$ such that $c \leq \frac{H'(\omega_e)}{\omega_e} \leq C$. Then, the solution v_e is Lyapounov stable regarding the norm $\|\Delta v\|_Q^2 = \iint_D \|\Delta v\|^2\, dx\, dy + c \iint_D (\Delta\omega)^2\, dx\, dy$.*

For the proof, see [HMRW 1985]. Let us observe that the theorem asserts the stability of the field of velocities and not of the motion of the fluid: it is possible that small variations in the velocity field originate increasing variations of the motion of the fluid.

3 Tridimensional problem

In [G 1990] it is considered an infinite cylinder with an ideal fluid moving with axial component of the velocity equal to 1, radial component equal to 0, and azimuthal component $\frac{K\xi}{2}$ between the radii $\xi = 0$ and $\xi = 1$, and $\frac{K}{2\xi}$ between the radii 1 and δ_A. It is supposed that the radius of the cylinder increases from δ_A to δ_B, and the variation of the velocity at each point is studied. The radius 1 varies reaching the value β solution of the equation

$$1 + \frac{K(1-\beta^2)J_0(K\beta)}{2\beta J_1(K\beta)} = \frac{\delta_A^2 - 1}{\delta_B^2 - \beta^2}$$

In [G 1990] it is observed that at increasing δ_B from δ_A, the radius of the nucleus β varies continuously from 1 until a certain value in which there not exists solution of the equation (at least, varying continuously). Experiments show that at increasing δ_B, the stationary solution with axial symmetry is lost, appearing helicoidal solutions. For the possible values β, the velocity at each point is the following (the subindex 1 stands for radial component, 2 tangential, 3 azimuthal):

In the nucleus $(0 \leq \xi \leq \beta)$

$$v_1 = 0; \quad v_2 = \frac{K\xi}{2} + \frac{K(1-\beta^2)J_1(K\xi)}{2\beta J_1(K\beta)}; \quad v_3 = 1 + \frac{K(1-\beta^2)J_0(K\xi)}{2\beta J_1(K\beta)}$$

In the ring $(\beta \leq \xi \leq \delta_B)$

$$v_1 = 0; \quad v_2 = \frac{K}{2\xi}; \quad v_3 = \frac{\delta_A^2 - 1}{\delta_B^2 - \beta^2}$$

It is supposed that $v_3 > 0$ for every ξ.

The stationary solutions form a family with several degrees of freedom: even with δ_B fixed, other solutions are obtained if the radius of the nucleus is changed (taken initially equal to 1), the axial component of the velocity (initially 1), the value of K (related with the initial component v_2), the value of δ_A, or the value of p_e.

4 Conjectures and results

It is conjectured that at increasing δ_B a value is obtained in which is not possible the continuous variation of β, or the stationary solution is unstable, for which the stationary solution falls in the basin of a stable solution corresponding to some element of the family described at the end of the former section, or near to it. The stability problem in tridimensional flows with the charactristics we are studying is open. Inspired in the problem, we study a plane case, obtained making void the axial component. The motion is then reduced to a bidimensional motion in a circle, verifying the bidimensional Euler equations. The idea is to apply the Rayleigh-Arnold theorem, the problem is that in the ring the vorticity is 0. For this reason we modify the velocity field in the ring for the tridimensional case, superposing a velocity field arbitrarily small corresponding to a rigid rotation, that is, we will consider in the ring,

$$v_1 = 0; \;\; v_2 = \frac{K}{2\xi} + \varepsilon\left(\xi - \frac{1}{\xi}\right); \;\; v_3 = 1$$

Reasoning as in [B 1967], supposing that the stream function ψ does not depend on z, we obtain the equation

$$\psi_{\xi\xi} - \frac{1}{\xi}\psi_\xi + 4\varepsilon^2\psi - \left(\frac{K}{2} - \varepsilon\right)\frac{\varepsilon}{\psi}\xi^2 = 2\varepsilon^2\xi^2 - 2\varepsilon\left(\frac{K}{2} - \varepsilon\right)$$

Considering the change of variable $\psi(\xi) = \frac{1}{2}\xi^2 + \xi F(\xi)$, it follows

$$\xi F_{\xi\xi} + F_\xi + \left(4\varepsilon^2 - \frac{1}{\xi}\right)F - \varepsilon(K - 2\varepsilon)\frac{1}{1 + \frac{2F}{\xi}} = -\varepsilon(K - 2\varepsilon)$$

For $\varepsilon = 0$ (case of the former section), and taking into account the conditions for the cylinder of radius δ_A, we have

$$F(\xi) = \frac{\delta_A^2 - \beta^2 + \delta_B^2 - 1}{2(\delta_B^2 - \beta^2)}\xi + \frac{\delta_B^2 - \delta_A^2\beta^2}{2\xi(\delta_B^2 - \beta^2)}$$

Using a numerical example of [G 1990], $K = 2$, $\delta_A = 3$, $\beta = 1.137$, $\delta_B = 3.093$, we obtain

$$\left|\frac{2F(\xi)}{\xi}\right| < .26$$

This result, and the factor $\varepsilon(K - 2\varepsilon)$ (which is small), justify the appproximation

$$\frac{1}{1 + \frac{2F}{\xi}} \approx 1 - \frac{2F}{\xi}$$

The equation is approximated by

$$\xi^2 F_{\xi\xi} + \xi F_\xi + [4\varepsilon^2\xi^2 - (1 - 2\varepsilon K + 4\varepsilon^2)]F = 0$$

whose solution gives

$$\psi(\xi) = \frac{1}{2}\xi^2 + C_1\xi J_E(2\varepsilon\xi) + C_2\xi Y_E(2\varepsilon\xi) \tag{1}$$

where

$$\begin{aligned} E &= \sqrt{1 - 2\varepsilon K + 4\varepsilon^2} \\ C_1 &= \frac{\beta(\delta_A^2 - \delta_B^2)Y_E(2\varepsilon\beta) + \delta_B(\beta^2 - 1)Y_E(2\varepsilon\delta_B)}{2\beta\delta_B[J_E(2\varepsilon\delta_B)Y_E(2\varepsilon\beta) - J_E(2\varepsilon\beta)Y_E(2\varepsilon\delta_B)]} \\ C_2 &= \frac{\delta_B(1 - \beta^2)J_E(2\varepsilon\delta_B) + \beta(\delta_B^2 - \delta_A^2)J_E(2\varepsilon\beta)}{2\beta\delta_B[J_E(2\varepsilon\delta_B)Y_E(2\varepsilon\beta) - J_E(2\varepsilon\beta)Y_E(2\varepsilon\delta_B)]} \end{aligned}$$

and Y_E is the Bessel function of second kind and order 0. Equaling the pressures in the radius β, we obtain the following equation (which we use to find β):

$$\frac{K(1-\beta^2)J_0(K\beta)}{2\beta J_1(K\beta)} = \frac{1-E}{\beta}[C_1 J_E(2\varepsilon\beta) + C_2 Y_E(2\varepsilon\beta)] + 2\varepsilon[C_1 J_{E-1}(2\varepsilon\beta) + C_2 Y_{E-1}(2\varepsilon\beta)] \quad (2)$$

If ε is near 0, then the right term in the equation 2 goes to $\frac{1-\delta_A^2}{\beta^2-\delta_B^2} - 1$, obtaining the same equation than in [G 1990] for $\mu = 1$.

The velocity field corresponding to this limit, taking the axial component of the velocity equal 0, is what we denoted $\tilde{v}_e$ in the statement of the theorem. If ϵ is different from 0, we obtain a nearby family, and is in this family (taking the axial component of the velocity equal 0) that we choose v_e, which is near the solucions of 1. If $\delta_B = \delta_A$, then $\beta = 1$ is solution of the equation 2. When we vary δ_B from δ_A, we consider the root β varying continuously. In [G 1990] it is shown that it is no possible to find β in this way for any value of δ_B: if δ_B is larger than a certain value, this solution disappears, so, we will suppose that δ_B is not larger than such value. The tangential component of the velocity in the ring is given by

$$v_2 = \frac{K}{2\xi} + \varepsilon\left(\xi - \frac{1}{\xi} + C_1 J_E(2\varepsilon\xi) + C_2 Y_E(2\varepsilon\xi)\right)$$

In [G 1990] it is shown that the tangential component of the velocity in the nucleus is

$$v_2 = \frac{K\xi}{2} + \frac{K(1-\beta^2)J_1(K\xi)}{2\beta J_1(K\beta)}$$

In the ring and in the nucleus, the radial component of the velocity is 0.

In the Rayleigh-Arnold theorem is essential the continuity of the vorticity. In our case, the vorticity is C^∞ only if we do not consider the circumference $\mathcal{C}_B$ of center the origin and radius β, but there exists a discontinuity in the derivatives on $\mathcal{C}_B$. This has some consequences, for instance, the existence of solutions $\forall t > 0$ is proved in the set of vector fields $C^{1+\alpha}$ for some $\alpha > 0$ (see [K 1967]). Therefore, the definition of Lyapounov stability must be understood in the interval of time where there exists solution. To prove the theorem, we need conserved functions, we will use:

$$H(v) = \frac{1}{2}\iint_D \|v\|^2\, dx\, dy$$

$$G_\Phi(v) = \iint_D \Phi(\omega)\, dx\, dy$$

where $\Phi : R \mapsto R$ is any C^∞ function. If $\mathcal{C}_A$ is the circumference centered in the origin and radius δ_A, and $\mathcal{C}_B(t)$ is the image in the time t of $\mathcal{C}_B$ (the circumference of radius δ_B at time 0) by the fluid motion, then also are conserved funcions:

$$\Gamma_A(v) = \int_{\mathcal{C}_A} v \cdot ds$$

$$\Gamma_B(v) = \int_{\mathcal{C}_B(t)} v \cdot ds$$

Let us define

$$H_G(v) = H(v) + G_\Phi(v) + a_1 \int_{\mathcal{C}_A} v \cdot ds + a_2 \int_{\mathcal{C}_B} v \cdot ds$$

where a_1, a_2 are real numbers. Computing at the stationary solution:

$$DH_G(v_e)\delta v = \int\int_D [v_e \cdot \delta v + \Phi'(\omega_e)\vec{k} \cdot \nabla \wedge \delta v]\, dx\, dy + a_1 \int_{\mathcal{C}_A} \delta v \cdot ds + a_2 \int_{\mathcal{C}_B} \delta v \cdot ds$$

Denoting ω_e^+ the vorticity at $\mathcal{C}_B$ continuously varying from the ring, and ω_e^-, varying continuously from the nucleus, after integrating by parts it is easy to conclude that $DH_G(v_e) = 0$ if

$$v_e + \nabla \wedge (\Phi'(\omega_e)\vec{k}) = 0 \tag{3}$$

$$a_1 = -\Phi'(\omega_e|_{\mathcal{C}_A}) \tag{4}$$

$$a_2 = \Phi'(\omega_e^+) - \Phi'(\omega_e^-) \tag{5}$$

If $\omega_e \neq 0$, supposing that the Bernoulli function B can be written as a function of ω_e, then (see [HMRW 1985]):

$$v_e = \frac{1}{\omega_e}\vec{k} \wedge \nabla B(\omega_e)$$

Therefore, taking into account 3,

$$-\omega_e \nabla \wedge (\Phi'(\omega_e)\vec{k}) = \vec{k} \wedge \nabla B(\omega_e),$$

what follows if

$$\omega_e \Phi''(\omega_e) = B'(\omega_e)$$

that is, al least for u in the set where the vorticity of the stationary solution varies

$$\Phi(u) = u \int^u \frac{B(\lambda)}{\lambda^2}\, d\lambda$$

We suppose that for the values of ω_e with which we work, $\frac{B'(\omega_e)}{\omega_e} > 0$. In fact, we work with ω_e varying in two intervals, one correspondig to the ring, with ω_e near 0, and other one, disjoint, corresponding to values of the vorticity in the nucleus. For the others values of ω_e, we define Φ''

being C^∞ and positive, and in this way, after integrating twice, we define Φ for any real number. Therefore, the velocity field v_e is a critical point of

$$H_G(v) = H(v) + G_\Phi(v) + a_1 \int_{\mathcal{C}_A} v \cdot ds + a_2 \int_{\mathcal{C}_B} v \cdot ds$$

where $\Phi(u) = u \int^u \frac{B(\lambda)}{\lambda^2}\, d\lambda$ for values of u corresponding to possible values of ω_e, B is the Bernoulli function for the stationary solution v_e, depending on ω_e, and a_1 and a_2 are given by 4 and 5. As H is quadratic, then

$$H(\delta v) = H(v_e + \delta v) - H(v_e) - DH(v_e)\delta v$$

Let us consider $v(t)$, the velocity field at t with $v(0) = v_e + \delta v$. Let $\delta v(t) = v(t) - v_e$ (Observe that $\delta v(0) = \delta v$). It follows

$$H(\delta v(t)) = \frac{1}{2}\iint_D \|v_e + \delta v(t)\|^2\, dx\, dy - \frac{1}{2}\iint_D \|v_e\|^2\, dx\, dy - \iint_D v_e \cdot \delta v(t)\, dx\, dy$$

As

$$0 = DH_G(v_e)\delta v(t) = \iint_D [v_e \cdot \delta v(t) + \Phi'(\omega_e)\delta\omega(t)]\, dx\, dy + a_1 \int_{\mathcal{C}_A} \delta v(t) \cdot ds + a_2 \int_{\mathcal{C}_B} \delta v(t) \cdot ds,$$

therefore

$$H(\delta v(t)) = \frac{1}{2}\iint_D \|v_e + \delta v(t)\|^2\, dx\, dy + a_1\Gamma_A(v_e + \delta v(t)) + a_2\Gamma_B(v_e + \delta v(t)) -$$

$$-\frac{1}{2}\iint_D \|v_e\|^2\, dx\, dy - a_1\Gamma_A(v_e) - a_2\Gamma_B(v_e) - \iint_D \Phi'(\omega_e)\delta\omega(t)\, dx\, dy = H_G(v_e + \delta v(t)) - H_G(v_e) -$$

$$- \iint_D [\Phi(\omega_e + \delta\omega(t)) - \Phi(\omega_e) - \Phi'(\omega_e)\delta\omega(t)]\, dx\, dy \leq H_G(v_e + \delta v(t)) - H_G(v_e)$$

because

$$\Phi(\omega_e + \delta\omega(t)) - \Phi(\omega_e) - \Phi'(\omega_e)\delta\omega(t) = \Phi''(\omega_e + \theta_1\delta\omega(t))\theta_2(\delta\omega(t))^2 \geq 0$$

with $0 < \theta_1 < 1$; $0 < \theta_2 < 1$. As H_G is a constant of the motion, $H_G(v_e + \delta v(t)) = H_G(v_e + \delta v)$ Then

$$\iint_D \|\delta v(t)\|^2\, dx\, dy = H(\delta v(t)) \leq \frac{1}{2}\iint_D (\|v_e + \delta v\|^2 - \|v_e\|^2)\, dx\, dy +$$

$$+ \iint_D [\Phi(\omega_e + \delta\omega) - \Phi(\omega_e)]\, dx\, dy + a_1 \int_{\mathcal{C}_A} \delta v \cdot ds + a_2 \left[\int_{\mathcal{C}_{B(-t)}} (v_e + \delta v) \cdot ds - \int_{\mathcal{C}_B} v_e \cdot ds\right]$$

Taking Φ'' of area as small as necessary between ω_e^- and ω_e^+, it follows that a_2 can be taken as small as wanted, and as the factor at which it multiplies is bounded by the Stokes theorem, the last term of the former addition can be taken as small as wanted. Having chosen Φ, if $\|\delta v\|_0$ and

$\delta\omega$ are small, the three first terms of the last addition also can be taken small. It follows that $\iint_D \|\delta v(t)\|^2\,dx\,dy$ can be taken as small as wanted for all t where it takes sense.

We have to verify that there exist values of the parameters where the hypotheses we have taken to obtain the former results are verified. Using some notation from [B 1967],

$$\frac{B'(\omega_e)}{\omega_e} = \frac{\xi\frac{dB}{d\psi}}{\frac{dC}{d\psi}\frac{d\omega}{d\xi}}$$

In the nucleus we have

$$\frac{dB}{d\psi} = \frac{K^2}{2};\quad \frac{dC}{d\psi} = K;\quad \frac{d\omega}{d\xi} = \frac{(\beta^2-1)K^2J_1(K\xi)}{2\beta J_1(K\beta)},$$

and so

$$\frac{B'(\omega_e)}{\omega_e} = \frac{\beta\xi J_1(K\beta)}{K(\beta^2-1)J_1(K\xi)}$$

As

$$\lim_{\xi\to 0}\frac{\xi}{J_1(K\xi)} = \frac{2}{K},$$

we obtain the wished bound of $\frac{B'(\omega_e)}{\omega_e}$ in the nucleus, at least if $K\beta < 3.8$ (for $K\beta = 3.8$ there is inversion of the axial component of the velocity).

In the ring

$$\frac{dB}{d\psi} = 2\epsilon^2 + \left(\frac{K}{2}-\epsilon\right)\frac{\epsilon}{\psi};\quad \frac{dC}{d\psi} = 2\epsilon$$

$$\frac{d\omega}{d\xi} = -\frac{4\epsilon^2(K-2\epsilon)}{\xi^2}[C_1J_E(2\epsilon\xi)+C_2Y_E(2\epsilon\xi)]+$$

$$+\frac{8\epsilon^2(1-E)}{\xi}[C_1J_{E-1}(2\epsilon\xi)+C_2Y_{E-1}(2\epsilon\xi)]+8\epsilon^3[C_1J_{E-2}(2\epsilon\xi)+C_2Y_{E-2}(2\epsilon\xi)]$$

If $|\epsilon|$ is small, it follows

$$\frac{d\omega}{d\xi} \approx \frac{2\epsilon^2K}{\xi^3(\beta^2-\delta_B^2)}[(\delta_A^2+\beta^2-1-\delta_B^2)\xi^2+\delta_B^2-\delta_A^2\beta^2]$$

Therefore

$$\frac{B'(\omega_e)}{\omega_e} \approx \frac{\xi^4(\beta^2-\delta_B^2)^2}{4\epsilon^2[(\delta_A^2+\beta^2-1-\delta_B^2)\xi^2+\delta_B^2-\delta_A^2\beta^2][(1-\delta_A^2)\xi^2+\delta_A^2\beta^2-\delta_B^2]}$$

For all the admissible values of the numerical example of [G 1990], (valid for $|\epsilon|$ small), that is, $K=2$, $\delta_A=3$, δ_B varying between 3 and 3.093, and β correspondingly varying between 1 and 1.137 that expression is larger than a positive constant, and is bounded from above. It is not difficult to verify that B is function of the vorticity, in fact, this assertion follows because there is neither inversion of the axial component of the velocity at any point, nor loss of the monotony of ω_e in the nucleus and in the ring. Therefore, at least for some value of the parameters, as near as wanted of the irrotational case in the ring, we find stable solutions.

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