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Systèmes dynamiques/*Dynamical Systems*

# The principal loop-bundle and dynamical systems

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**Abstract.** The purpose of this Note is to announce the proof of the non-existence of expansive homeomorphisms on simply connected closed manifolds, provided one assumes locally connectedness of local stable and unstable sets. We also show the non-existence of homoclinic points in the universal covering of Anosov diffeomorphisms. These results are obtained by studying the lifted dynamics to the principal loop-bundle, thus obtaining a link between dynamical systems and universal gauge theory. © 1999 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

***Fibré principal des lacets et systèmes dynamiques***

**Résumé.** *Le but de cette Note est d'annoncer la preuve de la non-existence d'homéomorphismes expansifs sur les variétés simplement connexes, compactes et sans bord, en supposant la connexité locale des ensembles stables et instables locaux. Aussi, nous montrons la non-existence de points homocliniques dans le revêtement universel d'un difféomorphisme d'Anosov. Ces résultats sont déduits de l'étude de la dynamique relevée au fibré principal des lacets ; de cette façon, nous établissons un lien entre les systèmes dynamiques et une théorie de gauge universelle.* © 1999 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

## *Version française abrégée*

### 1. Introduction

Le but de cette Note est d'annoncer les résultats suivants :

THÉORÈME 1. – *Il n'y a pas d'homéomorphismes expansifs sur les variétés compactes simplement connexes en supposant la conéxité locale des ensembles stables et instables locales.*

THÉORÈME 2. – *Le relèvement d'un difféomorphisme d'Anosov au revêtement universel n'a pas de point homoclinique.*

Ces résultats pourraient contribuer à la classification des difféomorphismes d'Anosov (*voir* [4], [7], [8], [13], [15]). Ces théorèmes sont obtenus en étudiant le relèvement de la dynamique au fibré principal de lacets. Ce dernier est défini de la façon suivante : soit $\Omega$ l'ensemble des applications continues

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$\alpha : I \to M$, soit $\Omega/\sim$ le quotient par la relation d'équivalence générée par les identifications : $\alpha\, a\, a^{-1}\, \beta \sim \alpha\, \beta$. Si l'on fixe un point base $x \in M$, l'espace total $\mathcal{E}_x$ est l'ensemble des classes des $\alpha \in \Omega$ qui satisfont $\alpha(0) = x$. On désigne par $\{\alpha\}$ la classe de $\alpha$. Le groupe $\mathcal{L}_x$ est formé par les classes des courbes $\alpha$ telles que $\alpha(0) = \alpha(1) = x$. La projection $\pi : \mathcal{E}_x \to M$ est définie par $\pi(\{\alpha\}) = \alpha(1)$. Cela donne un fibré principal muni d'une connection dont le relèvement horizontal est le suivant : si $p \in \mathcal{E}_x$ et $\alpha$ est une courbe dans la varieté de base telle que $\pi(p) = \alpha(0)$, le relèvement horizontal de $\alpha$ à partir de $p$ est la courbe $\widetilde{\alpha}(t) = p\,\{\alpha_t\}$, où $\alpha_t(s) = \alpha(ts)$. L'utilisation du fibré principal universel dans l'étude des systèmes dynamiques est motivée par l'observation que deux idées clés de [12] peuvent être interprétées plus facilement au moyen des lacets :

1) On considère les courbes « courtes » (*voir* définition de $A_1$ ci-dessous) dont les points initial et final sont séparés par itération dans le futur. Alors, les itérées dans le futur des classes d'équivalence de telles courbes demeurent loin (dans la topologie quotient) des classes des courbes courtes. Cette propriété semble une forme faible d'expansivité infinie dans le fibré des lacets.
2) Fers à cheval longs et fins qui arrivent dans les applications pseudo-Anosov généralisées de $S^2$ (lesquelles sont « longitude-expansive » mais non expansives) correspondant aux classes qui sont proches des classes de courbes courtes.

## 2. Définitions et résultats

DÉFINITION 1. – L'homéomorphisme $f$ est *expansif* s'il existe $r > 0$ tel que si $\operatorname{dist}\big(f^n(x), f^n(x)\big) < r$ pour tout $n$ entier, alors $x = y$.

DÉFINITION 2. – Soit $0 < \varepsilon < r$. L'ensemble stable local est défini par :

$$W_\varepsilon^s(x) = \big\{y : \operatorname{dist}\big(f^n(x), f^n(x)\big) < \varepsilon,\ \ n \geq 0\big\}$$

et l'ensemble instable local par :

$$W_\varepsilon^u(x) = \big\{y : \operatorname{dist}\big(f^n(x), f^n(x)\big) < \varepsilon,\ \ n \leq 0\big\}.$$

Soient $0 < \rho < r_1 < r_2 < r$ tels que si $\operatorname{dist}(x, y) < \rho$ et il existe $N$ tel que $\operatorname{dist}\big(f^N(x), f^N(y)\big) < \rho$ et $\operatorname{dist}\big(f^i(x), f^i(y)\big) \leq r_2$ pour tout $0 \leq i \leq N$, alors $\operatorname{dist}\big(f^i(x), f^i(y)\big) \leq r_1$ pour tout $0 \leq i \leq N$.

DÉFINITION 3. – Soit $X(N)$ l'ensemble des $(x, y)$ qui satisfont

$$\operatorname{dist}(x, y) < \rho \ \text{ et } \ \operatorname{dist}\big(f^N(x), f^N(y)\big) < \rho.$$

Soit $X_0(N) \subset X(N)$ l'ensemble des $(x, y)$ dans $X(N)$ vérifiant $\operatorname{dist}\big(f^i(x), f^i(y)\big) \leq r_1$ pour tout $0 \leq i \leq N$. Soit $X_1(N) = X(N) - X_0(N)$.

DÉFINITION 4. – Soit $A_i(N)$, $i = 0, 1$, l'ensemble des $\alpha \in \Omega$ qui satisfont $\alpha(I) \subset \mathrm{B}(\alpha(0), \rho)$ et $(\alpha(0), \alpha(1)) \in X_i(N)$.

DÉFINITION 5. – Soit $K_i \subset \mathcal{L}_x$ l'ensemble des éléments $p\,\{f^N \circ \alpha\}\{\beta\}\,p^{-1}$, où $\alpha$, $\beta \in A_i(N)$. Soit $\mathcal{L}_x^0$ la composante connexe de l'identité.

Le théorème suivant implique le théorème 1 et le théorème 2.

THÉORÈME 3. – *On a $K_1 \cap \mathcal{L}_x^0 = \phi$, en supposant la connexité locale des ensembles stables et instables locaux.*

## 1. Introduction

The aim of this Note is to announce the following results:

THEOREM 1.1. – *There are no expansive homeomorphisms on simply connected closed manifolds provided local stable and unstable sets are locally connected.*

THEOREM 1.2. – *The lifting of an Anosov diffeomorphism to the universal covering has no homoclinic points.*

These results might contribute to the classification of Anosov diffeomorphisms (*see* [4], [7], [8], [13], [15]).

These theorems are obtained by studying the lifted dynamics to the principal loop-bundle. The later is defined in the following way: let $\Omega$ be the set of continuous $\alpha : I \to M$. Let $\Omega/\sim$ be the quotient set by the equivalence relation generated by the identifications $\alpha\, a\, a^{-1}\, \beta \sim \alpha\, \beta$. Chosen a base point $x \in M$, the total space $\mathcal{E}_x$ is the set of classes of those $\alpha$ in $\Omega$ satisfying, $\alpha(0) = x$. We denote by $\{\alpha\}$ the class of $\alpha$. The group $\mathcal{L}_x$ consists of the classes of curves with $\alpha(0) = \alpha(1) = x$. The projection $\pi : \mathcal{E}_x \to M$ is given by $\pi(\{\alpha\}) = \alpha(1)$. This gives a principal bundle with a connection whose horizontal lifting is given as follows: if $p \in \mathcal{E}_x$ and $\alpha$ is a curve in the base manifold such that $\pi(p) = \alpha(0)$, the horizontal lifting of $\alpha$ through $p$ is the curve $\widetilde{\alpha}(t) = p\,\{\alpha_t\}$, where $\alpha_t(s) = \alpha(ts)$. (The horizontal lifting of regular curves arises from a connection 1-form, [16]).

A piece-wise linear version of this principal bundle was introduced by Milnor in [14]. It has been rediscovered by Gambini and Trias [6]. They used it to obtain the so called loop-representation of gauge theories (*see* also [1], [2], [5], [10], [16], [17], [18]).

On the other hand we recall the classification theorem of expansive homeomorphisms by Lewowicz [12], and independently, Hiraide [9]. They show that expansive homeomorphisms on surfaces are conjugate to pseudo-Anosov (*see* [19] for the corresponding 3-dimensional results).

The use of the universal principal bundle in the study of dynamical systems is motivated by observing that two key ideas in [12] can be interpreted more easily in terms of loops:

1) Consider short curves (*see* the definition of $A_1$ below) whose endpoints separate by iteration in the future. Then, all future iterates of equivalence classes of such curves stay far away (in the quotient topology) from classes of short curves.
   This property resembles a mild form of infinite expansivity in the loop-bundle.
2) Thin and long horseshoes arising in the generalized pseudo-Anosov maps of $S^2$ (which are length expansive but not expansive maps) correspond to classes which are close to classes of short curves.

## 2. Definitions and results

DEFINITION 2.1. – The homeomorphism $f$ is said to be *expansive* if there exists $r > 0$ such that if $\operatorname{dist}\big(f^n(x), f^n(x)\big) < r$ for every integer $n$, then $x = y$.

DEFINITION 2.2. – Let $0 < \varepsilon < r$. Define local stable sets by:

$$W^s_\varepsilon(x) = \big\{y :\ \operatorname{dist}\big(f^n(x), f^n(x)\big) < \varepsilon,\ \ n \geq 0\big\}$$

and local unstable sets by:

$$W^u_\varepsilon(x) = \big\{y :\ \operatorname{dist}\big(f^n(x), f^n(x)\big) < \varepsilon,\ \ n \leq 0\big\}.$$

The expansivity implies the existence of a suitable Lyapunov function (*see* [11]). This Lyapunov function gives a kind of convexity: there are numbers $0 < \rho < r_1 < r_2 < r$ such that if $\operatorname{dist}(x, y) < \rho$, dist $(f^N(x), f^N(y)) < \rho$ for some $N$ and dist $(f^i(x), f^i(y)) \leq r_2$ for every $0 \leq i \leq N$, then dist $(f^i(x), f^i(y)) \leq r_1$ for every $0 \leq i \leq N$.

DEFINITION 2.3. – Define $X(N)$ as the set of pairs $(x, y)$ such that

$$\text{dist}\,(x, y) < \rho \ \text{ and } \ \text{dist}\,(f^N(x), f^N(y)) < \rho.$$

Let $X_0(N) \subset X(N)$ be the set of pairs $(x, y)$ in $X(N)$ such that dist $(f^i(x), f^i(y)) \leq r_1$ for every $0 \leq i \leq N$. Let $X_1(N) = X(N) - X_0(N)$.

DEFINITION 2.4. – Define $A_i(N)$, $i = 0, 1$, as the set of $\alpha \in \Omega$ such that $\alpha(I) \subset \mathrm{B}(\alpha(0), \rho)$ and $(\alpha(0), \alpha(1)) \in X_i(N)$.

DEFINITION 2.5. – Let $K_i \subset \mathcal{L}_x$ be the set of elements of the form $p\{f^N \circ \alpha\}\{\beta\}p^{-1}$, where $\alpha$, $\beta \in A_i(N)$.

The following theorem implies Theorems 1.1 and 1.2 above.

THEOREM 2.1. – *The set $K_1$ does not meet the connected component of the identity in $\mathcal{L}_x$, provided local stable and unstable sets are locally connected.*

## 3. Outline of proofs

Let $L_\varepsilon(x)$ be the minimum $n$ such that there is a decomposition $x = x_1 x_2 \ldots x_n$ with $x_i(I) \subset B_i$, where $B_i$ are balls of radius $\varepsilon$.

This length allows us to define a diameter $diam_\varepsilon(M)$ and it is also useful to state the length-expansivity property, which follows from (but it is not equivalent to) the expansivity:

LEMMA 3.1. – *There are constants $C > 0$ and $\lambda > 1$ such that if $L_\rho(\alpha) = 2$, then either $L_\rho(f^n \circ (\alpha)) > C\lambda^n$ for every $n \geq 0$, or $L_\rho(f^n \circ \alpha) > C\lambda^{-n}$ for all $n \leq 0$.*

Choose $R > 2\,\mathrm{diam}_\rho(M)$ and bigger than the length of a representative of a fixed element in $K_1 \subset \mathcal{L}_x$.

Choose $N > 0$ such that $C\lambda^N > R$. Let $F$ be the set of classes in $\mathcal{E}_x$ for having a representative $\alpha$ with $L_\rho(f^{-2N} \circ \alpha) \leq R$.

Let $D$ be the set of those classes in $\mathcal{E}_x$ for which there is a representative $\alpha$ such that $L_\rho(\alpha) \leq R$. Let $T(D)$ be the set of classes in $\mathcal{L}_x$ having a representative $\alpha$ with $L_\rho(\alpha) \leq 2R$ and $T(F)$ the set of classes in $\mathcal{L}_x$ having a representative $\alpha$ with $L_\rho(f^{-2N} \circ \alpha) \leq 2R$. Observe that $T(F)$ and $T(D)$ generate the group $\mathcal{L}_x$. Let $k$ be a group element satisfying $k\,F \cap D \neq \phi$, and let $U$, $W$ be suitable neighborhoods of the identity. Then we have the following basic lemma which is a topological consequence of length expansivity.

LEMMA 3.2. – *Let $h_0, h_1, \ldots, h_n$ be a sequence of group elements with $h_i\,F \cap h_{i+1}\,F \neq \phi$ for every $0 \leq i \leq n-1$, $h_i\,F \cap D \neq \phi$ for every $i$ and $h_i \in T(D)k\,W$ for every $i$. Then $h_n\,h_0^{-1} \in U\{\gamma\}$, where $L_\rho(f^{-N} \circ \gamma) \leq 2$.*

This lemma and the the fact (mentioned in the introduction), that curves in $A_1$ stay far away from neighborhoods of short horizontal curves, imply the following:

COROLLARY 3.1. – *There is no sequence $h_0, \ldots, h_n$ as above, joining the identity to a an element of $K_1$.*

This corollary is closely related to the following well known fact about Anosov diffeomorphisms: if $u\,s$ is a loop, where $u$ is an unstable curve in $A_1$ and $s$ is a stable one, then $u$ is very long provided $s$ is short enough. Nevertheless, the above mentioned corollary can be applied also to many sequences arising in length-expansive homeomorphisms of $S^2$, which, of course, are not expansive. In order to obtain information on the fundamental group of $M$, we need an extra ingredient involving the locally connectedness of local stable and unstable sets, namely:

LEMMA 3.3. – *Let* $h_0, h_1, \ldots, h_n$ *be a sequence of group elements satisfying* $h_i F \cap h_{i+1} F \neq \phi$ *for* $0 \leq i \leq n-1$, $h_i F \cap D \neq \phi$, $h_i \in T(D)^2 k W^2$ *for all* $i$, *and* $h_0$, $h_n \in T(D) k W$. *Then, there is another sequence with the same endpoints and moreover, contained in* $T(D) k W$.

The idea for the proof of this lemma is as follows, by Lemma 3.2 the elements $h_0, h_1, \ldots, h_n$ stay close to the iterates of short curves, and thus, they can be approximated by elements in $K_0$ (small horseshoes). The argument, now becomes local: the locally connectedness hypothesis implies the connectedness of $X_0$, which in turn, permits to modify again the sequence in order to be contained in $T(D) k W$.

With this lemmas the idea of the proof of the main theorem is the following: we classify the pairs of group elements in $T(D) k W$ into two classes, according to whether they can be joined by a sequence as above inside $T(D) k W$, or not. By using Lemma 3.3, we can show that both classes are open and closed. They are also non-void due to the corollary. In this way we separate the elements of $K_1$.

## References


[1] Ashtekar A., Lewandowski J., Representation theory of analytical holonomy C*-algebras, in: Knots and quantum gravity, J. Baez (Ed.), Oxford University Press, 1994.

[2] Barret J.W., Holonomy and path structures in general relativity and Yang–Mills theory, Int. J. Theor. Phys. 30 (1991) 1171–1215.

[3] Bonatti C., Langevin R. et al., Diffeomorphismes de Smale des surfaces, Astérisque, 1996.

[4] Franks J., Anosov diffeomorphisms, Proc. of Symposia in Pure Math. XIV (1970).

[5] Gambini R., Pullin J., Loops knots, gauge theories and quantum gravity, Cambridge Monogr. Math. Phys., Cambridge Univ. Press, 1996.

[6] Gambini R., Trias A., Gauge dynamics in the $C$ representation, Nucl. Phys. B 278 (1986) 436–448.

[7] Ghys É., Holomorphic Anosov systems, Invent. Math. 119 (3) (1995) 585–614.

[8] Gromov M., Groups of polynomial growth and expanding maps, Inst. Hautes Études Sci. Publ. Math. 53 (1981) 53–73.

[9] Hiraide K., Expansive homeomorphisms of surfaces are pseudo-Anosov, Osaka J. Math. 27 (1990) 117–162.

[10] Lewandowski J., Group of loops, holonomy maps, path bundle and path connection, Class. Quantum Grav. 10 (1993) 879–904.

[11] Lewowicz J., Lyapunov functions and topological stability, J. Differ. Eq. 38 (1980) 192–209.

[12] Lewowicz J., Expansive homeomorphisms of surfaces, Bol. Soc. Bras. Mat. 20 (1990) 113–133.

[13] Manning A., There are no new Anosov diffeomorphisms on tori, Amer. J. Math. 96 (1974) 422–429.

[14] Milnor J., Construction of Universal Bundles, I, Ann. Math. 63 (2) (1956) 272–284.

[15] Newhouse S., On codimension one Anosov diffeomorphisms, Amer. J. Math. 92 (3) 761–770.

[16] Reiris M., Spallanzani P., The loop derivative as a curvature, (to appear).

[17] Spallanzani P., The groups of loops and hoops, Preprint.

[18] Teleman C., Généralisation du groupe fondamental, Ann. Sci. École Norm. Sup. 77 (3) (1960) 195–234.

[19] Vieitez J., Expansive homeomorphisms and hyperbolic diffeomorphisms on 3-manifolds, Ergon. Theor. and Dynam. Syst. 16 (1996) 591–622.