Universidad Centro Occidental Lisandro Alvarado. Departamento de Matemática. Decanato de Ciencias. Apartado Postal 400. Barquisimeto, Venezuela.
E-mail address: nromero@delfos.ucla.edu.ve

Universidad de La República. Facultad de Ciencias. Centro de Matemática. Montevideo, Uruguay.
E-mail address: leva@cmat.edu.uy

Universidad Politécnica de Cataluña. Departament de Matemática Aplicada 2. Escola Técnica Superior d'Enginyers Industrials. Colom 11, 08222. Terrasa, Barcelona, España.
E-mail address: vilamajor@ma2.upc.es

union of the $J_{i-1}(a')$ for $a' \in \partial a$. This shows how to prove the induction. Finally, once $J_{k-1}(a)$ was constructed for every $a \in A_{k-1}$, one defines $J = J_G$ equal to the union of the j-skeletons of G for $1 \leq j \leq k-1$. This is homeomorphic to S^{k-1} and constitutes the boundary of the immediate basin of ∞.

Given a neighborhood N of the origin, it holds that the superorbit of any point in J intersects N, and as in the proof of the two dimensional case, it follows now that J is the boundary of the immediate basin of ∞.

It remains to prove the last assertion in part c) of theorem A. If $\lambda > |\lambda_0|$ for every other eigenvalue λ_0 of $DG_{\mathcal{O}}$, then there exists a strong unstable manifold $W^{uu}(\mathcal{O})$ associated to λ; each one of the separatrices of W^{uu} is invariant under G, but the above construction shows that there are not G-invariant curves in J. This implies that J is tangent at $\mathcal{O}$ to the codimension one submanifold $W = 0$ and it follows that it is of class C^1.

The first corollary follows immediately from the proof of the teorem. To prove the second, let μ_∞ be the Feigenbaum parameter for the family $\pi_\mu^{(1)}$. If $\mu < \mu_\infty$, then there exists an attracting periodic orbit Λ_μ which attracts all but a finite set of $[0, \mu]$. Then $\Lambda_\mu^{k-1} = \Lambda_\mu \times \cdots \times \Lambda_\mu$ is an invariant set for $\pi_\mu^{(k-1)}$, and its orbits are periodic of the same period of Λ_μ, and attracting. Moreover all but a finite set of points of Q_μ^{k-1} is attracted to Λ_μ^{k-1}. This implies that $\pi_\mu^{(k-1)}$ is structurally stable. It follows that G^k restricted to J' is conjugated to $\pi_\mu^{(k-1)}$.

If $\mu > \mu_\infty$, there exists a neighborhood N of the critical point of $\pi_\mu^{(1)}$, such that the following holds: The set of points of $[0, \mu]$ which positive orbit does not intersect N, is a Cantor expanding set, denoted Λ_μ. Then Λ_μ^{k-1} is a Cantor expanding set for $\pi_\mu^{(k-1)}$. Such an invariant set is persistent for small C^1 perturbations, and this implies the result. □

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2. $||D\Gamma_0(v_1, v_2)|| \leq \delta$ at every point (x_1, x_2) at a distance greater than ρ from the bondary of $Q_2(*, *, 0)$. The number ρ was chosen so that the closed ρ-neighborhod of the boundary of $Q_2(*, *, 0)$ is containd in an open set V; any point in V either has no preimage or its succesive G^3 preimages converge to Δ.

If U is a neighborhood of $Q_2(*, *, 0)$ and $\gamma_1 = G^{-3}(\gamma_0) \cap U$, then γ_1 is connected because the intersection of the boundary of of the image of G^3 with U is a two dimensional submanifold close to the corresponding set for F_μ, namely: $\{(x_1, x_2, x_3) : \max\{x_1, x_2\} = \mu^2/4\}$. It follows analogously that γ_1 is the graph of a function satisfying 1. and 2. Then we can proceed as before to obtain (as the limit of graphs) a two dimensional manifold γ, which is close to the piece of $Q_2(*, *, 0)$ that is ρ-bounded away from its boundaries. Now denote by $J_2(*, *, 0)$ the union of those components of the succesive G^3 preimages of γ that are contained in U.

Claim: *The boundary of $J_2(*, *, 0)$ is Δ.*

Indeed, as γ is G^3-forward invariant, then each G^3-preimage of it is also a two dimensional manifold that contains γ. Moreover, the points of γ which preimage does not belong to γ are contained in the neighborhood V of the boundary of $Q_2(*, *, 0)$, and thus its succesive G^3 preimages converge to Δ. As Δ is normally expanding (outside the ρ-neighborhood V of $\mathcal{O}$ and its preimages), it follows that $J_2(*, *, 0)$ is a two dimensional manifold outside this ρ-neighborhood. As Δ is forward invariant, the claim is proved.

It follows that

$$J = J_G = \left(\bigcup_{i=1}^{3} G^{-i} J_2(*, *, 0)\right) \cup \left(\bigcup_{i=1}^{3} G^{-i}(J_1(*, 0, 0))\right) \cup G^{-3}(\mathcal{O}),$$

is a two dimensional manifold homeomorphic to S^2, because each J_2 is homeomorphic to an open piece of plane with boundary formed up by preimages of the $J_1(*, 0, 0)$. It is also clear that $G(J) \subset J$. This proves the theorem in the case $k = 3$.

It remains to treat the general case of dimension $k > 3$. It will be done by induction. The first step consists in the construction of so called 0 faces of G. These are precisely the preimages of $\mathcal{O}$, assumed as always to be the fixed point in the boundary of the immediate basin of ∞. These 0-faces are denoted by $J_0(a)$ for $a \in A_0$.

The induction hypothesis is:

For every $1 \leq j < i$ and $a \in A_j$, there exists a j-dimensional manifold $J_j(a)$, forward invariant under G^k, and which boundary is formed up by the union of the $J_{j-1}(a')$ for $a' \in \partial a$. The union of the $J_j(a)$ for $a \in A_j$ is called the j-skeleton of G.

The induction is completed fixing first some $a \in A_i$ for which the last component of a is the symbol 0 and the others are 0 or $*$. Then, as in the previous dicussion, neighborhoods U of $Q_i(a)$ and V of the boundary of $Q_i(a)$, and a number $\rho > 0$ are fixed, so that:

1. For a point X in V, either X has no G^k preimage or its succesive preimages converge to the union of $J_{i-1}(a')$ for $a' \in \partial a$.
2. At every point in U at a distance greater than ρ from the boundary of $Q_i(a)$ the differential of G expands those directions associated with the axis for which $a_l = *$.

Using this, the method developed above for the construction of a G^k invariant manifold of dimension i close to the piece of $Q_i(a)$ at a distance ρ from its boundary also works. Then the union of the G^k preimages of this manifold, denoted $J_i(a)$ is shown to be homeomorphic to an open set in R^i and having boundary equal to the

∂a. It follows that the boundary of $Q_j(a)$ is $\bigcup\{Q_{j-1}(a') : a' \in \partial a\}$. The set of j-dimensional faces of the cube is called the j-skeleton of Q_μ.

The submanifold J will be constructed by induction in k steps; in each one of these steps a set of j-dimensional manifolds is created. These are denoted by $\Gamma_j(a_1, \cdots, a_k)$, for some sequence $(a_1, \cdots, a_k)$ and $a_i \in \{0, \mu, *\}$. The manifold $\Gamma_j(a_1, \cdots, a_k)$ is called a j-dimensional face of G and is close the corresponding face of the cube Q_μ.

To clarify the proof of this part of the theorem, we will first give the complete construction for $k = 3$, and then explain why the arguments work in the general case.

The 0-dimensional faces of G are simply the preimages $G^{-3}(\mathcal{O})$. For the construction of the 1-dimensional faces, begin taking the sequence $a = (*, 0, \cdots, 0)$. Observe that $Q_1(a)$ is F_μ^3-invariant. As shown in proposition 2, there exist neighborhoods U of $Q_1(a)$, V of $\mathcal{O}$ and $\mathcal{U}_1$ of F_μ such that for every $G \in \mathcal{U}_1$, a), b), c) and d) in the proof of proposition 2 hold and also e) if the number ρ is taken sufficiently small. Finally, condition f) can be substituted by:
f') For every $(x_1, x_2, x_3) \in U \setminus G^{-1}(D(0; \rho))$:

$$\frac{|\partial_1 g_1|}{|\partial_2 g_2|} \leq \tau < 1 \quad \text{and} \quad \frac{|\partial_1 g_1|}{|\partial_3 g_3|} \leq \tau < 1,$$

where $G^3 = (g_1, g_2, g_3)$. Then, always as in the referred proposition, take a function $\Gamma_0 : [\rho, \mu - \rho] \to R^2$, $\Gamma_0 = (\Gamma_0^2, \Gamma_0^3)$ satisfying $|(\Gamma_0^i)'(t)| < \delta$ for $i = 2, 3$.

Define as before γ_1 as the preimage of the graph of Γ_0 under G^3 that is contained in U. The proofs of the claims suffer small modifications: In claim 2 one has now to estimate the quotients $|v/u|$ and $|w/u|$ where

$$(u, v, w) = (DG_{(x_1,x_2,x_3)})^{-3}(1, (\Gamma_0^2)'(t), (\Gamma_0^3)'(t)), \quad \text{and}$$
$$G^3(x_1, x_2, x_3) = (t, \Gamma_0^2(t), \Gamma_0^3(t)).$$

It is easy to see that these quotients can be obtained smaller than δ if the perturbation is small. Similar corrections can be done in the proof of claim 3. Thus we have proved that there exists a curve γ, G^3-forward invariant and close to the piece of $Q_1(a)$ contained in $\rho < x < \mu - \rho$. Now consider the curve $J_1(*, 0, 0) = U \cap (\bigcup_{n \geq 0} G^{-2n}(\gamma))$. This curve is G^3-forward invariant and is the component of its G^3-preimage contained in U. Consider the components of the G and G^2 preimages of $J_1(*, 0, 0)$ that are close to $Q_2(*, *, 0)$. The union of these curves with the preimages of $\mathcal{O}$, give a closed G^3-forward invariant curve, denoted by Δ. It depends on the eigenvalues at $\mathcal{O}$ if the curves collapse smoothly or not. It is also possible for $J_1(*, 0, 0)$ to be a spiral around $\mathcal{O}$. By the moment we are only interested on the fact that Δ is homeomorphic to the circle; this closed curve will be the boundary of a two dimensional G^3 forward invariant manifold close to $Q_2(*, *, 0)$.

Now we will show how to construct the 2-dimensional faces of G. Begin with that associated to $Q_2(*, *, 0)$. At the points $(x_1, x_2, 0)$ the differential of G^3 is close to the matrix:

$$\begin{pmatrix} -2x_1 + \mu & 0 & 0 \\ 0 & -2x_2 + \mu & 0 \\ 0 & 0 & \mu \end{pmatrix}$$

Again the same reasoning of proposition 2 can be applied. Begin taking a function Γ_0 defined in the set of $X \in Q_2(*, *, 0)$ at a distance $\geq \rho$ from the boundary of $Q_2(*, *, 0)$, and values in R satisfying:
1. γ_0, the graph of Γ_0, is contained in a neighborhood of $Q_2(*, *, 0)$.

Thus both curves have to be tangent to the less expanding vector $(1, \lambda^-)$. It follows that this union is of class C^1. Then the curves $\tilde{\gamma}$ collapse smoothly at the points of intersection, and it results that J is of class C^1.

We will leave without proof the case where $\lambda^+ < \lambda^-$. In this case, the fact that the positive eigenvalue is the smaller one implies that J has a cusp at $\mathcal{O}$ and this cusp is repeated at the preimages of $\mathcal{O}$. □

6. Proof of Theorem A

Proof. Part (a) follows from proposition 1. The fact that there is some eigenvalue $\lambda > 1$ is a consequence of its proof. As G is close to F_μ, the characteristic polynomial of G at $\mathcal{O}$ is close to $(-1)^k(x^k - \mu)$; if the perturbation is small, then the other roots are close to the other k^{th} roots of μ, and cannot be positive.

The proof of part (b) is similar to the proof for rational mappings in the Riemann sphere. The following lemma is part (b).

Lemma 2. *Let $G \in \mathcal{U}$ be a delay endomorphism such that the set of critical points ℓ_1 of G is contained in B_∞. Then B_∞ is connected and its complementary set A has uncountably many components.*

Proof. As B_∞ contains the curve ℓ_1 of critical points and B_∞ is invariant, then it also contains the image P of ℓ_1, which is the boundary of the image of G.

Define $A_0 = G(A)$. A_0 is contained in the interior of $\tilde{P}$ (the image of G) and is not connected because the fixed points of G belong to A_0 and are contained in different components of $\tilde{P} \setminus \ell_1$. It follows that we can choose a compact connected set K such that $A_0 \subset K$, K is contained in the interior of $\tilde{P}$ and $G^{-1}(K) \cap A_0 \subset K$. Observe that the preimage of K has two connected components K_1 and K_2, one located at each side of ℓ_1. It is also clear that G restricted to any K_i is a diffeomorphism onto K. Then the preimage of K_1 has also two connected components, $K_{11} \subset K_1$ and $K_{21} \subset K_2$. The same holds for K_2 giving preimages $K_{12} \subset K_1$ and $K_{22} \subset K_2$. These preimages are also compact. Taking succesive preimages and labeling as above, one has, for any finite sequence $a_0, a_1, \cdots, a_n$ of numbers 1 and 2, a compact connected set $K_{a_0,\cdots,a_n}$ satisfying, for every $0 \le j \le n$:

$$G^j(K_{a_0,\cdots,a_n}) \subset K_{a_j,\cdots,a_n}$$

and $K_{a_0,\cdots,a_n} \subset K_{a_0,\cdots,a_{n-1}}$. Furthermore, for any infinite sequence $\{a_0, a_1, \cdots\}$ of numbers 1 and 2, the nested sequence of compacts sets associated have a nonempty intersection which is also compact and connected. It is clear that $A_0 = \bigcap_{n \ge 0} G^{-n}(K)$. This implies the lemma. □

Beginning with the proof of part (c) of theorem A, we introduce some more notation. Label the j-dimensional faces of the k-dimensional cube $Q_\mu = [0, \mu]^k$ with a sequence $(a_1, \cdots, a_k)$. Each a_i is one of the three symbols $0, \mu$ and $*$. When $a_i = *$ this means that the $i-th$ coordinate varies between 0 and μ; when $a_i = 0$ (resp. μ) this means that the $i-th$ variable is constant equal to 0 (resp. μ). If the face is j- dimensional, then the number of i such that $a_i = *$ is equal to j. Any j-dimensional face is denoted by $Q_j(a_1, \cdots, a_k)$.

Denote by A_j the set of sequences a such that exactly j of the entries of a is a $*$. The boundary of the j-dimensional face $Q_j(a)$ given by the sequence $a = (a_1, \cdots, a_k)$ is the union of $2j$ faces of dimension $j-1$. Each one of these faces is given by the sequence obtained substituting one of the symbols $*$ in $(a_1, \cdots, a_k)$ by a 0 or a μ. The set of sequences obtained from a in this manner is denoted by unit circle because each of the curves $\tilde{\gamma}_i$ is homeomorphic to an interval and do not intersect the others, excepting at their extreme points. See figure 2.

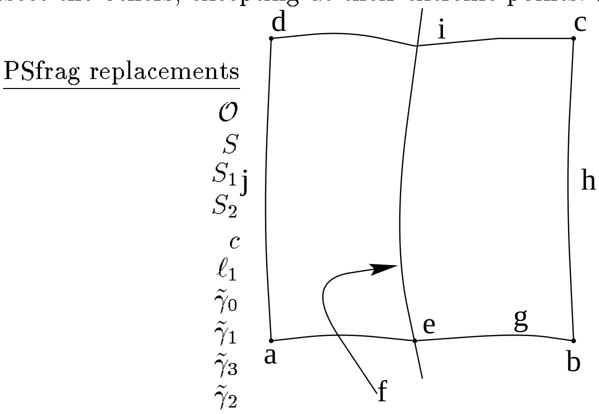

Figure 2

It remains to prove that J is equal to the boundary of the immediate basin of ∞, B^0_∞. Take a small neighborhood N of the origin such that J separates N into two components, one of which is contained in B_∞ (this is a consequence of proposition 2). As J is forward invariant, it follows that it does not intersect B_∞ and then $J \cap N$ is contained in the boundary of B^0_∞. Moreover, the superorbit of any X in J (that is, the set of points Y such that for some positive n and m it holds that $G^n(Y) = G^m(X)$) intersects N. This clearly implies that the whole J is a subset of ∂B^0_∞. The other inclusion is obvious since J separates the plane. □

For $k = 2$, the restrictions of G and G^2 to the curve J are shown in figure 3.

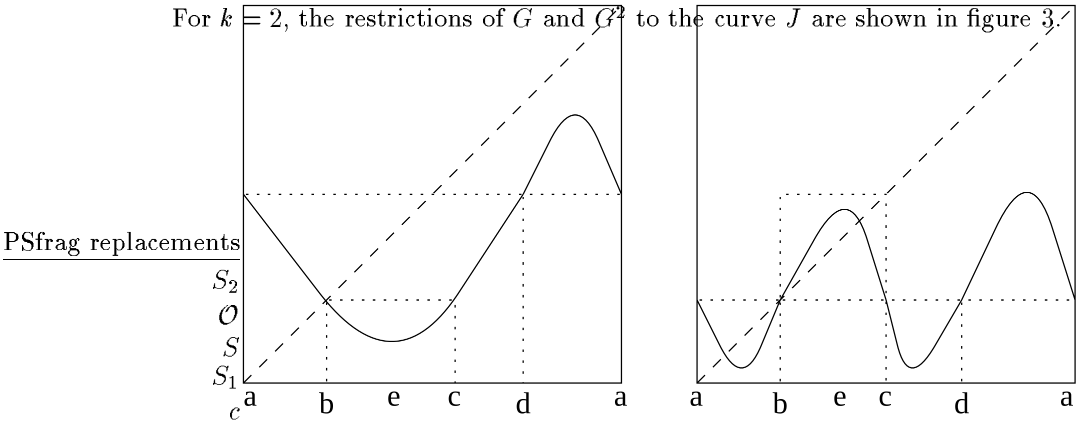

Figure 3

Corollary 4. *Let G be as the above proposition, and let λ^+, λ^- be the eigenvalues at the fixed point $\mathcal{O}$.*

1. *If $-\lambda^- < \lambda^+$ then the curve J is of class C^2.*
2. *If $-\lambda^- > \lambda^+$ then the curve J is C^2 except at $\mathcal{O}$ and its preimages, where it has cusps.*

Proof. Suppose first that $\lambda^+ > \lambda^-$. Then the strong unstable manifold W^{uu} of $\mathcal{O}$ is tangent to the eigenvector $(1, \lambda^+)$ associated to $\lambda^+ > 0$. It follows that each of the strong unstable separatrices is forward invariant. Therefore it is not possible that $\tilde{\gamma}_0$ be coincident with a separatrix of W^{uu}. Recall that the union of $\tilde{\gamma}_0$ and $\tilde{\gamma}_2$ is a curve forward invariant under G, but the image of one is the other and viceversa.

By claim 1 and properties (d) and (e), it follows that if γ_1 is the graph of a function, then the domain of this function must contain $[\rho, \mu - \rho]$.

Beginning with the proof of (2), take any t such that $G^2(x,y) = (t, \Gamma_0(t))$, $(x,y) \in U$ and $x \in [\rho, \mu-\rho]$. Denote the tangent vector to γ_1 at (x,y) by (u,v). It follows from equation (11) that

$$\begin{aligned} \left|\frac{v}{u}\right| &= \left|\frac{-\partial_1 h + \partial_1 g.\Gamma_0'}{\partial_2 h - \partial_2 g.\Gamma_0'}\right| \leq \frac{\epsilon + |\partial_1 g.\Gamma_0'|}{|\partial_2 h| - \epsilon.|\Gamma_0'|} \\ &\leq \frac{|\partial_1 g|}{|\partial_2 h| - \epsilon.|\Gamma_0'|}|\Gamma_0'| + \frac{\epsilon}{|\partial_2 h| - \epsilon.|\Gamma_0'|} \\ &\leq \tau\delta + \epsilon < \delta, \end{aligned} \tag{12}$$

if ϵ is small, that is, if the perturbation is chosen small with respect to τ, and hence to ρ. This implies (2) and also (1), and proves the claim.

It will be clarifying to see what happens when the point (x,y) does not belongs to $x \in [\rho, \mu-\rho]$. If $x > \mu - \rho$, then the G^2 preimage of (x,y) does not exist in U (by (d) and (e)). If $x < \rho$, then either $G^{-2n}(x,y) \in V$ for every $n > 0$ and then (c) implies that the sequence converges to $\mathcal{O}$, or there exists a first n_0 such that $G^{-2n_0}(x,y)$ belongs to V_0 and again has no preimage.

Consider now the space $\mathcal{C}$ of curves defined in $[\rho, \mu-\rho]$ satisfying (i) and (ii), endowed with the C^1 topology in $[\rho, \mu-\rho]$. Define the operator Φ carrying Γ_0 to Γ_1 as above.

Claim 3: Φ *is a contraction of* $\mathcal{C}$.

Note first that the above claims imply that Φ is well defined and carries $\mathcal{C}$ into $\mathcal{C}$. Take curves Γ_0 and α_0 in $\mathcal{C}$ and apply G^{-2}. Denote the curves obtained by Γ_1 and α_1. Observe that

$$\begin{aligned} |G^2(t,x) - G^2(t,y)| &= |(g(t,x) - g(t,y), h(t,x) - h(t,y))| \\ &= |(\partial_2 g(\xi).(x-y), \partial_2 h(\eta).(x-y))| \\ &\approx \partial_2 h.|x-y| \approx \mu.|x-y|. \end{aligned}$$

This implies that $|(t, \Gamma_1(t)) - (t, \alpha_1(t))| \leq \frac{1}{\mu}||\Gamma_0 - \alpha_0||$. Moreover,

$$|\Gamma_1' - \alpha_1'| = \left|\frac{-\partial_1 h_0 + \partial_1 g_0 \Gamma_0'}{\partial_2 h_0 - \partial_2 g_0 \Gamma_0'} - \frac{-\partial_1 h + \partial_1 g \alpha_0'}{\partial_2 h - \partial_2 g \alpha_0'}\right|, \tag{13}$$

where the subindices 0 indicate that the derivatives of the first member are evaluated at $(t, \Gamma_1(t))$ and those of the second at $(t, \alpha_1(t))$. As was shown above these points are at a distance less than $||\Gamma_0 - \alpha_0||/\mu$. One has to prove that the derivatives of the curves Γ_1 and α_1 are close; a simple calculation similar to that done in equation (12) gives that the norm of $\Gamma_1 - \alpha_1$ is less than a constant < 1 times $||\Gamma_0 - \alpha_0||$. This finishes the proof of the claim.

Thus Φ has a unique fixed point in $\mathcal{C}$; this is a function $\Gamma : [\rho, \mu-\rho] \to \mathbb{R}$, satisfying $|\Gamma'(t)| < \delta$. Denote by γ its graph. It is clear that $G(\gamma) \subset \gamma$ and $G^{-2}(\gamma) \cap U \supset \gamma$. Define $\tilde{\gamma}_0 = \bigcup_{n \geq 0} G^{-2n}(\gamma) \cap U$.

As γ intersects V, then $\tilde{\gamma}_0$ is a curve joining $\mathcal{O}$ to S, is C^0 close to the segment joining $\mathcal{O}$ to $(\mu, 0)$, and is homeomorphic to an open interval. Observe that $G^{-1}(\tilde{\gamma}_0)$ has two connected components, one of them, $\tilde{\gamma}_2$, is a curve joining $\mathcal{O}$ to S_2 and the other one, $\tilde{\gamma}_1$, is a curve joining S with S_1. Note also that $G^{-1}(\tilde{\gamma}_2) = \tilde{\gamma}_0$ and that $G^{-1}(\tilde{\gamma}_1)$ is a curve $\tilde{\gamma}_3$ joining S_2 with S_1. Thus the curve $J = J_G = \bigcup_{i=0}^{i=3} \tilde{\gamma}_i$ is a closed curve satisfying $G(J) \subset J$ and $G^{-1}(J) \supset J$. J is homeomorphic to the

(a) The closure of the connected component of $G^{-2}(U)$ containing $\mathcal{O}$ is contained in U.
(b) The closure of the connected component of $G^{-2}(V)$ containing $\mathcal{O}$ is contained in V.
(c) $G^{-2n}(X) \cap V \to \mathcal{O}$ for every $X \in V$, when $n \to +\infty$.
(d) Note that $G^{-2}(V)$ has connected components V_1 and V_2 containing S_1 and S_2 respectively. Take V small such that $V_i \cap \tilde{P}_G = \emptyset$ for $i = 1, 2$ (this is possible because $\mu < 4$ and if $\mathcal{U}_1$ is small enough). Remember that $\tilde{P}_G$ is the set of points having exactly two preimages under G. It follows that the component V_0 of $G^{-1}(V)$ containing S, has no preimage under G^2.

Take also $\rho > 0$ such that:
(e) The disc $D(0; \rho)$ centered at $\mathcal{O}$ and radius ρ, is contained in V.
(f) There exists τ such that for every $(x, y) \in U \setminus G^{-1}(D(0; \rho))$ it holds that:

$$\frac{|\partial_1 g|}{|\partial_2 h|} \leq \tau < 1,$$

where $G^2 = (g, h)$. This last item is true because the differential of G at (x, y) is close to $\begin{pmatrix} -2x+\mu & 0 \\ 0 & \mu \end{pmatrix}$. It should be necessary to shrink U once again. See figure 1.

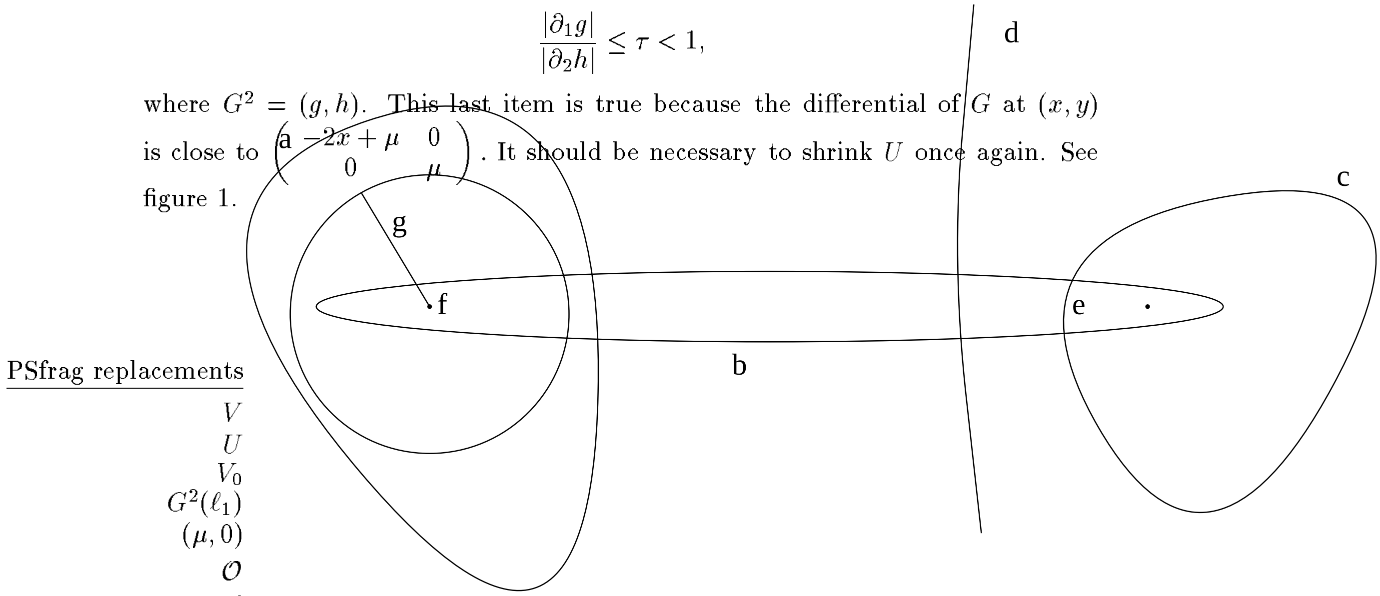

Figure 1

Suppose that $\Gamma_0 : [\rho, \mu - \rho] \to \mathbb{R}$ is a function satisfying, for some $\delta > 0$:

(i) γ_0, the graph of Γ_0, is contained in U.
(ii) $|\Gamma_0'(t)| < \delta$ for every $t \in [\rho, \mu - \rho]$.

Define $\gamma_1 = G^{-2}(\gamma_0) \cap U$.

Claim 1: γ_1 *is connected.*

Observe that S does not belong to $G^2(\mathbb{R}^2)$, see item (d) of the properties of V. Now γ_0 is almost horizontal and $G^2(\ell_1)$ is the boundary of $G^2(\mathbb{R}^2)$ and its intersection with U is almost vertical . Then these two curves have a unique point of transverse intersection; clearly this point has first coordinate in $[\rho, \mu - \rho]$, so it can be denoted by $(t_0, \Gamma_0(t_0))$. Observe that for $t > t_0$ the point $(t, \Gamma_0(t))$ has no G^2-preimage in U, and for $t < t_0$ the point $(t, \Gamma_0(t))$ has two G^2-preimages in U, one located at each side of ℓ_1. This implies that the preimage of γ_0 contains a pair of curves; as these curves collapse at the unique point preimage of $(t_0, \Gamma_0(t_0))$, we conclude that γ_1 is a continuous curve. This proves the claim.

Claim 2: γ_1 *contains the graph of a function* Γ_1 *defined in* $[\rho, \mu - \rho]$*, and satisfying the properties (i) and (ii) stated above for* Γ_0*.*

Observe that this is a general fact for convex delay endomorphisms: we do not ask the mapping F to be a perturbation of F_μ. However, the above proof also works in this case: The fact that ∞ is an attractor is trivial in the convex case. The presence of a fixed point of φ with derivative greater than or equal to 1 now follows from the convexity of the function φ, and the fact that F has fixed points. Finally, the Lyapunov function can be defined in this case as $W(x) = V(x)$, (that is, $\alpha_i = 0$) making easier the proof, because now the function $q(x)$ is automatically negative for every nonzero x. Observe that it is not needed the fact that the derivative of φ at the fixed point is greater than one, now it suffices ≥ 1. Note also that the proof for q positively definite is similar.

5. The two dimensional case

We now make a disgression to consider in more detail the case $k = 2$. If G is a C^2 small delay perturbation of $F_\mu(x,y) = (y, -x^2 + \mu x)$, $\mu > 1$, and $\mathcal{O}$ is fixed point of G, then the eigenvalues of $DG_{\mathcal{O}}$, λ^+ and λ^-, satisfy: $\lambda^+ \approx -\lambda^- \approx \sqrt{\mu}$. Since the matrix of G at this fixed point is $\begin{pmatrix} 0 & 1 \\ d & e \end{pmatrix}$, then the condition $e < 0$ is equivalent to $\lambda^+ < -\lambda^-$.

Let $G^2 = (g,h)$, $(x,y) \in U$ such that $G^2(x,y) = (x_1,y_1)$ and let (u,v) be a vector in $\mathbb{R}^2_{(x_1,y_1)}$. Then, denoting by (u_1,v_1) the vector $DG^{-2}_{(x_1,y_1)}(u,v)$, ($DG^{-2}_{(x_1,y_1)}$ considered as the inverse of $DG^2_{(x,y)}$), one has:

$$u_1 = \frac{1}{d}(\partial_2 h.u - \partial_2 g.v), \; v_1 = \frac{1}{d}(-\partial_1 h.u + \partial_1 g.v), \tag{11}$$

where d is the Jacobian of G^2 at (x,y) and the derivatives are calculated at the point (x,y).

Let S be the another preimage of the fixed point $\mathcal{O}$ of G and S_1, S_2 the preimages of S under G. S has preimages because G is close to F_μ. Let S_2 (resp. S_1) be the preimage of S which falls near to $(0,\mu)$ (resp. (μ,μ)).

We now present the two dimensional version of the proof of existence of invariant manifolds in the boundary of the basin of ∞. The proof we give now will be the first induction step for the higher dimensional case.

Proposition 2. *Let G be a C^2 small delay perturbation of F_μ with $1 < \mu < 4$. Then there exists a curve $J = J_G$ satisfying:*

1. $G^{-1}(J) \supset J$
2. $G(J) \subset J$
3. *J is homeomorphic to S^1.*
4. *J is the boundary of the immediate basin of ∞.*

Proof. The proof cannot apply directly the usual method of graph transforms (see for example [HPS]) because Q is not normally expanding for F_μ. Recall that the line $\{(x,0) : 0 \leq x \leq \mu\}$ is F^2_μ-invariant and that the vertical vector is more expanded by DF than the horizontal one, excepting at the points $\mathcal{O}$ and its preimage (see item 5 in section 2). However, a slight modification of this method together with the previous proposition give a proof.

Begin taking connected neighborhoods U of the line joining $\mathcal{O}$ to $(0,\mu)$, V of $\mathcal{O}$ and $\mathcal{U}_1$ of F_μ such that for every $G \in \mathcal{U}_1$:

$$W(F(x)) - W(x) < q(x) + \sum_{i=1}^{k-1}((\lambda-1)\alpha_i x_i^2 - \alpha_i x_{i+1}^2 + \alpha_i x_i^2)$$
$$= q(x) + \lambda\alpha_1 x_1^2 + \sum_{i=2}^{k-1}(\lambda\alpha_i - \alpha_{i-1})x_i^2 - \alpha_{k-1}x_k^2 \quad (10)$$

Note that depending only on the eigenvalue λ (greater and far from 1 for any perturbation) it is possible to choose an $\epsilon > 0$ small and values α_i such that:

$$\lambda\alpha_1 < 2-\epsilon,\ \lambda\alpha_i - \alpha_{i-1} < -\epsilon,\ \text{ for } 2 \leq i \leq k-1 \text{ and } \alpha_{k-1} > \epsilon.$$

Fix some choice of the values α_i satisfying the above properties and consider

$$\beta_1 = \lambda\alpha_1,\ \beta_i = \lambda\alpha_i - \alpha_{i-1} \text{ for } 2 \leq i \leq k-1 \text{ and } \beta_k = \alpha_{k-1}.$$

Next define a function T by the last member of equation (10). We have just proved that $W(F(x)) - W(x) < T(x)$, whenever $W(x) < 0$.

Observe that T clearly satisfies: $T(\mathcal{O}) = 0$, $DT_{\mathcal{O}} = 0$; moreover, by the choice of the numbers α_i and equation (8) it holds that $T''(x) < 0$ for every $x \in \mathbb{R}^k$. Obviously these conditions imply $T(x) < 0$ for every $x \in \mathbb{R}^k \setminus \{\mathcal{O}\}$, consequently $W(F(x)) - W(x) < 0$ if $W(x) < 0$.

It follows that the set $\{W < 0\}$ is F-invariant and that $F^n(x) \to \infty$ when $n \to \infty$, that is, $\{W < 0\} \subset B_\infty$. Finally, the fact that $\mathcal{O}$ belongs to the boundary of $\{W < 0\}$ implies that this fixed point belongs to the boundary of B_∞. It remains to prove that ∞ is an attractor. This have not been proved until now because $\{W < 0\}$ does not contain a neighborhood of ∞. We will be done after proving the following:

Claim: *The preimage of $\{W = 0\}$ is a compact subset of $\mathbb{R}^k$.*

We will prove that $\{W = 0\}$ separates P, which implies the claim, by the remark at the end of section 2. Recall from lemma 1 that P is the image of the set of critical points ℓ_1 and that the latter is the graph of a function $\bar{x}_1 : \mathbb{R}^{k-1} \to \mathbb{R}$. It follows that P is the graph of the function $p : \mathbb{R}^{k-1} \to \mathbb{R}$ given by

$$p(x_1, \cdots, x_{k-1}) = f(\bar{x}_1(x_1, \cdots, x_{k-1}), x_1, \cdots, x_{k-1}).$$

Observe that $p(0) = f(\bar{x}_1(0), 0) \approx f(\mu/2, 0) \approx \mu^2/4$, and that $p''(0)$ can be taken uniformily small if F is close to F_μ. On the other hand, note that $\{W = 0\}$ is also the graph of a function defined in $\{x_k = 0\}$, because $V(e_k) = 1$. Denote by $\bar{\alpha}$ this function. Clearly $\bar{\alpha}(0) = 0$. In addition, when $F \to F_\mu$, the matrix associated to the second derivative of f approachs that of f_μ (see equation (8)); then the possibility of choices of the sequence of numbers α_i includes $\alpha_i \geq \delta > 0$ for every i and some $\delta > 0$. Finally, observe that the second derivative of $\bar{\alpha}$ is associated to the diagonal matrix with entries $2\alpha_i$. Therefore, if the perturbation F is sufficiently small, then the function $\bar{\alpha} - p$ is negative at 0 and has positive definite second derivative. This implies that $\{W = 0\}$ separates P, and proves the claim and the theorem. □

The following corollary will not be used in the remaining of the paper, but it seems interesting.

Corollary 3. *Let f be C^2-convex function, and F the delay endomorphism associated. If F has fixed points, then there exists a fixed point in the boundary of the immediate basin of ∞. And, as was mentioned above, if F has no fixed points, then $B_\infty = \mathbb{R}^k$.*

and $\sum_{i=1}^{k} a_i > 1$, because $\varphi'(0) > 1$. Moreover, the Hessian matrix H_x associated to the second derivative of q at any point $x \in \mathbb{R}^k$ satisfies:

$$H_x \approx \begin{pmatrix} -2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

The following remark is crucial:
Given $\epsilon > 0$, the perturbation G of F can be taken sufficiently small in such a way that for every diagonal matrix H with entries $(\beta_1, \cdots, \beta_k)$ satisfying $\beta_1 < 2 - \epsilon$ and $\beta_i < -\epsilon$ for every $i > 1$ it holds that

$$H + H_x \tag{8}$$

is negatively definite for every x.

Consider the linear transformation

$$A(x_1, \cdots, x_k) = (x_2, \cdots, x_k, L(x)),\ x = (x_1, \cdots, x_k).$$

Obviously $F(x) = A(x) + q(x)e_k$, where e_k is the last vector of the cannonical basis of $\mathbb{R}^k$ and $x = (x_1, \cdots, x_k)$. The characteristic polynomial of A is

$$P_A(y) = (-1)^k[y^k - (a_k y^{k-1} + \cdots + a_2 y + a_1)].$$

As $\sum_{i=1}^{k} a_i > 1$, it follows that A has an eigenvalue $\lambda > 1$. Then the adjoint A^* acting in the dual space of $\mathbb{R}^k$ has λ as an eigenvalue. Let V be an eigenvector of A^* associated to the eigenvalue λ. Then the kernel of the functional V is invariant under A. As the vector e_k of the cannonical basis of $\mathbb{R}^k$ is cyclic for A, this means that $\{A^j(e_k)\ ;\ 0 \le j \le k-1\}$ is a basis of $\mathbb{R}^k$, and V cannot be zero, it follows that $V(e_k) \neq 0$. Take for example $V(e_k) = 1$.

Define the following Lyapunov function:

$$W(x) = W(x_1, \cdots, x_k) = V(x) - \alpha_1 x_1^2 - \cdots - \alpha_{k-1} x_{k-1}^2,$$

the numbers α_i, $1 \le i \le k-1$ to be determined.

Observe that:

$$\begin{aligned} W(F(x)) - W(x) &= V(A(x) + q(x)e_k) - \alpha_1 x_2^2 - \cdots - \alpha_{k-1} x_k^2 - \\ &- V(x) + \alpha_1 x_1^2 + \cdots \alpha_{k-1} x_{k-1}^2. \end{aligned} \tag{9}$$

Now take any $x = (x_1, \cdots, x_k)$ such that $W(x) < 0$; using the linearity of V and the fact that it is an eigenvector of A^* with eigenvalue $\lambda > 1$, then:

$$\begin{aligned} V(A(x) + q(x)e_k) - V(x) &= A^*V(x) + q(x)V(e_k) - V(x) \\ &= (\lambda - 1)V(x) + q(x) \\ &= (\lambda - 1)W(x) + q(x) + (\lambda - 1)\sum_{i=1}^{k-1} \alpha_i x_i^2 \\ &< q(x) + (\lambda - 1)\sum_{i=1}^{k-1} \alpha_i x_i^2 \end{aligned}$$

Therefore, following with equation (9), it comes that:

$y = (y_1, \cdots, y_k) \in \mathbb{R}^k$ there exists a unique x_k such that (4) holds. So ϕ is bijective, with inverse

$$\phi^{-1}(y_1, \cdots, y_k) = (y_1, \cdots, y_{k-1}, \tau(y_1, \cdots, y_k)),$$

where τ is determined by

$$y_k = h_1(y_1, \cdots, y_{k-1}, \tau(y)). \tag{5}$$

The smoothness of the inverse of ϕ follows immediately as the differential of ϕ is never singular. It is clear also that ϕ is C^r close to the identity map of $\mathbb{R}^k$.

Now compute $\phi H \phi^{-1}$ and use equation (5) to conclude that it is a delay endomorphism, obviously C^r close to F.

Denote by H_j any mapping of the form:

$$H_j(x_1, \cdots, x_k) = (x_2, \cdots, x_j, h_1(x), \cdots, h_{k-j+1}(x)),\ x = (x_1, \cdots, x_k).$$

The same reasoning as above shows that if an H_{j-1} is C^r close to an H_j, then

$$\phi_j(x_1, \cdots, x_k) = (x_1, \cdots, x_{j-1}, h_1(x), x_{j+1}, \cdots, x_k)$$

conjugates H_{j-1} with a mapping of the form H_j, that is close to the initial one. In this way the proof of the theorem can be done by induction. □

4. The boundary of B_∞

The point at ∞ is clearly an attracting fixed point for any delay endomorphism F for which the associated function f is C^2-convex (a function f is C^2-convex if it is C^2, and the quadratic form associated to its second derivative is uniformily strictly definite), see [RRV]. It is also known that if F has no fixed points, then $B_\infty = \mathbb{R}^k$, see [RV]. We begin with an extension of this result, proving that for the (not necessarily convex) perturbations of the family F_μ, the point at ∞ is an attractor, and a fixed point point belongs to the boundary of its basin. As above, the strong topology is considered.

Proposition 1. *Let F be a small C^2 perturbation of*

$$F_\mu(x_1, \cdots, x_k) = (x_2, \cdots, x_k, -x_1^2 + \mu x_1)$$

Then ∞ is an attractor for F and the analytic continuation of the fixed point $\mathcal{O}$ belongs to the boundary of the basin of ∞.

Proof. It suffices to consider delay endomorphisms. Let

$$F(x_1, \cdots, x_k) = (x_2, \cdots, x_k, f(x_1, \cdots, x_k)), \tag{6}$$

where f is a C^2 perturbation of f_μ with $f_\mu(x_1, \cdots, x_k) = -x_1^2 + \mu x_1$, $\mu > 1$. We will construct a Lyapunov function $W : \mathbb{R}^k \to \mathbb{R}$ such that every point in the set $\{W < 0\}$ has its positive orbit divergent to ∞.

The function $\varphi : \mathbb{R} \to \mathbb{R}$ defined by $\varphi(t) = f(t, \cdots, t)$ is C^2 close to $t \to -t^2 + \mu t$, so that φ has two fixed points. Without loss of generality, assume that 0 is the smallest one and observe that then $\varphi'(0)$ is close to μ, greater than 1.

The function f can be expressed as:

$$f(x_1, \cdots, x_k) = q(x_1, \cdots, x_k) + L(x_1, \cdots, x_k)$$

where $L : \mathbb{R}^k \to \mathbb{R}$ is the (gradient) differential of f at $\mathcal{O} = (0, \cdots, 0)$, and q satisfies $q(\mathcal{O}) = 0$, $Dq_{\mathcal{O}} = 0$ and $q''(x) = f''(x)$ for every x. It is clear that there exist $a_1, \cdots, a_k$ such that

$$L(x_1, \cdots, x_k) = a_1 x_1 + \cdots + a_k x_k \tag{7}$$

Lemma 1. *Let G be any delay small C^2 perturbation of an F_μ. Then the set ℓ_1 of critical points of G is the graph of a function defined in the hyperplane $\{x_1 = 0\}$. This graph is C^1 close to $\{x_1 = \mu/2\}$.*

Proof. Let $d = \partial_1 g$ be the determinant of DG. It is clear that $\partial_1 d = \partial_{11} g$ and this function is uniformily close to -2, because g is a perturbation of $f_\mu(x_1, \cdots, x_k) = -x_1^2 + \mu x_1$. This implies that the equation $d = 0$ defines implicitly a function $\bar{x}_1$ of the variables $x_2, \cdots, x_k$. Moreover, this function satisfies

$$\partial_j \bar{x}_1 = \frac{\partial_{1j} g}{\partial_{11} g},$$

which uniformily converges to zero as G converges to F_μ. □

Notation: Let P be the image of ℓ_1 under G. Then P is the set of points which have exactly one preimage under G. P separates $\mathbb{R}^k$ into two components. One of them, which closure will be called $\tilde{P}$, is the set of points having exactly two preimages. In the other component no point has preimage. Lemma 1 implies that the intersection of P with a compact set is almost horizontal, if the perturbation is taken sufficiently small.

An important fact about delay endomorphisms is the following:
Remark. Let G be a delay endomorphism in dimension k and let γ be a C^2, $(k-1)$-dimensional submanifold intersecting transversally and separating P. Then $G^{-1}(\gamma)$ is a compact submanifold of $\mathbb{R}^k$ without boundary, of class C^2, and codimension one.

For example, this fact makes it possible for a stable manifold to be homeomorphic to a sphere.

3. Openness of conjugacy classes of delay endomorphisms

Consider the space of C^r endomorphism of $\mathbb{R}^k$ ($r \geq 1$), endowed with the strong C^r topology. Consider the quotient space of equivalence classes under conjugation. The next result says that the projection of the set of C^r delay endomorphisms is open in the quotient space.

Theorem 1. *If F is a C^r delay endomorphism then every small C^r perturbation of F is conjugated to a delay endomorphism C^r close to F.*

Proof. Take any delay endomorphism $F : \mathbb{R}^k \to \mathbb{R}^k$:

$$F(x_1, \cdots, x_k) = (x_2, \cdots, x_k, f(x)),\ x = (x_1, \cdots, x_k).$$

Let $H(x) = (x_2, \cdots, x_{k-1}, h_1(x), h_2(x))$ be C^r close to F. We will show how to conjugate H with a delay endomorphism also C^r close to F.

Consider the following change of coordinates:

$$(y_1, \cdots, y_k) = \phi(x_1, \cdots, x_k) = (x_1, \cdots, x_{k-1}, h_1(x)). \tag{3}$$

We claim that the transformation ϕ is a diffeomorphism C^r close to the identity map of $\mathbb{R}^k$. Clearly $y_i = x_i$ for every $1 \leq i \leq k-1$. What remains to prove is that for any $y = (y_1, \cdots, y_k) \in \mathbb{R}^k$, there exists a unique solution x_k of:

$$y_k = h_1(y_1, \cdots, y_{k-1}, x_k). \tag{4}$$

Fix $y_1, \cdots, y_{k-1}$ and consider h_1 as a function of one real variable (the last one). By the strong proximity of H and F, it follows that the derivative of h_1 with respect to the last variable is close to one. For that h_1 is bijective when considered as a function of the last variable and the first $k-1$ are fixed. Therefore, for any

Corollary 2. *Let μ_∞ be the Feigenbaum parameter for the one dimensional quadratic family. If $\mu < \mu_\infty$, and G is a perturbation of F_μ satisfying the hypothesis of Theorem A, part c, then the restriction of G^k to J' is conjugated to $\pi_\mu^{(k-1)}$. On the other hand, if G is a perturbation of some F_μ with $\mu \in (\mu_\infty, 4)$, then $G^k \mid J'$ has infinitely many recurrences.*

There exist two fundamental ideas in the proof of Theorem A: a global one, the fact that ∞ is an attractor and that there is a fixed point in the boundary of its basin, and a local one, which states the persistence of invariant manifolds (not always smooth nor normally expanding). The combination of these features gives the starting point to the study of the dynamics of the transit through the two stable and different settings explained above.

With the same techniques of these proofs, it can be stated analougous results for perturbations of other one parameter families of delay endomorphisms, which for some value of the parameter preserve a measure equivalent to Lebesgue in a compact set. For example

$$(x, y) \to (y, x^2 + 4\lambda x - \lambda y)$$

preserves a measure equivalent to Lebesgue in a bounded set for $\lambda = 2$. It is possible to extend to its perturbations all the results in this work.

2. Starting point example

For $\mu > 1$, consider the family:

$$F_\mu(x_1, \cdots, x_k) = (x_2, \cdots, x_k, -x_1^2 + \mu x_1), \ \mu > 1.$$

The following facts are easily verified:

1. For every $1 < \mu \leq 4$, the cube
$$Q = Q_\mu^k = [0, \mu]^k$$
is completely invariant, that is, $F_\mu^{-1}(Q) = Q$.
2. For every $\mu \leq 4$, the basin of ∞ is the complementary set of Q.
3. For every $\mu > 4$ the basin of ∞ is dense and its complement is an invariant expanding Cantor set.
4. For every $\mu < 4$, the boundary J of Q is completely invariant but $F(J)$ is a proper subset of J.
5. The fixed point $\mathcal{O} = (0, \cdots, 0)$ belongs to J and the differential of F_μ at $\mathcal{O}$ has characteristic polynomial $p(x) = (-1)^k(x^k - \mu)$ (thus, J cannot be normally expanding, it is not even smooth). However, J is in certain sense normally expanding under F_μ^k: for every point in the F^k-invariant face $J' = \{(x_1, \cdots, x_{k-1}, 0) : 0 < x_j < \mu\} \subset J$, the vertical vector $e_k = (0, \cdots, 0, 1)$ is more expanded by DF^k than any vector tangent to J'.

Now we will consider perturbations of F_μ and describe some simple geometric features. All perturbations in this paper are considered in the Whitney (or strong) topology. For example, a C^2-Whitney (or strong) neighborhood of a function g is determined by continuous functions $\epsilon_i(x) > 0$, $i = 0, 1, 2$ and given by the set of functions h such that

$$||D_x^2 g - D_x^2 h|| \leq \epsilon_2(x), ||D_x g - D_x h|| \leq \epsilon_1(x) \text{ and } |g(x) - h(x)| < \epsilon_0(x) \text{ for every } x.$$

For a delay endomorphism of $\mathbb{R}^k$ with associated function $g : \mathbb{R}^k \to \mathbb{R}$, the set of critical points is clearly $\{x \in \mathbb{R}^k : \partial_1 g(x) = 0\}$.

This work is a first step to explain the dynamics around the bifurcation which makes:

- the open region A collapse, giving rise to an expanding Cantor set,
- the basin of ∞ become dense in $\mathbb{R}^k$, and
- the boundary of the basin of ∞ become a fractal set.

Theorem A. *There exists a C^2 neighborhood $\mathcal{U}$ of the family $\{F_\mu : \mu > 1\}$ in the space of C^2 endomorphisms of $\mathbb{R}^k$ satisfying the following properties:*

(a) *For every $G \in \mathcal{U}$ the point at infinity is attracting and there is a fixed point (in the sequel denoted $\mathcal{O}$) in the boundary of its basin B_∞. The differential of G at $\mathcal{O}$ has a unique positive eigenvalue $\lambda > 1$.*

(b) *If the set of critical points of $G \in \mathcal{U}$ is contained in the basin of ∞, then B_∞ is connected and its complementary set A has uncountably many components.*

(c) *If $\mu < 4$ and G is a perturbation of F_μ, then the boundary of the immediate basin of ∞ is a codimension one submanifold J_G. Moreover, if $\lambda > |\lambda_0|$ for any other eigenvalue λ_0 of the differential of G at $\mathcal{O}$, then J_G is normally expanding, and hence of class C^1.*

We will show in section 3 that any perturbation of a delay endomorphism is conjugated to a delay endomorphism: then it suffices to consider *delay* perturbations of F_μ.

For endomorphisms in $\mathcal{U}$ the set ℓ_1 of critical points is a codimension one submanifold of $\mathbb{R}^k$ of class C^1. Denote by $\mathcal{H}_0$ the set of endomorphisms for which B_∞ is dense and its complementary set A is an expanding Cantor set. As was said above, perturbations of F_μ with $\mu > 4$ belong to $\mathcal{H}_0$.

It is natural to ask whether $G \in \mathcal{U}$ and $B_\infty \supset \ell_1$ imply that $G \in \mathcal{H}_0$. This is certainly true (not immediate) for $k = 1$. Theorem A (part b), gives a partial affirmative answer to the higher dimensional case. In a forthcoming paper, we will show that the condition $\ell_1 \subset B_\infty$ determines an open set of endomorphisms which boundary constitutes a codimension one C^0 submanifold of $\mathcal{U}$.

If the conditions of part c of Theorem A are satisfied, then there exists a C^1 codimension one submanifold J_G equal to the boundary of the immediate basin of ∞, B^0_∞. We want to say something about the restriction of G to J_G.

Corollary 1. *For any G satisfying the hypothesis of part c of Theorem A, there exists an open subset J' of J containing the fixed point $\mathcal{O}$ in its boundary, such that the following properties are satisfied:*

1. *$G^k(J') \subset J'$, and if $\bar{J}'$ denotes the closure of J', then $\bigcup_{j=0}^{k} G^{-j}(\bar{J}') \supset J$.*
2. *The intersection of $G^j(J')$ with J' is empty, for every $1 \leq j \leq k-1$.*
3. *J' is homeomorphic and close to a $(k-1)$-dimensional open cube Q_{k-1} and G^k restricted to any compact subset of J' is C^1 close to the mapping $\pi_\mu^{(k-1)}$ in Q_{k-1}.*

This means that the dynamical properties of the mapping G restricted to J, are determined by its behaviour on J'. More precisely, as the first $k-1$ images of J' are disjoint and $G^k(J') \subset J'$, then the mapping carrying all the information is G^k restricted to the closure of J'. The third part of the corollary says that this mapping is a perturbation of a $(k-1)$-dimensional product of quadratic maps of the interval. Even in dimension one, the C^1 proximity with a product of quadratic maps of the interval does not imply the conjugacy with any map of the family.

However, as an immediate corollary of Theorem A , we have:

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ROMERO, N., ROVELLA, A., AND VILAMAJÓ, F.

SRC. Let $F_\mu(x_1, \cdots, x_k) = (x_2, \cdots, x_k, -x_1^2 + \mu x_1)$. For any G in a C^2 neighborhood $\mathcal{U}$ of the family F_μ, the point at ∞ is an attractor (with basin denoted by B_∞), and there exists a repelling fixed point in the boundary of B_∞. This gives the initial step to the study of the whole boundary of B_∞ and the changes it suffers: for perturbations of F_μ with μ small, the boundary of B_∞ is an invariant codimension one manifold, while for large values of μ the basin B_∞ is dense and its complementary set an expanding Cantor set. The techniques developed will be applied to delay endomrphisms.

1. Introduction

Let $f : \mathbb{R}^k \to \mathbb{R}$ be any continuous function. The delay endomorphism associated to f is the mapping:

$$F(x_1, \cdots, x_k) = (x_2, \cdots, x_k, f(x_1, \cdots, x_k)) \tag{1}$$

In this paper we will study perturbations of the one parameter family of delay endomorphisms F_μ associated to the function

$$f_\mu(x_1, \cdots, x_k) = -x_1^2 + \mu x_1 \tag{2}$$

Let $\pi_\mu^{(1)}(x) = -x^2 + \mu x$ be the quadratic family in $\mathbb{R}$. By $\pi_\mu^{(k)}$ we will denote the mapping of $\mathbb{R}^k$ given as the product of k times $\pi_\mu^{(1)}$. The delay endomorphism associated to a function as in equation (2), is a k^{th} root of the mapping $\pi_\mu^{(k)}$. In this sense, this work generalizes some of the dynamical properties of the perturbations of the quadratic family to higher dimensions.

It is well known the importance of the one dimensional quadratic family in the study of homoclinic bifurcations for two dimensional diffeomorphisms, see for example [N] and [PT]. On the other hand, there exists a result of Mora, see [Mo], which states that the transformations $\pi_\mu^{(k)}$ appear as limit of renormalizations associated to critical homoclinic orbits of a repelling fixed point.

We will show that any perturbation G of an F_μ can be extended with continuity to the compactification $\mathbb{R}^k \cup \{\infty\}$ of $\mathbb{R}^k$, and that the point ∞ is an attracting fixed point of G. The basin of ∞ will always be denoted by $B_\infty = B_\infty(G)$, and the complementary set of B_∞ by $A = A(G)$.

It is clear (see section 2) that for small values of μ (precisely $\mu < 4$), $A(F_\mu)$ contains an open region, and that for any μ large ($\mu > 4$), $B_\infty(F_\mu)$ is dense and the restriction of F_μ to $A(F_\mu)$ is an expanding map. These conditions hold for any small perturbation of the family. Note that if $\mu > 4$, then well known results about the dynamics of endomorphisms imply that F_μ is C^2 structurally stable and then any perturbation has the same structure. As a reference for these statements see for example the works of Przytycki [P] and of Mañé and Pugh [MP].

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