ON THE APPROXIMATION OF TIME ONE MAPS OF ANOSOV FLOWS BY AXIOM A DIFFEOMORPHISMS

NANCY GUELMAN

Abstract. Let M be a smooth compact Riemannian manifold without boundary, and $\phi : M \times \mathbb{R} \to M$ a transitive Anosov flow.

In 1975, Palis and Pugh wondered whether the time one map of a transitive Anosov flow could be approximated by hyperbolic or Axiom A diffeomorphisms.

It is a well known fact that in the case when the flow arises from the suspension of an Anosov diffeomorphism $g : N \to N$ such an approximation can be carried out with Axiom A diffeomorphisms.

In the case of $dim(M) = 3$ or 4, we prove that if the time one map of a transitive Anosov flow is C^1-approximated by Axiom A diffeomorphisms, then it is flow equivalent to a suspension of an Anosov diffeomorphism.

Introduction

Throughout this paper M denotes a smooth compact Riemannian manifold without boundary, and $\phi : M \times \mathbb{R} \to M$ a C^r flow, with $r \geq 1$.

Let us consider that ϕ is an Anosov flow (see Definition 1.1) and let $f_\tau(x) = \phi(x, \tau)$, $\forall x \in M$ be the flow ϕ at time τ. Although f_τ is not an Anosov diffeomorphism (see Definition 1.2), there exists a Df_τ-invariant splitting of TM

$$TM = E^s \oplus E^c \oplus E^u,$$

such that $Df_\tau|E^s$ is uniformly contracting, $Df_\tau|E^u$ is uniformly expanding, and E^c is a nonhyperbolic central direction, i.e. f_τ is a partially hyperbolic diffeomorphism.

The object of our study are *transitive* Anosov flows (i.e. the case when the non-wandering set is the whole manifold).

An interesting question is what kind of dynamical system can appear under perturbations of a time one map of a transitive Anosov flow.

Palis and Pugh (see [8]) wondered whether the time one map of a transitive Anosov flow could be approximated by hyperbolic or Axiom A diffeomorphisms.

We shall give a precise answer to this problem in case $dim(M) = 3$ or 4.

It is a well known fact that in the case when the flow arises from the suspension of an Anosov diffeomorphism $g : N \to N$ such an approximation can be carried out with Axiom A diffeomorphisms. Let us explain it:

The suspension manifold N_g is obtained from the direct product $N \times [0, 1]$ by identifying pairs of points of the form $(x, 0)$ and $(g(x), 1)$ for $x \in N$. The suspension flow $\varphi(x, t)$ is determined by the vector field $\frac{\partial}{\partial t}$. We have that the suspension of an Anosov diffeomorphism is an Anosov flow in the corresponding manifold. Besides, if the diffeomorphism is transitive; so is its suspension.

The manifold N_g is fibered over S^1 and the projection of the time one map onto S^1 is the identity map. Let f be a diffeomorphism preserving fibers, C^1- close to $\varphi(x, 1)$ such that the projection of f over S^1 is a Morse-Smale map. We have that f is an Axiom A diffeomorphism.

On the other hand, Bonatti and Díaz (see [1]) proved that if τ is a period of a periodic orbit of a transitive Anosov flow, then there exist an open set $\mathcal{U}$ of nonhyperbolic and transitive diffeomorphisms, and a sequence $(g_n)_{n\in\mathbb{N}}$, $g_n \in \mathcal{U}$ such that $g_n \to f_\tau$.

Recall that two flows ρ and ψ are conjugated if there exists a homeomorphism H such that H maps orbits of the flow ρ onto orbits of the flow ψ preserving the orientation given by the positive time direction. Both flows are flow equivalent if H preserves the time, i.e. $\rho_t(x) = H^{-1} \circ \psi_t \circ H(x)\ \forall x,\ \forall t \in \mathbb{R}$.

Our main result is:

Theorem 1. *Let M be a smooth compact riemannian manifold without boundary, $dim(M) = 3$ or 4. If the time one map of a transitive Anosov flow is C^1-approximated by Axiom A diffeomorphisms, then it is flow equivalent to a suspension of an Anosov diffeomorphism .*

A codimension one Anosov flow defined on an n-manifold M is an Anosov flow such that for all $x \in M$, $dimE^s(x) = 1$ or $dimE^u(x) = 1$. It is worthwhile to note that Verjovsky (see [10]) proved that if $n > 3$ any codimension one Anosov flow is transitive (see [3] for a counterexample in dimension 3). Then we have the following

Corollary 0.1. *Let $dim(M) = 4$. The time one map of an Anosov flow ϕ can be approximated by Axiom A diffeomorphisms if and only if ϕ is flow equivalent to a suspension of an Anosov diffeomorphism.*

In a previous work we have proved that if the time one map of a transitive codimension one Anosov flow is C^1-approximated by Axiom

A diffeomorphisms verifying a technical property related to periodic points, then the flow is conjugated to a suspension of an Anosov diffeomorphism (see [4]).

Now we can remove this technical property asked for the Axiom A diffeomorphism before.

Besides, we prove flow equivalence now, instead of conjugacy showed earlier.

Let us give a rough outline of the proof of our theorem. The basic idea is that if the time one map of the flow can be approximated by Axiom A diffeomorphisms, then we can find a (topological) global section. This is done through the following steps:

We study attractors of Axiom A diffeomorphism close to the time one map of a transitive Anosov flow (see Section 1).

We examine the projection along the central foliation to conclude the following fact:
Let $\mathcal{A}$ be an attractor set of f and let $W^s(\mathcal{A})$ be its stable set, then there exists a residual set Q of $W^s(\mathcal{A})$ such that $\forall x \in Q$, $\forall y \in W^s(x)$ there exists y_x in the connected component of the central leaf of y intersection $W^s(\mathcal{A})$ verifying that $y_x \in \mathcal{A}$, i.e. $\forall y \in W^s(x)$, the connected component of ($W^c(x) \cap W^s(\mathcal{A})) \ \cap \mathcal{A} \neq \emptyset$ (see Section 2).

We use this fact to construct, in Section 3, a connected and closed set, E, included in the universal covering of M, $\widetilde{M}$, satisfying that $\mathcal{A} \subset \Pi(E)$, where $\Pi : \widetilde{M} \to M$ is the canonical projection. We prove that E is Γ-invariant or $\Gamma(E) \cap E = \emptyset$, for every covering transformation Γ and $\Pi(E)$ is a compact set in M.
In Section 4, we define an equivalence relation in $\widetilde{M}$, $\sim$, and we deduce that $E/\sim$ is a closed hypersurface and $\Pi(E/\sim)$ is a compact hypersurface in M.

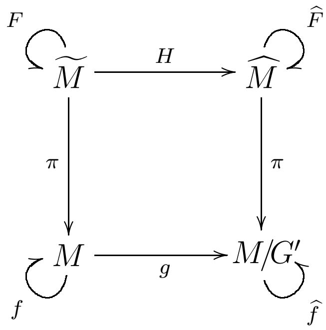
Let $\widehat{M} = \widetilde{M}/\sim$

Besides there exists $\widehat{F} : \widehat{M} \to \widehat{M}$, the canonical projection of F, where F is a lifting of f, and it holds that the central foliation of $\widehat{F}$, $\mathcal{F}^c_{\widehat{F}}$ is topologically transversal to $E/\sim$. Analogously there exists $\widehat{f} : M/G' \to M/G'$, the canonical projection of f, where M/G' is the projection of $\widehat{M}$, and it holds that the central foliation of $\widehat{f}$, $\mathcal{F}^c_{\widehat{f}}$ is topologically transversal to $\Pi(E/\sim)$.

The key of this section is to find a homeomorphism $H : \widetilde{M} \to \widehat{M}$ satisfying that H is Γ-invariant, for every covering transformation Γ and $H(W^c_F(x)) = W^c_{\widehat{F}}(H(x))$.

Hence, we obtain the existence of a homeomorphism $g : M \to M/G'$ verifying $g(W^c_f(x)) = W^c_{\widehat{f}}(g(x))$.

We have the following diagram:



Therefore $\mathcal{F}^c_f$ is topologically transversal to the compact hypersurface $g^{-1}(\Pi(E/\sim))$.

From this fact, we can find a global section for the flow and we show that the flow is topologically conjugated to a suspension of a codimension one Anosov diffeomorphism.

The flow equivalence is proved in Section 5. This proof is based on the rotation numbers of f restricted to periodic orbits.

1. Properties of basic sets.

Some of the statements of the present and the following sections have already appeared in ([4]).

We include them for completeness and because some of their proofs have been simplified.

We begin recalling some basic definitions about flows and diffeomorphisms.

Definition 1.1. *A compact ϕ_t-invariant set, $\Lambda \subset M$, is called a* **hyperbolic set for the flow** ϕ *if there exist a Riemannian metric on an open neighborhood $\mathcal{U}$ of Λ, and $\lambda < 1 < \mu$ such that for all $x \in \Lambda$ there is a decomposition*

$$T_x(M) = E^s_x \oplus E^u_x \oplus E^0_x$$

such that $\partial_t \phi(x,t)|_{t=0} \in E^0_x - \{0\}$, $dim(E^0(x)) = 1$, $D_x\phi_t(x)(E^i_x) \subset E^i_{\phi(x,t)}$, with $i = s, u$, and

$$\|D_x\phi(x,t)|_{E^s(x)}\| \leq \lambda^t \text{ with } t \geq 0$$

$$\|D_x\phi(x,t)|_{E^u(x)}\| \leq \mu^t \text{ with } t \leq 0.$$

A C^r flow $\phi : M \times I\!R \to M$, is called an **Anosov flow** *if M is a hyperbolic set for ϕ.*

Let $f : M \to M$ be a C^r diffeomorphism .

Definition 1.2. *An f-invariant set Λ is called* **hyperbolic** *if there exists a Df-invariant decomposition of $T_\Lambda M$ such that*

$$T_\Lambda M = E^s \oplus E^u$$

and $Df|E^s$ is uniformly contracting and $Df|E^u$ is uniformly expanding. More precisely, there are $c > 0$, λ, with $0 < \lambda < 1$ such that for all $x \in \Lambda$

$$\|D_x f^n | E^s(x)\| < c\lambda^n$$

and

$$\|D_x f^{-n} | E^u(x)\| < c\lambda^n.$$

A diffeomorphism $f : M \to M$ is called an **Anosov diffeomorphism** *if M is a hyperbolic set for f.*

Let $f_1 : M \to M$, the time one diffeomorphism of ϕ defined as

$$f_1(x) = \phi(x, 1),\ \forall x \in M,$$

where $\phi : M \times I\!R \to M$ is a codimension one Anosov flow if $dim(M) > 3$ (In the case that $dim(M) = 3$, codimension one property is replaced by transitivity.) Without loss of generality we may assume $dimE^s_x = n-2$ and $dimE^u_x = 1$ for all $x \in M$.

Since ϕ has no singularities, it follows that there exist f_1-invariant foliations $\mathcal{F}^{cs}$, $\mathcal{F}^{cu}$, $\mathcal{F}^{ss}$, $\mathcal{F}^{uu}$ and $\mathcal{F}^c$. Notice that the leaf of $\mathcal{F}^c$ through x is the same as the ϕ-orbit of x, and we denote it by $F^c(x)$ or $W^c_\phi(x)$ or $\mathcal{O}_\phi(x)$.

By well known properties of transitive Anosov flows, we have that

$$\{F^c(x) | F^c(x) \text{ is a closed set }\} \text{ is dense in } M.$$

$$\{F^c(x) | F^c(x) \text{ is dense in } M\} \text{ is a residual set.}$$

If $\mathcal{O}$ is a periodic orbit of ϕ, then $W^s(\mathcal{O})$ consists of all points whose foward ϕ orbits never stay far from $\mathcal{O}$ and $W^u(\mathcal{O})$ of all points whose reverse ϕ orbits never stay far from $\mathcal{O}$. Both of them are dense in M, and so are $F^{cs}(x)$ and $F^{cu}(x)$ $\forall x \in \mathcal{O}$.

Since f_1 is C^r, we have that the leaves of $\mathcal{F}^{cs}$, $\mathcal{F}^{cu}$ and $\mathcal{F}^c$ are C^r. Let $f : M \to M$ be a diffeomorphism C^1-close to f_1. The map f is plaque expansive (see [6]), there exist $\mathcal{F}^{cs}_f$, $\mathcal{F}^{cu}_f$ and $\mathcal{F}^c_f$ and there is a homeomorphism $h : M \to M$ close to the identity such that if $h(x) = x'$, then $F^c_f(x')$ is C^1-close to $F^c_{f_1}(x)$ in compact sets and the manifolds $F^{cs}_f(x')$ and $F^{cs}_{f_1}(x)$ are C^1-close in compact sets. In addition,

$$hof_1(F^c_{f_1}(x)) = foh(F^c_{f_1}(x)).$$

Therefore every leaf of $\mathcal{F}_f^c$ is invariant and every periodic point of f is in a closed leaf of $\mathcal{F}_f^c$.

According to what was mentioned above we have that

$$\{F_f^c(x)|F_f^c(x) \text{ is a closed set}\} \text{ is dense in } M$$

and

$$\{\mathcal{F}_f^c(x)|F_f^c(x) \text{ is dense in } M\} \text{ is a residual set.}$$

Let us denote by $F_f^c(x)$ or by $W^c(x)$ the leaf of the central foliation through the point x.

The metric induced by the Riemannian metric on the leaves of $\mathcal{F}_f^c$ will be denoted d^c. Analogously we define d^s and d^u.

We recall that a diffeomorphism $f : M \to M$ satisfies Axiom A if the non-wandering set $\Omega(f)$ is hyperbolic and the set of periodic points is dense in $\Omega(f)$.

From now on we will assume that f is an Axiom A diffeomorphism C^1- close to f_1.

Let $\mathcal{O} = F_f^c(x)$ where $F_f^c(x)$ is a closed curve.

The rotation number of f must be rational, because if it were irrational, there would be an hyperbolic minimal set $I \subset \mathcal{O}$ and it would be included in a basic set Λ.
If $\mathcal{O} \subset \Omega(f)$ then $\mathcal{O}$ would be in a basic set and $f|_{\mathcal{O}}$ would be expansive which leads to a contradiction with the nonexistence of one dimensional expansive diffeomorphism. Let $y \in \mathcal{O}$ then $\alpha(y) = \omega(y) = I$, hence

$$y \in W^s(I) \cap W^u(I) \subset W^s(\Lambda) \cap W^u(\Lambda) \subset \Lambda,$$

therefore $y \in \Omega(f)$ which is a contradiction.

Furthermore, there exist at least two periodic orbits in $\mathcal{O}$ because f is an Axiom A diffeomorphism. All the points in $\Omega(f) \cap \mathcal{O}$ must be periodic because if there were a nonperiodic point, $x \in \Omega(f) \cap \mathcal{O}$ then the invariance of $\Omega(f) \cap \mathcal{O}$ implies that $\alpha(x)$ and $\omega(x)$ would be periodic points of different indices so they would be in different basic sets. Summarizing we have proved

Lemma 1.1. *If $\mathcal{O} = F_f^c(x)$ is a closed curve then*

- *the rotation number is rational*
- *there exists at least two periodic orbits in $\mathcal{O}$.*

From now on, we choose an orientation for $\mathcal{F}^c$, and denote C_b^a the curve included in a central foliation leaf, between a and b.

We will consider the connected component of $\mathcal{F}^c(a)$ between a and b in the positive direction from a, in the case that $\mathcal{F}^c(a)$ is a closed curve.

We will assume that $C^x_{f(x)}$ is the connected component of $W^c(x)$ between x and $f(x)$, in such a way that length of $C^x_{f(x)}$ is close to the length of the ϕ-orbit of x, between x and $\phi(x,1)$. This implies that in finitely many cases of closed central manifold, $C^x_{f(x)}$ winds around itself more than once. Notice that in these cases the definition of $C^x_{f(x)}$ does not agree with the previous one.

Let us recall that there exists a finite number of attractors (repellers) whose basin of attraction (repulsion) are open since f is Axiom A.
Let us show some elementary properties of attractor basic sets.
Let $\mathcal{A}$ denote an attractor basic set of the spectral decomposition of f. Notice that $\mathcal{A} \neq M$ because f can not be an Anosov diffeomorphism. There is no loss of generality if we consider that $\mathcal{A}$ is connected.

Lemma 1.2. $Dim(W^s(x)) = n-1, \forall x \in \mathcal{A}$

We have assumed that $dim(E^s_\phi) = n-2$, then as f is C^1-close to f_1 we have that $dim(W^s(x)) = n-1$ or $dim(W^s(x)) = n-2$ for all $x \in \Omega(f)$.

Let $x \in \mathcal{A} \cap per(f)$, where $per(f)$ is the set of f-periodic points.

Suppose that $dim(W^s(x)) = n-2$.
Since $\mathcal{A}$ is an attractor, $W^u(x) \subset \mathcal{A}$; hence $F^c_{loc}(x) \subset W^u(x) \subset \mathcal{A}$. The set $\mathcal{A}$ is closed and f-invariant so there exists $x' \in F^c(x) \cap \mathcal{A} \cap per(f)$. But $dim(W^s(x')) = n-1$ since $dim(W^s(x)) = n-2$. It follows that there exist two periodic points of different indices in $\mathcal{A}$, which is impossible. ■

Lemma 1.3. *For every closed curve $\mathcal{O}$ in $\mathcal{F}^c$ there exists a periodic point $p \in \mathcal{A} \cap \mathcal{O}$.*

Since $\mathcal{O}$ is closed, $W^s(\mathcal{O})$ is dense in M and $W^s(\mathcal{A})$ is an open set, there exist y in $W^s(\mathcal{O}) \cap W^s(\mathcal{A})$ and $y' \in W^{ss}(y) \cap \mathcal{O}$ such that $y' \in W^s(\mathcal{A})$.
As $y' \in \mathcal{O}$, $y' \in W^s(p)$ for a periodic point $p \in \mathcal{O}$. Then $p \in \mathcal{A} \cap \mathcal{O}$. ■

Remark 1.1. *Let $K = \max_{x \in M} length(C^x_{f(x)})$. K is finite because M is compact and the map $g : M \to I\!R$ such that every $x \in M$ is mapped into the length of $C^x_{f(x)}$ is continuous.*

The previous lemma implies that in every segment γ of central closed curve with $length(\gamma) \geq K$, there exists a periodic point $p \in \gamma \cap \mathcal{A}$.

Analogously we have that in every segment γ of central closed curve with $length(\gamma) \geq K$, there exists a periodic point $p \in \gamma \cap \Lambda$, where Λ is a repeller set.

Corollary 1.1. *Every leaf of $\mathcal{F}^c$ intersects $\mathcal{A}$.*

Let $\gamma \subset \mathcal{F}^c$ with length$(\gamma) \geq K$. Since

$$\{F_f^c(x) | F_f^c(x) \text{ is a closed set}\} \text{ is dense in } M,$$

we can choose arcs γ_n such that γ_n are included in closed leaves of $\mathcal{F}^c$, $\gamma_n \to \gamma$, and length$(\gamma_n) \geq K$. Then, there exists a sequence (p_n) such that $p_n \in \mathcal{A} \cap \gamma_n$, and any of its limit points $p \in \gamma \cap \mathcal{A}$. ■

Lemma 1.4. *In every leaf of $\mathcal{F}_f^c$ there exists at least one point outside of $W^s(\mathcal{A})$.*

If $F_f^c(x)$ is closed, by the remark of lemma 1.3 we have that in every segment γ of central closed curve with length$(\gamma) \geq K$, there exists a periodic point $p \in \gamma$ such that $p \notin W^s(\mathcal{A})$.

Suppose that there exists a curve $\gamma \subset F_f^c(x)$ such that $\gamma \subset W^s(\mathcal{A})$ and length$(\gamma) \geq K + 1$.
Then there exists an open set V, $V \subset W^s(\mathcal{A})$ and $\gamma \subset V$. There exists $y \in V$ such that $W^c(y)$ is closed, and $W^c(y) \cap V$ has length greater or equal than K. This gives the existence of a point $p \in W^c(y) \cap V$, such that $p \notin W^s(\mathcal{A})$ which is a contradiction. ■

Note that we have proved that every leaf of the central foliation "goes away " from the basin of attraction of any attractor.

Lemma 1.5. *No arc γ, γ included in $F_f^c(x)$ for any x, satisfies $\gamma \subset \mathcal{A}$.*

Suppose the statement is false, i.e. there exists $\gamma \subset W_{loc}^c(x)$ such that $\gamma \subset \mathcal{A}$. Since $\gamma \subset \mathcal{A} \subset W^s(\mathcal{A})$, then the negative iterates of γ are included in $\mathcal{A}$ and the length of them grow exponentially.
Let $z \in \alpha(x)$ then $z \in \mathcal{A}$ and by the proof of lemma 1.4, $W^c(z)$ has to intersect $\partial(W^s(\mathcal{A}))$, but $W^c(z) \subset \mathcal{A} \subset W^s(\mathcal{A})$, which yields a contradiction. ■

Remark. *All the above lemmas admit versions for repeller basic sets and the proofs are analogous. In fact, if Λ is a repeller basic set, then for $x \in \Lambda$, $Dim(W^s(x)) = n-2$, every leaf of $\mathcal{F}_f^c$ intersects Λ, in every leaf of $\mathcal{F}_f^c$ there exists a point outside of $W^u(\Lambda)$, and no γ included in $F_f^c(x)$ satisfies $\gamma \subset \Lambda$.*

2. Properties of the projection along the central foliation.

Let us introduce the following maps:

Definition 2.1. *Let $S_A : W^s(\mathcal{A}) \to \partial W^s(\mathcal{A})$ be a map such that, for every x in the basin of the attractor $\mathcal{A}$, $S_A(x)$ is the nearest point in its central leaf in the positive direction verifying that it is not in the basin of attraction of $\mathcal{A}$.*

Definition 2.2. *Let $\tilde{S}_A : W^s(\mathcal{A}) \to \partial W^s(\mathcal{A})$ be the map analogous to S_A, but in the negative direction of the central foliation .*

Definition 2.3. *Let $S : \mathcal{A} \to \partial W^s(\mathcal{A})$ be the restriction of S_A to $\mathcal{A}$ and $\tilde{S} : \mathcal{A} \to \partial W^s(\mathcal{A})$ the restriction of $\tilde{S}_A$ to $\mathcal{A}$.*

By lemma 1.4 we have that the previous maps are well defined.

Let $\widetilde{W^c(x)} = C_{S_A(x)}^{\tilde{S}_A(x)}$ denote the connected component of $W^c(x) \cap W^s(\mathcal{A})$ which contains x.

Let $l : \mathcal{A} \to I\!\!R$, $l(x) = length(C^x_{S(x)})$.

Lemma 2.1. *l is lower semicontinuous.*

We have that $C^x_{S(x)} - \{S(x)\} \subset W^s(\mathcal{A})$ and $W^s(\mathcal{A})$ is an open set. The central foliation is a C^1- lamination because f is C^1-close to the time one map of an Anosov flow (see [6]), hence for all $\epsilon > 0$ there exists a neighborhood U_x of x such that if $y \in U_x$ then the curve $C^y_{y'}$ included in $\mathcal{F}^c(y)$ with $length(C^y_{y'}) = l(x) - \epsilon$ is included in $W^s(\mathcal{A})$. Then $l(y) \geq l(x) - \epsilon$ which proves that l is a semicontinuous map. ■

Since $l : \mathcal{A} \to I\!\!R$ is semicontinuous, the set R of points of continuity of l is a residual set. Let $\Phi : M \times I\!\!R_{\geq 0} \to M$ such that $\Phi(x, l) = z$, if $z \in W^c(x)$, z is in the positive direction of $W^c(x)$ and $length(C^x_z) = l$. Φ is a continuous map then

$$S(x) = \Phi(x, l(x))$$

is continuous over R.

Without loss of generality we can assume that R is a residual set of continuity for both S and $\tilde{S}$.
Analogously there exists a residual set Q in $W^s(\mathcal{A})$ such that Q is a set of continuity for S_A and $\tilde{S}_A$.

Let us prove some properties of the map S. They are verified by $\tilde{S}$ and the proofs are analogous.

Lemma 2.2. *$S(R)$ is f-invariant.*

Let $x \in R$, $y = S(x)$. For all $z \in C^x_y - \{y\}$, we have that $z \in W^s(\mathcal{A})$, $f(z) \in W^c(f(x))$ and $f(z) \in W^s(\mathcal{A})$. Since $f(y) \in \partial W^s(\mathcal{A})$ it follows that $f(y) = S(f(x))$. Replacing f by f^{-1} we conclude that

$$f(S(R)) = S(R).$$

■

Lemma 2.3. *$\overline{S(R)}$ is transitive and $\overline{S(R)} \subseteq \Omega(f)$.*

Since the set of dense orbits is a residual set in $\mathcal{A}$, we have that there exists $x \in R$ such that its orbit is dense in R.

By continuity of $\mathcal{F}^c$, we conclude that the image of a dense orbit is dense in $S(R)$ therefore $\forall y \in S(R)$ we have that $y \in w(S(x))$ then $y \in \Omega(f)$.

We have proved that $S(R) \subseteq \Omega(f)$. ∎

Corollary 2.1. *From the above properties we conclude that $\overline{S(R)}$ is included in Λ, a basic set of the spectral decomposition of f.*

Lemma 2.4. *For all $y \in S(R)$, $dim(W^s(y)) = n-2$.*

Let $y = S(x)$ with $x \in \mathcal{A}$; since $dim(W^{ss}(y)) = n-2$ and $dim(W^{uu}(y)) = 1$, $dim(W^s(y)) = n-1$ or $n-2$, but by lemma 1.2 if $z \in C_y^x - \{y\}$ then $z \in W^s(x)$. Then

$$W_\epsilon^c(y) = \{z \in W^c(y) \text{ such that } d^c(z,y) < \epsilon\}$$

can not be included in $W^s(y)$ and we can assert that $dim(W^s(y)) = n-2$. ∎

Lemma 2.5. *The set of periodic points in $\mathcal{A} \setminus R$ is nowhere dense in $\mathcal{A}$.*

In order to prove the lemma it is enough to prove:

Let $(p_n)_{n\in N}$ be a sequence of periodic points such that S is not continuous at p_n and $p_n \to x$. Then S is not continuous at x.

Let $q_n = S(p_n)$.

Since p_n is a point of discontinuity, there exist $\alpha > 0$ and $(r_{n_k}) \subset \mathcal{A}$ such that $\lim_{k\to\infty} r_{n_k} = p_n$ and

$$length(C_{S(r_{n_k})}^{r_{n_k}}) > length(C_{S(p_n)}^{p_n}) + \alpha$$

and for any ϵ with $0 < \epsilon < \frac{\alpha}{2}$ there exist $(s_{n_k}) \subset R$ such that $\lim_{k\to\infty} s_{n_k} = p_n$ and

$$length(C_{S(s_{n_k})}^{s_{n_k}}) \geq length(C_{S(r_{n_k})}^{r_{n_k}}) - \epsilon > length(C_{S(p_n)}^{p_n}).$$

It follows that there exists a periodic limit point of $S(s_{n_k})$, q'_n, in $W^c(p_n)$.

Both q_n and q'_n are in $W^c(p_n) \cap \overline{S(R)}$, are periodic and $dim(W^s(q_n)) = dim(W^s(q'_n)) = n-2$. Since q_n and q'_n are in the same closed leaf of $\mathcal{F}^c$, it follows that there exists a periodic point p'_n, such that $p'_n \in C_{q'_n}^{q_n}$ and $dim(W^s(p'_n)) = n-1$.

Suppose, contrary to our claim, that S is continuous at x.

From $p_n \to x$ we conclude that $q_n \to S(x)$ by the continuity of S at x.(See Figure 1)

Besides $q'_n \to S(x)$ *because there exist* $(s_{n_k}) \subset R$ *such that* $\lim_{k\to\infty} s_{n_k} = p_n$ *and* $\lim_{k\to\infty} S(s_{n_k}) = q'_n$. *Letting a convenient subsequence* $k(n)$, *we can assert that*

$$\lim_{n\to\infty} s_{n_{k(n)}} = x \text{ and } \lim_{n\to\infty} S(s_{n_{k(n)}}) = S(x)$$

by the continuity of S *at* x. *This gives* $q'_n \to S(x)$.

Then $dist(q_n, q'_n) \to 0$ *when* $n \to \infty$ *and* $d^c(q_n, q'_n) \to 0$ *when* $n \to \infty$.

But $d^c(q_n, q'_n) > \min\{d^c(p'_n, q'_n), d^c(p_n, q'_n)\}$ *and this leads to a contradiction because* p'_n *and* q'_n *(or* p_n *and* q'_n *) are in different basic sets because they have different indices.*

We have proved that S *is not continuous at* x. ■

Observe that as a consequence we have that for all $x \in \mathcal{A}$ *there exists a sequence of periodic points* $(p_n)_{n\in N} \subset R$ *such that* $p_n \to x$.

Lemma 2.6. $S(W^s(x)) \subset W^s(S(x))$.

Let $x \in \mathcal{A}$, $y \in W^s(x) \cap \mathcal{A}$. *Suppose that* $S(y) \notin W^s(S(x))$.
Since $S(y) \in F^{cs}(x)$ *there exists* $z = W^s(S(y)) \cap W^c(x)$. *We have that* $\forall w \in \partial(W^s(\mathcal{A})), W^s(w) \subset \partial(W^s(\mathcal{A}))$, *then* $W^s(S(x)) \subset \partial(W^s(\mathcal{A}))\ \forall x \in \mathcal{A}$, *and* $z \in \partial(W^s(\mathcal{A}))$, *but this contradicts the definition of* S. ■

Lemma 2.7. *If* x *is a point of continuity of* S, *then all the points in* $W^s(x) \cap \mathcal{A}$ *are continuity points of* S.

Let x *be a point of continuity of* S, $y \in W^s_{loc}(x) \cap \mathcal{A}$. *We first prove that* y *is a continuity point of* S.
Let $\{y_n\}_{n\in N} \subset \mathcal{A}$, *such that* $\lim_{n\to\infty} y_n = y$. *There exists* $x_n = W^s_{loc}(y_n) \cap W^u(x)$ *and* $y_n \in W^s(x_n)$. *By continuity of the stable foliation, we have* $\lim_{n\to\infty} x_n = x$, *and by continuity of* S *at* x *we conclude that* $\lim_{n\to\infty} S(x_n) = S(x)$.
From $y_n \in W^s(x_n)$, *and the above lemma, it follows that* $S(y_n) \in W^s(S(x_n))$, *hence* $S(y_n) = W^s_{loc}(S(x_n)) \cap W^c(y_n)$.
By the continuity of W^s *and* W^c *we have that:*
$\lim_{n\to\infty} W^s_{loc}S(x_n) = W^s_{loc}S(x)$ *and* $\lim_{n\to\infty} W^c(y_n) = W^c(y)$; *hence*

$$\lim_{n\to\infty} S(y_n) = W^s_{loc}S(x) \cap W^c(y) = S(y).$$

We have proved that $\forall y \in W^s_{loc}(x) \cap \mathcal{A}$, S *is continuous at* y *i.e.* $S|_{W^s_{loc}(x)\cap\mathcal{A}}$ *is continuous.*
Now, if $z \in W^s(x) \cap \mathcal{A}$ *there is* $N > 0$ *such that* $f^N(z) \in W^s_{loc}(f^N(x)) \cap \mathcal{A}$ *and the previous argument still applies.* ■

Remark. *Note that lemmas 2.6 and 2.7 are verified not only by* S *and* $\tilde{S}$ *but also by* S_A *and* $\tilde{S}_A$. *The proofs are analogous.*

Lemma 2.8. *If $x \in \mathcal{A}$, then x is a point of continuity of S if and only if x is a point of continuity of S_A.*

We only have to prove that if $x \in \mathcal{A}$ is a point of continuity of S then it is a continuity point of S_A.
Let y be a point close to x, then $y' = W^u_{loc}(x) \cap W^s_{loc}(y)$ is a point in $\mathcal{A}$ such that $S(y')$ is close to $S(x)$ and

$$S_A(y) = W^s(S(y')) \cap W^c(y) \text{ is close to } S_A(y') = S(y').$$

Hence $S_A(y)$ is close to $S_A(x) = S(x)$. ■

Let us prove the next lemma

Lemma 2.9. *Let x be a continuity point of S_A and $\tilde{S}_A$,(i.e. $x \in Q$) then for all $y \in W^s(x)$, $\widetilde{W^c(y)} \cap \mathcal{A} \neq \emptyset$.*

Recall that $W^s_\epsilon(x) = \{y \in W^s(x)$ such that $d^s(x, y) < \epsilon\}$. Let $\epsilon > 0$ be such that $\cup_{x\in\mathcal{A}} W^s_\epsilon(x) \subset W^s(\mathcal{A})$.
Let $x \in Q$ and U_x be a neighborhood of x such that for all $y \in U_x$ we have that length$(\widetilde{W^c(y)})$ is close enough to length$(\widetilde{W^c(x)})$, and let $y \in U_x \cap W^s(x)$. Since $\widetilde{W^c(y)} \subset W^s(\mathcal{A})$ and $W^s(\mathcal{A})$ is open, there exists a neighborhood of $\widetilde{W^c(y)}$, $\mathcal{V}$, such that $\mathcal{V} \subseteq W^s(\mathcal{A})$ and $\mathcal{V} \subset \cup_{z\in\mathcal{U}_x}(\widetilde{W^c(z)})$, in such a way that if $z \in \mathcal{V}\cap\mathcal{A}$ then length$(\widetilde{W^c(z)})$ is close enough to length$(\widetilde{W^c(y)})$.
By the density of the closed leaves in the central foliation, there exists a curve ζ in $\mathcal{V}$, included in a closed leaf of the central foliation, $\mathcal{O}$ such that $\zeta = \mathcal{O} \cap W^s(\mathcal{A})$.

There exists a periodic point p such that $p \in \zeta \cap \mathcal{A}$, $\zeta = \widetilde{W^c(p)}$ and since S_A and $\tilde{S}_A$ are continuous at y by the remark of lemma 2.7, the lengths of $\widetilde{W^c(y)}$ and ζ are close; and the lengths of the curves $C^p_{S_A(p)}$, and $C_p^{\tilde{S}_A(p)}$ are greater than the ϵ previously defined.

Then, considering open sets $\mathcal{V}_n$ such that $\mathcal{V}_n \to \widetilde{W^c(y)}$, we can assert that there exist curves $\zeta_n \subset \mathcal{V}_n$ and periodic points $p_n \in \zeta_n \cap \mathcal{A}$ such that the lengths of $\widetilde{W^c(y)}$ and ζ_n are close; and the lengths of the curves $C^{p_n}_{S_A(p_n)}$, and $C_{p_n}^{\tilde{S}_A(p_n)}$ are greater than ϵ.

Since ζ_n converges to $\widetilde{W^c(y)}$ and the distance of p_n to $\partial(W^s(\mathcal{A}))$ is bounded away from 0, there exists a limit point p of p_n such that $p \in \mathcal{A} \cap \widetilde{W^c(y)}$.

We have proved that if $x \in Q$ then

$$\forall y \in W^s_{loc}(x), \exists p \in \widetilde{W^c(y)} \cap \mathcal{A}.$$

Successive applications of this proceeding enables us to conclude that if $x \in Q$

$$\forall y \in W^s(x), \exists p \in \widetilde{W^c(y)} \cap \mathcal{A}.$$

■

Remark. *By this lemma we have that if $x \in \mathcal{A}$ is a continuity point of S and $\widetilde{S}$, then*

$$\forall y \in W^s(x), \widetilde{W^c(y)} \cap \mathcal{A} \neq \emptyset.$$

Corollary 2.2. $\Lambda = \overline{S(R)}$ *is a repeller set.*

Let $x \in Q \cap \mathcal{A}$, $z \in W^s(S(x))$ and $z' = W^c(z) \cap \widetilde{W^{ss}(x)}$. Since $z' \in W^s(x)$ with $x \in Q$, then by lemma 2.9 there exists $q \in \widetilde{W^c(z')} \cap \mathcal{A}$; hence $S(q) = z$ and $z \in S(R)$. Then

$$\forall x \in Q \cap \mathcal{A}, W^s(S(x)) \subseteq S(R).$$

We have proved that $\overline{S(R)}$ is included in a basic set Λ. Now, if $y = S(x)$ with $x \in \mathcal{A} \cap Q$ then

$$W^s(y) \subseteq S(R) \subseteq \overline{S(R)} \subseteq \Lambda \subseteq \overline{W^s(y)}.$$

It follows that $\overline{S(R)}$ is a basic set, and since it contains a stable manifold we have that $\Lambda = \overline{S(R)}$ is a repeller set. ■

Lemma 2.10. *Let Λ be a basic set and $x \in \Lambda$.*

1. *If $dim(W^s(x)) = n-1$ then there is a finite number of points of Λ in the connected component of $W^c(x) \cap W^s(\Lambda)$ that contains x.*
2. *If $dim(W^s(x)) = n-2$ then there is a finite number of points of Λ in the connected component of $W^c(x) \cap W^u(\Lambda)$ that contains x.*

We will prove just the first statement.
Suppose that it is false. Then we can choose $\{x_i\}$ in $\Lambda \cap W^s(\Lambda) \cap W^c(x)$, such that $x_1 < x_2 < \ldots < x_l < \ldots$ in the given orientation of $W^c(x)$. There exists $k > 0$ such that $f^{-1}|_{W^c_k(x)}$ "expands", $\forall x \in \mathcal{A}$. Then there exists $n_1 \in I\!N$ verifying that length$(f^{-n_1}(C^x_{x_1})) > k$, for all $n \geq n_1$. There exists $n_2 \in I\!N$ such that length$(f^{-n_2}(C^{x_1}_{x_2})) > k$, for all $n \geq n_2$. Let l_0 such that $kl_0 > K + 1$, where $K = \max_{x \in M}$ length$(C^x_{f(x)})$

We continue in this way obtaining $n_3, \ldots, n_{l_0}$
Let $N = max\{n_1, \ldots, n_{l_0}\}$, *then*

$$length(f^{-N}(C^x_{x_{l_0}})) > kl_0 > K + 1$$

Hence, as in the proof of lemma 1.4 we conclude that there exists $p \in f^{-N}(C^x_{x_{l_0}})$ *such that* $p \in \partial W^s(\Lambda)$ *and therefore* $f^N(p) \in \partial W^s(\Lambda)$ *and* $f^N(p) \in C^x_{x_{l_0}} \subseteq W^s(\Lambda)$; *which is a contradiction.*
We have actually proved that there are no more than $[\frac{K+1}{k}]$ *points of* Λ *in the connected component of* $W^s(\Lambda) \cap W^c(x)$. ∎

3. Properties of the set E

If $x \in \mathcal{A}$, $f(x) \in \mathcal{A}$ *then there exists* $z \in W^c(x)$ *such that* $z \in \mathcal{A}$, *and* $C^x_z \cap \mathcal{A} = \{x, z\}$.

Let $\widetilde{M}$ *be the universal covering of* M, *and* $\Pi : \widetilde{M} \to M$ *the canonical projection. It is a well known fact that* $\widetilde{M}$ *is homeomorphic to* $\mathbb{R}^n$ *(See, for instance* [11] *).*

Let p *be a fixed point of* f *(or* f^k*) verifying that* p *is a continuity point of* S *and* $\widetilde{S}$, *then by the remark of lemma 2.9 we have that* $\forall y \in W^s(p)$ *there exists at least* $z \in W^s(p) \cap \widetilde{W^c(y)}$ *such that* $z \in \mathcal{A}$. *Let* $\widetilde{p} \in \widetilde{M}$ *such that* $\Pi(\widetilde{p}) = p$.

Let F *be a lifting of* f *such that* $F(\widetilde{p}) = \widetilde{p}$.

Let $q, q' \in \Lambda$ *where* Λ *is a repeller, with* $p \in C^q_{q'}$, $C^q_{q'} \cap \mathcal{A} = \{p\}$ *and* $C^q_{q'} \cap per(f) = \{q, p, q'\}$. *We will call* $\widetilde{\mathcal{A}} \subset \widetilde{M}$ *the set such that* $\Pi(\widetilde{\mathcal{A}}) = \mathcal{A}$ *and* $\widetilde{\Lambda} \subset \widetilde{M}$ *the set such that* $\Pi(\widetilde{\Lambda}) = \Lambda$.
Let D *a Riemannian metric in* $\widetilde{M}$ *induced by* d, *where* d *is the Riemannian metric in* M. *We define*

$$W^s_F(\psi) = \{\eta \in \widetilde{M} | D(F^n(\eta), F^n(\psi)) \to 0, \text{ for } n \to \infty\}$$

and

$$W^s_{F,\epsilon}(\psi) = \{\eta \in \widetilde{M} | D(F^n(\eta), F^n(\psi)) < \epsilon, \text{ for } n \geq 0\}$$

Analogously we define $W^u_F(\psi)$ *and* $W^u_{F,\epsilon}(\psi)$. *We denote by* $W^c_F(x')$ *the connected component of* $\Pi^{-1}(W^c_f(x))$ *that contains* x' *and by* $W^c_{F,\epsilon}(x') = \{y \in W^c_F(x') | D(y, x') < \epsilon\}$.

We will call $\overline{C^a_b}$ *the connected component of* $W^c_F(a)$ *that contains* a *such that* $\Pi(\overline{C^a_b}) = C^{\Pi(a)}_{\Pi(b)}$.

Let $\widetilde{q'}, \widetilde{q} \in W^c_F(\widetilde{p})$ *verifying* $\Pi(\widetilde{q'}) = q'$ *and* $\Pi(\widetilde{q}) = q$.

Let

$$B(\widetilde{q}) = \cup_{x \in W^s_F(\widetilde{q})} W^c_{F,\delta}(x),$$

let $D_{\widetilde{p}}$ *be the connected component of* $\Pi^{-1}(W^s(p))$ *that contains* $\widetilde{p}$*, and* $D^n_{\widetilde{p}} = D_{\widetilde{p}} \setminus \cup_{k=0}^n F^k(B(\widetilde{q}) \cup B(\widetilde{q'}))$ *(see Figure 2).*

Since $W^s_F(\widetilde{q}), W^s_F(\widetilde{q'}) \subset \widetilde{\Lambda}$ *and the local central manifold is expanding on* Λ *by the last remark of section 1, we have that there exists a sequence* $((n_i))$ *such that* $((D^{n_i}_{\widetilde{p}}))$ *is a sequence of decreasing closed and connected sets, then we define* $C(\widetilde{p}) = \cap_{n\in\mathbb{N}} D^n_{\widetilde{p}}$*.*

It follows that $C(\widetilde{p})$ *is an* F*-invariant, closed and connected set.*

Besides, we have that if $y \in D_{\widetilde{p}} \cap \widetilde{\mathcal{A}}$ *then* $y \in C(\widetilde{p})$ *and* $\forall y \in W^s_F(\widetilde{q})$*, if* $y' = W^c_F(y) \cap W^s_F(\widetilde{q'})$ *then there exists at least* $\overline{z} \in \overline{C^y_{y'}} \cap \widetilde{\mathcal{A}}$ *. It follows that* $\forall y \in D_{\widetilde{p}}$*,* $W^c_F(y) \cap W^s_F(\widetilde{p}) \cap C(\widetilde{p})$ *is a point in* $\widetilde{\mathcal{A}}$*, or it is a segment with end points in* $\widetilde{\mathcal{A}}$*.*

Let us denote by E *the* F*-invariant set*

$$E = \cup_{x\in C(\widetilde{p})} W^u_F(x).$$

Notice that E *is a connected and closed set.*

The interior of $C(\widetilde{p})$ *is empty; so is the interior of* E*.*

Lemma 3.1. *Let* $\Gamma : \widetilde{M} \to \widetilde{M}$ *be a covering transformation. Then, either* $\Gamma(E) \cap E = \emptyset$ *or* E *is* Γ*-invariant.*

Suppose that there exist $x, y \in E$ *such that* $\Gamma(x) = y$*. Since* Γ *preserves* $\mathcal{F}^u_F$*, we have that* $\Gamma(W^u_F(x)) = W^u_F(y)$*.*

Let a *be a point in* E *such that* a *is close to* x*, then there exists* $\alpha \in C(\widetilde{p})$ *such that* $a \in W^u_F(\alpha)$*. Therefore* $\Gamma(a) \in \Gamma(W^u_F(\alpha)) = W^u_F(\Gamma(a))$ *and since* $W^u_F(\Gamma(a))$ *is close to* $W^u_F(y)$*, it follows that* $W^u_F(\Gamma(a)) \cap D_{\widetilde{p}} \neq \emptyset$*. Let* $z = W^u_F(\Gamma(a)) \cap D_{\widetilde{p}}$*.*

In the case that $a \in \widetilde{\mathcal{A}}$ *we will prove that* $\Gamma(a) \in E$*.*

Suppose that $z \notin C(\widetilde{p})$*, then there exists* $n \in \mathbb{N}$ *such that* $F^{-n}(z) \in B(\widetilde{q}) \cup B(\widetilde{q'})$ *and* $z \in W^u_F(\Lambda)$*. It follows that* $\Gamma(a) \in W^u_F(\widetilde{\Lambda})$*, but* $\Pi(\Gamma(a)) = \Pi(a) \in \mathcal{A}$*, which is a contradiction.*

In the case that $a \notin \widetilde{\mathcal{A}}$*, there exist* $\mu, \nu \in C(\widetilde{p}) \cap \widetilde{\mathcal{A}}$*, such that* $\alpha \in \overline{C^\mu_\nu}$*, and* $a \in \overline{C^{\mu'}_{\nu'}}$ *with* $\mu' \in W^u_F(\mu)$ *and* $\nu' \in W^u_F(\nu)$*. Since* Γ *preserves* $\mathcal{F}^c_F$*, we have that* $\Gamma(a) \in \overline{C^{\Gamma(\mu')}_{\Gamma(\nu')}}$*, and from* $\Gamma(\mu'), \Gamma(\nu') \in \widetilde{\mathcal{A}}$ *we conclude that* $z \in C(\widetilde{p})$ *and finally* $\Gamma(a) \in E$*.*

We have proved that $E_\Gamma = \{x \in E \text{ such that } \Gamma(x) \in E\}$ *is an open set.*

Let (x_n) *be a sequence of points in* E *such that* $\Gamma(x_n) \in E$ *and* $x_n \to x$*. Since* Γ *is continuous we have that* $\Gamma(x_n) \to \Gamma(x)$ *and from the closedness of* E *we have that* $\Gamma(x) \in E$*. We have proved that* E_Γ *is an* *open and closed set, then the connectedness of E implies that $E_\Gamma = E$ or $E_\Gamma = \emptyset$.* ■

Proposition 3.1. *$\Pi(E)$ is a compact set.*

Since $C^q_{q'} \cap \mathcal{A} = \{p\}$, we have that there exist γ, $\delta > 0$, such that if $y \in \mathcal{A} \cap W^s(p)$ and $d(\widetilde{W^c(p)}, \widetilde{W^c(y)}) < \gamma$ then $d(p, y) < \delta$.
Let $B(p, \gamma, \delta) = \{x \in W^s(\mathcal{A}) | d(\widetilde{W^c(p)}, \widetilde{W^c(x)}) < \gamma, d(p, x) < \delta\}$.

Let us prove that there exists $L > 0$ such that for all $x \in \Pi(E)$, $W^{uu}_{f,L}(x) \cap B(p, \gamma, \delta) \neq \emptyset$, where $W^{uu}_{f,L}(x) = \{y \in W^{uu}_f(x) | d^u(y, x) \leq L\}$.

We begin by proving that there exists $L > 0$ such that for all $x \in \mathcal{A}$, $W^{uu}_{f,L}(x) \cap B(p, \gamma, \delta) \neq \emptyset$. Suppose that for all L_n there exist $x_n \in \mathcal{A}$ such that $W^{uu}_{f,L_n}(x_n) \cap B(p, \gamma, \delta) = \emptyset$. Then there exists a limit point of x_n, y, such that $y \in \mathcal{A}$ and $W^{uu}_f(y) \cap B(p, \gamma, \delta) = \emptyset$. This contradicts the density of $W^{uu}_f(y)$.
Let $x \in \Pi(E)$ such that $x \notin \mathcal{A}$. Then there exist $a, b \in \mathcal{A}$ such that $x \in C^a_b$. We have proved that there exists $a' \in W^{uu}_{f,L}(a) \cap B(p, \gamma, \delta)$. Let $b' = W^{uu}_{f,L}(b) \cap W^c_{f,loc}(a')$. We claim that $C^{a'}_{b'} \subset B(p, \gamma, \delta)$. We have that $b' \in \mathcal{A}$, and $b' \in W^s(p)$ because if not, there were $w \in C^{a'}_{b'} \cap \Lambda$ then there were $\widetilde{w} \in C(\widetilde{p}) \cap \widetilde{\Lambda}$ which is absurd. Since $d(\widetilde{W^c(p)}, \widetilde{W^c(a')}) = d(\widetilde{W^c(p)}, \widetilde{W^c(b')}) < \gamma$ then $d(b', p) < \delta$, hence $b' \in B(p, \gamma, \delta)$; therefore $W^{uu}_{f,L}(x) \cap C^{a'}_{b'} \in B(p, \gamma, \delta)$.

We have proved that there exists $L > 0$ such that for all $x \in \Pi(E)$, $W^{uu}_{f,L}(x) \cap B(p, \gamma, \delta) \neq \emptyset$. In fact, there exists $L > 0$ such that for all $x \in \Pi(E)$, $W^{uu}_{f,L}(x) \cap B(p, \gamma, \delta) \cap W^s_{loc}(p) \neq \emptyset$. Then

$$\Pi(E) \subset \bigcup_{x \in \overline{B(p,\gamma,\delta)}} W^{uu}_{f,L}(x),$$

where $\overline{B(p, \gamma, \delta)}$ is the closure of $B(p, \gamma, \delta)$. In fact,

$$\Pi(E) \subset \bigcup_{x \in \overline{B(p,\gamma,\delta)} \cap \Pi(C(\widetilde{p}))} W^{uu}_{f,L}(x),$$

and since $C(\widetilde{p})$ is closed and Π is a local homeomorphism, we have that $\Pi(E)$ is included in a compact set. Besides

$$\bigcup_{x \in \overline{B(p,\gamma,\delta)} \cap \Pi(C(\widetilde{p}))} W^{uu}_{f,L}(x) \subseteq \Pi(E),$$

then $\Pi(E)$ is a compact set. ■

4. Existence of a global section of ϕ

We will define an equivalence relation on $\widetilde{M}$ in the following way:
If $x, y \in E$, we say that $x \sim y$ if $x, y \in \overline{C_b^a} \subset E$, where $a, b \in \widetilde{\mathcal{A}}$.
If $x', y' \in \Gamma(E)$ then $x' = \Gamma(x)$ and $y' = \Gamma(y)$, we say that $x' \sim y'$ if $x \sim y$.

Since $C(\widetilde{p})/\sim$ is connected and for all $y \in D_{\widetilde{p}}$ we have that $(W_F^c(y) \cap C(\widetilde{p}))/\sim$ is a point, it follows that $C(\widetilde{p})/\sim$ is a curve if $dim(M) = 3$, or $C(\widetilde{p})/\sim$ is a surface if $dim(M) = 4$.
Hence $E/\sim$ is homeomorphic to $\mathbb{R} \times \mathbb{R}$ or $\mathbb{R}^2 \times \mathbb{R}$ respectivelly.

Let $[x] = \{y \in \widetilde{M} | x \sim y\}$, and $G = \{[x]/x \in \widetilde{M}\}$. Let $\widehat{M} = \widetilde{M}/\sim$. There exists $\widehat{F} : \widehat{M} \to \widehat{M}$, the canonical projection of F, such that $\widehat{F}$ preserves the dynamical properties of F. We claim that G is an upper semicontinuous decomposition of $\widetilde{M}$.

We have that $\forall x \in \widetilde{M},\ [x] = x$ or $[x]$ is a closed arc $\overline{C_b^a}$, then $[x]$ is a compact set. Let U be an open set in $\widetilde{M}$ such that $[x] \subset U$.

In the case that $x \in E$, we suppose, contrary to our claim, that there exists a sequence $(y_n)_{n \in \mathbb{N}}$ such that $y_n \to x$ and $[y_n]$ is not included in U, then there exist $a_{n_k} \in [y_{n_k}] \cap \widetilde{\mathcal{A}} \cap E$ verifying $a_{n_k} \notin U$ and $a_{n_k} \to a$ because lengths of $[y_{n_k}]$ are locally bounded.

It follows that $a \in W_F^c(x) \cap \widetilde{\mathcal{A}} \cap E$, but $a \notin U$ and therefore $a \notin [x]$ which is a contradiction.

The case that $x \in \Gamma(E)$ is equivalent because Γ is a homeomorphism; otherwise there exists a neighbourhood V of x such that for all $y \in V,\ [y] = y$. So, we have proved that there exists an open set, V, such that $[x] \subset V \subset U$, verifying that $\forall y \in V,\ [y] \subset U$.

We have proved that G is an upper semicontinuous decomposition of $\widetilde{M}$ then $\widehat{M} = \widetilde{M}/G$ is metrizable (therefore there exists a distance $\widetilde{D}$ compatible with the quotient topology $\widetilde{M}/G$) . Besides $\Pi_1 : \widetilde{M} \to \widehat{M}$ is a closed map, then $\Pi_1(E) = E/\sim$ is a closed hypersurface.

Let $G' = \{\Pi[x]\ /\ [x] \in G\}$. By abuse of notation, we continue to write $\Pi : \widehat{M} \to M/G'$, for a covering projection.

Since Π_1 is Γ-invariant, if Γ is a covering transformation, we can consider $\Pi_1 : M \to M/G'$ to simplify notation.
It holds that $\Pi \circ \Pi_1(x) = \Pi_1 \circ \Pi(x),\ \forall x \in \widetilde{M}$.

It follows that $\Pi(\Pi_1(E)) = \Pi(E/\sim) = \Pi_1(\Pi(E))$ is a compact set in M/G' because Π_1 is continuous and $\Pi(E)$ is compact.

Moreover, $\Pi(E/\sim)$ is a hypersurface because Π is a local homeomorphism and $\Gamma(E) \subset E$ or $\Gamma(E) \cap E = \emptyset$.

Proposition 4.1. *There exists a homeomorphism* $H : \widetilde{M} \to \widehat{M}$ *such that* $H(W^c_F(x)) = W^c_{\widehat{F}}(H(x))$

For every covering map $\Gamma : \widetilde{M} \longrightarrow \widetilde{M}$*, there exists* $\widetilde{\Gamma} : \widehat{M} \longrightarrow \widehat{M}$*, the canonical projection of* Γ*.*

Let $\Gamma_E = \{\widetilde{\Gamma} : \widehat{M} \longrightarrow \widehat{M}$ *such that* $\widetilde{\Gamma}$ *is a covering map verifying that* $\widetilde{\Gamma}(E/\sim) \subset E/\sim\}$*.*

Since $\Pi(E/\sim)$ *is a compact manifold, then* Γ_E *is finitely generated (see* [9]*)*

Let $\{\widetilde{g_1}, ..., \widetilde{g_k}\}$ *be generators of* Γ_E*. Note that* $\widetilde{g_i}$ *is the canonical projection of a covering map* $g_i : \widetilde{M} \to \widetilde{M}$*.*

Since $\Pi(E/\sim)$ *is a compact hypersurface in* M*, and* $dim(M) = 3$ *or* 4*, we have that there exists* E'*, a compact differential manifold homeomorphic to* $\Pi(E/\sim)$*.*

Let $j : E' \to \Pi(E/\sim)$ *be a homeomorphism,* $\widetilde{E'}$ *the universal covering of* E' *and* $\Pi' : \widetilde{E'} \to E'$ *the canonical projection.*

We will prove that there exists a fundamental domain $\overline{D}$ *of* $\Pi'|_{E'}$ *(i.e.* $\widetilde{\Gamma}(D) \cap D = \emptyset$*, for all* $\widetilde{\Gamma}$*, where* $\widetilde{\Gamma} : \widetilde{E'} \to \widetilde{E'}$ *is any covering map different from the identity. Besides,* $\Pi'(\overline{D}) = \Pi'(\widetilde{E'})$ *and* $\widetilde{E'} = \bigcup_{\widetilde{\Gamma}} \widetilde{\Gamma}(\overline{D})$*), such that* $\overline{D}$ *is a simplicial complex.*

Since every differential manifold is triangulable, there exist a finite simplicial complex K *and a homeomorphism* $\rho : K \to E'$*.*

We can subdivide $\rho(K)$ *in such a way that if* K_i *is a simplex of* K*,* $J_i = \rho(K_i)$*, and* $\widetilde{J_i}$ *is a connected component of* $\Pi'^{-1}(J_i)$*, then* $\Pi' : \widetilde{J_i} \to J_i$ *is a homeomorphism. Since* E' *is connected, we can denote by* $J_1, ...J_m$ *the simplexes of* $\rho(K)$ *verifying* $J_i \cap J_{i+1} \neq \emptyset$*, for* $i = 1, m-1$*.*

Let $\overline{J_1}$ *be a simplex in* $\widetilde{E'}$ *such that* $\Pi'(\overline{J_1}) = J_1$*, let* $\overline{J_2}$ *be a simplex in* $\widetilde{E'}$ *such that* $\Pi'(\overline{J_2}) = J_2$*, and* $\overline{J_1} \cap \overline{J_2} \neq \emptyset$*. This is possible because* $J_1 \cap J_2 \neq \emptyset$*. We continue in this fashion obtaining* $\overline{J_3}....\overline{J_m}$*.*

Then $\overline{D} = \cup_1^m \overline{J_i}$ *is a fundamental domain of* $\Pi'|_{E'}$ *and it is a simplicial complex.*

Let $\widetilde{j} : \widetilde{E'} \to E/\sim$ *be a lifting of* j *and let* $\overline{P} = \widetilde{j}(\overline{D})$*. We have that* P *is a finite simplicial complex and it is a fundamental domain of* $\Pi|_{E/\sim}$*.*

It follows that

$$E/\sim = \cup_{\widetilde{\Gamma} \in \Gamma_E} \widetilde{\Gamma}(\overline{P})$$

We will define an open neighborhood U *of* $\overline{P}$ *by*

$$U = \cup_{x \in \overline{P}} W^c_{\widehat{F}, loc}(x),$$

where $\widetilde{g}_i(W^c_{\widehat{F},loc}(x)) = W^c_{\widehat{F},loc}(\widetilde{g}_i(x))$, *in the case that* $x, \widetilde{g}_i(x) \in \overline{P}$ *for* $i = 1, ..., k$.

We begin by defining $W^c_{\widehat{F},loc}(x)$ *in such a way that* $\widetilde{g}_i(W^c_{\widehat{F},loc}(x)) = W^c_{\widehat{F},loc}(\widetilde{g}_i(x))$, *in the case that* $x, \widetilde{g}_i(x)$ *are vertices (0- dimensional simplexes) of* $\overline{P}$, *then we define* $W^c_{\widehat{F},loc}(x)$ *verifying that* $\widetilde{g}_i(W^c_{\widehat{F},loc}(x)) = W^c_{\widehat{F},loc}(\widetilde{g}_i(x))$, *in the case that* $x, \widetilde{g}_i(x)$ *are in an edge (1- dimensional simplex) of* $\overline{P}$, *and so on.*

By construction we have that there exists a homeomorphism $j_1 : \overline{U} \to \overline{P} \times [0,1]$ *such that* $j_1(W^c_{\widehat{F},loc}(x) \cap \overline{U}) = \{x\} \times [0,1] \ \ \forall x \in \overline{P}$.

Let $V = \cup_{\widetilde{\Gamma} \in \Gamma_E} \widetilde{\Gamma}(U)$. *We have that* V *is an open neighborhood of* $E/\sim$ *and* V *is* $\widetilde{g}_i$*-invariant for* $i = 1, ..., k$ *and then* V *is* $\widetilde{\Gamma}$*-invariant,* $\forall \widetilde{\Gamma} \in \Gamma_E$.

Since Π_1 *is a continuous map, we have that* $W = \Pi_1^{-1}(V)$ *is an open neighborhood of* E. *Besides* $W = \cup_{x \in E} \overline{C^{\alpha_x}_{\beta_x}}$, *where* $\Pi_1(\overline{C^{\alpha_x}_{\beta_x}}) = W^c_{\widehat{F},loc}(\pi_1(x)) \cap V$.

Let us define a continuous map $H : \widetilde{M} \to \widehat{M}$, *in the following way:*

In the case that $y \in \Pi_1^{-1}(U)$, *there exists* $x \in W^c_{F,loc}(y) \cap E$, *and* $y \in \overline{C^{\alpha_x}_{\beta_x}}$, *where* Π_1 *maps* $\overline{C^{\alpha_x}_{\beta_x}}$ *in* $W^c_{\widehat{F},loc}(\pi_1(x)) \cap U$.

Besides, $\forall z \in W^c_F(x) \cap \Pi_1^{-1}(U)$ *we have that* $\alpha_z = \alpha_x$ *and* $\beta_z = \beta_x$. *It follows that* $\Pi_1^{-1}(U) = \bigcup_{x \in E \cap \Pi_1^{-1}(\overline{P})} \overline{C^{\alpha_x}_{\beta_x}}$.

Let

$$\alpha = \bigcup_{x \in E \cap \Pi_1^{-1}(\overline{P})} \alpha_x \text{ and } \beta = \bigcup_{x \in E \cap \Pi_1^{-1}(\overline{P})} \beta_x.$$

Since $\alpha_x, \beta_x \notin E$ *we have that* $\Pi_1(\alpha) = \alpha$ *and* $\Pi_1(\beta) = \beta$.

Let $L_\alpha : \alpha \to \overline{P}$ *be the map such that* $L_\alpha(\alpha_x) = \Pi_1(x)$.

We have that L_α *is well defined and if* $\alpha_x \neq \alpha_y$, *then there exist* $x, y \in E$, *with* $x \in \overline{C^{\alpha_x}_{\beta_x}}$ *and* $y \in \overline{C^{\alpha_y}_{\beta_y}}$; *therefore* $[x] \neq [y]$, *i.e.* $L_\alpha(\alpha_x) \neq L_\alpha(\alpha_y)$. *Besides,* $\forall z \in \overline{P}$, $\exists y \in E$ *verifying* $\Pi_1(y) = z$. *Then* $L_\alpha(\alpha_y) = z$. *We have proved that* L_α *is a biyective map and we will prove that it is a homeomorphism.*

Suppose that $\alpha_{x_n} \to \alpha_x$, *then there exist* $x_n \in \overline{C^{\alpha_{x_n}}_{\beta_{x_n}}} \cap E$ *and* $x \in \overline{C^{\alpha_x}_{\beta_x}} \cap E$. *Let* y *be a limit point of* x_n *then* $\Pi_1(x_{n_k}) \to \Pi_1(y)$, *but* $y \in E \cap [x]$, *therefore* $\Pi_1(x_{n_k}) \to \Pi_1(x)$ *and the continuity of* L_α *follows.*

Suppose that $p_n, p \in \overline{P}$ *with* $p_n \to p$. *There exist* $x_n, x \in E$ *satisfying* $\Pi_1(x_n) = p_n$ *and* $\Pi_1(x) = p$, *we dont know if* $x_n \to x$, *but we have that* $W^c_{\widehat{F},loc}(p_n)) \cap U$ *is close to* $W^c_{\widehat{F},loc}(p) \cap U$, *and their end points are close. Since the end points of* $W^c_{\widehat{F},loc}(\Pi_1(x_n)) \cap U$ *are* $\Pi_1(\alpha_{x_n})$ *and* $\Pi_1(\beta_{x_n})$, *and*

α_{x_n} and β_{x_n} are in $\Pi_1^{-1}(U) - E$, it follows that $\Pi_1(\alpha_{x_n}) = [\alpha_{x_n}] = \alpha_{x_n}$ and $\Pi_1(\beta_{x_n}) = [\beta_{x_n}] = \beta_{x_n}$.

Analogously the end points of $W^c_{\widehat{F},loc}(\Pi_1(x)) \cap U$ are α_x and β_x; hence $\alpha_{x_n} \to \alpha_x$ and $\beta_{x_n} \to \beta_x$.

Then, we have proved that L_α is a homeomorphism.

Analogously we prove that $L_\beta : \beta \to \overline{P}$, defined in the obvious way is a homeomorphism.

The lengths of $\overline{C^{\alpha_x}_{\beta_x}}$ vary continuously, then we can define a map $j_2 : \Pi_1^{-1}(\overline{U}) \to \overline{P} \times [0,1]$ in the following way : $\forall y \in \Pi_1^{-1}(\overline{U})$, there exists $x \in W^c_{F,loc}(y) \cap E$, so let

$$j_2(y) = (p, \lambda) \text{ where } p = L_\alpha(\alpha_x) \text{ and } \lambda = \frac{length(\overline{C^{\alpha_x}_{y}}))}{length(\overline{C^{\alpha_x}_{\beta_x}})}$$

We have that $j_2(W^c_F(y) \cap \Pi_1^{-1}(\overline{U})) = \{\Pi_1(x)\} \times [0,1] \quad \forall y \in \Pi_1^{-1}(\overline{U})$.

It follows that $j_2 : \Pi_1^{-1}(\overline{U}) \to \overline{P} \times [0,1]$ is a homeomorphism.

We define $H : \Pi_1^{-1}(\overline{U}) \to \overline{U}$ by

$$H = j_1^{-1} \circ j_2$$

H is a homeomorphism and H satisfies

$$H(W^c_F(y) \cap \Pi_1^{-1}(\overline{U})) = W^c_{\widehat{F}}(\Pi_1(y)) \cap \overline{U}$$

We extend $H : \widetilde{M} \to \widehat{M}$ in the following way:

If $z = \Gamma(y) \in W$ then $z \in \overline{C^{\alpha_{\Gamma(x)}}_{\beta_{\Gamma(x)}}}$; since $\widetilde{\Gamma}(W^c_{\widehat{F},loc}(\pi_1(x))) \cap V = W^c_{\widehat{F},loc}(\widetilde{\Gamma}(\pi_1(x))) \cap V$, we define $H(z) = \widetilde{\Gamma}(H(y))$. It follows that $H \circ \Gamma = \widetilde{\Gamma} \circ H$.

If $y = \Gamma(x)$ with $x \in W$, we define $H(y) = \widetilde{\Gamma}(H(x))$; and $H(y) = [y]$ otherwise.

By construction we have that $H(W^c_F(x)) = W^c_{\widehat{F}}(H(x))$.

It is easy to see that H is a biyective map and by construction we have that H is a homeomorphism. ■

The previous proposition implies that $\mathcal{F}^c_F$ is homeomorphic to $\mathcal{F}^c_{\widehat{F}}$ and $\widehat{M}$ is homeomorphic to $\widetilde{M}$.

We have that $\mathcal{F}^c_{\widehat{F}}$ is topologically transversal to $E/\sim$.

Recall that $G' = \{\Pi[x] \ / \ [x] \in G\}$.

There exists $\widehat{f} : M/G' \to M/G'$, the canonical projection of f, such that $\widehat{f}$ preserves the dynamical properties of f. We have that $\Pi(\mathcal{F}^c_{\widehat{F}}) = \mathcal{F}^c_{\widehat{f}}$ is topologically transversal to the compact hypersurface $\Pi(E/\sim)$.

Since $H \circ \Gamma = \widetilde{\Gamma} \circ H$, we have that there exists a map $g : M \to M/G'$ verifying that $g(W^c_f(x)) = W^c_{\widehat{f}}(g(x))$. We have that g is a *homeomorphism. Therefore there exists a compact hypersurface* $\Sigma = g^{-1}(\Pi(E/G))$ *such that* $\mathcal{F}_f^c$ *is topologically transversal to* Σ.

Proposition 4.2. *The flow* ϕ *is conjugated to a suspension of an Anosov diffeomorphism.*

We have proved that $\{F_f^c(x)\}_{x\in M}$ *is topologically transversal to* Σ.

Recall that as f *is* C^1 *close to* f_1, *where* $f_1(x) = \phi(x,1)$ *there exists a homeomorphism* $h : M \to M$ *close to the identity such that* $h(x) = x'$, *and* $F_f^c(x')$ *is* C^1*-close to* $F_{f_1}^c(x)$ *in compact sets.*
Moreover

$$h(F_{f_1}^c(x)) = F_f^c(x').$$

Since $h^{-1}(\Sigma)$ *is a topological hypersurface we have that* $\{F_{f_1}^c(x)\}_{x\in M}$ *is topologically transversal to* $h^{-1}(\Sigma)$, *i.e.* $\forall x \in M$ *there exists* $T_x > 0$ *such that* $\phi(x, T_x) \cap h^{-1}(\Sigma)$ *"transversally ".*

Then ϕ, *may be reparametrized in such a way that it becomes a suspension, i.e. the Anosov flow is conjugated to a suspension which is an Anosov flow, too.* ∎

Remark 4.1. *The flow* ϕ *is conjugate to the suspension of an Anosov diffeomorphism and the hypersurface* $\Pi(E/G)$ *is homeomorphic to the torus* T^{n-1}.

Let $l : h^{-1}(\Sigma) \to h^{-1}(\Sigma)$ *be the map defined by* $l(x) = \phi(x, T_x)$.

For every $x \in h^{-1}(\Sigma)$ *there exist an* l*-stable and an* l*-unstable sets of* x, *where the* l*- stable (unstable) set of* x *is the intersection of the* ϕ *stable (unstable) manifold of* x *with* $h^{-1}(\Sigma)$. *Since* ϕ *is a transitive flow, it follows that* l *is an transitive diffeomorphism.*

We have that $l|\ h^{-1}(\Sigma)$ *is a hyperbolic diffeomorphism. If* $h^{-1}(\Sigma)$ *were a smooth manifold,* $l|\ h^{-1}(\Sigma)$ *would be an Anosov codimension one diffeomorphism and we could apply Franks result to conclude that* $l|\ h^{-1}(\Sigma)$ *is topologically conjugated to a hyperbolic toral automorphism (See* [2]*). Although* $h^{-1}(\Sigma)$ *is just a topological manifold, the Franks proof remains valid but, in this case we need to use a* C^0 *version of the classical theorem of Haefliger. This can be found in Chapter 7 of* [5].

Let $A : T^{n-1} \to T^{n-1}$ *be an Anosov diffeomorphism such that* $l|\ h^{-1}(\Sigma)$ *is conjugated to* $A|T^{n-1}$; *it follows that* $h^{-1}(\Sigma)$ *and* $\Pi(E/\sim)$ *are homeomorphic to* T^{n-1}. *Besides, if* ψ *is the suspension of* A, ϕ *is conjugated to* ψ. *Hence the flow* ϕ *is conjugated to the suspension of an Anosov diffeomorphism* A *under a function* ρ. *Then there exists a homeomorphism* $\overline{H} : M \to (T^{n-1})_A^\rho$, *such that*

$$\overline{H}(\mathcal{O}_\phi(x)) = \mathcal{O}_{X_{A,\rho}}(\overline{H}(x))$$

where $X_{A,\rho}$ is the flow under a function ρ built over $A : T^{n-1} \to T^{n-1}$ on the manifold $(T^{n-1})^{\rho}_{A}$ (See [7]).

5. Flow equivalence

Let y be a k-periodic point of f with $y \in \Pi(E)$. There exists $\widetilde{y} \in E$ such that $\Pi(\widetilde{y}) = y$. Let x be a periodic point, verifying that $x \in W^c_f(y)$.

Let $\eta_f(x)$ be the rotation number of x. We have that $\eta_f(x) = \eta_f(y)$.

Suppose that there exist $\widetilde{z} \neq \widetilde{w}$ such that $\Pi(\widetilde{z}) = z$, $\Pi(\widetilde{w}) = w$ and $[\widetilde{y}] = \overline{C^{\widetilde{w}}_{\widetilde{z}}}$. The points z and w are periodic ones.

We claim that for all $n \in I\!N$, $F^n(\widetilde{w}) \neq \widetilde{z}$.

Suppose that there exists $k \in I\!N$ such that $F^k(\widetilde{w}) = \widetilde{z}$, then $\overline{C^{\widetilde{w}}_{F^{lk}(\widetilde{z})}} \subset [\widetilde{y}]$, $\forall\ l \in I\!N$, hence $W^c_{\widehat{f}}(w)$ consist of only one point, but $W^c_{\widehat{F}}(\widetilde{w}) = \Pi^{-1}(W^c_{\widehat{f}}(w))$ is homeomorphic to $W^c_F(\widetilde{w})$ which is homeomorphic to $I\!R$. This is a contradiction.

The same argument shows that $F^n(\widetilde{w}) \notin \overline{C^{\widetilde{w}}_{\widetilde{z}}}$, $\forall n \in I\!N$.

Let x be a periodic point in $C^z_{f(w)}$ such that $\Pi^{-1}(x) \notin [F^n(\widetilde{y})]$, $\forall n \in I\!N$ (This point exists because f is hyperbolic and $z, f(w) \in \mathcal{A}$ and $W^c_{\widehat{f}}$ is homeomorphic to $I\!R$).

Then the f-period of x is equal to the $\widehat{f}-$ period of x, hence $\eta_f(x) = \eta_{\widehat{f}}(x)$.

So there is no loss of generality if we consider the rotation numbers of $\widehat{f}$ instead of those of f.

This is advantageous because $\mathcal{F}^c_{\widehat{f}}$ is topologically transversal to $\Pi(E/G)$ and $\Pi(E/G)$ is a hypersurface $\widehat{f}$-invariant.

From now on, we will suppose that $G = \widetilde{M}$, then $\widehat{F} = F$, $\widehat{f} = f$. In the same way that every leaf of $\mathcal{F}^c_f$ verifies that $F^c(x) \cap \mathcal{A} \neq \emptyset$, we have that $F^c(x) \cap \Pi(E) \neq \emptyset$ because $\Pi(E)$ is a connected component of $\mathcal{A}$. Therefore, $\mathcal{F}^c_f$ is topologically transversal to the hypersurface $\Pi(E)$.

Since $\phi : M \times I\!R \to M$ is an Anosov flow with a global transversal section, ϕ is conjugated to a suspension of the first return map, $l : h^{-1}(\Sigma) \to h^{-1}(\Sigma)$. Since l is conjugated to an Anosov diffeomorphism $A : T^{n-1} \to T^{n-1}$, we have that there exists $y \in h^{-1}(\Sigma)$ such that $l(y) = y$ and card$\{u \in \mathcal{O}_{\phi}(y) \cap h^{-1}(\Sigma)\} = 1$. Let $y' = h(y)$, we have that $y' \in \Pi(E)$ and $\mathcal{F}^c_f$ is topologically transversal to the hypersurface $\Pi(E)$. Besides, card$\{u \in F^c_f(y') \cap \Pi(E)\} = 1$.

Let $k = period(y')$ and let $G(x) = f^k(x)$. We have that $\Pi(E)$ is G-invariant.

Let $c : \Pi(E) \to I\!N$ be the map given by

$$c(x) = card\{v \in \widetilde{M} | v \in \text{connected component of } (\Pi^{-1}(C^x_{G(x)})) \cap \Pi^{-1}(\Pi(E))\}$$

Let $x \in \Pi(E)$ and $\overline{x} \in \Pi^{-1}(\Pi(E))$ such that $\Pi(\overline{x}) = x$. We have that the connected component of $\Pi^{-1}(C^x_{G(x)})$ is included in $W^c_F(\overline{x})$; the "end points" of the connected component of $\Pi^{-1}(C^x_{G(x)})$ are in different connected components of $\Pi^{-1}(\Pi(E)) \subset \widetilde{M}$, and $\Pi^{-1}(\Pi(E))$ is topologically transversal to $\mathcal{F}^c_F$.

From transversality we have the continuity of c, then there exists $m \in I\!N$ such that $c(x) = m$ for all $x \in \Pi(E)$.

We can assert that the segment of the central curve between y' and $G(y')$ "winds around itself" m times.

Hence, the rotation number of $G|W^c(y')$,

$$\eta_G(W^c(y')) = \frac{m}{1} \cong 1 \quad mod\ z$$

and the rotation number of $f|W^c(y')$,

$$\eta_f(W^c(y')) = \frac{m}{k}.$$

If $x \in \Pi(E)$ is an f-periodic point, then there exists $l > 0$ verifiyng $G^l(x) = x$, and we define $j(x) = card\{u \in C^x_{G^l(x)} \cap \Pi(E)\}$.

Since $\Pi(E)$ is a connected component of $\mathcal{A}$, we have that $F^c(x) \cap \Pi(E) \neq \emptyset$. Therefore,it will cause no confusion if we use $j(x)$ for any l-periodic point of G.

We claim the following

Proposition 5.1. *Let x be a periodic point of f, then the rotation number of x,*

$$\eta_f(W^c(x)) = \frac{m}{kj(x)}, \tag{1}$$

where m, k, and $j(x)$ are the above defined.

Let $x \in \Pi(E)$ a periodic point of G, and suppose that there exists $n \in I\!N$ such that $j(x) = mn$. It follows that $period_G(x) = n$ and

$$\eta_G(W^c(x)) = \frac{1}{n},$$

and

$$\eta_f(W^c(x)) = \frac{1}{nk} = \frac{m}{j(x)k}$$

In the case of $j(x) = mn + r$ with $0 < r < m$, let $s = \min\{l \in I\!N / rl = \dot{m}\}$ and let $\alpha \in I\!N$ such that $rs = m\alpha$. We have that

$$sj(x) = smn + sr = m(sn + \alpha)$$

then $period_G(x) = sn+\alpha$ *and the segment of the central curve between* x *and* $G^{sn+\alpha}(x)$ *"winds around itself"* s *times, it follows that*

$$\eta_G(W^c(x)) = \frac{s}{sn+\alpha} = \frac{1}{n+\frac{r}{m}} = \frac{m}{mn+r} = \frac{m}{j(x)}$$

Hence

$$\eta_f(W^c(x)) = \frac{m}{j(x)k}$$

Then every periodic point in $\Pi(E)$ *has rotation number of the form (1), and since every closed central leaf of* $\mathcal{F}^c_f$ *intersects* $\Pi(E)$ *then every* f*- periodic point of* M *has rotation number of the form (1).* ∎

Let x' *be a* f*-periodic point, and* $h^{-1}(x') = x$*, we have that* $F^c_f(x')$ *is* C^1*-close to* $F^c_{f_1}(x)$ *and* $\mathcal{F}^c_{f_1}$ *is topologically transversal to* $h^{-1}(\Pi(E)) = h^{-1}(\Sigma)$*. We have proved that for every Axiom A diffeomorphism,* f*,* C^1*- close to* f_1*, there exists* $\Sigma_f = h^{-1}(\Sigma)$ *such that* $\mathcal{F}^c_{f_1}$ *is topologically transversal to* Σ_f*.*

Recall that $\forall x \in \Sigma_f$ *there exists* $T_x > 0$ *such that*

$$T_x = min\{t > 0 | \phi(x,t) \in \Sigma_f\}$$

Let Λ *be a transversal section of* $\mathcal{F}^c_{f_1}$*.*

For every $x \in \Sigma_f$ *let*

$$e(x) = card\{v | v \in \phi(x,t) \cap \Lambda \text{ with } 0 \leq t < T_x\}$$

Since ϕ *is topologically transversal to* Λ *and* Σ_f *is connected we have that there exists* $e \in \mathbb{N}$ *such that* $e(x) = e$ *for all* $x \in \Sigma_f$*.*

It follows that $e.j(x') = j_{f_1}(x)$*, where* $j_{f_1}(x) = card\{u | u \in \mathcal{O}_\phi(x) \cap \Lambda\}$*. Hence*

$$\eta_f(W^c(x')) = \frac{m.e}{j_{f_1}(x)k}$$

By the continuity of the rotation number, it follows that the rotation numbers of f_1 *are of the form*

$$\eta_{f_1}(W^c(x)) = \frac{\beta}{j_{f_1}(x)}, \tag{2}$$

where $j_{f_1}(x)$ *is the above defined.*

Proposition 5.2. *If the flow* ϕ *is conjugated to a suspension of an Anosov diffeomorphism A under a function* ρ*, and the rotation numbers of* $f_1(x) = \phi(x,1)$ *are of the form (2), then* ϕ *is flow equivalent to the suspension of A.(Here the time of the first returned map is constant).*

Since

$$period(\phi(x,t)) = \frac{1}{\eta_{f_1}(W^c(x))},$$

and

$$\eta_{f_1}(W^c(x)) = \frac{\beta}{j_{f_1}(x)},$$

we have that

$$period(\phi(x,t)) = \frac{j_{f_1}(x)}{\beta},$$

and if $y \in M$ *is such that* $j_{f_1}(y) = 1$ *then* $period(\phi(y,t)) = \frac{1}{\beta}$.

Since $\phi : M \times I\!R \to M$ *is an Anosov flow conjugated to the suspension of an Anosov diffeomorphism* A*, there exists a homeomorphism* $\overline{H} : M \to (T^{n-1})^{\rho}_{A}$*, such that*

$$\overline{H}(\mathcal{O}_{\phi}(x)) = \mathcal{O}_{X_{A,\rho}}(\overline{H}(x)).$$

Let $\psi : T^{n-1} \to I\!R > 0$ *be the map such that* $\psi(x) \equiv \frac{1}{\beta}$.

Let $X_{A,\psi}$ *be the flow under a function* ψ *built over* $A : T^{n-1} \to T^{n-1}$ *on the manifold* $(T^{n-1})^{\psi}_{A}$.

We have that $X_{A,\psi}$ *is conjugated to* $X_{A,\rho}$ *and therefore to* ϕ.

Let $\underline{H} : M \to (T^{n-1})^{\psi}_{A}$ *be the homeomorphism such that*

$$\underline{H}(\mathcal{O}_{\phi}(x)) = \mathcal{O}_{X_{A,\psi}}(\underline{H}(x))$$

It follows that if x *is a* l*-periodic point of* A *we have that*

$$period\mathcal{O}_{X_{A,\psi}(x)} = \frac{l}{\beta}$$

If x *is a periodic point of* ϕ *with* $period(\mathcal{O}_{\phi}(x)) = \frac{j_{f_1}(x)}{\beta}$*, and* $\underline{H}(\mathcal{O}_{\phi}(x))$ *is a periodic orbit of* $X_{A,\psi}$.

We have that

$$j_{f_1}(x) = card\{u | u \in \mathcal{O}_{\phi}(x) \cap \Lambda\} = card\{u | u \in \mathcal{O}_{X_{A,\psi}}(\underline{H}(x)) \cap \underline{H}(\Lambda)\}$$

and we claim that

$$card\{u | u \in \mathcal{O}_{X_{A,\psi}}(\underline{H}(x)) \cap \underline{H}(\Lambda)\} =$$

$$card\{u | u \in \mathcal{O}_{X_{A,\psi}}(\underline{H}(x)) \cap T^{n-1}\}$$

For every $x \in T^{n-1}$ *let*

$$d(x) = card\{v | v \in \eta(x,t) \cap \underline{H}(\Lambda) \text{ with } 0 \le t < \frac{1}{\beta}\}$$

where $\eta(x,t)$ *is the flow defined by* $\dot{\eta}(x,t) = X_{A,\psi}(\eta(x,t))$.

Since η *is topologically transversal to* $\underline{H}(\Lambda)$ *and* T^{n-1} *is connected we have that there exists* $d \in I\!N$ *such that* $d(x) = d$ *for all* $x \in T^{n-1}$.

If $d \geq 2$*, then*

$$2 \leq card\{v|v \in \eta(\underline{H}(y), t) \cap \underline{H}(\Lambda) \text{ with } 0 \leq t < \frac{1}{\beta}\} =$$

$$card\{u|u \in \mathcal{O}_\phi(y) \cap \Lambda\} = 1$$

which is a contradiction. Then

$$card\{u|u \in \mathcal{O}_{X_{A,\psi}}(\underline{H}(x)) \cap \underline{H}(\Lambda)\} =$$

$$card\{u|u \in \mathcal{O}_{X_{A,\psi}}(\underline{H}(x)) \cap T^{n-1}\} = period_A(\underline{H}(x))$$

and it follows that

$$period(\mathcal{O}_\phi(x)) = period(\mathcal{O}_{X_{A,\psi}}(\underline{H}(x))).$$

Since ϕ *and* $X_{A,\psi}$ *are conjugated and the periods of corresponding periodic orbits agree, then* ϕ *and* $X_{A,\psi}$ *are flow equivalent (See* [7]*, Ch.19).*
■

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IMERL, Facultad de Ingeniería, Universidad de La República, C.C. 30, Montevideo, Uruguay

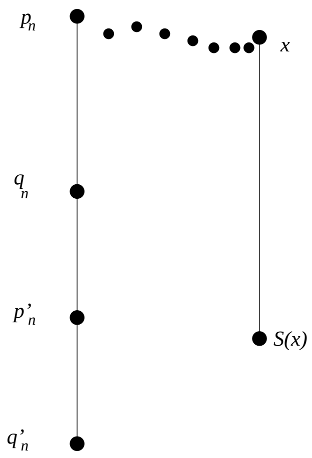

Figure 1.

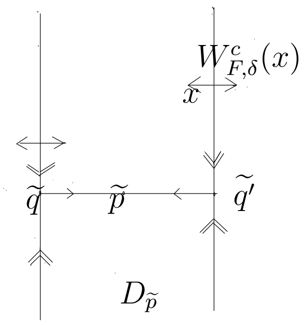

Figure 2