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On homoclinic tangencies, hyperbolicity, creation of homoclinic orbits and variation of entropy

Enrique R Pujals† and Martín Sambarino‡

† Instituto de Matemática, Universidade Federal do Rio de Janeiro, CP 68.530, CEP 21.945-970 Rio de Janeiro, RJ, Brazil
‡ IMERL-Fac de Ingenieria, Universidad de la República, CC 30, Montevideo, Uruguay

E-mail: enrique@impa.br and samba@fing.edu.uy

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Abstract. We show that an assertion made in Gambaudo *et al* (1999 *Nonlinearity* **12** 443) is not correct and, as a consequence, we confirm and even generalize previous results by those authors (Gambaudo and Rocha 1994 *Nonlinearity* **7** 1251–9). Another main result states that variation of the topological entropy of surface diffeomorphisms is always associated with the unfolding of homoclinic tangencies.

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1. Introduction

In [GR1], Gambaudo and Rocha gave sufficient conditions for a diffeomorphism on the 2-sphere to be C^1 approximated by another exhibiting a homoclinic point, using a theorem of Araújo and Mañé. The proof of this theorem was never published, and Gambaudo and Rocha have recently published an erratum [GR2] pointing out this fact. There, they also assert that our own results in [PS] are weaker than those claimed by Araújo and Mañé, and not sufficient to establish the claims in [GR1].

Our first goal here is to show that the methods in [PS] do immediately yield a strong version of the claim of Araújo and Mañé as required for [GR1]. In fact, we can prove that the conclusions of Gambaudo and Rocha hold not only in the 2-sphere but, indeed, in any two-dimensional smooth compact manifold (see the corollary below).

Before we state our first main theorem, let us recall that the stable and unstable sets

$$W^s(p, f) = \{y \in M : \mathrm{dist}(f^n(y), f^n(p)) \to 0 \quad \text{as} \quad n \to \infty\}$$

$$W^u(p, f) = \{y \in M : \mathrm{dist}(f^n(y), f^n(p)) \to 0 \quad \text{as} \quad n \to -\infty\}$$

of a hyperbolic periodic point p of C^r diffeomorphisms $f : M \to M$ are C^r-injectively immersed submanifolds of M. A point of intersection of these submanifolds is called a *homoclinic point*. We say that a diffeomorphism exhibits a *homoclinic tangency* if the stable and unstable manifolds of some hyperbolic point have some non-transverse intersection.

Recall also that a set Λ is called hyperbolic for f if it is compact and f-invariant, and there exists a decomposition $T_\Lambda M = E^s \oplus E^u$ of the tangent bundle $T_\Lambda M$ into two Df-invariant sub-bundles E^u and E^s, and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|Df^n|_{E^s(x)}\| \leqslant C\lambda^n \qquad \text{and} \qquad \|Df^{-n}|_{E^u(x)}\| \leqslant C\lambda^n$$

for all $x \in \Lambda$ and for every positive integer n.

Theorem A. *Let $f : M \to M$ be a C^2 diffeomorphism of a compact surface M whose periodic points are all hyperbolic and which is not C^1 approximated by a diffeomorphism exhibiting a homoclinic tangency.*

Then, the non-wandering set $\Omega(f)$ can be decomposed into two compact invariant sets, $\Omega(f) = \Lambda_1 \cup \Lambda_2$, such that Λ_1 is hyperbolic and Λ_2 consists of a finite union of periodic simple closed curves $\mathcal{C}_1, \ldots, \mathcal{C}_n$, normally hyperbolic and such that each $f^{m_i} : \mathcal{C}_i \to \mathcal{C}_i$ is conjugate to an irrational rotation (m_i denotes the period of $\mathcal{C}_i$).

In particular, almost every point in M (with respect to the Lebesgue measure) is in the basin of attraction of some hyperbolic attractor or some normally attracting periodic simple closed curve.

As an immediate consequence, we obtain the following result that generalizes [GR1].

Corollary B. *Let $f : M \to M$ be a C^2 diffeomorphism of a compact surface having infinitely many periodic points, all of them hyperbolic. Then f is C^1 approximated by a diffeomorphism exhibiting a homoclinic point.*

Indeed, suppose otherwise. Then, in particular, f cannot be approximated by a diffeomorphism exhibiting a homoclinic tangency. It follows that f satisfies the hypotheses of theorem A, and so its non-wandering set can be written as a union of hyperbolic sets and closed curves supporting irrational rotations. Since f is assumed to have infinitely many periodic points, there must exist some non-trivial hyperbolic set in $\Omega(f)$. Thus, f itself has a homoclinic orbit, which is a contradiction.

Next, we state another consequence of the methods we developed in [PS]: *surface diffeomorphisms such that the topological entropy is not constant in a C^∞ neighbourhood can be C^1 approximated by others exhibiting homoclinic tangencies*. Let us first recall the definition of the topological entropy.

Given a metric space X and a transformation $T : X \to X$, we say that a subset $S \subset X$ is an *(n, ϵ)-generator* if for every $x \in X$ there is $y \in S$ such that $d(T^j(x), T^j(y)) \leqslant \epsilon$ for all $0 \leqslant j \leqslant n$. Let

$$r(n, \epsilon) = \min\{\mathrm{card}(S) : S \text{ is an}(n, \epsilon)\text{-generator}\}.$$

The *topological entropy* $h_{\mathrm{top}}(T)$ is defined as

$$h_{\mathrm{top}}(T) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r(n, \epsilon).$$

It is well known that this limit exists if T is continuous, see, e.g., [M]. We say that $f \in \mathrm{Diff}^\infty(M)$ is a *point of entropy variation* if for any neighbourhood $\mathcal{U} \subset \mathrm{Diff}^\infty(M^2)$ there is $g \in \mathcal{U}$ such that $h_{\mathrm{top}}(f) \neq h_{\mathrm{top}}(g)$.

Theorem C. *Let $f \in \mathrm{Diff}^\infty(M^2)$ be a point of entropy variation. Then, f is C^1 approximated by a diffeomorphism exhibiting a homoclinic tangency.*

We first proved this theorem as a corollary of some general results describing the dynamics of systems with a dominated splitting that we obtained in [PS2]. Then, after a lecture given by the first author at IMPA, Moreira and Ávila observed that theorem C can also be deduced directly from theorem A and [S], as described in the last section of this paper.

2. Proof of theorem A

We make extensive use of results and arguments from [PS].

For $f \in \mathrm{Diff}^1(M)$, denote by $\mathrm{Per}_h(f)$ the set of hyperbolic periodic points of saddle type of f, by $P_0(f)$ the set of sinks (periodic attractors), and by $F_0(f)$ the set of sources (periodic repellers). Let $\Omega(f)$ be the non-wandering set of f, and define

$$\Omega_0(f) = \Omega(f) - \big(P_0(f) \cup F_0(f)\big).$$

Clearly, $\Omega_0(f)$ is a compact invariant set. Let $\mathcal{U}$ be the set of diffeomorphisms that are not approximated by others exhibiting a homoclinic tangency, i.e.

$$\mathcal{U} = \mathrm{Diff}^1(M) - \overline{\{f \in \mathrm{Diff}^1(M) : f \text{ exhibits a homoclinic tangency}\}}.$$

An f-invariant set Λ is said to have a *dominated splitting* if it is possible to decompose its tangent bundle into two invariant sub-bundles $T_\Lambda M = E \oplus F$, and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|Df^n_{/E(x)}\| \|Df^{-n}|_{F(f^n(x))}\| \leqslant C\lambda^n \qquad \text{for all} \quad x \in \Lambda \quad n \geqslant 0. \tag{1}$$

Lemma 2.1. *Let $f \in \mathcal{U}$ with all its periodic points hyperbolic. Then, there exists $\mathcal{U}_0(f)$ such that for any $g \in \mathcal{U}_0(f)$, the set $\mathrm{Per}_h(g)$ has a dominated splitting. Moreover, constants C and λ as in definition (1) may be chosen to be independent of $g \in \mathcal{U}_0(f)$.*

Proof. This is a direct consequence of lemmas 2.2.2 and 2.0.1 of [PS]. Indeed, due to the hyperbolicity of the periodic points of f and the fact that f cannot be C^1 approximated by a diffeomorphism exhibiting a homoclinic tangency, the angle between the stable and unstable subspaces of the hyperbolic periodic points of saddle type of any diffeomorphism in a suitable neighbourhood of f is uniformly bounded away from zero (see lemma 2.2.2 of [PS]). This implies, as in lemma 2.0.1 of the same paper, the conclusion of our lemma. □

Corollary 2.1. *Let $f \in \mathcal{U}$ with all its periodic points hyperbolic. Then $\Omega_0(f)$ has a dominated splitting.*

Proof. As we argue in [PS], for any $x \in \Omega_0(f)$ there exists $g_n \in \mathcal{U}_0(f)$ converging to f and $p_n \in \mathrm{Per}_h(g_n)$ converging to x. Take a set $K \subset \Omega_0(f)$ such that any orbit in $\Omega_0(f)$ has only one point in K. Now, for $x \in K$ take $g_n \in \mathcal{U}_0(f)$ converging to f and $p_n \in \mathrm{Per}_h(g_n)$ converging to x. Taking subsequences if necessary, we may suppose that the stable (respectively, unstable) subspace $E^s_{p_n}$ (respectively, $E^u_{p_n}$) converges to a subspace $E(x)$ (respectively, $F(x)$) of T_xM. Note that, since the angle between $E^s_{p_n}$ and $E^u_{p_n}$ is uniformly bounded away from zero, $T_xM = E(x) \oplus F(x)$ holds. Extend this decomposition to the whole orbit of x by iteration under Df^n. It follows, by the uniformity of the constants C and λ, that $T_{\Omega_0(f)} = E \oplus F$ is a dominated splitting. Indeed, let $y \in \Omega_0$ and let $x \in K$ and p be such that $f^p(x) = y$. Then

$$E(y) = Df^p E(x) = \lim_{n\to\infty} Dg_n^p E^s_{p_n} = \lim_{n\to\infty} E^s_{g_n^p(p_n)}$$

and

$$F(y) = Df^p F(x) = \lim_{n\to\infty} Dg_n^p E^u_{p_n} = \lim_{n\to\infty} E^u_{g_n^p(p_n)}.$$

Therefore, for any $m \geqslant 0$,

$$\|Df^m_{/E(y)}\| \|Df^{-m}|_{F(f^m(y))}\| = \lim_{n\to\infty} \|Dg^m_{n/E^s_{g_n^p(p_n)}}\| \|Dg^{-m}_{n/E^u_{g_n^m(g_n^p(p_n))}}\| \leqslant C\lambda^m$$

holds. In particular, if $x \in K$ is periodic, then $E(x)$ (respectively, $F(x)$) must coincide with the stable (respectively, unstable) subspace of the periodic point. Otherwise, under iteration they would converge exponentially fast to the unstable subspace (or to the stable subspace in the past) and this would contradict the last inequality above for sufficiently large p and m. So the extension by iteration is indeed well defined. □

Proof of theorem A. Let f be as in the hypothesis of the theorem. Then, by the previous corollary, $\Omega_0(f)$ has a dominated splitting. Applying theorem B of [PS], we conclude that $\Omega_0(f)$ can be decomposed into the union of two disjoint compact invariant sets, one hyperbolic (say Λ_1) and the other (Λ_2) a finite union of normally hyperbolic periodic simple closed curves supporting irrational rotations. In particular, this implies that there are finitely many sinks and sources: otherwise, their accumulation points would be in $\Omega_0(f)$, and hence in Λ_1 (there are no periodic points in a neighbourhood of Λ_2). Since Λ_1 is hyperbolic, the maximal invariant set in an admissible neighbourhood of Λ_1 is also hyperbolic, but it would contain infinitely many sinks or sources, which is a contradiction.

Thus, $\Omega(f)$ can be decomposed as in the conclusion of the theorem. Moreover, the Lebesgue measure of the normally repelling simple closed curves is zero and, since f is C^2, the stable manifolds of the hyperbolic pieces of $\Omega(f)$ that are not attractors are also zero [B]. This concludes the proof of theorem A. □

3. Proof of theorem C

First, we need the following.

Lemma 3.1. *Let f be as in theorem A and let $\Omega(f) = \Lambda_1(f) \cup \Lambda_2(f)$ be the stated decomposition. Assume that f is a Kupka–Smale diffeomorphism. Then, there exists a C^1-neighbourhood $\mathcal{U}(f)$ such that for any $g \in \mathcal{U}(f)$ we have that $\Omega(g) \subseteq \Lambda_1(g) \cup \Lambda_2(g)$ where $\Lambda_1(g)$ is hyperbolic and $\Lambda_2(g)$ is finite union of normally hyperbolic simple periodic curves. Furthermore, $f_{/\Lambda_1(f)}$ and $g_{/\Lambda_1(g)}$ are conjugate.*

Proof. Let $L(f)$ be the limit set of f. Since $L(f) \subset \Omega(f)$ and $\Lambda_2(f) \subset L(f)$ we have that $L(f) = L_1(f) \cup \Lambda_2(f)$ and $L_1(f)$ is a hyperbolic set. Hence $\overline{\mathrm{Per}(f)} = L_1(f)$ (see [Sh]). Therefore, we have a spectral decomposition for $L_1(f) = L_1^1 \cup \ldots \cup L_1^r$ where L_1^j, $j = 1, \ldots, r$ are locally maximal invariant transitive (and hyperbolic) sets. On the other hand, $\Lambda_2(f)$ is a finite union of normally hyperbolic simple closed curves $\mathcal{C}_1 \cup \ldots \cup \mathcal{C}_p$ supporting irrational rotations. Thus $L(f) = L_1^1 \cup \ldots \cup L_1^r \cup \mathcal{C}_1 \cup \ldots \cup \mathcal{C}_p$. Since f is Kupka–Smale, the no-cycle condition holds (since in dimension two, cycles can occur only among pieces of saddle type). This yields a filtration on M adapted to f (see [Sh]). Therefore, $\Omega(f) = L(f)$. Moreover, by the local stability of the locally maximal transitive hyperbolic sets and the persistence of the normally hyperbolic periodic simple closed curves, we conclude the proof of the lemma. □

Recall that, by the variational principle (see [M]), $h_{\mathrm{top}}(f) = h_{\mathrm{top}}(f/\Omega(f))$ holds for any diffeomorphism f. Moreover, if $\Omega(f) \subseteq \Lambda_1 \cup \Lambda_2$ where Λ_1 and Λ_2 are two compact disjoint invariant sets, then

$$h_{\mathrm{top}}(f/\Omega(f)) = \max\{h_{\mathrm{top}}(f/\Lambda_1), h_{\mathrm{top}}(f/\Lambda_2)\}.$$

Thus, we obtain the the following.

Corollary 3.1. *Let f be a C^2 Kupka–Smale diffeomorphism of a compact surface which is not C^1 approximated by one exhibiting a homoclinic tangency. Then, there exists a C^1 neighbourhood $\mathcal{U}(f)$ such that for any $g \in \mathcal{U}(f)$, $h_{\mathrm{top}}(g) = h_{\mathrm{top}}(f)$ holds.*

Proof. Take $\mathcal{U}(f)$ from the previous lemma. Since the entropy on an invariant closed curve is always zero, we obtain

$$\begin{aligned} h_{\mathrm{top}}(g) &= h_{\mathrm{top}}(g/\Omega(g)) = \max\{h_{\mathrm{top}}(g/\Lambda_1(g)), h_{\mathrm{top}}(g/\Lambda_2(g))\} \\ &= h_{\mathrm{top}}(g/\Lambda_1(g)) = h_{\mathrm{top}}(f/\Lambda_1(f)) \\ &= \max\{h_{\mathrm{top}}(f/\Lambda_1(f)), h_{\mathrm{top}}(f/\Lambda_2(f))\} = h_{\mathrm{top}}(f/\Omega(f)) \\ &= h_{\mathrm{top}}(f). \end{aligned}$$

□

Proof of theorem C. Arguing by contradiction, assume that f cannot be C^1 approximated by a diffeomorphism exhibiting a homoclinic tangency. Take a small neighbourhood in the C^∞ topology $\mathcal{U}(f)$ such that no diffeomorphism in this neighbourhood is C^1 approximated by a diffeomorphism exhibiting a homoclinic tangency. Since f is a point of entropy variation, and the entropy function on compact surfaces is continuous in the C^∞ topology, by results of Newhouse [Ne], Yomdin [Y] and Katok [K], there exist two diffeomorphisms g_1, $g_2 \in \mathcal{U}(f)$ such that $h_{\mathrm{top}}(g_1) < h_{\mathrm{top}}(g_2)$. We may assume, again by the continuity of the entropy function, that g_1 and g_2 are Kupka–Smale diffeomorphisms. Note that, by the corollary above, the entropy is constant in a neighbourhood $\mathcal{U}_1$ (respectively, $\mathcal{U}_2$) of g_1 (respectively, g_2.) By a result of Sotomayor [S], we can take a continuous path $G : [1, 2] \to \mathrm{Diff}^\infty(M)$ such that:

- $G(1) \in \mathcal{U}_1$,
- $G(2) \in \mathcal{U}_2$,
- $G(t) \in \mathcal{U}(f)$ for any $t \in [1, 2]$,
- $\{t \in [1, 2] : G(t)$ is not Kupka–Smale$\}$ is at most countable.

From the previous corollary and the fact that the set of values of t such that $G(t)$ is not Kupka–Smale is at most countable, we conclude that there exists an open subset $U \in [1, 2]$ such that

- $h_{\mathrm{top}}(G(t))$ is constant in each connected component of U
- the complement of U is at most enumerable.

Since the function $h_{\mathrm{top}}(G(t))$ is continuous, we conclude that it must be constant on the whole of $[1, 2]$, which contradicts the assumption that $h_{\mathrm{top}}(g_1) \neq h_{\mathrm{top}}(g_2)$. □

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