ON AXIOM A DIFFEOMORPHISMS C^0-CLOSE TO PSEUDO-ANOSOV MAPS

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ABSTRACT. Axiom A diffeomorphisms g, C^0-close to pseudo-Anosov maps f on a surface M and on its universal covering $\mathcal{M}$ are considered. It is shown that for "large" basic sets B of g, a necessary and sufficient condition to have the property that each trajectory in B is shadowed by a trajectory of f is to have a lifting to $\mathcal{M}$ of one of its stable (unstable) manifolds at a bounded distance from some stable (unstable) manifold of the lifting of f.

1. INTRODUCTION.

Let f be a pseudo-Anosov map of the compact connected oriented riemannian surface M. Let $g : M \to M$ be a homeomorphism $C^\circ-$close to f ; the set K (J) of points of M so that their g-orbit ($f-$ orbit) is shadowed by a close $f-$orbit ($g-$ orbit)is a compact $g-$invariant (resp. $f-$invariant) subset of M. It is known (see[H1],[L1]) that while $J = M$ always, K is, in general, strictly contained in M. In [H2] the elements of K are characterized in terms of liftings of their $g-$orbit to the universal cover $\mathcal{M}$ of M. In this paper we consider mainly, the case when g is an Axiom A diffeomorphism, and characterize the "large" basic sets of g included in K, in terms of the liftings to $\mathcal{M}$ of their stable (unstable) manifolds.

Let F be a lifting of f to $\mathcal{M}$. Then there exist (see [T], [FLP]) equivariant pseudo-metrics D_S, D_U, and $\lambda > 1$ such that, for $\xi, \eta \in \mathcal{M}$,

$$D_S\left(F^{-1}(\xi), F^{-1}(\eta)\right) = \lambda D_S(\xi, \eta),$$

$$D_U(F(\xi), F(\eta)) = \lambda D_U(\xi, \eta);$$

$D = D_S + D_U$ is an equivariant metric on $\mathcal{M}$. Let

$$W_S^F(\xi) = \{\eta \in \mathcal{M} : D(F^n(\xi), F^n(\eta)) \to 0, \text{for } n \to \infty\}$$

$$W_U^F(\xi) = \{\eta \in \mathcal{M} : D(F^n(\xi), F^n(\eta)) \to 0, \text{for } n \to -\infty\}$$

denote the stable and unstable sets of ξ. Let G be the lifting of g to $\mathcal{M}$, $C^\circ-$close to F. Analogously, denote $W_S^G(\xi)$, $W_U^G(\xi)$ the $G-$stable (resp. $G-$unstable) set of $\xi \in \mathcal{M}$. When g is Axiom A, for $\xi \in \mathcal{M}$, B a basic set of g, $\pi(\xi) = x \in B$, we shall denote

$${}^B W_S^G(\xi) \subset W_S^G(\xi) \;\; \left({}^B W_U^G(\xi) \subset W_U^G(\xi)\right)$$

the part of the $G-$stable (resp. $G-$unstable) manifold that projects under π onto the intersection of the $g-$stable (resp. $g-$unstable) manifold of x with B; here , $\pi : \mathcal{M} \to M$, denotes the canonical projection.

We shall say that a basic set B of g is large if ${}^{B}W_{S}^{G}(\xi)$ is $D-$unbounded for some ξ, $\pi(\xi) \in B$. It is easy to show, on account of the density properties of stable manifolds on basic sets, that if this unboundedness property is true for some ξ then it holds also for every ${}^{B}W_{S}^{G}(\zeta)$,$\pi(\zeta) \in B$. g has large basic sets and the liftings to $\mathcal{M}$ of the stable (unstable) manifolds of the "small" basic sets of g have arbitrarily small $D-$diameter, provided g is sufficiently $C^{\circ}-$ close to f . These small basic sets project into a finite set in the quotient space considered in [C-S], obtained identifying two points of M when, for each $n \in Z$, the distance of their $n-$iterates is much less than the expansivity constant of f (see section 3).

We show that a large basic set B of g is contained in K if and only if there exist $\xi, \eta \in \mathcal{M}$, $\pi(\xi) \in B$, and $V \geq 0$ such that

$$D\left(\zeta, W_{S}^{F}(\eta)\right) \leq V,$$

for all $\zeta \in {}^{B}W_{S}^{G}(\xi)$. Clearly, if the condition is fulfilled, we may take η to be an $F-$ singular point, provided we choose an adequate V. For $B \subset K$, there exists a continuous function $h : B \to M$, such that $h \circ g/B = f \circ h$. In case h is surjective, it is called a semi-conjugacy between g/B and f.

The proof of this and some other connected results, that use some of the ideas in [L2, section 6], follow from Proposition 2.1 , that concerns $G-$semitrajectories shadowed by $F-$semitrajectories, and simple fundamental properties of basic sets of Axiom A diffeomorphisms.

Finally, we remark that the results obtained from the proved ones by changing past for future, stable for unstable, attractor by repeller, etc., and conversely, will be tacitly assumed.

2. Large stable sets.

In this section we look for F - trajectories shadowing G - trajectories, at a D - distance less than $p > 0$, during the first N- positive iterates. . Given $r > 0$, it is possible to find a C°-neighbourhood of f such that for g in that neighbourhood, there are liftings F and G that satisfy

$$D\left(F(\xi), G(\xi)\right) < r,\ D\left(F^{-1}(\xi), G^{-1}(\xi)\right) < r$$

for every $\xi \in \mathcal{M}$. Take p, q, $r > 0$ such that

$$q > 3r(\lambda-1)^{-1},\ q > r \tag{2.1}$$

$$p > (\lambda q + 2r)(\lambda-1)^{-1}, \tag{2.2}$$

$$p > 2\left(\lambda+\lambda^{-1}-2\right)^{-1} r, \tag{2.3}$$

and choose g in the neighbourhood mentioned before. Take $N > 0$, $N \in Z$, and

$$\rho > \lambda^{N+1} p + 3r(\lambda^{N+1} - 1)(\lambda-1)^{-1}.$$

For $\xi \in \mathcal{M}$, let $B_{d}(\xi)$ denote the closed $D-$disk of center ξ and radius d. Let $C \subset B_{p}(\xi)$,

$$C = C_N = \{\eta \in \mathcal{M} : F^{\nu}(\eta) \notin B_p(G^{\nu}(\xi)) \text{ for some } \nu, 0 < \nu \leq N\}$$

Proposition 2.1. *Let $\xi \in \mathcal{M}$ be such that $W_S^G(\xi)$ contains an arc A, $\xi \in A$, whose end- points do not belong to $B_\rho(\xi)$. Then $C = C_0 \cup C_1$, where $C_o \cap C_1 = \emptyset$, and C_0, C_1 are open subsets of C.*

Proof. Let $\Xi \in \mathcal{M}$, $\Xi \notin W_S^G(\xi)$, have the property that $G^n(\Xi) \in G^n(B_\rho(\xi))$ for $0 \leq n \leq N$. Let $\eta \in B_p(\xi)$ be such that for some ν, $0 < \nu \leq N$,

$$F^{\nu}(\eta) \notin B_p(G^{\nu}(\xi))$$

and assume moreover, that ν is the first positive integer such that this occurs. Then, since

$$D\left(G^{\nu-1}(\xi), F^{\nu-1}(\eta)\right) \leq p,$$

$$D\left(G^{\nu}(\xi), F^{\nu}(\eta)\right) \leq \lambda p + r$$

and therefore,

$$F^{\nu}(\eta) \in G^{\nu}(B_\rho(\xi)).$$

So, we may define $i(\eta)$ to be the *mod* $2-$ intersection number of an arc, within $G^{\nu}(B_\rho(\xi))$, joining $F^{\nu}(\eta)$ to $G^{\nu}(\Xi)$, with the arc $G^{\nu}(A)$. This intersection number is well-defined since the end points of $G^{\nu}(A)$ do not belong to the topological disk $G^{\nu}(B_\rho(\xi))$ and $F^{\nu}(\eta) \notin A$, as shown through the following lemmas.

Lemma 2.2. *Assume that $D_U(\xi', \eta') > q$. Then, $\eta' \notin W_S^G(\xi')$.*

Proof. Let ξ', η' be such that $D_U(\xi', \eta') > q$. Then,

$$D_U(G(\xi'), G(\eta')) \geq$$

$$D_U(F(\eta'), G(\xi')) - D_U(F(\eta'), G(\eta')) \geq$$

$$D_U(F(\eta'), F(\xi')) - D_U(F(\xi'), G(\xi')) -$$

$$\begin{aligned} D_U(F(\eta'), G(\eta')) \quad &\geq \\ \lambda D_U(\xi', \eta') - 2r \quad &> \quad \lambda q - 2r > q \end{aligned}$$

by (2.1). It follows that for all $k > 0$,

$$D_U\left(G^k(\eta'), G^k(\xi')\right) > q$$

and therefore $\eta' \notin W_S^G(\xi')$. □

Lemma 2.3. $F^{\nu}(\eta) \notin W_S^G(G^{\nu}(\xi))$.

Proof. We have that $D\left(G^{\nu-1}(\xi), F^{\nu-1}(\eta)\right) \leq p$, and $D(F^{\nu}(\eta), G^{\nu}(\xi)) > p$. If we had

$$D_S(F^{\nu}(\eta), G^{\nu}(\xi)) \geq p - q,$$

we would have

$$\begin{aligned} D_S\left(G^{\nu-1}(\xi), F^{\nu-1}(\eta)\right) &\geq D_S\left(F^{-1}(G^{\nu}(\xi)), F^{\nu-1}(\eta)\right) - \\ D_S\left(F^{-1}(G^{\nu}(\xi)), G^{-1}(G^{\nu}(\xi))\right) &\geq \end{aligned}$$

$$\lambda D_S(F^{\nu}(\eta), G^{\nu}(\xi)) - r \geq \lambda(p-q) - r.$$

Since by (2.2), $\lambda(p-q) - r > p$, this is absurd. Thus, $D_S(F^{\nu}(\eta), G^{\nu}(\xi)) < p - q$, and consequently,

$$D_U(F^{\nu}(\eta), G^{\nu}(\xi)) > q.$$

Therefore $F^{\nu}(\eta) \notin W_S^G(G^{\nu}(\xi))$, as it follows from the previous lemma. □

Now, we continue with the proof of the proposition. Let C_0, C_1 denote respectively the subsets of C of points η for which $i(\eta) = 0$, $i(\eta) = 1$. Clearly, $C = C_0 \cup C_1$, $C_0 \cap C_1 = \emptyset$. We show now that C_0, C_1 are open subsets of C.

Let $\eta \in C$. As before, call ν the first positive integer for which

$$D(F^{\nu}(\eta), G^{\nu}(\xi)) > p,$$

and assume that for $0 \leq n < \nu$,

$$D(F^{\nu}(\eta), G^{\nu}(\xi)) < p.$$

Then the continuity of F and G together with the compactness of $G^{\nu}(A)$, imply readily that for ζ close enough to η we have $i(\zeta) = i(\eta)$.

Assume then that for some n, $0 \leq n < \nu$, $D(F^n(\eta), G^n(\xi)) = p$. We show that $n = \nu - 1$; since for $\nu = 1$ this is trivial, we consider $\nu \geq 2$. For α, $\beta \in \mathcal{M}$ let

$$\Delta(\alpha, \beta) = D(F(\alpha), G(\beta)) - 2D(\alpha, \beta) + D\left(F^{-1}(\alpha), G^{-1}(\beta)\right)$$

Lemma 2.4. $\Delta(\alpha, \beta) \geq (\lambda + \lambda^{-1} - 2) D(\alpha, \beta) - 2r$.

Proof. $\Delta(\alpha, \beta) \geq$

$$D(F(\alpha), F(\beta)) - D(F(\beta), G(\beta)) - 2D(\alpha, \beta) +$$

$$D\left(F^{-1}(\alpha), F^{-1}(\beta)\right) - D\left(F^{-1}(\beta), G^{-1}(\beta)\right) \geq$$

$$D(F(\alpha), F(\beta)) - 2D(\alpha, \beta) + D\left(F^{-1}(\alpha), F^{-1}(\beta)\right) - 2r \geq$$

$$D_S\left(F\left(\alpha\right), F(\beta)\right) + D_U\left(F\left(\alpha\right), F(\beta)\right) - 2D\left(\alpha, \beta\right) +$$

$$D_S\left(F^{-1}\left(\alpha\right), F^{-1}\left(\beta\right)\right) + D_U\left(F^{-1}\left(\alpha\right), F^{-1}\left(\beta\right)\right) - 2r \geq$$

$$\lambda D\left(\alpha, \beta\right) + \lambda^{-1} D\left(\alpha, \beta\right) - 2D\left(\alpha, \beta\right) - 2r.$$

□

If $D\left(\alpha, \beta\right) = p$, we get on account of (2.3) that $\Delta\left(\alpha, \beta\right) > 0$; thus if

$$D\left(F^{-1}\left(\alpha\right), G^{-1}\left(\beta\right)\right) \leq p,$$

$$D\left(F\left(\alpha\right), G(\beta)\right) > p.$$

Applying this to $\alpha = F^n\left(\eta\right), \beta = G^n\left(\xi\right)$ we obtain easily that $n = \nu - 1$.

Let $\eta' = F^n\left(\eta\right)$, and let $\left(\mathcal{M}^*, \Gamma\right)$ denote the suspension flow of G. Since for any t, $0 \leq t \leq 1$,

$$\Gamma\left(\eta', t\right) \in \Gamma\left(B_\rho\left(\xi\right), n + t\right),$$

$$\Gamma\left(\eta', t\right) \notin \Gamma\left(A, n + t\right),$$

the end points of $\Gamma\left(A, n + t\right)$ do not belong to $\Gamma\left(B_\rho\left(\xi\right), n + t\right)$, and, finally, because $\Gamma\left(\Xi, t\right) \in \Gamma\left(B_\rho\left(\xi\right), t\right)$ for $0 \leq t \leq N$, the *mod* 2 intersection number of $\Gamma\left(A, n + t\right)$ and an arc within $\Gamma\left(B_\rho\left(\xi\right), n + t\right)$joining $\Gamma\left(\eta', t\right)$ to $\Gamma\left(\Xi, n + t\right)$ is well defined, and furthermore, it varies continuously with t. Denote also by i this intersection number. We get, in this way, that

$$i\left(\eta'\right) = i\left(F^n\left(\eta\right)\right) = i\left(G\left(F^n\left(\eta\right)\right)\right).$$

The proof of the proposition will be complete if we show that

$$i\left(G(F^n\left(\eta\right))\right) = i\left(F^{n+1}\left(\eta\right)\right) = i\left(F^\nu\left(\eta\right)\right) = i\left(\eta\right),$$

for, in that case, for ζ in a sufficiently small neighbourhood of η, $i\left(\zeta\right) = i\left(\eta\right)$, no matter whether $D\left(G^n\left(\xi\right), F^n\left(\zeta\right)\right) > p$ or $D\left(G^n\left(\xi\right), F^n\left(\zeta\right)\right) \leq p$. To that aim we prove that

$$B_r\left(G\left(F^n\left(\eta\right)\right)\right) \subset \Gamma\left(B_\rho\left(\xi\right), \nu\right)$$

and that

$$B_r\left(G\left(F^n\left(\eta\right)\right)\right) \cap \Gamma\left(A, \nu\right) = \emptyset,$$

which, on account of $F^\nu\left(\eta\right) \in B_r\left(G\left(F^n\left(\eta\right)\right)\right)$ and of the arcwise connectedness of $B_r\left(G\left(F^n\left(\eta\right)\right)\right)$ implies the conclusion we wanted to obtain. If $\zeta \in B_r\left(G\left(F^n\left(\eta\right)\right)\right)$,

$$D\left(\zeta, G^{\nu}\left(\xi\right)\right) \leq \lambda p + 3r$$

since $D\left(F^{n}\left(\eta\right), G^{n}\left(\xi\right)\right) = p$; thus

$$D\left(G^{-\nu}\left(\zeta\right), \xi\right) \leq \lambda^{N+1} p + 3r\left(\lambda^{N+1} - 1\right)\left(\lambda - 1\right)^{-1} = \rho.$$

On the other hand, as shown before in this proof, $D_U\left(F^{n}\left(\eta\right), G^{n}\left(\xi\right)\right) > q$ which implies

$$D_U\left(G\left(F^{n}\left(\eta\right)\right), G^{\nu}\left(\xi\right)\right) > \lambda q - 2r.$$

Then, if $\zeta \in B_r\left(G\left(F^{n}\left(\eta\right)\right)\right)$,

$$D_U\left(\zeta, G^{\nu}\left(\xi\right)\right) \geq D_U\left(G\left(F^{n}\left(\eta\right)\right), G^{\nu}\left(\xi\right)\right) - D_U\left(\zeta, G\left(F^{n}\left(\eta\right)\right)\right) \geq \lambda q - 3r > q$$

by (2.1). This implies that $\zeta \notin W_S^G\left(G^{\nu}\left(\xi\right)\right)$, as we have already shown. This completes the proof of the proposition. □

Let us state, for further use, the following

Lemma 2.5. *If* $\zeta \in W_S^G\left(\xi\right)$, *then*

$$\limsup_{n \to \infty} D_S\left(G^{n}\left(\xi\right), G^{n}\left(\zeta\right)\right) \leq 2r\lambda\left(\lambda - 1\right)^{-1}.$$

Proof. Assume $D_S\left(\xi, \zeta\right) \leq Q$, then $D_S\left(G\left(\xi\right), G\left(\zeta\right)\right) \leq \lambda^{-1} Q + 2r$; for any positive integer k,

$$D_S\left(G^{k}\left(\xi\right), G^{k}\left(\zeta\right)\right) \quad \leq \quad \lambda^{-k} Q + 2r\left(\lambda^{-k} - 1\right)\left(\lambda^{-1} - 1\right)^{-1}.$$

□

3. Persistence and expansiveness

In this section we shall briefly 1) recall persistence properties of pseudo-Anosov maps f of M, 2) comment expansivity properties of homeomorphisms g, C^0- close to f and 3), apply 1) and 2) to the case when g is Axiom A.

1) In [H1], [L1], it is shown that given $\varepsilon > 0$ there exists a C^0- neighbourhood N_ϵ of f such that for $g \in N_\epsilon$, and $x \in M$, there exists $y \in M$, such that $dist\left(f^{n}\left(x\right), g^{n}\left(y\right)\right) \leq \varepsilon$ for every $n \in Z$. Here $dist$ denotes the riemannian distance on M. Let $K = K_g$ denote the compact subset of M,

$$K = \left\{y \in M : dist\left(f^{n}\left(x\right), g^{n}\left(y\right)\right) \leq \varepsilon,\ n \in Z,\ \text{for some } x \in M\right\}$$

and take $\varepsilon < \alpha/3$, α being an expansivity constant for f, i.e,

$$dist\left(f^{n}\left(x\right), f^{n}\left(y\right)\right) \leq \alpha, \quad \forall n \in Z, \ \text{implies } x = y.$$

Then $h : K \to M$, $h\left(y\right) = x$ (where x is the unique point of M with the property that $dist\left(f^{n}\left(x\right), g^{n}\left(y\right)\right) \leq \varepsilon$, $n \in Z$) is a continuous surjective function and, moreover, a semi-conjugacy between g/K and f.

2) In [L1] it is proved that for $0 < \delta < \alpha$, there is a C^0- neighbourhood , N_δ, of f, such that for $g \in N_\delta$, either

$$dist\left(g^{n}\left(x\right), g^{n}\left(y\right)\right) > \alpha \text{ for some } n \in Z$$

or

$$dist\left(g^{n}\left(x\right), g^{n}\left(y\right)\right) \leq \delta \text{ for all } n \in Z \ .$$

Let

$$R_{\delta} = \left\{(x,y) \in M \times M : dist\left(f^{n}\left(x\right), f^{n}\left(y\right)\right) \leq \delta,\ n \in Z\right\}.$$

Then, if $\delta < \frac{\alpha}{3}$, R_δ is an equivalence relation.

3) Call π_δ the canonical projection, $\pi_\delta : M \to M/R_\delta$ (See [C-S] in connection with the space M/R_δ, and the dynamics induced by g on it). If g is an axiom A diffeomorphism, and B is a basic set of g we shall say that B is *large* (see section 1) if for some $x \in B$ and some lifting ξ of x to $\mathcal{M}$, ${}^{B}W_{S}^{G}\left(\xi\right)$ is $D-$unbounded.

Let f, g, F, G, p, q, r, be as in the previous section and choose, furthermore, q and r, to satisfy also

$$2r\lambda\left(\lambda-1\right)^{-1}+q \quad < \quad \delta.$$

Lemma 3.1. *Let $g \in N_\varepsilon \cap N_\delta$, be axiom A. Then*

a) If $\pi\left(\xi\right)$belongs to a basic set B and ${}^{B}W_{S}^{G}\left(\xi\right)$ is $D-$bounded, then $\pi_\delta\left(B\right)$ is finite.

b) g has a large basic set.

Proof. a) We have, according to lemmas 2.3 and 2.5, that for n sufficiently large, the $D-$diameter of ${}^{B}W_{S}^{G}\left(G^{n}(\xi)\right)$ is less than δ, since $2r\lambda\left(\lambda-1\right)^{-1}+q < \delta$. Well known properties of axiom A basic sets imply then that $\pi_\delta\left(B\right)$ is finite.

b) Let $x \in M$ be such that $clos\left\{f^{n}\left(x\right), n \in Z\right\} = M$. Then either $\omega_f\left(x\right) = M$, or $\alpha_f\left(x\right) = M$. Choose $y \in M$ so that $h\left(y\right) = x$; then $\omega_g\left(y\right) \subset K$, $\alpha_g\left(y\right) \subset K$ and

$$h\left(\omega_g\left(y\right)\right) \supset \omega_f\left(x\right),$$

$$h\left(\alpha_g\left(y\right)\right) \supset \alpha_f\left(x\right).$$

Since both, $\omega_g\left(y\right)$and $\alpha_g\left(y\right)$ are included in g basic sets, the proof of b) follows from a) because on K, R_δ coincides with

$$R_{\delta} = \left\{(x,y) \in K \times K : h\left(x\right) = h\left(y\right)\right\},$$

as it is easy to see on account of the fact that we have chosen $\varepsilon, \delta < \frac{\alpha}{3}$. □

4. Axiom A diffeomorphisms

Let f, g, F, G be as before, g an axiom A diffeomorphism, $g \in N_\varepsilon \cap N_\delta$; $\varepsilon, \delta, p, q, r$ as in the previous sections, and furthermore, choose p such that $D\left(\xi, \eta\right) \leq p$, implies

$$dist\left(\pi\left(\xi\right), \pi\left(\eta\right)\right) \leq \varepsilon, \quad \xi,\ \eta \in \mathcal{M}.$$

For technical reasons we shall assume in the proof of the next theorem that $\lambda > 5$; as it is easy to check this implies no loss of generality.

Theorem 4.1. *Let B be a large basic set of g. Then $B \subset K$, if and only if there exist $\xi, \eta \in \mathcal{M}$, $\pi(\xi) \in B$, and $V > 0$ such that for every $\zeta \in {}^{B}W_S^G(\xi)$,*

$$D\left(\zeta, W_S^F(\eta)\right) < V.$$

Proof. Since for $x \in B$, $h\left({}^{B}W_S^g(x)\right) \subset W_S^f(h(x))$, the necessity of the condition follows easily from the fact that $dist(x, h(x)) < \varepsilon$, for all $x \in B$.

We prove then the "if" part, applying Proposition 2.1 with an adequate choice of p in terms of V and showing, furthermore, than in the present conditions, both, C_0 and C_1, $C_0 \cup C_1 = C = C_N$, are non-void for $N \geq 1$, those $C_N \neq B_p(\xi)$.

Since B is large, ${}^{B}W_S^G(\xi)$ is $D-$ unbounded. For each positive integer n, choose arcs $a_n \subset W_S^G(\xi)$ with end-points p_n, q_n, such that $D(p_n, q_n) > n$, and a point η_n' such that $W_S^F(\eta_n')$contains no singularity and such that for $\zeta \in a_n$,

$$0 < D\left(\zeta, W_S^F(\eta_n')\right) \leq V.$$

It is easy to see that we may choose such arcs a_n and points η_n' .

Let $p_n', q_n' \in W_S^F(\eta_n')$,

$$D(p_n', p_n) \leq V, D(q_n', q_n) \leq V,$$

and let a_n' be the arc of $W_S^F(\eta_n')$ whose end-points are p_n', q_n'; $D(p_n', q_n') > n - 2V$. Assume $V > 4q$, take N to be a positive integer and let

$$\rho' > \lambda^{N+1} 2V + 3r \frac{\lambda^{N+1} - 1}{\lambda - 1}.$$

Take $n > \max\{100\rho', 100V\}$. Choose p_n'', q_n'' on a_n' such that

$$D\left(p_n', p_n''\right) = \rho' + V = D\left(q_n', q_n''\right);$$

then $D(p_n'', q_n'') > n - 2\rho' - 4V$. Call b_n' the sub-arc of a_n' between p_n'' and q_n''. Consider $W_U^F(p_n''), W_U^F(q_n'')$(we assume that no one of them contains singular points); p_n', q_n' lie in different components of the complement of $W_U^F(p_n'')$ and of the complement of $W_U^F(q_n'')$, and since $D\left(p_n', p_n''\right) > V$ $\left(D(q_n', q_n'') > V\right)$, $p_n(q_n)$ lies in the same components as p_n' (resp. q_n') of both complements. Moreover we have that $W_U^F(q_n'')$ $\left(W_U^F(p_n'')\right)$ lies in the same component of the complement of $W_U^F(p_n'')$ $\left(W_U^F(q_n'')\right)$ as q_n, q_n' (resp. p_n, p_n'). We may then choose a sub-arc $b_n \subset a_n$ with end-points both in $W_U^F(p_n'')$ or both in $W_U^F(q_n'')$ or one in $W_U^F(p_n'')$ and the other in $W_U^F(q_n'')$, and lying between both $F-$unstable manifolds (see Figure 1). Moreover it is not difficult to see that the arcs b_n may be chosen through points $\zeta_n \in b_n$ with $\pi(\zeta_n) \in B$, and such that ζ_n separates b_n in two arcs, both of arbitrarily large $D-$diameter. The $D-$distance between the end-points of b_n is larger or equal to $D_S(p_n'', q_n'')$ $> n - 2\rho' - 4V$, and the D_S-distance (and, consequently, the $D-$distance) between the points of b_n and each one of the end-points of a_n is larger than ρ', as it is easy to see on account of the choice of p_n', q_n' and of p_n'' , q_n''.

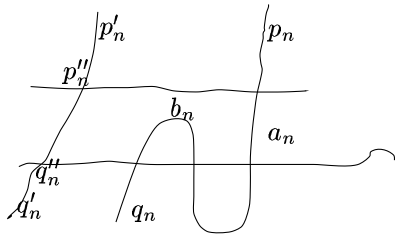

FIGURE 1

Since the end-points of a_n do not belong to the balls of $D-$radius ρ' centered at any point ζ of b_n, Proposition 1 may be applied letting $\xi = \zeta$, $p = 2V$, $\rho = \rho'$ and $A = a_n$. So the corresponding sets C_0, C_1, are open. We prove now that they are non-void.

Consider the subset c_n of b_n that consists of points ξ_n such that $W_U^F(\xi_n)$ cuts b'_n. For these points, since $W_U^F(\xi_n)$ cuts only once b'_n, lies between $W_U^F(p''_n)$ and $W_U^F(q''_n)$ and separates $\mathcal{M}$, we have that the *mod* 2$-$intersection number of $W_U^F(\xi_n)$ and a_n is 1. Therefore, the arc of $W_U^F(\xi_n)$ contained in $B_{2V}(\xi_n)$ has *mod* 2$-$intersection number with a_n equal to 1; recall that any two points in a_n are at a D_U distance less than $q < V$. Call τ_n the point of intersection of $W_U^F(\xi_n)$ with b'_n; since this intersection is transversal, whenever the arc of $W_U^F(\xi_n)$ with end-points ξ_n and τ_n has no singular points, ξ_n has a neighbourhood in b_n included in c_n.

Let us show that c_n is closed in b_n. In fact, if $\xi_n \in d_n = b_n - c_n$, all arcs in $W_U^F(\xi_n)$ through ξ_n, with both end-points at a D_U distance $V' = \max_{\theta \in b_n} D_U(\theta, b'_n)$, from ξ_n do not intersect b'_n, and therefore, as it is easy to see, no matter whether ξ_n is singular or not, there is a neighbourhood of ξ_n contained in d_n, since the fact that the points in b_n are at a D_U distance from $W_S^F(\eta'_n)$ less than V' implies that the D_U-distance between a point in b_n and the point of intersection of its $F-$unstable manifold with $W_S^F(\eta'_n)$ is less than V'.

Thus, we have that c_n is closed and that if ξ_n belongs to the boundary of c_n in b_n it has the following property : $W_U^F(\xi_n)$contains a singular point σ that lies in the interior of the arc of $W_U^F(\xi_n)$ whose end-points are ξ_n and the unique intersection point of $W_U^F(\xi_n)$ and $W_S^F(\eta'_n)$ (or coincides with ξ_n). Since only one prong of σ, say l, cuts $W_S^F(\eta'_n)$, and since the union of any two prongs of σ separate $\mathcal{M}$ and lies between $W_U^F(p''_n)$ and $W_U^F(q''_n)$, all other prongs must cut a_n at points of c_n (to prove it, consider the arc that consists of the union of l and another prong). Moreover the *mod*2$-$intersection number of this arc with the arc a_n, is 1.

Take an open arc in d_n whose end-points are in c_n. Call as before ξ_n one of its end-points, and σ the corresponding singularity. Clearly all the points $\zeta \in {}^B W_S^G(\xi)$ in this open arc lie in a sector limited by two consecutive prongs of σ, none of them being l. The $D-$ distance between ζ and the union of the two consecutive prongs of σ limiting the sector is less than V because the $D-$distance to $W_S^F(\eta'_n)$ is less than V and the sector lies in one of the components of the complement of $W_S^F(\eta'_n)$ and the border of the sector separates the sector and $W_S^F(\eta'_n)$. Take μ in the union of the two consecutive prongs, such that $D(\zeta, \mu)$ is the $D-$distance of ζ to this union. Since $D_S(\zeta, \mu) < V$ and $D_S(\mu, \xi_n) = 0$, we have that $D_S(\zeta, \xi_n) < V$; on the other hand, as ζ and ξ_n belong to a_n, $D_U(\zeta,\xi_n) < q$, according to Lemma 2.3. Thus, $D(\zeta,\xi_n) < V+q$.

Consider the closed arc of $W_U^F(\sigma)$ lying in the union of l and the prong that contains μ (if $\mu=\sigma$, take any one of the consecutive prongs) and included in the interior of $B_{2V}(\zeta)$ except for its end-points. Denote by α this arc. The D_U-distance (which coincides with the $D-$ distance) between ξ_n and any one of the end-points of this arc is larger than $V-q$ because $D(\zeta,\xi_n) < V+q$ and the $D-$distance between ζ and any end-point is $2V$. Since $V-q>q$, and the *mod* $2-$ intersection number of the union of the whole prongs with a_n is 1, the *mod* $2-$intersection number of α with a_n is also 1, because, again by Lemma 2.3, the intersection of the union of the whole prongs with a_n is contained in $B_q(\xi_n)$, and therefore coincides with $\alpha\cap a_n$.

Since the D_U-distance between ζ and any one of the end- points of the mentioned arc α is larger than $V-q-D_U(\zeta,\xi_n) > q$, the fact that C_0 and C_1 are not empty is a consequence of the arguments in the two last paragraphs of the proof of Proposition 2.1 and of the following remarks (that permit to conclude readily that the end-points of α belong to C, and, moreover, that they belong to different C_i, $i=0,1$) : I) $B_{2V}(\zeta)$ contains an arc whose *mod* 2-intersection number with a_n is 1. II) the $D-$disks are homotopically trivial, III) each one of the end-points of the arcs mentioned in I) have an $F-$image that lies outside $B_{2V}(G(\zeta))$,which may be shown as follows. Let υ be such an end-point; then $D_U(\upsilon,\zeta) > V-2q$ as it follows from arguments in the previous paragraph, and therefore,

$$D_U(F(\upsilon),G(\zeta)) > \lambda(V-2q)-r > 2V,$$

since $\lambda>5$, $V>4q$, and $q>r$.

Thus, since $B_{2V}(\zeta)$ is connected, for all $\zeta\in b_n$, $C\neq B_{2V}(\zeta)$, and consequently for each such ζ there exists η such that $D(F^\nu(\eta),G^\nu(\zeta))\leq 2V$ if $0\leq\nu\leq N$. In this way we get arcs b_n of arbitrarily large $D-$diameter whose points have $G-$trajectories shadowed (at a $D-$distance less than $2V$) by $F-$trajectories during the first N iterates.

These arcs project to M - under π- homeomorphically onto arcs of arbitrarily large length. Assume that $\pi(\zeta_n)\to x\in B$. Then, every point in the $B-$stable manifold of x is the limit of points on B with the property of having liftings to $\mathcal{M}$ that are shadowed, (at a $D-$distance less or equal than $2V$) by $F-$trajectories, during the first $N-$iterates. Therefore every point in the $B-$stable manifold of x, and, consequently, every point in B has the same property. Since this is true for every N, we conclude that any point $x\in B$ has a lifting ξ to $\mathcal{M}$ such that there exists $\eta\in\mathcal{M}$ with the property that

$$D(F^\nu(\eta),G^\nu(\xi))\leq 2V,\ \nu\geq 0.$$

Thus,

$$D_S(F^\nu(\eta),G^\nu(\xi))\leq 2V,\nu\geq 0$$

and therefore

$$\limsup_{\nu\to\infty} D_S(F^\nu(\eta),G^\nu(\xi))\leq \cdot\frac{\lambda}{\lambda-1}r.$$

On the other hand, $D_U(\xi,\eta) \le q$,for,otherwise

$$D_U(F^\nu(\eta), G^\nu(\xi)) > \lambda^\nu q - \frac{\lambda^\nu - 1}{\lambda - 1} r \to \infty$$

when $\nu \to \infty$, as follows immediately from $i)$. Since the same argument shows that

$$D_U(F^\nu(\eta), G^\nu(\xi)) \le q,$$

for every $\nu \ge 0$, we have then that

$$\limsup_{\nu\to\infty} D(F^\nu(\eta), G^\nu(\xi)) \le \cdot\frac{\lambda}{\lambda - 1} r + q \le p$$

by (2.2). Consequently, if $x \in B$,for each point $z \in \omega(x)$ there exists $y \in M$ such that

$$D(g^n(z), f^n(y)) \le p, \ n \in Z.$$

Since for many $x \in B$, $\omega(x) = B$, this completes the proof of the theorem. □

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