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Advances in Mathematics 182 (2004) 278–306
ADVANCES IN Mathematics
http://www.elsevier.com/locate/aim

Genericity of wild holomorphic functions and common hypercyclic vectors

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Received 6 June 2002; accepted 25 November 2002

Communicated by Virgil Voiculescu

Abstract

Let T_α be the translation operator by α in the space of entire functions $\mathscr{H}(\mathbb{C})$ defined by $T_\alpha(f)(z) = f(z+\alpha)$. We prove that there is a residual set G of entire functions such that for every $f \in G$ and every $\alpha \in \mathbb{C}\setminus\{0\}$ the sequence $T_\alpha^n(f)$ is dense in $\mathscr{H}(\mathbb{C})$, that is, G is a residual set of common hypercyclic vectors (functions) for the family $\{T_\alpha : \alpha \in \mathbb{C}\setminus\{0\}\}$. Also, we prove similar results for many families of operators as: multiples of differential operator, multiples of backward shift, weighted backward shifts.

MSC: 47A16(47B38); 30D20

Keywords: Entire functions; Hypercyclic operators; Hypercyclic and universal vectors; Residual

1. Introduction

During the first half of the last century, Birkhoff [B] and MacLane [M] showed that certain entire functions can approximate *any* entire one under a suitable limiting process. Specifically, Birkhoff constructed an entire function f so that any entire one can be obtained as the limit (uniformly on compact sets) of translates $f(z+c_n)$ for some sequence c_n; and MacLane proved the existence of an entire function so that the sequence of its derivatives is dense in the space $\mathscr{H}(\mathbb{C}) =$

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0001-8708/$ - see front matter
doi:10.1016/S0001-8708(03)00079-3

$\{f : \mathbb{C} \to \mathbb{C}$ holomorphic$\}$ endowed with the usual topology of uniform convergence on compact sets.

Both results can be regarded as examples of a phenomenon that has been a concern of operator theorists during the last decades, namely the notion of hypercyclicity. Let recall the definition: a continuous linear operator $T : X \to X$ acting on a topological vector space X is called *hypercyclic* provided there is a vector x whose orbit $\{T^n(x),\ n = 1, 2, \ldots\}$ is dense in X, and such a vector x is said to be a hypercyclic vector for T. It turns out, by a simple Baire's categorical argument that the set of hypercyclic vectors (denoted by $\mathscr{HC}(T)$) for a hypercyclic operator T is residual, i.e. is G_δ-dense (see the survey [G]).

From the above point of view, Birkhoff and MacLane's theorems can be restated as follows: the translation operator $T_\alpha : \mathscr{H}(\mathbb{C}) \to \mathscr{H}(\mathbb{C}),\ \alpha \neq 0$ defined by $T_\alpha(f)(z) = f(z + \alpha)$ and the differentiation operator $D : \mathscr{H}(\mathbb{C}) \to \mathscr{H}(\mathbb{C}),\ D(f) = f'$ are hypercyclic (let us point out that hypercyclic functions—vectors—for these operators are also called universal functions).

In particular, in case of Birkhoff's result, we have a noncountable family of operators $\{T_\alpha : \mathscr{H}(\mathbb{C}) \to \mathscr{H}(\mathbb{C}),\ \alpha \in \mathbb{C} \setminus \{0\}\}$ where each one of these operator is hypercyclic. A natural question is whether this family share a hypercyclic vector. In other words: does there exist an entire function f so that $\{T_\alpha^n(f),\ n \geqslant 1\}$ is dense in $\mathscr{H}(\mathbb{C})$ for *every* $\alpha \in \mathbb{C} \setminus \{0\}$? The next theorem gives a positive answer. Moreover, it says that the set of functions enjoying this property is in fact generic (i.e. contains a G_δ-dense set).

Theorem 1. *There is a G_δ-dense set $G \subset \mathscr{H}(\mathbb{C})$ such that $G \subset \mathscr{HC}(T_\alpha)$ for every $\alpha \in \mathbb{C} \setminus \{0\}$.*

Let us observe that any function $f \in G$ has a "wild behavior at ∞" in any direction, more precisely any entire function g can be approximated on any compact set by translating f in any direction. Furthermore, by Hurwitz's theorem, it holds that $f(\{w + z : \theta - \varepsilon < \arg(z) < \theta + \varepsilon\}) = \mathbb{C}$ for every $w \in \mathbb{C}, 0 \leqslant \theta \leqslant 2\pi$ and $\varepsilon > 0$ (in particular every ray is a Julia ray). Also, the range of any half line is dense in $\mathbb{C}$ and the sequence $\{f(z + n\alpha); n \geqslant 0\}$ is dense in the complex plane for every $z \in \mathbb{C}, \alpha \in \mathbb{C} \setminus \{0\}$ as well.

Motivated by the above theorem, we may ask if a similar phenomenon can occur in case of other noncountable families of hypercyclic operators. As we mentioned before, MacLane's theorem says that the differentiation operator $D : \mathscr{H}(\mathbb{C}) \to \mathscr{H}(\mathbb{C})$ is hypercyclic. It turns out that the operators $\lambda D : \mathscr{H}(\mathbb{C}) \to \mathscr{H}(\mathbb{C})$ defined by $\lambda D(f)(z) = \lambda f'(z)$ are hypercyclic for $\lambda \in \mathbb{C} \setminus \{0\}$ as well (see [GoS]). We will show that the above family λD of differentiation operators admits a G_δ-dense set of common hypercyclic vectors. However, we would like to view it from another perspective. Consider the family of modulations on $\mathscr{H}(\mathbb{C})$ defined by $R_\lambda : \mathscr{H}(\mathbb{C}) \to \mathscr{H}(\mathbb{C}),\ R_\lambda(f)(z) = f(\lambda z)$ and we ask if there is an entire function f so that $R_\lambda(f)$ is universal (hypercyclic) for the operator D for every $\lambda \in \mathbb{C} \setminus \{0\}$. In other words, is there a function f so that the sequence $\{D^n \circ R_\lambda(f) : n \geqslant 1\}$ is dense in $\mathscr{H}(\mathbb{C})$ for every $\lambda \in \mathbb{C} \setminus \{0\}$? We will provide a positive answer to this question but before let us introduce some notation. A vector $x \in X$ will be called universal for a sequence of operators $T_n : X \to X$, if the sequence $\{T_n(x),\ n = 1, 2, \dots\}$ is dense in X. The set of universal vectors will be denoted by $\mathscr{U}(\{T_n : n \geqslant 1\})$.

Theorem 2. *For every $\lambda \in \mathbb{C}\backslash\{0\}$ consider the sequence of operators $T_{n,\lambda} = D^n \circ R_\lambda$. Then, there is a G_δ-dense set $F \subset \mathscr{H}(\mathbb{C})$ such that $F \subset \mathscr{U}(\{T_{n,\lambda} : n \geqslant 1\})$ for any $\lambda \in \mathbb{C}\backslash\{0\}$. In other words for any $\lambda \in \mathbb{C}\backslash\{0\}$ and $f \in F$ the sequence of (entire) functions $h_n(z) = \lambda^n f^{(n)}(\lambda z)$ is dense in $\mathscr{H}(\mathbb{C})$.*

We observe that any automorphism R in $\mathscr{H}(\mathbb{C})$ sends a dense set to a dense one. Therefore a G_δ dense set of common hypercyclic vectors for the family of operators λD is obtained from the above theorem and the identity $(\lambda D)^n = R_\lambda^{-1} \circ D^n \circ R_\lambda$. It is also interesting to note that a common hypercyclic entire function f for the family of operators λD has the next property: not only the sequence $\{f^{(n)}(z)\}$ is dense in the complex plane for any point z, but the sequence $\{\lambda^n f^{(n)}(z)\}$ is dense in $\mathbb{C}$ for any $\lambda \in \mathbb{C}\backslash\{0\}$ as well.

Notice that if f is hypercyclic for λD (respec. T_α) then f' is also hypercyclic for λD (respec. T_α). Therefore, by Baire's category theorem and the above two theorems we conclude the following corollary about genericity of "wild entire functions":

Corollary 3. *There is a G_δ dense set $G \subset \mathscr{H}(\mathbb{C})$ so that for any $f \in G$ and any $j \in \mathbb{N}$ we have*

$$f^{(j)} \in \bigcap_{\alpha, \lambda \in \mathbb{C}\backslash\{0\}} \mathscr{HC}(T_\alpha) \cap \mathscr{HC}(\lambda D).$$

Equivalently, for any $f \in G$, for any $\lambda, \alpha \in \mathbb{C}\backslash\{0\}$ and every $j \in \mathbb{N}$ the following hold:

(1) $\overline{\{f^{(j)}(z + n\alpha) : n \geqslant 0\}} = \mathscr{H}(\mathbb{C})$.
(2) $\overline{\{\lambda^n f^{(j+n)}(z) : n \geqslant 0\}} = \mathscr{H}(\mathbb{C})$.

Although the proofs of Theorems 1 and 2 share some similar ideas, they are based in quite different tools. A basic tool in the proof of Theorem 1 is Runge's theorem. On the other hand, to prove Theorem 2 we use essentially the fact that the family shares a common set where the hypercyclicity criterion applies (see [GS] for a statement of this criterion). In fact, we will derive Theorem 2 from a more general theorem that can be applied to other families of operators in order to obtain common hypercyclic vectors. For instance, we will apply it to families of multiples backward shift (see Theorem 4 below) and weighted backward shifts (see Section 6). However, for the sake of clarity and simplicity of the introduction, we postpone the statement of it to Section 3.

In the context of linear operators in Banach spaces, Rolewicz [R] was the first who realized that hypercyclic operators exist in this setting. Although the backward shift T acting on $\ell^2 = \ell^2(\mathbb{N})$ (the Hilbert space of square summable sequences), defined by $T(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots)$ is not hypercyclic, Rolewicz proved that the multiple of the backward shift λT, $|\lambda| > 1$ is indeed hypercyclic. Salas [S] asked if this family admits a common hypercyclic vector. Recently Abakumov and Gordon [AG] settled this question. Their proof consists of a clever construction of a common hypercyclic vector for this class of operators (A. Peris informed us that this result has been obtained independently by him—unpublished—[P]). As a consequence of our methods we give a nonconstructive proof of this fact by showing that the set of common hypercyclic vectors is residual. Let us point out an interesting aspect of the proof: if x is hypercyclic for λT then it is also hypercyclic for $\lambda e^{2\pi i\theta} T$ for every θ (see Theorem 16).

Theorem 4. *There is a G_δ-dense set $R \subset \ell^2$ such that $R \subset \mathscr{HC}(\lambda T)$ for every complex number λ, $|\lambda| > 1$.*

Let us turn our attention back to entire functions. In view of Theorem 1 above, the function f described in the following theorem can be considered "pathological":

Theorem 5. *Let $T_1 : \mathscr{H}(\mathbb{C}) \to \mathscr{H}(\mathbb{C})$ be the translation operator by* 1, *that is $T_1(f)(z) = f(z+1)$. Then, there exists an entire function $f \in \mathscr{H}(\mathbb{C})$ hypercyclic for T_1 which also satisfies*

$$\lim_{r \to \infty} f(z + re^{2\pi i\theta}) = 0 \quad \forall \theta, \ 0 < \theta < 1, \ \forall z \in \mathbb{C}.$$

Moreover, the above limit holds uniformly on $\{\theta : \varepsilon \leqslant \theta \leqslant 1 - \varepsilon\}$ and z in a compact set.

We may ask if a function as in the above theorem can be universal for the differentiation operator. We conjecture that this is not the case. Moreover we ask if a universal (i.e. hypercyclic) entire function for the operator D can be bounded on a sector. Also, does there exist a universal function for D for which not every ray is a Julia ray?

However, in the opposite direction we can specify the behavior along a "large" set of rays:

Theorem 6. *Denote by S^1 the unit circle. Given any closed nowhere dense set $E \subset S^1$, such that $1 \notin E$, there exists an entire function $f \in \mathscr{H}(\mathbb{C})$ which is hypercyclic for both T_1 and D and satisfies*

$$\lim_{r \to \infty} f(re^{2\pi i\theta}) = 0 \quad \textit{for every } e^{2\pi i\theta} \in E.$$

The paper is organized as follows: Theorem 1 is proved in Section 2; a sufficient criterion for the existence of common hypercyclic vectors is given in Section 3; in Section 4 we prove Theorem 2; in Section 5 we include the proof of Theorem 4 and in Section 6 we deal with families of weighted backward shifts. Theorems 5 and 6 are proved in Section 7. In the last section we show that the set of common hypercyclic vectors for a wide class of families is either empty or residual and we also provide some remarks and problems.

2. Proof of Theorem 1

The proof of Theorem 1 consists of two steps. First, by using categorical arguments we show that there is a G_δ-dense set of common hypercyclic functions for a "one dimensional" subfamily. Secondly, we show that the above G_δ-dense set satisfies the conclusion of Theorem 1. This is done by an argument which is interesting by itself and a main tool is the minimality of the irrational rotation. Indeed, we will prove the following two theorems:

Theorem 7. *There is a G_δ-dense set $G \subset \mathscr{H}(\mathbb{C})$ such that $G \subset \mathscr{HC}(T_{e^{2\pi i\theta}})$ for every $\theta, 0 \leqslant \theta \leqslant 1$.*

Theorem 8. *Let f be hypercyclic for $T_{e^{2\pi i\theta}}$ for some θ. Then f is also hypercyclic for $T_{re^{2\pi i\theta}}$ for any positive real number r.*

Before proving these two theorems, let us explain how to obtain Theorem 1. In fact, the set G provided by Theorem 7 satisfies the conclusion of Theorem 1, since any function $f \in G$ is hypercyclic for $T_{e^{2\pi i\theta}}$ for every $\theta, 0 \leqslant \theta \leqslant 1$ and, by Theorem 8, it is also hypercyclic for $T_{re^{2\pi i\theta}}$ where r is any real positive number, that is, f is hypercyclic for T_α for any $\alpha \in \mathbb{C} \setminus \{0\}$.

2.1. *Proof of Theorem 7*

Let $\{\phi_j : j \geqslant 1\}$ be dense in $\mathscr{H}(\mathbb{C})$. Consider the set

$$E(s,j,k,m) = \left\{ f \in \mathscr{H}(\mathbb{C}) : \forall \theta, 0 \leqslant \theta \leqslant 1 \quad \exists n = n(\theta) \leqslant m \right.$$
$$\left. \text{so that } \sup_{|z| \leqslant k} |f(z + ne^{2\pi i\theta}) - \phi_j(z)| < \frac{1}{s} \right\}.$$

The proof of Theorem 7 will be based on the following two lemmas:

Lemma 9. *The set $E(s,j,k,m)$ is open in $\mathscr{H}(\mathbb{C})$ for every s,j,k,m.*

Lemma 10. *The set $\bigcup_{m \geqslant 1} E(s,j,k,m)$ is dense in $\mathscr{H}(\mathbb{C})$ for every s,j,k.*

Before proving these two lemmas, let us explain how to conclude Theorem 7. It follows, by Baire's category theorem, that the set

$$G = \bigcap_s \bigcap_j \bigcap_k \bigcup_m E(s,j,k,m)$$

is G_δ-dense in $\mathscr{H}(\mathbb{C})$. It is easy to see that G satisfies the conclusion of Theorem 7. However, for the sake of completeness we include the proof. Let $f \in G$ and fix $\theta,\ 0 \leqslant \theta \leqslant 1$. We want to show that f is hypercyclic for $T_{e^{2\pi i\theta}}$. Thus, let any $g \in \mathscr{H}(\mathbb{C})$, a compact set L and $\varepsilon > 0$ be given. Take $s \geqslant 1, \frac{1}{s} < \frac{\varepsilon}{2}$ and k such that $L \subset \{|z| \leqslant k\}$. Moreover, choose ϕ_j so that $\sup_{|z| \leqslant k} |g(z) - \phi_j(z)| < \frac{\varepsilon}{2}$. Since $f \in G$ then $f \in E(s,j,k,m)$ for some m. Therefore, there is $n = n(\theta) \leqslant m$ such that

$$\begin{aligned}\sup_{|z|\leqslant k} |T^n_{e^{2\pi i\theta}}(f)(z) - g(z)| &= \sup_{|z|\leqslant k} |f(z + ne^{2\pi i\theta}) - g(z)| \\ &\leqslant \sup_{|z|\leqslant k} |f(z + ne^{2\pi i\theta}) - \phi_j(z)| + \sup_{|z|\leqslant k} |\phi_j(z) - g(z)| < \varepsilon.\end{aligned}$$

Let us proceed with the proof of the lemmas.

Proof of Lemma 9. Let $f \in E(s,j,k,m)$ and denote by S^1 the unit circle. Consider the sets

$$C_l = \left\{ \alpha \in S^1, \text{ so that } \sup_{|z|\leqslant k} |f(z + l\alpha) - \phi_j(z)| < \frac{1}{s} \right\}, \quad l = 1, \dots, m.$$

It is easy to see that C_l is open and $S^1 \subset \bigcup_{l=1}^m C_l$ (since $f \in E(s,j,k,m)$). That is, $\bigcup_{l=1}^m C_l$ is an open and finite covering of S^1. Hence, there are compact sets $I_l \subset C_l$ such that $S^1 \subset \bigcup_{l=1}^m I_l$. We will find ε so that the set $\{g \in \mathscr{H}(\mathbb{C}) : \sup_{|z|\leqslant k+m} |g(z) - f(z)| < \varepsilon\}$ (which is open in $\mathscr{H}(\mathbb{C})$) is contained in $E(s,j,k,m)$. For each $l = 1, \dots, m$, since I_l is compact we may find $\varepsilon_l > 0$ so that

$$\text{if } \sup_{|z|\leqslant k+m} |g(z) - f(z)| < \varepsilon_l \text{ and } \alpha \in I_l \text{ then } \sup_{|z|\leqslant k} |g(z + l\alpha) - \phi_j(z)| < \frac{1}{s}.$$

So, choosing $\varepsilon < \min\{\varepsilon_l, 1 \leqslant l \leqslant m\}$ the proof of Lemma 9 is finished. □

Proof of Lemma 10. Let s,j,k be fixed. And let $g \in \mathscr{H}(\mathbb{C})$, a compact set C and $\varepsilon > 0$ be given. We want to show that there is $f \in \mathscr{H}(\mathbb{C})$ and $m \geqslant 1$ such that

$$f \in E(s,j,k,m) \tag{1}$$

and

$$\sup_{z \in C} |f(z) - g(z)| < \varepsilon. \tag{2}$$

Without loss of generality we may assume that $C \subset \{|z| \leqslant k\}$. In order to simplify the notation, set $\phi = \phi_j$. Let us choose $\delta < \frac{1}{2}$ such that

$$\text{if } |z| \leqslant k \text{ and } |z - w| < \delta \text{ then } |\phi(z) - \phi(w)| < \frac{1}{2s}. \tag{3}$$

Consider a partition $0=\theta_0<\theta_1<\cdots<\theta_l=1$ (to be chosen later). Set $t=2k+1$ and $B=\{|z|\leqslant k+\delta\}$. For $d=0,1,\dots,l$ define the sets

$$B_d = B + e^{2\pi i\theta_d}(d+1)t.$$

Notice that $B, B_0, B_1, \dots, B_l$ are pairwise disjoint. Define the function h on the compact set $R = B \cup \bigcup_{d=0}^{l} B_d$ (having connected complement) by

$$h(z) = \begin{cases} g(z) & z \in B, \\ \phi(z - e^{2\pi i\theta_d}(d+1)t) & z \in B_d \ \ d=0,1,\dots,l. \end{cases} \tag{4}$$

By Runge's theorem, there is an entire function f (polynomial) such that

$$\sup_{z\in R} |f(z)-h(z)| < \min\left\{\frac{1}{2s}, \varepsilon\right\}. \tag{5}$$

We will choose l and the partition $0=\theta_0<\theta_1<\cdots<\theta_l=1$ so that f is the desired function. First, by (4) and (5) we get

$$\sup_{z\in C} |f(z)-g(z)| \leqslant \sup_{z\in B} |f(z)-g(z)| < \varepsilon,$$

which implies (2) (no matter the length of the partition and the partition itself are). It remains to show (1). To this end, we will find l and $0=\theta_0<\theta_1<\cdots<\theta_l=1$ such that for $\theta,\ \theta_d\leqslant\theta<\theta_{d+1}$, we get that

$$\sup_{|z|\leqslant k} |f(z+(d+1)te^{2\pi i\theta}) - \phi(z)| < \frac{1}{s}. \tag{6}$$

If this is the case, setting $m=(l+1)t$ we get (1). In order to establish (1) let $\theta, \theta_d\leqslant\theta<\theta_{d+1}$ and assume for the moment that

$$|e^{2\pi i\theta_{d+1}} - e^{2\pi i\theta_d}|(d+1)t < \delta. \tag{7}$$

Then, for $|z|\leqslant k$, it follows that $z+(d+1)te^{2\pi i\theta}\in B_d$. Hence, for $|z|\leqslant k$,

$$\begin{aligned} &|f(z+(d+1)te^{2\pi i\theta}) - \phi(z)| \\ &\quad \leqslant |f(z+(d+1)te^{2\pi i\theta}) - \phi(z+(d+1)te^{2\pi i\theta} - (d+1)te^{2\pi i\theta_d})| \\ &\qquad + |\phi(z+(d+1)t(e^{2\pi i\theta} - e^{2\pi i\theta_d})) - \phi(z)|. \end{aligned}$$

Let us estimate the right-hand side of the above inequality. Firstly, by (4) and (5) we get

$$|f(z+(d+1)te^{2\pi i\theta}) - \phi(z+(d+1)te^{2\pi i\theta} - (d+1)te^{2\pi i\theta_d})| < \frac{1}{2s}.$$

And secondly, by (7) and (3) we have

$$|\phi(z+(d+1)t(e^{2\pi i\theta}-e^{2\pi i\theta_d}))-\phi(z)|<\frac{1}{2s}.$$

Thus, (6) follows. Therefore, it is only left to find l and $0=\theta_0<\theta_1<\cdots<\theta_l=1$ so that (7) holds for any $d=0,1,\dots,l-1$. Observe that

$$2\pi(\theta_{d+1}-\theta_d)(d+1)t<\delta \tag{8}$$

implies (7). Moreover, (8) is equivalent to

$$\theta_{d+1}-\theta_d<\frac{\delta}{2\pi(d+1)t}. \tag{9}$$

Setting $\beta_d=\theta_{d+1}-\theta_d$, (9) is transformed to

$$\beta_d<\frac{\delta}{2\pi(d+1)t}. \tag{10}$$

Thus, we need to find l positive numbers $\beta_0,\dots,\beta_{l-1}$ so that (10) holds and

$$\beta_0+\beta_1+\cdots+\beta_{l-1}=1. \tag{11}$$

So, choose $l\geqslant 1$ such that

$$\eta=\frac{\delta}{2\pi t}\left(1+\frac{1}{2}+\cdots+\frac{1}{l}\right)>1 \tag{12}$$

and, for $d=0,1,\dots,l-1$ set

$$\beta_d=\frac{1}{\eta}\left(\frac{\delta}{2\pi(d+1)t}\right).$$

Hence (10), (11) hold and the proof of Lemma 10 is finished. This completes the proof of Theorem 7. $\square$

2.2. *Proof of Theorem 8*

Let f be hypercyclic for $T_{e^{2\pi i\theta}}$ for some θ and let $r>0$. We shall conclude that f is also hypercyclic for $T_{re^{2\pi i\theta}}$. Let $g\in\mathscr{H}(\mathbb{C})$, a compact set L and $\varepsilon>0$ be given. It suffices to prove that there is some n so that

$$\sup_{z\in L}|f(z+nre^{2\pi i\theta})-g(z)|<\varepsilon.$$

We shall use a result due to Ansari (Theorem 1 in [A]). We would like to state it since we will use it repeatedly.

Theorem 11. *Let T be a hypercyclic operator and let n be any positive integer. Then T^n is also hypercyclic and moreover T, T^n have the same hypercyclic vectors.*

Continuing with the proof, we will distinguish two cases: r is rational or r is irrational.

In case $r = \frac{p}{q}$ is rational we consider $T^p_{e^{2\pi i\theta}}$. It follows from the above theorem that f is also hypercyclic for $T^p_{e^{2\pi i\theta}}$. Thus, there exists some positive integer m so that

$$\sup_{z\in L} |f(z + mpe^{2\pi i\theta}) - g(z)| < \varepsilon$$

and the result follows by the equality $mqre^{2\pi i\theta} = mpe^{2\pi i\theta}$.

Now, assume that r is irrational.

Let δ be such that

$$\text{if } |z| \in L \text{ and } |z - w| < \delta \text{ then } |g(z) - g(w)| < \frac{\varepsilon}{2}. \tag{13}$$

Set $L_\delta = \{z : d(z, L) \leqslant \delta\}$, and choose an integer $k > 2\sup_{z\in L_\delta}|z|$. In the sequel we denote by $[x]$ the integer part of the real number x and by $\{x\} = x - [x]$ the fractional part of x. It follows by the minimality of the irrational rotation by r/k [E] that there is a sequence of positive integers $n_1 < n_2 < \cdots$ such that

$$0 \leqslant \left\{ n_j \frac{r}{k} \right\} < \frac{\delta}{k} \tag{14}$$

and

$$\sup |n_{j+1} - n_j| < \infty. \tag{15}$$

Set $m_j = [n_j \frac{r}{k}]$, the integer part of $n_j \frac{r}{k}$. Then,

$$0 \leqslant n_j r - m_j k < \delta \tag{16}$$

and

$$\sup |m_{j+1} - m_j| < m \quad \text{for some integer } m. \tag{17}$$

Set $L^l_\delta = L_\delta + lke^{2\pi i\theta}$, $l = 1, \ldots, m-1$ and consider the compact set

$$K = L_\delta \cup L^1_\delta \cup \cdots \cup L^{m-1}_\delta.$$

Define the function h on K by

$$h(z) = \begin{cases} g(z), & z \in L_\delta, \\ g(z - lke^{2\pi i\theta}), & z \in L^l_\delta,\ l = 1, \ldots, m-1. \end{cases}$$

By Runge's theorem, there is an entire function $\xi(z)$ so that

$$\sup_{z\in K} |\xi(z) - h(z)| < \frac{\varepsilon}{4}. \tag{18}$$

Since f is hypercyclic for $T_{e^{2\pi i\theta}}$, then, by Theorem 11, is also hypercyclic for $T^k_{e^{2\pi i\theta}}$. That is, there exists some integer n so that

$$\sup_{z\in K} |f(z + nke^{2\pi i\theta}) - \xi(z)| < \frac{\varepsilon}{4}. \tag{19}$$

By (17) there is j such that $nk \leqslant m_j k \leqslant nk + (m-1)k$. Moreover, there is some $l, 0 \leqslant l \leqslant m-1$ such that $m_j k = nk + lk$. Set $w = (n_j r - m_j k)e^{2\pi i\theta}$. Notice that, by (16), $|w| < \delta$. Using (13), (18) and (19), for $z \in L$ we have

$$\begin{aligned}
&|f(z + n_j r e^{2\pi i\theta}) - g(z)| \\
&\quad = |f(z + (n_j r - m_j k)e^{2\pi i\theta} + m_j k e^{2\pi i\theta}) - g(z)| \\
&\quad \leqslant |f(z + w + (lk + nk)e^{2\pi i\theta}) - \xi(z + w + lke^{2\pi i\theta})| \\
&\qquad + |\xi(z + w + lke^{2\pi i\theta}) - g(z + w)| \\
&\qquad + |g(z + w) - g(z)| \\
&\quad < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

This completes the proof of Theorem 8.

3. A common universality criterion

Let $I \subset \mathbb{R}^+$ be an open interval of the positive real line. Recall that a F-space is a topological vector space whose topology is induced by a complete invariant metric ρ. Although, in general, ρ is not induced by a norm, in order to simplify the notation we write $||x|| = \rho(x, 0)$. In the next theorem we give conditions that guarantee the existence of common universal (hypercyclic) vectors for some families of (sequences of) operators.

Theorem 12. *Let X be a separable F-space and let $\{T_{n,\lambda} : n \in \mathbb{N},\ \lambda \in I\}$ be a family of operators acting on X, such that for fixed n the map $I \ni \lambda \to T_{n,\lambda}$ is continuous. Assume that there is a dense set $\{x_j : j \geqslant 0\}$ in X and a family of operators $\{S_{n,\lambda} : n \in \mathbb{N},\ \lambda \in I\}$ such that $T_{n,\lambda} \circ S_{n,\lambda} = Id$ and*

(1) *Given x_j and a compact set $K \subset I$ there is a sequence of positive numbers c_k such that*
  (a) $\sum_k c_k < \infty$,
  (b) $||T_{n+k,\lambda} \circ S_{n,\alpha}(x_j)|| \leqslant c_k$ *for any $n, k \geqslant 0$ and $\lambda, \alpha \in K$,*

(c) $||T_{n,\lambda}\circ S_{n+k,\alpha}(x_j)||\leqslant c_k$ *for any* $n,k\geqslant 0$ *and* $\lambda,\alpha\in K,\ \lambda\leqslant\alpha$.
Notice in particular that $||T_{n,\lambda}(x)||\to_n 0$ *and* $||S_{n,\lambda}(x)||\to_n 0$ (*and uniformly on* K).

(2) *Given* ε, x_j *and a compact set* $K\subset I$, *there exists* $0<C(\varepsilon)<1$ *such that*, *for* $\lambda,\ \alpha\in K$ *the following holds*:

$$\text{if } 1\geqslant\frac{\lambda}{\alpha}>C(\varepsilon)^{\frac{1}{n}}\quad\text{then } ||T_{n,\lambda}\circ S_{n,\alpha}(x_j)-x_j||<\varepsilon.$$

Then, *there exists a residual set* $G\subset X$ *such that*

$$\overline{\{T_{n,\lambda}(x);n\geqslant 0\}}=X\quad\forall\lambda\in I,\ \forall x\in G.$$

In other words, $G\subset\mathcal{U}(\{T_{n,\lambda}(x);n\geqslant 0\})\ \forall\lambda\in I$.

Proof. Let $K=[\lambda_1,\lambda_2]\subset I$. Define the set

$$E_K(s,j,m)=\left\{x\in X:\forall\lambda\in K\ \exists n=n(\lambda)\leqslant m\ \text{so that}\ ||T_{n,\lambda}(x)-x_j||<\frac{1}{s}\right\}.$$

It is enough to show that $E_K(s,j,m)$ is open and $\bigcup_m E_K(s,j,m)$ is dense, because then write I as a countable union of compact intervals K_n and the set

$$G=\bigcap_n\bigcap_s\bigcap_j\bigcup_m E_{K_n}(s,j,m)$$

satisfies the conclusion of the Theorem 12.

The openness of $E_K(s,j,m)$ can be established along the same lines as in the proof of Lemma 9 and we leave it to the reader. It remains to prove the density of $\bigcup_m E_K(s,j,m)$. Let $w=x_p$ for some positive integer p and $\delta>0$. We will find some m and $y\in E_K(s,j,m)$ so that $||y-w||<\delta$.

Fix k large enough such that

$$||T_{n,\lambda}(w)||<\frac{1}{4s}\quad\forall n\geqslant k,\ \forall\lambda\in K \tag{20}$$

and

$$\sum_{n\geqslant k}c_n<\min\left\{\delta,\frac{1}{4s}\right\}. \tag{21}$$

Consider a partition $\lambda_1=\alpha_0<\alpha_1<\cdots<\alpha_l=\lambda_2$ of the interval $[\lambda_1,\lambda_2]$ and define the vector

$$y=w+S_{k,\alpha_0}(x_j)+S_{2k,\alpha_1}(x_j)+\cdots+S_{(l+1)k,\alpha_l}(x_j). \tag{22}$$

We will choose l, and $\lambda_1 = \alpha_0 < \alpha_1 < \cdots < \alpha_l = \lambda_2$ so that $\|y - w\| < \delta$ and $y \in E_K(s, j, m)$ for some m. In fact, we will show that if $\alpha_{i-1} < \lambda \leqslant \alpha_i$, $i = 1, \ldots, l$, then $\|T_{(i+1)k,\lambda}(y) - x_j\| < \frac{1}{s}$ (for α_0 just take T_{k,α_0}). Thus, if $m = (l+1)k$ we get that $y \in E_K(s, j, m)$.

Let us first estimate $\|y - w\|$:

$$\begin{aligned}\|y - w\| &= \|S_{k,\alpha_0}(x_j) + S_{2k,\alpha_1}(x_j) + \cdots + S_{(l+1)k,\alpha_l}(x_j)\| \\ &\leqslant c_k + c_{2k} + \cdots + c_{(l+1)k} \leqslant \sum_{n \geqslant k} c_n < \delta,\end{aligned}$$

where the last inequality holds by (21).

Let λ, $\alpha_{i-1} < \lambda \leqslant \alpha_i$. Then,

$$\begin{aligned}T_{(i+1)k,\lambda}(y) = {} & T_{(i+1)k,\lambda}(w) + T_{(i+1)k,\lambda} \circ S_{k,\alpha_0}(x_j) + \cdots \\ & + T_{(i+1)k,\lambda} \circ S_{(i+1)k,\alpha_i}(x_j) + \cdots + T_{(i+1)k,\lambda} \circ S_{(l+1)k,\alpha_l}(x_j).\end{aligned}$$

We proceed estimating $\|T_{(i+1)k,\lambda}(y) - x_j\|$:

$$\begin{aligned}&\|T_{(i+1)k,\lambda}(y) - x_j\| \\ &\quad \leqslant \|T_{(i+1)k,\lambda}(w)\| + \|T_{(i+1)k,\lambda} \circ S_{k,\alpha_0}(x_j) + \cdots + T_{(i+1)k,\lambda} \circ S_{ik,\alpha_{i-1}}(x_j)\| \\ &\qquad + \|T_{(i+1)k,\lambda} \circ S_{(i+1)k,\alpha_i}(x_j) - x_j\| \\ &\qquad + \|T_{(i+1)k,\lambda} \circ S_{(i+2)k,\alpha_{i+1}}(x_j) + \cdots + T_{(i+1)k,\lambda} \circ S_{(l+1)k,\alpha_l}(x_j)\|.\end{aligned}$$

Firstly, notice that by (20) $\|T_{(i+1)k,\lambda}(w)\| < \frac{1}{4s}$ holds. Secondly, using (21) and assumption (b) we get

$$\begin{aligned}&\|T_{(i+1)k,\lambda} \circ S_{k,\alpha_0}(x_j) + \cdots + T_{(i+1)k,\lambda} \circ S_{ik,\alpha_{i-1}}(x_j)\| \\ &\quad \leqslant \|T_{(i+1)k,\lambda} \circ S_{k,\alpha_0}(x_j)\| + \cdots + \|T_{(i+1)k,\lambda} \circ S_{ik,\alpha_{i-1}}(x_j)\| \\ &\quad \leqslant c_{ik} + c_{(i-1)k} + \cdots + c_k < \frac{1}{4s}.\end{aligned}$$

Similarly, by (21) and assumption (c) the following holds:

$$\begin{aligned}&\|T_{(i+1)k,\lambda} \circ S_{(i+2)k,\alpha_{i+1}}(x_j) + \cdots + T_{(i+1)k,\lambda} \circ S_{(l+1)k,\alpha_l}(x_j)\| \\ &\quad \leqslant \|T_{(i+1)k,\lambda} \circ S_{(i+2)k,\alpha_{i+1}}(x_j)\| + \cdots + \|T_{(i+1)k,\lambda} \circ S_{(l+1)k,\alpha_l}(x_j)\| \\ &\quad \leqslant c_k + c_{2k} + \cdots + c_{(l-i)k} < \frac{1}{4s}.\end{aligned}$$

Therefore, we only have to find l and $\lambda_1 = \alpha_0 < \alpha_1 < \cdots < \alpha_l = \lambda_2$ so that for λ, $\alpha_{i-1} < \lambda \leqslant \alpha_i$,

$$||T_{(i+1)k,\lambda} \circ S_{(i+1)k,\alpha_i}(x_j) - x_j|| < \frac{1}{4s} \tag{23}$$

holds for every $i = 1, \dots, l$. Set $\varepsilon = \frac{1}{4s}$ and let $C(\varepsilon)$ be as in item (2) of the theorem. Thus, (23) holds provided

$$\frac{\lambda}{\alpha_i} > C(\varepsilon)^{\frac{1}{(i+1)k}}. \tag{24}$$

Since $\alpha_{i-1} < \lambda \leqslant \alpha_i$ it is enough to show that

$$\frac{\alpha_{i-1}}{\alpha_i} > C(\varepsilon)^{\frac{1}{(i+1)k}}, \quad i = 1, \dots, l. \tag{25}$$

Setting $\beta_i = \frac{\alpha_{i-1}}{\alpha_i}$, (25) is equivalent to

$$\beta_i > C(\varepsilon)^{\frac{1}{(i+1)k}}, \quad i = 1, \dots, l. \tag{26}$$

Hence, it is enough to find l positive numbers $\beta_1, \dots, \beta_l < 1$ such that (26) holds and such that

$$\prod_{i=1}^{l} \beta_i = \frac{\lambda_1}{\lambda_2} \tag{27}$$

because, if this is the case, define

$$\alpha_l = \lambda_2, \quad \alpha_i = \lambda_2 \prod_{r=i+1}^{l} \beta_r, \quad i = 0, \dots, l-1$$

and we get the desired partition verifying (25).

Choose $l \geqslant 1$ such that

$$\eta = \frac{\lambda_1}{\lambda_2} (C(\varepsilon))^{-\frac{1}{k}(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{l+1})} > 1$$

and setting $N = \frac{1}{k}(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{l+1})$, define

$$\beta_i = \eta^{\frac{N}{(i+1)k}} (C(\varepsilon))^{\frac{1}{(i+1)k}}.$$

Therefore (26) and (27) are satisfied and this completes the proof of the theorem. $\square$

4. Proof of Theorem 2

Recall that the usual metric in $\mathscr{H}(\mathbb{C})$ is given by

$$\rho(f,g)=\sum_{n\geqslant 1}\left(\frac{\sup_{|z|\leqslant n}|f(z)-g(z)|}{1+\sup_{|z|\leqslant n}|f(z)-g(z)|}\right)\frac{1}{2^n}$$

for $f,g\in\mathscr{H}(\mathbb{C})$. We will write $||f||=\rho(f,0)$ as we did in the previous section.

We will pursue an argument similar to the proof of Theorem 1. In fact, Theorem 2 is a consequence of the next two theorems as we already did in the proof of Theorem 1.

Theorem 13. *There is a G_δ-dense set $F\in\mathscr{H}(\mathbb{C})$ such that for any $f\in F$ and any real positive number λ, the function h defined as $h(z)=f(\lambda z)$ belongs to $\mathscr{HC}(D)$.*

Theorem 14. *If a function $f\in\mathscr{HC}(D)$ then, for any θ, $0\leqslant\theta\leqslant 1$ the function defined by $h(z)=f(e^{2\pi i\theta}z)$ also belongs to $\mathscr{HC}(D)$.*

4.1. *Proof of Theorem 13*

We will apply Theorem 12 for the family $T_{n,\lambda}=D^n\circ R_\lambda$ where $\lambda\in\mathbb{R}^+$.

Define the operators $S_{n,\lambda}:\mathscr{H}(\mathbb{C})\to\mathscr{H}(\mathbb{C})$, by $S_{n,\lambda}(f)(z)=f^{(-n)}(\frac{z}{\lambda})$ where $f^{(-1)}$ is the antiderivative of f such that the value at 0 is 0 and $f^{(-n)}=(f^{(-(n-1))})^{(-1)}$. Let us check that the conditions of Theorem 12 are satisfied. First, observe that

$$T_{n,\lambda}\circ S_{n,\lambda}(f)(z)=T_{n,\lambda}\left(f^{(-n)}\left(\frac{z}{\lambda}\right)\right)=\lambda^n\left(\frac{1}{\lambda^n}f\left(\lambda\left(\frac{z}{\lambda}\right)\right)\right)=f(z),$$

i.e., $T_{n,\lambda}\circ S_{n,\lambda}=Id$. To verify items (1) and (2) of Theorem 12 we will use as a dense subset in $\mathscr{H}(\mathbb{C})$ the set of polynomials $\{p_n\}$ with coefficients in $\mathbb{Q}+i\mathbb{Q}$. So, fix such a polynomial p. Then

$$T_{n+k,\lambda}\circ S_{n,\alpha}(p)=0\quad\text{for } k>deg(p).$$

On the other hand

$$T_{n,\lambda}\circ S_{n+k,\alpha}(p)(z)=\frac{\lambda^n}{\alpha^n}p^{(-k)}\left(\frac{\lambda}{\alpha}z\right).$$

Then for $\lambda\leqslant\alpha$,

$$||T_{n,\lambda}\circ S_{n+k,\alpha}(p)||\leqslant||p^{(-k)}||$$

holds. Let us define $c_k=||p^{(-k)}||$ and set $m=deg(p)$. It is not difficult to show that there is a constant $C=C(p)$ (for instance, we may choose C to be the maximum of the modulus of the coefficients of p) so that

$$\sup_{|z|\leqslant n} |p^{(-k)}(z)| \leqslant \frac{(m+1)}{k!}\, Cn^{m+k}.$$

Therefore

$$c_k \leqslant \sum_{n\geqslant 1}\left(\frac{\frac{(m+1)}{k!}\, Cn^{m+k}}{1+\frac{(m+1)}{k!}\, Cn^{m+k}}\right)\frac{1}{2^n}.$$

We need to show that $\sum_k c_k < \infty$. For this, it is enough to prove that

$$\sum_k \sum_{n\geqslant 1}\left(\frac{\frac{(m+1)}{k!}\, Cn^{m+k}}{1+\frac{(m+1)}{k!}\, Cn^{m+k}}\right)\frac{1}{2^n} < \infty.$$

Set $l=[2\log k]$ the integer part of $2\log k$. Then

$$\begin{aligned} c_k &\leqslant \sum_{n\geqslant 1}\left(\frac{\frac{(m+1)}{k!}\, Cn^{m+k}}{1+\frac{(m+1)}{k!}\, Cn^{m+k}}\right)\frac{1}{2^n} \\ &\leqslant \sum_{n=1}^{l}\left(\frac{\frac{(m+1)}{k!}\, Cn^{m+k}}{1+\frac{(m+1)}{k!}\, Cn^{m+k}}\right)\frac{1}{2^n} + \sum_{n\geqslant l+1}\left(\frac{\frac{(m+1)}{k!}\, Cn^{m+k}}{1+\frac{(m+1)}{k!}\, Cn^{m+k}}\right)\frac{1}{2^n} \\ &\leqslant \frac{C(m+1)}{k!}(2\log k)^{m+k+1} + \frac{2}{2^{2\log k}}. \end{aligned}$$

Notice that $\sum_k \frac{2}{2^{2\log k}} < \infty$. Let us show that

$$\sum_k \frac{(m+1)}{k!}(2\log k)^{m+k+1} < \infty.$$

We apply the ratio test and we obtain

$$\begin{aligned} &\frac{(2\log(k+1))^{m+k+2}}{(k+1)!}\,\frac{k!}{(2\log k)^{m+k+1}} \\ &\quad = \frac{2\log(k+1)}{k+1}\left(\frac{\log(k+1)}{\log k}\right)^{m+k+1} \end{aligned}$$

$$\leqslant \frac{2\log(k+1)}{k+1}\left(\frac{\log(k+1)}{\log k}\right)^{m+1}\left(\frac{\log(k+1)}{\log k}\right)^{k}$$

$$\leqslant \frac{2\log(k+1)}{k+1}\left(\frac{\log(k+1)}{\log k}\right)^{m+1}\left(1+\frac{\log(1+\frac{1}{k})}{\log k}\right)^{k}$$

$$\leqslant \frac{2\log(k+1)}{k+1}\left(\frac{\log(k+1)}{\log k}\right)^{m+1}\left(1+\frac{1}{k}\right)^{k}$$

$$\leqslant \frac{2\log(k+1)}{k+1}\left(\frac{\log(k+1)}{\log k}\right)^{m+1} e \to_k 0.$$

Hence $\sum_k c_k < \infty$. This completes the proof of item (1).

It remains to prove item (2). Let p (polynomial) and $\varepsilon > 0$ be given. Choose ε_1, l and afterwards δ such that

$$||\varepsilon_1 p|| < \frac{\varepsilon}{2}, \tag{28}$$

$$\sum_{n \geqslant l} \frac{1}{2^n} < \frac{\varepsilon}{4} \tag{29}$$

and

$$\text{if } |w-z| < \delta,\ |z|, |w| \leqslant l \text{ then } |p(w)-p(z)| < \frac{\varepsilon}{4}. \tag{30}$$

Finally, let $0 < C(\varepsilon) < 1$ be so that

$$0 < 1 - C(\varepsilon) < \varepsilon_1 \quad \text{and} \quad (1 - C(\varepsilon))l < \delta. \tag{31}$$

We are ready to prove item (2). Let $\lambda \leqslant \alpha$ be such that $1 \geqslant \frac{\lambda}{\alpha} > C(\varepsilon)^{\frac{1}{n}}$. Set $p_1(z) = p(\frac{\lambda}{\alpha} z)$. Notice that $T_{n,\lambda} \circ S_{n,\alpha}(p)(z) = (\frac{\lambda}{\alpha})^n p_1(z)$. Then

$$\begin{aligned} ||T_{n,\lambda}\circ S_{n,\alpha}(p) - p|| &\leqslant ||T_{n,\lambda}\circ S_{n,\alpha}(p) - p_1|| + ||p_1 - p|| \\ &\leqslant \left|\left|\left(\frac{\lambda}{\alpha}\right)^n p_1 - p_1\right|\right| + ||p_1 - p|| \\ &\leqslant \left|\left|\left(1 - \left(\frac{\lambda}{\alpha}\right)^n\right) p_1\right|\right| + ||p_1 - p|| \\ &\leqslant ||(1 - C(\varepsilon))p|| + ||p_1 - p|| \leqslant \frac{\varepsilon}{2} + ||p_1 - p||, \end{aligned}$$

where in the last inequality we used (31) and (28). Moreover, using (31), for $|z| \leqslant l$ we have that

$$\left|\frac{\lambda}{\alpha} z - z\right| = \left(1 - \frac{\lambda}{\alpha}\right)|z| < (1 - C(\varepsilon))l < \delta.$$

Therefore, by (30) and (29) we conclude that

$$\begin{aligned} ||p_1 - p|| &\leqslant \sup_{|z| \leqslant l} |p_1(z) - p(z)| + \sum_{n \geqslant l} \frac{1}{2^n} \\ &\leqslant \sup_{|z| \leqslant l} \left|p\left(\frac{\lambda}{\alpha} z\right) - p(z)\right| + \frac{\varepsilon}{4} < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}. \end{aligned}$$

This completes the proof of Theorem 13.

4.2. Proof of Theorem 14

Let $f \in \mathscr{HC}(D)$ and fix $\theta, 0 \leqslant \theta \leqslant 1$. We want to show that the function $h(z) = f(e^{2\pi i\theta} z)$ is also hypercyclic for the operator D.

Let $g \in \mathscr{H}(\mathbb{C})$, a compact set L and $\varepsilon > 0$ be given. It suffices to find n so that

$$\sup_{z \in L} |h^{(n)}(z) - g(z)| < \varepsilon.$$

Notice that $h^{(n)}(z) = e^{2\pi i n\theta} f^{(n)}(e^{2\pi i\theta} z)$. We will consider two cases: θ is rational or irrational. Assume that $\theta = \frac{p}{q}$ is rational. Since f is hypercyclic for D, it follows by Theorem 11 that f is also hypercyclic for D^q, that is, the sequence of functions $\{D^{nq}(f), n \geqslant 1\}$ is dense in $\mathscr{H}(\mathbb{C})$ and so is the sequence $\{R_{e^{2\pi i\theta}} \circ D^{nq}(f),\ n \geqslant 1\}$. Choose n so that $\sup_{z \in L} |R_{e^{2\pi i\theta}} \circ D^{nq}(f)(z) - g(z)| < \varepsilon$. Since

$$D^{nq}(h)(z) = e^{2\pi i nq\theta} f^{(nq)}(e^{2\pi i\theta} z) = f^{(nq)}(e^{2\pi i\theta} z) = R_{e^{2\pi i\theta}} \circ D^{nq}(f)(z)$$

the result follows.

Next, we treat the case θ irrational. Let B_1 be a closed ball centered at the origin, $L \subset B_1$ and set $g_1(z) = g(e^{-2\pi i\theta} z)$.

Let p be a polynomial satisfying

$$\sup_{z \in B_1} |p(z) - g_1(z)| < \frac{\varepsilon}{4}. \tag{32}$$

Choose $l > deg(p)$ so that

$$\sum_{n \geqslant l} \sup_{z \in B_1} |p^{(-n)}(z)| < \frac{\varepsilon}{4}. \tag{33}$$

Let $\delta > 0$ be such that for any $0 \leqslant \beta \leqslant \delta$

$$|e^{2\pi i\beta} - 1| \sup_{z \in B_1} |g_1(z)| < \frac{\varepsilon}{4}. \tag{34}$$

By the minimality of the rotation by $l\theta$, there is a sequence of positive integers $n_1 < n_2 < \dots$ such that

$$0 \leqslant \{n_k l\theta\} < \delta \tag{35}$$

and

$$\sup_k |n_{k+1} - n_k| < m \quad \text{for some integer } m. \tag{36}$$

Let B be a closed ball so that $B_1 \subset B^\circ$. By Cauchy estimates, we can find $\varepsilon_1 > 0$ so that

$$\text{if } \xi, \psi \in \mathscr{H}(\mathbb{C}) \text{ and } \sup_{z \in B} |\xi(z) - \psi(z)| < \varepsilon_1 \text{ then}$$

$$\sup_{z \in B_1} |\xi^{(j)}(z) - \psi^{(j)}(z)| < \frac{\varepsilon}{4}, \quad j = 0, 1, \dots, ml. \tag{37}$$

Set $\xi(z) = p(z) + p^{(-l)}(z) + p^{(-2l)}(z) + \cdots + p^{(-(m-1)l)}(z)$. As before, by Theorem 11 the sequence $\{D^{nl}(f), n \geqslant 0\}$ is dense in $\mathscr{H}(\mathbb{C})$. It follows that for some n we have

$$\sup_{z \in B} |D^{nl}(f)(z) - \xi(z)| < \varepsilon_1.$$

Since the sequence $\{n_k; k \geqslant 1\}$ satisfies (36), we may find n_k so that $n_k l = nl + jl$ for some $j, 0 \leqslant j \leqslant (m-1)$. By (37), (32), (33) and the definition of ξ we get

$$\begin{aligned}
\sup_{z \in B_1} |f^{(n_k l)}(e^{2\pi i\theta} z) - g_1(z)| &= \sup_{z \in B_1} |D^{n_k l}(f)(z) - g_1(z)| \\
&\leqslant \sup_{z \in B_1} |D^{n_k l}(f)(z) - \xi^{(jl)}(z)| + \sup_{z \in B_1} |\xi^{(jl)}(z) - g_1(z)| \\
&= \sup_{z \in B_1} |D^{nl+jl}(f)(z) - \xi^{(jl)}(z)| + \sup_{z \in B_1} |\xi^{(jl)}(z) - g_1(z)| \\
&\leqslant \frac{\varepsilon}{4} + \sup_{z \in B_1} |p(z) - g_1(z)| + \sum_{n \geqslant l} \sup_{z \in L} |p^{(-n)}(z)| \\
&< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{3\varepsilon}{4}.
\end{aligned}$$

Finally, by using the above estimate, (35) and (34) we conclude, setting $w = e^{2\pi i\theta}z$, that

$$\sup_{z \in L} |D^{n_k l}(h)(z) - g(z)| = \sup_{z \in L} |e^{2\pi i n_k l\theta} f^{(n_k l)}(e^{2\pi i\theta}z) - g(z)|$$

$$\leqslant \sup_{z \in B_1} |e^{2\pi i n_k l\theta} f^{(n_k l)}(e^{2\pi i\theta}z) - e^{2\pi i n_k l\theta} g(z)| + \sup_{z \in B_1} |e^{2\pi i n_k l\theta} g(z) - g(z)|$$

$$\leqslant \sup_{w \in B_1} |f^{(n_k)}(w) - g_1(w)| + \sup_{w \in B_1} |e^{2\pi i n_k\theta} - 1||g_1(w)| < \frac{3\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.$$

The proof of Theorem 14 is completed.

5. Proof of Theorem 4

For the proof of Theorem 4 we rely on the next two theorems, as we did in previous sections.

Theorem 15. *There is a G_δ-dense set $G \subset \ell^2$ such that $G \subset \mathscr{HC}(\lambda T)$ for every real number λ, $\lambda > 1$.*

Theorem 16. *Let x be hypercyclic for λT for some $\lambda > 1$. Then x is also hypercyclic for $\lambda e^{2\pi i\theta} T$ for any $\theta, 0 \leqslant \theta \leqslant 1$.*

5.1. Proof of Theorem 15

We will derive it from Theorem 12. Let $T_{n,\lambda} = (\lambda T)^n$ and $S_{n,\lambda} = (\lambda S)^n$ where S is the forward shift $S(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$. Notice that $T_{n,\lambda} \circ S_{n,\lambda} = Id$. Consider the following denumerable and dense set on ℓ^2:

$$D = \{(x_n) \in \ell^2 : x_n \in \mathbb{Q} + i\mathbb{Q} \text{ and } x_n = 0 \text{ eventually}\}.$$

Let us check item (1) of Theorem 12. Fix $x = \{x_n\} \in D$ and a compact interval $[\lambda_1, \lambda_2] \subset (1, \infty)$. Let k_0 be such that $x_k = 0$ for $k \geqslant k_0$. Observe that $T_{n+k,\lambda} \circ S_{n,\alpha}(x) = 0$ for $k \geqslant k_0$. Secondly, for $\lambda \leqslant \alpha$ we get

$$\begin{aligned} ||T_{n,\lambda} \circ S_{n+k,\alpha}(x)|| &\leqslant \left(\frac{\lambda}{\alpha}\right)^n \frac{1}{\alpha^k} ||S^k(x)|| \\ &\leqslant \frac{1}{\alpha^k} ||x|| \leqslant \frac{1}{\lambda_1^k} ||x||. \end{aligned}$$

Hence, setting $c_k = \frac{1}{\lambda_1^k}||x||,\ k \geqslant k_0$, conditions (a)–(c) of item (1) hold. We proceed by verifying item (2) of Theorem 12. Let $\varepsilon > 0$ be given. Then,

$$||T_{n,\lambda} \circ S_{n,\alpha}(x) - x|| = \left| \left(\frac{\lambda}{\alpha}\right)^n - 1 \right| ||x||.$$

Choose $C(\varepsilon),\ 0 < C(\varepsilon) < 1$ so that $C(\varepsilon) > 1 - \frac{\varepsilon}{||x||}$. Therefore, it is straightforward to obtain item (2). This completes the proof of Theorem 15.

5.2. *Proof of Theorem 16*

Let x be hypercyclic for λT and fix $\theta,\ 0 \leqslant \theta \leqslant 1$. We want to show that the orbit of x under $\lambda e^{2\pi i\theta} T$ is dense. If $\theta = \frac{p}{q}$ is rational it follows that the sequence $(\lambda e^{2\pi i\theta} T)^n(x)$ is dense. Indeed, by Theorem 11, the sequence $\{(\lambda T)^{nq}(x), n \geqslant 0\}$ is dense and the result follows since $(\lambda e^{2\pi i\theta} T)^{nq}(x) = (\lambda T)^{nq}(x)$.

Assume that θ is irrational and let z be any point in ℓ^2 and $\varepsilon > 0$ be given. Let $\delta > 0$ be such that

$$\text{if } ||y - z|| < \frac{\varepsilon}{2} \text{ and } 0 \leqslant \beta < \delta \text{ then } ||e^{2\pi i\beta} y - z|| < \varepsilon. \tag{38}$$

Let $y = \{y_j\} \in \ell^2$ be such that $||y - z|| < \frac{\varepsilon}{6}$ and y_j is eventually zero. Take a positive integer l satisfying

$$y_j = 0 \quad \text{for } j \geqslant l \quad \text{and} \quad \sum_{j \geqslant l} (\lambda S)^j(y) < \frac{\varepsilon}{6}.$$

There is a sequence of positive integers $n_1 < n_2 < \cdots$ such that

$$0 \leqslant \{n_k l\theta\} < \delta \quad \text{and} \quad \sup|n_{k+1} - n_k| < m. \tag{39}$$

Let $w = y + (\lambda S)^l(y) + \cdots + (\lambda S)^{(m-1)l}(y)$. Let ε_1 be such that if $||v - u|| < \varepsilon_1$ then $||(\lambda T)^j(v) - (\lambda T)^j(u)|| < \frac{\varepsilon}{6},\ 0 \leqslant j \leqslant ml$. Using Theorem 11, choose some n so that

$$||(\lambda T)^{nl}(x) - w|| < \varepsilon_1.$$

There is some n_k so that $n_k l = nl + jl$ for some $j, 0 \leqslant j \leqslant (m-1)$. Therefore,

$$\begin{aligned} ||(\lambda T)^{n_k l}(x) - z|| &\leqslant ||(\lambda T)^{nl+jl}(x) - (\lambda T)^{jl}(w)|| \\ &\quad + ||(\lambda T)^{jl}(w) - y|| + ||y - z|| \\ &\leqslant \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{2}. \end{aligned}$$

Hence, by (39) and (38) we get

$$||(\lambda e^{2\pi i\theta}T)^{n_k l}(x) - z|| = ||e^{2\pi i n_k l\theta}(\lambda T)^{n_k l}(x) - z|| < \varepsilon.$$

The proof of Theorem 16 is completed.

6. Weighted backward shifts

In this section we will make use of Theorem 12 to obtain common hypercyclic vectors for families of weighted backward shifts. Theorem 12 can be applied for a wide class of such operators. However, conditions (1) and (2) of Theorem 12 must be checked in each particular case. In the sequel we just deal with a specific family of weighted backward shifts. Recall that a weighted backward shift with weighted sequence $\{a_i,\ i\geqslant 1\}$ (bounded and positive) is the (bounded) operator $T : \ell^2 \to \ell^2$ defined by

$$T(x_1, x_2, x_3, \dots) = (a_1x_2, a_2x_3, \dots).$$

Salas [S1] proved that such an operator is hypercyclic if and only if $\sup_n \prod_{j=1}^n a_i = \infty$. Our family of weighted sequence is $a_i(\lambda) = 1 + \frac{\lambda}{i}$ where $\lambda \in \mathbb{R},\ \lambda > 1$. Let us denote by T_λ the weighted backward shift with weighted sequence $\{a_i(\lambda) : i = 1, 2, \dots\}$.

Theorem 17. *There is a G_δ-dense set $R \subset \ell^2$ such that $R \subset \mathscr{HC}(T_\lambda)$ for every real number $\lambda,\ \lambda > 1$.*

Proof. As we already mentioned, we shall verify items (1) and (2) of Theorem 12.

As before, consider the following denumerable and dense set on ℓ^2:

$$D = \{(x_n) \in \ell^2 : x_n \in \mathbb{Q} + i\mathbb{Q} \text{ and } x_n = 0 \text{ eventually}\}.$$

For each $\lambda > 1$ define the weighted forward shift S_λ by

$$S_\lambda(x_1, x_2, \dots) = \left(0, \frac{x_1}{a_1(\lambda)}, \frac{x_2}{a_2(\lambda)}, \dots\right).$$

Also, set $T_{n,\lambda} = T_\lambda^n$ and $S_{n,\lambda} = S_\lambda^n$. It follows that $T_{n,\lambda} \circ S_{n,\lambda} = Id$. Fix $x = \{x_l\} \in D$ and a compact interval $[\lambda_1, \lambda_2] \subset (1, \infty)$. Then, there is k_0 such that $T_{k,\lambda}(x) = 0$ $\forall k \geqslant k_0,\ \forall \lambda > 1$. For $k \geqslant k_0$ set $c_k = \frac{1}{\prod_{i=1}^k a_i(\lambda_1)} ||x||$. Let $\{e_n\}$ be the canonical basis of ℓ^2. Now, for $\lambda \leqslant \alpha$ we have

$$\begin{aligned}||T_{n,\lambda}\circ S_{n+k,\alpha}(x)|| &= \left|\left|\sum_{j=k}^{k+k_0}\left(\frac{\prod_{i=j+1}^{n+j} a_i(\lambda)}{\prod_{i=1+j-k}^{n+j} a_i(\alpha)}\right)x_{1+j-k}e_{j+1}\right|\right| \\ &= \left|\left|\sum_{j=k}^{k+k_0}\left(\frac{\prod_{i=j+1}^{n+j} a_i(\lambda)}{\prod_{i=1+j-k}^{j} a_i(\alpha)\prod_{i=1+j}^{n+j} a_i(\alpha)}\right)x_{1+j-k}e_{j+1}\right|\right| \\ &\leqslant \left|\left|\sum_{j=k}^{k+k_0}\left(\frac{1}{\prod_{i=1+j-k}^{j} a_i(\alpha)}\right)x_{1+j-k}e_{j+1}\right|\right| \\ &\leqslant \frac{1}{\prod_{i=1}^{k} a_i(\alpha)}\left|\left|\sum_{j=k}^{k+k_0} x_{1+j-k}e_{j+1}\right|\right| \\ &\leqslant \frac{1}{\prod_{i=1}^{k} a_i(\lambda_1)}||x|| = c_k.\end{aligned}$$

We have to prove that $\sum_k c_k < \infty$. In what follows $a_n \sim b_n$ means $\lim_n \frac{a_n}{b_n} = 1$. By standard calculus [K] we get

$$\log\left(\prod_{i=1}^{k} a_i(\lambda_1)\right) = \sum_{i=1}^{k}\log\left(1+\frac{\lambda_1}{i}\right) \sim \sum_{i=1}^{k}\frac{\lambda_1}{i} \sim \lambda_1\log(k),$$

so $c_k \sim \frac{1}{k^{\lambda_1}}||x||$ and $\sum_k c_k < \infty$ follows. Therefore, items (a), (b) and (c) of (1) are satisfied.

Let us verify item (2). Notice that

$$||T_{n,\lambda}\circ S_{n,\alpha}(x) - x|| \leqslant \left|\frac{\prod_{i=1}^{n} a_i(\lambda)}{\prod_{i=1}^{n} a_i(\alpha)} - 1\right|||x||.$$

Let $\varepsilon > 0$ and fix a, $0 < a < 1$. Observe that $\lim_n (1 - a^{1/n})\log n = 0$. Hence, for n large enough

$$|n^{\lambda_2(1-a^{1/n})} - 1| < \frac{\varepsilon}{||x||}.$$

Thus, if n is large enough and $\lambda, \alpha \in [\lambda_1, \lambda_2]$ satisfy $1 > \frac{\lambda}{\alpha} > a^{1/n}$ then

$$\left|\frac{n^\lambda}{n^\alpha} - 1\right| \leqslant \left|\frac{n^\alpha}{n^\lambda} - 1\right| = \left|n^{\alpha(1-\frac{\lambda}{\alpha})} - 1\right| < \frac{\varepsilon}{||x||}.$$

Since $\prod_{i=1}^{n} a_i(\lambda) \sim n^\lambda$ it follows that item (2) holds for n large enough (say $n \geqslant n_0$) with $1 > C(\varepsilon) > a$. Therefore, choose $C(\varepsilon)$ such that item (2) also holds for $n \leqslant n_0$.

7. Proofs of Theorems 5 and 6

We shall use the following powerful approximation theorem due to Arakeljan (see [Ga, p. 161]):

Theorem 18. *Denote by $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ the extended complex plane. Let $F \subset \mathbb{C}$ be a closed set such that $\bar{\mathbb{C}} \backslash F$ is connected and locally connected at ∞. Suppose that $\varepsilon(t)$ is a continuous and positive function for $t \geqslant 0$ and satisfies*

$$\int_1^\infty t^{-\frac{3}{2}} \log \varepsilon(t)\, dt > -\infty. \tag{40}$$

Then for every function $g : F \to \mathbb{C}$, continuous on F and holomorphic in its interior, there is an entire function f such that

$$|f(z) - g(z)| < \varepsilon(|z|), \quad \forall z \in F.$$

For instance, we will use the "error" function $\varepsilon(t) = \exp(-t^{\frac{1}{4}})$ which satisfies (40). Observe that $\lim_{t \to +\infty} \varepsilon(t) = 0$. In particular, if the set F in the above theorem is unbounded, then the approximation function f for a given function g on F is "tangent to g at ∞ through F."

7.1. Proof of Theorem 5

We start by defining a set S that eventually contains any ray emanating from any point z except the one in the direction of the positive real line:

$$S = \mathbb{C} \backslash \{z = x + iy : x > 1, \ -\log x < y < \log x\}.$$

Denote by $B_k = \{|z| \leqslant k\}$. Let n_1 be such that $B_1^1 = B_1 + n_1$ does not intersect S, that is $B_1^1 \cap S = \emptyset$ (for instance take $n_1 = 4$). For $k \geqslant 2$ let n_k be such that $B_k^k = B_k + n_k$ does not intersect $S \cup B_1^1 \cup \cdots \cup B_{k-1}^{k-1}$. Define the set

$$F = S \cup \bigcup_k B_k^k.$$

It is not difficult to see that F satisfies the conditions of Theorem 18, that is, F is closed and $\bar{\mathbb{C}} \backslash F$ is connected and locally connected at ∞.

Let $\{p_k : k \geqslant 1\}$ be an enumeration of the polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$. Consider a function $h : S \to \mathbb{C}$ continuous on S and holomorphic in the interior of S, such that

$$\lim_{z \to \infty, z \in S} h(z) = 0$$

(for instance take $h(z) = 0$). Finally let $g : F \to \mathbb{C}$ defined by

$$g(z) = \begin{cases} h(z) & z \in S, \\ p_k(z - n_k) & z \in B_k^k. \end{cases}$$

By Theorem 18, there is an entire function f such that

$$|f(z) - g(z)| < \varepsilon(|z|), \quad \forall z \in F.$$

One can easily check that the function f satisfies the conclusion of Theorem 5.

7.2. *Proof of Theorem 6*

Let $E \subset S^1$ be a closed nowhere dense (for instance any Cantor set) such that $1 \notin E$. First, we claim that there is an entire function ϕ which is hypercyclic for T_1 and satisfies

$$\lim_{r \to \infty} \phi(re^{2\pi i\theta}) = 0 \quad \text{where } e^{2\pi i\theta} \in E. \tag{41}$$

This can be done in a similar way as the proof of Theorem 5 by making some minor modifications. Indeed, let

$$S = \{re^{2\pi i\theta} : r \geqslant 0,\ e^{2\pi i\theta} \in E\}$$

(observe that S has empty interior) and for each $B_k = \{|z| \leqslant k\}$ let n_k be such that $B_k^k = B_k + n_k$ does not intersect $S \cup B_1^1 \cup \cdots \cup B_{k-1}^{k-1}$. Define the set

$$F = S \cup \bigcup_k B_k^k.$$

As before, define $g : F \to \mathbb{C}$ by

$$g(z) = \begin{cases} 0 & z \in S, \\ p_k(z - n_k) & z \in B_k^k. \end{cases}$$

Applying Theorem 18 with the error function $\varepsilon(t) = \exp(-t^{\frac{1}{4}})$ we get a function ϕ, which is hypercyclic for T_1 and satisfies (41). This proves our claim. Now we will find a function f which is hypercyclic for D and it is close enough to ϕ so that it is also hypercyclic for T_1 and satisfies (41). In order to do this, consider the set

$$A = \{g \in \mathscr{H}(\mathbb{C}) : |g(z) - \phi(z)| < \varepsilon(|z|) \text{ for } z \in F\}.$$

Notice that any function $g \in A$ is hypercyclic for T_1 and satisfies (41). Thus, our aim is to find a function $f \in A$ which is also hypercyclic for D.

The set A is a G_δ nonempty set, because $\phi \in A$ and $A = \bigcap_n A_n$ where

$$A_n = \bigcap_n \{g \in \mathscr{H}(\mathbb{C}) : |g(z) - \phi(z)| < \varepsilon(|z|) \text{ for } z \in F \cap \{|z| \leqslant n\}\}.$$

This means that A is a Baire space and we can apply Baire's category arguments. Although $A_n \cap \mathscr{U}(D)$ is residual in A_n (since A_n is open) it is not true, a priori, that $A \cap \mathscr{U}(D)$ is even nonempty. However, we will show that this is not the case, in fact the set $A \cap \mathscr{U}(D)$ is residual in A. For this, it is enough to prove that the set

$$E(s,j,m) = \left\{ g \in A : \text{ for some } n \geqslant 0, \sup_{|z| \leqslant m} |g^{(n)}(z) - p_j(z)| < \frac{1}{s} \right\}$$

is open and dense in A. It is straightforward to check that is open. We shall prove the denseness. In other words, given a function $g_1 \in A$, a compact set L and $\delta > 0$ we must find n and an entire function $\psi \in A$ so that

$$\sup_{z \in L} |\psi(z) - g_1(z)| < \delta \text{ and } \sup_{|z| \leqslant m} |\psi^{(n)}(z) - p_j(z)| < \frac{1}{s}. \tag{42}$$

We may assume without loss of generality that $L \subset \{|z| \leqslant m\}$. Let d, η be positive numbers so that $d - \eta > m$ and $\{d - \eta \leqslant |z| \leqslant d + \eta\} \cap B_k^k = \emptyset \ \forall k$. Let γ and β be as follows:

$$\gamma = \inf_{|z| \leqslant d + \eta} \{\varepsilon(|z|) - |g_1(z) - \phi(z)|\}, \tag{43}$$

$$0 < \beta < \min\left\{ \inf_{|z| \leqslant d + \eta} \varepsilon(|z|), \delta, \gamma \right\}. \tag{44}$$

Notice that, by the definitions of β and γ, the following holds:

$$\text{if } \theta \in \mathscr{H}(\mathbb{C}) \text{ and } |\theta(z) - g_1(z)| < \beta, \ |z| \leqslant d + \eta \text{ then } |\theta(z) - \phi(z)| < \varepsilon(|z|). \tag{45}$$

By using Runge's theorem, we may find a polynomial q_1 such that

$$\sup_{|z| \leqslant d} |q_1(z) - g_1(z)| < \frac{\beta}{4}. \tag{46}$$

Let $n_0 = deg(q_1)$ and fix $n > n_0$ so that $\sup_{|z| \leqslant d} |p_j^{(-n)}(z)| < \frac{\beta}{4}$. Set $q = q_1 + p_j^{(-n)}$ and observe that $q^{(n)} = p_j$ and

$$\sup_{|z| \leqslant d} |q(z) - g_1(z)| < \frac{\beta}{2}. \tag{47}$$

Using Cauchy estimates, we may find a positive number β_1 such that

$$\text{if } \theta \in \mathscr{H}(\mathbb{C}) \text{ and } \sup_{|z| \leqslant d} |\theta(z) - q(z)| < \beta_1$$

$$\text{then } \sup_{|z| \leqslant d-\eta} |\theta^{(n)}(z) - q^{(n)}(z)| < \frac{1}{s}. \tag{48}$$

Consider the set $S_1 = \{|z| \leqslant d\} \cup S$. Let k_0 be such that $B_k^k \cap S_1 = \emptyset$ for $k \geqslant k_0$ and set $F_1 = S_1 \cup \bigcup_{k \geqslant k_0} B_k^k$. It follows that F_1 satisfies the condition of Theorem 18. Let $\xi : F \to \mathbb{C}$ be the function

$$\xi(z) = \begin{cases} q(z) & |z| \leqslant d, \\ \phi(z) & z \in B_k^k, k \geqslant k_0, \\ \phi(z) & \{|z| \geqslant d + \eta\} \cap S_1, \\ (1-t)q(z) + t\phi(z) & \{|z| = d + t\eta\} \cap S_1, \ 0 \leqslant t \leqslant 1. \end{cases}$$

Observe that ξ is continuous on F_1 and holomorphic in its interior.

Let $\varepsilon_1(t), \ t \geqslant 0$ be a continuous positive function such that

$$\varepsilon_1(t) \leqslant \varepsilon(t) \quad \forall t \geqslant 0; \qquad \varepsilon_1(t) < \min\left\{\frac{\beta}{2}, \beta_1\right\} \quad \text{for } 0 \leqslant t \leqslant d + \eta \tag{49}$$

and satisfying (40). Applying Theorem 18 to F_1, ε_1 and ξ we obtain an entire function ψ so that

$$|\psi(z) - \xi(z)| < \varepsilon_1(|z|), \quad z \in F_1. \tag{50}$$

Let us check that ψ satisfies (42) and $\psi \in A$. By (47), (49) and (50) we get

$$\begin{aligned} \sup_{z \in L} |\psi(z) - g_1(z)| &\leqslant \sup_{|z| \leqslant d} |\psi(z) - q(z)| + \sup_{|z| \leqslant d} |q(z) - g_1(z)| \\ &< \sup_{|z| \leqslant d} \varepsilon_1(|z|) + \frac{\beta}{2} < \beta < \delta \end{aligned}$$

and using (48) and (50) it follows that

$$\begin{aligned} \sup_{|z| \leqslant m} |\psi^{(n)}(z) - p_j(z)| &\leqslant \sup_{|z| \leqslant d-\eta} |\psi^{(n)}(z) - p_j(z)| \\ &= \sup_{|z| \leqslant d-\eta} |\psi^{(n)}(z) - q^{(n)}(z)| < \frac{1}{s}. \end{aligned}$$

Thus, (42) is valid.

Let us verify that $\psi \in A$. For $z \in F$ and $|z| \geqslant d + \eta$ we have

$$|\psi(z) - \phi(z)| = |\psi(z) - \xi(z)| < \varepsilon_1(|z|) \leqslant \varepsilon(|z|).$$

For $z \in F$ and $|z| \leqslant d$, as before

$$|\psi(z) - g_1(z)| < \beta$$

which, by (45), implies $|\psi(z) - \phi(z)| < \varepsilon(|z|)$. Finally, for $z \in F$ and $d \leqslant |z| \leqslant d + \eta$, using (43), (44) and (49) we get

$$\begin{aligned}
|\psi(z) - \phi(z)| &\leqslant |\psi(z) - \xi(z)| + |\xi(z) - \phi(z)| \\
&\leqslant \varepsilon_1(|z|) + |(1-t)q(z) + t\phi(z) - \phi(z)| \\
&\leqslant \frac{\beta}{2} + (1-t)|q(z) - \phi(z)| \\
&\leqslant \frac{\beta}{2} + (1-t)|q(z) - g_1(z)| + (1-t)|g_1(z) - \phi(z)| \\
&\leqslant \frac{\beta}{2} + \frac{\beta}{2} + |g_1(z) - \phi(z)| < \gamma + |g_1(z) - \phi(z)| \\
&< \varepsilon(|z|).
\end{aligned}$$

Therefore, $\psi \in A$ and the proof is finished.

8. Final remarks

(1) As we mentioned in the introduction, let us now show that the set of common universal vectors for a "wide" class of families of sequence of operators is either residual or empty. Indeed, let $\{T_{n,\gamma} : n \geqslant 1 \ \gamma \in \Gamma\}$ be a family of sequence of operators acting on a separable F-space X, where Γ is a locally compact metric separable space and the map $\Gamma \ni \gamma \to T_{n,\gamma}$ is continuous for every $n \geqslant 1$. For a compact set $K \subset \Gamma$ define the set $E_K(s, j, m)$ as in Section 3. The same proof as in Lemma 9 shows that $E_K(s, j, m)$ is open (in fact, the crucial point is that any open set in the topological space Γ can be written as an exhaustive sequence of compact sets). Assume that the set of common universal vectors for the family $T_{n,\gamma}$ is nonempty. Then, it follows trivially that $\bigcup_{m \geqslant 1} E_K(s, j, m)$ is dense, and our assertion follows as in the conclusion of Theorem 12.

(2) Theorem 12 gives sufficient conditions in order to obtain common universal vectors for families indexed by real numbers. If one might want to prove a similar theorem for families indexed by an open set of $\mathbb{R}^p, p \geqslant 2$ then condition (2) should be

replaced by one of the form

$$\text{if } 1 > |\lambda - \alpha| > C(\varepsilon)^{\frac{1}{n^{1/p}}} \text{ then } ||T_{n,\lambda} \circ S_{n,\alpha}(x_j) - x_j|| < \varepsilon.$$

However, none of the families treated in our paper satisfy this kind of condition.

(3) The family of weighted backward shifts T_λ we dealt in Section 6 satisfies

$$\lim_{n \to \infty} \prod_{i=1}^{n} a_i(\lambda) = \infty$$

and

$$\lim_{n \to \infty} a_n(\lambda) = 1.$$

In [LM] it is proved that the above conditions imply the existence of an infinite dimensional closed subspace of hypercyclic vectors for T_λ. We ask if there is a *common* infinite dimensional closed subspace of hypercyclic vectors for all T_λ.

Acknowledgments

We would like to thank Julia Gordon for useful discussions during an informal seminar at the University of Maryland.

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