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Lyapunov functions and expansive diffeomorphisms on 3D-manifolds

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Dedicated to Emilia Barreiro de Vieitez, my mother, in memoriam

Abstract. Let M be a three-dimensional compact connected oriented manifold and $f : M \to M$ an expansive diffeomorphism. We prove that non-wandering points have local stable or unstable sets locally separating M. This property allows us to prove that if $\Omega(f) = M$ then f is conjugate to a linear Anosov diffeomorphism and $M = T^3$, the three-dimensional torus.

1. *Introduction*

This paper is intended to be a contribution to the classification of expansive homeomorphisms in the three-dimensional case. Using Lyapunov functions and the property that the topological entropy of an expansive homeomorphism is positive plus topological properties of a 3-manifold, we are able to prove that there are no 'pseudo-Anosov' diffeomorphisms defined on 3-manifolds. That is, we cannot have a diffeomorphism acting on a 3-manifold M such that for it we have local stable and unstable foliations defined on an open dense set of M and a set of codimension two of singularities (see [**Vi2**]) for which the local stable and unstable sets are not Euclidean balls. Moreover, this is a special property of dimension three, as we can give examples of diffeomorphisms with foliations with singularities in dimension $n \geq 4$ by the simple device of taking the product $f \times a$ of a pseudo-Anosov diffeomorphism f defined on a surface S of genus $g > 1$ and a linear Anosov diffeomorphism $a : T^{n-2} \to T^{n-2}$ defined on a $n-2$ torus in the obvious way. Then it is clear that $f \times a : S \times T^{n-2} \to S \times T^{n-2}$ is expansive and $\Omega(f \times a) = S \times T^{n-2}$ but the singularities of the pseudo-Anosov map f prohibit the existence of a uniform local product structure (see definitions later) in $S \times T^{n-2}$ which is an essential device to prove in our case that $M = T^3$. Still we have an open dense set in which we have well-defined local stable and unstable foliations. Moreover, the non-wandering set is the whole manifold. An example given by Franks and Robinson has shown (see [**FrRo**]) that the hypothesis $\Omega(f) = M$ cannot be dropped.

That example is a quasi-Anosov diffeomorphism defined in the amalgamated sum of two three-dimensional tori. It is known that quasi-Anosov maps are Axiom A and that they are the C^1 interior of the expansive diffeomorphisms. In this case the non-wandering set is composed of two basic hyperbolic parts, an expanding attractor A and a shrinking repeller R.

Hence it follows, again taking products, that there are examples in dimension greater than or equal to five of expansive diffeomorphisms with singularities such that for them the non-wandering set has at least two basic parts.

Why do these expansive diffeomorphisms in the three-dimensional case (assuming $\Omega(f) = M$) behave in this way? Roughly speaking the reason is that, if we assume that $\Omega(f) = M$ and that f is expansive, one of the local sets, say for instance the stable one, has to separate a small neighbourhood $V \equiv \mathbb{R}^3$ in M while the other one, say the unstable one, contains a non-trivial compactum but it cannot separate any neighbourhood (otherwise we would have a non-trivial intersection between a local stable set and a local unstable one and that would contradict expansivity). Some reflection shows that the asymmetry between the stable sets (they separate locally) and the unstable ones (they cannot separate but are non-trivial continua passing through the points) gives 'no place' for there to be more than 'one leaf' in the local stable set of a point (this is what occurs in the singularities of pseudo-Anosov maps defined on two-dimensional surfaces of genus greater than one). In a similar way there cannot be more than one unstable 'leaf' which (proving that it is locally connected) has to be an arc.

However, without assuming that the non-wandering set is all of M, it may be that none of these local sets separates at all. In Franks and Robinson's [**FrRo**] example this is exactly what happens. In this case $\Omega(f) = \mathcal{R} \cup \mathcal{A}$ where $\mathcal{R}$ is a shrinking repeller and $\mathcal{A}$ is an expanding attractor. Hence, for points in the shrinking repeller, the stable manifolds locally separate and the unstable manifolds are arcs. In the expanding attractor the situation reverses and outside the basic part both the unstable and stable manifolds are arcs immersed in M. Thus for points $x \notin \Omega(f)$, in the cited example, neither the stable manifolds nor the unstable ones locally separate the space.

Let us point out that the stable and unstable sets are manifolds in the cited example, while in this paper we have no right to assume this and most of our effort is aimed at proving that we still have Euclidean balls as local stable and unstable sets.

In order to achieve this end we assume that f is smooth. The relation between differentiability and the regularity of stable and unstable sets is not well understood but roughly speaking its use in this article is as follows. Expansive homeomorphisms have positive topological entropy and for them there exists a measure which is maximal. Using Pesin's theory we have recurrent points x with strong stable and unstable manifolds which, at least, are arcs passing through x.

From this point, assuming that the local unstable sets do not separate the space, we take the iterates of a disc transversal to the unstable arc through x and which has the recurrent point x in its interior in order to obtain, in the α-limit of x, two-dimensional continua which will be part of the local stable set of those limit points. Using the fact that $\Omega(f) = M$ we extend the separation property of the local stable sets of the points $v \in \alpha(x)$ to all points in M.

Finally using the fact that M is three-dimensional we find that the local stable sets are discs and the local unstable sets are arcs. This provides us with C^0 foliations by planes and lines transversal to each other.

From this we immediately have the Pseudo-Orbit Tracing Property (POTP) and that $M \sim T^3$. A result of Hiraide (see [**H1**]) states that expansive homeomorphisms with POTP defined on tori are conjugated to linear Anosov diffeomorphisms. So we are done.

The following questions arise.

- Are there expansive diffeomorphisms with singularities defined on manifolds of dimension four such that the non-wandering set has more than one part? Is an expansive diffeomorphism defined on T^4 always conjugate to an Anosov diffeomorphism?
  Dimension four has many complications. Most of them can be overcome by the fact that if $n \geq 4$, then we can have $\mathbb{R}^n \equiv H \times l$ with $l \equiv \mathbb{R}$, but with H a set that is not a manifold at all (a wild set). If $n \leq 3$ this cannot occur, that is, H has to be homeomorphic to $\mathbb{R}^{n-1}$.
- Without assuming that $\Omega(f) = M$, what can we say in dimension three? Can we have a sort of pseudo-Anosov map without assuming in this case that the non-wandering set is all of M? In other words, can we have points with local stable and unstable sets that are not manifolds for $f : M \to M$, an expansive diffeomorphism defined on a three-dimensional manifold M? What kind of 3-manifolds support the action defined by an expansive diffeomorphism without assuming $\Omega(f) = M$? Partial results in this direction have been obtained by the author in collaboration with Ma. Alejandra Rodríguez Hertz and R. Ures for quasi-Anosov maps (preprint).
- Can we obtain differentiable models (for dimension three or more) for all the expansive homeomorphisms? Moreover, are there always analytical models? The answer is yes in the two-dimensional case (see [**H2**] or [**Le2**]).
- In the three-dimensional case dropping differentiability, can we obtain local stable (or unstable) separating sets? If so, are they always locally connected? As pointed out previously, differentiability is used here to apply Pesin's theory on the existence of strong stable and unstable manifolds. This, in turn, is used to find local stable separator sets. It seems possible to try to obtain the same results without assuming differentiability.

Let us briefly describe the contents of this paper. Lyapunov functions are constructed in §2. They are used as analytical tools to obtain the separation properties for the local stable sets which is done in §3. In §4 all the parts needed to ensure that we fulfil the hypotheses of §3 are collected. In §5 the basic property that there is a uniform local product structure in the whole manifold is proved. Finally in §6 we use some of my previous results (see [**Vi2**] and [**Vi3**]) and the main thesis of this paper, assuming the existence of a uniform local product structure, is proved. In [**Vi5**] a more detailed proof of the fact that the existence of a uniform local product structure implies that $M \sim T^3$ is given.

2. *Lyapunov functions*

Let us assume here that $f : M \to M$ is an expansive diffeomorphism and that M is a 3-manifold. Fix a point $x \in M$ and assume, moreover, that we have some knowledge about the local unstable manifold $W^u_\epsilon(x)$ of x. Assume explicitly assume that $W^u_\epsilon(x)$ does not locally separate M, i.e. there is no open connected subset $A \subset M$ such that $A\backslash W^u_\epsilon(x)$ is not connected. Let us also assume that there is an arc U contained in $W^u_\epsilon(x)$, $x \in U$ such that the first homology group $H_1(B(x,r)\backslash U, \mathbb{Z})$ does not vanish for a small $r > 0$. In this case we will show that the procedure used by Lewowicz in [**Le1**] and Ures in [**Ur**] to construct Lyapunov functions may be improved to obtain a circle $\mathcal{C}$ in $B(x,r)$ and a Lyapunov function U for f such that, for all $y \in \mathcal{C}$, $\Delta U(x,y) < 0$. Moreover, $\mathcal{C}$ cannot be shrunk to a point in $B(x,r)\backslash U$. We will follow closely the exposition of Ures [**Ur**] in the construction of U.

Let $f : M \to M$ be as before and $\alpha > 0$ a constant of expansivity for f. Let $0 < \beta < \frac{1}{2}\alpha$. We begin by defining

$$C^+ = \{(x,y) \in M \times M / d(f^n(x), f^n(y)) \le \beta, n \ge 0\}$$
$$C^- = \{(x,y) \in M \times M / d(f^n(x), f^n(y)) \le \beta, n \le 0\}.$$

LEMMA 2.1. *Endowing $M \times M$ with the topology inherited by the metric d of M, C^+ and C^- are closed subsets of $M \times M$.*

Proof. See Ures' article [**Ur**]. □

Let $0 < \delta < \beta$ and let us define C^∞ functions $h : M \times M \to \mathbb{R}^+$ such that

$$\begin{aligned} h(x,y) &= 0 \quad \text{if } (x,y) \in C^- \text{ or } d(x,y) > \beta \\ 1 \ge h(x,y) &> 0 \quad \text{elsewhere} \end{aligned}$$

and $H : M \times M \to \mathbb{R}^+$ such that

$$\begin{aligned} H(x,y) &= 1 \quad \text{if } d(x,y) < \delta \\ H(x,y) &= 0 \quad \text{if } d(x,y) > \beta \\ 1 \ge H(x,y) &> 0 \quad \text{elsewhere.} \end{aligned}$$

These functions clearly exist.

The following function is used to control the number of iterates for which two given points remain neighbours:

$$\prod_{j=0}^{n} H(f^j(x), f^j(y)).$$

If $x, y \in M$ are rather close then the product should be one. However, if there is an N such that $d(f^N(x), f^N(y)) > \beta$ then for any $n \ge N$ we have $\prod_{j=0}^{n} H(f^j(x), f^j(y)) = 0$. Let us define

$$a_n(x,y) = h(f^n(x), f^n(y)) \prod_{j=0}^{n} H(f^j(x), f^j(y)).$$

LEMMA 2.2. *The sequence $\{a_n(x,y)\}$ is uniformly convergent to zero. That is, given $\beta > 0$ there is an $N \in \mathbb{N}$ such that if $n \ge N$ then $|a_n(x,y)| = a_n(x,y) < \beta$ for all $(x,y) \in M \times M$.*

Proof. See [**Ur**]. □

LEMMA 2.3. *There exists a C^∞ function $e : \mathbb{R}^+ \to \mathbb{R}^+$, $e(0) = 0$, $e'(x) > 0$ for all $x \in \mathbb{R}^+$, such that the series*

$$\sum_{n=0}^{+\infty} e(a_n(x, y))$$

converges uniformly for all $(x, y) \in M \times M$.

Proof. See [**Ur**]. □

From Lemma 2.3 we have that $V^- : M \times M \to \mathbb{R}^+$ given by

$$V^-(x, y) = \sum_{n=0}^{+\infty} e(a_n(x, y))$$

is a uniformly continuous function such that $V^-(x, y) \geq 0$ and vanishes iff $(x, y) \in C^-$ or $d(x, y) \geq \beta$.

Remark 2.4. A refinement of the arguments used in [**Ur**] may be used to prove that V^- can be assumed to be C^r where r is the degree of differentiability of f.

Let $\delta \geq \gamma > 0$ be such that if $d(x, y) < \gamma$, $x, y \in M$, then $d(f(x), f(y)) < \delta$ and let us compute

$$\Delta V^-(x, y) = V^-(f(x), f(y)) - V^-(x, y)$$

for $x, y \in M$ such that $d(x, y) < \gamma$. We have

$$\Delta V^-(x, y) = V^-(f(x), f(y)) - V^-(x, y) = \sum_{n=0}^{+\infty} e(a_n(f(x), f(y))) - \sum_{n=0}^{+\infty} e(a_n(x, y)).$$

As $d(x, y) < \delta$, $d(f(x), f(y)) < \delta$ we have that $H(x, y) = H(f(x), f(y)) = 1$ and therefore

$$a_n(f(x), f(y)) = h(f^n(f(x)), f^n(f(y))) \prod_{j=0}^{n} H(f^j(f(x)), f^j(f(y))) = a_{n+1}(x, y).$$

It follows that

$$\Delta V^-(x, y) = -e(a_0(x, y)) \leq 0.$$

Moreover, $\Delta V^-(x, y) = -e(a_0(x, y)) = 0$ iff

$$a_0(x, y) = h(x, y)H(x, y) = h(x, y) = 0$$

iff $(x, y) \in C^-$.

In a similar way, changing C^- to C^+, $n \geq 0$ to $n \leq 0$, and redefining the functions h, H, a_n and e in a convenient way we may find $V^+(x, y)$ such that $\Delta V^+(x, y) = e(a_0(x, y)) \geq 0$ and $\Delta V^+(x, y) = 0$ iff $(x, y) \in C^+$ or $\operatorname{dist}(x, y) \geq \beta$.

Let us now define

$$b_n(x, y) = V^-(f^n(x), f^n(y)) \prod_{j=0}^{n} H(f^j(x), f^j(y))$$

then as in Lemma 2.2 we may prove that $b_n(x, y)$ converges uniformly, and as in Lemma 2.3, we may prove that the function

$$U^-(x, y) = \sum_{n=0}^{+\infty} g(b_n(x, y))$$

is uniformly continuous, where $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^∞ increasing function analogous to the function e in Lemma 2.3.

If we compute $\Delta U^-(x, y)$ when $d(x, y) < \gamma$, we obtain

$$\Delta U^-(x, y) = U^-(f(x), f(y)) - U^-(x, y) = -g(b_0(x, y)) \leq 0.$$

Moreover, $\Delta U^-(x, y) = 0$ iff $(x, y) \in C^-$.

Let us define

$$N = \{(x, y) \in M \times M / d(x, y) < \nu\} \tag{$**$}$$

where $\nu > 0$ is chosen such that $d(x, y) < \nu$ implies $d(f(x), f(y)) < \gamma$. Hence if $(x, y) \in N$,

$$\begin{aligned}\Delta^2 U^-(x, y) &= \Delta(\Delta U^-(x, y)) \\ &= [U^-(f^2(x), f^2(y)) - U^-(f(x), f(y))] - [U^-(f(x), f(y)) - U^-(x, y)] \\ &= -g(b_0(f(x), f(y))) + g(b_0(x, y)) \\ &= -g(V^-(f(x), f(y))) + g(V^-(x, y)) \geq 0.\end{aligned}$$

The last assertion follows from the facts that $\Delta V^-(x, y) \leq 0$ and g is an (strictly) increasing function. Once again $\Delta^2 U^-(x, y) = 0$ iff $(x, y) \in C^-$. In a similar way we may find a $U^+(x, y)$ defined on N such that $U^+(x, y)$, $\Delta U^+(x, y)$, $\Delta^2 U^+(x, y)$ are greater than or equal to zero and vanish iff $(x, y) \in C^+$. Pick some $\lambda > 0$ and define on N the function $U : N \to \mathbb{R}^+$ by

$$U(x, y) = \lambda U^+(x, y) + U^-(x, y).$$

We will refine the choice of λ later (see the choice of λ below Lemma 2.12).

The function U has the following properties.

(1) Both $\Delta^2 U(x, y)$ and $U(x, y)$ are greater than or equal to zero and both vanish iff $(x, y) \in C^- \cap C^+$, i.e. $x = y$ (expansive properties of f).

(2) $\Delta U(x, y) = \lambda \Delta U^+(x, y) + \Delta U^-(x, y) = \lambda \Delta U^+(x, y) > 0$ if $(x, y) \in C^-$, $x \neq y$.

(3) $\Delta U(x, y) = \lambda \Delta U^+(x, y) + \Delta U^-(x, y) = \Delta U^-(x, y) < 0$ if $(x, y) \in C^+$, $x \neq y$.

(4) Remark 2.4 holds for U, ΔU and $\Delta^2 U$, so we may assume that these functions are C^r where r is the degree of differentiability of f.

Remark 2.5. We will use Lyapunov functions in the suspension $(\hat{M}, \varphi)$ of (M, f). More precisely, we want to define Lyapunov functions in

$$N = \bigcup_{t \in \mathbb{R}} M_t \times M_t$$

where as usual $M_t = p(M \times \{t\})$, $p : M \times \mathbb{R} \to \hat{M}$ being the suspension projection, $p(x, t + n) = p(f^n(x), t)$ for all $n \in \mathbb{Z}$, $x \in M$, $t \in \mathbb{R}$. We sketch the construction of these functions from the Lyapunov function $U(x, y)$ and $\Delta U(x, y)$, $\Delta^2 U(x, y)$. The reader may see a detailed proof in [**Le1**, p. 203, Theorem 5.1].

Let $q(s)$ be a C^∞ real increasing function defined in $0 \le s \le 1$ such that $q(0) = 0$, $q(1) = 1$, and $q^{(n)}(0) = q^{(n)}(1) = 0$ for all $n = 1, 2, \ldots$. If $x, y \in M$, $s \in [0, 1)$, we define

$$V(p(x, s), p(y, s)) = (1 - q(s))\Delta U(x, y) + q(s)\Delta U(f(x), f(y)).$$

As in [**Le1**, p. 204], it follows that if $x \ne y$ and $d(x, y) < \nu$, $\nu > 0$ as in $(**)$, then

$$V(p(x, s+1), p(y, s+1)) - \mathcal{V}(p(x, s), p(y, s)) > 0.$$

Therefore, defining

$$\mathcal{V}(p(x, t), p(y, t)) = \int_0^1 V(\varphi(p(x, t), s), \varphi(p(y, t), s))\, ds$$

and taking into account that $\varphi(p(x, t), s) = p(x, t + s)$ we have that $\dot{\mathcal{V}}(p(x, t), p(y, t))$ is positive (see [**Le1**, p. 204]). Moreover, if $p(y, t) \in W^s_\epsilon(p(x, t))$ then $\mathcal{V}(p(x, t), p(y, t)) > 0$ and if $p(y, t) \in W^u_\epsilon(p(x, t))$ then $\mathcal{V}(p(x, t), p(y, t)) < 0$. These follow from the analogous properties of $V(x, y)$.

To finish, define

$$\begin{aligned}\mathcal{U}(p(x, t), p(y, t)) &= U(x, y) + \int_0^t \mathcal{V}(p(x, s), p(y, s))\, ds \\ &= U(x, y) + \int_0^t ((1 - q(s))\Delta U(x, y) + q(s)\Delta U(f(x), f(y)))\, ds.\end{aligned}$$

As $\Delta^2 U(x, y) > 0$, if $x \ne y$, $\Delta U(f(x), f(y)) > \Delta U(x, y)$ and therefore if $\Delta U(f(x), f(y)) < 0$

$$\begin{aligned}&U(x, y) + \int_0^t ((1 - q(s))\Delta U(x, y) + q(s)\Delta U(f(x), f(y)))\, ds \\ &\quad > U(x, y) + \int_0^1 ((1 - q(s))\Delta U(x, y) + q(s)\Delta U(x, y))\, ds \\ &\quad = U(f(x), f(y)) > 0.\end{aligned}$$

Hence $\mathcal{U}(p(x, t), p(y, t))$ is always positive if $x \ne y$ and vanishes in the diagonal.

It is easy to see that

$$\dot{\mathcal{U}}(p(x, t), p(y, t)) = \mathcal{V}(p(x, t), p(y, t))$$

and this finishes the construction of the Lyapunov functions on $(\hat{M}, \varphi)$.

Let us assume that we have C^r Lyapunov functions $\mathcal{U}, \dot{\mathcal{U}}, \ddot{\mathcal{U}}$ as constructed earlier for $(\hat{M}, \varphi)$ and define, for $\sigma > 0$,

$$K_t(x, \sigma) = \{y \in \hat{M}_t / U(\varphi_t(x), y) \le \sigma\}$$

as the t-section of the σ-tube around x.

We define the σ-tube around x as

$$K(x,\sigma)=\bigcup_{t\in\mathbb{R}} K_t(x,\sigma).$$

The boundary of the t-section of the σ-tube around x is

$$\partial K_t(x,\sigma)=\{y\in \hat{M}_t/U(\varphi_t(x),y)=\sigma\}$$

and the boundary of the σ-tube is

$$\partial K(x,\sigma)=\bigcup_{t\in\mathbb{R}} \partial K_t(x,\sigma).$$

For $y\in K_t(x\sigma)$, $\psi>0$, we also define

$$t^+(x,y,t)=\sup\{\tau\geq t/\varphi_s(y)\in K_s(x,\sigma+\psi)\ \forall s: t\leq s\leq\tau\}-t$$
$$t^-(x,y,t)=\inf\{\tau\leq t/\varphi_s(y)\in K_s(x,\sigma+\psi)\ \forall s: \tau\leq s\leq t\}-t.$$

$t^+(x,y,t)$, $t^-(x,y,t)$ are the times needed to leave the slightly greater tube $K(x,\sigma+\psi)$ in the future and in the past for the y-trajectory, respectively, measured from t, $t^+>0$, $t^-<0$ (cf. [**Ft**]). It follows from the expansive properties of φ that $t^+=+\infty$ and $t^-=-\infty$ iff $x=y$. Moreover, $t^+=+\infty$ iff $y\in W^s_\epsilon(\varphi_t(x))$ and $t^-=-\infty$ iff $y\in W^u_\epsilon(\varphi_t(x))$, where $\epsilon>0$ is so small that $y\in K_t(x\sigma)$ implies $d(y,\varphi_t(x))<\epsilon$, $t\in\mathbb{R}$.

LEMMA 2.6. *t^+ and t^- are continuous functions.*

Proof. This follows from the fact that $\ddot{\mathcal{U}}$ is bounded away from zero on $\partial K(x,\sigma+\psi)$. □

Remark 2.7. For small values of $s\in\mathbb{R}$ we have that

$$t^+(\varphi_s(x),\varphi_s(y),t+s)=T^+(x,y,t)-s$$
$$t^-(\varphi_s(x),\varphi_s(y),t+s)=T^-(x,y,t)-s.$$

We will write $t^+(x,y)$ and $t^-(x,y)$ for $t^+(x,y,0)$ and $t^-(x,y,0)$, respectively.

LEMMA 2.8. *Let $r_0>0$ be such that* $\operatorname{dist}(x,y)<r_0$ *implies $\mathcal{U}(x,y)<\sigma$. For all $r_0>r>0$, there is a $T>0$, depending on r, such that if $d(\varphi_t(x),\varphi_t(y))\geq r$ then $\mathcal{U}(\varphi_{t+s}(x),\varphi_{t+s}(y))=\sigma+\psi$ for some $s\in[-T,T]$.*

Proof. Let $m=\min\{\ddot{\mathcal{U}}(y,z)/\operatorname{dist}(y,z)\geq r\}$ then $m>0$. Assume that $\dot{\mathcal{U}}(\varphi_t(x),\varphi_t(y))\geq 0$. As $\ddot{\mathcal{U}}$ is positive we have that $\dot{\mathcal{U}}(\varphi_{t+s}(x),\varphi_{t+s}(y))>0$, $s>0$ as long as the previous expression makes sense. Let $\mu>0$ be such that $\operatorname{dist}(y,z)>r$ implies $\mathcal{U}(y,z)\geq\sigma-\mu>0$. Hence, $\mathcal{U}(\varphi_{t+s}(x),\varphi_{t+s}(y))>\sigma-\mu$, $s>0$. By Taylor's expansion we have that

$$\mathcal{U}(\varphi_{t+s}(x),\varphi_{t+s}(y))\geq\sigma-\mu+m\tfrac{1}{2}s^2.$$

Thus it suffices to make $T=\sqrt{2(\mu+\psi)/m}$. If we now have $\dot{\mathcal{U}}(\varphi_t(x),\varphi_t(y))\leq 0$ taking $s<0$ we find the same bound. □

COROLLARY 2.9. *A necessary and sufficient condition to have $t^+(x,y)$ and $t^-(x,y)$ both finite is the existence of a t_0 such that $\dot{\mathcal{U}}(\varphi_{t_0}(x),\varphi_{t_0}(y))=0$.*

Remark 2.10. We remark here the trivial fact that for $t^+(x, y)$ (respectively $t^-(x, y)$) we have $\dot{\mathcal{U}} > 0$ (respectively $\dot{\mathcal{U}} < 0$) provided $t^+(x, y)$ (respectively $t^-(x, y)$) is finite.

PROPOSITION 2.11. *The functions t^+ and t^- are smooth.*

Proof. We prove smoothness for t^+, the same proof is valid for t^-. Let us assume that $t^+(x, y)$ is finite. We have

$$\mathcal{U}(\varphi_{t^+(x,y)}(x), \varphi_{t^+(x,y)}(y)) = \sigma + \psi$$
$$\dot{\mathcal{U}}(\varphi_{t^+(x,y)}(x), \varphi_{t^+(x,y)}(y)) > 0.$$

As $\mathcal{U}$ is smooth as a function of (x, y, t), we may apply the Implicit Function Theorem to conclude that $t^+(x, y)$ is smooth. □

Assume now that, for a certain $x \in M$, $W^u_\epsilon(x)$ does not locally separate M and that it contains a continuum γ such that, for some $r > 0$, $B(x, r)\backslash\gamma$ is not homotopically trivial.

Observe that the subset $\{x\} \times W^u_\epsilon(x) \cap N \subset C^-$. Reciprocally, if $(x, y) \in C^-$ then $y \in W^u_\epsilon(x)$.

LEMMA 2.12. *There is a simple closed curve $\mathcal{C} \subset B(x, r)\backslash\gamma$, such that $\mathcal{C} \cap W^u_\epsilon(x) = \emptyset$. Moreover, $\mathcal{C} \not\simeq 0$ in $B(x, r)\backslash\gamma$.*

Proof. Otherwise for every simple closed curve $\mathcal{C} \not\simeq 0$ in $B(x, r)\backslash\gamma$ there is a point $y \in \mathcal{C} \cap W^u_\epsilon(x)$. From this it is not difficult to see that $W^u_\epsilon(x)$ locally separates M, contradicting our assumptions. □

Choose a simple closed curve $\mathcal{C}$ as in Lemma 2.12. Then there is an $R > 0$ such that $\text{dist}(\mathcal{C}, W^u_\epsilon(x)) > R > 0$. Therefore, for any point $y \in \mathcal{C}$, $t^-(x, y)$ is finite. By the compactness of $\mathcal{C}$ and the continuity of $t^-(x, y)$ and $t^+(x, y)$ we may find $\lambda > 0$ in order to ensure that

$$\dot{\mathcal{U}}(x, y) = \lambda\dot{\mathcal{U}}^+(x, y) - \dot{\mathcal{U}}^-(x, y) < 0, \quad \text{for all } y \in \mathcal{C}.$$

Moreover, we can choose $\mathcal{C}$ to have as small a diameter as we wish.

LEMMA 2.13. *Given $r > 0$ there is an $s > 0$ such that, for all $x \in M$,*

$$\text{dist}(W^s_\epsilon(x)\backslash B(x, r), W^u_\epsilon(x)\backslash B(x, r)) \geq s.$$

Proof. See [**Vi2**]. □

3. *Separation properties*

In the two-dimensional case it has been proved by Lewowicz (see [**Le2**]) that both $W^s_\epsilon(x)$ and $W^u_\epsilon(x)$ locally separate the surface on which they live. In the three-dimensional case it happens that none of them has this property, for instance, in the quasi-Anosov diffeomorphism constructed by Franks and Robinson (see [**FrRo**]).

Quasi-Anosov maps are Ω-stable and for them Smale's Spectral Decomposition Theorem holds, so it is clear that if x is a non-wandering point then $W^s_\epsilon(x)$ and $W^u_\epsilon(x)$ are transverse. We may wonder whether the same is true in the more general setting of assuming only expansivity and differentiability. The question makes sense since it has been proved by Mañé (see [**Ma1**]) that quasi-Anosov diffeomorphisms are the C^1 interior of expansive diffeomorphisms.

The answer is 'No', as examples given by Lewowicz and Enrich show. Lewowicz's example (see [**Le1**]) is given by

$$F(x, y) = \left(2x - \frac{1}{2\pi}\sin 2\pi x + y, x - \frac{1}{2\pi}\sin 2\pi x + y\right)$$

in T^2. Here $(0, 0)$ is a non-hyperbolic fixed point, and $F(x, y)$ is conjugate to a linear Anosov diffeomorphism.

In his paper, Enrich (see [**En**]) gives an example with a pair of periodic points p, q, such that the unstable manifold of one of them intersects the stable one of the other in a point x exhibiting a cubic tangency. Moreover, this example is in the two-dimensional torus and is conjugate to a linear Anosov diffeomorphism too. Hence x is non-wandering.

But the answer might be 'Yes' from the topological point of view. A partial answer in this direction is given in [**Vi4**]. There it is shown that a periodic point of an expansive diffeomorphism $f : M \to M$, $\dim(M) = 3$, that loses hyperbolicity in a single direction is topologically hyperbolic.

In this section we assume that we have a point x such that $W^u_\epsilon(x)$ does not locally separate M and such that there is an $r_0 > 0$ such that, for all $n \in \mathbb{Z}$, there is an arc $u_n \subset W^u_\epsilon(f^n(x))$ with the property that for all n there is a disc D_n separating $B(f^n(x), r_0)$ in two connected components B^+_n and B^-_n and such that if we put $u^+_n \cup u^-_n = u_n \backslash \{f^n(x)\}$ then u^+_n and u^-_n are arcs joining $f^n(x)$ with points of the boundary of $B(f^n(x), r_0)$, say p_n and q_n, one of them in ∂B^+_n and the other in ∂B^-_n. Moreover, we assume that $f^{-1}(u_n) \subset u_{n-1}$. From these assumptions it follows that there is an $r_1 > 0$ such that $W^s_\epsilon(v)$ separates $B(v, r)$ for any $0 < r \leq r_1$ for a point $v \in \alpha(x)$ and $W^u_\epsilon(v)$ has points in at least two components of $B(v, r)\backslash W^s_\epsilon(v)$.

In the rest of this section let us assume that $x \in M$ is a point in the manner just described.

We will work in the suspension $(\hat{M}, \varphi)$ of (M, f) under the constant function 1. It is known that, for a fixed $t \in \mathbb{R}$, $\hat{M}_t$ is homeomorphic to M. Hence, we will usually identify M with $\hat{M}_0$. As we have proved in the previous section, we may assume that, for $r > 0$, we have a simple closed curve $\mathcal{C} \subset B(x, r) \cap \hat{M}_0$ and Lyapunov functions $\mathcal{U}, \dot{\mathcal{U}}, \ddot{\mathcal{U}}$ defined on $N \subset \hat{M}$, N, with the following properties.

(1) $\mathcal{U}(x, y) \geq 0$ for all $x, y \in N \subset \hat{M}$ and $\mathcal{U}(x, y) = 0$ iff $x = y$.
(2) $\dot{\mathcal{U}}(x, y) > 0$ if $y \in W^u_\epsilon(x)$, $y \neq x$, $\dot{\mathcal{U}}(x, y) < 0$ if $y \in W^s_\epsilon(x)$, $y \neq x$, $\dot{\mathcal{U}}(x, y) = 0$ if $y = x$.
(3) $\dot{\mathcal{U}}(x, y) < 0$ if $y \in \mathcal{C}$.
(4) $\ddot{\mathcal{U}}(x, y) \geq 0$ for all $x, y \in N \subset \hat{M}$ and $\ddot{\mathcal{U}}(x, y) = 0$ iff $x = y$.
(5) $\mathcal{C} \not\sim 0$ in $B(x, r_0)\backslash u_0$.

From (5) we have that any 2-disc $D \subset \hat{M}_0$ such that $\partial D = \mathcal{C}$ has an intersection mod 2 equal to 1 with u_0. This means that the intersection between a 2-disc D is in a general position and such that $\partial D = \mathcal{C}$ with u_0 is an odd number of points.

Let u_t be analogous in the suspension to u_n. From the properties of u_n it is not difficult to derive the analogous properties of u_t.

There are two possibilities: either the connected component of

$$W_\epsilon^u(x) \cap \Big[\bigcup_{t \geq 0} \varphi_t(u_{-t}) \Big]$$

containing u_0 is locally connected (lc) or it is not. In the first case this connected component is an arc and we may choose $\mathcal{C}$ so small that there is a disc D, $\partial D = \mathcal{C}$, intersecting this component just in x.

In the second case for any D passing by x such that $\partial D = \mathcal{C}$ we will have an infinity of arcs in $W_\epsilon^u(x) \cap \big[\bigcup_{t \geq 0} \varphi_t(u_{-t}) \big]$ accumulating on u_0 such that all of these arcs intersect D. Nevertheless we may still suppose that the *arcwise-connected component* of $W_\epsilon^u(x) \cap \big[\bigcup_{t \geq 0} \varphi_t(u_{-t}) \big]$ containing $u_0 \ni x$ intersects D at a single point. We will continue to speak of u_0 when referring to the connected component (arcwise-connected component) in the lc case (respectively the non-lc case) of $W_\epsilon^u(x) \cap \big[\bigcup_{t \geq 0} \varphi_t(u_{-t}) \big]$ containing x.

For $k > 0$, $t \in \mathbb{R}$, $x \in \hat{M}_0$, let us define

$$K_t(x, k) = \{ y \in \hat{M}_t / \mathcal{U}(\varphi_t(x), y) \leq k \}$$

as the t-section of the k-tube around x.

We define the k-tube around x as

$$K(x, k) = \bigcup_{t \in \mathbb{R}} K_t(x, k),$$

$\mathcal{U}(x, y)$ being a uniformly continuous function that vanishes iff $x = y$, given $k_1 > 0$ there is an $r_1 > 0$ such that the connected component of $K_t(x, k_1)$, $K_t'(x, k_1)$, that contains $\varphi_t(x)$ also contains $B(\varphi_t(x), r)$ for all $0 < r \leq r_1$, for all $t \in \mathbb{R}$, $x \in \hat{M}_0$. There is also an $r_2 > 0$ such that $K_t(x, k_1) \subset B(\varphi_t(x), r)$, for all $t \in \mathbb{R}$, $r \geq r_2$, $x \in \hat{M}_0$. We choose $0 < k_1 < k_2 < \nu$, where $\nu > 0$ is as in $(**)$ of the previous section, and $0 < r_1 < r_0 < \epsilon$ such that

$$B(\varphi_t(x), r_1) \subset K_t'(x, k_1) \subset K_t(x, k_1) \subset B(\varphi_t(x), r_0) \subset B(\varphi_t(x), \epsilon) \subset K_t(x, k_2).$$

As we have previously stated we will identify $\hat{M}_0$ with M.

LEMMA 3.1. *For all $y \in K_0(x, k_1)$ it holds that $y \in W_\epsilon^s(x)$ (respectively $y \in W_\epsilon^u(x)$) iff for all $t \geq 0$ (respectively $t \leq 0$) we have $\dot{\mathcal{U}}(\varphi_t(x), \varphi_t(y)) \leq 0$ (respectively $\dot{\mathcal{U}}(\varphi_t(x), \varphi_t(y)) \geq 0$).*

Proof. If for all $t \geq 0$ we have $\dot{\mathcal{U}}(\varphi_t(x), \varphi_t(y)) \leq 0$ then, for all $t \geq 0$, $U(\varphi_t(x), \varphi_t(y)) \leq k_1$ and, consequently, $\varphi_t(y) \in B(\varphi_t(x), \epsilon)$. However, if there is $t_0 \geq 0$ such that $\dot{\mathcal{U}}(\varphi_t(x), \varphi_t(y)) > 0$, using the fact that $\ddot{\mathcal{U}}(x, y) > 0$ as long as $d(\varphi_t(x), \varphi_t(y)) \leq \nu$ the sign of $\dot{\mathcal{U}}(\varphi_t(x), \varphi_t(y))$ will be positive and, moreover, $\dot{\mathcal{U}}(\varphi_t(x), \varphi_t(y)) \geq \dot{\mathcal{U}}(x, y) = h > 0$. It follows that

$$\mathcal{U}(\varphi_t(x), \varphi_t(y)) = \mathcal{U}(x, y) + \int_0^t \dot{\mathcal{U}}(\varphi_s(x), \varphi_s(y))\, ds \geq \mathcal{U}(x, y) + ht.$$

The last expression is greater than k_2 if $t > [k_2 - U(x, y)]/h$. So there is a $t > 0$ such that $y \notin K_t(x, k_2)$ and hence $y \notin W_\epsilon^s(x)$. □

Recall that $\partial D = \mathcal{C} \not\sim 0$ in $B(x,r)\backslash u_0$ and $\dot{\mathcal{U}}(x,y) < 0$ for all $y \in \mathcal{C}$. Let us assume that $D \subset B(x,r) \subset K_0'(x,k_1)$.

As we have stated in the previous section we may assume that $\mathcal{U}(x,y)$ as well as $\dot{\mathcal{U}}(x,y)$ and $\ddot{\mathcal{U}}(x,y)$ are C^1 functions.

Let us define

$$\partial K_t(x,k) = \{y \in \hat{M}_t / \mathcal{U}(\varphi_t(x), y) = k\}$$

as the boundary of the t-section of the k-tube $K_t(x,k)$ around x, and

$$\dot{K}_t(x,k) = \{y \in \hat{M}_t / \mathcal{U}(\varphi_t(x), y) < k\}$$

as the interior of the t-section of the k-tube $K_t(x,k)$ around x. We define

$$\partial K(x,k) = \bigcup_{t\in\mathbb{R}} \partial K_t(x,k) \quad \text{and} \quad \dot{K}(x,k) = \bigcup_{t\in\mathbb{R}} \dot{K}_t(x,k)$$

as the boundary and interior of the k-tube around x, respectively.

Let $y \in \mathcal{C}$ and let x be joined by an arc $\beta \subset D$ to y; $\beta : [0,1] \to D$, $\beta(0) = x$, $\beta(1) = y$. As $\dot{U}(x,y) < 0$ and $\ddot{U}$ is positive, there is a first $T(y) < 0$ such that $\varphi_{T(y)}(y) \notin \dot{K}'_{T(y)}(x,k_1)$. Otherwise for all $t < 0$ $d(\varphi_t(x), \varphi_t(y)) \le \epsilon$. Hence $y \in W^u_\epsilon(x)$ and, by Lemma 3.1, we will have $\dot{\mathcal{U}}(x,y) > 0$, contradicting $y \in \mathcal{C}$. By the compactness of $\mathcal{C}$ there is a fixed $T < 0$ such that for any $y \in \mathcal{C}$, $T(y) \in [T,0)$. As $\ddot{\mathcal{U}}$ is positive, $\dot{\mathcal{U}}(\varphi_{T(y)}(x), \varphi_{T(y)}(y)) < 0$ and therefore, for $t < T(y)$ small enough, $\mathcal{U}(\varphi_t(x), \varphi_t(y)) > k_1$.

We claim that for all $t < T(y)$, for all possible arcs $\beta \subset D$ joining x to y, there is a point $z \in \beta$ such that $\mathcal{U}(\varphi_t(x), \varphi_t(z)) \ge k_1$. If this were not true, a $t_0 < T(y)$ would exist such that for all $z \in \beta$, $\mathcal{U}(\varphi_{t_0}(x), \varphi_{t_0}(z)) < k_1$.

Let $A = \{s \in [0,1] / \forall t \in [t_0, 0],\ \mathcal{U}(\varphi_t(x), \varphi_t(\beta(s))) < k_1\}$. $A \ne \emptyset$ for $x \in A$. Let $s^* = \sup(A)$. It holds that $s^* < 1$ because $y \notin A$. Hence $\mathcal{U}(\varphi_t(x), \varphi_t(\beta(s^*))) \le k_1$ and as s^* is the supremum of A there exists $t_1 \in (t_0, 0)$ such that $\mathcal{U}(\varphi_{t_1}(x), \varphi_{t_1}(\beta(s^*))) = k_1$. It follows that $\dot{\mathcal{U}}(\varphi_{t_1}(x), \varphi_{t_1}(\beta(s^*))) = 0$, but as $\ddot{H} > 0$ for small $\tau > 0$, $\mathcal{U}(\varphi_{t_1\pm\tau}(x), \varphi_{t_1\pm\tau}(\beta(s^*))) > k_1$ which is a contradiction.

As a consequence, for all $t \le T$, the image by φ_t of every arc joining x to the boundary $\mathcal{C}$ of D has points in the interior of $B(\varphi_t(x), r_1)$ (for instance x) and points in the exterior of $B(\varphi_t(x), r)$ (as they remain outside $K_t(x,k_1)$).

Now, for all $t \le T$, $D_t(x) = D_t$, the connected component of $\varphi_t(D) \cap B(\varphi_t(x), r)$ containing $\varphi_t(x)$, separates $B(\varphi_t(x), r)$. A proof of this fact will be given at the end of this section (see also [**Vi1**, §2]). There it is proved that if we have a point z such that $\varphi_t(z) \in K_t(x,k)$ for all $t \le 0$ then the connected component $D_t(z)$ of $\varphi_t(D) \cap B(\varphi_t(x), r)$ containing $\varphi_t(z)$, separates $B(\varphi_t(x), r)$ too. Of course, it may happen that $D_t(z) = D_t$.

In fact we may prove that the separation property holds for a neighbourhood W such that $W \sim B^3$ and $\partial W \sim S^2$ assuming that the image by φ_t of every arc joining x to the boundary $\mathcal{C}$ of D has points in the interior of W and points in the exterior of W.

Let u_t^+ and u_t^- be the connected components of $u_t \backslash \varphi_t(x)$ where u_t is an arc in the section $M_t \subset \hat{M}$ analogous to u_n.

PROPOSITION 3.2. *There is a divergent sequence* $\{t_n \in \mathbb{N}\}$ *such that if* q_n^+ *and* q_n^- *are the first intersection points of* $u_{t_n}^+$ *and* $u_{t_n}^-$ *with* $\partial B(\varphi_{t_n}(x), r)$ *respectively, then* D_t *separates* q_n^+ *from* q_n^- *in* $\operatorname{clos}(B(\varphi_{t_n}(x), r))$.

To prove the proposition we need the following lemma.

LEMMA 3.3. *For all* $z \in D_t$, *for all* $t \geq s \geq 0$, $U(\varphi_s(x), \varphi_{s-t}(z)) \leq k_1$.

Proof. The arguments are rather similar to those we have just used to prove that, for all $t < T(y)$ and all arcs in D joining x with y, there are points outside $K_t(x, k_1)$. If this thesis were false, there would be a point $w = \varphi_t(z) \in D_t$ and there would be a t_1, $t > t_1 > 0$ such that $U(\varphi_{t_1}(x), \varphi_{t_1}(z)) > k_1$. As D_t is arcwise-connected we may join $\varphi_t(x)$ with $\varphi_t(z)$ within D_t by an arc β.

Defining as before

$$A = \{\lambda \in [0, 1]/\forall s \in [t, 0], U(\varphi_s(x), \varphi_s(\beta(\lambda))) < k_1\},$$

we find again a point $\beta(\lambda^*)$ for it such that $U(\varphi_s(x), \varphi_s(\beta(\lambda^*))) \leq k_1$ for all $t \geq s \geq 0$, but with the property that there is a $t^* \in (t, 0)$ such that $U(\varphi_{t^*}(x), \varphi_{t^*}(\beta(\lambda^*))) = k_1$. Again this forces $\dot{\mathcal{U}}$ to vanish, contradicting that $\ddot{\mathcal{U}}$ is positive. □

Proof of Proposition 3.2. Assume first that the connected component of $W_\epsilon^u(x) \cap \left[\bigcup_{t\geq 0} \varphi_t(u_{-t})\right]$ containing u_0 intersects D just in $\{x\}$. As $\varphi_t(u_0) \subset u_t$, $t \leq 0$, and $u_0 \cap D = \{x\}$ and is transversal to D, we have that D_t locally separates u_t in a small neighbourhood of $\varphi_t(x)$. Assume that there is a pair of points $z \in u_t^+$ and $w \in u_t^-$ such that they can be joined by an arc within $B(\varphi_t(x), r)$ without intersections with D_t. Hence D_t cuts u_t in another point $y' = \varphi_t(y) \neq \varphi_t(x) = x'$. As both are in u_t we have that $\operatorname{dist}(\varphi_s(x'), \varphi_s(y')) \leq \epsilon$ for all $s \leq 0$. As both belong to D_t, by Lemma 3.3, we also have that, for all $s \in [0, |t|]$, $\operatorname{dist}(\varphi_s(x'), \varphi_s(y')) \leq r_0$. Thus $y \in W_\epsilon^u(x)$ and belongs to $\bigcup_{\tau\geq 0} \varphi_\tau(u_{-\tau})$. But $u_0 \cap D = \{x\}$ so $y = x$ and we have arrived at a contradiction. The same argument proves that two points $z, w \in u_t^+$ ($z, w \in u_t^-$) cannot be separated by D_t. This finishes the proof in the lc case and also in the non-lc case if there is a 2-disc D such that $\partial D = \mathcal{C}$ and the connected component of $W_\epsilon^u(x) \cap \left[\bigcup_{t\geq 0} \varphi_t(u_{-t})\right]$ containing u_0 intersects D only in x.

Assume now that we are in the non-lc case and that for any 2-disc D, $\partial D = \mathcal{C}$, for any simple closed curve $\mathcal{C} \not\sim 0$ in $B(x, r_0)\backslash u_0$ we have infinitely many intersection points between the connected component of $W_\epsilon^u(x) \cap \left[\bigcup_{t\geq 0} \varphi_t(u_{-t})\right]$ containing u_0 and D. Observe that, in this case, local connectedness is lost for all $t_0 \in \mathbb{R}$. That is, $W_\epsilon^u(\varphi_{t_0}(x)) \cap \left[\bigcup_{t\geq t_0} \varphi_t(u_{-t})\right]$. Hence there is a sequence $t_n \to +\infty$ such that

$$W_\epsilon^u(x) \cap \left[\varphi_{t_n}(u_{-t_n}) \Bigg\backslash \left(\bigcup_{0\leq t\leq (t_n-1)} \varphi_t(u_{-t}) \right)\right] \neq \emptyset.$$

Otherwise we will have a time t_0 from which $W_\epsilon^u(x) \cap \left[\bigcup_{t\geq t_0} \varphi_t(u_{-t})\right]$ is locally connected and therefore the same is true for $W_\epsilon^u(x) \cap \left[\bigcup_{t\geq 0} \varphi_t(u_{-t})\right]$. We claim that if there is a point $y \in W_\epsilon^u(x) \cap \left[\bigcup_{t\geq 0} \varphi_t(u_{-t})\right]$, y in the interior of the k_1-tube around x, not in the arcwise-connected component u_0 of x then the arcwise-connected component of y contains a whole arc $\gamma \ni y$ such that both components of $\gamma\backslash\{y\}$ reach the boundary of the tube.

For $y \in \bigcup_{t\geq 0}\varphi_t(u_{-t})$ and $\varphi_t(u_t) \supset u_{t+1}$, so there is $t_0 > 0$ such that $\varphi_{-t_0}(y) \in u_{-t_0}$ and, moreover, we may take t_0 so large that all the arc γ is mapped by φ_{-t_0} into u_{-t_0}. So if there are points in the connected component γ of $\varphi_{t_0}(u_{-t_0}) \cap K_0(x, k_1)$ containing y that are not in $W^u_\epsilon(x)$, there would be a point $z \in \gamma$ and a time $-t_1 \in (-t_0, 0)$ such that $\varphi_{-t_1}(z) \notin K_{-t_1}(x, k_2)$ while $\varphi_t(y) \in K_t(x, k_2)$ for all $t \in [-t_2, 0]$. Thus the arc $[z, y] \subset \gamma$ will contain a point w such that the φ-trajectory through w will be tangent to the k_2-tube around x exhibiting a maximum and thus contradicting the fact that $\ddot{\mathcal{U}} > 0$. Therefore for all $t \leq 0$ we have infinitely many arcs $\gamma_{t,n} \subset W^u_\epsilon(\varphi_t(x))$ accumulating in points of u_t and with diameters bounded away from zero.

We will argue with the arcwise-connected component u_0 in the non-lc case choosing D so that $u_0 \cap D = \{x\}$ and their intersection is transversal. Observe that the non-lc hypothesis implies that $\bigcup_{t\geq 0}\varphi_t(u_{-t})$ accumulates in a non-trivial subcontinuum of the connected component of $W^u_\epsilon(x)$ containing x. However, our assumption about the non-separation property of $W^u_\epsilon(x)$ implies that we may assume that there is a curve $\mathcal{C}$ such that infinitely many of the arcs $\gamma_{0,n}$ cut a disc D such that $\partial D = \mathcal{C}$ in such a way that their intersection index mod 2 is 1. Hence, changing the point x for another point if necessary we may assume that there are infinitely many arcs intersecting D as we have previously stated and with the property that, for any $\gamma_{t,n}$, there is a point x_n accumulating in x. Moreover, we may assume that for all points z in u_0, no matter how we choose a disc D passing through z and transversal to u_0, we always will have infinitely many intersection points between $W^u_\epsilon(z) \cap \left[\bigcup_{t\geq 0}\varphi_t(u_{-t})\right]$ and D. Otherwise we may argue with z instead of x and reduce everything to the case where we have a single intersection point. In fact, x and z have the same α-limit points. Hence we also have arcs like $\gamma_{(0,n)}$ for all $z \in u_0$.

Take an orientation in D and in $\bigcup_{t\in\mathbb{R}^+}\varphi_t(u_{-t})$. We may define an index of oriented intersection between

$$W^u_\epsilon(x) \cap \left[\bigcup_{0\leq t\leq T}\varphi_t(u_{-t})\right] = W^u_\epsilon(x) \cap \varphi_T(u_{-T})$$

and the disc D. From our previous remarks there are infinitely many $t_n \in \mathbb{N}$ such that $\varphi_{t_n}(u_{-t_n}) \cap D$ is odd. We have that there are $q_n^+, q_n^- \in u_{-t_n}$ in different connected components of $u_{-t_n}\backslash\{\varphi_{-t_n}(x)\}$. We also claim that there is a $T > 0$ such that if $-t \leq -T$ then for all $z \in W^u_\epsilon(x) \cap D$ and $y \in \partial D = \mathcal{C}$, the image by φ_{-t} of any arc β joining y to z has points very close to $\varphi_{-t}(x)$ (for instance $\varphi_{-t}(z)$) and outside the tube $K_{-t}(x, k)$ (see the arguments prior to Proposition 3.2) also holds. Hence the proof that D_t separates $B(\varphi_t(x), r)$ can be repeated to prove that the connected component $D_t(z)$ of $\varphi_{-t}(D) \cap B(\varphi_t(x), r)$ containing z also separates as was pointed out earlier. If $z \in W^u_\epsilon(x) \cap \varphi_{t_n}(u_{-t_n}) \cap D$, the union of all $D_{t_n}(z)$ separates in $B(\varphi_{-t_n}(x), r)$ as clearly u_{-t_n} intersects this union in an odd number of points. This finishes the proof in the non-lc case. □

LEMMA 3.4. *Assume that D_t separates q_t^+ from q_t^- in $\partial B(\varphi_t(x), r)$, $r > 0$. Then there exists $\mu > 0$ such that for all $t \leq 0$, $\mathrm{dist}(q_t^+, q_t^-) > \mu$.*

Proof. If this were not the case, then, as D_{t_n} separates $q_{t_n}^+$ from $q_{t_n}^-$, we will have a point $z_{t_n} \in D_{t_n}$ with $\lim_{n\to\infty} \mathrm{dist}(q_{t_n}^+, z_{t_n}) = 0$. Let us assume without losing generality that

$x_\infty = \lim_{n\to\infty} \varphi_{t_n}(x)$ and that $y_\infty = \lim_{n\to\infty} q^+_{t_n} = \lim_{n\to\infty} z_{t_n}$. They will have the property that, for all $t \in \mathbb{R}$, $\mathrm{dist}(\varphi_t(x_\infty), \varphi_t(y_\infty)) \le \epsilon$. This follows from Lemma 3.3 and the fact that $u_t \subset W^u_\epsilon(\varphi_t(x))$. But this contradicts the fact that f is expansive. □

PROPOSITION 3.5. *Let $\{\varphi_{-t_n}(x)\}$ be a convergent subsequence to $v \in M$. Clearly v is an α-limit point for x. Then there is $r_0 > 0$ such that $D(v) \subset W^s_\epsilon(v)$, the connected component of $B(v, r) \cap W^s_\epsilon(v)$ containing v, separates $B(x, r)$ for all $0 < r \le r_0$. Moreover, $C(v)$, the connected component of $B(v, r) \cap W^u_\epsilon(v)$ containing v, has points in two components of $B(v, r)\backslash D(v)$ and in both components reaches the boundary $\partial B(v, r)$ for all $0 < r \le r_0$.*

Proof. Let $t = t_n \in \mathbb{Z}$, $n \le 0$ and let v be a limit point for $\{f^{t_n}(x)\}$. Then $\varphi_{t_n}(x) = f^{t_n}(x)$, where we again identify $\hat{M}_0$ with M. As in [**Vi2**, Proposition 1.5, p. 595], we may find $D(v)$ and $C(v)$ taking the Hausdorff limit from convergent subsequences from $\{D_{t_n}\}$ and $\{u_{t_n}\}$ respectively. As $u_{t_n} \subset W^u_\epsilon(f^{t_n}(x))$ the proof that $C(v) \subset W^u_\epsilon(v)$ is the same as that in the mentioned proposition of [**Vi2**].

In order to prove that $D(v) \subset W^s_\epsilon(v)$ we observe from Lemma 3.3 that, for all $t_n \le 0$ and $y \in D_{t_n}$, $U(\varphi_s(x), \varphi_{s-t_n}(y)) \le k_1$, for all $s \in [t_n, 0]$. This means that $\mathrm{dist}(\varphi_s(x), \varphi_{s-t}(y)) \le r_0 < \epsilon$. Take $t = n_h$, $s = k$, $n_h \in \mathbb{Z}^-$, $k \in \mathbb{N}$ such that $f^{n_h}(x) \to v$, and assume that $z \in D(v)$. Hence, from the definition of the Hausdorff limit, there exists a sequence $\{y_h\}$, $y_h \in D_{n_h}$, such that $y_h \to z$. Therefore we obtain that

$$\mathrm{dist}(f^k(v), f^k(z)) = \lim_{h\to\infty} \mathrm{dist}(f^{k+n_h}(x), f^k(y_h)) \le r_0.$$

Thus $D(v) \subset W^s_\epsilon(v)$. The separation property of $D(v)$ and the properties of $C(v)$ with respect to $D(v)$ follow from Proposition 3.2 as in [**Vi2**, Proposition 1.5]. □

Remark 3.6. In our construction we have that $C(v)$ is the Hausdorff limit of a sequence of local unstable arcs. Every local unstable arc intersects $D(v)$ in a single point close to v. Changing v by one of these points of intersection we may assume that $C(v)$ is itself an arc.

Definition 3.1. A compact n-dimensional topological space, $n \ge 1$ is called an n-dimensional Cantor-manifold if it cannot be disconnected by a subset of dimension less or equal than $n - 2$.

Remark 3.7. An n-dimensional Cantor-manifold need not be a manifold at all. Aleksandrov (see [**Al**, ch. VI, Definition 5.21]) calls it an n-dimensional strongly connected compactum.

It follows from the definition that an n-dimensional Cantor-manifold has dimension n at all of its points.

THEOREM 3.8. *Suppose $\mathcal{C}$ is a compact subset of $\mathbb{R}^n$ which separates p and q while no proper closed subset of $\mathcal{C}$ does so (an irreducible separating set). Then $\mathcal{C}$ is an $(n - 1)$-dimensional Cantor-manifold.*

Proof. See [**HW**, Theorem VI 11]. □

Let p and q be points in $C(v)$ separated in $B(v, r_0)$ by $D(v)$. Assume also that p, q are in the interior of $B(v, r_0)$. As $C(v)$ is separated by $D(v)$ in $B(v, r_0)$ it follows that there are points $p_1, q_1 \in \partial B(v, r_0) \cap C(v)$, in the same component of $C(v)\backslash D(v)$ that p and q respectively, separated in $\partial B(v, r_0)$ by $D(v) \cap \partial B(v, r_0)$. Let us call $\partial B^+(v, r_0)$ the component of $\partial B(v, r_0)\backslash D(v)$ containing p_1.

PROPOSITION 3.9. *There exists $\hat{D}(v) \subset D(v)$, $v \in \hat{D}(v)$ such that $\hat{D}(v) \cup \partial B^+(v, r_0)$ is an irreducible separating set of p and q and therefore a two-dimensional Cantor-manifold.*

Proof. Let us define $\mathcal{D}$ as

$$\mathcal{D} = \{D \subset D(v)/D \text{ a continuum, } D \cup \partial B^+(v, r_0) \text{ separating } p \text{ from } q, v \in D\}.$$

As $D(v) \in \mathcal{D}$ it holds that $\mathcal{D} \neq \emptyset$. We partially order $\mathcal{D}$ by inclusion $D' > D$ if $D' \subset D$, $D' \neq D$, $D', D \in \mathcal{D}$. Let $D(v) < D_1 < \cdots < D_\mu < \cdots$ be a chain in $\mathcal{D}$. Then $D_\infty = \cap_\mu D_\mu$ also belongs to $\mathcal{D}$. Therefore by Zorn's lemma there is a maximal element in $\mathcal{D}$, say $\hat{D}(v)$. If D is a proper compact subset of $\hat{D}(v)$ then $D \cup \partial B^+(v, r_0)$ cannot separate p from q. If this were not the case then D would contain a subcontinuum D' with the same property. D' should contain v because every subset of $D(v)\backslash\{v\}$ does not separate p from q. Hence $D' \in \mathcal{D}$ and $D' > \hat{D}(v)$ contradicting that $\hat{D}(v)$ is maximal. □

Remark 3.10. It follows that $\hat{D}(v) \cup \partial B^+(v, r_0)$ is the common boundary of two disjoint open connected subsets, one containing p and the other containing q (see [**HW**, remark following Theorem VI 11]).

In the following let us assume that $D(v) \cup \partial B^+(v, r_0)$ is itself an irreducible separating set that separates p from q on $B(v, r_0)$.

Let $w \in D(v)$ be a point in the interior of $B(v, r_0)$, i.e. there exists $\nu > 0$ such that $B(w, \nu) \subset \text{int}(B(v, r_0))$. Let $S(w)$ also be the connected component of $D(v) \cap B(w, \nu)$ containing w. Let $w_j \to w$ when $j \to +\infty$, $w_j \in D_j$. There is a residual set of $\nu > 0$ such that for all j the connected component S_j of $D_j \cap B(w, \nu)$ containing w_j has the property that $S_j \cap \partial B(w, \nu)$ is a one-dimensional (curved) polyhedron contained in $\partial B(w, \nu)$.

PROPOSITION 3.11. *Given w and ν as above and r, $0 < r < \alpha/4$ there exists $N > 0$ such that $f^{-N}(S(w))$ separates $B(f^{-N}(w), r)$.*

Proof. Let w_j be a converging sequence to w and γ_j an arc joining w_j with $y_j \in D_j \cap \partial B(w, \nu)$. Here D_j are the surfaces with boundary converging to $D(v)$ in the Hausdorff sense.

We claim that there are $N, J > 0$ such that, for all $j \geq J$,

$$\sup_{s\in[0,1]} \text{dist}(f^{-N}(\gamma_j(0)), f^{-N}(\gamma_j(s))) > 2r.$$

Let y be a point in $\partial B(w, \nu) \cap D(v)$ such that $\text{dist}(y_j, y) < \delta_j$ and $\mathcal{U}$ a Lyapunov function for f. Because $y \in W^s_\epsilon(w)$ we have that $\dot{\mathcal{U}}(w, y) < 0$ and by continuity the same is true for (w_j, y_j) if Δ_j is sufficiently small. Moreover, $\text{dist}(w, y) = \nu$ and therefore $\dot{\mathcal{U}}$ is bounded away from zero in $W^s_\epsilon(w) \cap \partial B(w, \nu)$ and so the same is true for (w_j, y_j) if $j \geq J$ for J sufficiently large. By the compactness of $\left(\left(\bigcup_{j\geq J} D_j\right) \cup D(v)\right) \cap \partial B(w, \nu)$ and from the previous remarks the claim follows. Let us fix N. As $j \to +\infty$ we have that the connected component S_j of $D_j \cap B(w, \nu)$ containing w_j converges to $S(w)$ and therefore $f^{-N}(S_j)$ converges to $f^{-N}(S(w))$. Hence, if we prove that $f^{-N}(S_j)$ separates $B(f^{-N}(w), r)$ we are done. This is the goal of the following lemma. □

LEMMA 3.12. *Let B be the canonical open ball of $\mathbb{R}^3$, S an orientable surface with boundary and $O \in \mathrm{int}(S)$. Assume that there is a continuous function $\varphi : S \to \mathbb{R}^3$ which is a homeomorphism onto its image. Assume, moreover, that $\varphi(O) = \vec{0} \in \mathbb{R}^3$ and that if $\gamma : [0, 1] \to S$ is a continuous curve joining $\gamma(0) = O \in S$ to a point $\gamma(1) \in \partial S$ then $\varphi(\gamma([0, 1])) \not\subset B$. Then if X denotes the connected component of $\varphi^{-1}(B)$ that contains O we have that $\varphi(X)$ disconnects B.*

Proof. (Suggested by M. Sebastiani) Let $F = \varphi(X)$ where X is the connected component of O in $\varphi^{-1}(B) \cap S$. As B is open and S is locally arcwise-connected, X is locally arcwise-connected and open in S. Therefore X is arcwise-connected and it follows that $X \cap \partial S = \emptyset$. Otherwise there exists $y \in X \cap \partial S$ and, by arcwise connectedness of X, an arc β joining O to y which has the property that $\varphi(\beta) \subset B$ contradicting the properties of φ. Hence X is open and $F = \varphi(X)$ is homeomorphic to an open surface. Moreover, X is closed in $\varphi^{-1}(B)$ being a connected component. As φ is a homeomorphism between S and its image and we have a homeomorphism between $\varphi^{-1}(B)$ and $B \cap \varphi(S)$. Hence $F = \varphi(X)$ is closed in $B \cap \varphi(S)$ which is closed in B. We conclude that F is closed in B. It follows from the following lemma that $B \backslash F$ is not connected. □

LEMMA 3.13. *Let B be the standard open three ball of $\mathbb{R}^3$ and F closed in B and homeomorphic to an open surface. Then F separates B.*

Proof. It is more or less the same proof as that in [**Vi1**, Lemma 4]. □

4. *Assembling the puzzle*

In this section we pick from different sources the results we need in order to satisfy the hypotheses of the previous section. Perhaps this might justify the name of this section.

THEOREM 4.1. *Every homeomorphism $f : M \to M$, M a compact metric space, has an ergodic invariant Borel probability measure μ.*

Proof. See for instance [**Ma2**, ch. I and II]. □

THEOREM 4.2. (Poincaré recurrence theorem) *Let μ be an ergodic invariant Borel probability measure for $f : M \to M$, M and f as in 4.1, and let $A \subset M$ be a Borel set. Then for any $N \in \mathbb{N}$*

$$\mu(\{x \in A/\{f^n(x)\}_{n \geq N} \subset (M \backslash A)\}) = 0.$$

Proof. See [**Ma2**, ch. I, §2]. □

COROLLARY 4.3. *Let f, M, μ be as before. Then* $\operatorname{supp} \mu \subset R(f)$ *where*

$$R(f) = \{x \in M/x \in \alpha(x) \cap \omega(x)\}$$

is the subset of M of bi-recurrent points.

Proof. See [**KH**, ch. 4, §f]. □

THEOREM 4.4. *Let $f : M \to M$ be an expansive homeomorphism, (M, d) a compact metric space. Then its topological dimension* $\operatorname{dimtop}(M)$ *is finite. Moreover if* $\operatorname{dimtop}(M) > 0$ *then* $0 < h_{\text{top}}(f) < \infty$*, where* $h_{\text{top}}(f)$ *is the topological entropy of* f.

Proof. That the topological entropy is finite is proved in [**Wa**, Corollary 7.11, p. 177]. However, in [**Ft**], Fathi proves that the following inequality holds:

$$HD(M) \leq 2\frac{h_{\text{top}}(f)}{\log k}$$

where $k > 1$ is a constant which depends on f, and $HD(M)$ is the Hausdorff dimension of M (see [**Fa**]). As $\operatorname{dimtop}(M) \leq HD(M)$ for any metric d defining the topology of M (see [**HW**]) we have that $\operatorname{dimtop}(M) < \infty$. By assumption we have that $\operatorname{dimtop}(M) > 0$. Hence we obtain

$$h_{\text{top}}(f) \geq \tfrac{1}{2}\operatorname{dimtop}(M) \cdot \log k > 0.$$ □

Let $\mathcal{M}(M)$ be the set of all Borel probability measures defined on M, and let $\mathcal{M}(M, f) = \{\mu \in \mathcal{M}(M)/f_*(\mu) = \mu\}$, i.e. $\mathcal{M}(M, f)$ is the subset of $\mathcal{M}(M)$ of those measures invariant by f.

THEOREM 4.5. (Variational principle) *Let $f : M \to M$ be a continuous map. Then*

$$h_{\text{top}}(f) = \sup\{h_\mu(f)/\mu \in \mathcal{M}(M, f)\}$$

where $h_\mu(f)$ denotes the measure theoretic entropy of f with respect to the measure $\mu \in \mathcal{M}(M, f)$.

Proof. See [**Wa**, ch. 8]. □

Let $\mathcal{M}_{\max}(M, f)$ be the subset of $\mathcal{M}(M, f)$ of measures μ such that $h_{\text{top}}(f) = h_\mu(f)$. There are examples in which $\mathcal{M}_{\max}(M, f) = \emptyset$. But if f is expansive then we have the following theorem.

THEOREM 4.6. *Let $f : M \to M$ be an expansive homeomorphism of a compact metric space M. Then the entropy map ($\mu \in \mathcal{M}(M, f) \mapsto h_\mu \in \mathbb{R}^+$) is upper semicontinuous and therefore $\mathcal{M}_{\max}(M, f) \neq \emptyset$. Moreover, $\mathcal{M}_{\max}(M, f)$ contains an ergodic measure.*

Proof. That the entropy map is upper semicontinuous when f is expansive is Theorem 8.2 of [**Wa**, p. 184]. As $\mathcal{M}(M, f)$ is compact in the weak*-topology, the result follows from the fact that an upper semicontinuous function defined on a compact space attains its supremum. That there is a member of $\mathcal{M}_{\max}(M, f)$ that is ergodic is a consequence of the fact that $h_{\text{top}}(f) < \infty$ and $\mathcal{M}_{\max}(M, f) \neq \emptyset$ (see [**Wa**, Theorem 8.7, part (iii), pp. 191–192]). □

Up to this moment differentiability has played no role at all in this section and one may wonder to what extent it is essential. We now will use it to find strong stable and unstable manifolds.

THEOREM 4.7. (Oseledec) *Let $f : M \to M$ be a C^1 diffeomorphism of a compact smooth manifold of dimension n and let $\mu \in \mathcal{M}_{\mathrm{erg}}(M, f)$ where $\mathcal{M}_{\mathrm{erg}}(M, f)$ denotes the subset of ergodic measures of $\mathcal{M}(M, f)$. Then there are numbers $\chi_1 > \chi_2 > \cdots \chi_k$, $\chi_j \in \mathbb{R}$ for all $j = 1, \dots, k$, $k \leq n$ a measurable splitting $T_xM = E_x^1 \oplus \cdots \oplus E_x^k$ such that $\dim(E_x^j = n_j)$ is independent of $x \in M$, μ-a.e. and such that $D_x f(E_x^j) = E_{f(x)}^j \mu$-a.e. such that*

$$\lim_{n\to\infty} \frac{1}{n} \log \|D_x f^n(v)\| = \chi_h$$

for $x \in M \mu$-a.e. whenever

$$v \in \sum_{j=1}^{h} E_x^j \Big\backslash \sum_{j=1}^{h-1} E_x^j.$$

The numbers χ_j are called the Lyapunov exponents associated with the pair (f, μ).

Proof. See [**Ma2**, ch. IV, §10]. □

THEOREM 4.8. (Ruelle's inequality) *Let $f : M \to M$ be a C^1 diffeomorphism defined on the smooth manifold M. Also let μ be an ergodic measure for f and $\chi_1, \dots, \chi_k$ be the Lyapunov exponents associated with (f, μ). Then the following inequality holds*

$$\sum_{\chi_j > 0} \chi_j \geq h_\mu(f).$$

Proof. See [**KH**, Theorem S.2.13, pp. 669–672]. □

COROLLARY 4.9. *If $f : M \to M$ is a C^1 expansive diffeomorphism then there is a measure $\mu \in \mathcal{M}_{\mathrm{erg}}(M, f) \cap \mathcal{M}_{\max}(M, f)$ such that μ-a.e. there is a positive Lyapunov exponent $\chi^+ > 0$ and a negative Lyapunov exponent $\chi^- < 0$ for the pair f, μ.*

Proof. By 4.4, $h_{\mathrm{top}}(f) > 0$. By Theorem 4.6 there exists a measure μ in $\mathcal{M}_{\max}(M, f)$ that is ergodic. Hence, using Theorem 4.8 we must have a positive Lyapunov exponent μ-a.e. To prove the existence of $\chi^- < 0$ it is enough to consider f^{-1} instead of f. □

In order to obtain strong stable (unstable) manifolds almost everywhere with respect to μ we assume here that f is $C^{1+\theta}$. As pointed out in the supplement of [**KH**, pp. 673ff], this assumption seems to be crucial in the proofs. However, the dynamical meaning of this assumption is not clear to the author of this article.

THEOREM 4.10. (Pesin, Ruelle) *Let M be a compact differentiable $3D$-manifold, and $f : M \to M$ an expansive diffeomorphism of class $C^{1+\theta}$, μ a Borel probability ergodic measure as before, χ^+, χ_0, χ^- the spectrum of (f, μ), $\chi^+ > 0$, $\chi^- < 0$ and E_x^+, C_x, E_x^- the respective associated subspaces in $T_xM = E_x^+ \oplus C_x \oplus E_x^-$ (if $\chi_0 = \chi^+$ or $\chi_0 = \chi^-$, C_x should be omitted from the equality).*

Then for $x \in M$ μ-a.e., the subsets of M defined as

$$W^{ss}(x) = \left\{ y \in M \Big/ \lim \sup_{n\to\infty} \frac{1}{n} \log d(f^n(x), f^n(y)) < \rho < 0 \right\}, \quad 0 > \rho > \chi^-$$

$$W^{uu}(x) = \left\{ y \in M \Big/ \lim \sup_{n\to-\infty} \frac{1}{n} \log d(f^n(x), f^n(y)) < -\rho < 0 \right\}, \quad 0 < \rho < \chi^+$$

are non-trivial smooth submanifolds immersed in M, tangent respectively to E_x^- and E_x^+ at the point x.

Proof. That $W^{ss}(x)$ and $W^{uu}(x)$ are smooth submanifolds, tangent respectively to E_x^- and E_x^+ is a consequence of [**Pe**, Theorem 2.2.1] or, alternatively, [**Ru**, Theorem 6.3.1]. The non-triviality of $W^{ss}(x)$ and $W^{uu}(x)$ follows from the existence of $\chi^- < 0$ and $\chi^+ > 0$. Another proof is given in [**FHY**]. □

Let us define Pesin's region as the subset of M where all Lyapunov exponents are different from zero.

Now there are two possible cases.

(1) Pesin's region is not empty, i.e. there exists $y \in M$ such that the Lyapunov exponents associated to y do not vanish. In this case it is not difficult to prove that there is a small ball $B(y, r)$, $r > 0$ such that either $S(y)$ the connected component of $W^{ss}_{\text{loc}}(y) \cap B(y, r)$ containing y separates $B(y, r)$ and $W^{uu}_{\text{loc}}(y)$ intersects $\partial B(y, r)$ in two different connected components of $B(y, r)\backslash S(y)$ or the situation is reversed. Thus in the first case, for instance, the local stable manifold of y contains a 2-disc and the local unstable manifold contains an arc transversal to it; both the disc and the arc have the point y in their interior.

(2) Pesin's region is void and we always have a Lyapunov exponent $\chi_0 = 0$. Now, by Theorems 4.2, 4.7 and 4.9 we are able to find a bi-recurrent point $x \in M$ such that for it we have a positive Lyapunov exponent χ^+ and a negative Lyapunov exponent χ^- and E_x^+, E_x^- are subspaces given by Oseledec's theorem. By our assumption we also have a zero Lyapunov exponent, and an associated subspace E_x^0. It is clear that E_x^-, E_x^0 and E_x^+ are all one-dimensional subspaces.

In the second case $W^u_\epsilon(x)$ may or may not still separate locally M. As we seek the separation property either for $W^u_\epsilon(x)$ or for $W^s_\epsilon(x)$ let us assume that $W^u_\epsilon(x)$ does not separate any neighbourhood $V \subset M$ and let us prove that, in this case, for certain $v \in \alpha(x)$ we have that $D(v)$ the connected component of $W^s_\epsilon(v)$ containing v has the separation property. Moreover, $C(v)$, the connected component of $W^u_\epsilon(v)$ containing v, joins v with the boundary of a neighbourhood $B(v, r)$ of v at least in two components of $B(v, r)\backslash D(v)$. In order to prove this we are going to see whether the hypotheses of Proposition 3.5 hold.

PROPOSITION 4.11. *Along the orbit of x, $W^u_\epsilon(f^n(x))$ contains an arc u_n which verifies the hypotheses in §3.*

Proof. If the orbit of the bi-recurrent point x is periodic then we have a periodic point with eigenvalues λ_1, 1 and λ_2 with $|\lambda_1| < 1$ and $|\lambda_2 > 1|$. We have proved in [**Vi4**] that, in this case, x is a topologically hyperbolic periodic point, so we are done.

Otherwise, the orbit $o(x)$ of x is infinite. Let μ be an invariant ergodic maximal probability measure associated with f. For x, μ-a.e. in M let us define the subspaces

$$E_x^+ = \left\{ u \in T_x M \Big/ \limsup_{n\to\infty} \frac{1}{n} \log \|D_x f^{-n} u\| \le -\chi^+ \right\}$$

$$E_x^+(k) = \left\{ u \in T_x M \Big/ \limsup_{n\to\infty} \frac{1}{n} \log \|D_x f^{-n} u\| \le -\chi^+ + \frac{1}{k} \right\}.$$

Clearly $E_x^+(k) \supset E_x^+$ and by the finiteness of the dimension of T_xM we have that there is $\mathbb{N} \ni k > 0$ such that $E_x^+(k) = E_x^+$. Let us also define

$$M_k = \{x \in M / E_x^+(k) = E_x^+\}$$

$E_x^+(k,m) = \{u \in T_xM / \|D_x f^{-n} u\| \leq m e^{-n(\chi^+ - 1/k)} \|u\|\}, \quad m \geq 1, n \geq 0.$ $E_x^+(k) = \bigcup_{m \in \mathbb{N}^+} E_x^+(k,m)$ so we may define

$$M_{k,m} = \{x \in M_k / E_x^+(k) = E_x^+(k,m)\}$$

and finally we define

$$M_{k,m,j} = \{x \in M_{k,m} / \dim(E_x^+) \geq j\}.$$

It is proved in [**Pe**, Theorem 2.3.1] that the sets $M_{k,m,j}$ are closed and that E_x^+ varies continuously on $x \in M_{k,m,j} \backslash M_{k,m,j+1}$. However, if the Lyapunov exponents are χ^-, 0 and χ^+ then $\dim(E_x^+) = 1$ and therefore $M_{k,m} = M_{k,m,1}$ so E_x^+ varies continuously on $M_{k,m}$.

Given $\epsilon' > 0$ there exist k, m such that $\mu(M_{k,m}) > 1 - \epsilon'$. Let

$$T_{M_{k,m}}(z) = \lim_{n \to \infty} \frac{1}{n} \#\{0 \leq i \leq n / f^{-i}(z) \in M_{k,m}\}.$$

By Birkhoff's theorem $T_{M_{k,m}}(z) = \mu(M_{k,m})$ μ-a.e. $= 1 - \epsilon'$. However, by Corollary 4.3, we have

$$\operatorname{supp} \mu \subset \{x \in M / x \in \alpha(x) \cap \omega(x)\}.$$

Therefore there is a bi-recurrent point x such that its orbit $o(x)$ is almost contained in $M_{k,m}$. Recall that $M_{k,m} = M_{k,m,1}$ which is closed. Let $z \in M_{k,m}$ be an accumulation point of $\{f^{n_j}(x)\}$ where $\forall j$, $f^{n_j}(x) \in M_{k,m}$, $n_j \to -\infty$. By Pesin's theory (see Theorem 4.10) strong local invariant manifolds $W^{uu}_{\text{loc}}(z)$ and $W^{uu}_{\text{loc}}(f^{n_j}(x))$ exist and $W^{uu}_{\text{loc}}(f^{n_j}(x))$ converges in the C^1 topology to $W^{uu}_{\text{loc}}(z)$. In particular the diameters of both components of $W^{uu}_{\text{loc}}(f^{n_j}(x)) \backslash \{f^{n_j}(x)\}$ are bounded away from zero by a number $\rho > 0$ for all $j \in \mathbb{N}$. We may assume that $x = f^{n_0}(x)$, i.e. $n_0 = 0$.

We will work in the suspension flow $(\hat{M}, \varphi)$ of (M, f) identifying M with $\hat{M}_0$. Moreover, we also identify $f^n(x)$ with $\varphi_n(x)$, $n \in \mathbb{Z}$.

Let $h \in \mathbb{Z}^-$, then there is $n_j \leq h$ such that $f^{n_j}(x)$ belongs to the sequence converging to z. Let $\mathcal{U}(z, w)$ be a Lyapunov function for f and let $k_1 > 0$ and $r > 0$ be such that if $\operatorname{dist}(z, w) \geq \rho$ then $\mathcal{U}(z, w) \geq k_1$ and if $\mathcal{U}(z, w) \geq k_1$ then $\operatorname{dist}(z, w) \geq r$, for all $z, w \in M_{n_j} \equiv M$. If $y \in W^{uu}_{\text{loc}}(f^{n_j}(x)) \backslash \{f^{n_j}(x)\}$ then the first derivative of the Lyapunov function $\dot{\mathcal{U}}(f^{n_j}(x), y) > 0$. Moreover, the second derivative is positive too and therefore $\mathcal{U}$ increases. There are points $z, w \in W^{uu}_{\text{loc}}(f^{n_j}(x)) \backslash \{f^{n_j}(x)\}$ at a distance at least ρ from $f^{n_j}(x)$ and in different components of $W^{uu}_{\text{loc}}(f^{n_j}(x)) \backslash \{f^{n_j}(x)\}$. Both are joined to $f^{n_j}(x)$ by unstable arcs, therefore for all t between n_j and h there are points in both arcs that are in the boundary of the k_1-tube around x. Take for the time t the connected component of $\varphi_{t-n_j}(W^{uu}_{\text{loc}}(f^{n_j}(x)) \cap K_t(x, k_1))$ containing $\varphi_t(x)$. This component is a local unstable arc. Let $t = h \in \mathbb{Z}$. The proof that the hypotheses in §3 hold needed to prove Proposition 3.5, is almost complete. To finish it we first find a circle $\mathcal{C}$ in $B(x, r)$ such that $\mathcal{C} \cap W^u_\epsilon(x) = \emptyset$ and such that $\mathcal{C} \not\simeq 0$ in $B(x, r) \backslash W^{uu}_{\text{loc}}(x)$; and, second, we define for $t \in \mathbb{R}$, $t \leq 0$, u_t the following construction.

If $t = n_j$ then we parameterize $W^{uu}_{\text{loc}}(f^{n_j}(x))$ by a homeomorphism $\gamma : [-1, 1] \to W^{uu}_{\text{loc}}(f^{n_j}(x))$. Let $t_{-1} < 0$ and $t_1 > 0$ be the points of $[-1, 1]$ such that $\gamma(-t_{-1}, t_1)$ is in the interior of the n_j section of the k_1-tube around x, and such that the $\gamma(t_{-1}), \gamma(t_1)$ belong to the boundary of the tube. Then take as u_{n_j} the subarc $\gamma([t_{-1}, t_1])$. If $t \neq n_j$ then there is a $j \leq 0$ such that t is between n_{j+1} and n_j. We repeat the same construction but with $\varphi_{t-n_{j+1}}(W^{uu}_{\text{loc}}(f^{n_{j+1}}(x)))$. □

4.1. *Half-plane case.* To finish this section we need to prove for x that when Pesin's region is void but $W^u_\epsilon(x)$ locally separates M, the connected component, $C(x)$, of $W^u_\epsilon(x) \cap B(x, r)$ containing x separates a neighbourhood of x and, moreover, that $W^s_\epsilon(x)$ contains subcontinua joining x with the boundary of $B(x, r)$ at least in two connected components of $B(x, r)\backslash C(x)$. We can prove this if we know that $W^s_\epsilon(x)$ does not separate any neighbourhood in M. In this case we interchange the roles played by $W^u_\epsilon(x)$ and $W^s_\epsilon(x)$.

The only case that remains is that in which both $W^s_\epsilon(x)$ and $W^u_\epsilon(x)$ separate in such a way that we cannot find a circle $\mathcal{C}$ with the desired homological properties. This implies that there are continua $K_s(x) \subset W^s_\epsilon(x)$ and $K_u(x) \subset W^u_\epsilon(x)$ such that $W^{uu}_{\text{loc}}(x) \subset K_u(x)$, $W^{ss}_{\text{loc}}(x) \subset K_s(x)$ (in particular $x \in K_s(x) \cap K_u(x)$) and $K_s(x)$ and $K_u(x)$ both separate neighbourhoods in M in such a way that any Jordan curve $\mathcal{C} \subset B(x, r_0)$ such that $\mathcal{C} \not\sim 0$ in $B(v, r_0)\backslash W^{ss}_{\text{loc}}(x)$ (let us assume that $W^{uu}_{\text{loc}}(x)$ and $W^{ss}_{\text{loc}}(x)$ are large enough) intersects $K_u(x)$ and any Jordan curve $\mathcal{C} \subset B(x, r_0)$ such that $\mathcal{C} \not\sim 0$ in $B(v, r_0)\backslash W^{uu}_{\text{loc}}(x)$ intersects $K_s(x)$. We note here that the same has to be true for all the iterates of x. Otherwise we may argue with the iterate instead of doing it with x. Let us state in this case that $K_s(x)$ and $K_u(x)$ semi-separate $B(x, r_0)$.

To have a picture we may imagine that x is the origin O in $\mathbb{R}^3$, K_s is the positive half-plane in the xOy plane bounded by Oy, K_u is the negative half-plane in the xOz plane bounded by Oz. In this picture the axes Oy and Oz play the role of $W^{ss}_{\text{loc}}(x)$ and $W^{uu}_{\text{loc}}(x)$ respectively.

As x is bi-recurrent we have an infinite subsequence $\{n_j\}$ such that $f^{n_j}(x)$ converges to x when $n_j \to \pm\infty$. From what we have stated earlier there exist $K_s(f^{n_j}(x))$ and $K_u(f^{n_j}(x))$ for every point $f^{n_j}(x)$ that semi-separate $B(f^{n_j}(x), r_0)$. If x is periodic then this cannot hold as we have proved in **[Vi4]**. So assume that the orbit of x is infinite. $W^{ss}_{\text{loc}}(f^{n_j}(x))$ converges in the C^1 topology to $W^{ss}_{\text{loc}}(x)$. This implies that the Hausdorff limit of a convergent subsequence of the corresponding $K_s(f^{n_j}(x))$ converge to a subset of the continuum $K_s(x)$ associated to x. Moreover, $W^{uu}_{\text{loc}}(f^{n_j}(x))$ converges in the C^1 topology to $W^{uu}_{\text{loc}}(x)$ and we may assume that we have chosen n_j such that the corresponding $K_u(f^{n_j}(x))$ also converge to the $K_u(x)$ associated with x.

LEMMA 4.12. *As x is bi-recurrent and not periodic $W^s(x) \cap W^s(f^k(x)) = \emptyset$ if $0 \neq k$.*

Proof. Otherwise, if $z \in W^s(x) \cap W^s(f^k(x))$ then, if w is an ω-limit point of z, $f^{n_h}(z)$ converges to w and the same should be true for $f^{n_h}(x)$ and for $f^{n_h+k}(x)$. Hence $f^k(w) = w$ and the ω-limit of x has to be a periodic orbit. But x is itself bi-recurrent and therefore it has to be periodic. Thus $W^s(x) \cap W^s(f^k(x)) = \emptyset$. □

We note that $f^n(x)$ is bi-recurrent for all n and this implies that for all $K_s(f^{n_j}(x))$ and all $K_u(f^{n_j}(x))$ infinitely many sets $K_s(f^n(x))$ and $K_u(f^n(x))$ are accumulating in $K_s(f^{n_j}(x))$ and $K_u(f^{n_j}(x))$ respectively. Hence it is not difficult to prove that there are points z in $K_s(x)$ (or in any iterate $K_s(f^{n_j}(x))$) such that $C(z)$, the connected component of $W^u_\epsilon(z)$ containing z that is a non-trivial continuum, intersects $K_s(f^{n_j}(x))$ in points y_j converging to z with j. Let z_j be another sequence converging to z such that $f^{n_j}(z_j)$ also converges to z with n_j diverging to infinity. This sequence exists because $z \in \Omega(f) = M$. We also have that $C(f^{n_j}(z_j))$ intersects $K_s(x)$ in points w_j. But $\text{dist}(f^{-n_j}(w_j), z_j)$ converges to zero because both points are in the same local unstable set. Therefore, denoting $f^{-n_j}(w_j)$ by $\hat{w}_j$ we find that these points $\hat{w}_j$ belong to sets of the form $K_s(f^{n_j}(x))$. Thus there are infinitely many sets of the form $K_s(f^n(x))$ accumulating on $K_s(x)$. We iterate all the subset $K_s(x)$ by f^{-n_j} where n_j is such that $f^{-n_j}(w_j)$ converges to z.

PROPOSITION 4.13. *The half-plane situation implies a contradiction.*

Proof. Let $B(z, r)$ be a ball such that $K_s(x)$ separates it. Let $L \subset K_s(x)$ be a continuum joining z with the boundary of $B(z, r)$. By arguments used in the previous section there is an N such that if $n_j > N$ then $f^{-n_j}(L)$ has points both in the interior and exterior of the Lyapunov tube of radius $\sigma > 0$ where σ is chosen such that if $\mathcal{U}(x, y) \leq \sigma$ then $\text{dist}(x, y) \leq \epsilon$ and if $\mathcal{U}(x, y) = \sigma$ then $\text{dist}(x, y) \geq r_0$. If now we take the connected component D containing $f^{-n_j}(w_j)$ of $f^{-n_j}(K_s(x)) \cap B(z, r_0)$ we find that it separates $B(z, r_0 - \psi)$ if $f^{-n_j}(w_j)$ is sufficiently close to z for a small $\psi > 0$.

If we had originally chosen z at a distance less than $r_0/2$ from x, D would intersect $W^{uu}_{\text{loc}}(x)$ or $W^{uu}_{\text{loc}}(f^{-n_k}(x))$ or both (we are assuming that z_j is between $K_s(x)$ and $K_s(f^{n_k}(x))$). Otherwise there has to be a neighbourhood of $W^{uu}_{\text{loc}}(x)$ which also does not intersect D. As D separates $K_s(f^{-n_j}(x)) \subset D$ and any local unstable continuum C that intersects $K_s(x)$ near z intersects $K_s(f^{-n_k}(x))$ for all k, C will intersect D at least twice (because D separates in $B(z, r_0 - \psi)$ but does not separate x from points in $K_s(f^{-n_k}(x))$ where we are assuming that $K_s(f^{-n_j}(x))$ is locally between $K_s(f^{-n_k}(x))$ and $K_s(x)$) and hence the number of intersections mod 2 of C has to be zero. However, it has an intersection point with D, namely $K_s(f^{-n_j}(x))$. Therefore $W^{uu}_{\text{loc}}(x)$ intersects D and by a standard argument we have (passing to the limit in the Hausdorff metric) that $K_s(x)$ separates a neighbourhood of x. Reversing arguments we will find that $K_u(x)$ also separates a neighbourhood of x. As $W^{uu}_{\text{loc}}(x) \subset K_u(x)$ and $W^{ss}_{\text{loc}}(x) \subset K_s(x)$ we have that $K_s(x) \cap K_u(x)$ has more than one intersection point contradicting the expansive properties of f. □

5. *Stable and unstable sets*

We assume from now on that we have a point $p \in M$ such that for the connected component $D = D(p) \subset W^s_\epsilon(p)$ containing p separates $B(p, r)$ for certain $r > 0$ and that the connected component $C(p) \subset W^u_\epsilon(p)$ containing p is an arc with points in two connected components of $B(p, r) \subset D(p)$.

In this section we prove that there are a sort of homoclinic points for p, i.e. points x such that $x \in W^s(o(p)) \cap W^u(p)$, $x \neq p$. From this it will follow that there is a neighbourhood of p with a local product structure. Here $o(p)$ denotes the orbit of p.

Definition 5.1. If there is a point $p \in M$ such that for it there exist $x \in M$ with the property that $x \in W^s(o(p)) \cap W^u(p)$, $x \neq p$ then we will say that p is a weak homoclinic point (whp).

Remark 5.1. If the stable and unstable sets are Euclidean spaces and are transversal then this definition has no sense (or all points verify it).

LEMMA 5.2. *There is an $r_3 > 0$ such that for all $n \in \mathbb{N}$, S_n the connected component of $f^{-n}(S) \cap B(f^{-n}(q), r_3)$ containing $f^{-n}(q)$ separates $B(f^{-n}(q), r_3)$ in at least two connected components, $B_n^+ = B^+(f^{-n}(q), r_3)$, $B_n^- = B^-(f^{-n}(q), r_3)$, and $S_n \subset W_\epsilon^s(f^{-n}(q))$.*

Proof. It is the same proof we have given in §3 in the proof of Proposition 3.11. □

Let us make the following two assumptions.

Assumption A. For all $\rho > 0$ there is a non-void open set $A \subset B(p, \rho)$ such that there is $N \subset A$, N dense in A, such that if $y \in N$ then the continuum $C(y) \subset W_\epsilon^u(y)$ cuts $D \subset W_\epsilon^s(p)$; i.e. given $\rho = 1/m$ there is a neighbourhood $U_m \subset B(p, 1/m)$, and N_m dense in U_m such that $\forall x \in N_m$, $C(x) \cap D \neq \emptyset$. This is a technical condition which will be used in the first part of this section to prove the existence of weak homoclinic points.

Remark 5.3. It is easy to prove that for all $y \in A \subset B(p, \rho)$ there is a continuum $C(y) \subset W_\epsilon^u(y)$ cutting $D \subset W_\epsilon^s(p)$. This may be accomplished by taking limits in the Hausdorff metric of $\{C(y_n)\}$, $\{y_n\} \subset N$, $y_n \to y$.

From the previous remark we may restate Assumption A as follows. For all $\rho > 0$ there is a non-void open set $A \subset B(p, \rho)$ such that if $y \in A$ then the continuum $C(y) \subset W_\epsilon^u(y)$ cuts $W_\epsilon^s(p)$.

LEMMA 5.4. *Under Assumption A there is a weak homoclinic intersection between $W^s(p)$ and $W^u(p)$.*

Proof. Let $y \in U_m \subset B(p, 1/m)$, $m \in \mathbb{N}^+$. If $y \in B(p, 1/m)$, $m \geq N > 0$, then $\text{dist}(y, W_\epsilon^s(p)) < N$. We will choose $N \in \mathbb{N}$ later. For a fixed $y \in U_m$ let $\{y_k\}_{k\in\mathbb{N}} \subset U_m$ be a sequence converging to y such that there is a sequence $n_k \to +\infty$ such that $\lim f^{-n_k}(y_k) = y$ too. Such sequences exist because $y \in \Omega(f) = M$. For every y_k we have that $C(y_k) \cap W_\epsilon^s(p) \neq \emptyset$ (here $C(y_k) \subset W_\epsilon^u(y_k)$). However, by the expansive properties of f there is a single intersection point, say, w_k. As in [**Vi3**, Lemma 4.2], we may prove that the function $h : \{y_k\}_{k\in\mathbb{N}} \to W_\epsilon^s(p)$ given by $h(y_k) = w_k$ is continuous and therefore we may find a compactum S included in $D \subset W_\epsilon^s(p)$ such that $w_k \in \text{int}(S)$ and such that $\text{dist}(w_k, \partial S)$ is bounded away from zero for all k. By [**Ma**, Lemma I, p. 315], we have that for every $x \in M$, for all $\epsilon > 0$, $\epsilon \leq \alpha$, $\lambda > 0$ there is an $N \in \mathbb{N}$ such that if $n \geq N$ then $f^{-n}(W_\epsilon^u(x)) \subset W_\lambda^u(f^{-n}(x))$. Thus, as $w_k \in C(y_k) \subset W_\epsilon^u(y_k)$ we have that $\lim f^{-n_k}(w_k) = y$. By Lemma 5.2 there is a sequence $\{S_k\}$ of compacta such that $S_k = f^{-n_k}(S)$.

It is clear that $S_k \subset W^s(f^{-n_k}(p))$. We claim that for m, k great enough S_k intersects $W_\epsilon^u(p)$. For, by fixing m, $f^{-n_k}(w_k)$ belongs to $U_m \subset B(p, 1/m)$ and, by Lemma 5.2, S_k is a subcontinuum of $W_\epsilon^s(f^{-n_k}(w_k))$ and separates $B(f^{-n_k}(w_k), r_3)$ if $k = k_m$ is great enough. Let us choose N so big that when $m \geq N$, S_{k_m} separates $B(p, r_3/2)$; increase N if necessary in order for ψ to be so small that $\text{dist}(W^s_\epsilon(y)\backslash\partial B(y, r_3/2), W^u_\epsilon(z)\backslash\partial B(y, r_3/2)) > 2\psi$ whenever $\text{dist}(y, z) < 1/N$. We may now adapt the ideas of [**Vi3**, Lemma 4.2 and Theorem 4.3 (Theorem B)], to find that S_{k_m} intersects $W^u_\epsilon(p)$ for sufficiently large k_m. Therefore there is a weak homoclinic intersection between $W^u(p)$ and $W^s(o(p))$. □

Assumption B. There are $\rho > 0$ and an open dense set $A \subset B(p, \rho)$ such that if $y \in A$ then any continuum $C(y) \subset W^u_\epsilon(y)$, does not intersect $W^s_\epsilon(p)$. It follows that if $0 < \rho' < \rho$ then there is an A' open and dense in $B(p, \rho')$ with the same property. In this case we cannot use the arguments of Lemma 5.4 to find a weak homoclinic intersection. However, we will show that under Assumption B there is an r_2, $0 < r_2 \leq r_1$ such that any point $x \in D_n = f^n(D) \subset W^s(f^n(p))$, $n \in \mathbb{Z}$ has a local unstable set, $W^u_\epsilon(x)$, which contains a continuum $C(x)$, $x \in C(x)$, such that $C(x)\backslash\{x\}$ is not connected and at least two connected components of $C(x)\backslash\{x\}$ reach the boundary of $B(x, r_2)$.

It seems that Assumption B is rather artificial principally if we observe that it implies, as we have just stated, that all points of D_n have 'two-sided' local unstable sets, but we cannot prove directly that it cannot hold. Instead of proving directly that Assumption B is not possible we use it to prove the 'two-sidedness' of the local unstable sets of points of D_n. As a result of this property we conclude that Assumption B is not possible, i.e. that Assumption A holds.

LEMMA 5.5. *If Assumption B takes place then there is an $r_2 > 0$ such that if $x \in D_n$ then there is a continuum $C(x) \subset W^u_\epsilon(x)$ such that $C(x)\backslash\{x\}$ has at least two connected components each of which reach $\partial B(x, r_2)$.*

Proof. Let $x \in D$ such that $S(x)$ the connected component of $W^s_\epsilon(x) \cap B(x, r_1)$ containing x separates $B(x, r_1)$ in at least two connected components B^+_x, B^-_x. As $C(x)$ is a non-trivial continuum such that $C(x) \cap \partial B(x, r_0) \neq \emptyset$ and $C(x) \cap S(x) = \{x\}$ there are points of $C(x)$ at least in a component of $W^s_\epsilon(x) \cap B(x, r_1)$, say B^+_x. Let us redefine the notation and call B^-_x the union of the other connected components different from B^+_x such that x is a point in their boundary. If there is no continuum $C'(x) \subset W^u_\epsilon(x)$ such that $C'(x) \cap \partial B(x, r_1) \cap B^-_x \neq \emptyset$ then assuming that $r_1 \leq r_0/2$, we have that for all $y \in B(x, \lambda)$, $C(y) \cap \partial B(x, r_1) \cap B^+_x \neq \emptyset$. This is true because for all sequences y_n converging to x we cannot have $C(y_n) \cap \partial B(x, r_1) \cap B^-_x \neq \emptyset$. Otherwise the same would be true for $C(x)$. Take $y \in B(x, \lambda)$, $y \in B^-_x$. Hence $C(y) \cap S(x) \neq \emptyset$, and by expansivity it is a single point w. Moreover, we may assume that y is such that $\text{dist}(y, w) < \lambda$, where $\lambda > 0$ has been chosen such that $w \in W^u_\epsilon(z)$ and $\text{dist}(z, w) < \lambda$ imply $w \in W^u_{\epsilon/k}(z)$. Thus $f^N(w) \in f^N(W^u_{\epsilon/k}(y))$. As N is fixed there is a k_0 such that for all $k \geq k_0$ $f^N(W^u_{\epsilon/k}(y)) \subset W^u_\epsilon(f^N(y))$ which contradicts Assumption B. Thus $C(x) \cap \partial B(x, r_1) \cap B^-_x \neq \emptyset$ and also $C(x) \cap \partial B(x, r_1) \cap B^+_x \neq \emptyset$. Now let x be any point of D_n. By Lemma 5.2 there is an n_1 such that the connected component $S(f^{-n_1}(x))$ of $W^s_\epsilon(f^{-n_1}(x)) \cap B(f^{-n_1}(x), r_1)$ containing x separates $B(f^{-n_1}(x), r_1)$. Hence the previous arguments apply to $f^{-n_1}(x)$ and we obtain a continuum $C(f^{-n_1}(x)) = C^+ \cup C^-$, $C^+ \subset B^+_{f^{-n_1}(x)}$, $C^- \subset B^-_{f^{-n_1}(x)}$,

$C^+ \cap \partial B(f^{-n_1}(x), r_1) \neq \emptyset$, $C^+ \cap \partial B(f^{-n_1}(x), r_1) \neq \emptyset$. Let $\mathcal{U}$ be a Lyapunov function for f. We have that for all $y \in C(f^{-n_1}(x))$, $y \neq f^{-n_1}(x)$, $\Delta\mathcal{U}(f^{-n_1}(x), y) > 0$. Thus in the connected component $\mathcal{C}^+$ of $f^{n_1}\big(C^+_{f^{-n_1}(x)}\big) \cap \mathrm{clos}(B(x, r_1))$ containing x the same is true. Using similar arguments as those in [**Vi3**, Lemma 4.1], we may prove that there is an $r_2 > 0$ such that $\mathcal{C}^+ \cap \partial B(x, r_2) \neq \emptyset$ and calling $\mathcal{C}^-$ to the connected component of $f^{n_1}\big(C^-_{f^{-n_1}(x)}\big) \cap \mathrm{clos}(B(x, r_1))$ containing x, we also have that $\mathcal{C}^- \cap \partial B(x, r_2) \neq \emptyset$. As $\mathcal{C}^+$ and $\mathcal{C}^-$ are locally separated by $S(x)$, we obtain $C(x) = \mathcal{C}^+ \cup \mathcal{C}^-$ with the desired properties. □

Remark 5.6. If B_x is any connected component of $B(x, r)\backslash S(x)$ with the point x in its boundary then the previous argument shows that there is a $C'(x)$ contained in the local unstable set of x and also in B_x. This implies, as in [**Vi2**], that the number of regions B_x is finite.

Definition 5.2. Let $D, D' \subset M$ be continua such that both separate $B(x, r)$ and such that D is contained in one region of $B(x, r)\backslash D'$ and D' is contained in one region of $B(x, r)\backslash D$. We say that a point z is between D and D' in $B(x, r)$ if it belongs to the region of $B(x, r')\backslash D$ containing D' intersected with that of $B(x, r')\backslash D'$ containing D. We say that a set $S \subset B(x, r)$ is between D and D' in $B(x, r)$ if every point in S is between D and D'.

LEMMA 5.7. *Assumption B implies a contradiction.*

Proof. Assume that Assumption B holds. Let $x \in D$ and let us call $x_n = f^{-n}(x)$. By Lemma 5.2 there are a continuum S, $x \in \mathrm{int}(S)$ and $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$, S_n, the connected component of $f^{-n}(S) \cap B(x_n, r_1)$ containing x_n separates $B(x_n, r_1)$ and $S_n \subset W^s_\epsilon(x_n)$. Moreover, by Assumption B, for all x_n there is a continuum $C_n \subset W^u_\epsilon(x_n)$ such that it is separated by S_n in at least two connected components C_n^+ and C_n^- and both components reach $\partial B(x_n, r_2)$. We choose a subsequence $n_k \to +\infty$ such that $\{x_{n_k}\}$ converges to y in the metric of M and $\{S_{n_k}\}$ and $\{C_{n_k}\}$ converge to $D(y)$ and $C(y)$ respectively in the Hausdorff metric. In order to simplify notation we assume that $\{x_n\}$, $\{S_n\}$ and $\{C_n\}$ converge. By Cauchy's condition of convergence given $\sigma > 0$ there is $N_1 \geq N_0$ such that if $n, m \geq N_1$ then $\mathrm{dist}(x_n, x_m) < \sigma$, and $\mathrm{Hdist}(S_n, S_m) < \sigma$, $\mathrm{Hdist}(C_n, C_m) < \sigma$. Let us choose $\sigma > 0$ so small that if z is in $B(y, 2r_1/3)\backslash B(y, r_1/3)$ and between S_n and S_m then a continuum $C(z) \subset W^u_\epsilon(z)$ intersects either S_n or S_m. Let us assume that if $\mathrm{dist}(w, z) < r_0$ and $w \in W^u_{2\epsilon}(z)$ ($w \in W^s_{2\epsilon}(z)$) then $w \in W^u_\epsilon(z)$ (respectively $w \in W^s_\epsilon(z)$). Choose z between S_n and S_m, with $n, m \geq N_1$ fixed. Assume $C(z) \cap S_n = \{w\}$. As $w \in W^s(p)$ there is a continuum $C(w) \subset W^u_\epsilon(w)$ such that $C(w)\backslash S_n = C^+ \cup C^-$ with C^+ and C^- continua locally separated by S_n and both reaching $\partial B(w, r_2)$. One of those pieces, C^+ or C^-, is in the same component with respect to S_n as z and S_m are. Let C^+ be that piece. Then $C^+ \cap S_m \neq \emptyset$ otherwise there will be points of $C(w)$ and S_n at a distance less than σ. But $S_n \subset W^s_\epsilon(w)$ and also $C(w) \subset W^u_\epsilon(w)$ which contradicts expansivity if $\sigma > 0$ is sufficiently small. Let us consider the closed set C given by the union of $C(z)$ with the closure of the subset of C^+ between S_n and S_m. We claim that $C \subset W^u_\epsilon(z)$. Let $v \in C$. The claim is clear if $v \in C(z)$. If $v \in C^+$ and is between S_n and S_m then for all $n \leq 0$, $\mathrm{dist}(f^n(w), f^n(z)) \leq \epsilon$ and $\mathrm{dist}(f^n(w), f^n(v)) \leq \epsilon$

so $\text{dist}(f^n(v), f^n(z)) \le 2\epsilon$. But every point between S_n and S_m is at a distance less than r_0 from z if σ and r_2 are sufficiently small. Therefore $v \in W^u_\epsilon(z)$. Also we have a subcontinuum $C' \subset C$ such that C' joins z to $u \in S_m$. Thus we have obtained the following property.

Every point z between S_n and S_m has a continuum $C' \subset W^u_\epsilon(z)$ which cuts both, S_n and S_m.

But with similar arguments to those used in Lemma 5.5 we see that this property implies a contradiction to Assumption B for we may take z so close to S_n that by applying f^n we have that $f^n(C')$ is in $W^u_\epsilon(f^n(z))$ and intersects D. Moreover, we may take a small neighbourhood V of z such that if $z' \in V$ then the same holds for z'. But this implies Assumption A thus contradicting Assumption B. □

THEOREM 5.8. *There are always weak homoclinic points for f.*

Proof. It follows from the fact that Assumption A always holds. □

We are now in the following position. There is a point p such that $D(p)$, the connected component of $W^s_\epsilon(p)$ containing it, separates $B(p, r)$ and $C(p)$, the connected component of $W^u_\epsilon(p)$ containing p, is such that $C(p)\backslash\{p\}$ contains two subcontinua C^+ and C^- in different connected components of $B(p, r)\backslash D(p)$ and joining p with the boundary of $B(p, r)$. Moreover, there is a sequence x_n converging to p, $x_n \in C(p)$, such that for them $D(x_n)$, the connected component of $W^u_\epsilon(x_n)$ containing x_n, separates $B(p, r)$ too. We may also assume that the sequence $D(x_n)$ converges to $D(p)$ in the Hausdorff metric defined on the compacta. From this we will prove that there is a neighbourhood A in M in which a local product structure is defined. Applying Theorem A of [**Vi3**] we conclude that there is a uniform local product structure. This implies that M is the 3-torus T^3 and f is conjugated to an Anosov diffeomorphism.

The idea of the proof is as follows.

(1) We prove that every point x in a suitable neighbourhood A of p has continua $D(x)$ and $C(x)$, $D(x)$ separating $B(p, r)$ and $C(x)$ such that $C(x)\backslash\{x\}$ contains two subcontinua C^+ and C^- in different connected components of $B(p, r)\backslash D(x)$ and joining x to the boundary of $B(p, r)$.

(2) The continua $C(x)$ are arcs. This will follow from the fact that if $y \in C(x)$ and we subtract y from $C(x)$ then we disconnect it unless y is an end-point of $C(x)$, and the fact that there are two end-points in $C(x)$.

(3) $D(x)$ is locally connected.

(4) The continua $D(x)$ are discs. This will follow from the characterization of the surfaces given by Young (see [**Br**, Theorem 16.32]). If D is a locally connected compact metric space X such that $\forall x \in X$ there exists $U(x)$ connected such that $U(x)\backslash\mathcal{C}$ is not connected for all simple closed curve $\mathcal{C} \subset U(x)$ then D is a 2-manifold possibly with a boundary.

(5) Using the fact that the non-wandering set is the whole manifold we prove that periodic points are dense in A.

(6) Finally we prove that these periodic points are topologically hyperbolic and we are in a position to apply Theorem A of [**Vi3**].

As most of the arguments are the same as those we have used in [**Vi2**] or [**Vi3**] detailed proofs are not usually given.

Taking into account that $\{D(x_n)\}$ converges in the Hausdorff metric to $D(p)$ we may assume that $D(x_n) \subset [D(p)]_\sigma$ for all $n \in \mathbb{N}$ where $\sigma > 0$ is so small that for every point z between $D(p)$ and $D(x_n)$ in $B(p, r)$ there is a continuum $C(z) \subset W^u_\epsilon(z)$ which intersects either $D(p)$ or $D(x_n)$ (or both).

LEMMA 5.9. *There is $r'' > 0$ such that for every z between $D(p)$ and $D(x_n)$ in $B(p, r'')$ there is a compact connected set $D(z) \subset W^s_\epsilon(z)$, $z \in D(z)$, which separates $B(p, r)$ and p and x_n are in different connected components of $B(p, r)\backslash D(z)$.*

Proof. Let z be as in the hypotheses of Lemma 5.9. As $\Omega(f) = M$, there are sequences $z_j \to z$ and $n_j \to +\infty$ such that $\lim_{j\to+\infty} f^{-n_j}(z_j) = z$. For sufficiently small $r'' > 0$ for every y between $D(p)$ and $D(x_n)$ in $B(p, r'')$ there is a continuum $C(y) \subset W^u_\epsilon(y)$ which intersects either $D(p)$ or $D(x_n)$. If $C(y)$ only intersects $D(p)$, then there is a neighbourhood $V(y)$ between $D(p)$ and $D(z_n)$ in $B(p, r'')$ such that every $y' \in V(y)$ has a continuum $C(y') \subset W^u_\epsilon(y')$ intersecting $D(p)$ too; otherwise there would be a sequence $y_m \to y$ such that the corresponding $C(y_m)$ intersects only $D(x_n)$ and taking a convergent subsequence of $\{C(y_m)\}$ we will have (see Theorem 5.8 and also [**Vi2**, §1]) a continuum $C(y) \subset W^u_\epsilon(y)$ intersecting $D(x_n)$. Therefore we may assume that for the sequence $\{z_j\}$, $C(z_j)$ intersects the same subcontinuum, say $D(p)$, in a point w_j. Thus $f^{-n_j}(C(z_j))$ intersects $f^{-n_j}(D(p))$ in $f^{-n_j}(w_j)$. By Lemma 5.5, given $\lambda > 0$ there is $L \in \mathbb{N}$ such that for all $n \geq L$, for all $w \in M$, $f^{-n}(W^u_\epsilon(w)) \subset W^u_\lambda(f^{-n}(w))$. Therefore the diameter $\text{diam}(f^{-n_j}(C(z))) \to 0$ uniformly with $j \to +\infty$, and, as $f^{-n_j}(z_j) \to z$, we also have $f^{-n_j}(w_j) \to z$. As in [**Vi1**, §2], there is an $N \in \mathbb{N}$ and a sequence of continua $S_j \subset W^s_\epsilon(f^{-n_j}(w_j))$, such that S_j separates $B(f^{-n_j}(w_j), 2r)$ if $n_j \geq N$. To see this observe that, as M is compact, $\{w_j\}$ converges to a point w. For if for a certain subsequence $\{w_{j_k}\}$ the $\lim_{k\to+\infty} w_{j_k} = w$ then $\{w\} = C(z) \cap S(p)$ by the uniqueness of the intersection point (the expansivity of f). Take $D \subset D(p)$ such that w$\in$ int(D) and $D \subset W^s_{\epsilon/4}(w)$. Therefore, there is a $J \in \mathbb{N}$ such that for all $j \geq J$, $w_j \in \text{int}(D)$ and $D \subset W^s_{\epsilon/2}(w_j)$. Without loss of generality we may assume that $w_j \in \text{int}(D)$ for all $j \in \mathbb{N}$. Given an arc $\gamma_j \subset D$ joining w_j with ∂D there is an $N_j \in \mathbb{N}$ such that $f^{-n}(\gamma_j)$ reaches the boundary of $B(f^{-n}(w_j), r)$ for all $n \geq N_j$ for all arcs γ_j (the compactness of D). It follows, by the compactness of $\{w_j\}_{j\in\mathbb{N}} \cup \{w\}$, that there is an $N \in \mathbb{N}$ such that for all arcs γ joining ∂D with a point w_j, $f^{-n}(\gamma)$ reaches $\partial B(f_{-n_j}(w_j), 2r)$ provided that $n \geq N$. Hence, if $n_j \geq N$, $f^{-n_j}(\gamma)$ reaches $\partial B(f^{-n_j}(w_j), 2r)$. From this, as in Lemma 5.7, we may prove that the connected component S_j of $W^s_\epsilon(f^{-n_j}(w_j)) \cap B(f^{-n_j}(w_j), 2r)$ containing $f^{-n_j}(w_j)$ separates $B(f^{-n_j}(w_j), 2r)$. As in Theorem 5.8, taking a convergent subsequence from $\{S_j\}$, there is a continuum $D(z) \subset W^s_\epsilon(z)$, $z \in D(z)$. Assume, without loss of generality, that $D(z) = \text{Hlim}_{j\to\infty}(S_j)$. We have that $D(z)$ separates $B(z, 2r)$ if z is sufficiently close to p, say $z \in B(p, r'')$. For with arguments similar to those used before we may prove that S_j intersects $C(p)$ in a single point if we take z near p, say $z \in V(p)$. Moreover, S_j separates $C(p)$. Then $D(z)$ will intersect $U(p, 2r)$ and as in Theorem 5.8 (see also [**Vi2**, Proposition 1.5]), we may prove that it separates $U(p)$ in $B(z, 2r)$. Therefore it separates $B(z, r')$ for $0 < r' \leq 2r$. Taking r'' such that if

$z \in B(p, r'')$ then $B(z, r/2) \subset B(p, r) \subset B(z, 2r)$ we may prove that $D(z)$ separates $B(p, r)$. It is clear that p and x_n are in different connected components of $B(p, r)\backslash D(z)$. This finishes the proof. □

LEMMA 5.10. *For every z between $D(p)$ and $D(x_n)$ in $B(p, r'')$, r'' as in Lemma 5.9, there is a continuum $C(z) \subset W^u_\epsilon(z)$, $z \in C(z)$, such that $C(z)$ is separated by $D(z)$ in $B(p, r)$. Moreover, $C(z)$ joins $D(p)$ with $D(x_n)$.*

Proof. By Lemma 5.9, for all $z \in B(p, r'')$, $D(z)$ intersects $C(p)$. As $\Omega(f) = M$ there are sequences $\{z_j\}$, $\lim_{j\to\infty} z_j = z$, $n_j \to +\infty$ such that $\lim_{j\to\infty} f^{n_j}(z_j) = z$. With arguments similar to those of Lemma 5.9 we find $C(z) \subset W^u_\epsilon(z)$ as the Hausdorff limit of a sequence of continua $\gamma_j \subset W^u_\epsilon(f^{n_j}(w_j))$, where $\{w_j\} = D(z_j) \cap U(p, r)$. Moreover, $D(z_j)$ separates $U(p)$ into two arcs $U^+(p)$ and $U^-(p)$ and from this we have two sequences $\{\gamma_j^+\}$ and $\{\gamma_j^-\}$, with $\gamma_j = \gamma_j^+ \cup \gamma_j^-$, $f^{n_j}(w_j) = \gamma_j^+ \cap \gamma_j^-$, $\gamma_j^+ \subset W^u_\epsilon(f^{n_j}(w_j)) \cap f^{n_j}(U^+(p))$ and $\gamma_j^- \subset W^u_\epsilon(f^{n_j}(w_j)) \cap f^{n_j}(U^-(p))$ such that both γ_j^+ and γ_j^- reach $\partial B(f^{n_j}(w_j), r)$ and $D(f^{n_j}(w_j))$ locally separates them from one another at $f^{n_j}(w_j)$. As $D(x_n) \subset [D(p)]_\sigma$, the σ-parallel body of $D(p)$, we have that γ_j^+ and γ_j^- must intersect $D(x_n)$ or $D(p)$. But if γ_j^+ intersects $D(p)$ then γ_j^- intersects $D(x_n)$ and *vice versa*; otherwise we violate expansivity. Thus $C(z)$, being the Hausdorff limit of $\{\gamma_j\}$, intersects both $D(x_n)$ and $D(p)$; therefore $C(z)$ joins $D(p)$ with $D(x_n)$. We have that $D(x_n) \cap D(p) = \emptyset$ and, by Lemma 5.9, $D(z)$ separates $U(p, r)$ so it separates $D(x_n)$ from $D(p)$. It follows that the points given by $C(z) \cap D(p)$ and $C(z) \cap D(x_n)$ are in different components of $B(p, r)$ with respect to $D(z)$. □

LEMMA 5.11. *The continua $C(x)$ are locally connected (lc).*

Proof. If this were not true then we would have a subcontinuum $X \subset C(x)$ and a neighbourhood V such that $X \cap V$ is not connected and that, moreover, there would be a sequence of continua X_k converging in the Hausdorff metric to a continuum X_∞ with k such that $X \cap V \supset X_\infty \cup X_k$ and such that $X_k \cap X_\infty = \emptyset$. Take $y \in X_\infty$. Then $D(y)$ intersects $C(x)$ in y and therefore it cannot cut X_k. Take $y_k \in X_k$ converging to y and the corresponding $D(y_k)$. It follows that $D(y_k)$ converges to $D'(y) \subset D(y)$, with $D'(y)$ separating $B(p, r)$ too. Thus, either $D(y_k)$ cuts X_∞ or there are points in the boundary of V of $D(y_k)$ and of X_k arbitrarily close. Both possibilities contradict expansivity. □

COROLLARY 5.12. *The continua $C(x)$ are locally arcwise-connected.*

Proof. This follows from the fact that a set that is connected and locally connected is arcwise-connected. □

LEMMA 5.13. *The continuum $C(p)$ is an arc.*

Proof. Remember that p is accumulated by points where the strong local stable manifolds and the strong local unstable manifolds are defined. The way we construct $D(x)$ defines a continuous bijection between $C(p)$ and these strong local unstable manifolds. Thus $C(p)$ reduces to an arc. □

LEMMA 5.14. *The continua $D(x)$ are lc.*

Proof. This is rather similar to the proof of Lemma 5.11. The main difference is that if V, X, X_k, X_∞, y and y_k are as before in Lemma 5.11 then we have that X_k separates a small neighbourhood of y_k. The proof of this fact is as follows: take a sequence of points y_{k_l} converging to y_k with l and such that $f^{-n_{k_l}}(y_{k_l})$ also converges to y_k with $n_{k_l} \in \mathbb{N}$ diverging. Let $w_{k_l} \in D(p) \cap C(y_{k_l})$. Repeating our previous arguments we may find a set $D'(y_k)$ passing through y_k that separates V and that is the connected component of $W^s_\epsilon(y_k) \cap V$ containing y_k. But $X_k \ni y_k$ is connected and by definition and our assumption that $D(x)$ is not locally connected at X it has to contain $D'(y_k)$. □

THEOREM 5.15. *There is an open set $A \subset M$ in which there is defined as a local product structure.*

Proof. Let p be as before, r'' be as in Lemma 5.9, $n \in \mathbb{N}$ such that $D(x_n) \subset [D(p)]_\sigma$ (which may be assumed; see the paragraph above Lemma 5.9), and $n' > n$ be such that $x_n \in B(p, r'')$. Therefore $D(x_{n'})$ is between $D(x_n)$ and $D(p)$. Consider an arc γ of points of $C(p)$ between $D(x_n)$ and $D(p)$ such that $x_{n'} \in \operatorname{int}(\gamma)$ and every point of γ is in $B(p, r'')$. Let D be a continuum in $D(x_{n'})$ such that $x_{n'} \in \operatorname{int}(D)$ and $D \subset B(p, r'')$. For every point $y \in D$ there is a $C(y)$, as in Lemma 5.10, joining $D(p)$ with $D(x_n)$; and for every point $z \in \gamma$ there is a $D(z)$, as in Lemma 5.9, such that $D(z)$ separates p from x_n in $B(p, r)$. Therefore $C(y)$ intersects $D(z)$ and, by the expansive properties of f, $C(y) \cap D(z)$ is a single point w. Let us define a function $h : D \times \gamma \to M$ by $h(y, z) = w$, it is clear that $h(x_{n'}, x_{n'}) = x_{n'}$. We claim that h is continuous, injective and onto. Therefore h is a homeomorphism and its image contains a neighbourhood A of $x_{n'}$. Therefore h defines an f-lps in A.

The proof of our claim is as follows. To see that h is continuous we consider a sequence $\{(y_n, z_n)\} \subset D \times \gamma$ such that it converges to $(y, z) \in D \times \gamma$ as $n \to \infty$. As M is compact, the sequence $\{w_n\}$ such that $w_n = h(y_n, z_n)$ has a convergent subsequence $\{w_{n_k}\}$, say to a point w_∞. But $w_{n_k} = C(y_{n_k}) \cap D(z_{n_k})$ hence $w_\infty \in W^u_\epsilon(y) \cap W^s_\epsilon(z)$. Thus $w_\infty = w = C(y) \cap D(z)$ by expansiveness. Therefore h is continuous. To prove that it is injective let (y, z) and (y', z') be points in $D \times \gamma$ such that $h(y, z) = h(y', z') = w$. This is the same as say was that $C(y') \cap D(z') = C(y) \cap D(z)$ and therefore $C(y) \cap D(z') = C(y) \cap D(z)$. As $w \in W^s_\epsilon(z)$ and $w \in W^s_\epsilon(z')$, for all $n \geq 0$, $\operatorname{dist}(f^n(z), f^n(z')) \leq 2\epsilon < \alpha$. As $z, z' \in \gamma \subset U(p, r)$ we also have that for all $n \leq 0$, $\operatorname{dist}(f^n(z), f^n(z')) < \alpha$. Therefore, by the expansive properties of f, $z = z'$. In a similar way we may prove that $y = y'$.

To see that it is onto it suffices to recall that given x in $B(p, r)$ we have that $C(x)$ joins $D(p)$ to $D(x_n)$ and $D(x)$ intersects $C(p)$. This finishes the proof of the claim and the theorem. □

COROLLARY 5.16. *For all $x \in B(p, r'')$ $C(x)$ is an arc.*

Proof. This follows from the existence of the local product structure and the fact that $C(p)$ is an arc. □

LEMMA 5.17. *It holds that $D(p)$ is a 2-disc.*

Proof. We have that

$$D(p) \times C(p) \overset{h}{\mapsto} A \supset B(p, r'')$$

so we may assume that $D(p) \times C(p) \sim \mathbb{R}^3$. Let $\mathcal{C}$ be any Jordan curve contained in $D(p)$. There is at least one because $D(p)$ is arcwise-connected and locally separates M. We consider the cylinder S defined by the curve $\mathcal{C}$ and the arcs $C(y)$ with $y \in \mathcal{C}$. Let us identify $C(p)$ with $\mathbb{R}$ and let us compactify $\mathbb{R}^3$ with a point, obtaining S^3. Then the cylinder S compactifies to a pinched torus K where a whole meridian is collapsed to a point (at infinity). K separates S^3 in two connected components and therefore $\mathcal{C}$ separates $D(p)$ in two connected components too. Hence $D(p)$ is locally a 2-manifold. It follows that it is a Euclidean 2-disc around p. □

LEMMA 5.18. *If there is an f-lps in A then periodic points are dense in A.*

Proof. See [**Vi3**, Proposition 2.15]. Although it is assumed there that $\Omega(f) = M$ the only thing we need is that $\Omega(f) \supset A$. □

LEMMA 5.19. *If $q \in A$ is a periodic point then it is topologically hyperbolic.*

Proof. See [**Vi3**, Proposition 2.16]. □

6. *Conclusion*

In view of the previous results we are able to apply Theorem A of [**Vi3**] to conclude the following theorem.

THEOREM 6.1. *There is a uniform local product structure on the whole manifold M.*

Proof. See [**Vi3**, §3]. □

Therefore the hypotheses of [**Vi2**] hold, i.e. topologically hyperbolic periodic points are dense in M. From this we have the following theorem.

THEOREM 6.2. *If M is a compact connected orientable smooth 3-manifold and $f : M \to M$ is an expansive homeomorphism such that* $\mathrm{Per}_H(f)$ *is dense in M then $M \cong T^3$ and f is conjugated to a linear Anosov diffeomorphism.*

Proof. See [**Vi2**, Theorems 10.5 and 10.6]. See also [**Vi5**] for a detailed proof of why M is a torus. □

Finally, taking into account the previous results we have the following theorem.

THEOREM 6.3. *If M is a compact connected orientable smooth 3-manifold and $f : M \to M$ is an expansive $C^{1+\theta}$-diffeomorphism,* $0 < \theta$, *with $\Omega(f) = M$ then $M \cong T^3$ and f is conjugated to a linear Anosov diffeomorphism.*

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