Perturbations of the quadratic family of order two

N Romero†, A Rovella‡and F Vilamajó¶

†Universidad Centro Occidental Lisandro Alvarado. Departamento de Matemática. Decanato de Ciencias y Tecnología. Apartado Postal 400. Barquisimeto, Venezuela. e-mail:nromero@uicm.ucla.edu.ve

‡Universidad de La República. Facultad de Ciencias. Centro de Matemática. Iguá 4225. Montevideo 11400, Uruguay. e-mail:leva@cmat.edu.uy

¶Universidad Politécnica de Cataluña. Departament de Matemática Aplicada 2. Escola Técnica Superior d'Enginyers Industrials. Colom 11, 08222. Terrasa, Barcelona, España. e-mail:vilamajor@ma2.upc.es

Abstract. Define the quadratic family of order two as $F_\mu(x,y) = (y, -x^2 + \mu x)$ where μ is a real parameter. The boundary of the basin of attraction of the fixed point at ∞ is an invariant curve for $\mu < 4$, and is a Cantor set for $\mu > 4$. Perturbations of F_μ with $\mu \neq 4$ were studied in [4] (also in higher dimension) where it was proved that these situations persist. Now we study perturbations of the bifurcation point $\mu = 4$, where the explosion of the basin, B_∞, occurs. We prove that either there exists a connected invariant curve J contained in the boundary of the basin, or the set of critical points is a subset of B_∞ and the boundary has uncountably many components accumulated by the preimages of the analytic continuation of the fixed point at the origin. The curve J undergoes a fractalization process until it ceases to exist.

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1. Introduction

To any delay equation of order $k \geq 1$

$$x_{n+k} = f(x_n, \cdots, x_{n+k-1}),\ n \geq 0,$$

where $f : \mathbb{R}^k \to \mathbb{R}$ is any function, it is defined a delay endomorphism of $\mathbb{R}^k$ by the mapping

$$F(x_1, \cdots, x_k) = (x_2, \cdots, x_k, f(x_1, \cdots, x_k)).$$

The quadratic family of order $k \geq 1$ is the family $\{F_\mu\}_{\mu \in \mathbb{R}}$ of delay endomorphisms associated with the equation

$$x_{n+k} = -x_n^2 + \mu x_n.$$

Observe that the k-th power of F_μ is the product of k times the one dimensional quadratic family. Therefore, a huge number of qualitative dynamical properties of the one dimensional quadratic family can be extended to F_μ^k. Meanwhile, the perturbations of the family F_μ can exhibit very complicated dynamical behaviour; the authors do not have references in the literature about this topic. However, perturbations of the quadratic family of order k take a fundamental role, for example, in the theory of higher codimension homoclinic bifurcations: by means of a rescaling argument, those endomorphisms appear as a limit of sequences of first return Poincaré maps near homoclinic points; see L. Mora [1] for details.

Any smooth and small perturbation of an F_μ has a fixed point near the origin, that fixed point will be supposed to be the origin 0. A simple stability argument (see proposition 1 in section 3) implies that for every small perturbation G of any F_μ, the point at infinity is an attractor and the fixed point 0 belongs to the boundary of the basin of attraction of ∞. Moreover, there it was proved that when G is close to an F_μ with $\mu < 4$, there exists an invariant set J, homeomorphic to the $(k-1)$-dimensional sphere S^{k-1}, which constitutes the boundary of the immediate basin of ∞. It is not difficult to see that F_μ is hyperbolic for $\mu > 4$; the basin of ∞ is, in this case, dense in $\mathbb{R}^k$ and its complementary set is an expanding Cantor set. This is a C^2 structurally stable property and hence it extends to perturbations of F_μ; see R. Mañé and C. Pugh [2] and F. Przytycki [3].

Our present purpose is to investigate the dynamics of perturbations of F_4 in the plane. We will first show that for every perturbation G of F_4, there exists a curve $J_0(G)$, homeomorphic to the circle, such that the unbounded component of its complementary set, is forward invariant and contained in $B_\infty(G)$. Using $J_0(G)$ and its preimages we will reproduce the boundary of $B_\infty(G)$.

Denote by $A(G)$ the complementary set of $B_\infty(G)$, and let d be the Hausdorff metric in the space $\mathcal{K}$ of nonempty compact subsets of the plane. Consider the C^2 strong topology on the space of endomorphisms of $\mathbb{R}^2$ and define the operator $G \to A(G)$ from a neighborhood $\mathcal{U}$ of F_4 to $\mathcal{K}$. It is well known and easy to prove that this operator is continuous in a residual subset of $\mathcal{U}$ (see proposition 2 in section 3).

Theorem A *There exists $\mathcal{U}$ such that for every $G \in \mathcal{U}$, point of continuity of $G \to A(G)$, it holds that*

$$G^{-n}(J_0(G)) \to \partial B_\infty(G)$$

in the metric space $(\mathcal{K}, d)$.

In order to describe the dynamics over the boundary of $B_\infty(G)$ for every perturbation G of F_4, we will give a simple algorithm defining a sequence of curves J_n, each one contained in $G^{-n}(J_0)$. This sequence of curves provides a criterion to determine whether the critical points of G are contained in $B_\infty(G)$.

Theorem B *Let G in $\mathcal{U}$. There exists $N \geq 0$ such that J_N is contained in the interior of the image of G if and only if the set of critical point of G is contained in $B_\infty(G)$.*

Motivated by the results for complex polynomials in the Riemann sphere we looked for global features that would explain the transit between the situations described above: the passage from a connected to a totally disconnected boundary. Either the hypothesis of theorem B is true or every J_n intersects the boundary of the image of G. It becomes necessary to analyze the sequence J_n in both cases.

Theorem C *(a) If every J_n intersects the boundary of the image of G, then the limit of the sequence $\{J_n\}$ exists in $\mathcal{K}$, and gives an invariant connected set J contained in $\partial B_\infty(G)$ (the boundary of $B_\infty(G)$) which contains critical points.*
(b) If some J_N is contained in the interior of the image of G, the $\partial B_\infty(G)$ has uncountably many components and is accumulated by preimages of the fixed point 0.

In dimension $k > 1$, it remains as an open problem to determine if every perturbation G of F_4 for which the critical points are contained in $B_\infty(G)$ is hyperbolic; for $k = 1$ we will give a proof of this in the next section.

In the sequel $\widetilde{H}_0$ will denote the set of endomorphisms G such that the set of critical points is contained in $B_\infty(G)$.

This paper is organized as follows. In section 2 we will construct the curve J mentioned above; its relationship with the basin of ∞ will be analyzed in section 3, arriving to the proofs of the theorems A, B and C at the end of the section. In the last section we will analyze a particular example and show some remarkable computer figures.

2. Construction of almost invariant curves

The purpose of this section is to prove, for every strong C^2 small perturbation G of F_4, the existence of a curve $J_0(G)$ homeomorphic to the circle, for which the unbounded component of its complementary set is contained in the basin of ∞. The study of the relationship of this curve with the boundary of $B_\infty(G)$ will be made in the next section. The curve $J_0(G)$ will be constructed at the end of this section. The first step towards the definition of $J_0(G)$ is to show the existence of a curve γ connecting the fixed point 0 with its other preimage. The curve γ is either G^2 forward invariant or the set of points in γ whose G^2-positive orbit is contained in γ is a Cantor set K_0. Moreover, in the latter case, $\gamma \setminus K_0$ is contained in B_∞. It is in this sense that we call γ an almost invariant curve.

Given $\epsilon > 0$, denote by $\mathcal{U}_\epsilon$ a strong C^2-neighborhood of F_4 such that for some compact set V containing $Q = [0, 4]^2$, for every $G \in \mathcal{U}_\epsilon$ and every $x \in V$:

$$\max\{\|G(x) - F_4(x)\|, \|DG(x) - DF_4(x)\|, \|D^2G(x) - D^2F_4(x)\|\} \leq \epsilon.$$

Recall that a strong (or Whitney) C^2-neighborhood of a C^2 map $F : \mathbb{R}^2 \to \mathbb{R}^2$ is given by continuous and positive functions $\epsilon_i : \mathbb{R}^2 \to \mathbb{R}$ $(i = 0, 1, 2)$ and is defined by:

$$\begin{aligned}\mathcal{V}(F, \epsilon_0, \epsilon_1, \epsilon_2) = \{g \in C^2(\mathbb{R}^2, \mathbb{R}^2) : & \|G(x) - F(x)\| < \epsilon_0(x), \\ & \|DG(x) - DF(x)\| < \epsilon_1(x) \text{ and} \\ & \|D^2G(x) - D^2F(x)\| < \epsilon_2(x) \text{ for every } x \in \mathbb{R}^2\}.\end{aligned}$$

Take $\epsilon > 0$ small enough. If $G \in \mathcal{U}_\epsilon$, then $DG(0)$ has eigenvalues $\lambda^+ > 0 > \lambda^-$ having modulus close to 2. It is clear that $\lambda^+ + \lambda^- = 0$ constitutes a codimension one submanifold which separates $\mathcal{U}_\epsilon$ into two components $\mathcal{U}_\epsilon^\pm$, depending on the sign of $\lambda^+ + \lambda^-$. Let $v^\pm = (1, t^\pm)$ be eigenvectors associated with the eigenvalues $\lambda^\pm$. Observe that $t^\pm = \lambda^\pm$ if G is a delay endomorphism, and then $|t^\pm - \lambda^\pm|$ is close to 0. Also let λ_0 be the eigenvalue of $DG(0)$ with smallest modulus, and $(1, t_0)$ an eigenvector associated.

Lemma 1 *Given ρ sufficiently small there are $\epsilon > 0$ and $\delta > 0$ such that, for any $G \in \mathcal{U}_\epsilon^\pm$, any point $(x, y) \in B_\rho = \{x : \|x\| < \rho\}$ with $x > 0$, $|y| < x$ and every vector $w = (w_1, w_2)$ satisfying $|w_2/w_1| < \delta$ it holds that $|\frac{w_2^{(n)}}{w_1^{(n)}}| < 2|t_0|$, where $D(G/B_\rho)^{-2n}(x, y)(w) = (w_1^{(n)}, w_2^{(n)}) = w^{(n)}$.*

Proof: Let g be the inverse of the restriction of G^2 to a small neighborhood of 0. Let A be the differential of g at 0. It is well known that g and A are C^1 conjugated in a small neighborhood of 0. Moreover, this conjugation is C^1 close to the identity map. Let Φ be a C^1 conjugation between g and A defined in B_ρ for ρ small. Then $Dg^n(x, y) = D\Phi^{-1} A^n D\Phi(x, y)$. For a vector $v = (v_1, v_2)$, define the slope of v as the quotient $|v_2/v_1|$. The hypothesis on (x, y) and w and the fact that Φ is close to the identity map, imply that the vector $D\Phi(x, y)(w)$ has a small slope and that the point $\Phi(x, y)$ is contained in the sector $Z = \{(x, y) : t^- x < y < t^+ x\}$. As Z is invariant under A^n, the slope of $w^{(n)}$ converges to $|t_0|$ for any vector tangent to $\mathbb{R}^2$ at a point of Z. □

In other words, the proof says nothing but the fact that the mapping $slope(w) \to slope(w^{(1)})$ is contrancting, having $|t_0|$ as an attractor. It follows that for every G there exists a τ_G such that for every $(x, y) \in Z$ and every vector w with slope less than τ_G, it holds that the slope of $w^{(1)}$ remains bounded by τ_G.

Next we analyze some simple geometrical features of the mappings G.

Let $Q(*, 0) = \{(x, y) : 0 \leq x \leq 4, y = 0\}$ and denote by $Q(0, *)$, $Q(4, *)$, and $Q(*, 4)$ the other sides of the square $Q = [0, 4]^2$. Let U be a small neighborhood of $Q(*, 0)$ and $\epsilon > 0$ such that for every $G \in \mathcal{U}_\epsilon$, the connected component of $G^{-2}(U)$ containing 0 is contained in U.

Consider the set Λ_{G^2} of points in U which never leave U under iterations of G^2. The following result is the technical part of the proof that this set is contained in the boundary of the immediate basin of ∞ and is contained in a smooth curve.

Let $G^2 = (g_1, g_2)$ such that G belongs to $\mathcal{U}_\epsilon^\pm$. Given any $\rho > 0$, there exists $\tau < 1$ such that the set U and ϵ can be made small so that the following holds:
if $(x, y) \in U$ with $\rho \leq x \leq 4 - \rho$, then:

$$\left| \frac{\partial_1 g_1(x, y)}{\partial_2 g_2(x, y)} \right| \leq \tau < 1. \tag{1}$$

Let S be the other preimage of 0 under G.

Lemma 2 *Given $G \in \mathcal{U}_\epsilon^\pm$, let γ_0 be a C^1 curve joining 0 with S and given by $\{(t, \Gamma_0(t)) : 0 \le t \le s\}$. There exists $\delta > 0$ such that if γ_0 satisfies:*
(i) γ_0 is contained in U,
(ii) $|\Gamma_0'(t)| < \delta$ for every $t \in [\rho, 4-\rho]$ and
(iii) $|\Gamma_0'(t)| < \tau_G$ for every t,
then $\gamma_1 = (G/U)^{-2}(\gamma_0)$ is the graph of a C^1 function Γ_1 defined in the horizontal axis and satisfying (i), (ii) and (iii).

Note that γ_1 contains 0 and S, but may be not connected. This will be discussed in detail below.

Proof: Property (i) is obviously true for γ_1 since U is $(G/U)^{-2}$-invariant.

If $G^2(x_1, y_1) = (t, \Gamma_0(t))$, then the vector (u, v) tangent to γ_1 at the point (x_1, y_1) satisfies $DG^2(x_1, y_1)(u, v) = (1, \Gamma_0'(t))$, and so:

$$\left|\frac{v}{u}\right| = \left|\frac{\partial_1 g_1.\Gamma_0' - \partial_1 g_2}{\partial_2 g_2 - \partial_2 g_1.\Gamma_0'}\right|. \tag{2}$$

The derivatives of g_i, $i = 1, 2$, are calculated at the point (x_1, y_1) and those of Γ_0 at the point t. We will estimate the quotient in equation 2 in three different cases.
First case: Suppose that x_1 and t belong to $[\rho, 4-\rho]$.
Then $|\Gamma_0'(t)| < \delta$, and hence, using equation 1:

$$\left|\frac{v}{u}\right| \le \frac{|\partial_1 g_1.\Gamma_0'|}{\partial_2 g_2 - \epsilon\delta} + \epsilon < \tau|\Gamma_0'| + \epsilon,$$

if δ is small. Moreover, ϵ can be taken small enough so that this last term is smaller than δ.
Second case: Suppose that $t > 4 - \rho$.
Then $|x_1 - 2| < c_{\rho,\epsilon}$ with $c = c_{\rho,\epsilon} \to 0$ when ρ and ϵ go to 0. This implies $|\partial_1 g_1| \le 3c_{\rho,\epsilon}$. Using equation 2 and the definition of Γ_0, it comes that:

$$\left|\frac{v}{u}\right| \le \frac{3c\tau_G + \epsilon}{|\partial_2 g_2| - |\epsilon\tau_G|} < \delta,$$

if c is small with respect to δ, which can be done by diminishing ϵ again.
Third case: It remains to consider either $x_1 \in (0, \rho)$ or $x_1 > 4 - \rho$.
If U and δ are small, then we have the hypothesis of lemma 1. The point (x_1, y_1) remains in the sector Z, and the application of that lemma implies that γ_1 satisfies property (iii). As the fixed point 0 is repeling, there are no more cases to consider. This proves the lemma. □

Observe that for any endomorphism G C^2-close to F_4, the critical points of G are located in an almost vertical curve (close to $x = 2$); this set will be denoted by $\ell_1 = \ell_1(G)$. The image $P = P(G)$ of this curve is close (in compact subsets of the plane) to the line $y = 4$ and constitutes the boundary of the image $Im(G)$ of the plane under the transformation G. P separates the plane into two components, one of them, located below P, is the set of points having exactly two preimages.

Given a bounded set D denote by $ext(D)$ the unbounded component of the complementary set of D, and by $int(D)$ the union of the bounded components of the complement of D. For every set A, $cl(A)$ will denote the closure of A.

Simple considerations as the given above imply the following result; that will be used mainly in the next section.

Lemma 3 *Let G be a C^2 small perturbation of some F_μ and γ a C^1 curve intersecting P transversally. Then:*
(a) If γ' is a connected component of $\gamma \cap Im(G)$ with two points in P, then $G^{-1}(\gamma')$ is a C^1 simple closed curve.
(b) Moreover, if $\gamma \cap Im(G)$ is connected and γ is homeomorphic to S^1, then $ext(G^{-1}(\gamma)) = G^{-1}(ext(\gamma))$.
(c) If γ is homeomorphic to S^1 and $\gamma_1 \cap Im(G) \subset int(\gamma)$, then $G^{-1}(\gamma_1) \subset int(G^{-1}(\gamma))$.

Recall that we are supposing that the perturbation G of F_4 has the origin 0 as fixed point; we have denoted its first preimage as S, which is close to $(4,0)$; denote the preimages of S by S_1 (close to $(4,4)$) and S_2 (close to $(0,4)$). Observe that if ρ is small, $G^{-1}(B_\rho)$ has a component, V_ρ, which does not intersects B_ρ. The preimage of V_ρ has two connected components, V_ρ^1 and V_ρ^2, neighborhoods of S_1 and S_2 respectively. If G is close to F_4, then the set P dissects both V_ρ^1 and V_ρ^2. Denote by P_1 (resp. P_2) the intersection of P with V_ρ^1 (resp. V_ρ^2). Obviously $G(P_1)$ and $G(P_2)$ intersect V_ρ, and by the proximity with F_4, they are almost vertical.

One can distinguish two possibilities:
(1) S is located at the left of $G(P_2)$ (this will be referred as $G \in \mathcal{G}_l$),
(2) S is located over, or at the right, of $G(P_2)$ (denoted $G \in \mathcal{G}_r$).

In case (2), the preimage $\gamma_1 = (G/U)^{-2}(\gamma_0)$ of lemma 2 is a curve connecting 0 with S. In case (1), $(G/U)^{-2}(S)$ consists of two points, the preimage of γ_0 has two components, one of them joining 0 with a preimage of S under G^2 and the other joining S with the other preimage. (Note also that this depends on the position of S with respect to $G(P_2)$, not being related with its position with respect to $G(P_1)$).

Lemma 4 *Given any $\rho > 0$ sufficiently small, there are a neighborhood U of $Q(*,0)$ and $\epsilon > 0$ such that:*
(a) for every $G \in \mathcal{G}_r \cap \mathcal{U}_\epsilon^\pm$, there exists a curve γ of class C^1, contained in U, joining 0 with S and such that $(G/U)^{-2}(\gamma) = \gamma$, and
(b) for every $G \in \mathcal{U}_\epsilon^\pm \cap \mathcal{G}_l$, there exists a curve γ of class C^1, contained in U and joining 0 with $G(P_2)$, such that $(G/U)^2(\gamma) \supset \gamma$ and $(G/U)^{-2}(\gamma) \subset \gamma$.

Proof: (a) Take $U, \epsilon > 0$ and $\rho > 0$ sufficiently small. Let $C(G)$ be the space of all C^1 functions $\Gamma_0 : [0,s] \to \mathbb{R}$ (s is the first coordinate of S) satisfying the conditions (i) to (iii) of the lemma 2. From the same lemma it follows clearly that if $\Theta_G(\Gamma_0) = \Gamma_1$ is the function obtained from $(G/U)^{-2}(\gamma_0)$ where $\gamma_0 = graph(\Gamma_0)$, then Θ_G defines an operator on $C(G)$. Since $G^2 = (g_1, g_2)$, then for every $t \in [0,s]$ and any $\Gamma_0, \widetilde{\Gamma}_0 \in C(G)$ it holds that

$$\Gamma_0(t) - \widetilde{\Gamma}_0(t) = \partial_2 g_2(t, \theta(t)) \left(\Gamma_1(t) - \widetilde{\Gamma}_1(t) \right),$$

for some $\theta(t)$ that belongs to the open segment joining $\Gamma_1(t)$ and $\widetilde{\Gamma}_1(t)$. Now, as $G \in \mathcal{U}_\epsilon^\pm$, the above identity implies that Θ_G is a contraction when the C^1 topology is considered. So the part (a) of the lemma follows.
(b) In order to obtain that the preimage of γ_0 is a curve also in case (1), we will add to γ_0 a segment T connecting S with $G(P_2)$.

Let T be a segment of line joining S with $G(P_2)$, tangent at S to the vector $DG(S)^{-1}(1, t_0)$ (see just before lemma 1 for the definition of t_0). Note that ϵ can be made smaller so that $G^{-2}(T)$ is almost horizontal (see the estimatives of case 2 in lemma 2).

Let γ_0 be a curve joining 0 with S satisfying (i) to (iii) of lemma 2; add T to γ_0. The G^2 preimage of this union is a curve γ_1 joining 0 with S and satisfying 1, 2 and 3 of the referred lemma. So, we have defined an operator Θ_G on $C(G)$ as in the above lemma. In similar form as in that lemma, the operator Θ_G is a contraction. The curve γ will be the graph of the fixed point of this operator. Note also that $\gamma \cup T$ is C^1 and satisfies $(G/U)^{-2}(\gamma \cup T) = \gamma$. This proves part (b) and the lemma. □

Let f_μ be the one dimensional quadratic family, $f_\mu(x) = -x^2 + \mu x$.

Corollary 1 *Let $G \in \mathcal{U}_\epsilon^\pm$ with ϵ small enough and let γ be the curve obtained in lemma 4. If $G^2 = (g_1, g_2)$, then $g_2 \circ \gamma$ converges to f_4 in the C^1 topology when G converges to F_4 in the C^2 topology.*

Proof: Given any $\epsilon_0 > 0$, one can choose $\rho > 0$ such that $g_2 \circ \gamma$ is ϵ_0-C^1 close to multiplication by t_0^2 in the interval $[0, \rho]$, and f_4 is ϵ_0-C^1 close to multiplication by 4 in the same interval; as $|t_0^2| \to 4$ when $\epsilon \to 0$, the assertion follows in this interval, and also in its preimage, close to S. Note also that the number δ of lemma 2 can be chosen as small as we wish by taking ρ and ϵ small. Thus γ is C^1 close to $Q(*, 0)$ in $[\rho, 4-\rho]$, and then G^2 restricted to the curve γ is C^1 close to F_4^2 restricted to $Q(*, 0)$, which is exactly f_4. □

Now we want to establish the relation between this curve γ and the set of points never leaving a neighborhood U of $Q(*, 0)$ under iterations of G^2. Clearly this set is given by $\Lambda_{G^2}(U) = \bigcap_{n \geq 0} G^{-2n}(U)$.

For U, a small neighborhood of $Q(*, 0)$, this set can be either a Cantor set contained in γ or the whole curve. To analyze these possibilities we first need a result stating that in dimension one any mapping in $\widetilde{\mathcal{H}}_0 \cap \mathcal{U}_\epsilon$ is hyperbolic. We include a proof of the following result because we have no reference for it, but is surely well known by one-dimensional dynamicists.

Theorem 1 *Let g be a C^1 perturbation of f_4 such that g has only one critical point c which satisfies:*
1. c is a quadratic critical point, i.e. $g''(c)$ exists and is not zero, and
2. the image of c does not belong to the interval I which extreme points are 0 and its preimage.
Then the set of points never leaving I is hyperbolic.

As an immediate consequence of theorem 1, one has:
If g is a C^2 perturbation of f_4 such that the critical point of g has unbounded orbit, the g is hyperbolic.

Proof: We apply the simple fact that the mapping f_4 is C^1 conjugated to the tent map via a function φ having derivative zero only at the points 0 and 4.
Claim 1: *Given $\delta_0 > 0$ there exist $\epsilon > 0$ and $m > 0$ such that, if g is C^1 close to f_4 and $|g^j(x) - c| \geq \delta_0$ for every $0 \leq j \leq m$ then*

$$|(g^m)'(x)| \geq (1.9)^m.$$

Observe that $h = \varphi^{-1} g \varphi$ is close to the tent map, from which one can conclude that it has derivative greater than 1.99 outside a neighborhood of the point c. Then

$$|(g^m)'(x)| \geq |\varphi'(h^m(x))|(1.99)^m |(\varphi^{-1})'(x)|.$$

The derivative of φ^{-1} is bounded away from zero. Observe also that if ϵ is small then $g^j(x)$ does not reach a neighborhood of 0 unless the whole orbit is very close to 0.

Therefore, if this does not happen, then $|\varphi'(h^m(x))| \geq d(\delta_0)$, where $d(\delta_0)$ is a function of δ_0 that goes to 0 when $\delta_0 \to 0$. It follows in this case that if m is large, then

$$|(g^m)'(x)| \geq Kd(\delta_0)(1.99)^m \geq (1.9)^m,$$

for some positive constant K. It remains to consider the case where the whole piece of orbit is contained in a neighborhood of 0. At this point, there is no need of passing to the conjugation; the conclusion is obvious.

Claim 2: *Given $\delta_0 > 0$ there exists $\epsilon > 0$ such that for any $g \in \mathcal{U}_\epsilon$, a positive integer n and a point x satisfying $|g^j(x) - c| \geq \delta_0$ for every $0 \leq j \leq n-1$ but $|g^n(x) - c| < \delta_0$, it follows that $|(g^n)'(x)| \geq (1.9)^n$.*

The proof of this claim is the same as above until the moment of considering the derivative of φ at the point $h^n(x)$. Now we know that this point is far from 0 because of the hypothesis on $g^n(x)$.

Claim 3: *There exists $\delta_0 > 0$ such that for any given x with positive orbit contained in I and $|x - c| < \delta_0$ there exists $l = l(x) > 1$ satisfying $|g^j(x) - c| \geq \delta_0$ for every $1 \leq j \leq l$ and $|(g^l)'(x)| \geq (1.9)^l$.*

Any point x as above has its second image close to 0. By the hypothesis of the theorem, we know that the critical point is quadratic, this implies that $\dfrac{|g'(x)|}{|x-c|}$ and $\dfrac{g(x) - g(c)}{(x-c)^2}$ are uniformily bounded for x in a neighborhood of c. Observe that $g^2(x)$ is close to 0, and that $\dfrac{g^2(x)}{(x-c)^2}$ is bounded. Let $l(x) > 2$ be the first integer such that $g^{l-2}(g^2(x)) > 1$. It follows that $|(g^{l-2})'(g^2(x))|$ has the order of $\left(g^2(x)\right)^{-1}$. Therefore $|(g^l)'(x)|$ has the order of $|x-c|^{-1}$, and this implies the claim.

Now we will conclude that there exist constants $C > 0$ and $\lambda > 1$ such that if $n > 0$, then $|(g^n)'(x)| \geq C\lambda^n$ for every x whose positive orbit is contained in I. Indeed, once a function g is fixed, there exists a number $\tau > 0$ such that $|g'(x)| \geq \tau$ for every x having its positive orbit contained in I. For the points that never enter the interval $(c - \delta_0, c + \delta_0)$, claim 1 gives the result, with $C = 1$ and $\lambda = 1.9$. For the other points we will obviously have, by using claims 2 and 3, that $|(g^n)'(x)| \geq \tau(1.9)^{n-1}$. This finishes the proof of the theorem 1. □

Remark 1 *The proof also works with minor modification in claim 3 if the condition 1 on the hypothesis is substituted by:*
1'. the critical point of g is non-flat.
One can also ask the critical point of g to be $C^{1+\epsilon}$, being equivalent to $x \to x^{1+\epsilon}$.

Corollary 2 *If U is a small neighborhood of $Q(*, 0)$, then there exists $\epsilon > 0$ such that for any $G \in \mathcal{G}_l \cap \mathcal{U}_\epsilon^\pm$, the set $\Lambda_{G^2}(U)$ is a Cantor set.*

Proof: Note first that $\Lambda_{G^2}(U)$ is contained in γ. Next, the facts that γ is nearly horizontal and the line of critical points of G, ℓ_1, nearly vertical, imply that they are transverse, from which it follows that the critical point c of G^2 in γ is quadratic (and unique). Finally, observe that for $G \in \mathcal{G}_l$, $G^2(c)$ leaves the interval whose extreme points are 0 and the first coordinate of S. Thus the sharper version of the theorem above implies the result. □

In lemma 4 we have constructed a curve γ joining 0 with its first preimage S. In both cases the curve γ can be enlarged (if necessary) to a curve reaching $G(P_1)$ and

$G(P_2)$, the piece added being a segment of line denoted by T. Call γ^0 the C^1 curve obtained. Then define:

$$J_0 = G^{-2}(\gamma^0) \cup \big(G^{-1}(\gamma^0) \setminus G^{-1}(T)\big)$$

Observe that the curve J_0 is homemorphic to the circle: indeed, $G^{-1}(\gamma^0)$ has two connected components, say $\tilde{\gamma}^1$ and $\tilde{\gamma}^2$: $\tilde{\gamma}^1$ joins S with P_1 and $\tilde{\gamma}^2$ connects 0 with P_2. $\tilde{\gamma}^2$ has a connected preimage, it is precisely $\gamma^0 \setminus T = \gamma$; the preimage of $\tilde{\gamma}^1$ is a curve connecting S_1 to S_2. Therefore, taking off $G^{-1}(T)$ from $G^{-1}(\gamma^0) \cup G^{-2}(\gamma^0)$ one obtains a simple closed curve. In these terms, the curve J_0 is given by $J_0 = \tilde{\gamma}^1 \cup \tilde{\gamma}^2 \cup G^{-1}(\tilde{\gamma}^1 \cup \tilde{\gamma}^2) \setminus G^{-1}(T)$, see Figure 1.

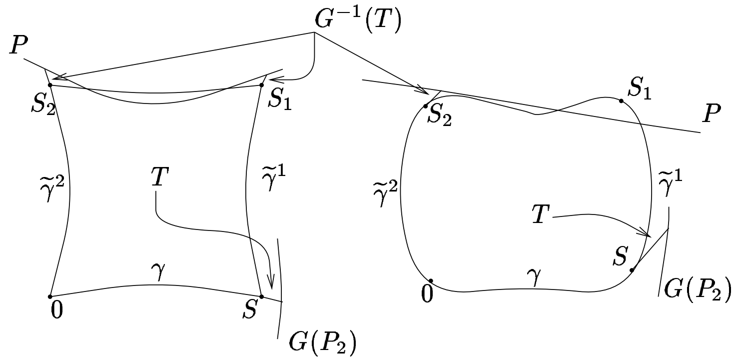

Figure 1. Two examples of the curve J_0

Observe that by construction the curves γ^0 and $G^{-1}(\tilde{\gamma}^1)$ are graphs of functions of the first coordinate, while $\tilde{\gamma}^1$ and $\tilde{\gamma}^2$ are graphs of functions of the second coordinate.

Recall that we have considered only the cases where the eigenvalues at the origin satisfy $\lambda^+ + \lambda^- \neq 0$. If λ^+ is greater than $-\lambda^-$ (that is, $G \in \mathcal{U}_\epsilon^+$), then the collapse of the curves γ and γ^2 occurring at 0 is smooth, while in the contrary case, when $G \in \mathcal{U}_\epsilon^-$, a cusp is created at the origin. The same holds at the other points $(S, S_1$ and $S_2)$ of intersection of the curves with its preimages.

Summing up, $J_0(G)$ is of class C^1 for every $G \in \mathcal{U}_\epsilon^+$ and it is C^1 except at 0 and its first two preimages, where it has cusps, when $G \in \mathcal{U}_\epsilon^-$.

3. The Boundary of B_∞

In this section we will analyze the boundary of B_∞ and its relationship with J_0 and its preimages. Beginning with J_0 itself, we have the following:

Proposition 1 *For any $G \in \mathcal{U}_\epsilon^\pm$, the curve $J_0 = J_0(G)$ satisfies:*
(a) $G(ext(J_0)) \subset ext(J_0)$.
(b) $G^{-1}(J_0) \subset cl(int(J_0))$
(c) $ext(J_0) \subset B_\infty(G)$.

Proof: The assertion of part (a) follows easily from the construction of J_0 and the proximity of G with F_4, and (b) is a trivial consequence of (a). For the proof of (c) observe first that the fact that 0 belongs to the boundary of $B_\infty(G)$ is easy to prove: there exists a neighborhood V of 0 such that G/V (the restriction of G to V)

has uniquely defined inverse and $(G/V)^{-n}(x) \to 0$ whenever $x \in V$ and $n \to +\infty$. Moreover, if $K \subset V \cap B_\infty(F_4)$ is a compact set, then $K \subset B_\infty(G)$ for every G close to F_4. These facts together imply that 0 belongs to the boundary of $B_\infty(G)$.

We will use a stronger version proved in [4] and which inplies the following assertion:

Claim: *For every $G \in \mathcal{G}_l \cap \mathcal{U}_\epsilon^\pm$ the segment T employed to construct J_0 is contained in $B_\infty(G)$.*

Recall that $v^\pm = (1, t^\pm)$ denotes eigenvectors associated to the eigenvalues $\lambda^\pm$ of $DG(0)$. Following the steps of the proof of proposition 1 in [4], one can verify that there exists an $\alpha > 0$ such that the function $W(x,y) = y - t^- x - \alpha x^2$ satisfies:

$$W(G(x,y)) - W(x,y) < 0 \text{ for every } (x,y) \text{ such that } W(x,y) < 0.$$

The following properties are easy consequences of this fact:

(i) $\{W < 0\}$ is G-invariant.

(ii) $\{W < 0\}$ is contained in $B^0_\infty(G)$ (the immediate basin of ∞).

(iii) $G^{-1}(\{W = 0\})$ is homeomorphic to S^1 and $ext(G^{-1}(\{W = 0\})) = G^{-1}(\{W < 0\})$ is contained in $B^0_\infty(G)$.

(iv) $G^{-1}(\{W = 0\})$ contains a curve joining 0 with S. This curve is the graph of a function of the first coordinate and G is C^2 close to F_4 it comes that it has positive derivative at S.

From these observations one can conclude that $T \subset B_\infty(G)$; indeed, it follows that T is tangent to $G^{-1}(\{W = 0\})$ at S, but T is a segment, while $G^{-1}(\{W = 0\})$ has positive concavity at S. Then $T \subset ext(G^{-1}(\{W = 0\})) \subset B_\infty(G)$. This proves the claim.

Let W^u be a G-invariant curve tangent to $(-1, -t^+)$ at 0 and beginning at 0 (W^u is a separatrix of the strong unstable manifold of 0 if $\lambda^+ + \lambda^- > 0$, or is a separatrix of any center stable manifold if $\lambda^+ + \lambda^- < 0$). Let W_1^u be the connected preimage of W^u that contains S, and consider the unbounded set E such that $E \cap int(J_0) = \emptyset$ and its boundary is the union of W^u, W_1^u and γ. To prove (c) it is necessary to consider the cases when $G \in \mathcal{G}_r$ and $G \in \mathcal{G}_l$. Suppose $G \in \mathcal{G}_r$. In this case the set E is G^2-forward invariant. Let U be a small neighborhood of $Q(*, 0)$ with $\gamma \subset U$, by a continuity argument, if ϵ is small enough, then every point $x \in E \setminus U$ belongs to $B_\infty(G)$. Each point x in U for which the G^2 positive orbit is contained in U must belong to γ; hence $E \subset B_\infty(G)$. Therefore it follows by construction of J_0 that $ext(J_0) \subset B_\infty(G)$ when $G \in \mathcal{G}_r$.

Now suppose that $G \in \mathcal{G}_l$; in this case E is not G^2-forward invariant, in fact $G^2(E \cap \ell_1) \cap E = \emptyset$. However, $G^2(E) = E \cup E'$ where E' is the unbounded set which boundary consists of $W_1^u, G^2(E \cap \ell_1)$ and the segment T used to construct the curve γ when $G \in \mathcal{G}_l$. Hence, if E' is contained in $B_\infty(G)$ then (c) will be proved also in this case. But the claim and (ii) implies that $E' \subset B_\infty(G)$. So the proof of the proposition is complete. □

Now we list some properties relating the invariance of J_0 with its location in respect to P.

Lemma 5 *For $G \in \mathcal{U}_\epsilon^\pm$ with ϵ sufficiently small, and $J_0 = J_0(G)$ it holds that:*
(a) J_0 is forward invariant if and only if S_1 and S_2 do not belong to the image of G, $Im(G)$. In this case, $J_0 = \partial B^0_\infty(G)$.
(b) J_0 is backward invariant if and only if it is forward invariant and $G^{-1}(\tilde{\gamma}^1) \cap P = \emptyset$. In this case $B_\infty(G)$ is connected, its boundary is J_0 and its complementary set has nonempty interior.

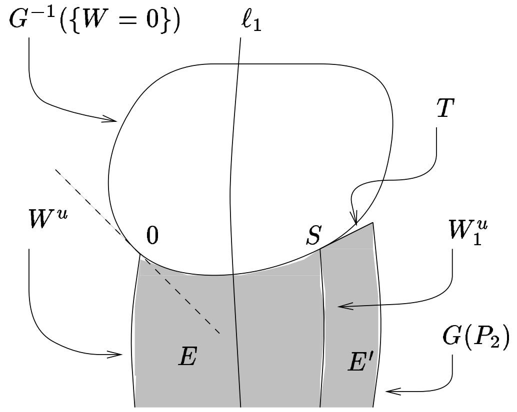

Figure 2. The curve $G^{-1}(\{W=0\})$ and the sectors E and E'

Proof: Beginning with the proof of (a), suppose that S_1 and S_2 do not belong to $Im(G)$, this implies that both connected preimages of γ intersect P, here is no need of considering the segment T. This implies that J_0 is forward invariant. Conversely, if any of the points S_1 or S_2 is not in the image of G, then it was necessary to add the segment T to γ in order to obtain that both connected preimages of γ reach P. Then, as J_0 contains preimages of T but does not contain T, it follows that it is not forward invariant.

If J_0 is forward invariant, then it has no intersection with $B_\infty(G)$. This, together with the previous proposition, implies the last assertion of part (a).
To prove (b), recall that $\tilde{\gamma}^1$ is the preimage of γ that joins S with S_1. If S_1 and S_2 do not belong to $Im(G)$ and $G^{-1}(\tilde{\gamma}^1)$ does not intersect P, then it does not intersect $Im(G)$. This implies that it has no preimage and it follows that J_0 is also backward invariant. Conversely, if $G^{-1}(\tilde{\gamma}^1)$ intersects P, then it has another preimage which cannot belong to J_0 and hence it is not backward invariant.

If J_0 is backward invariant, then the immediate basin of $B_\infty(G)$ has no preimage and then $B_\infty(G)$ is connected. □

The approach we have chosen to study the basin of ∞ is the creation (sometimes artificially) of a curve J_0 whose $ext(J_0)$ is contained in $B_\infty(G)$. Sometimes this curve J_0 is not the whole boundary of $B_\infty(G)$, so we will now consider its preimages.

Proposition 2 *Let $G \in \mathcal{U}_\epsilon^\pm$ with ϵ sufficiently small. Then:*
(a) $\bigcup_{n\geq 0} G^{-n}(ext(J_0)) = B_\infty(G)$.
(b) $\partial B_\infty(G) \subset \liminf G^{-n}(J_0)$.
(c) If $\partial B_\infty(G)$ is backward invariant, then $\limsup G^{-n}(J_0) \subset \partial B_\infty(G)$.
(d) If G_0 is a continuity point of the mapping $G \to A(G)$, then $\partial B_\infty(G_0)$ is backward invariant.

The concept of convergence in statements (b) and (c) is related to the Hausdorff metric of nonempty compact subsets of the plane. Observe that a sequence K_n converge to K if and only if the following conditions hold:
(i) For every neighborhood U of K there exists n_0 such that $K_n \subset U$ for every $n \geq n_0$.
(ii) For every open set U such that $U \cap K \neq \emptyset$, there exists n_0 such that $K_n \cap U \neq \emptyset$ for every $n \geq n_0$.

Associated with each of the conditions (i) and (ii) one can define lim inf and lim sup of a sequence K_n of compact sets as follows: $\liminf K_n$ is the set of limits of sequences $\{x_n\}$ where each x_n belongs to K_n, and $\limsup K_n$ is the set of limits of subsequences of sequences as above. Observe that $\liminf K_n$ and $\limsup K_n$ always exist and are unique. Moreover, if $\liminf K_n = K$ then (ii) holds, and if $\limsup K_n = K$ then (i) holds. Obviously, $\liminf K_n = \limsup K_n$ implies that K_n converges to K in $\mathcal{K}$.

Proof: (a) Is obvious since $ext(J_0) \subset B_\infty(G)$.
(b) Suppose that $x \in \partial B_\infty(G)$. Take a positive integer p. Let V_p the open ball with center at x and radius $\frac{1}{p}$. If $G^n(V_p) \subset int(J_0)$ for every $n \geq 0$, then $V_p \subset A(G)$ and so $x \notin \partial B_\infty(G)$. If there exists $n > 0$ such that $G^n(V_p) \subset ext(J_0)$, then $V_p \subset B_\infty(G)$ and so $x \notin \partial B_\infty(G)$. In conclusion, there exists some $n_p \geq 1$ such that $G^n(V_p) \cap J_0 \neq \emptyset$ for all $n \geq n_p$. This defines a sequence $\{n_p\}_{p\geq 1}$. Take, for every j with $n_p \leq j < n_{p+1}$, a point $x_j \in V_p$ such that $G^j(x_j) \in J_0$. Then the sequence $\{x_j\}_{j\geq 1}$ satisfies $x_j \in G^{-j}(J_0)$ and $x_j \to x$ when $j \to +\infty$. This implies that $x \in \liminf G^{-n}(J_0)$.
(c) Let $x \notin \partial B_\infty(G)$. If $x \in B_\infty(G)$ then there exists a neighborhood U of x contained in $B_\infty(G)$ and hence there exists n_0 such that $G^n(U)$ is contained in $ext(J_0)$ for every $n \geq n_0$. As $J_0 \subset cl(B_\infty(G))$ and $\partial B_\infty(G)$ is backward invariant, it follows that $G^{-n}(J_0) \subset cl(B_\infty(G))$ for every $n \geq 0$. Therefore no point interior to $A(G)$ can be accumulated by preimages of J_0. This proves part (c).
(d) Suppose that $\partial B_\infty(G_0)$ is not backward invariant. Then there exists $x \in \partial B_\infty(G_0))$ and a point $y \notin \partial B_\infty(G_0))$ such that $G_0(y) = x$. Since $y \notin \partial B_\infty(G_0))$, there exists V_δ neighborhood of y such that $V_\delta \cap B_\infty(G_0)) = \emptyset$. We will prove that for every $\epsilon > 0$ there exists G ϵ-close to G_0 such that $y \in \partial B_\infty(G))$. This contradicts the continuity of $G \to A(G)$ at G_0, and proves the assertion. So, given $\epsilon > 0$ a point z in $B_\infty(G_0))$ as close to x as we wish can be chosen in such a way that if one perturbs G_0 only in a small neighborhood V' of y, then the perturbation G satisfies $G(y) = z$. The neighborhood V' can be taken contained in V_δ, therefore the G_0-positive orbit of z does not intersect V'. It follows that $G_0^n(z) = G^n(z)$ for every $n \geq 0$ and then z, and therefore y, belong to $B_\infty(G)$. □

Remark 2 *(i) Observe that the proof of this proposition is purely topological. Neither the proximity of G to F_4 nor the fact that the dimension is two are relevant; indeed nothing changes in the proof if one begins with a C^0 endomorphism G and a set J_0 satisfying properties (a) and (c) of proposition 1.*

(ii) The operator $G \to A(G)$ is lower semicontinuous: if K is a compact set contained in $B_\infty(G_0)$, then $K \subset B_\infty(G)$ for every C^0 perturbation G of G_0. Therefore, it follows that the operator is continuous in a residual set. For that reason, proposition 2 implies that:
For generic $G \in \mathcal{U}_\epsilon^\pm$ it holds that $\lim_{n\to+\infty} G^{-n}(J_0) = \partial B_\infty(G)$.

The preceding proposition gives a description of $\partial B_\infty(G)$; the problem is that the boundary of $B_\infty(G)$ has a lot of components and it is not clear where the interesting dynamics is. We will now construct a subset J of the boundary that is forward invariant.

Definition 1 *The definition is by recurrence. $J_0 = J_0(G)$ was defined above. Suppose defined curves $J_i = J_i(G)$ for every $0 \leq i \leq n-1$. Let J_{n-1}^0 be the closure of* *the connected component of the intersection of J_{n-1} with the topological interior of the image of G that contains 0. Then define $J_n(G)$ as the connected component of $G^{-1}(J^0_{n-1}(G))$ that contains 0.*

Also define $\widetilde{J}^0_{n-1}(G)$ as the closure of $int(J^0_{n-1}) \cup P$.

Observe that the boundary of $\widetilde{J}^0_{n-1}$ is the union of J^0_{n-1} and a connected subset of P. It is not true that $ext(J_n)$ is a subset of $B_\infty(G)$, there can be components of $A(G)$ contained in $ext(J_n)$; however, these are "small" components and the determining dynamics occur within $int(J_n)$. We will show later (proposition 5) that if some J_n is contained in the interior of $Im(G)$, then ℓ_1 is contained in $B_\infty(G)$, and if this is not true, then the limit of the J_n exists and is a connected set J.

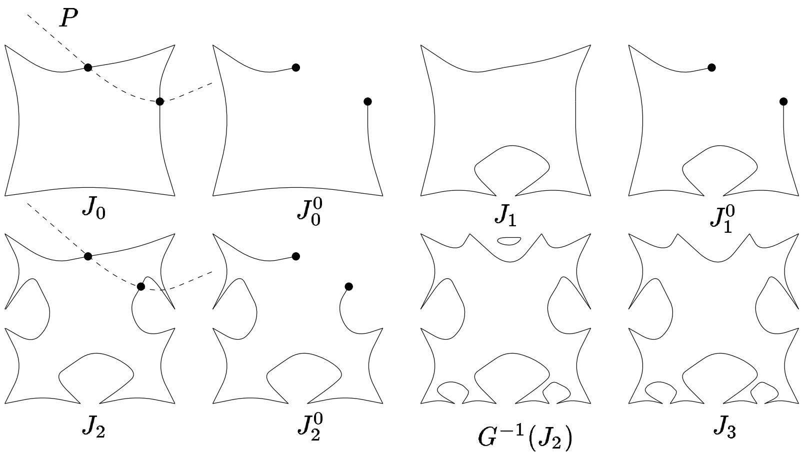

Figure 3. The curves $J_0, J^0_0, J_1, J^0_1, J_2, J^0_2, G^{-1}(J_2)$ and J_3

We will first show that $\{ext(J_n)\}$ is an increasing sequence.

Proposition 3 *Let $G \in \mathcal{U}^{\pm}_\epsilon$ with ϵ small enough, then for every $n \geq 1$ it holds that:*
(E_n) : $ext(J_{n-1}(G)) \subset ext(J_n(G))$.
(E^0_n) : $ext(\tilde{J}^0_{n-1}) \subset ext(\tilde{J}^0_n)$.

Proof: Proposition 1 implies (E_1) and (E^0_1).
Suppose that (E_{n-1}) and (E^0_{n-1}) hold. Lemma 3 implies that $G(ext(J_{n-1})) \subset ext(\tilde{J}^0_{n-2})$. If $x \in J_n$ then $G(x) \in J^0_{n-1}$, and then it does not belong to $ext(\tilde{J}^0_{n-1})$, which, by (E^0_{n-1}) and the above, implies that $x \notin ext(J_{n-1})$. Thus $J_n \subset cl(int(J_{n-1}))$ which is equivalent to (E_n).
Now let $x \in ext(\tilde{J}^0_{n-1})$. If $x \in ext(J_{n-1})$, then (E_n) implies that $x \in ext(J_n)$ and so $x \in ext(\tilde{J}^0_n)$. If x does not belong to $ext(J_{n-1})$, then x belongs to a component of $cl(int(J_{n-1})) \cap Im(G)$ that does not contain 0. But by (E_n), J_n is contained in $cl(int(J_{n-1}))$, so the above implies that the component of $J_n \cap Im(G)$ that contains x, does not contain 0. That is, $x \in ext(\tilde{J}^0_n)$. This proves (E^0_n) and completes the induction. □

Proposition 4 *Suppose that $G \in \mathcal{U}^{\pm}_\epsilon$ with ϵ sufficiently small, and that for every $n \geq 0$ the curve $J_n = J_n(G)$ is not contained in $Im(G)$. Then:*
(a) For every $n \geq 0$, J_n is homeomorphic to S^1.

(b) For every $n \geq 0$, $J_n \subset cl(B_\infty(G))$.
(c) $J = \lim J_n$ *exists in the Hausdorff topology.*
(d) J *is forward invariant and contained in the boundary of* $B_\infty(G)$.
(e) J *is connected and* $int(J)$ *is simply connected.*
(f) Let $\widetilde{A} = A_0 \cup int(A_0)$, *where* A_0 *is the connected component of* $A(G)$ *that contains* 0. *Then* $\tilde{A} \supset \bigcap_{n\geq 0} cl(int(J_n))$.
(g) $J \subset \partial A_0$ *and then* A_0 *connects* 0 *with* $P = P(G)$.

Proof: (a) For $n = 0$, (a) holds by construction of J_0. Assume by induction that J_{n-1} is homeomorphic to S^1. Then lemma 3 implies that $J_n = G^{-1}(J^0_{n-1})$ is homeomorphic to S^1 because by hypothesis J^0_{n-1} intersects P.
(b) Again by induction. For $n = 0$, the result follows from proposition 1, part (c). The hypothesis of the proposition and part (a) imply that J^0_{n-1} intersects P in two points. Now induction hypothesis implies that $J^0_{n-1} \subset cl(B_\infty(G))$, and it becomes clear that J_n is contained in $cl(B_\infty(G))$. Observe that $cl(B_\infty(G))$ is not backward invariant, as we saw in proposition 2, but for every point x in $cl(B_\infty(G))$ and interior to the image of G, it holds that $G^{-1}(x) \subset cl(B_\infty(G))$; in J_{n-1} there are exactly two points not belonging to the interior of the image of G, but they are accumulated by other points in the interior of $Im(G)$.

It remains to prove that actually, J_n is contained in the closure of $B^0_\infty(G)$: this follows from the above together with the facts that $J^0_{n-1} \subset cl(B^0_\infty(G))$, J_n is homeomorphic to S^1 and $J_n \cap J_{n-1} \neq \emptyset$.
(c) **Claim:** *Let* $\{H_j\}$ *be an increasing sequence of open sets; denote the boundary of* H_j *by* h_j, *each* h_j *homeomorphic to* S^1. *Then* $\lim h_j$ *exists and is equal to* $h = \partial(\bigcup_{j\geq 0} H_j.)$
The proof of the claim is easy and we leave it. Then part (c) follows taking $H_j = int(J_j)$.
(d) If $x \in J$, then there exists a sequence $\{x_n\}$, each $x_n \in J_n$, such that $x_n \to x$. As $G(x_n) \in J^0_{n-1} \subset J_{n-1}$, it follows that $G(x) \in J$. $J \subset cl(B_\infty(G))$ because $J_n \subset cl(B_\infty(G))$. As J is bounded and forward invariant, it follows that actually, $J \subset \partial B_\infty(G)$.
(e) As J is the boundary of an increasing sequence of simply connected sets, it follows that $int(J)$ is simply connected and hence J is connected.
(f) $\widetilde{A}$ is the minimal simply connected set containing A_0. As J contains 0, is contained in $\partial B_\infty(G) \subset A(G)$, and is connected, it follows that $\tilde{A}$ contains $cl(int(J))$.
(g) By part (d), J is contained in $\partial B_\infty(G)$ which is a subset of $A(G)$; furthermore, J is connected and contains 0. It follows that $J \subset A_0$, in fact, as $J \subset \partial B_\infty(G)$, we conclude that $J \subset \partial A_0$.
As each J_n intersects P, one can take a sequence $\{x_n\}$, each $x_n \in J_n \cap P$, and then any convergent subsequence has its limit in $J \cap P$. □

Lemma 6 *Let* $G \in \mathcal{U}^\pm_\epsilon$ *with* ϵ *small enough. Then:*
(a) $ext(J_n) \cap G^{-1}(J_{n-1}) = \emptyset$ *for every* $n \leq N$ *implies that* $ext(J_N) \subset B_\infty(G)$.
(b) $ext(J_N) \setminus \bigcup_{p=0}^{N-1} G^{-p}(ext(J_{N-p}) \cap G^{-1}(cl(int(J_{N-p-1})))) \subset B_\infty(G)$.

Proof: (a) Observe that $ext(J_n) \cap G^{-1}(J_{n-1}) = \emptyset$ if and only if $G^{-1}(J_{n-1}) \subset cl(int(J_n))$. By lemma 3 part (b), this is equivalent to $J_{n-1} \cap Im(G) \subset \tilde{J}^0_{n-1}$. But $G^{-1}(\tilde{J}^0_{n-1}) = cl(int(J_n))$, hence the property above implies that $G(ext(J_n)) \subset ext(J_{n-1})$. We can then conclude from the hypothesis of part (a) that $G^N(ext(J_N)) \subset ext(J_0) \subset B_\infty(G)$.

(b) Let $x \in ext(J_N) \setminus \bigcup_{p=0}^{N-1} G^{-p}(ext(J_{N-p}) \cap G^{-1}(cl(int(J_{N-p-1}))))$. For $p = 0$ this implies that $G(x) \notin cl(int(J_{N-1}))$, thus $G(x) \in ext(J_{N-1})$; then for $p = 1$ the above expression implies that $G(x) \notin G^{-1}(cl(int(J_{N-2})))$, or, which is the same, $G^2(x) \in ext(J_{N-2})$. This shows how a simple induction argument implies that $G^N(x) \in ext(J_0)$, completing the proof. □

Proposition 5 *Let $G \in \mathcal{U}_\epsilon^\pm$ with ϵ sufficiently small. If there exists N such that $J_N \cap P = \emptyset$, then $ext(J_N) \subset B_\infty(G)$.*

Proof: The proof will be made in several steps. Suppose that N be the minimal natural number such that $J_n \cap P = \emptyset$. If for some value of $n \leq N$, J_n contains 0, S, S_1 and S_2, then define $\gamma_n^0, \gamma_n^1, \gamma_n^2$ and γ_n^3 as the connected component of $J_n \setminus \{0,\ S,\ S_1,\ S_2\}$ that join 0 with S, S with S_1, 0 with S_2 and S_1 with S_2 respectively.
Step 1: *For every $n \leq N$ the curve J_n contains $0, S, S_1$ and S_2.*
The proof is by induction on $n \leq N$. For $n = 0$ it is clear by definition. Suppose it is true for all $0 \leq j \leq n < N$; it is enough to prove that $S \in J_n^0$, because this implies $S_1, S_2 \in J_{n+1}$. In fact, it is enough to prove that $\gamma_n^0 \cap P = \emptyset$. Denote by $\widetilde{\gamma}_n^3$ the connected component of $\gamma_n^3 \cap Im(G)$ that contains S_2; if S_2 does not belong to $Im(G)$, take $\widetilde{\gamma}_n^3 = \emptyset$. Clearly, $\gamma_n^0 = G^{-1}(\widetilde{\gamma}_{n-1}^3 \cup \gamma_{n-1}^2)$. Let (z_{n-1}, w_{n-1}) be the point of intersection of $\widetilde{\gamma}_{n-1}^3$ with P. Suppose that $|z_{n-1} - s_1| \geq 1$ where s_1 is the first coordinate of S_1. Then, as G is close to a delay endomorphism, $G^{-1}(\widetilde{\gamma}_{n-1}^3)$ cannot have points with second coordinate greater than $s_1 - 1$, and then cannot intersect P, which is close to $\{(x, y) : y = s_1\}$ in compact sets. On the other hand, if $|z_{n-1} - s_1| \leq 1$, then the same conclusion follows from the fact that $G^{-1}(\widetilde{\gamma}_{n-1}^3)$ cannot intersect $\widetilde{\gamma}_{n-1}^3$. The proof that $G^{-1}(\gamma_{n-1}^2)$ cannot intersect P is similar. This proves the claim.

In particular, it follows that $S_1, S_2 \in Im(G)$ because the contrary assumption implies $J_n \cap P \neq \emptyset$ for every n. Moreover, if $x \in J_n \cap \gamma_0$ for all $n \leq N$, then $x \in J_n^0$.
Step 2: *Suppose that for some $p < N$ it holds that $x \in J_p^0$ and $x \in J_n$ for every $p \leq n \leq N$. Then $x \in J_n^0$ for every $p \leq n \leq N$.*
If the assertion in step 2 is false, there exists some minimal natural number m such that $x \notin J_m^0$ and $p < m \leq N$. Then $x \notin \widetilde{J}_m^0$ and this implies that $x \notin \widetilde{J}_N^0$ by proposition 3. As $x \in J_N$, the preceding argument also implies that $x \notin J_k^0$ for $k \geq m$. In particular, $x \notin J_N^0$, but J_N is connected and contains the points 0 and x, so $J_N \cap P \neq \emptyset$, which is a contradiction. This proves step 2.

Note that $S_2 \in Im(G)$ was obtained as a consequence of step 1. This implies $G \in \mathcal{G}_l$.

Recall from corollary 2 that if U is a small neighborhood of $Q(*, 0)$, then $\Lambda_{G^2}(U)$ is a Cantor set contained in γ_0. Denote by K_0 this Cantor set. Moreover, $x \in K_0$ implies $x \in J_n^0$ for every $n \leq N$, and $\gamma_0 \setminus K_0 \subset \bigcup_{n \geq 1} G^{-2n}(T)$, where T is the segment employed to construct the curve γ_0, see lemma 4. Define $K = G^{-1}(K_0) \cup G^{-2}(K_0)$; then $K_0 \subset K \subset J_0$ and $K = \{x \in J_0 : G^n(x) \in J_0\ \forall n \geq 0\}$.
Step 3: *If $x \in J_m$ for some $m \leq N$ and $G^m(x) \in K$, then $x \in J_N$.*
Note first that by definition of J_m, $G(x) \in J_{m-1}^0$ and then $G^j(x) \in J_{m-j}^0$ for every $1 \leq j \leq m$. The fact that $G^m(x) \in K$ implies that $G^m(x) \in J_n^0$ for every $n \leq N$. It follows that $G^{m-1}(x) \in J_{n+1}$ for every $n \leq N - 1$ and as $G^{m-1}(x) \in J_1^0$, then the assertion in step 2 implies that $G^{m-1}(x) \in J_n^0$ for every $1 \leq n \leq N$. Now we will show that $G^{m-2}(x) \in J_n^0$ for every $2 \leq n \leq N$: indeed, as $G^{m-1}(x) \in J_n^0$ for every $1 \leq n \leq N$ then $G^{m-2}(x) \in J_{n+1}$ for every $1 \leq n \leq N$. This together with $G^{m-2}(x) \in J_2^0$ imply, again by step 2, that $G^{m-2}(x) \in J_n^0$ for every $2 \leq n \leq N$. So

we can proceed by induction to obtain $x \in J_n^0$ for every $m \leq n \leq N$, in particular, $x \in J_N$.

Now we analyze the components of $J_k \setminus J_{k-1}$ for $k \leq N$. The next assertion is trivial.

Step 4: *If β is a connected component of $J_k \setminus J_{k-1}$, then $G^k(\beta) = \tilde{\gamma}_0^3$ and there is a component α of $J_{k-1} \setminus J_k$ such that $\alpha \cup \beta$ is connected and $G^k(\alpha)$ is a component of $T' = G^{-1}(T)$. We call β the substitution of α in J_{k-1}. The extreme points, p_1 and p_2, of the substitution β (β is an open interval) belong both to J_{k-1} and J_k; moreover, as these points are preimage of S_1 or S_2, it follows that they are preimage of K. By step 3 this implies that p_1 and p_2 belong to J_N, and hence belong to $Im(G)$.*

Let β be a substitution of some $\alpha \subset J_{k-1}$.

Step 5: *If $\beta \cap P \neq \emptyset$, then $\alpha \cap P \neq \emptyset$ and β cannot intersect an unbounded component of $P \setminus \alpha$.*

Obviously the second assertion of step 5 implies the first one. Suppose that the second assertion is not true. Then at least one of the extreme points p_i of β is not contained in J_k^0. This implies that p_i is not contained in J_N^0. However, it is known that $p_i \in J_N$ and this is a contradiction because $J_N \cap P = \emptyset$.

Step 6: *$G^{-1}(\tilde{\gamma}_0^3)$ is the union of the graphs of two functions φ_1^+ and φ_1^- defined in an interval $d_1 \subset \ell_1$.*

As $\tilde{\gamma}_0^3$ and P are graphs of functions defined in the first coordinate axis, the vertical line through any point of P intersects $\tilde{\gamma}_0^3$ in at most one point. The preimage of a vertical segment joining a point of P with a point of $\tilde{\gamma}_0^3$ is an almost horizontal curve joining two points of $G^{-1}(\tilde{\gamma}_0^3)$, one located at each side of ℓ_1. Each of these curves intersects ℓ_1, thus defining an interval $d_1 \subset \ell_1$. Then the function φ_1^+ (resp. φ_1^-) assign, to a point $x \in d_1$ the corresponding point of $G^{-1}(\tilde{\gamma}_0^3)$ located at the right (resp. left) of ℓ_1.

Step 7: *β is the union of the graphs of two functions φ_k^+ and φ_k^- defined in an interval $d_k \subset G^{-(k-1)}(\ell_1)$. Moreover, $\varphi_k^{\pm}(x) - x$ is almost horizontal for every $x \in d_k$ (see figure 4(a)).*

Observe that $\varphi_1^{\pm}(x) - x$ is almost horizontal. If U is a small neighborhood of the boundary of the square Q, then there exists a neighborhood $\mathcal{U}$ of F_4 such that $DG^{-2}(x)$ mantains the horizontal directions whenever x and $G^{-2}(x)$ belong to U. Perhaps there is a finite number m of iterates needed to have $G^{-m}(\tilde{\gamma}_0^3)$ contained in U, but certainly this m is uniformily bounded for $G \in \mathcal{U}$. This permits to establish the fact that $DG^{-m}(\varphi_1^{\pm}(x) - x)$ (the derivatives calculated at points close to β) is almost horizontal. The assertion of the step follows easily.

Now we study the intersection of P and β; as P is almost horizontal one can give a P-order in the set $P \cap \beta$: we say $x < y$ in $P \cap \beta$ if the first coordinate of x is less than that of y. Using φ_k^+ and the fact that d_k is an interval, one can naturally assign a φ_k^+-order in $P \cap graph(\varphi_k^+)$. It is clear now that the identity map of $P \cap graph(\varphi_k^+)$ is order preserving (or order reversing, dependingon the choise of the first point of d_k). The same can be done with $P \cap graph(\varphi_k^-)$. We have almost obtained:

Step 8: *No component of $\beta \cap P$ is contained in an unbounded component of $P \setminus \gamma_0^3$.*

As we saw in step 5, β cannot intersect P in an unbounded component of $P \setminus \alpha$. So the assertion in step 8 can only be false if there exists an intersection point of P and β with first coordinate less than z_k. This would contradict the fact that the orders determined by β and by P on $P \cap \beta$ are coincident. See figure 4(b).

Step 9: *Every component of $J_k \cap Im(G)$ is contained in $\tilde{J}_k^0$.*

We have proved in the preceding step that no component of J_k has points in P with

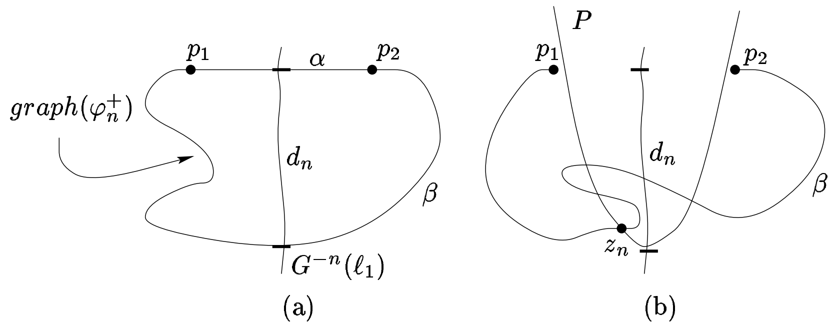

Figure 4. The curve β and the graph of φ_n^+

first coordinate less than z_k. Analougous to the definition of z_k one can define, for each J_k the other point of $J_k^0 \cap P$ (that is, the extreme point of J_k^0 closest to S_1); denote it by (x_k, y_k). The same proof we did above implies that no point of a substitution β of J_k can have intersection with a point with first coordinate greater than x_k. The assertion in step 9 follows.

Now the lemma 6 implies the result. □

Proposition 6 *Suppose that for some n, $J_n \cap P = \emptyset$. Then $G^{-m}(J_n)$ has 2^m connected components, and $ext(G^{-m}(J_n))$ is contained in $B_\infty(G)$, so $B_\infty(G)$ is connected. Moreover, $A(G)$ has uncountably many components.*

Proof: The only fact that deserves attention is that as a consequence of step 3 of the last proposition, no component of $A(G)$ is contained in $cl(ext(J_n))$. The rest of the proof is very simple. □

Proof of the theorems A, B and C.

Theorem A was proved in proposition 2. For the proofs of theorems B and C suppose first that every J_n intersects the boundary P of $Im(G)$. Then the proposition 4 implies that the limit of the sequence J_n exists and is a forward invariant curve J intersecting P; it follows that J contains critical points. This proves one of the directions in theorem B and part (a) of theorem C. To prove the other direction of theorem B, assume now that there exists some N such that $J_N \cap P = \emptyset$. The proposition 5 implies that $ext(J_n) \subset B_\infty(G)$, and it follows that P, and consequently ℓ_1, is contained in $B_\infty(G)$.

To prove the first assertion in (b) of theorem C we use proposition 6. It remains to show that in this case, every point in $\partial B_\infty(G)$ is accumulated by preimages of 0 and this is consequence of the facts that for every k, $J_k \cap Im(G)$ is contained in $\widetilde{J}_k^0$ (step 9 of proposition 5) and that the Hausdorff limit of J_n is $\{0\}$. □

4. The symmetric family

In this section we will consider an example of application of the preceding constructions. Let $g_{\mu,b}(x,y) = -x^2 - by^2 + \mu x + b\mu y$ and define $G_{\mu,b}$ as the delay endomorphism associated to the delay equation of order two $x_{n+2} = g_{\mu,b}(x_n, x_{n+1})$, $n \geq 0$.

Observe that $G_{\mu,0}$ is the quadratic family of order two. The advantage in considering this particular family is the symmetry it shows; this permits to avoid a large number of (possible) complicated cases and permits to illustrate better the results of the last sections.

Taking μ near 4 and b near 0, and calculating the eigenvalues of $DG_{\mu,b}(0)$ it follows that $G_{\mu,b} \in \mathcal{U}_\epsilon^+$ if and only $b > 0$ and $G_{\mu,b} \in \mathcal{U}_\epsilon^-$ if and only if $b < 0$, for some ϵ. Clearly every curve J_n is C^1 when $b > 0$ and every J_n has cusps at the preimages of 0 it contains when $b < 0$. On the other hand, the line $\ell_1(\mu)$ of critical points of $G_{\mu,b}$ is $x = \mu/2$ and the boundary, $P(\mu)$, of the image of $G_{\mu,b}$ is the parabola with equation $y = -bx^2 + b\mu x + \mu^2/4$. Observe that $P(\mu)$ is symmetric with respect to $\ell_1(\mu)$.

The another preimage of the origin is $S = (\mu, 0)$, which has preimages $S_1 = (\mu, \mu)$ and $S_2 = (0, \mu)$. It becomes clear that $G_{\mu,b} \in \mathcal{G}_r$ if and only if $\mu \leq 4$. By the symmetry it follows that the distance from S_1 and S_2 to $P(\mu)$ is the same. Therefore lemma 5 implies that J_0 is forward invariant if and only if $\mu \leq 4$.

We will consider only the case $b < 0$, leaving the case of $b > 0$, for which the properties are quite similar. Next we state some properties for this family.

1.- *There exists some $\mu_0 \in (0,4)$ such that J_0 is backward invariant only for $\mu \in (0, \mu_0)$. Moreover μ_0 is the first point of discontinuity of the mapping $\mu \to A(G_{\mu,b})$; for every $\mu \in (0,\mu_0)$, $B_\infty(G_{\mu,b})$ is connected and $A(G_{\mu,b})$ is the closure of $int(J_0)$.*

Observe that by theorem 5 we have only to prove that γ^3 intersects $P(\mu)$ also for some $\mu < 4$. Indeed, consider the less expanding vector $(1, \lambda^+)$ at the origin. Its second preimage is a vector tangent to γ^3 at S_2; this vector is of the form (u, v) with $u > 0$ and $v < 0$ and trivially transverse to the parabola $P(\mu)$. So there must exist some intersection beetwen P and γ^3 before $\mu = 4$. This proves the first assertion. Now define μ_0 as the least value of μ for which γ^3 and $P(\mu)$ have nonempty intersection. This intersection must be a tangency because γ^3 is of class C^1. By the claim of proposition 2 it follows that μ_0 is a discontinuity point of $\mu \to A(G_{\mu,b})$. The last asssertion follows trivially from lemma 5.

2.- *Let $\mu_1 = 4(1-b)$. Then μ_1 is the greater value of the parameter for which $P(\mu)$ intersects J_0. Consequently, $G_{\mu,b} \in \widetilde{\mathcal{H}}_0$ if and only if $\mu > \mu_1$.*

It is not difficult to prove that the set $ext(Q_\mu)$, (where Q_μ is the square of vertices 0, S, S_1, S_2) is forward invariant. This implies that J_0 is contained in the closure of $int(Q_\mu)$. Moreover the preimages of S_1 have second coordinate equal to μ, and belong to γ^3. It follows that the last parameter μ for which $P(\mu)$ and γ^3 have nonempty intersection is that for which $P(\mu)$ contains the preimages of S_1 and this easy calculation gives the value of μ_1.

3.- *$B_\infty(G_{\mu,b})$ is connected iff μ does not belong to the interval (μ_0, μ_1). For μ in this interval $B_\infty(G_{\mu,b})$ has infinitely many components.*

The first assertion is a trivial consequence of the two properties above. It remains to prove the second one. Observe that for $\mu \in (\mu_0, \mu_1)$, there is at least a nontangencial intersection beetwen $P(\mu)$ and γ^3. It follows that there is a component J_0^1 of $J_0 \cap Im(G)$ contained in γ^3. Then $int(G^{-1}(J_0^1))$ is contained in $B_\infty(G_{\mu,b})$ and also in $int(J_0)$. Therefore $B_\infty(G_{\mu,b})$ is not connected for these parameters. Moreover this new component of $B_\infty(G_{\mu,b})$ has infinitely many disjoint preimages.

Now we show some figures illustrating the different situations described above for the symetric family with $b < 0$.

The figures 5 and 6 were constructed taking the preimages of 0. In the figure 5, $g_{\mu,b}(x,y) = -x^2 + 0.3y^2 + 4.1x - 1.23y$, so $\mu > 4$ and J has this fractal aspect (observe that for $\mu \leq 4$ $J = J_0$, hence it is C^1 except at the points $0, S, S_1$ and S_2). One can also infer, from the figure, that $\mu < \mu_0$ because $B_\infty(G_{\mu,b})$ is connected (if this were not the case, then there would be preimages of the origin in $int(J)$). It follows that $J = \partial B_\infty(G_{\mu,b})$. In the figure 6, $g_{\mu,b}(x,y) = -x^2 + 0.3y^2 + 4.8x - 1.44y$. It is clear that $B_\infty(G_{\mu,b})$ is not connected; the white components of $int(J)$ are all contained in $B_\infty(G_{\mu,b})$ and its boundaries are preimages of J. As J still exists (and contains critical points) the parameter μ necessary lies between μ_0 and μ_1.

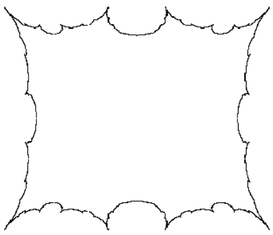

Figure 5.

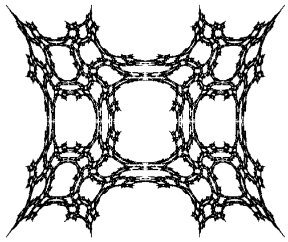

Figure 6.

The figures 7 and 8 were also obtained from the plotting of the preimages of the fixed point at the origin. In the first case, figure 7, $\mu = 4.4$ and $b = -0.1$. Observe that the parameter μ satisfies the equation $\mu = 4(1-b)$, that is, $\mu = \mu_1$. So this is the last point of existence of the curve J. Again, the white components are contained in $B_\infty(G_{\mu,b})$. It follows that there exists only two point of ℓ_1 that are not contained in $B_\infty(G_{\mu,b})$. In the figure 8 we have considered $g_{\mu,b}(x,y) = -x^2+0.1y^2+4.47x-0.447y$, so $\mu > \mu_1$, $\ell_1 \subset B_\infty(G_{\mu,b})$, J has disappeared and it seems that the complementary set of $B_\infty(G_{\mu,b})$ is a Cantor set. Now $B_\infty(G_{\mu,b})$ is a connected set.

In all the figures 5 to 8 the preimages of the fixed point 0 seem to be dense in the curve J. In the figures 9 and 10 we leave the symmetric family to obtain a clear example where J is not accumulated by the preimages of 0. In this last figures $g_{\mu,b}(x,y) = -x^2 + 0.35y^2 + 5x - 1.966y$. In this case the curve J still exists and it is connected.

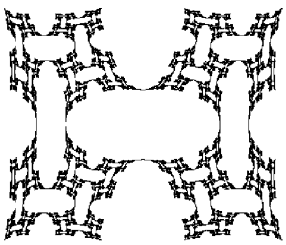

Figure 7.

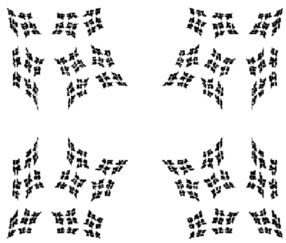

Figure 8.

In figure 9 we have plotted the preimages of the fixed point 0; in figure 10 we took some point in $int(J)$ and plotted its preimages. Now it is possible to observe all the curve J and its preimages. The fact that unables the preimages of the origin to be dense in J is that a periodic point of saddle type belongs to J. So $G_{\mu,b}$ restricted to J has an attracting periodic orbit that cannot contain preimages of the origin. It follows also that the immediate stable manifold of the saddle is contained in J, therefore J contains smooth curves and is fractal elsewhere.

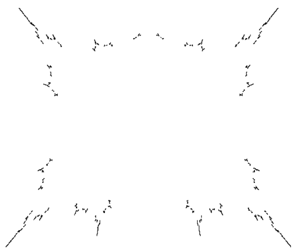

Figure 9.

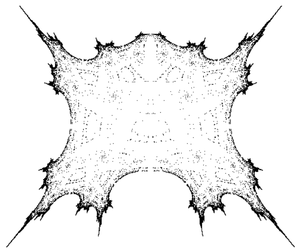

Figure 10.

Acknowledgments

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