Bernoulli Elliptical Stadia

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Abstract

Let $Q_{a,h}$ be a convex region of the plane whose boundary consists of two semiellipses joint by two (straight) lines parallel to the major axis of the semiellipses (elliptical stadium). The axis of the semiellipses have length 2 and $2a, a > 1$, and the lines have length $2h$. For $1 < a < \sqrt{4-2\sqrt{2}}$ and $h > 2a^2\sqrt{a^2-1}$, we give a complete proof of the following result: the billiard map in the elliptical stadium $Q_{a,h}$ is ergodic, K-mixing and Bernoulli with respect to the natural billiard measure.

1 Introduction

Let $Q = Q_{a,h}$ be a convex region of the plane bounded by two semiellipses with axis of length 2 and $2a, a > 1$ and joint by two straight lines of length $2h$ which are parallel to the major axis of the semiellipses. The region Q is called *elliptical stadium*. Let T be the dynamical system describing the free motion of a point mass in Q with elastic reflections at the boundary according to the law: the angle of incidence equals the angle of reflection.

In [MOP], it was showed that the map T has non-zero Lyapunov exponents (L.E.) for $1 < a < \sqrt{4-2\sqrt{2}}$ and $h > 2a^2\sqrt{a^2-1}$ by constructing a suitable T-invariant cone field. In this paper, for the same values of the parameters a, h, we give a complete proof of the following result: the billiard map T of the elliptical stadium $Q_{a,h}$ is ergodic, K-mixing and Bernoulli with respect to the natural billiard measure.

The scheme of the proof of the ergodicity of T is the same as the one used in [DelM] for truncated ellipses and consists of two main steps. First we introduce a restricted phase space and the first return map Φ on it induced by T. Then we show that Φ is locally ergodic (i.e., each ergodic component of Φ is open (mod 0)) by using the version of the Fundamental Theorem proved in [LW]. In this part of the proof, the main difference between this paper and [DelM] is the proof of the noncontraction property for blocks of type 2 (Lemma 18). The second step in the proof of the ergodicity consists in proving that Φ has only one ergodic component. We prove that Φ has only a finite number of ergodic components (Lemma 21), and construct a finite collection of trajectories (possibly disjoint) with the property that starting from any ergodic component we can reach any other ergodic component by traveling along these trajectories (Theorem 22).

The Kolmogorov and Bernoulli properties are proved at the end of the paper using the general results contained in [Pe77] and [CH, OW].

2 Background material on Billiards

Let Q be an open bounded and connected subset of the plane whose boundary consists of a finite number of closed C^{k+1}-curves $\Gamma_i, k \geq 2$. The billiard in Q is the dynamical system describing the free motion of a point mass inside Q with elastic reflections at the boundary $\Gamma = \cup_i \Gamma_i$.

The study of billiards turns out to be simpler if we consider the so called billiard map associated to the billiard in Q. Let $n(q)$ be the unit normal of the curve Γ at the point $q \in \Gamma$ pointing toward the interior of Q. Consider the set

$$M' = \{(q, v) : \ q \in \Gamma, \ |v| = 1, \ \langle v, n(q)\rangle \geq 0\}.$$

Let π denote the projection of M' onto Q, i.e., $\pi(q, v) = q$. We introduce the set of coordinates (s, θ) on M' where s is the arc length parameter along Γ and θ is the angle between v and the tangent to the boundary at q. Clearly $0 \leq \theta \leq \pi$ and $\langle n(q), v\rangle = \sin\theta$. A natural probability measure on M' is $d\nu = c \sin\theta \, ds d\theta$ where $c = (2|\Gamma|)^{-1}$ is the normalizing factor and $|\Gamma|$ stands for the total length of Γ. The *billiard map* $T : M' \to M'$ is defined by $T(q_0, v_0) = (q_1, v_1)$ where q_1 is the point of Γ hit first by the oriented line through (q_0, v_0) and v_1 is the velocity vector after the reflection at q_1. Formally, $v_1 = v_0 - 2\langle n(q_1), v_0\rangle n(q_1)$. We denote by $z_i = (q_i, v_i) \in M'$, $i \in \mathbb{N}$, the ith iterate of $z_0 = (q_0, v_0)$ under the map T. These points represent the successive collisions with the Γ of a trajectory beginning at $z_0 = (q_0, v_0)$ so that $T(q_i, v_i) = (q_{i+1}, v_{i+1})$. The angle between v_i and the boundary at q_i is denoted by θ_i, and the Euclidean distance between the bouncing points q_i and q_{i+1} is denoted by t_i. Since the speed of the point mass is one, then t_i is also the time between q_i and q_{i+1}. The negative iterates $z_i = (q_i, v_i), i < 0$ of z_0 are defined analogously. The main relations are $Tz_i = z_{i+1}$ and $q_{i+1} = q_i + t_i v_i$ with $i \in \mathbb{Z}$.

The map T is piecewise C^k. It is not well defined at z_0 if $n(q_1)$ is not defined or if the oriented line through z_0 is tangent to some Γ_k ($\theta_1 = 0, \pi$). Finally T is continuous but not differentiable at z_0 if Γ is C^1 but not C^2 at q_1. The measure ν is preserved by T. The sets of points $x = (q, v) \in M'$ whose forward or backward trajectory is tangent to Γ or ends in $\Gamma_i \cap \Gamma_j$ have μ-measure zero.

If T is well defined and differentiable at $\tilde{z}_0 = (\tilde{q}_0, \tilde{v}_0)$, then for all $z_0 = (q_0, v_0)$ in a small neighborhood of $\tilde{z}_0$ the derivative matrix of T is given by

$$D_{z_0}T = \begin{pmatrix} \frac{t_0 K_0 - \sin\theta_0}{\sin\theta_1} & \frac{t_0}{\sin\theta_1} \\ K_1 \frac{t_0 K_0 - \sin\theta_0}{\sin\theta_1} - K_0 & \frac{K_1 t_0}{\sin\theta_1} - 1 \end{pmatrix} \tag{1}$$

where $K_i = K(z_i)$, $i = 0, 1$, is the curvature of Γ at q_i. If both q_0, q_1 do not belong to straight lines, then (1) can be rewritten as

$$\begin{pmatrix} \frac{t_0 - d_0}{r_0 \sin\theta_1} & \frac{t_0}{\sin\theta_1} \\ \frac{t_0 - d_0 - d_1}{r_0 d_1} & \frac{t_0 - d_1}{d_1} \end{pmatrix} \tag{2}$$

where $r_i = 1/K_i, i = 0, 1$, is the radius of curvature of Γ at q_i and $d_i = r_i \sin\theta_i, i = 0, 1$. Note that if $K_i > 0$ (focusing component), then d_i is the length of the subsegment of $\overline{q_0 q_1}$ contained in the disk $D(q_i)$ tangent to Γ at q_i with radius $r_i/2$ (*half-osculating disk*).

We remark the main differences with other usual conventions related with these formulas (see, for example [CM]): the curvature of the ellipse is positive, the angle θ, measured from the boundary to v, is always positive and increases counterclockwise.

2.1 Elliptical Billiards

In this section, we recall some elementary geometrical properties of elliptical billiards which we will use extensively in the sequel.

Consider an ellipse with semiaxis of length 1 and $a > 1$ given by

$$x(u) = a\cos u,\ y(u) = \sin u, \qquad 0 \le u \le 2\pi.$$

The curvature of the ellipse at the point $(x(u), y(u))$ is given by

$$K(u) = \frac{a}{(a^2\sin^2 u + \cos^2 u)^{3/2}}.$$

An ellipse may be also parameterized by the coordinate φ which is the angle made by the line tangent to the ellipse at the point (x, y) with the horizontal line $y = 0$. The equations of our ellipse and its curvature parameterized by φ are given by

$$x(\varphi) = \sqrt{\frac{a^2\tan^2\varphi}{1 + a^2\tan^2\varphi}}, \quad y(\varphi) = \pm\frac{1}{\sqrt{1 + a^2\tan^2\varphi}},$$

$$K(\varphi) = \frac{(1 + a^2\tan^2\varphi)^{1/2}(1 + \tan^2\varphi + a^2\tan^2\varphi + a^2\tan^4\varphi)}{a^2(1 + \tan^2\varphi)^{5/2}}.$$

The relation between the coordinates s and φ reads $K(s)ds = d\varphi$ from which we immediately obtain that $d\theta/d\varphi = K(s)^{-1}d\theta/ds$.

Now consider the billiard in our ellipse. Its phase space may be parameterized by the two sets of coordinates (s, θ) and (φ, θ). For the moment we use the second set of coordinates. The function

$$G(\varphi, \theta) = \frac{\cos^2\theta - \epsilon^2\cos^2\varphi}{1 - \epsilon^2\cos^2\varphi} \quad [\epsilon = \frac{\sqrt{a^2 - 1}}{a} \text{ is the eccentricity of the ellipse}], \tag{3}$$

is a first integral for the billiard map of the ellipse, meaning that G is constant (≤ 1) along the orbits of the billiard map. The phase space of the elliptical billiard is foliated by the level curves of G.

It is important to compute the slope of the tangent lines to these level curves. We will give the answer in coordinates (s, θ). The slope of the level curve $G = G(z)$ at $z = (s, \theta)$ is given by

$$p(z) := \frac{d\theta}{ds} = \frac{K(s)\epsilon^2\sin 2\varphi}{\sin 2\theta}(1 - G). \tag{4}$$

Level curves in the phase space are associated to curves in the plane called *caustics*. A caustic Σ is characterized by the property: if a segment (or its continuation) of a trajectory is tangent to Σ, then any other segment (or its continuation) of the trajectory will be tangent to Σ. It is well known that an ellipse E has two continuous families of caustics consisting of ellipses and hyperbolae confocal to E. Hence we can divide the level curves of G in two classes: elliptic and hyperbolic level curves.

Definition 1. *Denote by $\mathcal{E}$ the subset of M' consisting of elliptic level curves and by $\mathcal{H}$ the subset of M' consisting of hyperbolic level curves. Clearly, points in $\mathcal{E}$ have trajectories with elliptical caustic, and points in $\mathcal{H}$ have trajectories with hyperbolic caustic.*

We have $0 < G < 1$ on $\mathcal{E}$, and $1 - a^2 \le G < 0$ on $\mathcal{H}$. Along trajectories that pass through the foci, whose union forms a saddle connection in the phase space, we have $G = 0$.

In the next Lemma, we restate the results of Lemma 2 and Corollary 1 of [MOP]. This Lemma illustrates an important property of the trajectories in ellipses on which the results of this paper rely.

Lemma 1. *If $1 < a < \sqrt{4-2\sqrt{2}}$ and q_{-1}, q_0, q_1, q_2 are successive bouncing points of a billiard trajectory in the ellipse with hyperbolic caustic such that q_0 and q_1 are on one semiellipse, then i) only q_0 and q_1 belong to the same semiellipse (there are at most two successive reflections at the same semiellipse), ii) the tangency points of the segments $\overline{q_{-1}q_0}, \overline{q_0q_1}, \overline{q_1q_2}$ with the hyperbolic caustic occur inside the ellipse, iii) the distance between the tangency point of $\overline{q_0q_1}$ with the hyperbolic caustic and the boundary of the semiellipse is bounded below by a positive constant independent of q_0, q_1.*

Proof. We only need to prove part (iii), since (i) is Lemma 2 and (ii) is Corollary 1 of [MOP]. Suppose that we can find a sequence of trajectories like those in the statement of the Lemma such that the distance between the tangency point of $\overline{q_0q_1}$ with the caustic (hyperbola) and the boundary of the elliptical billiard is arbitrarily close to 0. When a tangency point is very close to the boundary of the elliptical billiard, let us say to the point q_1, then Reflection Formula (6) implies that the tangency point between $\overline{q_1q_2}$ with the caustic occurs outside the ellipse, contradicting part (ii). □

2.2 Elliptical Stadia

We study now the billiard in the elliptical stadium. Let us denote by Γ_1, Γ_2 the two semiellipses of the stadium with length m and semiaxis of length 1 and $a > 1$. In this representation Γ_1 is the semiellipse contained in the half-plane $\{x \geq 0\}$ and the major semiaxis of the ellipses lies on the x-axis of the plane. The lines joining the two semiellipses have length $2h$. We denote the *focusing component* of the boundary of the stadium by $\Gamma_+ = \Gamma_1 \cup \Gamma_2$ and the *neutral component* consisting of the two lines by Γ_0. Let a_1, a_2, a_3, a_4 be the intersections of the semiellipses with the lines starting from the bottom right and moving counterclockwise, their s-coordinates are, respectively, $0, m, m+2h, 2m+2h$ (the total length of the Γ is $2m+4h$). We call such points *corners* of Γ.

We partition M' in the two rectangles $M_+ = \pi^{-1}(\Gamma_+)$ and $M_0 = \pi^{-1}(\Gamma_0)$ which are the subset of the phase space corresponding to the focusing and neutral components of Γ. The set M_+ is partitioned in two subsets $M_1 = \pi^{-1}(\Gamma_1)$ and $M_2 = \pi^{-1}(\Gamma_2)$. For $i = 1, 2$, we define

$$U_i = \{z \in M_i : T^{-1}z, Tz \notin M_i\}, \qquad V_i = M_i \setminus U_i = \{z \in M_i : T^{-1}z \in M_i \text{ or } Tz \in M_i\}$$

and $Y_i = \{z \in M_i : Tz \notin M_i\}$. In words, U_i is the set of points $z \in M_i$ which reflect only once at Γ_i, while V_i is the set of points $z \in M_i$ which have at least two consecutive reflections at Γ_i, and Y_i is the set of points $z \in M_i$ which may have several consecutive reflections backward in time at Γ_i but their next bounce is not at Γ_i. Note that Y_i includes U_i and the points of V_i which leave M_i. We might say that $V_i \setminus Y_i$ is the set of "sliding" trajectories along a semiellipse. We define $U = U_1 \cup U_2$, $V = V_1 \cup V_2$ and $Y = Y_1 \cup Y_2$.

Let B_N, $N \geq 0$, be the subset of M_0 consisting of vectors not perpendicular to Γ_0 which have at least N consecutive reflections at Γ_0 both in the past and in the future before reaching Γ_+. Clearly $B_0 = M_0$. The need for defining all these sets is explained in Remarks 16.

For a fixed $N \geq 0$, we define $M = M_+ \cup B_N$. The set M is the reduced phase space on which we will work. The actual value of N will be chosen in the next section (see Remark 16).

Let T denote the billiard map on M'. The map $\Phi : M \to M$ is the first return map on M induced by T, $\Phi(y) = T^{A(y)}y$ where $A(y) = \inf\{i \geq 1 : T^i y \in M\}$. The measure $\mu = (\nu(M))^{-1}\nu_{|M}$ is an invariant probability measure for Φ.

Remark 2. *Oseledec's Theorem can be applied to T (see [KS]) and gives the existence ν-a.e. of L.E. for T. It follows immediately that Φ has L.E. μ-a.e. and these are proportional to the L.E. of T with constant of proportionality equal to the average (with respect to μ) of the return time $A(y)$ over M (see [Wo85], Lemma 2.2). Also note that $\cup_{j=0}^{\infty} T^j M = M'$ which implies that T is ergodic if and only if Φ is ergodic (see [CFS], Chapter 1, §5).*

The maps T and Φ are piecewise analytic diffeomorphisms. We denote by S^+ the subset of M where Φ fails to be C^2 and call it the *singular set of* Φ. It is easy to see that S^+ consists of points whose next reflection is at corners of the stadium or at the boundary of B_N. The singular set S^- for the map Φ^{-1} is defined similarly but this time we consider reflections in the past. The set $S_n^+ = S^+ \cup \Phi^{-1}S^+ \cup \cdots \cup \Phi^{-n+1}S^+, n \geq 1$ is the singularity set for Φ^n, and $S_n^- = S^- \cup \Phi S^- \cup \cdots \cup \Phi^{n-1}S^-$ is the singularity set of Φ^{-n}. Let $S_\infty^+ = \cup S_n^+$ and $S_\infty^- = \cup S_n^-$. Then $S_\infty = S_\infty^+ \cap S_\infty^-$ is the set of points whose trajectories hit a corner or the boundary of B_N in the future and in the past.

We denote by $\mathcal{S}^+$ and $\mathcal{S}_-$ the singular sets of T and T^{-1} which are subsets of M' consisting of points whose next reflection forward and backward in time is at a corner of the stadium. Similarly $\mathcal{S}_n^+$ and $\mathcal{S}_n^-$ are the singular sets of T^n and T^{-n} for every $n \geq 0$.

Remark 3. *The singularity sets $S_n^\pm$ of Φ are directly related to the singularity sets $\mathcal{S}_m^\pm$ of T. In fact, if $z \in S^+ \cap M_+$, then we have $z \in \mathcal{S}^+$ (if the are no bounces on M_0) or $z \in \mathcal{S}_{2N+1}^+$ (if $T^i z \in M_0$ for $i = 1, \ldots, 2N$). If $z \in S^+ \cap B_N$, then $z \in \mathcal{S}_{N+1}^+$. Finally, if $z \in S_n^+ \cap M_+$, then $z \in \mathcal{S}_{(2N+1)n}$. Hence $S_n^\pm \subset \mathcal{S}_{(2N+1)n}^\pm$.*

3 Hyperbolicity

3.1 Geometric optics

In order to understand the dynamics of a billiard, we study the dynamics of infinitesimal families of billiard trajectories (variations). For a planar billiard, it turns out that variations are parameterized by a projective quantity called *focusing time*. A complete description of the dynamics of a variation is given by the *law of reflection* which explains how the focusing time of a variation changes after reflecting at the boundary of the billiard table.

A variation $\eta(\alpha, t)$ is an one-parameter smooth family of lines in $\mathbb{R}^2$. We say that $\eta(\alpha, t)$ focuses if $\partial\eta/\partial\alpha|_{\alpha=0} = 0$ for some $t \in \mathbb{R}$ which we call the focusing time of $\eta(\alpha, t)$. Let $u \in \mathcal{T}_z M, z \in M$ and $\xi : (-\varepsilon, \varepsilon) \to M$ be a curve such that $\xi(0) = z$ and $\xi'(0) = u$. We associate with u the variations

$$\eta_+(\alpha, t) = q(\alpha) + tv(\alpha), \ t \in \mathbb{R}$$
$$\eta_-(\alpha, t) = q(\alpha) + t\tilde{v}(\alpha), \ t \in \mathbb{R}$$

where $\xi(\alpha) = (q(\alpha), v(\alpha))$ and $\tilde{v}(a)$ is obtained by reflecting $v(\alpha)$ at $q(\alpha) \in \Gamma$. Although for each vector $u \in \mathcal{T}_z M$, we can construct infinitely many distinct variations $\eta_+(\alpha, t)$ ($\eta_-(\alpha, t)$), all of them focus and their focusing time is the same. We will call forward (backward) focusing time of u the focusing time of the variation $\eta_+(\alpha, t)$ ($\eta_-(\alpha, t)$).

Let $X_s = \partial/\partial s$ and $X_\theta = \partial/\partial\theta$. Given a vector $u = u_s X_s + u_\theta X_\theta \in \mathcal{T}_z M$, $u_s, u_\theta \in \mathbb{R}$, its forward and backward focusing times $f_+(u), f_-(u)$ are given by (see [Wo86])

$$f_\pm(u) = \begin{cases} \frac{\sin\theta}{K \pm \frac{u_\theta}{u_s}} & \text{if } u_s \neq 0 \\ 0 & \text{if } u_s = 0 \end{cases} \tag{5}$$

where $K \geq 0$ is the curvature of Γ at $\pi(z)$.

Reflection Law. Let $z_0 = (q_0, v_0) \in M \setminus S^+$ and $u \in \mathcal{T}_{z_0} M$. If f_0 is the forward focusing time of u and f_1 is the forward focusing time of $D_{z_0} T u$, then the relation between f_0 and f_1 is given by

$$\frac{1}{f_1} + \frac{1}{t_0 - f_0} = \frac{2K_1}{\sin\theta_1} \tag{6}$$

where K_1 is the curvature of Γ at q_1 (the other symbols are explained at the beginning of Section 2).

Now we compare the evolution of two vectors after a reflection. The following Lemma, which follows easily from (6), shows that the focusing time is monotone (under some conditions). This property will be very useful in the proof of the hyperbolicity of Φ.

Lemma 4. *Same notations as above. Let $u_1, u_2 \in \mathcal{T}_{z_0}M$. Denote by $f_0^{(1)}, f_0^{(2)}$ the forward focusing times of u_1, u_2 and by $f_1^{(1)}, f_1^{(2)}$ the forward focusing times of $D_{z_0}Tu_1, D_{z_0}Tu_2$. Furthermore, assume that $0 < f_0^{(1)} < f_0^{(2)} < t(z)$ and $0 < f_1^{(2)}$. Then $0 < f_1^{(1)} < f_1^{(2)}$.*

3.2 Invariant Cone Field

We restrict ourselves to the case $1 < a < \sqrt{4 - 2\sqrt{2}}$ and $h > 2a^2\sqrt{a^2 - 1}$. At this point N is any positive number. Later on in this section and in section 4.2 (see Lemma 15), we will impose some conditions on N.

Our final goal is to prove the ergodicity of the map Φ which, in turn, will give the ergodicity of T. The first step we take is the construction of a cone field on M which is eventually strictly Φ-invariant and has some additional properties required by the Fundamental Theorem (see Remark 16). As a consequence of the existence of this cone field, we have that Φ and therefore T have a.e. non-zero L.E. This property implies, by Pesin's theory ([KS]), that T has a.e. local stable and unstable manifolds which are absolute continuous.

We define the cone field $\{C\}$ on M as follows. For $z \in V$ the outgoing and incoming segments of z are both tangent to a caustic (ellipse or hyperbola). Let $P_+(z)$ and $P_-(z)$ be the corresponding points of tangency. We define $C(z)$ as the set of all $u \in \mathcal{T}_zM$ which focus between $\pi(z)$ and $P_+(z)$

$$C(z) = \{u \in \mathcal{T}_zM : 0 \leq f_+(u) \leq d(\pi(z), P_+(z))\}.$$

Given $z \in M_+$, call $Q_+(z)$ and $Q_-(z)$ the points where the outgoing and the incoming segments of z intersect the boundary of the half-osculating disk of Γ at $\pi(z)$. For $z \in U$, we define

$$C(z) = \{u \in \mathcal{T}_zM : 0 \leq f_+(u) \leq d(\pi(z), Q_+(z))\}.$$

Finally, for $z \in B_N$, we define $C(z)$ as the collection of all "divergent" vectors $u \in \mathcal{T}_zM$

$$C(z) = \{u \in \mathcal{T}_zM : f_+(u) \leq 0\}.$$

This definition is equivalent to the following one. $C(z)$ is the set of vectors whose slopes satisfy :

$$\frac{d\theta}{ds}(z) \geq 0, \text{ if } z \in U; \quad \frac{d\theta}{ds}(z) \geq p(z), \text{ if } z \in V; \quad \text{and } \frac{d\theta}{ds}(z) \leq 0, \text{ if } z \in B_N.$$

In [MOP], a similar cone field was defined (it differs on M_0) and proved to be eventually strictly invariant.

One edge of $C(z)$ is the line $\{u \in \mathcal{T}_z : u_s = 0\}$, while the second edge depends on $z \in M$. The next lemma shows that the slope of the second edge is uniformly bounded from $-1/r(z)$ for $z \in M_+$. Note that $-1/r(z)$ is the slope of a variation consisting of parallel rays.

Lemma 5. *There exists a $0 < \delta < 1/r_{\max}$ such that $u_\theta/u_s \geq -1/r(z) + \delta$ for all $z \in M_+$ and all $u \in C(z)$.*

Proof. We need an estimate of the slope $m(z)$ of the non-vertical edge of the cones $C(z), z \in M_+$. For $z \in U$, we have $m(z) = 0$, and the lemma is true for any fixed $0 < \delta < 1/r_{\max}$ on U. Now let $z = (s, \theta) \in V$. In this case $m(z) = p(z)$. Using formula (4), we see that $|m(z)| \leq B(\epsilon) \tan\theta/r(z)$, where $B(\epsilon) = \epsilon^2/2(1 - \epsilon^2)$. Then it is clear that we can find $\bar{\theta}$ and $0 < \delta_1 < 1/r_{\max}$ for which the lemma is true on the set $\{(s, \theta) \in V : \theta \in (0, \bar{\theta}) \cup (\pi - \bar{\theta}, \pi)\}$. It remains to prove the lemma on the complementary set $\{(s, \theta) \in V : \theta \in [\bar{\theta}, \pi - \bar{\theta}]\}$. For a z belonging to this set, formula (5) and the fact that $f_+(u) > 0$ imply that $m(z) \geq -1/r(z) + \delta_2$ where $\delta_2 = \sin\bar{\theta}/m_+ > 0$ (m_+ is finite). The proof is finished by taking $\delta = \min(\delta_1, \delta_2)$. □

Let $C'(z)$ be the closure of the complementary cone of $C(z)$ in $\mathcal{T}_z M$. Denote by $F_+(z)$ ($F_-(z)$) the supremum of the forward (backward) focusing times among all vectors in $C(z)$ ($C'(z)$) for every $z \in M_+$

$$F_+(z) = \sup_{u \in C(z)} f_+(u) \qquad \text{and} \qquad F_-(z) = \sup_{u \in C'(z)} f_-(u).$$

Now let
$$m_+ = \sup_{z \in M_+} F_+(z) \qquad \text{and} \qquad m_- = \sup_{z \in M_+} F_-(z).$$

By Lemma 5 or directly by construction of $\{C(z)\}$, it follows that m_+ and m_- are finite and that $m_+ = m_-$ because of the symmetry of the billiard table.

The value of N. We choose the value of N in such a way that the minimum length $l(N)$ of all trajectories starting at M_+ and ending in B_N is greater than m_+. In fact, this condition needs to be satisfied only by trajectories which start at points of U, because for the trajectories starting at V the vectors in C (C') focuses forward (backward) inside the table (it is obvious for points in $\mathcal{E}$, and use Lemma 1 for points in $\mathcal{H}$). For trajectories starting at U an upper bound for m_+ is given by $r_{\max} = a^2 < 2$. Since the distance between the segment is 2, then it is clear that we can take $N = 1$. This value is enough to guarantee that $\{C\}$ is eventually strictly invariant. However, we may need a larger N for the noncontraction property to hold (as explained in the proof of Lemma 15).

Proposition 6. *If $1 < a < \sqrt{4 - 2\sqrt{2}}$ and $h > 2a^2\sqrt{a^2 - 1}$, the cone field $\{C(z)\}$ is eventually strictly Φ-invariant.*

Proof. First we note that $\{C(z)\}$ is measurable because it is piecewise continuous by construction.

We make now an observation which will simplify the proof of the strict invariance of C. Let $X_1(z)$ and $X_2(z)$ be two vectors belonging to the edges of $C(z)$ such that $0 = f_+(X_1(z)) < f_+(X_2(z))$. By construction of $C(z)$, it is easy to see that $f_+(X_2(z)) < t(z)$. Now if we apply Lemma 4 to the vectors of $C(z)$, we obtain that $D_z\Phi X_2(z) \in C(\Phi z)$ implies $D_z\Phi u \in \text{int}(C(\Phi z))$ for any $u \in C(z)$ and $u \neq X_2(z)$. In other words, to check that C is (strictly) invariant, we only need to check that the boundary vectors X_2 are mapped (strictly) inside C. We have to check the invariance of C in following cases: 1) $z, \Phi z \in V_i$, 2) $z \in V_i$ and $\Phi z \in V_j$ with $i \neq j$, 3) $z, \Phi z \in U$, 4) $z \in V$ and $\Phi z \in U$, 5) $z \in U$ and $\Phi z \in V$, 6) $z \in M_+, \Phi z \in B_N$ or $z \in B_N, \Phi z \in M_+$, 7) $z, \Phi z \in B_N$. With the exception of the cases 6 and 7, these verifications were done in the Appendix of [MOP]. So here we only need to check that we have invariance (actually strict invariance) in the cases 6 and 7. The proof is easier if we recall that C is (strict) Φ-invariant if and only if C' is (strict) Φ^{-1}-invariant. By our choice of N, it follows that a $N > 0$ such that for every $z \in M_+ \cap \Phi^{-1}B_N$ ($z \in M_+ \cap \Phi B_N$), the vector $u \in C(z)$ ($u \in C'(z)$) focuses forward (backward) before x reflects at $\pi(\Phi x)$ ($\pi(\Phi^{-1}x)$). In other words $D_z\Phi C(z)$ ($D_z\Phi^{-1}C'(z)$) consists of divergent (convergent) vectors so that $D_z\Phi C(z)$ ($D_z\Phi^{-1}C'(z)$) is strictly contained in $C(\Phi z)$ ($C'(\Phi^{-1}(z))$).

Let A be the set of points $z \in M$ for which $\{C(z)\}$ is not eventually strictly invariant. We can write $A = A_1 \cup A_2$ where $A_1 \subset S_\infty^+$ and $A_2 \cap S_\infty^+ = \emptyset$. Clearly $\mu(A_1) = 0$. From Propositions 2-6 in the Appendix of [MOP], we see that the positive semitrajectory of a point in A_2 does not cross the table from one semiellipse to the other and reflects only at one arc. Hence, it must be the periodic trajectory along the minor axis of the semiellipses. So $\mu(A_2) = 0$, and therefore $\mu(A) = 0$. ☐

By Theorem 1 in [Wo86], we conclude that

Theorem 7. *If $1 < a < \sqrt{4 - 2\sqrt{2}}$ and $h > 2a^2\sqrt{a^2 - 1}$, then Φ has non-zero Lyapunov exponents μ-a.e. on M.*

We show now that for the values of a and h considered in this paper, trajectories in the elliptical stadium have *uniform defocusing time* when they cross the table from one arc to the other.

Consider a trajectory which crosses the stadium only once and whose endpoints are on distinct arcs. More precisely, consider a finite orbit $\{z, Tz, \dots, T^n z\}, n > 0$ such that $Tz, \dots, T^{n-1}z \in M_0$ if $n > 1$

and assume, without loss of generality that, $z \in M_1$ and $T^n z \in M_2$. We denote by $L(z)$ the length of the trajectory, i.e., $L(z) = \sum_{i=0}^{n-1} t_i$.

Lemma 8. *If $1 < a < \sqrt{4 - 2\sqrt{2}}$ and $h > 2a^2\sqrt{a^2 - 1}$, then there exists a positive number $\tilde{d}$ such that $L(z) - F_+(z) - F_-(T^n z) \geq \tilde{d}$.*

Proof. Let $z' = T^n z$. If $z \in U_i, z' \in U_j$, then $F_+(z) = d_1(z), F_-(z') = d_2(z)$ and the Lemma follows from Proposition 6 in [MOP]. If $z \in V_i, z' \in V_j$, then $F_+(z) = d(\pi(z), P_+(z)), F_-(z') = d(\pi(z'), P_-(z'))$. It is easy to see that Propositions 2 and 3 in [MOP] implies that $P_+(z)$ precedes $P_-(z')$ along the piece of trajectory $[z, z']$ and that their distance $d(P_+(z), P_-(z'))$ is bounded below by a positive constant not depending on z. In this case and in the following one, recall Lemma 1. Finally if $z \in V_i$ and $z' \in U_i$, the Lemma follows from Proposition 4 and 5 in [MOP]. The symmetric case can be proved in the same way. □

4 Local Ergodicity

4.1 General setting

We are going to use the version of the "Fundamental Theorem" proved in [LW] so the first thing to do is to "redefine" the map Φ and its phase space in agreement with the formalism introduced in [LW].

Phase Space and Symplectic Boxes. It is easy to see that each set $U_i, i = 1, 2$, is path-connected, and that each set $V_i, i = 1, 2$, has two path-connected components $V_{i,j}, j = 1, 2$, where $V_{i,1} = \{(s, \theta) : s \in V_i \text{ and } \theta < \pi/2\}$ and $V_{i,2} = \{(s, \theta) : s \in V_i \text{ and } \theta > \pi/2\}$. It also easy to see that B_N has four disjoint path-connected components $B_{N,j}, j = 1, \dots, 4$, each consisting of vectors with base point at the same segment of Γ_0 and forming with it an angle which is either greater than $\pi/2$ or less than $\pi/2$. The closure of the sets $V_{i,j}, i, j = 1, 2$, $U_i, i = 1, 2$, and $B_{N,j}, j = 1, \dots, 4$ are the symplectic boxes (see [LW] for the definition of a symplectic box) forming the phase space of Φ. We will denote them by $\mathcal{A}_i, i = 1, \cdots, 10$ and simply refer to them as *boxes* of M.

The map Φ is an analytic diffeomorphism from then interior of each box to its image, but it might not be well defined at points belonging to the boundary of several boxes. However we can extend Φ from the interior of each box up to its boundary. By doing so, Φ becomes a multivalued map at points belonging to the boundary of several boxes. From now on, when we refer to Φ we will have in mind this multivalued map (not defined at S^+).

The singular set S^+ divides each box $\mathcal{A}_i$ into a finite collections of sets. We obtain a new partition of M, and we denote the closure of its elements by $\mathcal{A}_n^+$. Similarly S^- decomposes M into subsets whose closure is denoted by $\mathcal{A}_m^-$ (note that the cardinality of these two partitions is the same).

We have that Φ maps the interior of each set $\mathcal{A}_n^+$ into the interior of a set $\mathcal{A}_m^-$. As we have done before, we can extend Φ from the interior of each set $\mathcal{A}_n^+$ up to its closure, and by doing so we obtain a multivalued map also on S^+. Similarly, by we extending Φ^{-1} from the interior of each set $\mathcal{A}_n^-$ up to its closure, we obtain a multivalued map on S^- and possibly on ∂M.

To summarize, we have decomposed the phase space M into boxes $\mathcal{A}_i$. Each box is decomposed in a finite number of sub-boxes $\mathcal{A}_n^+$ ($\mathcal{A}_m^-$) whose boundary consists of subsets of S^+ (S^-) and $\partial \mathcal{A}_i$, Φ is a diffeomorphism from the interior of any $\mathcal{A}_n^+$ to the interior of a $\mathcal{A}_m^-$ and Φ is a multivalued map on S^+. Finally the cone field $\{C\}$ is continuous in the interior of each $\mathcal{A}_i$.

Let $X_1(z), X_2(z)$ be the unitary boundary vectors of $C(z) = \{u \in \mathcal{T}_z M : u = aX_1(z) + bX_2(z), ab \geq 0\}$. To each cone $C(z)$, there is an associated quadratic form given by $Q_z(u) = ab$. The quantity

$$\sigma(D_z\Phi) = \inf_{u \in \text{int}(C(z))} \sqrt{\frac{Q_{\Phi z}(D_z \Phi u)}{Q_z(u)}}.$$

measures the amount of expansion generated by $D_z\Phi$. Similarly we can define σ for Φ^{-1} by replacing $C(z)$ with its complementary cone.

Definition 2. *A point $z \in M$ is called sufficient if there exists a $n > 0$ such that $z \notin S_n^+ \cap S_n^-$ and $\sigma(D_z\Phi^n) > 3$ or $\sigma(D_z\Phi^{-n}) > 3$.*

Remark 9. *Since $\{C\}$ is eventually strictly Φ-invariant, then the set of sufficient points of M has full μ-measure* [LW, §7F].

In the remaining part of this section, we prove the following theorem.

Theorem 10. *Every sufficient point in a box $\mathcal{A}_i$ of M has a neighborhood in $\mathcal{A}_i$ that belongs (mod 0) to an ergodic component of Φ.*

To prove this result we use the "Fundamental Theorem" [LW].

Remark 11. *The proof of the Fundamental Theorem for Φ, which is based on the Hopf's argument, relies on the existence μ-a.e. of local stable and unstable manifolds (LM in brief) of Φ and their absolute continuity. If a piecewise smooth map satisfying some general conditions (see Part I, [KS]) has non-zero L.E. with respect to some invariant Borel probability measure λ, then Pesin's theory ([KS]) implies the existence λ-a.e. of LM of the map and their absolute continuity. Note that, although the maps T and T^{-1} satisfy the conditions of [KS] (this is proved in Part V of [KS] for a very general class of billiard maps to which T belongs), the map Φ (or any power of T) does not necessarily. However, we do not need to check that Φ satisfies these conditions to prove the existence of LM for Φ and their absolute continuity. In fact, the LM of T are LM of Φ (and therefore they are absolutely continuous). We give only the proof that the local unstable manifolds (LUM) of T are LUM of Φ, the argument for local stable manifolds is the same. Consider a LUM W^u of T at $z \in M'$. Suppose that W^u is not a LUM of Φ. Then S_∞^- must cut W^u , and therefore, by remark 3, $\mathcal{S}_\infty^-$ cut W^u as well. But this is impossible, since W^u is a LUM of T. The proof that LM are also LM for $T^n, n \in \mathbb{Z}$ is identical.*

To apply the Fundamental Theorem, we need to verify the following conditions.

1. (Monotonicity) The cone field $\{C(z)\}$ is eventually strictly Φ-invariant, and the restriction of C to the interior of any box of M is continuous.

2. (Proper Alignment) The tangent space of S^- at any point $z \in S^-$ is strictly contained in $C(z)$, and the tangent space at any point $z \in S^+$ is strictly contained in the complementary cone $C'(z)$.

3. (Regularity) For any $n \geq 1$, the sets S_n^+ and S_n^- are regular, i.e., they are finite unions of smooth arcs (closed) which only intersect at their endpoints.

4. (Noncontraction property) Let $||\cdot||$ be the standard Riemannian metric on M in the coordinates (s,θ). There exists a constant $0 < \rho$ such that for every $n \geq 1$ and every $z \in M \setminus S_n^+$

$$||D_z\Phi^n u|| \geq \rho||u||$$

for every $u \in C(z)$.

5. (Sinai-Chernov Ansatz) Let μ_S be the 1-dimensional Riemannian volume on $S^- \cup S^+$. For μ_S-a.e. $z \in S^-(S^+)$

$$\lim_{n\to+\infty(-\infty)} \sigma(D_z\Phi^n) = +\infty.$$

Remark 12. *Note that the standard Riemannian metric in coordinates (s,θ) does not generate the invariant area element as it is required by the Fundamental Theorem [LW] (see § 7, page 36). However the symplectic area $\sin\theta ds d\theta$ is smaller than the Riemannian area $ds d\theta$, and this fact makes the Fundamental Theorem work also in this situation (see also the remark in § 14.A, page 73 of [LW]).*

The proof of the Sinai-Chernov Ansatz is exactly the same as in Lemmas 14 and 15 of [DelM].

Lemma 13. *If $1 < a < \sqrt{4-2\sqrt{2}}$ and $h > 2a^2\sqrt{a^2-1}$, then Conditions 1 and 2 are satisfied.*

Proof. Condition 1 follows from the definition of $\{C(z)\}$ and Proposition 6. We now prove Condition 2.

As a consequence of Remark 3 and the invariance of the cone field, it is suffice to prove Condition 2 for the singular sets $\mathcal{S}^\pm$ of T. Let $z \in \mathcal{S}^+$, and assume that z belongs only to one arc of $\mathcal{S}^+$. Pick a vector $v \in T_z\mathcal{S}^+$. The positive semitrajectory of every point contained in a small neighborhood of z in $\mathcal{S}^+$ hits a corner a_i. It follows that v focuses at a_i. But every $u \in C(z)$ focuses forward before a_i (this is certainly true if either $\pi(z), a_i$ belong to distinct arcs or $z \in M_0$, and if $\pi(z), a_i$ belong to the same arc then note that either $z \in \mathcal{E}$ or z belongs to the boundary of a box U_i and in this second case $z \in \mathcal{S}^+$), therefore v must be strictly contained in the complement of $C(z)$. Note that if z belongs to several arcs of $\mathcal{S}^+$, then the same argument works for the tangent space of each arc. Similarly we can prove that the tangent space of $\mathcal{S}^-$ at any point $z \in \mathcal{S}^-$ is strictly contained in $C(z)$. ☐

Lemma 14. *If $1 < a < \sqrt{4-2\sqrt{2}}$ and $h > 2a^2\sqrt{a^2-1}$, then Condition 3 is satisfied.*

Proof. Exactly as explained in the proof of the previous lemma, it is enough to prove the lemma for the singularity sets of T (recall Remark 3).

We only need to prove that $\mathcal{S}_n^+$ is regular. In fact, the regularity of $\mathcal{S}_n^-$ follows from the regularity of $\mathcal{S}_n^+$ because $\mathcal{S}_n^- = R\mathcal{S}_n^+$ where R is the map given by $R(s,\theta) = (s, \pi-\theta)$.

Let us note that $\mathcal{S}_m^-$ and $\mathcal{S}_n^+$ intersect transversally for every $m, n \geq 0$. In fact by Proper Alignment and invariance of $\{C\}$, it follows that $\mathcal{T}_z(\Phi^m\mathcal{S}^-)$ is strictly contained in $C(z)$ for every $z \in \Phi^m\mathcal{S}^-$, and $\mathcal{T}_z(\Phi^{-n}\mathcal{S}^+)$ is strictly contained in $C'(z)$ for every $z \in \Phi^{-n}\mathcal{S}^+$. Hence we easily conclude that $\mathcal{S}_m^-$ and $\mathcal{S}_n^+$ intersect transversally for all $m, n \geq 0$.

To prove the regularity of $\mathcal{S}_n^+$ we use induction. The sets $\mathcal{S}^+$ and $\mathcal{S}^-$ are regular. Now let $n > 1$, and assume that $\mathcal{S}_n^+$ is regular. Since $\mathcal{S}^-$ and $\mathcal{S}_n^+$ are transversal, then their intersection consists of finitely many points. Thus Φ^{-1} is continuous everywhere on S_n^+ except for a finite number of points, and so $\Phi^{-1}\mathcal{S}_n^+$ is a finite union of smooth arcs that intersect only at their endpoints, i.e., $\Phi^{-1}\mathcal{S}_n^+$ is regular. The same conclusion clearly holds for $\mathcal{S}_{n+1}^+ = \mathcal{S}^+ \cup \Phi^{-1}\mathcal{S}_n^+$. ☐

4.2 Noncontraction property

Definition 3. *Given a finite orbit $\gamma = \{z, \Phi z, \dots, \Phi^n z\}$ with $n > 0$, we say that the noncontraction property holds along γ if there is a constant $\lambda > 0$ such that $\|D_z\Phi^n u\| \geq \lambda\|u\|$ for every $u \in C(z)$.*

Consider the following finite orbits where $z \notin S_n^+, n > 0$:

1. $\{z, \Phi z, \dots, \Phi^n z\} \in B_N$,
2. $\{z, \Phi z, \dots, \Phi^n z\} \in M_+$,
3. $\{z, \Phi z\}$ with $z \in B_N$ and $\Phi z \in M_+$,
4. $\{z, \Phi z\}$ with $z \in M_+$ and $\Phi z \in B_N$,
5. $\{z, \Phi z, \dots, \Phi^n z\}$ with $z, \Phi^n z \in Y$.

We call these orbits *blocks*. It is not difficult to see that every finite orbit consists at most of 7 blocks. Namely given a finite orbit

$$\gamma = \{z, \Phi z, \Phi^2 z, \dots, \Phi^n z\}, n > 0,$$

there exists a $1 \leq \bar{k} \leq 7$ such that

$$\gamma = \bigcup_{i=1}^{\bar{k}} \{\Phi^{n_{i-1}} z, \dots, \Phi^{n_i} z\}$$

where $n_0 = 0 < n_1 < \ldots < n_{\bar{k}} = n$, and $\{\Phi^{n_{i-1}}z, \ldots, \Phi^{n_i}z\}$ is one of the five blocks listed above. Assume for the moment that the noncontraction property holds along each block with the constant $0 < \alpha_j \leq 1, j = 1, \ldots, 5$. If $\lambda = (\min_{1 \leq i \leq 5} \alpha_i)^7$, then we have

$$\|D_z \Phi^n u\| \geq \lambda \|u\|$$

for every $u \in C(z)$. The constant λ does not depend on the orbit considered and we conclude that the noncontraction property is satisfied.

It remains to prove the noncontraction along the five blocks.

Lemma 15. *If $1 < a < \sqrt{4 - 2\sqrt{2}}$ and $h > 2a^2\sqrt{a^2 - 1}$, then the noncontraction property holds along the blocks of type 1,3, 4 and 5.*

Proof. The proof of the noncontraction property for blocks of type 5 is the same as the proof of Lemma 13 in [DelM].

Let $\{z, \Phi z\}$ be a block of type 4. As $z \in M_+$ and $\Phi z \in B_N$, then, by (1), we have

$$D_z \Phi = \begin{pmatrix} \frac{L - d_0}{r_0 \sin \theta_1} & \frac{L}{\sin \theta_1} \\ -\frac{1}{r_0} & -1 \end{pmatrix}$$

where L is the length of the segment $[z, \Phi z]$, $r_0 = r(z)$, $d_0 = r(z) \sin \theta(z)$ and $\theta_1 = \theta(\Phi z)$. By Lemma 5, there exists $0 < \delta < 1/r_{\max}$ such that $-1/r(z) + \delta \leq u_\theta / u_s$ for all $u = (u_s, u_\theta) \in C(z)$ and all $z \in M_+$. If $u_\theta / u_s \geq 0$, then

$$\|D_z \Phi u\| \geq |\frac{1}{r(z)} u_s + u_\theta| \geq \min\{1, \frac{1}{r(z)}\} \|u\| . \tag{7}$$

If $-1/r(z) + \delta \leq u_\theta / u_s < 0$, then we have

$$\|D \Phi_z u\| \geq |\frac{1}{r(z)} u_s + u_\theta| \geq |u_s| \cdot |\frac{1}{r(z)} + \frac{u_\theta}{u_s}| \geq |u_s| \delta. \tag{8}$$

Now from $-1/r(z) + \delta \leq u_\theta / u_s < 0$, we easily obtain

$$\frac{r(z)}{\sqrt{1 + r^2(z)}} \leq \frac{|u_s|}{\|u\|} \leq 1$$

so that (8) becomes

$$\|D \Phi_z u\| \geq \frac{\delta r(z)}{\sqrt{1 + r^2(z)}} \|u\| . \tag{9}$$

If $r_{\max} = a^2$ and $r_{\min} = 1/a$ are the maximum and the minimum of the radius of curvature of Γ_+, then, combining (7) and (9), we conclude that

$$\|D \Phi u\| \geq \min \left\{ \frac{1}{r_{\max}}, \delta \frac{r}{\sqrt{1 + r_{\min}^2}} \right\} \|u\| .$$

Consider now a block $\{z, \Phi z\}$ of type 3. This time we have

$$D_z \Phi = \begin{pmatrix} -\frac{\sin \theta_0}{\sin \theta_1} & \frac{L}{\sin \theta_1} \\ -\frac{\sin \theta_0}{d_1} & \frac{L - d_1}{d_1} \end{pmatrix}$$

where L is the length of $[z, \Phi z]$, $\theta_0 = \theta(z)$, $\theta_1 = \theta(\Phi z)$ and $d_1 = r(\Phi z) \sin \theta_1$. The cone $C(z)$ consists of divergent vectors ($u_s u_\theta \leq 0$ for every $u = (u_s, u_\theta) \in C(z)$). Hence we have

$$\|D \Phi u\| \geq \sqrt{(-\frac{\sin \theta_0}{\sin \theta_1} u_s + \frac{L}{\sin \theta_1} u_\theta)^2} \geq \min\{\sin \theta_0, L\} \|u\| \tag{10}$$

for every $u \in C(z)$. For $N \geq 1$ there exist two positive numbers t and $\bar{L}$ for which $t \leq \sin\theta(z)$ and $\bar{L} \leq L(z)$ on $\{z \in B_N : \Phi z \in M_+\}$. We conclude that

$$\|D\Phi_z u\| \geq \min\{t, \bar{L}\}\, \|u\|\,. \tag{11}$$

Finally consider a block $\{z, \Phi z, \ldots, \Phi^n z\}$ of type 1. For this case, we have

$$D_z\Phi^n = \begin{pmatrix} -1 & \frac{L}{\sin\theta_1} \\ 0 & -1 \end{pmatrix}$$

where L is the length of the segment $[z, \Phi^n z]$ and $\theta_1 = \theta(\Phi z)$. All cones $C(\Phi^k z)$, $k = 1, \ldots, n$, consist of divergent vectors so that

$$\|D\Phi^n_z u\| \geq \sqrt{(-u_s + \frac{L}{\sin\theta_1} u_\theta)^2} \geq \min\{1, L\}\, \|u\| \geq \|u\|\,. \tag{12}$$

for every $u \in C(z)$ (in this case $L(z) > 2$). □

Remark 16. *We introduced the set B_N (and the map Φ) for two reasons. The first one is that in order to apply the Fundamental Theorem we need a continuous cone field on every symplectic box of M. Since m_+, m_- are finite, the cone field C on B_N can be chosen to be the constant cone field consisting of divergent vectors. The second reason is that the noncontraction property does not hold along trajectories that start at M_0, end at M_+ and are almost horizontal to the segments Γ_0. By introducing B_N with $N > 0$, we only need to consider trajectories which make an angle uniformly large with the horizontal direction and for these trajectories the noncontraction property holds as proved in Lemma 15.*

We introduce a new set of coordinates (J, J') in the tangent planes of M which are connected to the coordinates (u_s, u_θ) by the formulas

$$\begin{aligned} J &= \sin\theta\, u_s, \\ J' &= -\tfrac{1}{r} u_s - u_\theta. \end{aligned}$$

J is the restriction to M of a transversal Jacobi field along a billiard trajectory and J' is its derivative (see [LW]). We recall briefly some properties of the coordinates (J, J') which we will use later.

- The evolution of (J, J') along a segment of trajectory of length τ between two consecutive collisions is given by

  $$\begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}. \tag{13}$$

  At a reflection, (J, J') is transformed by the map

  $$\begin{pmatrix} -1 & 0 \\ \frac{2}{d} & -1 \end{pmatrix}. \tag{14}$$

  where $d = r\sin(\theta)$.

- In coordinates (J, J'), the cone $C(z), z \in M_+$ is given by

  $$C(z) = \{(J, J') : J' \leq -J/F_+(z)\}.$$

- The function $u = (J, J') \mapsto |J'|$ defines a seminorm on M. We will denote it by $|\cdot|$ to distinguish it from the standard norm $\|\cdot\|$. In the next Lemma we prove that these seminorms are equivalent on $C(z), z \in M_+$.

Lemma 17. *The seminorms $|\cdot|$ and $\|\cdot\|$ are equivalent on $C(z)$ for $z \in M_+$.*

Proof. We need to show that there are two positive numbers $c_1 \leq c_2$ such that $c_1\|u\|^2 \leq J'^2 \leq c_2\|u\|^2$ for all $u \in \cup_{z\in M_+} C(z)$. One inequality can be proved easily. We have $J'^2 = u_s^2/r^2 + u_\theta^2 + 2u_s u_\theta/r \leq 2\max(1/r_{\min}^2, 1)(u_s^2+u_\theta^2)$, and therefore we can take $c_2 = 2\max(1/r_{\min}^2, 1)$. To prove the other inequality we consider two cases. If $u_s u_\theta \geq 0$, then we obtain $J'^2 \geq u_s^2/r^2 + u_\theta^2 \geq \min(1/r_{\max}^2, 1)(u_s^2 + u_\theta^2)$. Lemma 5 tells us that the other case we need to consider is $u_s > 0$ and $0 > u_\theta/u_s \geq -1/r + \delta$. This time, we get $J'^2 = (u_s/r^2 + u_\theta)^2 = (1/r + u_\theta/u_s)^2 u_s^2 \geq \delta^2 u_s^2$. Since $u_\theta^2 \leq (-1/r+\delta)^2 u_s^2 \leq u_s^2/r_{\min}^2$ (and writing $u_s^2\delta^2 = 2u_s^2\delta^2/2$), we obtain $J'^2 \geq (u_s^2 + u_\theta^2)\delta^2 \min(1, r_{\min}^2)/2$. Therefore we can take $c_1 = \min(1, 1/r_{\max}^2, \delta^2/2, \delta^2 r_{\min}^2/2)$. □

Lemma 18. *If* $1 < a < \sqrt{4-2\sqrt{2}}$ *and* $h > 2a^2\sqrt{a^2-1}$*, then the noncontraction property holds along the blocks of type 2.*

Proof. We may think of blocks of type 2 as finite trajectories of the elliptical billiard because each subset $M_i, i = 1, 2$, of the elliptical stadium is naturally identified to the phase space of a semiellipse. So it is enough to prove the Lemma for finite trajectories of the elliptical stadium which have consecutive reflections along one of the two semiellipses. In the sequel, when we refer to an element originally defined on M_+ (T, Φ, C, etc) we will think of it as defined on the phase space of the elliptical billiard as well.

In virtue of the equivalence of $\|\cdot\|$ and $|\cdot|$ on M_+, it is enough to show that the noncontraction property is verified with respect to the semi-norm $|J'|$.

The proof consists of two parts. In the first part, we prove the statement for θ close to 0 or π (small θ), and in the second part, we prove it for θ bounded away from 0 and π (large θ).

Small θ. For $z_0 = (s_0, \theta_0)$ in the phase space of the elliptical billiard, define $n(z_0) \geq 0$ as the number of consecutive reflections of z_0 along the semiellipse to which s_0 belongs. Let $z_n = (s_n, \theta_n) = T^n z, 0 \leq n \leq n(z_0)$. Finally denote by $\theta_{\max}$ and $\theta_{\min}$ be the maximum and the minimum of θ along the invariant curve $G = G(s, \theta)$ containing z_0.

The next lemma is formulated only for $\theta \leq \pi/2$ (close to 0) but a similar results holds for $\theta > \pi/2$ (close to π).

Sublemma 1. *There exist* $\bar{\theta} > 0$ *and* $\beta > 1$ *such that if* $\theta_k < \bar{\theta}$ *for some* $0 \leq k \leq n(z_0)$*, then*

$$\theta_n \leq \beta\theta_k, \qquad 0 \leq n \leq n(z_0).$$

Proof. If the block is in $\mathcal{H}$, the result is obvious (in fact, $n = 1$ and θ is close to $\pi/2$). If the block is in $\mathcal{E}$, consider the function $G = G(s, \theta), s = s(\phi)$ defined in (3). For $0 < \theta < \pi/2$ and $G > 0$, the invariant curve $G = G(s, \theta)$ is the graph of the function

$$\theta = \cos^{-1}\sqrt{G + \epsilon^2\cos^2\varphi(1-G)}.$$

Thus

$$\theta_{\max}(G) = \cos^{-1}\sqrt{G} = \sin^{-1}\sqrt{1-G}$$

and

$$\theta_{\min}(G) = \cos^{-1}\sqrt{G + \epsilon^2(1-G)} = \sin^{-1}\sqrt{(1-G)(1-\epsilon^2)}.$$

As G goes to 1, we obtain

$$\frac{\theta_{\max}(G)}{\theta_{\min}(G)} \to \frac{1}{\sqrt{1-\epsilon^2}}.$$

Now note that G is strictly monotone in $\theta_{\min}$ so that there exist $\bar{\theta} > 0$ and $\beta > 1$ such that if $\theta_{\min} < \bar{\theta}$, then $\theta_{\max}/\theta_{\min} \leq \beta$. This easily implies $\theta_n \leq \theta_{\max} \leq \beta\theta_{\min} \leq \beta\theta_k$. □

Let $z = (s, \theta)$ and $\bar{z} = (\bar{s}, \bar{\theta}) = Tz$. Given $(J, J') \in C(z)$, let $(\bar{J}, \bar{J}') = D_z T(J, J')$. According to the construction of $C(z)$, we have $|\bar{J}/J'| \geq F_-(\bar{z})$ and $|\bar{J}/\bar{J}'| \leq F_+(\bar{z})$ so that

$$\frac{\bar{J}'}{J'} \geq \frac{F_-(\bar{z})}{F_+(\bar{z})}.$$

Sublemma 2. *For every $z=(s,\theta)$ with small θ and every $(J,J')\in C(z)$, we have*

$$\left|\frac{\bar{J}'}{J'}\right| \geq 1 - A\tan\bar{\theta}, \qquad A = A(\epsilon) = 2\epsilon^2/(1-\epsilon^2). \tag{15}$$

Proof. By the previous remarks we have (see formulas (5) and (4))

$$\frac{F_-(z)}{F_+(z)} = \frac{1+\frac{\epsilon^2\sin 2\varphi}{\sin 2\theta}(1-G)}{1-\frac{\epsilon^2\sin 2\varphi}{\sin 2\theta}(1-G)}.$$

Let

$$f(s,\theta) = \frac{\epsilon^2 \sin 2\varphi}{\sin 2\theta}(1-G(s,\theta))$$

then

$$\frac{F_-(z)}{F_+(z)} - 1 = 2\frac{f(s,\theta)}{1-f(s,\theta)}.$$

For every s, we have

$$1-G(s,\theta) = \frac{1-\cos^2\theta}{1-\epsilon^2\cos^2\varphi} \leq \frac{1-\cos^2\theta}{1-\epsilon^2}$$

so that

$$|f(s,\theta)| \leq \frac{\epsilon^2}{1-\epsilon^2}\frac{1-\cos^2\theta}{\sin 2\theta} = \frac{\epsilon^2}{2(1-\epsilon^2)}\tan\theta.$$

Note that for θ small, we have

$$2\left|\frac{f(s,\theta)}{1-f(s,\theta)}\right| \leq 4|f(s,\theta)|.$$

Therefore, provided that θ is small enough, we have

$$\frac{F_-(z)}{F_+(z)} \geq 1 - 2\left|\frac{f}{1-f}\right| \geq 1-4|f| \geq 1 - A\tan\theta. \qquad \square$$

Let $z_0 = (s_0,\theta_0)$, and $z_n = (s_n,\theta_n) = T^n z_0$ for $0<n\leq n(z_0)$. Pick a $(J_0,J_0')\in C(z)$, and let $(J_n,J_n') = D_zT^n(J_0,J_0')$ for $0<n\leq n(z_0)$.

It follows easily from Sublemma 1, that if $\theta_0 \leq \bar{\theta}/\beta$ ($\bar{\theta}$ and β are the same as in Sublemma 1), then $\theta_n \leq \bar{\theta}$ for all $0\leq n \leq n(z_0)$. Therefore we can apply sublemma 2 to each factor of the expression

$$\left|\frac{J_n'}{J_0'}\right| = \left|\frac{J_n'}{J_{n-1}'}\right| \cdots \left|\frac{J_1'}{J_0'}\right|,$$

obtaining

$$\left|\frac{J_n'}{J_0'}\right| \geq \Pi_{i=1}^n (1-A\tan\theta_i) \qquad \text{for all} \quad 0<n\leq n(z_0).$$

By Sublemma 1, then we have

$$\left|\frac{J_n'}{J_0'}\right| \geq (1-A\tan\beta\theta_0)^{n(z_0)}.$$

In [Do91, page 242], Donnay gave an estimate on the number of consecutive reflections along an absolute focusing arc (a semiellipse is absolute focusing if $a<\sqrt{2}$) for small angles. He proved that there exists a $C>0$ such that for all (s_0,θ_0) with θ_0 sufficiently small we have $n(s_0,\theta_0) \leq C/\sin\frac{\theta_0}{2}$. Thus we get

$$\left|\frac{J_n'}{J_0'}\right| \geq (1-A\tan\beta\theta_0)^{C\left(\sin\frac{\theta_0}{2}\right)^{-1}}.$$

It is easy to see that

$$\lim_{\theta\to 0^+}(1-A(\epsilon)\tan\beta\theta)^{C(\sin\frac{\theta}{2})^{-1}} = \exp(-2CA\beta). \tag{16}$$

Hence given a small $0<\delta<\exp(-2CA\beta)$, there exists a $\widetilde{\theta}\le\bar{\theta}/\beta$ such that for any (s_0,θ_0) with $\theta_0\le\widetilde{\theta}$ we have

$$\left|\frac{J'_n}{J'_0}\right| \ge \exp(-2CA\beta)-\delta>0 \qquad \text{for all} \quad 0<n\le n(s_0,\theta_0).$$

We conclude the first part of proof by noticing that the lower bound $\exp(-2CA\beta)-\delta$ is independent on n.

Large θ. Now let $\widetilde{\theta}<\theta_0<\pi-\widetilde{\theta}$. By using contradiction and Sublemma 1, we obtain that $\widetilde{\theta}/\beta<\theta_n$ for all $0\le n\le n(z_0)$. We claim that F_- is uniformly bounded away from 0 along blocks of type 2 with $\widetilde{\theta}<\theta_0<\pi-\widetilde{\theta}$. This is quite obvious for blocks of type 2 in $\mathcal{E}$, while it follows from part (iii) of Lemma 1 for blocks of type 2 in $\mathcal{H}$. Also it is easy to see that the number of reflections $n(z_0)$ is bounded below, i.e., there exists a $\bar{n}$ such that $n(z_0)\le\bar{n}$. Since m_+ is finite, then we conclude that there is $a<1$ such that $F_-/F_+\ge a$. Hence we have

$$\left|\frac{J'_n}{J'_0}\right| = \left|\frac{J'_n}{J'_{n-1}}\right|\cdots\left|\frac{J'_1}{J'_0}\right| \ge a^n \ge a^{\bar{n}} \qquad \text{for all} \quad 0<n\le n(z_0).$$

□

Conditions 1-5 are verified so that Theorem 10 is proved.

Remark 19. *Consider the Riemannian metric ρ given by $J^2+J'^2$ on the tangent bundle of M'. It can be proved that the symplectic form $\omega=\sin\theta ds\wedge d\theta$ becomes $J\wedge J'$ in coordinates (J,J'), thus the volume form induced by the metric ρ coincides with the symplectic form ω. Also it can be checked that the noncontraction property holds also for the metric ρ. Therefore, by introducing the metric ρ from the very beginning, we could have applied directly the Local Ergodic Theorem [LW] to Φ without using the remark 12.*

5 Ergodicity, Kolmogorov and Bernoulli properties

In this section we conclude the proof of the ergodicity for the map Φ. In the previous section, we have showed that each sufficient point has a neighborhood contained in a box of M which belongs (mod 0) to an ergodic component of Φ. To finish our proof, we need first to show that each box belongs (mod 0) to an ergodic component, and then that the orbits of any pair of boxes are not disjoint (mod 0). The first part is accomplished by proving that set of sufficient points contained in a box is topologically rich (path-connected in our case), while for the second part we need to construct special trajectories which have the property that starting from one box and traveling along these trajectory we can reach every other box.

Lemma 20. *If $1<a<\sqrt{4-2\sqrt{2}}$ and $h>2a^2\sqrt{a^2-1}$, then the subset of sufficient points contained in a box of M is path-connected.*

Proof. Let $\mathcal{A}_i$ be a box of M. The proof of the Sinai-Chernov Ansatz (Lemma 14 in [DelM]) includes the following result: the subset of non-sufficient points of $\mathcal{A}_i$ is contained in $S^+_\infty\cap S^-_\infty$. From the regularity of the sets $S^\pm_n$, it follows immediately that the set $S^+_\infty\cap S^-_\infty$ is at most countable. If we remove a countable set from a path-connected set, we still obtain a path-connected set. □

By a standard argument (see for instance Corollary 4.3 part (a) of [M93]) involving Theorem 10, we obtain that each box of M belongs to an ergodic component of Φ.

Lemma 21. *If* $1 < a < \sqrt{4 - 2\sqrt{2}}$ *and* $h > 2a^2\sqrt{a^2 - 1}$, *then every box of* M *belongs (mod 0) to an ergodic component of* Φ.

We can finish now the proof of the ergodicity of Φ.

Theorem 22. *If* $1 < a < \sqrt{4 - 2\sqrt{2}}$ *and* $h > 2a^2\sqrt{a^2 - 1}$, *then the map* Φ *is ergodic.*

Proof. We have to show that all boxes of M belong (mod 0) to the same ergodic component of Φ. To do this it is enough to check that the three boxes $U_1, U_2, V_{2,2}$ belong (mod 0) to the same ergodic component. If this is true, then, by the symmetry of Q, any other triple $U_1, U_2, V_{i,j}, 1 \leq i, j \leq 2$, will belong (mod 0) to the same ergodic component. We also need to show that B_N belongs to the same ergodic component of U and V. This easily follows from $\mu((\Phi B_N) \cap M_+) > 0$.

Two boxes $\mathcal{A}_i, \mathcal{A}_j, i \neq j$ belong (mod 0) to the same ergodic component if there is an orbit γ which intersects $\mathcal{A}_i$ and $\mathcal{A}_j$. If such an orbit exists, then there will be an open set in $\mathcal{A}_i$ which is mapped into an open set in $\mathcal{A}_j$, and open sets contain sets of sufficient points of positive measure.

It is easy to see that, practically, what we need to do is to show that there are two orbits such that one intersects U_1 and U_2, and the other one intersects $V_{2,2}$ and U_1. The first orbit is given by the periodic orbit along the x-axis. To construct the second orbit, we proceed as follows. We claim that if $z_0 \in M_0, z_1 = Tz_0 \in M_+,\ q_i = \pi(z_i), 0 \leq i \leq 1$ and $\overline{q_0 q_1}$ intersects the x-axis, then $z_1 \in Y$. In fact, assume, without loss of generality, that $z_1\ in M_2$ and suppose that $z_1 \notin Y$. If $z_2 = T^2 z_0,\ q_2 = \pi(z_2)$, then q_1 and q_2 belong to the same semiellipse. Let t be the line containing $\overline{q_0 q_1}$, and t' be the line obtained by reflecting t about the x-axis. The segments $\overline{q_0 q_1}$ and $\overline{q_1 q_2}$ are tangent to the same confocal ellipse E' and $\overline{q_1 q_2}$ is contained in the set bounded by t' and the semiellipse. E' is tangent to t, so E' must be tangent to t' as well by symmetry. However, the segment $\overline{q_1 q_2}$ lies to the left side of t', and therefore we conclude that E' intersects t' transversally, obtaining a contradiction.

Consider the orbit $\gamma = \{z_0, z_1, \cdots, z_n\}, z_i = T^i z_0, n > 2$ such that $\pi(z_0) = a_4;\ z_1 \in M_2;\ \pi(z_2) = a_3;\ z_i \in M_0, 2 < i < n;\ z_n \in M_1$. Note that the alternate segments of γ are parallel. By the claim it is easy to see that for any neighborhood W of z_0, we have $TW \cap Y_2 \neq \emptyset$ and $TW \cap {Y_2}^c \neq \emptyset$. We analyze several cases:

a) if $z_n \in Y_1$, then, by continuity of $T^k, 1 \leq k \leq n$, we can find a $z_0' \in M_2$ close to z_0 such that i) $z_0', z_1' \in M_2$, ii) $z_1' \in Y_2$, iii) $z_i' \in M_0, 1 \leq i < n$ and $i = n+1$, and iv) $z_n' \in Y_1$. Hence we have $z_0' \in V_2$ and $z_n' \in U_1$.

b) If $z_n \notin Y$, then we can find a $z_0' \in M_0$ close to z_0 such that i) $z_1' \in Y$, ii) $z_n', z_n \in M_1$ and iii) $z_n' \notin Y$ (again by the continuity of $T^k, 1 \leq k \leq n$). Thus $z_1' \in U_2$ and $z_n' \in V_1$.

c) If $\pi(z_n)$ is a corner, then $z_{n+1} \in M_1$ and we can find a $z_0' \in M_0$ close to z_0 such that i) $z_1' \in Y_2$ and ii) $z_n' \notin Y$. The last fact follows from the continuity of $T^k, 1 \leq k \leq n$. We have again $z_1' \in U_2$ and $z_n' \in V_1$. □

As an easy corollary, we obtain the ergodicity of the billiard map T.

Corollary 1. *If* $1 < a < \sqrt{4 - 2\sqrt{2}}$ *and* $h > 2a^2\sqrt{a^2 - 1}$, *then* T *is ergodic.*

The next theorem shows that T has indeed stronger ergodic properties.

Theorem 23. *If* $1 < a < \sqrt{4 - 2\sqrt{2}}$ *and* $h > 2a^2\sqrt{a^2 - 1}$, *then the billiard map* T *is a K-system and Bernoulli.*

Proof. The Bernoulli property follows from K-property by the general results in [CH] or [OW]. Thus we only need to show that T is a K-system. Since the billiard map T has non-zero Lyapunov exponents ν-a.e. and is ergodic, then Theorem 7.9 in [Pe77] (see also Theorem 13.1 in [KS]) implies the existence of a finite partition $\{\mathcal{C}_1, \ldots, \mathcal{C}_m\}$ of M' with the following properties: 1) $\mu(\mathcal{C}_i) > 0$ for $i = 1, \ldots, m$, 2) $T\mathcal{C}_i = \mathcal{C}_{i+1}, i = 1, \ldots, m-1$, and $T\mathcal{C}_m = \mathcal{C}_1$, 3) $T^m|_{\mathcal{C}_i}$ is K-system for $i = 1, \ldots, m$.

It is clear that T is a K-system if T^n is ergodic for every integer $n > 0$. Now the proof of the ergodicity of T^n is identical to the one of T. We consider the same reduced phase space $M = M_+ \cup B_N$ and the induced first return time map $\widetilde{\Phi} : M \to M$, $\widetilde{\Phi}(z) = (T^n)^{B(z)}z$, where $B(z) = \inf\{j > 0 : (T^n)^j z \in M\}$. Clearly $\widetilde{\Phi}$ preserves the measure μ. Recalling Remark 11, we see that $\widetilde{\Phi}$ has LSM and LUM μ-a.e. which are absolute continuous. Furthermore note that the cone field C is eventually strict invariant for $\widetilde{\Phi}$. Now the local ergodicity for $\widetilde{\Phi}$ can be proved exactly as we did for Φ.

We obtain that each box $\mathcal{A}_i$ belongs (mod 0) to an ergodic component of $\tilde{\Phi}$. To show that $\tilde{\Phi}$ has only one ergodic component, we show, as in Lemma 18, that for any pair of boxes of M there is a trajectory of $\widetilde{\Phi}$ intersecting both. In fact, we only need to check the existence of the following trajectories: 1) a trajectory intersecting $V_{1,i}$ and $V_{2,j}$ for each $i, j = 1, 2$; 2) a trajectory intersecting U_i and $V_{j,k}$ for each $i, j, k = 1, 2$; 3) a trajectory intersecting each connected component of B_N and $V_{i,j}$ for each $i = 1, 2$.

We only prove the existence of trajectories intersecting $V_{1,1}$ and $V_{2,1}$ because the existence of the other trajectories can be proved in the same way. The ergodicity of T implies that ν-a.e. trajectory of T covers densely M. Hence we can find a $z_0 \in V_{1,1}$ with the property that for some $m > 0$ we have $z_m = (s_m, \theta_m)$ with $s_m \in \Gamma_2$ and θ_m so small that the next n reflections take place at Γ_2, i.e., $s_{m+1}, \ldots, s_{m+n} \in \Gamma_2$. By the definition of $\widetilde{\Phi}$, we conclude that there exists a $k > 0$ such that $\widetilde{\Phi}^k z_0 \in V_{2,1}$. □

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References

[CH] N.I. Chernov & C. Haskell, *Nonuniformly hyperbolic K-systems are Bernoulli*, Ergod. Th. & Dynam. Sys., **16** (1996), 19-44.

[CM] N. I. Chernov & R. Markarian, *Introduction to the ergodic theory of chaotic billiards*, IMCA, Lima (2001).

[CFS] I. P. Cornfeld, S. V. Fomin & Ya. G. Sinai, *Ergodic Theory*, Springer, Berlin (1982) (Original Ed. 1980).

[DelM] G. Del Magno, *Ergodicity of a class of truncated elliptical billiard*, Nonlinearity **14** (2001), 1761–1786.

[Do91] V. Donnay, *Using Integrability to Produce chaos: Billiards with Positive Entropy*, Commun. Math. Phys. **141** (1991), 225–257.

[KS] A. Katok and J.-M. Strelcyn, *Invariant manifolds, entropy and billiards; smooth maps with singularities*, Lect. Notes Math., **1222**, Springer, New York, 1986.

[LW] C. Liverani & M. Wojtkowski, *Ergodicity in Hamiltonian Systems* Suny Stony Brook Preprint # 1992/16] [Dynamics reported, Dynam. Report. Expositions Dynam. Systems (N.S.) **4**, Springer, Berlin (1995), 130–202].

[M93] R. Markarian, *New ergodic billiards: exact results*, Nonlinearity **6** (1993), 819–841.

[MOP] R. Markarian, S. Oliffson Khamporst & S. Pinto de Carvalho, *Chaotic Properties of the Elliptical Stadium*, Comm. Math. Phys. **174** (1996), 661–679.

[OW] D. S. Ornstein & B. Weiss, *On the Bernoulli nature of systems with hyperbolic structure*, Ergod. Th. & Dynam. Sys. **18** (1998), 441–456.

[Pe77] Ya. B. Pesin, *Characteristic Lyapunov exponents and smooth ergodic theory*, Russ. Math. Surv. **32** (1977), 55–114.

[Wo85] M. Wojtkowski, *Invariant families of cones and Lyapunov exponents*, Ergod. Th. & Dynam. Sys. **5** (1985), 145–161.

[Wo86] M. Wojtkowski, *Principles for the design of billiards with nonvanishing Lyapunov exponents*, Commun. Math. Phys. **105** (1986), 391–414.

[Wo94] M. Wojtkowski, *Two applications of Jacobi fields to the billiard ball problem*, J. Differential Geom. **40** (1994), 155–164.