Observable invariant measures.

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June 30th., 2005

Abstract

For continuous maps on a compact manifold M, particularly for those that do not preserve the Lebesgue measure m, we define the *observable invariant measures* as a generalization of the Sinai-Ruelle-Bowen (SRB) measures. We prove that any continuous map has observable measures and characterize those that are SRB in terms of the observability.

We also define the *generalized ergodic attractors* and construct a decomposition in up to countable many of them, such that their basins are pairwise m-almost disjoint and cover m-almost all points in M.

1 Introduction

Let $f : M \mapsto M$ be a continuous map in a compact, finite-dimensional manifold M. Let m be the Lebesgue measure normalized to verify $m(M) = 1$, and not necessarily f-invariant. We denote $\mathcal{P}$ the set of all Borel probability measures in M, provided with the weak* topology, and a metric structure inducing this topology.

For any point $x \in M$ we denote $p\omega(x)$ to the set of the Borel probabilities in M that are the partial limits of the (not necessarily convergent) sequence

$$\left\{ \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)} \right\}_{n \in I\!N} \qquad [1]$$

where δ_y is the Delta de Dirac probability measure supported in $y \in M$.

The set $p\omega(x) \subset \mathcal{P}$ is the collection of the spatial probability measures describing the asymptotic time average (given by (1)) of the system states, provided the initial state is x. If the sequence [1] converges then the set $p\omega(x)$ has a unique element μ_x. But we shall consider also the case in which it does not converge. There exist maps such that, for Lebesgue a.e. $x \in M$, the sequence [1] is not convergent. (We provide the example 4.6 in Section 4.)

It is unknown how large is the family of maps for which [1] does not converge for a positive Lebesgue set of initial states $x \in M$. There is some evidence that it may be a small family, because all known examples are not robust and form a negligible part of the known scenario of

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dynamical systems. The phenomena exhibited when [1] is not convergent are similar to the time-delayed specification properties studied by Bowen in ([BR75]), but here they are seen on the time average probabilities, instead on the points along the orbit. If the time average sequence [1] is not convergent, it still seems to stabilize on a certain spatial probability measure μ, and maintains the approach to μ for a long period of time, say T. But then it changes, through very slow variations, to a different spatial distribution $\nu \neq \mu$, that shall be seen from a time scale much larger than T. (See Theorem 2.4 and its proof in Appendix 7.3.) These changes should occur aperiodically, slower each time, and infinitely many times in the future.

To include those cases for which the sequence [1] is not convergent we consider, for a given measure μ, the set of points $x \in M$ such that the minimum distance between μ and the tail of the sequence [1] is small, instead of considering the maximum distance. We define:

Definition 1.1 (Observable measures.) A probability measure $\mu \in \mathcal{P}$ is *observable* if for all $\epsilon > 0$ the set $A_\epsilon = \{x \in M : \text{ dist } (p\omega(x), \mu) < \epsilon\}$ has positive Lebesgue measure $m(A_\epsilon) > 0$.

We note that the definition above is independent of the choice of the distance in $\mathcal{P}$, provided that the metric structure induces its weak* topology. We also remark that observable measures are f-invariant, and that usually at most a few part of the space of invariant measures for f are observable measures (see the examples in Section 4).

In paragraph 7.1 of Section 7, we explain the mathematical and physical reasons that lead us to use the $\epsilon > 0$ condition in Definition 1.1 instead of using $\epsilon = 0$ at once. Be aware that this condition must be verified not only *for some* but *for all* $\epsilon > 0$ at each observable measure μ.

In our opinion the results about the observable measures that we state in this paper may be for use mostly on Lebesgue non preserving maps, to study the asymptotic behavior of time averages from a wide viewpoint.

The Sinai-Ruelle-Bowen (SRB) measures ([BR75], [R76], [S72]), also the usually known as physical or nature measures, are widely studied occupying a relevant interest in the ergodic theory of dynamical systems. ([A67], [PS82], [PS04], [V98], [BV00]).

In this paper we characterize the SRB measures as a particular class of observable measures and study the relations between them.

For further uses we call *generalized ergodic attractors,* not to the support in M of a SRB measure ν as done by Pugh and Shub in [PS89], but to the set $\{\nu\} \subset \{\mathcal{P}\}$. This is a kind of Milnor attractor [Mi85], in the space of probability measures instead of the manifold M.

Definition 1.2 (SRB measures and ergodic attractors.) A Borel probability measure ν in M is a *SRB measure* if the set $C = \{x \in M : p\omega(x) = \{\nu\}\}$ has positive Lebesgue measure $m(C) > 0$. The set $C = C(\nu) \subset M$ is called the basin of attraction of ν.

The set $\{\nu\} \subset \mathcal{P}$ is an *ergodic attractor* if ν is a SRB measure (even when ν is not ergodic).

From the definitions above note that SRB measures and observable measures are f-invariant. All SRB measure is observable but not all observable measure is SRB (we provide examples in Section 4).

The following open question refers to the existence and finiteness of SRB measures and to the convergence of the sequence [1] of time averages of the system for a set of initial states with total

Lebesgue measure. It is possed in [P99] and leads to a global understanding of the dynamics from an ergodic viewpoint:

1.3 Palis Conjecture *Most dynamical systems have up to finitely many SRB measures (or ergodic attractors) such that their basins of attraction cover Lebesgue almost all points.*

This conjecture admits the following equivalent statement, that seems weaker. (In fact, the definition 1.1 of observability is certainly weaker than the definition 1.2 of SRB measures.)

1.4 Equivalent formulation of Palis Conjecture: *Most dynamical systems have some and up to finitely many observable measures.*

Note: To prove the equivalence of statements 1.3 and 1.4 it is enough to join the thesis of Theorem 2 in Theorem 2.4 in section 2 of this paper.

We prove the following starting results:

THEOREM 1 (Existence of observable measures and SRB measures.)

For any continuous map f, the space $\mathcal{O}$ of all observable measures for f is non-empty and weak*-*compact.*

Even more, $\mathcal{O}$ (or some proper reduction of $\mathcal{O}$, as defined in 1.7) *has finite or countably infinite many elements if and only if there exist SRB measures for f.*

The first statement of this theorem is proved in paragraph 2.6 and the second one in paragraph 5.1.

Due to the second assertion of Theorem 1, if the map f (or the restriction of f to an invariant set with positive Lebesgue measure) does not have a SRB measure, then there exist non-countable infinitely many observable measures. They are dense in some region $\mathcal{O}_1 \subset \mathcal{O} \subset \mathcal{P}$.

Recalling the ideas of Mañé to state the ergodic closing lemma in [M82]; and working with the partial limits of the sequence [1] of time averages in $\mathcal{P}$ (instead the orbits in M): is it possible to perturb the map f to get a SRB measure, i.e. to "close" some part of such abundant set of observable measures into an isolated measure? We do not have an answer, and it seems a difficult question.

Notice the following consequence of the first part of Theorem 1:

Corollary 1.5 (Observable measures for families of maps.)

Given any continuous k-parameter family of maps $f_y : M \mapsto M$, $y \in Y \subset \mathbb{R}^k$, there exist observable measures in $M \times Y$ for $F : M \times Y \mapsto M \times Y$, defined as $F(x,a) = (f_a(x), a)$.

Therefore, for all $\epsilon > 0$, there exists a positive Lebesgue set of parameter values $y \in Y$ such that the f_y-invariant sets $A_{\epsilon, f_y} \subset M$ defined in 1.1, all have positive Lebesgue measure.

On the other hand, the $p\omega(x)$ limit sets may have many different partial limit measures. Nevertheless, we prove that $p\omega(x)$ is formed only with observable measures, for Lebesgue almost all $x \in M$.

Definition 1.6 (Basin of attraction.)

The *basin of attraction* $C(\mathcal{K})$ of a compact subset $\mathcal{K}$ of the space $\mathcal{P}$ of all the Borel probability measures in M, is the (maybe empty) subset of M defined as:

$$\{x \in M : p\omega(x) \subset \mathcal{K}\}$$

If the purpose is to study the asymptotic to the future time average behaviors of Lebesgue almost all points in M, then the set $\mathcal{O}$ of all observable measures for f is enough and besides the minimal compact subset of probability measures to be considered. In fact we have the following:

THEOREM 2 (Attracting property of the set of observable measures.)
The set $\mathcal{O}$ of all observable measures for f is the minimal compact subset of the space $\mathcal{P}$ whose basin of attraction has total Lebesgue measure.

We prove this theorem in paragraph 3.1.

Due to the conjecture in 1.3 and Theorem 1, we are interested in partitioning the set $\mathcal{O}$ of observable measures, or to reduce it as much as possible, into different compact subsets whose basins of attractions have positive Lebesgue measure. (But due to Theorem 2, no proper compact part of $\mathcal{O}$ shall have a total Lebesgue basin). We define:

Definition 1.7 (Generalized Ergodic Attractors - Reductions of the space $\mathcal{O}$.)
A *generalized ergodic attractor,* (or a *reduction* of the space $\mathcal{O}$ of all observable measures for f), is a compact subset $\mathcal{A}$ of the space $\mathcal{O}$ of observable measures such that its basin of attraction $C(\mathcal{A})$ has positive Lebesgue measure.

Remark 1.8 We refer to them as *ergodic* because the attraction is on the time averages distributions and not in the phase space, but these generalized ergodic attractors are not formed necessarily with ergodic measures. (We provide the example 4.6 in Section 4.)

From Definition 1.7 we note that a generalized ergodic attractor $\mathcal{A}$ is an ergodic attractor (according to Definition 1.2) if and only if its diameter is zero, and in this case the (unique) measure in $\mathcal{A}$ is a SRB measure.

1.9 Generalized Pugh-Shub Ergodic Attractors.

In [PS89] Pugh and Shub call Ergodic Attractor to (A, μ) where μ is a SRB measure and $A = \text{supp}(\mu) \subset M$.

We observe that it is enough to give the SRB measure μ (in fact the Pugh-Shub ergodic attractor is its support in M). This is the first reason of our Definition 1.2 of *ergodic attractor* in $\mathcal{P}$ instead M. Other reasons are related to the difference between topological and measurable supports of a measure, as follows:

We could define the support of μ as the compact support (i.e. the minimal compact subset A in M such that $\mu(A) = 1$). But in a measurable sense, (for instance to look at some ergodic attractors as supported on the non-compact unstable manifold of a hyperbolic set, as in [Pe77] and [PS89]), we will define the support of μ as the equivalent class of Borel measurable sets in M that have total μ-measure. The equivalence relation is defined as follows: a pair of Borel measurable sets are μ-equivalent if their symmetric difference has zero μ-measure. This last definition of support avoids some misunderstandings between topological and ergodic attractors.

Along this paper we will work, for reasons of simplicity, only with generalized ergodic attractors that live in the space $\mathcal{P}$ (i.e. in the sense of Definition 1.7), without reference to its support in M.

In spite a system could not exhibit a SRB measure, still the reductions of the space of observable measures divide the manifold in the basins of *generalized ergodic attractors.* Each reduction $\mathcal{A}$ has a basin $C = C(\mathcal{A})$ with positive Lebesgue measure and is minimal respect to C. We state this result as follows:

THEOREM 3 (Minimality of generalized ergodic attractors.)

Any generalized ergodic attractor $\mathcal{A}$ is the minimal compact set of observable measures attracting its basin $C(\mathcal{A})$. More precisely

$$m(C(\mathcal{A}) \setminus C(\mathcal{K})) > 0$$

for all compact subset $\mathcal{K} \subset \mathcal{A}$.

We prove Theorem 3 in paragraph 3.6.

The following result is much weaker but related with the Palis' conjecture stated in paragraph 1.3:

THEOREM 4 (Decomposition Theorem)

For any continuous map $f : M \mapsto M$ there exist a collection of (up to countable infinitely many) generalized ergodic attractors whose basins of attraction are pairwise Lebesgue-almost disjoint and cover Lebesgue- almost all M.

We prove the last theorem in paragraph 5.9.

In Theorem 5.7 and Corollary 5.8 we characterize those maps whose generalized ergodic attractors (as in the thesis of Theorem 4) are ergodic attractors (i.e. SRB measures), as asked in the statement of Palis conjecture.

Generalized ergodic attractors and observable measures, due to Theorems 1 and 4, do always exist for continuous maps.

On the other hand, SRB measures and ergodic attractors do not always exist.It is largely known the difficulties to characterize, or just find, non hyperbolic maps that do have SRB measures. This is a very difficult problem even in some systems whose iterated topological behavior is known. ([C93], [E98], [H00], [HY95]).

The difficulties appear when applying the known techniques for constructing the SRB measures in a hyperbolic setting ([Pe77], [S72], [A67]). The main obstruction when having a weak hyperbolic setting usually resides in the irregularity of the invariant manifolds, which technically translate the non trivial relations between topologic, measurable and differentiable properties of the system ([PS04]).

In fact, these difficulties appear for non-hyperbolic maps, even for maps possed in a very regular setting as Lewowicz maps in the two-torus (see [Le80]) For them, the differentiable regularity of the given transformation, (they are analytic maps), and the topological known behavior of the iterated system (they are conjugated to Anosov maps), are not enough to prove the existence of SRB measures and ergodic attractors [CE01].

1.10 Observability and ergodicity for maps preserving the Lebesgue measure.

For systems preserving the Lebesgue measure the main question is their ergodicity, and most results of this work translate, for those systems, as equivalent conditions to be ergodic. In fact we prove the following:

THEOREM 5 (Observability and ergodicity.)

If f preserves the Lebesgue measure m then the following assertions are equivalent:
(i) *f is ergodic respect to m.*
(ii) *There exists a unique observable measure μ for f.*
(iii) *There exists a unique SRB measure ν for f attracting Lebesgue a.e.*
Besides, if the assertions above are verified, then $m = \mu = \nu$

We prove this theorem in paragraphs 6.2, 6.4 and 6.5 of Section 6.

Given a map f that preserves the Lebesgue measure, the question if f has a SRB measure is mostly open even for differentiable maps that do not have some kind of uniform hyperbolicity [V98].

For Lebesgue preserving maps the existence of a SRB measure attracting Lebesgue a.e. is equivalent to the ergodicity of the map. The ergodicity of most maps that preserve the Lebesgue measure is also an open question. ([PS04], [BMVW03]).

Due to Theorem 5 both open questions above are equivalent to the *unique observability.*

Definition 1.11 Irreducibility

A generalized ergodic attractor $\mathcal{A} \subset \mathcal{P}$ is *irreducible* if it does not contain proper compact subsets that are also generalized ergodic attractors.

It is *trivial* or *trivially irreducible* if its diameter in $\mathcal{P}$ is zero , or in other words, if $\mathcal{A}$ has a unique observable measure μ.

Note that SRB measures are trivially irreducible and conversely.

THEOREM 6 (Irreducible generalized ergodic attractors.)

Let $f : M \mapsto M$ be a continuous map that preserves the Lebesgue measure m in M.
Any generalized ergodic attractor $\mathcal{A}$ for f is irreducible if and only if it is trivial.
Thus, the unique measure $\mu \in \mathcal{A}$ is a SRB measure.
Besides, all SRB measures for f are ergodic components of m.

We prove this theorem in paragraph 6.7.

1.12 Generalized Ergodic Attractors and Milnor Attractors.

Definition 1.13 Milnor attractor. [Mi85]

A Milnor attractor is a compact set $A \subset M$ such that its basin of attraction

$$C(A) = \{x \in M : \omega(x) \subset A\}$$

has positive Lebesgue measure, and for any compact proper subset $K \subset A$ the set

$$C(A) \setminus C(K) = \{x \in M : \omega(x) \subset A,\ \omega(x) \not\subset K\}$$

also has positive Lebesgue measure.

Note: Here $\omega(x)$ is the ω-limit set in M of the orbit of x, i.e. the set of limit points in M of the orbit with initial state x.

The generalized ergodic attractors are defined in $\mathcal{P}$ instead of M, but were inspired in definition 1.13. They play in $\mathcal{P}$ a similar topological role that Milnor attractors play in M.

We look at a SRB measure as a trivially irreducible generalized ergodic attractor, or a reduction with null diameter in the space of the observable measures. It is in $\mathcal{P}$ the limit of the convergent time average sequence [1], for a set of initial states with positive Lebesgue measure.

On the other hand, a sink in M is the limit of the convergent orbits for a set of initial states with positive Lebesgue measure. Roughly speaking, the SRB measures in $\mathcal{P}$ play the role that sinks do in M, those first considered as punctual attractors of time averages, and these last considered as punctual attractors of the orbits in M.

The supports and basins of attraction of the generalized ergodic attractors, and those of the Milnor attractors in M, are related as follows:

For any given Milnor attractor A in M with basin C, there exists a (unique minimal respect to C) generalized ergodic attractor $\mathcal{A}$ in $\mathcal{P}$ whose basin contains Lebesgue a.e. the basin C of A. This $\mathcal{A}$ is constructed as the space of all observable measures of the map f restricted to C (see Proposition 3.5).

On the other hand (but not conversely) for any given generalized ergodic attractor $\mathcal{A}$ in $\mathcal{P}$ with basin C, there exists a (unique minimal respect to C) Milnor attractor A in M whose basin contains Lebesgue a.e. the basin C of $\mathcal{A}$. This Milnor attractor A is constructed as the intersection of all compact sets in M containing $\omega(x)$ for Lebesgue a.e. $x \in C$.

Besides, the Milnor attractor A such constructed from a given generalized ergodic attractor $\mathcal{A}$, supports all the probability measures μ in $\mathcal{A}$ in the sense that $\mu(A) = 1$ for all $\mu \in \mathcal{A}$. This last fact was observed by Ashwin-Terry in [AT00] for the case of existing SRB measures. But, as remarked by Cao in [Ca04], the compact support of $\mathcal{A}$ and the Milnor attractor A may be very different subsets in M, from the topological viewpoint.

The topological phenomena of riddled basins of different Milnor attractors as in [AT00], can still be present for the basins of different generalized ergodic attractors. In Theorem 4 we prove that the basins, for any continuous dynamical system f, form a m-a.e. partition of M, of medible sets. But we do not say nothing about their possible topological riddling. Nevertheless, the riddling may be present for generalized ergodic attractors as for Milnor attractors, due to the constructions above.

2 The set of the observable invariant measures.

Definition 2.1 (Weak* topology in the space of probability measures.)
The weak* topologic structure in the space $\mathcal{P}$ is defined as:

$$\mu_n \to \mu \text{ in } \mathcal{P} \quad \text{iff} \quad \lim \int g \, d\mu_n = \int g \, d\mu \quad \text{for all } g \in C(M)$$

where $C(M)$ denotes the space of continuous real functions in M.

A classic basic theorem on Topology states that the space $\mathcal{P}$ is compact and metrizable when endowed with the weak* topology. [M89]

Let us denote as $\mathcal{P}_f \subset \mathcal{P}$ the set of the Borel probability measures in M that are f-invariant, that is $\mu(f^{-1}(B)) = \mu(B)$ for all Borel set $B \subset M$. Note that the Lebesgue measure m does not necessarily belong to $\mathcal{P}_f$.

Fix any metric in $\mathcal{P}$ giving its weak* topology structure. We denote as $B_\epsilon(\mu)$ to the open ball in $\mathcal{P}$ centered in $\mu \in \mathcal{P}$ and with radius $\epsilon > 0$.

2.2 The $p\omega$- limit sets.

At the beginning of this paper we defined, for each initial state $x \in M$, the set $p\omega(x)$ in the space $\mathcal{P}$. It is the set of the probabilities measures obtained as the limits, in the weak* topology in $\mathcal{P}$, of the convergent subsequences of the sequence [1] of time averages up to time $n \to +\infty$.

In other words:

$$p\omega(x) = \{\mu \in \mathcal{P} : \lim \frac{1}{n_i} \sum_{j=0}^{n_i - 1} g(f^j(x)) = \int g \, d\mu \ \forall \, g \in C(M) \ \text{ for some } n_i \to +\infty\}$$

Remark 2.3 Note that all $\mu \in p\omega(x)$ are f-invariant. That is $p\omega(x) \subset \mathcal{P}_f \subset \mathcal{P}$ although the sequence [1] contained in $\mathcal{P}$ may not intersect $\mathcal{P}_f$. Due to the compactness of $\mathcal{P}$, the set $p\omega(x)$ is compact and non-empty, for all $x \in M$.

For further uses we state here the following property for the $p\omega$-limit sets:

Theorem 2.4 Convex-like property.

For any point $x \in M$

i) *If $\mu, \nu \in p\omega(x)$ then for each real number $0 \leq \lambda \leq 1$ there exists a measure $\mu_\lambda \in p\omega(x)$ such that*

$$dist\,(\mu_\lambda, \mu) = \lambda \, dist\,(\mu, \nu)$$

ii) The set $p\omega(x)$ *either has a single element or non-countable infinitely many.*

Proof: See 7.3 in the appendix.

Definition 2.5 Given a real number $\epsilon > 0$ we say that a (non-necessarily f-invariant) probability measure μ in M *is ϵ-observable* if the set of points $x \in M$ such that $p\omega(x) \cap B_\epsilon(\mu) \neq \emptyset$ has positive Lebesgue measure.

Theorem 2.6 (Existence of observable measures.) *Given any continuous map f, the set $\mathcal{O}_f$ of the probability measures that are ϵ-observable for all $\epsilon > 0$ is a non-empty compact subset of the (weak* topologic) space $\mathcal{P}_f$ of the f-invariant measures.*

Proof: The key question is that $\mathcal{O}_f$ is not empty, which we prove at the end.

Let us first prove that $\mathcal{O}_f \subset \mathcal{P}_f$. In fact, given $\mu \in \mathcal{O}_f$ then, for any $\epsilon = 1/n > 0$ there exists some $\mu_n \in B_\epsilon(\mu)$ which is the limit of a convergent subsequence of [1] for some $x \in M$. As the limits of all convergent subsequences of [1] are f-invariant, we have that $\mu_n \in \mathcal{P}_f \subset \mathcal{P}$ for all natural number n, and $\mu_n \to \mu$ with the weak* topologic structure of $\mathcal{P}$. The space $\mathcal{P}_f$ is a compact subspace of $\mathcal{P}$ with the weak* topologic structure, so $\mu \in \mathcal{P}_f$ as wanted.

Second, let us prove that $\mathcal{O} = \mathcal{O}_f$ is compact. The complement $\mathcal{O}^c$ of $\mathcal{O}$ in $\mathcal{P}$ is the set of all probability measures μ(not necessarily f-invariant) such that for some $\epsilon = \epsilon(\mu) > 0$ the set $\{x \in M : p\omega(x) \cap B_\epsilon(\mu) \neq \emptyset\}$ has zero Lebesgue measure. Therefore $\mathcal{O}^c$ is open in $\mathcal{P}$, and $\mathcal{O}$ is a closed subspace of $\mathcal{P}$. As $\mathcal{P}$ is compact we deduce that $\mathcal{O}$ is compact as wanted.

Third, let us prove that $\mathcal{O}$ is not empty. Suppose by contradiction that it is empty. Then $\mathcal{O}^c = \mathcal{P}$, and for every $\mu \in \mathcal{P}$ there exists some $\epsilon = \epsilon(\mu) > 0$ such that the set $A = \{x \in M : p\omega(x) \subset (B_\epsilon(\mu))^c\}$ has total Lebesgue probability.

As $\mathcal{P}$ is compact, let us consider a finite covering of $\mathcal{P}$ with such open balls $B_\epsilon(\mu)$, say $B_1, B_2, \ldots B_k$, and their respective sets $A_1, A_2, \ldots A_k$ defined as above. As $m(A_i) = 1$ for all $i = 1, 2, \ldots, k$ we have that the intersection $B = \cap_{i=1}^k A_i$ is not empty. By construction, for all $x \in B$ the $p\omega$-limit of x is contained in the complement of B_i for all $i = 1, 2 \ldots, k$, and so it would not be contained in $\mathcal{P}$, that is the contradiction ending the proof. $\Box$

We reformulate Definition 1.1 of observability of measures, in the following equivalent terms:

Definition 2.7 (Observability revisited.) We say that $\mu \in \mathcal{P}_f$ is *observable* for f, if it is ϵ-observable (see 2.5) for all $\epsilon > 0$.

We denote as $\mathcal{O}_f$, or simply as $\mathcal{O}$, to the (compact and non empty, see Theorem 2.6) *set of all the probability measures that are observable for f.*

Definition 2.8 (Observability Size.) If μ is an observable measure (see Definitions 2.5 and 2.7), we call *observability size of* μ to the positive real function $o = o_\mu : I\!R^+ \mapsto I\!R^+$ defined as

$$o_\mu(\epsilon) = m(A(\epsilon, \mu))$$

where m is the Lebesgue measure in M and $A(\epsilon, \mu)$ is the set

$$A(\epsilon, \mu) = \{x \in M : p\omega(x) \cap B_\epsilon(\mu) \neq \emptyset\}$$

Remark 2.9 For any observable measure μ, its observability size function $o(\epsilon)$ *is positive and decreasing with* $\epsilon > 0$.

Then $o(\epsilon)$ has always a non-negative limit value when $\epsilon \to 0^+$.

3 Generalized ergodic attractors. Reductions of the set of observable measures.

The characterization of those continuous maps having SRB-measures as those whose sets of observable measures, or some reductions of them, are finite or countable infinite (Theorem 5.1), derives the attention to try to define and find sufficient conditions to reduce as much as possible the set of observable measures.

Besides, the *reductions* of the space of observable measures will work as *Generalized Ergodic Attractors,* even in the case that this reduction can not be done as much as to obtain SRB measures.

We first prove that the reducibility of the set $\mathcal{O}$ of observable measures for f must be defined carefully, because in the following sense, this set $\mathcal{O}$ is minimal.

Theorem 3.1 (Reformulation of Theorem 2. Attracting property of the set of observable measures.)

Let $f : M \mapsto M$ be any given continuous map in the compact manifold M.

The set $\mathcal{O}_f$ of all its observable measures belongs to the family

$$\aleph = \{\mathcal{K} \subset \mathcal{P} : \mathcal{K} \textit{ is compact and } p\omega(x) \subset \mathcal{K} \textit{ for Lebesgue almost every point } x \in M\}$$

Moreover

$$\mathcal{O}_f = \bigcap_{\mathcal{K} \in \aleph} \mathcal{K}$$

and thus $\mathcal{O}_f$ is the unique minimal set in $\aleph$.

Proof: For simplicity let us denote $\mathcal{O} = \mathcal{O}_f$. Given any subset $\mathcal{K} \subset \mathcal{P}$ (this $\mathcal{K}$ is not necessarily in $\aleph$ neither is necessarily compact), let us consider:

$$(2) \qquad A(\mathcal{K}) = \{x \in M : p\omega(x) \cap \mathcal{K} \neq \emptyset\}, \qquad C(\mathcal{K}) = \{x \in M : p\omega(x) \subset \mathcal{K}\}$$

It is enough to prove that $m(C(\mathcal{O})) = 1$ and that $\mathcal{K} \supset \mathcal{O}$ for all $K \in \aleph$. Let us first prove the second assertion.

To prove that $\mathcal{K} \supset \mathcal{O}$ it is enough to show that $\mu \notin \mathcal{O}$ if $\mu \notin \mathcal{K} \in \aleph$.

If $\mu \notin \mathcal{K}$ take $\epsilon = \text{dist}\,(\mu, \mathcal{K}) > 0$. For all $x \in C(\mathcal{K})$ the set $p\omega(x) \subset \mathcal{K}$ is disjoint with the ball $B_\epsilon(\mu)$. But almost all Lebesgue point $x \in C(\mathcal{K})$, because $\mathcal{K} \in \aleph$. Therefore $p\omega(x) \subset \mathcal{K} \cap B_\epsilon(\mu) = \emptyset$ Lebesgue a.e. This last assertion and Definitions 1.1 and 2.7 imply that $\mu \notin \mathcal{O}$, as wanted.

Now let us prove that $m(C(\mathcal{O})) = 1$, which is the key matter of this theorem. We know $\mathcal{O}$ is compact and not empty. So, for any $\mu \notin \mathcal{O}$ it is defined the distance $\text{dist}\,(\mu, \mathcal{O}) > 0$. Observe that the complement $\mathcal{O}^c$ of $\mathcal{O}$ in $\mathcal{P}$ can be written as the increasing union of compacts sets $\mathcal{K}_n$ (not in the family $\aleph$) as follows:

$$(3) \qquad \mathcal{O}^c = \bigcup_{n=1}^{\infty} \mathcal{K}_n, \qquad \mathcal{K}_n = \{\mu \in \mathcal{P} : \text{dist}\,(\mu, \mathcal{O}) \geq 1/n\} \supset \mathcal{K}_{n+1}$$

Let us take the sequence $A_n = A(\mathcal{K}_n)$ of sets in M defined in (2) at the beginning of this proof, and denote $A_\infty = A(\mathcal{O}^c)$. We deduce from (2)and (3) that:

$$A_\infty = \bigcup_{n=1}^{\infty} A_n, \quad m(A_n) \to m(A_\infty), \quad A_\infty = A(\mathcal{O}^c)$$

To finish the proof is thus enough to show that $m(A_n) = 0$ for all $n \in I\!N$.

In fact, $A_n = A(\mathcal{K}_n)$ and $\mathcal{K}_n$ is compact and contained in $\mathcal{O}^c$. By definitions 1.1 and 2.7 there exists a finite covering of $\mathcal{K}_n$ with open balls $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k$ such that

$$(4) \qquad m(A(\mathcal{B}_i)) = 0 \quad \text{for all } i = 1, 2, \ldots, k$$

By (2) the finite collection of sets $A(\mathcal{B}_i)$; $i = 1, 2, \ldots, k$ cover A_n and therefore (4) implies $m(A_n) = 0$ ending the proof. □

Recall Definition 1.6 of basin of attraction and Definition 1.7 of generalized ergodic attractor or reduction of the space $\mathcal{O}$ of the observable measures. Note that the basins of attraction of the non-empty compact subsets of $\mathcal{O}$, if non-empty, are f-invariant sets. They could have Lebesgue measure zero.

In theorem 3.1 we showed that the unique compact subset $\mathcal{K} \subset \mathcal{O}$ whose basin of attraction has total Lebesgue probability measure is $\mathcal{O}$. So is itself a generalized ergodic attractor.

But there may exist other compact subsets of $\mathcal{O}$ still having basins of attractions with positive Lebesgue measure. If they exist, they are the proper reductions or generalized ergodic attractors properly contained in $\mathcal{O}$.

3.2 Reducible, irreducible and trivially irreducible attractors. Let us recall Definitions 1.7 and 1.11. The space $\mathcal{O}$ of the observable measures for f *is reducible* if there exists some proper reduction of it. If $\mathcal{O}_1$ is a generalized ergodic attractor (i.e. a reduction of $\mathcal{O}$), then $\mathcal{O}_1$ *is reducible* if there exists some reduction properly contained in $\mathcal{O}_1$.

Then, $\mathcal{O}_1$ *is irreducible,* if there does not exist proper reductions of $\mathcal{O}_1$. In other words, $\mathcal{O}_1$ is irreducible if for all proper compact subset $\mathcal{K}$ of $\mathcal{O}_1$ the basin of attraction of $\mathcal{K}$ has zero Lebesgue measure.

We say that a generalized ergodic attractor $\mathcal{O}_1$ is trivially irreducible if its diameter is zero, i.e. it has a single element.

From Definition 1.2 the trivially irreducible generalized ergodic attractors, if there exist some, are the ergodic attractors, or in brief, the SRB measures.

As shown in the examples of Section 4, there exist maps whose spaces $\mathcal{O}$ of observable measures are irreducible and maps for which they are reducible. Also there exist maps that do not have irreducible subsets in $\mathcal{O}$.

In section 5.2 we define chains and co-chains of reductions. Those systems having SRB measures can be characterized also according to the existence of adequate chains of generalized ergodic attractors, each one containing the following.

Let $\mathcal{O}$ be the set of the observable measures for f. Let $\mathcal{O}_1$ be a reduction or generalized ergodic attractor of $\mathcal{O}$ (see definition 1.7). For further uses we define:

Definition 3.3 (Attracting size.)
The diameter of $\mathcal{O}_1$ is $\max\{dist(\mu,\nu) : \mu,\nu \in \mathcal{O}_1\}$.
The attracting size of $\mathcal{O}_1$ is $m(C(\mathcal{O}_1))$, where $C(\mathcal{O}_1)$ is the basin of attraction of $\mathcal{O}_1$ (see definition 1.6).

If the purpose is to find the generalized ergodic attractors and classify the phase space according to which basin each point belongs to, then we should try to know when a given compact subset of the space of observable measures is a reduction. That is why we state, for a seek of completeness and for further use, some results that characterize the reductions.

If the basin of attraction C of some compact subspace of $\mathcal{O}$ has positive Lebesgue measure, then there exists a compact set $K \subset C$ with positive Lebesgue measure. Thus we obtain the following characterization of all the reductions of $\mathcal{O}$, as a consequence of Egoroff Theorem:

Proposition 3.4 (Generalized ergodic attractors and uniform convergence.)
The subspace $\mathcal{O}_1$ is a reduction of the space $\mathcal{O}$ of the observable measures for f, *(i.e.* $\mathcal{O}_1$ *is a generalized ergodic attractor for* f*),* if and only if there exists a positive Lebesgue measure set $K \subset M$ such that

$$\text{dist}\left(\frac{1}{n}\sum_{j=0}^{n-1}\delta_{f^j(x)},\ \mathcal{O}_1\right) \to 0 \ \text{ uniformly in } x \in K$$

Note: The set $K \subset M$ is not necessarily f-invariant.

Proof: Let us call C to the basin of attraction of $\mathcal{O}_1$ (see definition 1.6). We have $m(C) > 0$ and therefore, the sequence in 3.4 converges to 0 m-a.e. $x \in C$. The direct result is now a straightforward consequence of Egoroff Theorem and its converse is obvious. □

The following is other characterization of the reductions of the space of observable measures for $f : M \mapsto M$, in terms of the invariant subsets in M that have positive Lebesgue measure:

Proposition 3.5 (Restricting the map to reduce the set of observable measures.) *The subspace $\mathcal{O}_1$ is a reduction of the space $\mathcal{O}_f$ of the observable measures for f (i.e. $\mathcal{O}_1$ is a generalized ergodic attractor for f) if and only if $\mathcal{O}_1 = \mathcal{O}_{f_1}$, where $\mathcal{O}_{f_1}$ is the set of all observable measures of the map $f_1 = f|_C$, obtained when f is restricted to some invariant set $C \subset M$ that has positive Lebesgue measure.*

Besides C can be chosen as the basin attraction $C(\mathcal{O}_1)$ of $\mathcal{O}_1$.

Proof: This Theorem is a corollary of Theorem 3.1. In fact, to prove the converse statement apply Theorem 3.1 to $f|_C$ instead of f, taking $C = C(\mathcal{O}_1)$ where $\mathcal{O}_1$ is the given reduction of $\mathcal{O}_f$. To prove the direct result apply also Theorem 3.1 to f_C, but now taking C as the given invariant subset in M with positive Lebesgue measure.□

3.6 - Proof of Theorem 2:

By Proposition 3.5 and Theorem 3.1 applied to $f|_C$ where $C = C(\mathcal{O}_1)$, we have that Lebesgue almost all $x \in C$ verifies $p\omega(x) \subset \mathcal{O}_1$, and any $\mu \in \mathcal{O}_1$ is observable for f_C. Take any $\mu \in \mathcal{O}_1 \setminus \mathcal{K}$. There exists a open ball $B_\epsilon(\mu)$ that does not intersect $\mathcal{K}$. As μ is observable for $f|C$, (see Definition 1.1), $p\omega(x)$ is not contained in $\mathcal{K}$ for a set of $x \in C$ with positive Lebesgue measure. Therefore $m(C \setminus C(\mathcal{K})) > 0$ as wanted. □

4 Examples.

Example 4.1 For a map with a single periodic point x_0, being a topological sink whose topological basin is M almost all point, the set $\mathcal{O}$ has a unique measure that is the δ-Dirac measure supported on x_0.

Example 4.2 For an ergodic Anosov diffeomorphism preserving the Lebesgue probability measure m the set $\mathcal{O}$ is irreducible containing uniquely the measure m. But there are also infinitely many other ergodic and non ergodic invariant probabilities, that are not observable (for instance the equally distributed Dirac delta measures combination supported on a periodic orbit). For any ergodic Anosov diffeomorphism the unique SRB measure μ is the unique observable measure.

Example 4.3 In [HY95] it is studied the class of diffeomorphisms f in the two-torus obtained from an Anosov when the unstable eigenvalue of a fixed point x_0 is weakened to be 1, maintaining its stable eigenvalue strictly smaller than 1 and uniform hyperbolicity outside a neighborhood of the fixed point. It is proved that f has a single SRB measure that is the Dirac delta supported on x_0 and that its basin has total Lebesgue measure. Therefore this is the single observable measure for f, although there are infinitely many other ergodic invariant measures.

Example 4.4 The diffeomorphism $f : [0,1]^2 \mapsto [0,1]^2; \quad f(x,y) = (x/2, y)$ has the set $\mathcal{O}$ of observable measures as the set of Dirac delta measures $\delta_{(0,y)}$ for all $y \in [0,1]$. In this case $\mathcal{O}$ coincides with the set of all ergodic invariant measures for f, it is infinitely reducible (i.e. $\mathcal{O}$ is reducible and any reduction of $\mathcal{O}$ is also reducible). Not all f-invariant measure μ for f

is observable: for instance, the one-dimension Lebesgue measure on the interval $[0] \times [0,1]$ is invariant and is not observable. This example shows that the set $\mathcal{O}$ is not necessarily closed on convex combinations.

Example 4.5 The maps exhibiting infinitely many simultaneous hyperbolic sinks, constructed from Newhouse's theorem ([N74])has a space $\mathcal{O}$ of observable measures that is reducible. But is has infinitely many reductions (the Dirac delta supported on the hyperbolic sinks) that are irreducible. Also the maps exhibiting infinitely Hénon-like attractors, constructed by Colli in [Co98]has a space of observable measures that is reducible, having infinitely many reductions (the SRB measures supported on the Hénon-like attractors) that are irreducible.

Example 4.6 The following example due to Bowen shows that the space of observable measures may be formed by measures that are partial limits of the sequences of time averages of the system states and that this sequence may be non convergent for Lebesgue almost all points. Consider a diffeomorphism f in a ball of $\mathbb{R}^2$ with two hyperbolic saddle points A and B such that the unstable global manifold $W^u(A)$ of A is a embedded arc that coincides (except for A and B) with the stable global manifold $W^s(B)$ of B, and the unstable global manifold $W^u(B)$ of B is also an embedded arc that coincides with the stable manifold $W^s(A)$ of A. Let us take f such that there exists a source C in the open ball U with boundary $W^u(A) \cup W^u(B)$, and all orbits in that ball U have α-limit set C and ω-limit set $W^u(A) \cup W^u(B)$. If the eigenvalues of the derivative of f at A and B are well chosen, then one can get that the time average sequences of the orbits in $U \setminus \{C\}$ are not convergent, have at least one subsequence convergent to the Dirac delta δ_A on A and have other subsequence convergent to the Dirac delta δ_B on B.

Due to Theorem 2.4, for each $x \in U \setminus \{C\}$ there are non countably many probability measures which are the limit measures of the time average sequence of the future orbit starting on x. All these measures are invariant under f and therefore, due to Poincaré Recurrence Theorem (see [M89]), all of them are supported on $\{A\} \cup \{B\}$. Due to this last observation and due to Theorem 2.4 all the convex combinations of δ_A and δ_B are limit measures of the sequence of time averages of any orbit starting at $U \setminus \{C\}$ and conversely.

Therefore the set $\mathcal{O}$ of observable measures for f coincides with the set of convex combinations of δ_A and δ_B. The set $\mathcal{O}$ is irreducible and formed by non-countable many probability measures. It is not the set of all invariant measures; in fact the measure δ_C is not observable.

This example also shows that the observable measures are not necessarily ergodic.

5 Relations between observability and SRB measures.

In this section we state some results that characterize the maps having SRB measures, in terms of the set of its observable measures.

Theorem 5.1 Cardinality of $\mathcal{O}$ and SRB measures.

Let $f : M \mapsto M$ be any continuous map in the compact manifold M.

If the set $\mathcal{O}$ of the observable measures for f, or some proper reduction $\mathcal{O}_1$ of $\mathcal{O}$, is finite or countable infinite then there exists a SRB measure μ for f.

Conversely, if there exists a SRB measure for f then, either the space $\mathcal{O}$ has a single element, or it is reducible and there exists a proper reduction $\mathcal{O}_2$ of $\mathcal{O}$ with a single element.

Proof: The converse assertion is immediate. In fact, if μ is SRB then the basin of attraction of μ has positive Lebesgue measure, and thus $\{\mu\}$ is a trivial reduction of $\mathcal{O}$. It is either a proper reduction or not. If not then $\mathcal{O} = \{\mu\}$.

Let us prove the direct assertion. Denote extensively as $\{\mu_n : n \in N\}$ the finite or countable infinite reduction $\mathcal{O}_1$ (proper or not) that is given in the hypothesis. (If it has only a finite cardinality, then repeat one or more of its elements in the extensive notation, but include all of them at least once).

By Proposition 3.5 the space $\mathcal{O}_1$ is the set of all observable measures for the restriction $f|C$ of f to some invariant set C with positive Lebesgue measure, say $m(C) > 0$. Rename if necessary $f|C$ as f, $\mathcal{O}_1$ as $\mathcal{O}$, and rename as m the Lebesgue measure in C, (i.e: the restriction to C of the Lebesgue measure in M, which is then renormalized to be a probability measure in C). Resuming:

$$[7] \qquad m\text{-almost every x } \in C : p\omega(x) \subset \mathcal{O} = \{\mu_n : n \in I\!N\}$$

Let us define $C_n \in C$ to be candidates of the basins of attraction for the measures μ_n, and relate their respective Lebesgue measures $m(C_n)$ as follows:

$$[8] \qquad C_n = \{x \in C : \mu_n \in p\omega(x)\}; \quad C = \bigcup_{n=1}^{\infty} C_n; \quad \sum_{n=1}^{\infty} m(C_n) \geq m(C) = 1$$

So $m(C_n) > 0$ for some $n \in I\!N$.

To end the proof we shall show that for all $x \in C_n : \{\mu_n\} = p\omega(x)$. Due to [7] and [8], it is enough to prove that $C_n \cap C_k = \emptyset$ if $\mu_n \neq \mu_k$.

By contradiction, suppose that for some $\mu_n \neq \mu_k$ there exists a point $x \in C_n \cap C_k \subset C$. Then, from [8] we have $\mu_n, \mu_k \in p\omega(x)$. Now we apply Theorem 2.4 and [7] to conclude that the space $\mathcal{O}$ is non-countably infinite. $\square$

Definition 5.2 (Chains of reductions.) *A chain of reductions* of the space $\mathcal{O}$ of the observable measures for f is a (finite or countable infinite) sequence $\{\mathcal{O}_n\}_{n \in I \subset I\!N}$ of reductions or *generalized ergodic attractors* (see Definition 1.7) such that $\mathcal{O}_i \subset_{\neq} \mathcal{O}_j$ if $i > j$ in the set I of natural indexes.

We call *length of the chain* to its finite or countable infinite cardinality $\#I$.

For any chain $\{\mathcal{O}_n;\}_{n \in I \subset I\!N}$ of reductions of the space of observable measures for f, let

$$d_n = \text{diam}(\mathcal{O}_n); \qquad s_n = \text{attrSize}(\mathcal{O}_n)$$

where diam and attrSize denote respectively the diameter and the attracting size, defined in 3.3.

We define $\lim_{n \to \sup I} d_n$ either to $\lim_{n \to +\infty} d_n$ if the chain length is infinite (in this case we assume $I = I\!N$), or to $d_{\sup I}$ when the chain has a finite length.

Analogously we define $\lim s_n$.

Theorem 5.3 (SRB measures and chains.) *There exists a SRB measure for f if and only if there exists a chain $\mathcal{O}_n$ of reductions of the space $\mathcal{O}$ of the observable measures of f, such that the sequence of its diameters converges to zero and the sequence of its attracting sizes converges to some $\alpha > 0$.*

Proof: The converse statement is immediate defining the length 1-chain $\{\{\mu\}\}$, where μ is the given SRB measure. The direct result is also immediate if the length of the given chain is finite. Let us see now the case when the chain $\mathcal{O}_n;\ n \in I\!N$ is infinite. As the sequence of its diameters converges to zero, then $\cap_{n\in I\!N}\mathcal{O}_n = \{\mu\}$ for some μ. It is enough to show that the attracting size of μ is positive. Note that from the construction of such μ we have that $C(\{\mu\}) = \cap_{n\in I\!N} C_n$ where C_n denotes the basins of attractions $C(\mathcal{O}_n)$. These basins are a countably infinite decreasing family of sets in M with positive Lebesgue measures s_n. Therefore, $attrSize(\{\mu\}) = m(C(\{\mu\})) = \lim s_n = \alpha > 0$, as wanted. □

Definition 5.4 (Independence of generalized ergodic attractors and chains.) We say that two generalized ergodic attractors or reductions of the space of observable measures *are independent if the basin of attraction of their intersection has zero Lebesgue measure.*

We note from Definition 1.6 that the basin of attractions of two reductions $\mathcal{O}_1$ and $\mathcal{O}_2$ intersect exactly in the basin of attraction of $\mathcal{O}_1 \cap \mathcal{O}_2$. Therefore:

Two ergodic attractors are independent if and only if the intersection of their basins has zero Lebesgue measure.

We say that *two chains of reductions are independent* if each one of the chains has a reduction that is independent with *some* reduction of the other chain.

Definition 5.5 (Co-chains of reductions.) *A co-chain of reductions* of the space $\mathcal{O}$ of the observable measures for f is a (finite or countable infinite) family $\{\mathcal{O}_n;\ n \in I \subset I\!N\}$ of reductions or *generalized ergodic attractors* (see Definition 3.2) *that are pairwise independent.*

We call *length of the co-chain* to its finite or countable infinite cardinality $\#I$.

Remark: If the space $\mathcal{O}$ of all the observable measures for f is irreducible, then $\{\mathcal{O}\}$ is the unique chain of reductions and also the unique co-chain.

Now we state a slightly generalized version of a known result in the theory of Discrete Mathematics, the Theorem of Dilworth ([Li95]), applied to the chains and co-chains of generalized ergodic attractors:

Theorem 5.6 (Reformulation of Dilworth Theorem.) *For any continuous map f the supreme k of the lengths of the co-chains of reductions in the space of observable measures for f, is equal to the supreme h of the number of pairwise independent chains.*

Moreover: for any co-chain of length l there is a family of l pairwise independent chains, and conversely.

Proof: Any co-chain $\{\mathcal{O}_j,\ j \in J\}$ with length $l = \#J$ can be seen as a collection $\{\{\mathcal{O}_j\},\ j \in J\}$ of l pairwise independent chains $P_j = \{\mathcal{O}_j\}$, each chain P_j with length one. So $k \leq h$.

Conversely, given any collection $\{P_j,\ j \in J\}$ of pairwise independent chains, take a reduction $\mathcal{O}_1 \in P_1$ independent to some $\widehat{\mathcal{O}}_2 \in P_2$, and take $\breve{\mathcal{O}}_2 \in P_2$ independent with $\widehat{\mathcal{O}}_3 \in P_3$. As both reductions $\widehat{\mathcal{O}}_2$ and $\breve{\mathcal{O}}_2$ belong to the chain P_2, one of them must be contained in the other; thus their intersection, say $\mathcal{O}_2$, is also a reduction of the chain P_2. Besides $\mathcal{O}_2$ is independent with $\mathcal{O}_1 \in P_1$ and with $\widehat{\mathcal{O}}_3 \in P_3$. Analogously construct by induction a (finite or infinite) sequence $\{\mathcal{O}_j : j \in J\}$ of pairwise independent reductions, such that $\mathcal{O}_j \in P_j$. This sequence of reductions is by definition a co-chain. Therefore, $h \leq k$. □

Theorem 5.7 (Co-Chains and SRB measures.) *A map $f : M \mapsto M$ has* (up to countable infinitely many) *SRB measures whose basins of attraction cover Lebesgue almost all point in M if and only if there exist a co-chain of reductions of the space $\mathcal{O}$ of the observable measures for f such that all of its diameters are zero and the* (finite or countable infinite) *sum of its attracting sizes is 1.*

Moreover, the (finite or countable infinite) *number of such SRB measures is equal to the supreme k of the lengths of the co-chains of reductions for f and to the supreme h of the number of independent chains for f.*

Proof: From Definitions 1.2 and 3.3 we obtain that f has SRB measures whose basins cover Lebesgue almost all points if and only if the following statement holds:

(9): There exist (up to countable many) $\mu_n \in \mathcal{O}$ such that $s_n = attrSize(\{\mu_n\}) > 0$ and $\sum s_n = 1$.

Note that two different trivial reductions of $\mathcal{O}$ are always mutually independent. Therefore, (9) is equivalent to the following:

(10): The family $\{\{\mu_n\}\}$ is a co-chain of trivial reductions such that $\sum s_n = 1$.

This last assertion is equivalent to the assertion of the fist part of Theorem 5.7, as wanted.

To prove the second part of Theorem 5.7 first observe that each reduction of any co-chain must contain at least one of the SRB measures μ_n because $\sum s_n = 1$, and that two different reductions of the same co-chain can not contain the same SRB measure because they must be independent. Then k is smaller or equal than the number of SRB measures. On the other hand, the set of all the different SRB measures form itself a co-chain, so k is greater or equal than the number of SRB measures. Finally apply Theorem 5.6 to show that the number of SRB measures is also equal to h. $\square$

Corollary 5.8 *A map $f : M \mapsto M$ has* (up to countable infinitely many) *SRB measures whose basins of attraction cover Lebesgue almost all point in M if and only if there exist a* (finite or infinite) *family*

$$\{\mathcal{O}_i^j,\ i \in I \subset I\!N,\ j \in J \subset I\!N\}$$

of generalized ergodic attractors $\mathcal{O}_i^j$ for f such that for all $i \in I$ and $j, k \in J,\ j \neq k$:

$$\mathcal{O}_{i+1}^j \subset \mathcal{O}_i^j, \quad \lim_{i \to \sup I} d_i^j = 0 \quad \lim_{i \to \sup I} s_i^{j,k} = 0 \quad \text{and} \quad \lim_{i \to \sup I} \sum_{j \in J} s_i^j \geq 1$$

where d_i^j and s_i^j denote respectively the diameter and attracting size of $\mathcal{O}_i^j$ and $s_i^{j,k}$ is the Lebesgue measure of the basin of attraction of $\mathcal{O}_i^j \cap \mathcal{O}_i^k$.

Proof: If there exist such SRB measures μ_j for $j \in J \subset N$, simply define the family $\mathcal{O}_i^j = \{\mu_j\}$ for $i = 1$ and $j \in J$. This family of ergodic attractors verify all stated conditions.

To prove the converse statement let us first apply Theorem 5.5 to the chains $\{\mathcal{O}_i, i \in \widehat{I} \subset I\}$, for each fixed $j \in J$ such that $\lim_{i \to supI} s_i^j > 0$. Then each of such chains has a intersection $\{\mu_j\}$ where μ_j is a SRB measure.

For $j \neq k$ the basins of attraction of μ_j and μ_k are Lebesgue almost disjoint, because its Lebesgue measure is $\lim s_i^{j,k} = 0$. Thus $\mu_j \neq \mu_k$. Finally consider the co-chain $\{\{\mu_j\}, j \in J\}$ and apply Theorem 5.7. $\square$

5.9 Proof of Theorem 4. We shall prove Theorem 4 in the following complete version, that gives an upper bound to the number of generalized independent ergodic attractors in which the space $\mathcal{P}$ can be decomposed, and besides states a sufficient condition for the decomposition be unique:

Theorem 5.10 (Decomposition into generalized independent ergodic attractors.)
Any continuous map $f : M \mapsto M$ has a collection S formed by (up to countable many) *pairwise independent generalized ergodic attractors* (that are not necessarily irreducible) *whose basins of attraction cover Lebesgue almost all points in M.*

The supreme a of the number of such generalized ergodic attractors verifies $a \leq k$ (where k is the supreme of the lengths of the co-chains of reductions in the space of the observable measures for f).

If there exists such a collection S whose generalized ergodic attractors are besides all irreducible, then such S is unique and besides $a = k = l$, where l is the cardinality of S.

Proof: To prove the first and second statements note that, by definition of the independence of the reductions, any co-chain P of reductions of the space $\mathcal{O}$ of the observable measures, verifies $\sum s_j \leq 1$, where s_j denotes the attracting size of the reduction $\mathcal{O}_j \in P$. Now take the family $\mathcal{F}$ of all the co-chains S such that $\sum s_j = 1$. (There exists always at least one such co-chain: in fact the length-1 co-chain $\{\mathcal{O}\}$ verifies $\sum s_j = s(\mathcal{O}) = 1$, due to Theorem 3.1). By construction each $S \in \mathcal{F}$ verifies the wanted conditions. As the family $\mathcal{F}$ is a subfamily of all the co-chains of reductions, we obtain $a \leq k$.

Let us prove now the last assertions of the theorem. If there exists in $\mathcal{F}$ a co-chain $S = \{\widehat{\mathcal{O}}_h : h \in H \subset I\!N\}$ with cardinality $l = \#H$ and whose reductions $\widehat{\mathcal{O}}_h$ are all irreducible, to prove that $a = k = l$ it is enough to show than $l \geq k$.

Take any co-chain $P = \{\mathcal{O}_j,\ j \in J \subset I\!N\}$ (P is not necessarily in $\mathcal{F}$).

It is enough to exhibit a injective application from each $j \in J$ to some $h \in H$.

In fact, let us fix any $j \in J$ and consider the basin of attraction $C(\mathcal{O}_j)$. By definition of reduction, this basin has positive Lebesgue measure $m(C(\mathcal{O}_j))$. But $S \in \mathcal{F}$, so $\sum_{h \in H} m(C(\widehat{\mathcal{O}}_h)) = 1$. Then we deduce that

$$0 < m(C(\mathcal{O}_j)) = \sum_{hinH} m\left(C(\mathcal{O}_j) \cap C(\widehat{\mathcal{O}}_h)\right) = \sum_{hinH} m\left(C(\mathcal{O}_j \cap \widehat{\mathcal{O}}_h)\right)$$

Therefore some of the intersections in the sum above at right has positive Lebesgue measure. We obtain that for all $j \in J$ there exist some $h = h(j) \in H$ such that $\mathcal{O}_j \cap \widehat{\mathcal{O}}_h$ is a reduction. But as $\widehat{\mathcal{O}}_h$ is irreducible then $\mathcal{O}_j \supset \widehat{\mathcal{O}}_h$.

To end the proof it remains to show that for $j \neq i \in J$ the sets $\widehat{\mathcal{O}}_{h(i)}$ and $\widehat{\mathcal{O}}_{h(j)}$ in S are different reductions. By contradiction, if they were the same reduction in S, they both would be contained in two different reductions $\mathcal{O}_i$ and $\mathcal{O}_j$ in the chain P and therefore these two last reductions would not be independent and P would not be a co-chain.

Let us prove now the unicity of S, if there exists one, such that $S = \{\widehat{\mathcal{O}}_h : h \in H \subset I\!N\} \in \mathcal{F}$ and $\mathcal{O}_h$ are all irreducible. If there were two such collections S_1 and S_2, then repeating the construction above in this proof with $S = S_1$ and $P = S_2$ we conclude any $\mathcal{O}_j \in S_2$ contains one and only one $\widehat{\mathcal{O}}_{h(j)} \in S_1$. But as all $\mathcal{O}_j \in S_2$ are irreducible we must have $\widehat{\mathcal{O}}_{h(j)} = \mathcal{O}_j \in S_1$. So $S_2 \subset S_1$. The symmetric relation is obtained taking $S = S_2$ and $P = S_1$. $\square$

6 Observable measures for Lebesgue preserving maps.

Let us consider $f : M \mapsto M$ preserving the Lebesgue measure m. We will prove Theorem 5.

Lemma 6.1 *If the Lebesgue measure m is f-invariant then $\mu \in \mathcal{P}$ is observable if and only for any $\epsilon > 0$ the set of points $x \in M$ such that $p\omega(x) = \{\mu_x\} \subset B_\epsilon(\mu)$ has positive Lebesgue measure.*

Proof: If the Lebesgue measure m is f-invariant, due to Birkhoff theorem and separability of the space $C(M)$, the sequence [1] is convergent to some measure μ_x for m-almost every point $x \in M$: in other words $p\omega(x) = \{\mu_x\} \subset \mathcal{P}$. Then $p\omega(x) \cap B_\epsilon(\mu) \neq \emptyset$ if and only if $p\omega(x) \subset B_\epsilon(\mu)$. $\square$

Theorem 6.2 (Ergodicity and observability I)
The Lebesgue measure m is f-invariant and ergodic if and only if m is the unique observable probability measure.

Proof: Due to the Theorems of Birkhoff and of Ergodic decomposition of invariant measures (see [M89]), the Lebesgue measure m is ergodic if and only if the sequence [1] converges to m for m-almost every point $x \in M$. This last condition occurs, due to Definition 2.5 and Lemma 6.1, if and only if m is the unique measure that is ϵ-observable for all $\epsilon > 0$. $\square$

Remark 6.3 In Section 3 we give the example 4.3 that does not preserve the Lebesgue measure m, but for which there exists a unique observable measure μ, that is singular respect to m. When f preserves the Lebesgue measure the situation of the example 4.3 can not occur. In fact, we have the following result:

Theorem 6.4 (Ergodicity and observability II)
If $f : M \mapsto M$ is continuous, preserves the Lebesgue probability measure m, and has a unique observable measure μ, then $\mu = m$ (and thus m is ergodic due to Theorem 6.2).

Proof: From Lemma 6.1, Definitions 2.5 and 2.7 and Theorem 3.1 we deduce that $p\omega(x) = \{\mu\} \quad m$-a.e..

The Ergodic Decomposition Theorem of invariant measures (see [M89]) applied to m states that

$$\text{(6)} \qquad \int g\, dm = \int_{x\in M} dm(x) \int_{y\in M} g(y)\, d\mu_x(y) \ \text{ for all } g \in L^1(m),$$

(in particular for all $g \in C(M)$), where $\mu_x = \lim \frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j(x)}$ with the weak* topology of the space $\mathcal{P}$ of Borel probabilities in M.

In our case $\mu_x = \mu$ m-a.e. and $m(M) = 1$. Therefore the integral at right in (6) is constantly m-a.e. equal to $\int g\, d\mu$. Therefore, from (6) we deduce:

$$\int g\, dm = \int g\, d\mu \qquad \text{for all } g \in C(M)$$

from where we deduce $g = \mu$. $\square$

6.5 End of the proof of Theorem 5:

It is a direct consequence of joining Definition 1.2 and Theorems 6.2 and 6.4. □

The following is a straightforward characterization of SRB measures in terms of its observability size, which coincides in the case of Lebesgue-preserving maps with the attracting size (see Definitions 2.8 and 3.3):

Theorem 6.6 (Observability size and SRB measures.)

Let f be a continuous map in a compact manifold M *preserving the Lebesgue measure* m.

A probability measure μ *is a SRB measure for* f *if and only if it is observable and its observability size function* $o(\epsilon)$ *verifies*

$$\lim_{\epsilon \to 0^+} o(\epsilon) > 0.$$

Besides, the limit above is the attracting size of $\{\mu\}$.

Proof: The converse assertion is immediate from Definitions 1.1, 2.8, 1.2 and 3.3.

To prove the direct result let us call $\alpha = \lim_{\epsilon \to 0^+} o(\epsilon) > 0$. Note that

$$m\{x \in M : \mu \in p\omega(x)\} = \alpha > 0 \qquad [7]$$

Then, by Lemma 6.1, a given μ verifies $\mu \in p\omega(x)$ if and only if $p\omega(x) = \{\mu\}$. Now from Definition 1.2 and from [7] we deduce that μ is a SRB measure. □

6.7 Proof of Theorem 6:

The SRB measures, if there exist some, are by Definition 3.2, (trivially) irreducible generalized ergodic attractors. Let us see that for Lebesgue preserving maps the converse assertion is also true: all irreducible attractors are trivial, and thus SRB measures.

If $\mathcal{A}$ is irreducible, then any proper compact subset $\mathcal{K} \subset \mathcal{A}$ verifies $m(C(\mathcal{K})) = 0$, where $C(\mathcal{K})$ denotes the basin of attraction of $\mathcal{K}$. Let us call C to the basin of attraction of $\mathcal{A}$ and apply 3.5: It is not restrictive to suppose that $m(C) = 1$ and that $\mathcal{A}$ is the space of all observable measures for f (if not, substitute f by $f|_C$ and rescale properly the measure m).

Take μ and ν in $\mathcal{A}$. We shall prove that $\mu = \nu$. By contradiction, suppose that $\epsilon = (1/2)\,\mathrm{dist}\,(\mu, \nu) > 0$. Then, by the definition of observability (see 2.5 and 2.7) and applying Lemma 6.1 and Proposition 3.5, we construct the following set $C(\epsilon, \mu)$ with positive Lebesgue measure:

$$C(\epsilon, \mu) = \{x \in C : p\omega(x) = \{\mu_x\} \subset B_\epsilon(\mu)\}\,; \qquad m(C(\epsilon, \mu)) > 0$$

Let us consider the compact set $\mathcal{K} \subset \mathcal{A}$ defined as: $\mathcal{K} = \mathcal{A} \cap B_\epsilon(\mu)$. We have that $\nu \notin \mathcal{K}$, and thus $\mathcal{K}$ is a compact proper subset of $\mathcal{A}$. Besides, the basin of attraction $C(\mathcal{K})$ verifies:

$$C(\mathcal{K}) = \{x \in A : p\omega(x) = \{\mu_x\} \subset \mathcal{A} \cap B_\epsilon(\mu)\} = C \cap C(\epsilon, \mu) = C(\epsilon, \mu),$$

which implies $m(C(\mathcal{K})) > 0$ contradicting that $\mathcal{A}$ is irreducible.

To prove that the unique $\mu \in \mathcal{A}$ is SRB and an ergodic component of m, just apply Proposition 3.5 and then, taking $f|C$ instead of f, apply Theorems 6.2 and 6.4.□

7 Appendix.

7.1 A physical meaning of observable and SRB measures.

For any open region $V \subset M$ in the phase space, the sequence of exact average time of staying in V in the future, from the initial state x is:

$$(2) \qquad \frac{\#\{0 \leq j < n : f^j(x) \in V\}}{n} = \frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j(x)}(V) =$$

$$= \frac{1}{n}\sum_{j=0}^{n-1} \chi_V(f^j(x)), \quad \text{for any open set } V \subset M$$

where χ_V denotes the characteristic function of V, i.e. $\chi_V(y) = 1$ if $y \in V$ and $\chi_V(y) = 0$ if $y \notin V$.

To take the weak* limit of the sequence [1] of time average distributions, stated at the beginning of this paper, we should integrate that sequence, not on the characteristic functions χ_V as done in (2), but on the continuous functions $g : M \mapsto R$. In fact, the limit in the weak* topology of the sequence [1] is given by the limits, when $n \to +\infty$, of the following sequence:

$$(3) \qquad \frac{1}{n}\sum_{j=0}^{n-1} g(f^j(x)) \quad \text{for any real continuous function } g \in C(M)$$

The sequence (2) of *exact* time average (average staying time in the region V of the space) gives the physical or nature meaning to SRB measures. But the exact sequence (2) *is different* from the sequence (3). And to define SRB measure (and also observable measure) is used the limit of (3), instead that of (2).

The problem above is usually irrelevant to the physical understanding of the SRB measures μ as nature measures, because of the following basic theorem of Measure Theory, that allows approximate χ_V with a continuous function g:

Theorem 7.2 (Regularity of the measures.) *For any given open set* $V \subset M$ (where M is for instance a finite dimensional manifold with Lebesgue measure m), *and for any given* $\epsilon > 0$, *there exists a continuous function* $g : M \mapsto [0, 1]$ *such that:*

$g(x) = 0$ if $x \notin V$, $g(x) = 1$ if $x \in K$ for some compact subset $K \subset V$ such that the interior of K is not empty, and

$$m(\{x \in M : |g(x) - \chi_V(x)| < \epsilon\}) > 1 - \epsilon$$

Proof: See Urysohn Lemma 4.32 and Theorem 7.8 of the book in [F84].

The inequality of Theorem 7.2 implies:

$$(4) \qquad m(\{x \in M : |g(f^j(x)) - \chi_V(f^j(x))| < \epsilon\}) > 0 \text{ for all } j \geq 0$$

because the function g and χ_V are both equal to 1 in the non-empty interior of some compact set $K \subset V$.

Therefore, the set

$$C_{\epsilon,V,n} = \left\{ x \in M : \left| \frac{1}{n} \sum_{j=0}^{n-1} g(f^j(x)) - \frac{1}{n} \sum_{j=0}^{n-1} \chi_V(f^j(x)) \right| < \epsilon \right\}$$

has positive Lebesgue measure $m(C_{\epsilon,V,n}) > 0$ for all $\epsilon > 0$. This is the mean reason to override the difference between sequences (2) and (3) and provide a nature or physical meaning to SRB measures as average time of staying in the regions V of the space.

But even when existing a SRB measure μ it may be verified that

$$m(C_{\epsilon,V,n}) \to 0^+ \quad \text{when } \epsilon \to 0$$

In fact, if the topological boundary of V supports the measure μ, then $\mu(V)$ may be very different from the limit of the exact sequence (2) of the time average staying in V, for Lebesgue almost all initial states $x \in M$. In example 4.3 the SRB measure is δ_{P_0}, supported in a point P_0. If V is a open set whose boundary contains P_0 then the sequence (2) converges to 1 for Lebesgue almost all points, while the sequence (3) converges to 0 (anyhow chosen the continuous map g approximating χ_V as in Theorem 7.2).

In resume: the set $C_{\epsilon,V,n}$ certainly has a positive Lebesgue measure for all $\epsilon > 0$, and this is the mathematical reason to assign a physical or nature meaning to the limit probability μ, in spite that may $m(C_{\epsilon,V,n}) \to 0$ when $\epsilon \to 0$.

Inspired on these arguments, we defined the *observable measures* in 1.1 using the $m(A_\epsilon) > 0$ condition for all $\epsilon > 0$ (instead $m(A_0) > 0$ for $\epsilon = 0^+$ at once). We think that the definition 1.1 of the observable measures generalizes the definition 1.2 of the SRB measures preserving the essence of their physical or nature meaning.

7.3 Proof of Theorem 2.4.

We shall prove the following: (i) If $\mu, \nu \in p\omega(x)$ then for each real number $0 \leq \lambda \leq 1$ there exists a measure $\mu_\lambda \in p\omega(x)$ such that

$$\text{dist}\,(\mu_\lambda, \mu) = \lambda\, \text{dist}\,(\mu, \nu)$$

(ii) The set $p\omega(x)$ either has a single element or non-countable infinitely many.

Proof: First let us deduce (ii) from (i): Suppose that $p\omega(x)$ has at least two different values μ and ν. It is enough to note that the application $\lambda \in [0,1] \mapsto \mu_\lambda \in p\omega(x)$ that verifies thesis (i), is injective. Therefore $p\omega(x)$ has non-countable infinitely many elements.

To prove (i) consider the sequence [1] $\mu_n = \left\{ \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)} \right\}_{n \in \mathbb{N}}$ of time averages. Either it is convergent, or has at least two convergent subsequences, say $\mu_{m_j} \to \mu$ and $\mu_{n_j} \to \nu$, with $\mu \neq \nu$.

(iii)It is enough to exhibit in the case $\mu \neq \nu$ a convergent subsequence of [1] whose limit μ_λ verifies the thesis (i).

Assertion A: *For any given $\epsilon > 0$ and $K > 0$ there exists a natural number $h = h(\epsilon, K) > K$ such that*

$$|\,\text{dist}\,(\mu_h, \mu) - \lambda\, \text{dist}\,(\nu, \mu)| \leq \epsilon$$

Let us first prove that Assertion A implies thesis (i): Take in assertion A: $h_0 = 1$ and by induction, for $j \geq 1$ take h_j given $\epsilon_j = 1/j$ and $K_j = h_{j-1}$. Then we obtain a sequence μ_{h_j}, subsequence of (1), that verifies $\text{dist}(\mu_{h_j}, \mu) \to \lambda(\text{dist}(\nu, \mu))$. Any convergent subsequence of μ_{h_j} (that do exist $\mathcal{P}$ is compact in the weak* topology) verify (iii).

Now, let us prove Assertion A:

As $\mu_{m_j} \to \mu$ and $\mu_{n_j} \to \nu$ let us choose first m_j and then n_j such that

$$m_j > K; \quad \frac{1}{m_j} < \epsilon/4; \quad \text{dist}(\mu, \mu_{m_j}) < \epsilon/4; \quad n_j > m_j; \quad \text{dist}(\nu, \mu_{n_j}) < \epsilon/4$$

To exhibit the computations let us explicit some metric structure giving the weak* topology of $\mathcal{P}$. We will use for instance the following distance:

$$\text{dist}(\rho, \delta) = \sum_{i=1}^{\infty} \frac{1}{2^i} \left| \int g_i \, d\rho - \int g_i \, d\delta \right|$$

for any $\rho, \delta \in \mathcal{P}$, where $\{g_i\}_{i \in \mathbb{N}}$ is a countable set of functions $g_i \in C(M)$ such that $|g_i| \leq 1$, dense in the unitary ball of $C(M)$.

Note from the sequence (1) that $|\int g \, d\mu_n - \int g \, d\mu_{n+1}| \leq (1/n)||g||$ for all $g \in C(M)$ and all $n \geq 1$. Then in particular for $n = m_j + k$, we obtain

$$(*) \qquad \text{dist}(\mu_{m_j+k}, \mu_{m_j+k+1}) \leq \frac{1}{m_j} < \epsilon/4 \quad \text{for all } k \geq 0$$

Now let us choose a natural number $0 \leq k \leq n_j - m_j$ such that

$$\left| \text{dist}(\mu_{m_j}, \mu_{m_j+k}) - \lambda \, \text{dist}(\mu_{m_j}, \mu_{n_j}) \right| < \epsilon/4 \quad \text{for the given } \lambda \in [0,1]$$

Such k does exist because inequality (*) is verified for all $k \geq 0$ and besides:

If $k = 0$ then $\text{dist}(\mu_{m_j}, \mu_{m_j+k}) = 0$ and if $k = n_j - m_j$ then $\text{dist}(\mu_{m_j}, \mu_{m_j+k}) = \text{dist}(\mu_{m_j}, \mu_{n_j})$

Now renaming $h = m_j + k$, joining all the inequalities above, and applying the triangular property, we deduce:

$$|\text{dist}(\mu_h, \mu) - \lambda \, \text{dist}(\nu, \mu)| \leq \left| \text{dist}(\mu_h, \mu_{m_j}) - \lambda \, \text{dist}(\mu_{m_j}, \mu_{n_j}) \right| +$$
$$+ \left| \text{dist}(\mu_h, \mu) - \text{dist}(\mu_h, \mu_{m_j}) \right| +$$
$$+\lambda \left| \text{dist}(\mu_{m_j}, \mu_{n_j}) - \text{dist}(\mu_{m_j}, \nu) \right| + \lambda \left| \text{dist}(\mu_{m_j}, \nu) - \text{dist}(\mu, \nu) \right| < \epsilon \qquad \Box$$

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