DYNAMICS OF VERTICAL DELAY ENDOMORPHISMS

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ABSTRACT. A vertical delay endomorphism F on $\mathbb{R}^k$, with $k \geq 2$, is the endomorphism associated to the difference equation $x_{n+k} = f(x_n, \cdots, x_{n+k-1})$, where the function f is C^2 and its partial derivative of second order with respect to the first variable is bigger than every other partial derivative of second order. The main goal of this paper is to describe the dynamical behavior of a huge class $\mathcal{F}$ of one-parameter families of vertical delay endomorphisms. We will prove that for any $\{F_\mu\}_{\mu \in \mathbb{R}}$ in $\mathcal{F}$ and every $|\mu|$ large enough, the nonwandering set $\Omega(F_\mu)$ of F_μ, is either the empty set or an expanding Cantor set and the restriction of F_μ to $\Omega(F_\mu)$ is conjugated to the unilateral shift on two symbols.

1. Introduction and statement of results

In the context of natural, physical or social sciences, discrete models are used to describe its dynamical phenomena. A large class of these models are expressed by means of difference equations of order $k \geq 1$, i.e. by equations with the following feature

$$x_{n+k} = f(x_n, \cdots, x_{n+k-1}), \ n \geq 0, \tag{1}$$

where f is a function defined on subsets of $\mathbb{R}^k$.

A standard way to study the limit behavior of any solution $\{x_n\}_{n \geq 1}$ of equation (1) consists in exploring the orbit structure of the discrete dynamical system given by the map

$$F(x_1, \cdots, x_k) = (x_2, \cdots, x_k, f(x_1, \cdots, x_k)). \tag{2}$$

In fact, the limit behavior of the solutions of (1) is inferred from the qualitative properties of the ω-limit set of F. The map given by (2) is known as the *delay endomorphism associated to f.*

In this paper we deal with delay endomorphisms such that the corresponding function belongs to the set of C^2 functions $f : \mathbb{R}^k \to \mathbb{R}$ satisfying

$$\inf_x \partial_{11} f(x) > \sum_{(i,j) \neq (1,1)} \sup_x |\partial_{ij} f(x)| \, ; \tag{3}$$

where $\partial_{ij} f(x)$ denotes $\dfrac{\partial^2 f(x)}{\partial x_i \partial x_j}$ for every i, j. Let $\mathcal{V}(\mathbb{R}^k)$ denote the set of functions defined on $\mathbb{R}^k$ and satisfying (3). Each function in this set is called a *vertical function* and the endomorphism F given by (2) will be called the *vertical delay endomorphism associated to f.*

Date: June 12, 2002.

Before stating the main result of this paper we introduce some notations and comments.

Let $\mathcal{H}_0$ be the set of C^1 endomorphisms $F : \mathbb{R}^k \to \mathbb{R}^k$ such that:

(1) ∞ is an attractor for F, this means that there exists $R > 0$ such that $\|F^n(x)\| \to +\infty$ when $n \to +\infty$ whenever $\|x\| > R$. If ∞ is an attractor for F, $B_\infty(F) = \{x : \|F^n(x)\| \to +\infty \text{ when } n \to +\infty\}$ is the basin of attraction of ∞;

(2) the nonwandering set of F, $\Omega(F)$, is either the empty set or a Cantor set which coincides with the complementary set of $B_\infty(F)$ and F restricted to $\Omega(F)$ is an expanding map.

From results showed by Mañé and Pugh [1] and Przytycki [2] it follows that every endomorphim in $\mathcal{H}_0$ is Axiom A and structuraly stable.

Now consider a vertical function $f : \mathbb{R}^k \to \mathbb{R}$ and the corresponding delay endomorphim F. It is clear that F has fixed points if and only if the graph of f intersects the diagonal of $\mathbb{R}^{k+1}$. On the other hand, the verticality condition on f implies that $t \to f^*(t) = f(t, \cdots, t)$ defines a convex function. Consequently there exists $\mu_0 \in \mathbb{R}$ such that for all $\mu < \mu_0$, the graph of $f_\mu = f - \mu$ does not intersect the diagonal of $\mathbb{R}^{k+1}$, in other words, for such value of the parameter μ the vertical delay endomorphism F_μ associated to f_μ has no fixed points. Moreover, F_μ has exactly two fixed points for all $\mu > \mu_0$.

Theorem 1. *Let $F : \mathbb{R}^k \to \mathbb{R}^k$ be a vertical delay endomorphism. If F has no fixed points, then the ω-limit set of any point is empty. In particular, $F \in \mathcal{H}_0$.*

The proof of this theorem is given in section 3. It is necessary to recall that under the same hypothesis (of absence of fixed points), the same qualitative property was proved for delay endomorphisms associated to convex functions with level sets more horizontal than vertical, see [3] for details. Since vertical functions are not necessarily convex, theorem 1 seems as a complementary result to the one obtained in [3].

The second and main result of this paper is related to the dynamical description of a large class of vertical delay endomorphims with fixed points. In fact, it will be proved the following theorem:

Theorem 2. *There exists an open subset $\mathcal{U}$ of $\mathcal{V}(\mathbb{R}^k)$ such that for every $f \in \mathcal{U}$ there exists $\mu_1 \in \mathbb{R}$ such that $\mu > \mu_1$ implies that the endomorphism F_μ associated to f_μ belongs to $\mathcal{H}_0$ and F_μ restricted to $\Omega(F_\mu)$ is conjugated to the unilateral shift on two symbols.*

Throughout this paper, the topology to be considered in $\mathcal{V}(\mathbb{R}^k)$ is the C^2 strong or Whitney topology.

The proof of this result will be given in section 4. We will define an open set $\mathcal{U}$ of vertical functions, such that every $f \in \mathcal{U}$ has the *attractor property*, that is: *there exists $\mu_1 \in \mathbb{R}$ such that, if F_μ is the delay endomorphisms associated to $f_\mu = f - \mu$, then F_μ has ∞ as an attractor for all $\mu > \mu_1$.*

Observe that if $f \in \mathcal{U}$ and f_μ and F_μ are as above, then combining theorem 1 and 2, one obtains an interval $[\mu_0, \mu_1] \subset \mathbb{R}$ such that for every $\mu \notin [\mu_0, \mu_1]$, F_μ belongs to $\mathcal{H}_0$. Before μ_0, B_∞ is the whole space, and after μ_1, it is a Cantor set. In the middle, a lot of changes occur in the dynamics of the mappings, some of which are explained in [4] and [5] for a particular class of vertical delay endomorphisms. See section 5 for numerical examples.

2. Image of vertical delay endomorphisms

We will now present some preliminaries on the image of vertical delay endomorphisms and some technical facts about C^2-convex functions; both are important to understand the dynamics of these endomorphisms.

For any vertical function $f : \mathbb{R}^k \to \mathbb{R}$ and its associated delay endomorphism F we will establish the relationship between the critical set of F and its image $Im(F) = F(\mathbb{R}^k)$. The set $crit(F) = \{x \in \mathbb{R}^k : \det D_x F = 0\}$ is called the *critical set of* F; here $D_x F$ denotes the differential matrix of F at x. It is easy to verify that $x \in crit(F)$ if and only if $\partial_1 f(x) := \dfrac{\partial f(x)}{\partial x_1} = 0$.

Recall that a function $g : \mathbb{R}^k \to \mathbb{R}$ is C^2*-convex* if it is C^2 and there is a positive constant α such that, for every unit vector $v \in \mathbb{R}^k$ and $x \in \mathbb{R}^k$ it holds $\langle H_g(x)v, v\rangle \geq \alpha$, where $H_g(x)$ denotes the Hessian matrix of g at x and $\langle \cdot, \cdot \rangle$ is the euclidian inner product. The following list contains some properties of a C^2-convex function g (see section 2 in [6]).

(a) g has a unique critical point, at this point g takes its minimun value: $\min g$;

(b) $\{g^{-1}(s) : s \geq \min g\}$ constitutes a partition of $\mathbb{R}^k$, where $g^{-1}(\min g)$ is the critical point of g, and $g^{-1}(s)$ with $s > \min g$ is a $(k-1)$-dimensional compact submanifold surrounding the critical point of g and has exactly two tangency points with hyperplanes $x_i = constant$;

(c) for each $i = 1, \cdots, k$, $\ell_i(g)$ denotes the set of all $x \in \mathbb{R}^k$ such that $\partial_i g(x) := \dfrac{\partial g(x)}{\partial x_i} = 0$. By the Implicit Function Theorem it follows that

$$\ell_i(g) = \{(x_1, \cdots, x_{i-1}, \tilde{x}_i(\widehat{x}_i), x_{i+1}, \cdots, x_k) : x_j \in \mathbb{R}\},$$

where $\widehat{x}_i = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_k)$ and $\tilde{x}_i : \mathbb{R}^{k-1} \to \mathbb{R}$ is a C^1 function satisfying

$$\frac{\partial \tilde{x}_i(\widehat{x}_i)}{\partial x_j} = -\frac{\partial_{ij} g(x_1, \cdots, x_{i-1}, \tilde{x}_i(\widehat{x}_i), x_{i+1}, \cdots, x_k)}{\partial_{ii} g(x_1, \cdots, x_{i-1}, \tilde{x}_i(\widehat{x}_i), x_{i+1}, \cdots, x_k)}, \tag{4}$$

for all $j \neq i$ and every $\widehat{x}_i$, where $\partial_{ij} g(x)$ denotes $\frac{\partial^2 g(x)}{\partial x_i \partial x_j}$. Clearly $\bigcap_{i=1}^k \ell_i(g)$ is the critical point of g and each tangengy point of $g^{-1}(s)$ $(s > \min g)$ with the hyperplane $x_i = constant$ belongs to $\ell_j(g)$ for all $j \neq i$. Similarly, if $C_i(g)$ denotes the set of all $x \in \mathbb{R}^k$ such that $\partial_j g(x) = 0$ for all $j \neq i$, then there exits a C^1 function $c_i(t) = (c_{1i}(t), \cdots, c_{i-1,i}(t), c_{i+1,i}(t), \cdots, c_{ki}(t))$ with $t \in \mathbb{R}$ such that, $x = (x_1, \cdots, x_k) \in C_i(g)$ if and only if there is $t \in \mathbb{R}$ with $x_i = t$ and $x_j = c_{ji}(t)$ for all $j \neq i$. Furthermore, the function c_i verifies

$$\sum_{j\neq i} \partial_{ji} g(\widehat{c}_i(t)) c'_{ji}(t) + \partial_{ii} g(\widehat{c}_i(t)) = \frac{\det H_g(\widehat{c}_i(t))}{\det H_g^i(\widehat{c}_i(t))} \geq \alpha, \tag{5}$$

where $H_g^i(x)$ is the matrix obtained from $H_g(x)$, taking off the i^{th} column and the i^{th} row, the constant α is taken from the C^2-convexity of g and $\widehat{c}_i(t) = (c_{1i}(t), \cdots, c_{i-1,i}(t), t, c_{i+1,i}(t), \cdots, c_{ki}(t))$.

Turning back to the vertical function f and its associated delay endomorphism F, it is always possible to find positive constants $a_1, \cdots, a_{k-1}$ such that

$$g(x_1, \cdots, x_k) = f(x_1, \cdots, x_k) + a_1 x_2^2 + \cdots + a_{k-1} x_k^2$$

defines a C^2-convex function. Obviously for such a function we have $\ell_1(g) = \ell_1(f)$, and so $crit(F) = \{(\tilde{x}_1(x_2, \cdots, x_k), x_2, \cdots, x_k) : x_j \in \mathbb{R}\}$ for some C^1 function $\tilde{x}_1 : \mathbb{R}^{k-1} \to \mathbb{R}$ satisfying

$$\frac{\partial \tilde{x}_1(\widehat{x}_1)}{\partial x_j} = -\frac{\partial_{1j} f(\tilde{x}_1(\widehat{x}_1), \widehat{x}_1)}{\partial_{11} f(\tilde{x}_1(\widehat{x}_1), \widehat{x}_1)}, \; j = 2, \cdots, k. \tag{6}$$

Denote by $P(F)$ the image under F of $crit(F)$. It is a trivial fact that $P(F)$ is the graph of $\varphi(x_2, \cdots, x_k) := f(\tilde{x}_1(x_2, \cdots, x_k), x_2, \cdots, x_k)$, and F is a one-to-one map from $crit(F)$ to $P(F)$. On the other hand, for every $\widehat{x} = (x_2, \cdots, x_k) \in \mathbb{R}^{k-1}$ define $\phi_{\widehat{x}} : \mathbb{R} \to \mathbb{R}$ by $\phi_{\widehat{x}}(t) = f(t, \widehat{x})$. It is immediate that $\phi'_{\widehat{x}}(t) = 0$ if and only if $t = \tilde{x}_1(\widehat{x})$; also $\phi''_{\widehat{x}}(t) \geq A(f) := \inf_x \partial_{11} f(x)$ for all $t \in \mathbb{R}$. Thus, for each $y_k > \phi_{\widehat{x}}(\tilde{x}_1(\widehat{x}))$ there are exactly two points $x_1^- < \tilde{x}_1(\widehat{x}) < x_1^+$ such that $\phi_{\widehat{x}}(t) = y_k$; while if $y_k < \phi_{\widehat{x}}(\tilde{x}_1(\widehat{x}))$, then there is not $t \in \mathbb{R}$ satisfying $\phi_{\widehat{x}}(t) = y_k$. In this way we conclude that

$$Im(F) = \{(y_1, \cdots, y_k) : y_k \geq \varphi(y_1, \cdots, y_{k-1})\},$$

and every point in $P^+(F) = \{(y_1, \cdots, y_k) : y_k > \varphi(y_1, \cdots, y_{k-1})\}$ has exactly two preimages under F located one at each side of the critical set of F, and the points in the set $P^-(F) = \{(y_1, \cdots, y_k) : y_k < \varphi(y_1, \cdots, y_{k-1})\}$ have no preimage under F.

3. On the absence of fixed points

Take a vertical function $f : \mathbb{R}^k \to \mathbb{R}$ and its associated delay endomorphism F, which has no fixed points. Denote by $\delta = \inf_x \partial_{11} f(x) - \sum_{(i,j) \neq (1,1)} \sup_x |\partial_{ij} f(x)| > 0$. In this section we describe the dynamics of F and therefore the limit behavior of the solutions of the difference equation of order k given by f.

Proof of Theorem 1:
Since F has no fixed point and the function $f^*(t) = f(t, \cdots, t)$ satisfies $(f^*)''(t) \geq \delta$ for all $t \in \mathbb{R}$, there exists $\mu_0 > 0$ such that $graph(f - \mu_0)$, graph of $f - \mu_0$, is tangent to the diagonal of $\mathbb{R}^{k+1}$ at a point $(t_0, \cdots, t_0)$, which by simplicity will be assumed to be the origin. Therefore, for every $x = (x_1, \cdots, x_k) \in \mathbb{R}^k$ one can write $f(x) = \mu_0 + \sum_{i=1}^k v_i x_i + \rho(x)$, where $\nabla f(0) = (v_1, \cdots, v_k)$ is the gradient vector of f at the origin and ρ is a C^2 function satisfying $\rho(0) = 0$, $\nabla \rho(0) = 0$ and $\partial_{ij} f(x) = \partial_{ij} \rho(x)$ for all $i, j = 1, \cdots, k$ and every $x \in \mathbb{R}^k$. Due to the geometry of the intersection between $graph(f - \mu_0)$ and the diagonal of $\mathbb{R}^{k+1}$ it follows that $\sum_{i=1}^k v_i = 1$. Recall that the tangent space of $graph(f - \mu_0)$ at $p = (a, f(a) - \mu_0)$ is the set of all vectors $(w, \langle \nabla f(a), w \rangle)$ with $w \in \mathbb{R}^k$.

Following the same strategy employed in [3] we will construct a continuous function $L : \mathbb{R}^k \to \mathbb{R}$ with positive orbital difference respect to F, that means $\Delta L(x) = (L \circ F - L)(x) > 0$ for all $x \in \mathbb{R}^k$. From this fact it follows that $\|F^n(x)\| \to +\infty$ when $n \to +\infty$ for every $x \in \mathbb{R}^k$, which proves the theorem.

Consider positive numbers $\epsilon_1, \cdots, \epsilon_{k-1}$ with $\epsilon_1 + \cdots + \epsilon_{k-1} = \frac{\delta}{2}$. For each $j = 1, \cdots, k-1$ define $\alpha_j = \frac{1}{2} \sum_{\ell=j}^{k-1} \left(\epsilon_\ell + \sum_{i=1}^k \sup_x |\partial_{\ell+1,i} f(x)| \right)$. Let $L : \mathbb{R}^k \to \mathbb{R}$

be the function given by $L(x_1, \cdots, x_k) = \sum_{i=1}^{k-1} \left(\alpha_i x_i^2 + \sum_{j=1}^{i} v_j x_i\right) + x_k$. Let us see that $\Delta L(x) > 0$ for every $x \in \mathbb{R}$. A simple verification shows that

$$\Delta L(x) = \mu_0 + \rho(x) - \alpha_1 x_1^2 + \sum_{j=2}^{k-1} (\alpha_{j-1} - \alpha_j) x_j^2 + \alpha_{k-1} x_k^2.$$

Obviously $\Delta L(0) = \mu_0$ and the gradient vector of ΔL vanishes at the origin; on the other hand, if $H_{\Delta L}(x) = [h_{ij}(x)]$ denotes the Hessian matrix of ΔL at x, then

$$h_{ij}(x) = \begin{cases} \partial_{ij} f(x) & \text{if } i \neq j \\ \partial_{11} f(x) - 2\alpha_1 & \text{if } i = j = 1 \\ \partial_{kk} f(x) + 2\alpha_{k-1} & \text{if } i = j = k \\ \partial_{jj} f(x) + 2(\alpha_{j-1} - \alpha_j) & \text{other wise} \end{cases}.$$

Recall that a real square matrix of order n, $B = [b_{ij}]$, is *diagonally dominated* if $b_{ii} > \sum_{j \neq i} |b_{ij}|$ for every $i = 1, \cdots, n$; see [7]. It is a simple exercise of Linear Algebra to show that if B is diagonally dominated, then $\det B > 0$. Now, for every $j = 2, \cdots, k$ let $H_{\Delta L}^{(j)}(x)$ be the square matrix obtained from $H_{\Delta L}(x)$ taking off the rows and columns indexed with $j, j+1, \cdots, k$. Finally, from the definition of the scalar α_j, $j = 1, \cdots, k-1$, it follows that $H_{\Delta L}(x), H_{\Delta L}^{(2)}(x), \cdots, H_{\Delta L}^{(k)}(x)$ are diagonally dominated, therefore $H_{\Delta L}(x)$ is positive definite for all $x \in \mathbb{R}^k$ and ΔL is a positive function. □

The obvious interpretation of this result in terms of difference equations of order k is expressed in the following corollary.

Corollary 1. *Let $f : \mathbb{R}^k \to \mathbb{R}$ be a vertical function. If the difference equation of order k given by f has no stationary solutions, then every solution has no limit points.*

We recall that in the context of difference equations, a stationary solution of the equation (1) is any constant sequence $\{x_n = \beta\}_{n \geq 0}$ such that $f(\beta, \cdots, \beta) = \beta$.

Remark 1. *Notice that in general the conclusion of theorem 1 is not true without the hypothesis of verticality. In fact, in the case $k = 2$ consider the function $f(x, y) = 2x^2 + y^2 + 4xy + 1$ which is not vertical; if $F(x, y) = (y, f(x, y))$ is the delay endomorphim associated to f, then it is easy to see that graph(f) has no points in the diagonal of $\mathbb{R}^3$, i.e. F has no fixed points; however, $\Omega(F)$ is not empty because F has periodic points of period two.*

4. Proof of Theorem 2

In this section we will prove the main theorem of this paper. Its proof will be contained in the propositions below. The first one gives a huge open set of vertical functions with the attractor property; the second one shows that for every vertical function f in the preceding open set, the nonwandering set of F_μ, (the delay endomorphism for $f_\mu = f - \mu$) is an expanding Cantor set which coincides with the complementary set of the basin of ∞ whenever the value of μ is large enough. In this case, the dynamics on that set is topologically conjugated to the unilateral shift on two symbols.

The next proposition contains the first part of theorem 2.

Proposition 1. *There exists an open set $\mathcal{U}$ in $\mathcal{V}(\mathbb{R}^k)$ such that every $f \in \mathcal{U}$ has the attractor property.*

Proof: Consider the function $L : \mathbb{R}^k \to \mathbb{R}$ given by

$$L(x_1, \cdots, x_k) = x_k + p(x_1, \cdots, x_{k-1}),$$

where $p(x_1, \cdots, x_{k-1}) = \sum_{i=1}^{k-1} \alpha_i x_i^2$. We will show that for every f in an open set $\mathcal{U}$, the numbers α_i's can be chosen in order to obtain, for every μ sufficiently large, the following facts:

(1) $L(x) > \mu$ implies $L(F_\mu(x)) - L(x) > 0$;
(2) $C(\mu) := F_\mu^{-1}(\{L \leq \mu\})$ is a compact set, where F_μ is the delay endomorphism associated to $f_\mu = f - \mu$.

The first item implies that if $x \in \{L > \mu\}$, then $F_\mu^n(x) \to \infty$, and the second one that if $x \in \mathbb{R}^k \setminus C(\mu)$, then $F_\mu^n(x) \to \infty$, so the proof will be complete. Observe that item 1 also implies that $C(\mu) \subset \{L \leq \mu\}$.

It is clear that

$$\begin{aligned} L(F_\mu(x)) - L(x) &= f_\mu(x) + p(x_2, \cdots, x_k) \\ &\quad -x_k - p(x_1, \cdots, x_{k-1}) \\ &= f(0) + \langle \nabla f(0), x \rangle - \mu \\ &\quad -x_k + ap(x_1, \cdots, x_{k-1}) + T(x), \end{aligned}$$

where $a > 1$ is an arbitrary constant and the function T is defined as

$$\begin{aligned} T(x) &= f_\mu(x) - f(0) - \langle \nabla f(0), x \rangle + \mu \\ &\quad +p(x_2, \cdots, x_k) - (1 + a)p(x_1, \cdots, x_{k-1}). \end{aligned}$$

Observe that $T(0) = \nabla T(0) = 0$ and if $H_T(x) = [t_{ij}]$ denotes the Hessian matrix of T at x, then

$$t_{ij}(x) = \begin{cases} \partial_{ij} f(x) & \text{if } i \neq j \\ \partial_{11} f(x) - 2\alpha_1(1 + a) & \text{if } i = j = 1 \\ \partial_{kk} f(x) + 2\alpha_{k-1} & \text{if } i = j = k \\ \partial_{jj} f(x) - 2\alpha_j(1 + a) + 2\alpha_{j-1} & \text{other wise} \end{cases}.$$

We will assume for a moment that with the choices of the values of the α_i's it holds that the function T satisfies $T(x) \geq \frac{b}{2}\|x\|^2$ for some positive number b and every $x \in \mathbb{R}^k$. Now suppose that $L(x) > \mu$, that is $p(x_1, \ldots, x_{k-1}) > \mu - x_k$, and substitute above on the formula for the orbital difference, to obtain:

$$\begin{aligned} L(F_\mu(x)) - L(x) &\geq \frac{b}{2}\|x\|^2 + \langle \nabla f(0), x \rangle \\ &\quad -(a + 1)x_k + (a - 1)\mu + f(0) \\ &\geq \frac{b}{2}\|x\|^2 - K_0\|x\| - K_1 + (a - 1)\mu, \end{aligned}$$

where K_0 and K_1 are constants not depending on x or μ. The right side of the last inequality is a polynomial in $\|x\|$, that becomes positive for every x if μ is large. In conclusion, $L(x) > \mu$ implies $L(F_\mu(x)) > L(x)$ if μ is large enough.

We will now define the α_i's and verify the remaining assertions. Recall that $A(f)$ denoted the infimum of the values $\partial_{11} f$; define $B(f) = \sum_{(i,j) \neq (1,1)} \sup_y |\partial_{ij} f(y)|$ and take any number $\epsilon > 3B(f)$. The definition begins as follows:

let $\alpha_{k-1} = \frac{1}{2}(\epsilon + \sup_y |\partial_{kk} f(y)|)$ and then define by recurrence for any $1 \le i \le k-2$, $\alpha_{k-1-i} = \frac{1}{2}(\epsilon + \sup_y |\partial_{k-i,k-i} f(y)| + 2(a+1)\alpha_{k-i})$.

It follows that:

$$\begin{array}{ll} t_{ij}(x) & = \partial_{ij} f(x) \text{ if } i \ne j, \\ t_{kk}(x) & = \partial_{kk} f(x) + \sup_y |\partial_{kk} f(y)| + \epsilon \ge \epsilon, \text{ and} \\ t_{jj}(x) & = \partial_{jj} f(x) + 2\alpha_{j-1} - 2(a+1)\alpha_j \\ & = \partial_{jj} f(x) + \epsilon + \sup_y |\partial_{jj} f(y)| \ge \epsilon, \text{ if } 1 < j < k \end{array}.$$

Finally, a simple recurrence implies that:

$$\begin{aligned} t_{11}(x) &= \partial_{11} f(x) - 2\alpha_1(a+1) \\ &= \partial_{11} f(x) - \sum_{s=1}^{k-1} (a+1)^s (\epsilon + \sup_y |\partial_{s+1,s+1} f(y)|). \end{aligned}$$

It is simple to see that $t_{11}(x)$ can be made bigger than ϵ if $A(f) > \bar{K} B(f)$, where $\bar{K} = 3 + 4\sum_{s=1}^{k-1}(a+1)^s$. Now, the condition $A(f) > \bar{K} B(f)$ defines an open set of vertical functions that will be denoted by $\mathcal{U}$.

Let $b = \epsilon - B(f)$ and observe that for every vector v with norm 1 it holds that:

$$\begin{aligned} \langle H_T(x) v, v \rangle &= \sum_{i=1}^{k} t_{ii} v_i^2 + \sum_{i \ne j} t_{ij} v_i v_j \\ &\ge \epsilon - \sum_{i \ne j} \sup_x |\partial_{ij} f(x)| \ge b > 0. \end{aligned}$$

This, together with $T(0) = 0$ and $\nabla T(0) = 0$, implies that $T(x) \ge \frac{b}{2}\|x\|^2$ for every x and so assertion 1 has been proved. To prove that $C(\mu)$ is a compact set, observe that this a consequence of the fact that the function $g_\mu(x) = f_\mu(x) + p(x_2, \ldots, x_k)$ is C^2-convex. Indeed, it is easy to verify that with this choice of the α_i's, the function g_μ has second derivative bounded below by ϵ. This finishes the proof of the proposition. □

Remark 2. *Consider a vertical function $f \in \mathcal{U}$. From the proof of the preceding proposition it follows that $\Lambda_\mu = \mathbb{R}^k \setminus B_\infty(F_\mu) = \bigcap_{n \ge 0} F_\mu^{-n}(C(\mu))$ for all μ large enough. As we mentioned at the end of the proof of the proposition, $C(\mu)$ is a compact set, and the estimates made to prove it, also imply that $C(\mu)$ is contained in the ball of center 0 and radius $a(\mu)$, with respect to the max norm in $\mathbb{R}^k$, and where $a(\mu)$ is a function satisfying*

$$\lim_{\mu \to \infty} \frac{a(\mu)}{\sqrt{\mu / b}} = 1.$$

Actually, $C(\mu)$ is contained in the interior of the image of F_μ, because the boundary of the image of F_μ is the image of the critical set of F_μ, and this can be parametrized as the image of a function $\Phi_\mu(x_1, \cdots, x_{k-1}) := f_\mu(\tilde{x}_1(x_1, \cdots, x_{k-1}), x_1, \cdots, x_{k-1}))$ satisfying $|\partial_{ij}\Phi| \le 2B(f)$.

Thus, it follows that $F_\mu^{-1}(C(\mu))$ does not intersect the set of critical points of F_μ and so it consists of the disjoint union of two sectors: $C^0(\mu)$ and $C^1(\mu)$, both contained in $C(\mu)$, see figure 1. In this way $\Lambda_\mu = \bigcap_{n \ge 0} F_\mu^{-n}(C_\mu^0 \cup C_\mu^1)$; consequently it is clear that for each $n \ge 1$ and $s_n \in \{0,1\}$, $F_\mu^{-n}(C^{s_n}(\mu))$ consists of 2^n subsectors $C^{s_0 \cdots s_n}(\mu)$, $s_0, \cdots, s_{n-1} \in \{0,1\}$, with the following properties:

(1) $C^{s_0\cdots s_n}(\mu) \subset C^{s_0\cdots s_{n-1}(\mu)}$,
(2) $F_\mu(C^{s_0\cdots s_n}(\mu)) \subset C^{s_1\cdots s_n}(\mu)$, *and*
(3) $x \in C^{s_0\cdots s_n}(\mu)$ *if and only if* $F_\mu^j(x) \in C^{s_j}(\mu)$ *for all* $j = 0, \cdots, n$.

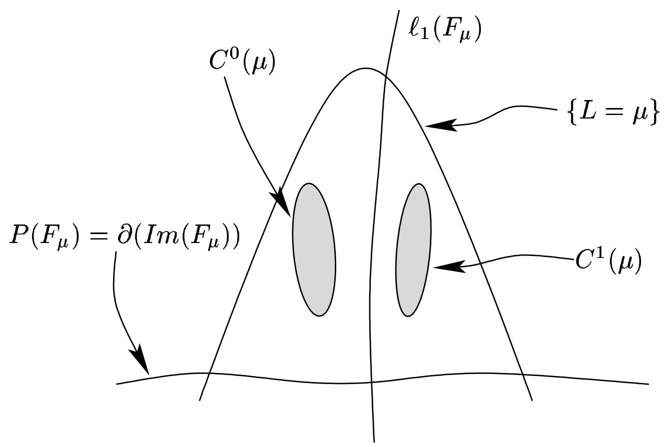

FIGURE 1. A two-dimensional picture of the localization of $C^0(\mu)$ and $C^1(\mu)$.

Obviously properties 1, 2 and 3 above imply that if $diam(C^{s_0\cdots s_n}(\mu)) \to 0$ when $n \to +\infty$ for all μ sufficiently large, then Λ_μ is a Cantor set and $F_\mu\mid_{\Lambda_\mu}$ is topologically conjugated to the unilateral shit on two symbols. Here $diam(B)$ denotes the diameter of the set B. To conclude the proof of theorem 2 we need to prove that $diam(C^{s_0\cdots s_n}(\mu)) \to 0$ when $n \to +\infty$ for all μ sufficiently large; but this a immediate consequence of the next proposition, which also implies that F_μ restricted to $\Lambda(\mu)$ is an expanding map.

Proposition 2. *If μ is large enough, there are constants $K > 0$ and $\rho > 1$ such that for every $x \in \Lambda_\mu$ and every $n \geq 1$ it holds that $\|D_x F_\mu^n\| \geq K\rho^n$.*

To prove this result we assume first the following lemma.

Lemma 1. *(i) If $x \in \Lambda_\mu$ and μ is large enough, then* $|\partial_1 f(x)| \geq 3\sum_{i\geq 2} |\partial_i f(x)|$.
(ii) Moreover, $|\partial_1 f(x)| \to +\infty$ when $\mu \to +\infty$ uniformily in $x \in \Lambda_\mu$.

Proof of Proposition 2: Consider on $\mathbb{R}^k$ the norm $\|v\|_\infty = \max_{1\leq i\leq k} |v_i|$. Observe that $D_x F_\mu(v_1, \ldots, v_k) = (v_2, \ldots, v_k, \sum_{i\geq 1} \partial_i f(x) v_i)$.

Take $x \in \Lambda_\mu$ and $v \in \mathbb{R}^k \setminus \{0\}$ with $\|v\|_\infty \leq 2$. If $|v_1| = 1$, then

$$\begin{aligned} \|D_x F_\mu(v)\|_\infty &\geq |\sum_{i\geq 1} \partial_i f(x) v_i| \\ &\geq |\partial_1 f(x)| \left(1 - \frac{\sum_{i\geq 2} |\partial_i f(x)|}{|\partial_1 f(x)|}\right); \end{aligned}$$

taking μ large enough it follows from lemma 1 that $\|D_x F_\mu(v)\|_\infty \geq 2$. A similar calculation shows that $\|D_x F_\mu(v)\|_\infty \geq 1$ for every $\|v\|_\infty = 1$ and μ large.

From this it follows that given $\|v\|_\infty = 1$, there exists j, $1 \leq j \leq k$ such that $|v_j| = 1$, and for this value of j it comes that $\|D_x F_\mu^j(v)\|_\infty \geq 2$. In conclusion, as DF never decreases norms, we obtain that $\|D_x F_\mu^k(v)\|_\infty \geq 2\|v\|_\infty$ for every vector v. Obviously this implies that there are constants $K > 0$ and $\rho > 1$ such that $\|D_x F_\mu^n(v)\| \geq K\rho^n\|v\|$ for all $v \in \mathbb{R}^k$, $n \geq 0$ and $x \in \Lambda_\mu$ with μ large enough. □

We conclude this section with the proof of lemma 1.

Proof of Lemma 1: Let $a(\mu)$ be as in remark 2. Let us see that for $i \in \{2, \cdots, k\}$ and all $\|x\| \leq a(\mu)$ it holds that:

$$|\partial_i f(x)| \leq |q_i| + a(\mu) \sum_{j \geq 1} \sup_x |\partial_{ij} f(x)|, \tag{7}$$

where q_i denotes $\partial_i f(0)$. In fact, take $\widehat{x} = (x_1, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_k)$ with $\|\widehat{x}\| \leq a(\mu)$, then

$$\partial_i f(\widehat{x}) = q_i + \int_0^1 \sum_{j \neq i} \partial_{ij} f(t\widehat{x}) dt,$$

therefore $|\partial_i f(\widehat{x})| \leq |q_i| + a(\mu) \sum_{j \neq i} \sup_x |\partial_{ij} f(x)|$. Now taking $x = (x_1, \cdots, x_k)$ with $\|x\| \leq a(\mu)$ and $\widehat{x}$ as above, one has:

$$|\partial_i f(x)| \leq |\partial_i f(\widehat{x})| + \left| \int_0^{x_i} \partial_{ii} f(\widehat{x} + te_i) dt \right|,$$

where e_i is the i^{th} canonical vector of $\mathbb{R}^k$; now (7) is immediate.

On the other hand, we claim that for $x \in \Lambda_\mu$ and μ large enough the following inequality is satisfied:

$$|\partial_1 f(x)| \geq 3D(f), \tag{8}$$

where

$$D(f) = \sum_{i > 1} \left[|q_i| + a(\mu) \sum_{j \geq 1} \sup_x |\partial_{ij} f(x)| \right].$$

Take any $x = (x_1, \cdots, x_k) \in \Lambda_\mu$ and let $\widetilde{x}$ be the point $(\tilde{x}_1(x_2, \cdots, x_k), x_2, \cdots, x_k)$ on the critical set of F_μ; clearly $\partial_1 f(x) = \int_{\tilde{x}_1(x_2, \cdots, x_k)}^{x_1} \partial_{11} f(t, x_2, \cdots, x_k) dt$ and

$$|\partial_1 f(x)| \geq A(f)\ |x_1 - \tilde{x}_1(x_2, \cdots, x_k)|. \tag{9}$$

Now we will take a minute to estimate the difference $f_\mu(x) - f_\mu(\widetilde{x})$. The points x and $\widetilde{x}$ only differ in the first coordinate, so assume that $x_1 > \tilde{x}_1(x_2, \cdots, x_k)$. As the critical set of F_μ does not depend on μ and it is almost vertical (close to a plane x_1 =constant, see equation (6)) we can take μ large in order to assure that $\tilde{x}_1(x_2, \cdots, x_k) \leq 2a(\mu)$. Furthermore, taking $\widetilde{x}_0 = (\tilde{x}_1(0, \cdots, 0), 0, \cdots, 0)$ and $\alpha(t) = (\tilde{x}_1(tx_2, \cdots, tx_k), tx_2, \cdots, tx_k)$, both in $crit(F_\mu)$, one has

$$\begin{aligned} |f(\widetilde{x}) - f(\widetilde{x}_0)| &= \left| \int_0^1 \langle \nabla f(\alpha(t)), \alpha'(t) \rangle dt \right| \\ &\leq \sum_{i=2}^k |x_i| \left| \int_0^1 \partial_i f(\alpha(t)) dt \right|; \end{aligned}$$

from this and (7) it follows that

$$|f(\tilde{x})| \leq |f(\tilde{x}_0)| + \sum_{i=2}^{k} a(\mu)(|q_i| + a(\mu)B(f)).$$

Therefore $f(\tilde{x})$ has order at most $B(f)a^2(\mu) \approx \frac{B(f)\mu}{b} \leq \frac{\mu}{2}$. We conclude then that

$$f_\mu(x) - f_\mu(\tilde{x}) = f_\mu(x) + \mu - f(\tilde{x}) \geq \mu/2 - a(\mu). \tag{10}$$

Suppose the opposite of (8); from

$$f_\mu(x) - f_\mu(\tilde{x}) = \int_{\tilde{x}_1(x_2,\cdots,x_k)}^{x_1} \partial_1 f(t, x_2, \cdots, x_k)dt \tag{11}$$

and (9) it is easy to conclude that

$$A(f)(\frac{\mu}{2} - a(\mu)) < 9(D(f))^2;$$

taking orders when $\mu \to \infty$, this implies that $A(f) < 9B(f)$, which contradicts the definition of $\mathcal{U}$. This proves item (i).

Finally, from equation (11) it comes that

$$f_\mu(x) - f_\mu(\tilde{x}) \leq \partial_1 f(x)|x_1 - \tilde{x}_1(x_2, \cdots, x_k)|;$$

so equation (10) implies that

$$|\partial_1 f(x)(x_1 - \tilde{x}_1(x_2, \cdots, x_k))| \geq \frac{\mu}{2} - a(\mu).$$

It follows that either $|\partial_1 f(x)| \to +\infty$ or $|x_1 - \tilde{x}_1(x_2, \cdots, x_k)| \to +\infty$ when $\mu \to +\infty$. In this case, using equation (9) we conclude $|\partial_1 f(x)| \to +\infty$ uniformily in $x \in \Lambda_\mu$. □

5. Examples and Conclusions

In this section we show some numerical examples of families of vertical delay endomorphisms in the light of the above theory.

Observe that for a quadratic function $f : \mathbb{R}^k \to \mathbb{R}$, the conditions of being vertical is easily seen to be equivalent to a relationship between the coefficients of the symmetric matrix associated with the quadratic form of f. For example in dimension two, if

$$f(x, y) = ax^2 + by^2 + cxy + dx + ey,$$

then f is vertical if $a > |b| + |c|$, and it has the attractor property whenever $a > 11(|b| + |c|)$, as can be deduced from the proof of proposition 1.

Recall that if f has the attracting property, then there are parameters μ_0 and μ_1, $\mu_0 < \mu_1$, such that for every $\mu < \mu_0$ the nonwandering set of the vertical delay endomorphism F_μ associated to $f_\mu = f - \mu$ is empty (because it has no fixed points, see theorem 1) and for every $\mu > \mu_1$ the nonwandering set of F_μ is an expanding Cantor set with chaotic dynamics, see theorem 2.

In figures 2 to 4 we follow the family F_μ for increasing values of μ where $f_\mu(x, y) = 4x^2 - 2y^2 + xy - \mu$. It is easy to see that for all $\mu < \mu_0 = -\frac{1}{12}$ the vertical delay endomorphism F_μ has no fixed points. On the other hand, although the criterion for the attracting property is not available to calculate explicity the number μ_1, numerical experiments show that for $\mu \gtrsim 0.4$ the nonwandering set of F_μ has uncountably many components because the set of critical points of F_μ is

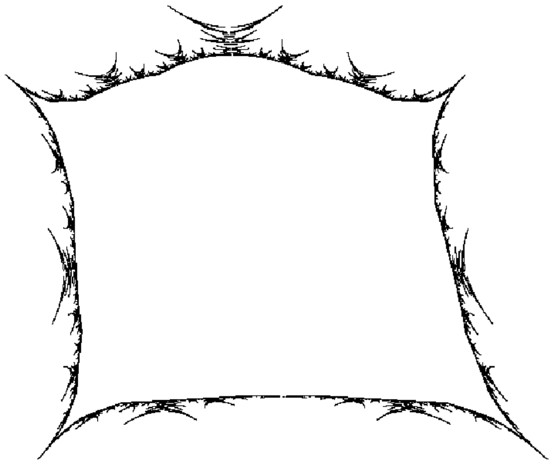

FIGURE 2. The picture represents the boundary of $B_\infty(F_0)$. It can be obtained plotting a large number of preimages under F_0 of points near the fixed point $(\frac{1}{3}, \frac{1}{3})$

contained in $B_\infty(F_\mu)$ (see theorem 1 of [5]). These conditions do not impliy that the nonwandering set of F_μ is expanding or a Cantor set.

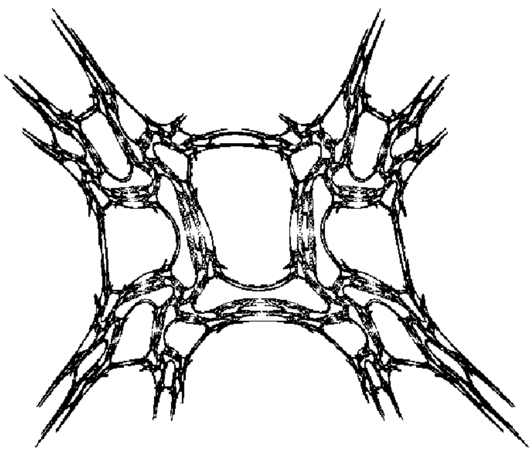

FIGURE 3. Same as in Fig. 2 this picture have been constructed by taking preimages of points near a fixed point of F_μ, in this case with $\mu = 0.3$. Here $B_\infty(F_{0.3})$ is not connected, in fact the forward orbit of every point at the white components is contained in $B_\infty(F_{0.3})$.

In figures 5 to 7 we consider the family of vertical functions $f_\mu = f - \mu$ and its associated delay endomorphisms F_μ where $f(x, y) = x^2 + 8x - 2y$. In this case it is obvious that the function f has the attracting property; in fact it is easy to verify that $\mu_0 = -\frac{25}{4}$ and $\mu_1 \gtrsim 4.83$.

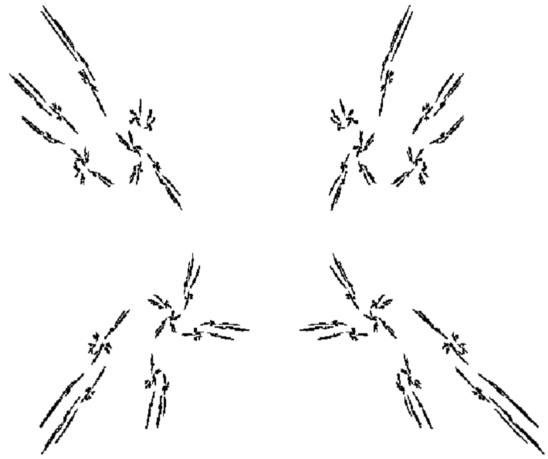

FIGURE 4. With $\mu = 0.48$ a large number of preimages of the fixed point $(0.6, 0.6)$ have been plotted. In this case the set of critical points of $F_{0.48}$ is contained in $B_\infty(F_{0.48})$. It seems that the complementary set of $B_\infty(F_{0.48})$ is a Cantor set and $B_\infty(F_{0.48})$ is a connected set.

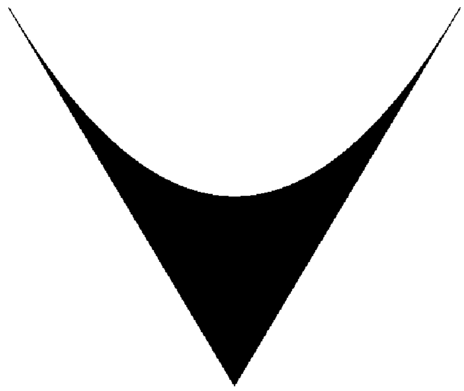

FIGURE 5. In this picture of the second example we have considered $\mu = 0$ and plotted a large number of preimages of the fixed point at the origin. This set is dense in the triangular region (black region). In this case the set J described in the last paragraph is just the boundary of this region and also the boundary of basin of attraction of ∞. This is a particular parameter where the mapping preserves an invariant measure absolutely continuous respect to Lebesgue measure on this triangular region.

With these examples we want to call atention on the fact that a huge number of changes in the dynamics of F_μ occur during the interval $[\mu_0, \mu_1]$, the nonwandering set of F_μ being empty for all $\mu < \mu_0$ ($B_\infty(F_\mu) = \mathbb{R}^2$) and a Cantor set for $\mu > \mu_1$.

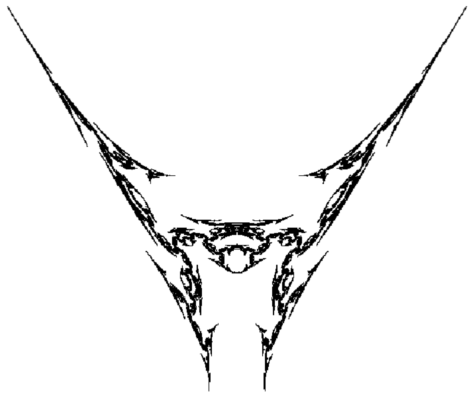

FIGURE 6. The same family of figure 5, now with $\mu = 0.3$. White regions correspond to the basin of attraction of ∞. Its complementary set is still connected.

FIGURE 7. The same family of the last two figures. For the value of $\mu = 6 > \mu_1$ preimages of the fixed point $(1,1)$ have been plotted. Observe that the basin of attraction of ∞ seems to be connected and its complementary set a Cantor set.

In [4] and [5] some of these changes were studied for pertubations of the two dimensional quadratic family $F_\mu(x,y) = (y, -x^2 + \mu x)$, where μ is a real parameter bigger than 1. It was proven there for this family, and it is conjectured for any vertical delay endomorphism, that for every μ such that the set of critical points of F_μ is not contained in the basin of attraction of ∞, there exists a connected set J containing one of the fixed points of F_μ and some critical points; moreover, J is forward invariant and it is contained in the boundary of $B_\infty(F_\mu)$. On the other hand, this connected set undergoes a fractalization process until it ceases to exist. The existence of this set J is present in the examples shown, see figures 2, 3, 5 and 6. We hope that the study and comprehension of the geometry and dynamics over it will be helpful in understanding some of the dynamical feature of the bifurcations occuring in the parameter interval $[\mu_0, \mu_1]$.

Acknowledgments This work was partially supported by Grant 004-DCT-2001 of CDCHT-UCLA. The first author thanks the hospitality of the Centro de Matemática at Universidad de la República where part of this research was done.

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