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On the Dynamics of n-Dimensional Quadratic Endomorphisms

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Dedicated to the memory of Ricardo Mañé and Wieslav Szlenk.

Abstract: Considering a convex endomorphism F (its n coordinates are convex functions) and the one parameter family $F_\mu = F - \mu\nu$, where ν is any vector of $\mathbb{R}^n$, we find sufficient conditions in order that for large values of the parameter, the dynamical behavior of F_μ is completely described: either the nonwandering set $\Omega(F_\mu)$ is empty or F_μ restricted to $\Omega(F_\mu)$ is an expanding map. These conditions are shown to be generic in the space of quadratic endomorphisms.

1. Introduction

Convexity seems to be a condition which when imposed on higher dimensional endomorphisms permits generalization of some parts of the theory of one dimensional dynamics. This occurs for delay equations (see [RV]) and in a more general context will be the subject of this work.

A real function f defined on $\mathbb{R}^n$ is C^2-*convex* if it is C^2 and there exists $\alpha > 0$ such that $q_x(v) = \langle H_f(x)v, v\rangle \geq \alpha$ for every unit vector $v \in \mathbb{R}^n$, where $H_f(x)$ denotes the Hessian matrix of f at the point x and $\langle\cdot,\cdot\rangle$ denotes the usual scalar product in $\mathbb{R}^n$. An endomorphism of $\mathbb{R}^n$ is called C^2-convex when all its coordinates are C^2-convex functions. The set of C^2-convex functions defined on $\mathbb{R}^n$ will be denoted by $CC^2(\mathbb{R}^n)$.

Next define the class $\mathcal{H}_0$ of C^1 endomorphisms of $\mathbb{R}^n$ containing the maps F which satisfy the following properties:

1. ∞ is an attractor for F (i.e. there exists $R > 0$ such that $||x|| > R$ implies that $F^k(x) \to \infty$ when $k \to \infty$). Denote by B_∞ the basin of attraction of ∞.
2. The nonwandering set $\Omega(F)$ is either empty or a Cantor set which coincides with the complement of the basin of ∞, and F restricted to $\Omega(F)$ is an expanding map.

Endomorphisms in $\mathcal{H}_0$ are always Axiom A (see Mañé and Pugh [MP]); by a theorem of Przytycki (see [P]) adapted to this case of noncompact manifolds, the structural stability of the endomorphisms in $\mathcal{H}_0$ also follows.

Let $F = (f_1, \cdots, f_n)$ be a C^2-convex endomorphism; for $\nu \in \mathbb{R}^n$ fixed, consider the one parameter family $F_\mu = F - \mu\nu$. We will find sufficient conditions on the geometry of intersections of the level sets of the functions f_i such that for large values of μ, the map F_μ belongs to $\mathcal{H}_0$ (see Proposition 1 in Sect. 3). We define $\mathcal{G}_\nu$ as the set of C^2 endomorphisms F of $\mathbb{R}^n$ for which there exists $\mu_0 \in \mathbb{R}$ such that F_μ belongs to $\mathcal{H}_0$ for every $|\mu| > \mu_0$. We will show in Sect. 3 that the intersection of $\mathcal{G}_\nu$ with the space of C^2-convex endomorphisms is open in the C^2-strong topology. However, in Example 3 of the last section we will show that there exists $F \in \mathcal{G}_\nu$ (F is not C^2-convex) which is not an interior point of $\mathcal{G}_\nu$ in the C^r-strong topology for any $r \geq 2$.

Observe that if $f : \mathbb{R} \to \mathbb{R}$ is a C^2-convex function then f_μ belongs to $\mathcal{H}_0$ for every μ large. We are trying to understand the situation in higher dimensions. Actually the same result does not hold in dimension $n \geq 2$; in fact, we will show in Sect. 5 that there are open sets of C^2-convex endomorphisms for which the families $\{F_\mu \,:\, \mu > 0\}$ do not intersect $\mathcal{H}_0$. (See Examples 1 and 2 of the last section). However, the situation for quadratic maps is quite different. Any quadratic endomorphism in $\mathbb{R}^n$ is determined by symmetric matrices $A_1, \cdots, A_n$, vectors of $\mathbb{R}^n$ $v_1, \cdots, v_n$, and real numbers $a_1, \cdots, a_n$, and given by

$$F(x) = (\langle A_1 x, x\rangle + \langle v_1, x\rangle + a_1, \cdots, \langle A_n x, x\rangle + \langle v_n, x\rangle + a_n).$$

Obviously the endomorphism F is C^2-convex if and only if each of the matrices A_i is positive. We will show that if at least one of the matrices A_i is positive, then ∞ is an attractor for F. There are quadratic endomorphisms for which this does not occur, as will soon become clear. In the space of quadratic endomorphisms it is more natural to consider the weak (compact-open) topology since the strong topology becomes discrete when induced in this space. Moreover, the weak topology coincides with the natural topology given by the immersion (via coefficients) of the quadratic space in euclidean space. With this topology, we will prove the following result:

Theorem 1. *For every $\nu \in \mathbb{R}^n \setminus \{0\}$, $\mathcal{G}_\nu$ is open and dense in the space of quadratic endomorphisms of $\mathbb{R}^n$.*

These kind of situations are also found in [BSV] and [RV], where delay endomorphisms were studied; these endomorphisms, which fail to be C^2-convex because they have $n - 1$ linear coordinates, "generically" display hyperbolic dynamics (including that of $\mathcal{H}_0$) when one parameter families are considered. In this sentence, "generically" has a different meaning, because the delay is required to be maintained. This will be explained in the first example of the last section.

2. Preliminaries

In this section we will describe some properties of a single C^2-convex function $f : \mathbb{R}^n \to \mathbb{R}$.

For each $i, j = 1, \cdots, n$ we denote the partial derivatives $\dfrac{\partial f}{\partial x_i}$ and $\dfrac{\partial^2 f}{\partial x_i \partial x_j}$ by $\partial_i f$ and $\partial_{ij} f$ respectively, the gradient vector of f at x by $\nabla f(x)$, and we define the sets

$$C_i(f) = \{x \in \mathbb{R}^n : \partial_j f(x) = 0 \text{ for } j \neq i\}, \; i \in \{1, \cdots, n\}.$$

Let $\alpha > 0$ be such that for every $v, x \in \mathbb{R}^n$:

$$q_x(v) = \langle H_f(x)v, v\rangle \geq \alpha\|v\|^2;$$

where $H_f(x)$ is the Hessian matrix of f at x.

Next we comment on the fundamental properties:

1. There exists $R > 0$ such that $f(x) \geq \frac{\alpha}{3}||x||^2$ if $||x|| \geq R$.
   Proof: Fix $x \in \mathbb{R}^n$ with norm 1 and define $\varphi_x(t) = f(tx)$ for positive t. Then

   $$\varphi_x''(t) = \langle H_f(tx)x, x\rangle \geq \alpha$$

   for every $t \geq 0$. It follows that

   $$\varphi_x(t) \geq \frac{\alpha}{2}t^2 + \varphi_x'(0)t + \varphi_x(0).$$

   As $|\varphi_x'(0)|$ is bounded above independently of x, this implies the assertion. It also follows that f is a proper function: preimages of bounded sets are bounded.
2. We claim that f has a unique critical point.

   Proof. The first item implies that f has an absolute minimum in the region $\|x\| \leq R$, that must be a critical point. Let x_0 be a point where f takes its absolute minimum, fix x with $\|x - x_0\| = 1$, and define $\psi_x(t) = f(x_0 + t(x - x_0))$ for $t \geq 0$. Then, as above, $\psi_x''(t) \geq \alpha$ for $t > 0$, which implies that $\psi_x(t) \geq \frac{\alpha}{2}t^2 + f(x_0)$ for $t > 0$, and the claims follows.

3. For $s \in \mathbb{R}$ the level sets $f^{-1}(s)$ are always compact; furthermore, when $s < \min f$, $f^{-1}(s) = \emptyset$; when $s = \min f$, $f^{-1}(s)$ is the critical point of f and if $s > \min f$, then $f^{-1}(s)$ is a compact set that separates $\mathbb{R}^n$ into two components, the bounded one being the strictly convex set $\{x \in \mathbb{R}^n : f(x) < s\}$, denoted in the sequel by $i(f^{-1}(s))$. The unbounded component will be denoted by $e(f^{-1}(s))$.
   Another simple consequence of the convexity is that every nonempty level set $f^{-1}(s)$ with $s > \min f$, has exactly two points of tangency with hyperplanes x_i = constant, $i = 1, \cdots, n$; these are the points of intersection of $f^{-1}(s)$ and $C_i(f)$.
4. The set $C_i(f)$ is the graph of a function defined in the i^{th} axis, that is, we claim that there exists $\tilde{x}_i : \mathbb{R} \to \mathbb{R}^{n-1}$ such that $\partial_j f(x_1, \cdots, x_n) = 0$ for every $j \neq i$ if and only if there exists $t \in \mathbb{R}$ satisfying $x_i = t$ and $(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) = \tilde{x}_i(t)$.

   Proof. Take $i = n$ to simplify the notation, and consider the map $g_n(x) = (\partial_1 f(x), \cdots, \partial_{n-1}f(x))$, where $x = (\tilde{x}, x_n) \in \mathbb{R}^n$ and $\tilde{x} = (x_1, \cdots, x_{n-1})$. It is easy to verify that $\partial_{\tilde{x}} g_n(\tilde{x}, x_n) = \widehat{H}_f(x)$, with $\widehat{H}_f(x)$ the matrix obtained from $H_f(x)$ if the last row and column are taken off. Since $H_f(x)$ is a positive matrix, $\widehat{H}_f(x)$ is nonsingular. As $g_n(x^0) = 0$, where $x^0 = (x_1^0, \cdots, x_n^0)$ is the critical point of f, then the implicit function theorem implies that there is a neighborhood V of x_n^0 and a function $\tilde{x}_n$ defined on V such that

   $$g_n(\tilde{x}_n(x_n), x_n) = 0$$

   for every $x_n \in V$. Moreover,

   $$\widehat{H}_f(\tilde{x}_n(x_n), x_n)\tilde{x}_n'(x_n) = -\widehat{\nabla}\partial_n f(\tilde{x}_n(x_n), x_n), \tag{1}$$

   where $\widehat{\nabla}\partial_n f = (\partial_{1n} f, \cdots, \partial_{(n-1)n} f)$.
   As $C_n(f)$ is the set of points where the level sets of f are tangent to the hyperplanes $x_n = const$, it follows that the domain of $\tilde{x}$ is all $\mathbb{R}$. The sets $C_i(f)$, $i = 1, \cdots, n$, are called the *critical lines of* f.

Now we separate in a lemma the main result of this section; it says that if μ is sufficiently large, then for each $1 \le i \le n$ there is a level set S_i of $f_\mu = f - \mu$, tangent to the hyperplane $x_i = f_\mu(S_i)$.

Lemma 1. *Let $f_\mu = f - \mu$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a C^2-convex function and $\mu \in \mathbb{R}$. Then there exists μ_0 such that for any $i = 1, \cdots, n$ and $\mu \ge \mu_0$ there are defined functions $s_i(\mu)$ and $\tilde{s}_i(\mu)$ with the following properties:*

1. *$f_\mu^{-1}(s_i(\mu))$ is tangent to $x_i = s_i(\mu)$ and to $x_i = \tilde{s}_i(\mu)$.*
2. *$s_i(\mu) \to +\infty$, $\tilde{s}_i(\mu) \to -\infty$, $\frac{s_i(\mu)}{\mu} \to 0$ and $\frac{\tilde{s}_i(\mu)}{\mu} \to 0$ as $\mu \to +\infty$.*
3. *$f_\mu^{-1}(s) \subset \{(x_1, \cdots, x_n) : x_i < s\}$ if $s > s_i(\mu)$. $f_\mu^{-1}(s) \cap \{(x_1, \cdots, x_n) : x_i > s\} \ne \emptyset$ if $s < s_i(\mu)$ and $f_\mu^{-1}(s)$ is not empty.*

Proof. We assume $i = n$, the proof for $i < n$ is similar; we denote by $x^0 = (x_1^0, \cdots, x_n^0)$ the point where f takes its minimum a. Fix μ large enough and define $\varphi_\mu(t) = f_\mu(\tilde{x}_n(t), t)$, where $(\tilde{x}_n(t), t) = (u_1(t), \cdots, u_{n-1}(t), t)$ is the parametrization of $C_n(f)$ given above. Observe that

$$\varphi'_\mu(t) = \partial_n f(\tilde{x}_n(t), t),$$

because for $1 \le j < n$, $\partial_j f = 0$ at points in $C_n(f)$. It follows that

$$\varphi''_\mu(t) = \sum_{i=1}^{n-1} \partial_{in} f_\mu(\tilde{x}_n(t), t) u'_i(t) + \partial_{nn} f_\mu(\tilde{x}_n(t), t).$$

Next we prove that φ''_μ is bounded below from 0. Developing the determinant of $H_f(\tilde{x}_n(t), t)$ by adjoints of the last row gives

$$det(H_f(\tilde{x}_n(t), t)) = \sum_{i=1}^{n} (-1)^{n-i} \partial_{in} f(\tilde{x}_n(t), t) A_i(t), \tag{2}$$

where $A_n(t) = det(\widehat{H}_f(\tilde{x}_n(t), t))$ and $A_i(t)$, for $i = 1, \cdots, n-1$, is the determinant of the matrix obtained from $H_f(\tilde{x}_n(t), t)$ taking off the i^{th} column and n^{th} row. Equation (1) says that

$$\widehat{H}_f(\tilde{x}_n(t), t)\tilde{x}'_n(t) = -\widehat{\nabla}\partial_n f(\tilde{x}_n(t), t).$$

Consider this a linear system with unknowns $\tilde{x}'_n(t) = (u'_1(t), \cdots, u'_n(t))$. By

Cramer's rule, $u'_i(t)$ times the determinant of $\widehat{H}_f(\tilde{x}_n(t), t)$ is equal to the determinant of the matrix obtained substituting the i^{th} column of $\widehat{H}_f(\tilde{x}_n(t), t)$ by $-\widehat{\nabla}\partial_n f(\tilde{x}_n(t), t) = -(\partial_{1n} f(\tilde{x}_n(t), t), \cdots, \partial_{(n-1)n} f(\tilde{x}_n(t), t))$. This last matrix is obtained from $H_f(\tilde{x}_n(t), t)$ taking off the last row and the i^{th} column and interchanging the last column with the i^{th} one. It follows that

$$A_i(t) = (-1)^{i-1} u'_i(t) det(\widehat{H}_f(\tilde{x}_n(t), t)).$$

In this way, from (2) we have

$$det(H_f(\tilde{x}_n(t), t)) = det(\widehat{H}_f(\tilde{x}_n(t), t)) \left(\sum_{i=1}^{n-1} \partial_{in} f(\tilde{x}_n(t), t) u'_i(t) + \partial_{nn} f(\tilde{x}_n(t), t) \right);$$

therefore $\varphi''_\mu(t) = \dfrac{det(H_f(\tilde{x}_n(t),t))}{det(\widehat{H}_f(\tilde{x}_n(t),t))}$; it is an exercise of linear algebra to prove that then $\varphi''_\mu \geq \alpha$.

On the other hand, it is clear that $\varphi'_\mu(x_n^0) = 0$ and $\varphi_\mu(x_n^0) = a - \mu$. From this we conclude that for every large value of μ there exists $s_n(\mu) > 0$ and $\tilde{s}_n(\mu) < 0$ with $x_n^0 \in (\tilde{s}_n(\mu), s_n(\mu))$ such that $\varphi_\mu(s_n(\mu)) = s_n(\mu)$, $\varphi_\mu(\tilde{s}_n(\mu)) = s_n(\mu)$, $\varphi_\mu(s) < s$ if $x_n^0 < s < s_n(\mu)$ and $\varphi_\mu(s) > s$ if $s > s_n(\mu)$. The lemma follows easily. □

Remark 1. – As an immediate consequence of the above lemma we have the following fact: if $F_\mu : \mathbb{R}^n \to \mathbb{R}^n$ is any endomorphism such that at least one of its coordinates (suppose the last one) is $f_\mu = f - \mu$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a C^2-convex function, then ∞ is an attractor for F_μ if μ is large enough. (This is also a consequence of the first property of C^2-convex functions stated above.) Moreover, if we define

$$C_n(\mu) = \{x \in \mathbb{R}^n : f_\mu(x) \in [\tilde{s}_n(\mu), s_n(\mu)]\},$$

$\tilde{s}_n(\mu)$ and $s_n(\mu)$ being as in the lemma and if B_∞ is the basin of infinity, then $B_\infty = \mathbb{R}^n \setminus \bigcap_{k\geq 0} F_\mu^{-k}(C_n(\mu))$. Now suppose that each coordinate f_i is C^2-convex and let $\tilde{s}_i(\mu), s_i(\mu)$ be as in the previous lemma when the i^{th} coordinate is considered.

If $C_i(\mu) = \{x \ : \ f_i(x) - \mu \in [\tilde{s}_i(\mu), s_i(\mu)]\}$ and $C(\mu) = \bigcap_{i=1}^{n} C_i(\mu)$, then $B_\infty(\mu) = \mathbb{R}^n \setminus \bigcap_{k\geq 0} F_\mu^{-k}(C(\mu))$.

– Observe that diminishing μ we can find a value $\tilde{\mu}$ such that $s_n(\tilde{\mu}) = \tilde{s}_n(\tilde{\mu})$. If $\mu < \tilde{\mu}$, then the basin of infinity for F_μ is equal to $\mathbb{R}^n$. Therefore, if for the one parameter family of C^2-convex endomorphisms $F_\mu = (f_1, \ldots, f_n) - \mu\nu$, if any of the entries of the vector ν is negative, then for every large positive μ, the F_μ-orbit of any point goes to ∞.

3. ϵ-Transversality

Now we will find conditions expressed in terms of the intersections of the level curves of $f_1, \cdots, f_n$ which will be sufficient to obtain that F_μ belongs to $\mathcal{H}_0$ for large values of μ. The precise statement is Proposition 1.

First we introduce some notation. By $[\{v_1, \cdots, v_k\}]$ we denote the linear subspace generated by $\{v_1, \cdots, v_k\} \subset \mathbb{R}^n$ and P_V (resp. $P_V^\perp$) denote the orthogonal projection of $\mathbb{R}^n$ onto the linear subspace V (resp. onto the orthogonal complement of V).

Lemma 2. *If $\{v_1, \cdots, v_k\}$ is a linearly independent set of vectors in $\mathbb{R}^n$ and $V = [\{v_1, \cdots, v_k\}]$, then for every $\epsilon > 0$ there exists $\delta > 0$ such that if $w_1, \cdots, w_k$ are linearly independent vectors in $\mathbb{R}^n$, $W = [\{w_1, \cdots, w_k\}]$ and $\|w_i - v_i\| < \delta$ for every $i = 1, \cdots, k$, then for any unit vector $v \in \mathbb{R}^n$ it holds that*

$$\|P_V(v) - P_W(v)\| < \epsilon.$$

Proof. Let $\{v'_1, \cdots, v'_k\}$ and $\{w'_1, \cdots, w'_k\}$ be orthonormal basis of the linear subspaces V and W obtained from $v_1, \cdots, v_n$ and $w_1, \cdots, w_n$ by the Gram Schmidt orthogonalization method. So for every vector $v \in \mathbb{R}^n$ we can write

$$P_V(v) = \sum_{i=1}^{k} \langle v, v'_i \rangle v'_i \quad \text{and} \quad P_W(v) = \sum_{i=1}^{k} \langle v, w'_i \rangle w'_i.$$

By continuity of scalar product, $\|w'_j - v'_j\|$ is small if $\|w_i - v_i\|$ is small for every $i \leq j$; so the lemma follows. □

Definition 1. *Given $\epsilon > 0$ we say that $\{v_1, \cdots, v_n\} \subset \mathbb{R}^n \setminus \{0\}$ is* **ϵ-transverse** *if for each $V_i = [\{v_1, \cdots, v_n\} \setminus \{v_i\}]$ with $i = 1, \cdots, n$, it holds that*

$$\|P^{\perp}_{V_i} v_i\| \geq \epsilon \|v_i\|.$$

Definition 2. *For any $\epsilon > 0$ we say that a set of n smooth hypersurfaces $S_1, \cdots, S_n$ in $\mathbb{R}^n$ is* **transverse** *if at each point of intersection $x \in \bigcap_{i=1}^n S_i$ the set of n normal vectors to the tangent spaces of the hypersurfaces is linearly independent.*
The set $\{S_1, \cdots, S_n\}$ is **ϵ-transverse** *if at each point $x \in \bigcap_{i=1}^n S_i$, the set of n normal vectors at x to the respective tangent spaces is ϵ-transverse.*

The following is an immediate corollary of Lemma 2.

Corollary 1. *If $\{v_1, \cdots, v_n\}$ is a set of unit vectors of $\mathbb{R}^n$ which is not ϵ-transverse, then there exists $\delta > 0$ such that if $w_1, \cdots, w_n$ are unit vectors satisfying $\|w_i - v_i\| < \delta$ for every $i = 1, \cdots, n$, then $\{w_1, \cdots, w_n\}$ is not ϵ-transverse.*

The following lemma is the basic tool to obtain expansivity.

Lemma 3 (*ϵ-transversality*). *Given $\epsilon > 0$ there exists $c(\epsilon) > 0$ such that if the set of unit vectors $\{v_1, \cdots, v_n\} \subset \mathbb{R}^n$ is ϵ-transverse, then the $n \times n$ matrix A whose rows are the vectors $v_1, \cdots, v_n$ satisfies:*

$$\|Av\| \geq c(\epsilon)\|v\|,$$

for every $v \in \mathbb{R}^n$.

Proof. Suppose by contradiction that there exists $\epsilon > 0$ such that for every positive integer k and all $i = 1, \cdots, n$ there exist unit vectors v_i^k and v^k such that the set $\{v_1^k, \cdots, v_n^k\}$ is ϵ-transverse and if A_k is the matrix whose rows are the vectors $v_1^k, \cdots, v_n^k$, then:

$$\|A_k v^k\| \leq \frac{1}{k}. \tag{3}$$

We can assume without loss of generality that the sequences $\{v_i^k : k \geq 1\}$ with $i = 1, \cdots, n$ and $\{v_k : k \geq 1\}$ converge to the unit vectors $v_1, \cdots, v_n$ and v. From the corollary above it follows that $\{v_1, \cdots, v_n\}$ is ϵ-transverse, hence linearly independent, but, on the other hand, if A is the matrix whose rows are the vectors $v_1, \cdots, v_n$, then pasing to the limit in Eq. (3) we have $Av = 0$. This contradiction proves the lemma. □

Remark 2. It can be proved that the number $c(\epsilon)$ in the preceding lemma can be chosen as a constant depending only on the dimension n times ϵ^{n-1}. We will not need this stronger version.

Proposition 1. *Let $F_\mu = (f_1 - \mu\nu_1, \cdots, f_n - \mu\nu_n)$ (each $\nu_i > 0$) be a C^2-convex endomorphism of $\mathbb{R}^n$ satisfying the following: Given $\epsilon > 0$ there exists μ_0 such that, if $\mu > \mu_0$ then*

$$\{f_i^{-1}(\mu\nu_i + s_i) \ : \ i = 1, \cdots, n\}$$

is ϵ-transverse whenever $s_i \in [\tilde{s}_i(\mu\nu_i), s_i(\mu\nu_i)]$ for each $i = 1, \cdots, n$. Then F_μ belongs to $\mathcal{H}_0$ for every μ sufficiently large.

Proof. Suppose first that $\nu_i = 1$ for each $i = 1, \cdots, n$. Since each component of F_μ is a C^2-convex function, Remark 1 implies that

$$\mathbb{R}^n \setminus B_\infty = \bigcap_{k\geq 0} F_\mu^{-k}(C(\mu)).$$

Take any $x \in \mathbb{R}^n \setminus B_\infty$. For each $i = 1, \cdots, n$ there exist $s_i \in [\tilde{s}_i(\mu), s_i(\mu)]$ such that $x \in \bigcap_{i=1}^{n} f_i^{-1}(s_i + \mu)$. The normal vector to $f_i^{-1}(s_i + \mu)$ at x is $\nabla f_i(x)$, so the hypothesis implies that the set $\{\nabla f_1(x), \cdots, \nabla f_n(x)\}$ is ϵ-transverse. On the other hand, it is clear that

$$\begin{aligned} \|(DF_\mu)_x(v)\|^2 &= \langle \nabla f_1(x), v\rangle^2 + \cdots + \langle \nabla f_n(x), v\rangle^2 \\ &\geq \min_{1\leq i\leq n} \|\nabla f_i(x)\|^2 \sum_{i=1}^{n} \langle \frac{\nabla f_i(x)}{\|\nabla f_i(x)\|}, v\rangle^2. \end{aligned}$$

The sum in the last member is equal to the square of the norm of $A(x)v$, where $A(x)$ is the matrix which rows are the vectors $\frac{\nabla f_i(x)}{\|\nabla f_i(x)\|}$. These are ϵ-transverse, so the ϵ-transversality lemma implies that

$$\|(DF_\mu)_x(v)\|^2 \geq c(\epsilon)^2 \min_{1\leq i\leq n} \|\nabla f_i(x)\|^2 \|v\|^2.$$

Therefore, if we prove that for every μ large $c(\epsilon)\min_{1\leq i\leq n} \|\nabla f_i(x)\| > 1$ for every $x \in C(\mu)$, then the result follows.

Let $x \in C_i(\mu)$, then $f_i(x) - \mu \geq \tilde{s}_i(\mu)$ and Lemma 1 implies that when $\mu \to \infty$, $\mu + \tilde{s}_i(\mu) \to \infty$. Then it follows that $\|x\| \to \infty$ and as f_i is a C^2-convex function, $\|\nabla f_i(x)\| \to \infty$ as $\mu \to \infty$. This proves the proposition in case $\nu_i = 1$ for each $i = 1, \cdots, n$. For the general case, define, instead of $C(\mu)$ the set

$$C_\nu(\mu) = \bigcap_{i=1}^{n}\{x \ : \ f_i(x) - \mu\nu_i \in [\tilde{s}_i(\mu\nu_i), s_i(\mu\nu_i)]\},$$

and then proceed as above. □

Remark 3. – Observe that if any $\nu_i \leq 0$ then for every μ large the nonwandering set of F_μ is empty. This is a consequence of Lemma 1.

– To give a simple example in which the conditions of the above proposition hold, take any C^2-convex endomorphisms $F = (f_1, f_2)$ of $\mathbb{R}^2$, such that, for any $i \in \{1, 2\}$ and $x \in \mathbb{R}^2$,

$$\partial_{ii} f_i(x) > \partial_{jj} f_i(x)$$

for every $j \neq i$.

Then the level curves of f_1 are more vertical than horizontal, and those of f_2 are more horizontal than vertical. This gives an idea why the level curves have to be transverse. The proof is similar to the one we give in the next section.

Now we make a digression to discuss some topologies in the space of C^2-convex functions of $\mathbb{R}^n$. The C^2-weak topology given by uniform convergence on compact subsets seems to be not useful because any C^2-convex function has arbitrary small perturbations which are not even convex functions. This represents a difficulty since we are dealing with the behaviour at infinity. A C^2-Whitney or strong neighborhood of a function f is given by continuous functions $\epsilon_i(x) > 0$, $i = 0, 1, 2$ and is defined by:

$$\begin{aligned}\mathcal{V}(f; \epsilon_0, \epsilon_1, \epsilon_2) = \{g \in C^2(R^n) : \; & \|H_f(x) - H_g(x)\| \leq \epsilon_2(x); \\ & \|\nabla f(x) - \nabla g(x)\| \leq \epsilon_1(x) \text{ and} \\ & |f(x) - g(x)| < \epsilon_0(x) \text{ for every } x\}.\end{aligned}$$

It is clear that $CC^2(\mathbb{R}^n)$ is open in $C^2(\mathbb{R}^n)$ when the strong topology is considered. This makes this topology more interesting in $CC^2(\mathbb{R}^n)$. Moreover, as $C^2(\mathbb{R}^n)$ is a Baire space (see [H]), it follows that also $CC^2(\mathbb{R}^n)$ is a Baire space. However, induced in the set of quadratic convex functions the Whitney topology is discrete, while the weak topology induces the natural topology of the norm which we will use in the next section. In the space of C^2-convex endomorphisms of $\mathbb{R}^n$ we will use product topologies. This means that a strong small perturbation of an endomorphism F of $\mathbb{R}^n$ is an endomorphism G such that each coordinate is close to the corresponding coordinate of F.

Remark 4. $\mathcal{G}_\nu$ is open under strong topology in the space of C^2-convex endomorphisms of $\mathbb{R}^n$.

Proof. Let F be a C^2-convex endomorphism in $\mathcal{G}_\nu$. Then $F_\mu = F - \mu\nu$ belongs to $\mathcal{H}_0$ for every $|\mu| > \mu_0$. By Remark 1, there is a continuous and increasing function $b(\mu)$ such that $b(\mu) \to +\infty$ as $\mu \to +\infty$ and the nonwandering set of F_μ is contained in the complementary set of the ball centered at 0 and with radius $b(\mu)$. As $\mathcal{H}_0$ is open, each F_μ has a neighborhood contained in $\mathcal{H}_0$. The family $\{F_\mu \; : \; \mu \geq \mu_0\}$ is not compact, but the nonwandering set of F_μ is determined by the restriction of F to a set of the form $\{x \; : \; b(\mu) \leq \|x\| \leq const.\sqrt{\mu}\}$, and there the values of a C^2-strong perturbation G can be chosen close to F. Then the nonwandering set of G_μ must be conjugated to that of F_μ. It is important to note that the C^2-convexity is crucial, because it makes the nonwandering set to go to ∞, when F and G are arbitrarily close. Compare this with the situation in Example 3 of the last section, where the distance from the nonwandering set of F_μ to 0 tends to 0 when $\mu \to +\infty$. □

In the following sections we will need to describe some perturbations of C^2-convex endomorphisms and the effect of these perturbations on the level sets of the functions. Recall that if L is the level set of a C^2-convex function, then $i(L)$ denotes the convex bounded region of the complementary set of L. If a is any point in $i(L)$ and S^{n-1} denotes the unit sphere of $\mathbb{R}^n$, then there exists a function $\varphi_L : S^{n-1} \to \mathbb{R}^+$ such that

$$\{a + \varphi_L(\theta)\theta \; : \; \theta \in S^{n-1}\} = L.$$

To prove the above, observe that each ray starting at $a \in i(L)$ must intersect L because L is compact. This intersection must be unique because $i(L)$ is strictly convex. We will call this function φ_L the parametrization of L. In this way it is clear that for each $g \in CC^2(\mathbb{R}^n)$, $t_0 > \min g$ and $a \in i(g^{-1}(t_0))$ there exists a function

$$\varphi^g \; : \; S^{n-1} \times (t_0, \infty) \to \mathbb{R}^+$$

such that for each $t > t_0$, the function $\varphi_t^g : S^{n-1} \to \mathbb{R}^+$ given by $\varphi_t^g(\theta) = \varphi^g(\theta, t)$ defines the parametrization of $g^{-1}(t)$.

In other words, φ^g is the unique function satisfying $g(a + \varphi^g(\theta, t)\theta) = t$ for every $\theta \in S^{n-1}$ and $t > t_0$. (Here we used polar coordinates in the domain of g.)

Suppose that g is as above and take a strong C^2-neighborhood $\mathcal{V}$ of g such that every $h \in \mathcal{V}$ is C^2-convex and satisfies $a \in i(h^{-1}(t_0))$. Then, for every $t > t_0$, we can define the parametrization φ_t^h of $h^{-1}(t)$. This defines an operator φ from $\mathcal{V}$ into $C^2(S^{n-1} \times (t_0, +\infty))$; i.e. $\varphi(h) = \varphi^h$. Considering the C^2-strong topology also in this space of functions we have:

Lemma 4. *The operator $\varphi : \mathcal{V} \to C^2(S^{n-1} \times (t_0, +\infty))$ is continuous.*

Proof. Let d be the distance from a to $h^{-1}(t_0)$ and define

$$\Phi_h : S^{n-1} \times (d, +\infty) \times (t_0, +\infty) \to \mathbb{R}$$

by

$$\Phi_h(\theta, s, t) = h(a + s\theta) - t.$$

Observe that

$$\frac{\partial^2}{\partial s^2} h(a + s\theta) = \langle H_h(a + s\theta)\theta, \theta\rangle \geq \alpha,$$

where $H_h(a + s\theta)$ is the Hessian matrix of h at the point $a + s\theta$. It follows that

$$\frac{\partial \Phi_h}{\partial s}(\theta, s, t) = \frac{\partial h}{\partial s}(a + s\theta) > 0$$

for every $s > d$. (Geometrically, $\frac{\partial h}{\partial s}(a + s\theta)$ is positive because for $s > d$ and any θ the line $a + s\theta$ is transverse to the level sets of h, and when s increases, $a + s\theta$ cuts higher level sets of h.) Thus the implicit function theorem provides a C^2 function $\varphi_h : S^{n-1} \times (t_0, +\infty) \to \mathbb{R}^+$ such that

$$\Phi_h(\theta, \varphi_h(\theta, t), t) = 0$$

and the dependence of φ_h on h is continuous because Φ_h depends continuously on h, by the parametrized implicit function theorem. (This follows from the parametrized version of the Inverse Mapping Theorem: *Let X be a topological space, M a manifold and $\psi : X \times M \to M$ such that for each $x \in X$, ψ_x is C^r and the map $x \to \psi_x$ is continuous. Fix $x \in X$, $p \in M$ and suppose that the differential $D_p\psi_x$ is invertible. Then there is a neighborhood N of x in X, such that for every $y \in N$, ψ_y is locally C^r-invertible and the inverses depend continuously on y.*) This proves the lemma. □

The advantage in considering φ^g instead of g is that the high level sets of g are images of the compact set S^{n-1} under φ_t^g, simplifying the work with level curves.

Corollary 2. *Let $g_1, \cdots, g_n$ be C^2-convex functions such that the set $\{g_i^{-1}(\mu) : i = 1, \cdots, n\}$ is $\epsilon(\mu)$-transverse for every $\mu > \mu_0$, where $\epsilon(\mu)$ is a continuous function of μ with range contained in an open interval I bounded away from 0. Then there exists a small neighborhood of $(g_1, \cdots, g_n)$ in the C^2-strong topology, such that for every $(h_1, \cdots, h_n)$ in that neighborhood, the set $\{h_i^{-1}(\mu) : 1 \leq i \leq n\}$ is $\epsilon'(\mu)$-transverse for every μ, where $\epsilon'(\mu)$ belongs to I for every μ.*

Proof. Let $h = (h_1, \cdots, h_n)$ be a small C^2-strong perturbation of $g = (g_1, \cdots, g_n)$; each level curve $g_i^{-1}(\mu)$ of g_i is the image under φ_μ^g of S^{n-1}. By continuity of φ, the functions h_i can be chosen so that $\varphi^{h_i}(S^{n-1} \times \{\mu\})$ and $\varphi^{g_i}(S^{n-1} \times \{\mu\})$ are located at a distance that converges to 0 arbitrarily fast when $\mu \to \infty$. Therefore, as ϵ-transversality for $\epsilon \in I$ is open, the result follows. □

4. Proof of Theorem 1

Consider $F = (f_1, \cdots, f_n)$ where each component is given by

$$f_i(x) = \langle A_i x, x \rangle + L_i(x) + a_i,$$

with A_i a symmetric matrix, L_i a linear function and $a_i \in \mathbb{R}$. We are not supposing that the matrices A_i are positive, so F is not convex necessarily.

Assume first that

1. $\{\langle A_i x, x \rangle = 0 \ : \ i = 1, \cdots, n\} \cap S^{n-1} = \emptyset$,
2. $\{\langle A_i x, x \rangle = \pm\nu_i\}$, $i = 1, \cdots, n$ is transverse for all possible choices of + and $-$,
3. A_i is invertible, $i = 1, \cdots, n$.

Under these conditions (that will be shown to be open and dense), we will show that:

(a) ∞ is an attractor for F.

(b) $F_\mu = F - \mu\nu$ belongs to $\mathcal{H}_0$ for every large value of $|\mu|$.

Proof of (a). Condition 1 and continuity imply that there exists $\delta > 0$ such that

$$\bigcap_{i=1}^{n} \{x : |\langle A_i x, x \rangle| < \delta\} \cap S^{n-1} = \emptyset.$$

Using Condition 1 we see that for every $x \in \mathbb{R}^n$ there exists some index i such that $|\langle A_i \frac{x}{||x||}, \frac{x}{||x||} \rangle| \geq \delta$, then we will have:

$$\|F(x)\|^2 = \sum_{j=1}^{n} (\langle A_j x, x \rangle + B_j(x))^2 \geq (\|x\|^2 \delta - |B_i(x)|)^2.$$

As each $B_i = L(x) + a_i$ is a polynomial of degree ≤ 1, it follows that there exist constants b_1, b_2 such that: $|B_i(x)| \leq b_1 \|x\| + b_2$, for every x. Then there exists $\delta_0 > 0$ such that:

$$\|F(x)\| \geq \delta_0 \|x\|^2 \tag{4}$$

for every $\|x\|$ large; this implies (a).

Proof of (b). Observe first that in the proof of (a) we use only Condition 1 and not the others, so ∞ is an attractor for every F_μ.

Let $D(r)$ be the open ball in $\mathbb{R}^n$ of radius r and centered at the origin.

Claim. *There exist numbers* $0 < r_1 < r_2$ *such that*

$$\mathbb{R}^n \setminus B_\infty(\mu) \subset D(r_2\sqrt{|\mu|}) \setminus D(r_1\sqrt{|\mu|})$$

for every $|\mu|$ *large.*

Proof of the Claim. Take $x \notin D(r_2\sqrt{|\mu|})$, r_2 to be fixed. Then using Condition 1 as in the proof of (a) we find that for some $1 \le i \le n$:

$$\begin{aligned}\|F_\mu(x)\| &\ge \delta_0\|x\|^2 - |\mu\nu_i| \ge \delta_0\|x\|^2 - |\mu| \max|\nu_i| \\ &\ge \delta_0\|x\|^2 - \max|\nu_i|\frac{\|x\|^2}{r_2^2} \ge \delta_1\|x\|^2\end{aligned}$$

for some $\delta_1 > 0$ and every x large, if r_2^2 is taken $\ge \max|\nu_i|/\delta_0$. (We used (4), where $\|x\|$ was required to be large; so begin taking $|\mu|$ large to assure this condition.) This implies that $\|F_\mu(x)\| \ge 2\|x\|$ if $x \notin D(r_2\sqrt{|\mu|})$ and μ is large. Now suppose that $x \in D(r_1\sqrt{|\mu|})$, r_1 to be fixed. It is clear that $|f_i(x)| \le K_1\|x\|^2 + K_2$ for some positive constants K_1, K_2, every $1 \le i \le n$ and $x \in \mathbb{R}^n$. Then

$$||F_\mu(x)||^2 = \sum_{i=1}^{n}(f_i(x) - \mu\nu_i)^2 \ge (f_i(x) - \mu\nu_i)^2$$

for each $1 \le i \le n$, in particular,

$$||F_\mu(x)|| \ge \max|\nu_i||\mu| - K_1 r_1^2|\mu| - K_2 \ge r_2\sqrt{|\mu|},$$

if r_1 is small and $|\mu|$ large.

Then, by the the first part of the proof of the claim, it follows that $F_\mu(x) \in B_\infty(\mu)$ and so $x \in B_\infty(\mu)$. The claim is proved.

Consequently, if $C(\mu) = D(r_2\sqrt{|\mu|}) \setminus D(r_1\sqrt{|\mu|})$, then:

$$\mathbb{R}^n \setminus B_\infty(\mu) = \bigcap_{k=1}^{\infty} F_\mu^{-k}(C(\mu)).$$

As each A_i is invertible by Condition 3, there exists a constant $d > 0$ such that $\|A_i x\| \ge d\|x\|$ for every $1 \le i \le n$ and $x \in \mathbb{R}^n$. Now fix $x_0 \notin B_\infty(\mu)$ and let's prove that $(DF_\mu)_{x_0}$ expands every nonzero vector v uniformly in x_0. For every $1 \le i \le n$ the level s_i defined by $s_i := f_i(x_0) - \mu\nu_i$ belongs to $(-r_2\sqrt{|\mu|}, r_2\sqrt{|\mu|})$ because the contrary assumption implies $\|F_\mu(x_0)\| \ge r_2\sqrt{|\mu|}$ and then $x_0 \in B_\infty(\mu)$. By Condition 2 plus continuity, it follows that there exists $\epsilon > 0$ such that $\{x : \langle A_i x, x\rangle = \nu_i\}$ for $1 \le i \le n$ is an ϵ-transverse set. Also, the intersection of these sets is compact, by Condition 1 and the proof of (a). This gives the ingredients necessary to apply the transversality lemma, as we did in Proposition 1. First observe that the level sets

$$\{x : f_i(x) = \mu\nu_i + s_i\} \text{ for } 1 \le i \le n$$

form an $\epsilon/2$-transverse set if μ is large, and

$$\begin{aligned}\{x : f_i(x) - \mu\nu_i = s_i\} &= \{x : f_i(x) - s_i = \mu\nu_i\} = \{x : \frac{f_i(x) - s_i}{\mu} = \nu_i\} \\ &= \{x : sgn(\mu)\left(\frac{\langle A_i x, x\rangle}{|\mu|} + \frac{L_i(x)}{|\mu|} + \frac{a_i - s_i}{|\mu|}\right) = \nu_i\} \\ &= \sqrt{|\mu|}\{x : \langle A_i x, x\rangle + \frac{L_i(x)}{\sqrt{|\mu|}} + \frac{a_i - s_i}{|\mu|} = sgn(\mu)\nu_i\},\end{aligned}$$

where $sgn(\mu)$ is the sign of μ. As the functions $x \to \frac{1}{\sqrt{|\mu|}} L_i(x) + \frac{a_i - s_i}{|\mu|}$ are very small in C^2-topology in compact sets when $|\mu|$ is large (recall that $\frac{\tilde{s}_i(\mu)}{\mu}$ and $\frac{s_i(\mu)}{\mu}$ go to 0 as μ goes to $+\infty$), and the level sets $\{\langle A_i x, x\rangle = \nu_i\}$ are regular and ϵ-transverse, then the family of level sets $\{f_i(x) = \mu\nu_i + s_i\}$ is $\epsilon/2$-transverse for every μ large, as was claimed.

Finally, for $x \notin B_\infty(\mu)$ and $1 \leq i \leq n$, $||A_i(x)|| \geq dr_1\sqrt{|\mu|}$; then, as a consequence of the ϵ-transversality lemma, F_μ is expanding outside $B_\infty(\mu)$. This proves (b). □

It remains to prove that Conditions 1, 2 and 3 are open and dense in the topology of the norm of the matrices (which corresponds with the weak topology). The first and third condition come from the fact that eigenvalues and eigenvectors depend continuously on the matrix, and for the second, take first generically a matrix A_2 such that the level sets corresponding to A_1 and A_2 are transverse (thus the intersection will be a manifold of dimension $n - 2$ or else the empty set). Then proceed by induction.

5. Examples

Example 1 (Delay endomorphisms). An endomorphism of $\mathbb{R}^2$ of the form $F(x, y) = (y, f(x, y))$, is called a delay endomorphism. Suppose that $f(x, y) = ax^2 + by^2$, with $a, b > 0$, and let $\nu = (0, 1)$. The function f is C^2-convex, so ∞ is an attractor for every $F_\mu = F - \mu(0, 1)$. If $b >> a$, it follows from [RV] that for every large $\mu > 0$, F_μ has 2 saddle type fixed points. The stable manifolds of these fixed points play an important rôle in the understanding of the dynamics of F_μ. (For a recent work on invariant manifolds of endomorphisms see [S].) Moreover the complemen of B_∞ is the closure of the stable manifold of these fixed points, which turns out to be homeomorphic to the product of a Cantor set and a circle. These endomorphisms are hyperbolic, and satisfy the conditions of Przytycki [P], so are also structurally stable. It follows that for every strong perturbation G of F, the family G_μ has the same dynamical behavior as F_μ. This shows that $\mathcal{G}_\nu$ is not dense in the strong topology. In addition, if only the second coordinate of F is perturbed within the quadratic functions, then the same results of [RV] can be applied, and the family perturbed is again not in $\mathcal{H}_0$. In sight of theorem 1 we conclude that both coordinates should be perturbed to obtain an endomorphism in $\mathcal{G}_\nu$. Moreover, Theorem 1 gives also sufficient conditions (1 to 3) at the beginning of Sect. 4 that are easy to check in general. For example, $G(x, y) = (y + \epsilon_1 x^2 + \epsilon_2 y^2, ax^2 + by^2)$ belongs to $\mathcal{G}_\nu$ whenever $\frac{\epsilon_1}{\epsilon_2} \neq \frac{a}{b}$.

Example 2. Next we will construct an example of a C^2-convex endomorphism such that the level curves have not transversality enough to obtain expansivity. Furthermore, every C^2-strong perturbation of this transformation gives rise to a one parameter family which is also nonexpanding for all parameters μ large. This should be compared with the situation in quadratic endomorphisms where the genericity holds but when other topology is considered.

There exists a C^2-convex endomorphism F in $\mathbb{R}^2$ such that for every small C^2-strong perturbation G of F, the family $\{G_\mu\,;\,\mu > \mu_0\}$ does not intersect $\mathcal{H}_0$.

In fact, let $b : \mathbb{R} \to \mathbb{R}$ be any C^2 function satisfying:

1. $b(0) = b'(0) = 0$,
2. $b''(x) > 1$ for every $x \in \mathbb{R}$, and
3. $1/2 < 2x - b'(x) < 3/4$ for every $x \geq 1$.

First we show some function b satisfying the conditions above. Take b such that $b(0) = b'(0) = 0$, $b''(x) = 3/2$ for $|x| \leq 1$, $b''(x) = 2$ for $|x| \geq 3/2$ and $b''(x) \in (1/2, 3/4)$ for $|x| \in (1, 3/2)$.

Then

$$2x - b'(x) = \int_0^x 2 - b''(t)dt \in (1/2, 3/4)$$

for every $x > 1$.

Define $f_\mu(x, y) = x^2 + b(y) - \mu$ and $g_\mu(x, y) = f_\mu(y, x)$. It follows that each element of the family $F_\mu = (f_\mu, g_\mu)$ is a C^2-convex endomorphism of $\mathbb{R}^2$.

The functions $x \to \phi_\mu(x) = x^2 + b(x) - \mu$ verify $\phi_\mu(0) = -\mu$, $\phi'_\mu(0) = 0$ and $\phi''_\mu(0) \geq 3$ for every x. It follows that ϕ_μ has a fixed point $x_\mu > 0$ such that $x_\mu \to +\infty$ when $\mu \to +\infty$. It is clear that the point $P_\mu = (x_\mu, x_\mu)$ is fixed for F_μ.

Observe that $\{\nabla f(P_\mu), \nabla g(P_\mu)\}$ is ϵ-transverse if and only if

$$\epsilon < \frac{4x_\mu^2 - b'^2(x_\mu)}{4x_\mu^2 + b'^2(x_\mu)}.$$

Using the third condition of the definition of b it comes that $4x_\mu^2 - b'^2(x_\mu) < 4x_\mu - 1$. Thus it follows that the set $\{\nabla f(P_\mu), \nabla g(P_\mu)\}$ is not $\frac{4x_\mu - 1}{4x_\mu^2 + b'^2(x_\mu)}$-transverse. P_μ is a saddle type fixed point, with one eigenvalue in $(0, 1)$.

Now consider the C^2-convex functions given by

$$\tilde{f}(x, y) = f(x, y) - x \quad \text{and} \quad \tilde{g}(x, y) = g(x, y) - y.$$

Observe that $\tilde{f}^{-1}(\mu) \cap \tilde{g}^{-1}(\mu)$ is the set of fixed points of F_μ. In addition, if $\tilde{F} = (\tilde{f}, \tilde{g})$, then $D\tilde{F}_{P_\mu} = DF_{P_\mu} - Id$ has an eigenvalue in $(-1, 0)$; so it follows that the transversality of $\{\tilde{f}^{-1}(\mu), \tilde{g}^{-1}(\mu)\}$ is

$$\epsilon(\mu) \in \left(0, \frac{|\det D\tilde{F}_{P_\mu}|}{\|\nabla \tilde{f}\| \|\nabla \tilde{g}\|}\right).$$

This, by Corollary 2 is preserved by small perturbations, and it follows that the family perturbating F_μ must have a saddle type fixed point $P'(\mu)$ for every μ. This enables the new family to belong to $\mathcal{H}_0$.

Example 3. $\mathcal{G}_\nu$ is not open in $C^r(\mathbb{R}, \mathbb{R})$ with the strong C^r-topology.

Let f be an even function having derivative $f'(x) > 2$ for every $x > 1$, having a unique critical point at $x = 0$, $f(0) = 0$ and negative Schwarzian derivative. Suppose also that for the family $f_\mu = f - \mu$, the following conditions hold:

(i) $x_\mu > 1$ is a fixed point of f_μ,
(ii) $f_\mu^2(0) > x_\mu$ and $f_\mu^2(0) - x_\mu \to 0$ as $\mu \to \infty$.

Then, as f has negative Schwarzian derivative and the critical orbit intersects $(x_\mu, +\infty) \subset B_\infty$, it follows that $f \in \mathcal{G}_1$. Now, if g is a small perturbation of f such that $f = g$ outside $|x| \leq 1$, g has it unique critical point at 0 and $g(0) = f(0) + \epsilon$, then $g_\mu^2(0) < x_\mu$ for every $\mu > 0$ large. This implies that the whole interval $[-x_\mu, x_\mu]$ is invariant and $g \notin \mathcal{G}_1$. To construct f, begin with $f(x) = 2x^2$ in $[-1, 1]$ and choose f' decreasing to 2 at infinity. It is easy to see that f can be taken C^∞ with negative Schwarzian ($f''' \leq 0$ for $x > 0$). The items are satisfied if a careful choice of the first derivative of f is made outside $[-1, 1]$. Observe that if f' were constant equal to 2 the first item does not hold, and if f' is constant > 2 then the second one is not true. Take for example $f(x) = 2x + \frac{1}{x}$ for $x > \beta > 1$.

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