Billiards with polynomial decay of correlations

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Abstract

We prove that the billiard map in Bunimovich Stadium has polynomial decay of correlations of order n^{-1}. Ergodic Bunimovich-type billiards (two arcs not bigger than half a circle) that do not have trajectories with infinitely many consecutive bounces on straight lines satisfy the same property. We also prove that a very particular class of these billiards have polynomial decay of correlations of order n^{-2}.

1 Introduction

Let Q be a convex region of the plane bounded by two arcs of circles (not bigger than half a circle) joined by straight lines that do not enter in the circles determined by the arcs (if the arcs are two semicircles joined by two parallel lines, we have the well-known Bunimovich stadium). Let T be the dynamical system describing the free motion of a point mass in Q, with elastic reflections on ∂Q (angle of incidence with the normal to the curve equal to the angle of reflection ϕ). The dynamical system T is defined more precisely in a rectangle M' in $\mathbb{R}^2$ with coordinates (r, ϕ) (position r in the boundary ∂Q and angle of reflection ϕ). A natural measure ν on the Borel sets $\mathcal{B}$ of M' is preserved by T.

We assume that $(M', \mathcal{B}, \nu, T)$ is hyperbolic (existence of local expanding and contracting manifolds) and mixing. We also assume that singularity lines accumulate in a well determined pattern (see Subsection 4 of Section 3).

We say that (T, ν) defined over M' has polynomial decay of correlations for Hölder continuous functions if there exists $\gamma > 0$ such that for every pair of Hölder functions f, g there exists $C_1 = C_1(f, g) > 0$ such that (for some $\omega \geq 0$)

$$C_n(f,g,T,\nu) = \left| \int_{M'} (f \circ T^n) g d\nu - \int_{M'} f d\nu \int_{M'} g d\nu \right| \leq C_1 |n|^{-\gamma} \log^{\omega} |n| \quad \forall n \in \mathbb{Z}. \tag{1}$$

In this paper we will prove that the billiard system (T, ν) on the stadium has polynomial decay of correlations with $\gamma = 1$. There is numerical evidence corroborating this result [VCG83]. The same result is obtained if there are not trajectories with infinitely many bounces on the straight lines. We will also prove that if, in addition, the arcs are as in the stadium (tangent lines at the endpoints of the semicircles coincide) then the system has polynomial decay of correlations with $\gamma = 2$.

A natural approach to understanding the speed of mixing and related statistical properties is to pick a suitable reference set, and to regard a part of the system as having "renewed" itself when it makes a "full" return to this set. The reference set is the union of a finite number of "rectangles" whose sides are parallel to unstable and stable manifolds. Each of these rectangles is described in terms of a long unstable manifold $W \subset M$. $M \subset M'$ is the set of points that are leaving each semicircle.

For a sufficiently large set of points of W the stable manifolds are long enough. In order to locate this kind of points we have to discard the points in the unstable fiber whose orbits come too close to singularity manifolds. Let W^1_∞ be the remaining set. We define a certain renewal time S from W^1_∞ to itself, as in Chernov's setting (cf [Ch99], Section 5), and such that T^S on the leaf W has good distortion properties. Our main aim is to obtain some "tail" bounds for the "length" $p\{S > N\}$.

Roughly speaking we separate the points of each trajectory into two classes: those that move "chaotically" in M and those that move "regularly" either sliding along the arcs of circles or in between the two parallel lines (if the arcs are bigger than half a circle a new source of "regularity" appears: those trajectories that stay for a long time bouncing almost perpendicular on the same circle; we do not study these regular trajectories, but our method can also be applied to them).

In order to study the first class of points ("chaotic" ones), an induced return map F is defined on M. F is uniformly hyperbolic, but has infinitely many singularity lines. The good expansion properties of F on unstable manifolds, in an appropriate metric p, prevail over bad properties coming from cutting by singularity lines.

For $x \in W^1_\infty$, let be $S = S(x)$; then its trajectory $T^i x,\ 0 \le i \le S(x)$ will stay $r(x)$ times in M and $b(x)$ times in $M' \setminus M:\ S(x) = r(x) + b(x)$. We have to take separately into account the time spent in M and in $M' \setminus M$. We show that the techniques introduced by N. Chernov and Lai-Sang Young to prove the exponential decay of correlations of Sinai billiards with non-zero angle can be applied to our return map $F : M \to M$, and that $p\{r > m\} \le C_2 \beta^m$. This implies the exponential decay of correlations of the induced map; this statement is one of the most important results of this work.

The use of some probabilistic standard methods allows to obtain $p\{S > N\} \le C_1 N^{-\gamma} \log^\omega N$ for some constants $\gamma > 0, \omega \ge 0$. This result allows to apply Young's theorems in [LSY99] and to obtain the polynomial decay of correlations of T.

We obtain upper bounds for the decay of correlations. In [Sa02], some results on lower bounds are proved. They can be applied to Bunimovich-type billiards with non-parallel straight lines, but some additional conditions should be proved.

Decay of correlations of billiards in polygons with pockets (arcs of circles larger than semicircles are admitted) and billiards bounded by arcs of circles smaller than a semicircle joined by parallel lines, whose ergodicity was proved in [CT98] and [DelM01], respectively, can be studied by the methods applied in this paper. However, we have not included the analysis of this type of billiards. We observe that if the arcs are larger than half a circle, a new type of "regular" trajectories appear close to diametral trajectories.

The paper is organized as follows. In Section 2 we study the hyperbolic structure of billiards whose focusing components are arcs of circles. In Section 3 we reduce the phase space and make a deep analysis of T^n and its singularity sets. In Section 4 we define the induced map and improve the results by Bunimovich, Sinai and Chernov [BSC91] proving the exponential decay of correlations of the induced map. Finally, Section 5 contains the proof of the tail bound for T and the main results.

2 Billiards. Hyperbolic structure.

A plane billiard is the dynamical system describing the free motion of a point mass inside an open bounded connected region Q of the plane, with elastic reflections at the boundary. The boundary consists of a finite set of closed C^{k+1}-curves $\partial Q_i, k \ge 2$. Let $n(q)$ be the unit normal to the curve ∂Q at the point q pointing to the interior of the billiard table. The phase space of such a dynamical system is

$$M' = \{(q, v);\ \ q \in \partial Q,\ \ |v| = 1,\ \ \langle v, n(q)\rangle\ \ge\ 0\}.$$

A coordinate system on M' is defined by the arc length parameter r along ∂Q and the angle ϕ between v and $n(q)$. Clearly $|\phi| \le \pi/2$ and $\langle n(q), v\rangle\ =\ \cos\phi$.

Consider the probability $d\nu = c\cos\phi\ drd\phi$, where $c = (2|\partial Q|)^{-1}$ is just a normalizing factor and $|\partial Q|$ stands for the total length of ∂Q. If m is the Lebesgue measure on M', then $d\nu = c\ \cos\phi\ dm$.

Let be $q_1 \notin \partial Q_i \cap \partial Q_j$, the point of ∂Q where the oriented line through (q_0, v_0) first hits ∂Q and $v_1 = v_0 - 2\langle n(q_1), v_0\rangle n(q_1)$, the velocity vector of the trajectory after reflection on $q_1 \in \partial Q$. Then, we define the map T by $T(x_0) = T(q_0, v_0) = (q_1, v_1)$. We will denote by $x_i = (q_i, v_i) \in M'$, $i \in \mathbb{N}$ the successive iterations (if defined) by T of $x_0 = (q_0, v_0)$; that is, $T(q_i, v_i) = (q_{i+1}, v_{i+1})$.

The angle between v_i and $n(q_i)$ will be ϕ_i, and t_i will indicate the Euclidean distance between bouncing points q_i and $q_{i+1}, i \in \mathbb{N}$. As velocity is one, t_i is also the time between successive bounces q_i, q_{i+1}. The backward orbit $x_i = (q_i, v_i)$, for negative $i \in \mathbb{Z}$, is analogously defined. The main relations are $T(x_i) = x_{i+1}, i \in \mathbb{Z}$, $q_{i+1} = q_i + t_i v_i$.

This map T will not be defined if $q_1 \in \partial Q_i \cap \partial Q_j$, and it will not be continuous in a neighbourhood of (q_0, v_0) if the oriented line through (q_0, v_0) is tangent to some ∂Q_k, $(\phi_1 = \pm\pi/2)$. In this case T is defined in a half open neighbourhood.

The map T is called the *billiard map.* It preserves the measure ν and it is of class C^k. The set of points $x_0 = (q_0, v_0) \in M'$ whose forward or backward trajectory is tangent to ∂Q for some $x_i, i \in \mathbb{Z}$, or it is in $\partial Q_i \cap \partial Q_j$, have ν-measure zero. T satisfies the following **involutive property**. For $x = (r, \phi) \in M$ let $-x = (r, -\phi)$; then $T^{-1}x = -T(-x)$.

If $\tilde{x}_1 = (\tilde{q}_1, \tilde{v}_1) = T(\tilde{x}_0)$ is defined for $\tilde{x}_0 = (\tilde{q}_0, \tilde{v}_0)$, then for all $x_0 = (q_0, v_0)$ in a small neighbourhood of $\tilde{x}_0$, the derivative matrix is given by:

$$DT(x_0) = -\begin{pmatrix} \frac{t_0 K_0 + \cos\phi_0}{\cos\phi_1} & \frac{t_0}{\cos\phi_1} \\ K_1 \frac{t_0 K_0 + \cos\phi_0}{\cos\phi_1} + K_0 & \frac{K_1 t_0}{\cos\phi_1} + 1 \end{pmatrix}, \tag{2}$$

where $K_i = K(x_i)$, $i = 0, 1$, is the curvature of ∂Q at q_i. $K < 0$ on focusing components.

We now study the hyperbolic structure of a general class of billiards whose boundaries include circular arcs, straight lines and dispersing components. Let be V_0 the union of the vertical lines in M' corresponding to the vertices $\partial Q_i \cap \partial Q_j, i \neq j$ and $V_n = T^n V_0, n \in \mathbb{Z}$. We will assume that parts of the curves V_n accumulate only as those singularity lines studied in Subsections 3 and 4 of Section 3. We recall that statistical properties of billiard of these type were studied in [Bu79, BSC90, BSC91, CT98].

For each $x = (q, v) = (r, \phi)$ we define unstable cones $C^u(x) \subset \mathcal{T}_x M'$

(i) by $d\phi/dr \geq -K$, if q lies on dispersing or neutral components of ∂Q, and

(ii) by $d\phi/dr \leq 0$, if q lies on a focusing component.

It is quite simple to prove (see for example [CT98]) that the derivative DT maps these **unstable cones** into themselves: $DT(C^u(x)) \subset C^u(Tx)$. Similarly DT^{-1} maps the complementary cones into themselves. They are called **stable cones** $C^s(x)$.

Let be now $x = (r, \phi) \in M'$ and $u = (dr, d\phi)$ be a tangent vector at x. We put

$$\mathcal{B}(x, u) = \frac{1}{\cos\phi}\left(\frac{d\phi}{dr} + K(r)\right) \tag{3}$$

The value of $\mathcal{B}(x, v)$ represents the curvature of the orthogonal cross-section of the bundle Σ_0 of the outgoing velocity vectors specified by the points $(r + \varepsilon dr, \phi + \varepsilon d\phi)$, $\varepsilon \approx 0$, see for example [BSC91]. Remember that the sign of $\mathcal{B}(x, u)$ has been set positive if the bundle Σ_0 is diverging and negative if it is convergent (focusing).

At the time the bundle Σ_0 reaches $q_1 \in \partial Q$ again, it reflects in ∂Q and a new bundle of rays Σ_1, goes back into Q. Put $x_1 = (r_1, \phi_1) = Tx$ and $u_1 = (dr_1, d\phi_1) = DT(u)$. The new bundle has curvature

$$\mathcal{B}(x_1, u_1) = \frac{2K(r_1)}{\cos\phi_1} + \frac{1}{t(x) + \mathcal{B}^{-1}(x, u)} \quad . \qquad \text{Hence}$$

$$\frac{d\phi_1}{dr_1} = K(r_1) + \cos\phi_1 \left(t(x) + \cos\phi \left(\frac{d\phi}{dr} + K(r) \right)^{-1} \right)^{-1} \tag{4}$$

If q, q_1 are in the same arc of circle Γ_i, their distance is $2R_i \cos\phi$; so, for any $q \in \Gamma_i$, $t(x) \geq 2R_i \cos\phi$. If $u \in C^u(x)$, using formula (3) we have that $\mathcal{B}(x,u) \leq -(R_i \cos\phi)^{-1}$. Then, $-\mathcal{B}^{-1}(x,u) > t(x)/2$, and in the important case (ii) above, the unstable bundle focuses before it reaches the midpoint between two collisions. After that it defocuses and becomes divergent (*defocusing property*). When it hits ∂Q again, at q_1, it already gets wider than it was near q. Obviously, in case (i), Σ_1 is also wider than Σ_0. The expansion of the bundle between collisions (with respect to the width measured in direction perpendicular to the rays) is the main property of unstable bundles.

From relations (3), (4), it is not difficult to obtain the analytical expressions of the differential equations satisfied by *local stable and unstable manifolds (LSM, LUM).*

Let be $S(y) = 2K(y)/\cos\phi(y)$, and

$$k^s(x) = \cfrac{1}{b_1(x) + \cfrac{1}{b_2(x) + \cfrac{1}{b_3(x) + \ldots}}}; \qquad b_{2k}(x) = S(T^k x), \quad b_{2k+1}(x) = t(T^k x), \ \ k \in \mathbb{N}. \tag{5}$$

This continued fraction converges if $K(T^k x) > 0$ and $\sum_{k=0}^{\infty} t(T^k(x)) = \infty$ (see [Si70]). For the class of billiards we are considering here, the continued fraction has positive terms b_{2k+1} (always), and b_{2k} if q is in a dispersing component; b_{2k} is null or negative if q lies in a neutral or focusing component. Convergence of this continued fraction was proved in [Bu79]. Therefore, for trajectories that in the future have infinitely many bounces, the LSM $(r, \phi^s(r))$ at $x_0 = (r_0, \phi_0)$ is defined by the differential equation

$$\frac{d\phi^s}{dr}(x) = -K(x) - k^s(x)\cos\phi^s(x), \qquad \phi^s(r_0) = \phi_0 \, .$$

The LUM through x_0 is the set of points $(r, \phi^u(r))$ with infinitely many bounces in the past, with $\phi^u(r)$ given by

$$\frac{d\phi^u}{dr}(x) = -K(x) + k^u(x)\cos\phi^u(x), \qquad \phi^u(r_0) = \phi_0 \, , \qquad \text{where}$$

$$k^u(x) = a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \cfrac{1}{a_4(x) + \ldots}}}; \quad a_{2k+1}(x) = S(T^{-k}x), \quad a_{2k}(x) = t(T^{-k}x), \tag{6}$$

$k \in \mathbb{N}$. $k^u(x)$ and $-k^s(x)$ are the curvatures of the beam (pencil) of trajectories forming the LUM $\gamma^u(x)$ and the LSM $\gamma^s(x)$ immediately after the reflection at q. We also denote by $k^u_-(x) = k^u(x) - S(x)$ and $-k^s_-(x) = -k^s(x) - S(x)$, the curvatures of these pencils before reflection at q.

Expansion rates in the p-metric. The expansion and contraction of tangent vectors can be conveniently described in a pseudometric that is usually called the p-metric. If $v = (dr, d\phi)$ is a vector in stable or unstable cones, then its p-norm is defined by

$$|v|_p = \cos\phi |dr|.$$

The expansion factor of unstable vectors $u = (dr, d\phi) \in C^u(x)$ is given by (see for example [BSC91])

$$\frac{|DT(u)|_p}{|u|_p} = |1 + t(x)\mathcal{B}(x,u)| = \left| 1 + \frac{t(x)}{\cos\phi} \left(\frac{d\phi}{dr} + K(r) \right) \right| . \tag{7}$$

So, the factor of expansion is $1 + t(x)\mathcal{B}(x,u)$ in case (i) [cones on dispersing or neutral components], and $-1 - t(x)\mathcal{B}(x,u)$ in case (ii). In either case it is greater or equal than 1; in case (ii) because

$t(x)\mathcal{B}(x,u) < -2$. We denote by $\lambda^u(x) = |1+t(x)k^u(x)|$ ($\lambda^s(x) = |1+t(x)k^s_-(Tx)$), the coefficient of expansion of the LUM $\gamma^u(x)$ (the coefficient of expansion of the LSM $\gamma^s(x)$) under the action of T, in the p-length.

We also introduce p-distances (see [Ma95]). If two points are on a LUM or a LSM then their p-distance is the integral of $\cos\phi|dr|$ along the curve. Given $y \in M$ and $A \subset M$, we consider the set $\mathcal{D}^u(y,A)$ of curves $\mathcal{C}$ joining y and some point of A, such that $\mathcal{T}_x\mathcal{C} \subset C^u(x)$ for every $x \in \mathcal{C}$. Let us define

$$p^u(y,A) = \inf_{\mathcal{C}\in\mathcal{D}^u(y,A)} \int_{\mathcal{C}} \cos\phi|dr|, \text{ if } \mathcal{D}^u(y,A) \neq \emptyset, \quad \text{and } p^u(y,A) = 1, \text{ otherwise.}$$

In the same way, we define $p^s(y,A)$. Finally $p(y,A) = \min\{p^u(y,A),\ p^s(y,A)\}$.

The new phenomenon, here, is the existence of series of consecutive reflections in neutral or focusing components. During these series, the curvatures are zero or the free paths are short, i.e. $t(x) \simeq 0$. Then, the expansion (and contraction) of unstable (stable) vectors is weak –see formula (7). We just have

$$|DT(v)|_p > |v|_p \ \ \forall v \in C^u, \qquad |DT^{-1}(v)|_p > |v|_p \ \ \forall v \in C^s. \tag{8}$$

To exclude the influence of these "disturbing factors" we go over a derived (return) map on a "smaller" restricted phase space that will be defined in Section 4.

3 Unstable manifolds. Singularities

We now restrict our study to billiards satisfying the following conditions. The boundary ∂Q consists of neutral components, and of two circular arcs Γ_i (not bigger than half a circle) of radius R_i such that there are no other components of the boundary in the circle determined by Γ_i. We recall that Bunimovich-type billiards are closely related with our class (their arcs may be bigger than half a circle). We will assume that our billiard systems are hyperbolic (non-vanishing Lyapunov exponents, existence of local invariant manifolds) and mixing.

We will study explicitly

(a) the Bunimovich stadium;

(b) billiards that do not have trajectories with infinitely many bounces on the straight lines. This condition on the bounces at straight lines is quite restrictive. For example, we recall that any poligonal convex billiard has periodic trajectories and it is not known if there is a polygon without periodic orbits . See for example [CT98], Section 2.

These are the billiards described in the Introduction. To simplify computation, we restrict ourselves to the billiard in the Stadium with two semicircles of radius 1 (Γ_1, designed at right, and Γ_2, at left)and parallel lines of length l. Its billiard map contains all the difficulties of the other types of billiards studied in this paper. *Differences will be marked in italics.*

Conventions. Throughout this paper, "const." will denote positive constant numbers that depend on l. The values of these "const." are irrelevant. As usual, $[\alpha]$ means the largest integer number equal or smaller than α.

1. Let a_1, a_2, a_3, a_4 be the vertices (intersections of the semicircles with the parallel lines) starting from the bottom right and moving counterclockwise (Γ_1 goes from a_1 to a_2); their r-coordinates are, respectively, $0, \pi, \pi+l, 2\pi+l$ (the total length of the boundary is $2\pi+2l$). We partition M' into three sets:
$M_0 = \{(q,v) : q \text{ is in the neutral part of the boundary}\}$,
$M = \{(q,v) : \langle v, n(q)\rangle > 0, q \text{ is in one of the semicircles}, \ q \neq a_i$, and the next reflection is in another component$\}$, and
$M_- = M' \backslash (M_0 \cup M)$.

M_- is the set of "sliding" trajectories along the semicircles. M is the region with uniform expansion on unstable manifolds, where the results in [BSC91] can be used. It is formed by two parallelograms M_1, M_2 corresponding to the right and left semicircles, respectively. The vertices of M_1 are (counterclockwise) $A(0,-\pi/2), B(\pi,0), C(\pi,\pi/2), D(0,0)$. Let be $\mu = \nu_{|M}/\nu(M)$, the normalized measure on M, induced by ν.

In the following two Subsections (specially in the last one) we prove some new results on the relative positions of invariant manifolds and the values of unstable derivatives. These results are crucial to applying (in Section 4) Chernov's theorem on exponential decay of correlations (Theorem 2.1 in [Ch99]).

2. Transversality. The angles between stable and unstable manifolds in M are bounded away from zero. We will study the edges of the cones $T(C^u)$; a similar analysis may be carried on for stable ones. This method avoids using expressions (5) and (6). We will distinguish three cases.

(a) If $x_1 \in M$, and $x = T^{-1}x_1 \in M_0$ (neutral component), from (4) we obtain $-1 \leq d\phi_1/dr_1 < -1/2$ (remember that $d\phi/dr > 0, K(r) = 0$, and $t(x) > 2\cos\phi_1$). Moreover, if $\phi, \phi_1 < \pi/3$ we have $d\phi_1/dr_1 < -5/6$ and if $\phi, \phi_1 \approx \pm\pi/2$ then $d\phi_1/dr_1 \approx -1$.

(b) If $x_1 \in M_1, x = T^{-1}x_1 \in M_2$ (different semicircles) then,

$$l + 2\sqrt{2} > t(x) \geq \max\{l, \cos\phi + \frac{l^2}{3}\cos\phi_1 + \cos\phi_1\} \tag{9}$$

and, in (4), we have $-1 \leq d\phi_1/dr_1 \leq -l^2/(l^2+3)$. The lower bound in (9) is obtained discussing different cases that depend on the number of successive bounces on M_1; for example, if this number is greater than two, $t(x) \geq \cos\phi + l/\sqrt{2} + \cos\phi_1$.

(c) If $x_n \in M$ and there are n successive hits on the same semicircle we can prove, by induction (see also [BSC91], p. 98), that

$$\frac{d\phi_n}{dr_n} = \frac{\cos\phi_n}{-(n-1)\tau + \cfrac{1}{-2/\tau + \cfrac{1}{t_0 + \cfrac{\cos\phi_0}{d\phi_0/dr_0 + K}}}} \quad , \quad \tau = 2\cos\phi_j, \;\; j = 1, \cdots, n. \tag{10}$$

If q_0 is in the same semicircle, we have $d\phi_n/dr_n \geq -1/(2n)$ which is not bounded away from zero; but, in this case, working as in the second case we have that, on stable cones, $1 - \frac{\cos\phi}{l+2\sqrt{2}} > d\phi/dr \geq 1 - \frac{3}{l^2+3}$ (using (4), $d\phi/dr$ can be written as a function of $d\phi_1/dr_1$).

3. Unstable derivatives. Let be V_0 the union of the four vertical lines in M' corresponding to the vertices. Put $V_n = T^n V_0|_M$ for all $n \in \mathbb{Z}$. Each V_n is a finite union of C^k-curves whose slope, in the (r,ϕ) coordinates, is negative (in unstable cones) for $n \geq 1$ and positive (in stable cones) for $n \leq -1$. This property is referred to as **alignment**.

The curves of V_n in the parallelogram M_1 (everything is symmetric in M_2) are of two types: a) those that are the images of the vertices a_2 and a_3 (or a_1 and a_4) after sliding with $n \geq 1$ bounces along the right semicircle; they are close to the vertex a_1 (or a_2); b) those that are the images of the vertices a_3, a_4, after $n \geq 1$ bounces on the parallel lines.

If lines are not parallel, the last type of singularities is not relevant to our present analysis: consider any trajectory beginning in a_4 or a_3 and arriving at Γ_1 –the arc of circle at right–; it has only a finite number of bounces on the non-parallel lines. Then the number of singularity curves of this type is finite.

3.1. Lines of type a) accumulate on A (or C). We will refer all the computations to those that accumulate on A. Their intersections with line $r = 0$ have ϕ-coordinates $-\pi/2 + \pi/(2n)$ and $-\pi/2 + \pi/(2n+1) + o(n^{-1})$, depending on whether they are images of a_2 or a_3, after having n consecutive bounces on the right-hand semicircle. The strips between these lines of order n and $n+1$ are the set of points that have had n consecutive bounces on the same semicircle. We call these strips n-**ccells** (there are two n-ccells for each n). Each n-ccell is a "quadrilateral", C_n, one of whose "horizontal" sides joins the points $(0, \frac{-\pi}{2} + \frac{\pi}{2n}), (\frac{\pi}{n+1}, \frac{-\pi}{2} + \frac{\pi}{2(n+1)})$; the other "horizontal" side is almost parallel to this one and contains the point $(0, \frac{-\pi}{2} + \frac{\pi}{2n\mp 1} + o(n^{-1}))$.

Let $\mathcal{C}$ be the curve $T^{-1}a_4$ in M_1; its coordinates $(\hat{r}, \hat{\phi})$ satisfy

$$\sin\hat{\phi} = l\sin\gamma - \cos\gamma\ , \quad \hat{r} = \gamma + \frac{\pi}{2} + \hat{\phi}$$

where $0 \le \gamma \le \arctan 2/l$ is the angle of the trajectory line with the line a_4a_1. The tangent line to $\mathcal{C}$ at $(0, -\pi/2)$, $\gamma = 0$ is $r = \pi/2 + \phi$ and the contact is of order 2. If $x = (r, \phi)$ is leaving a n-ccell, let $x_1 = (r_1, \phi_1)$ be the first bounce on the other semicircle. If $r \approx 0$, then $(\phi_1 + \frac{\pi}{2})^2 \approx 2l(\phi + \frac{\pi}{2}) + (\phi + \frac{\pi}{2})^2$, $|r_1 - \tilde{r}| \approx (\phi + \frac{\pi}{2}) + (\phi_1 + \frac{\pi}{2})$, where $\tilde{r}$ is the r-coordinate of a vertex of the stadium. In particular, these relations prove that a regular component of $T^{-m}a_3\,(m > 0)$, close to A, is an increasing curve joining the point $B_m(0, b_m)$, $b_m \approx \frac{-\pi}{2} + \frac{\pi^2}{2l(2m-1)^2}$, with the curve $\mathcal{C}$ at the point with ϕ-coordinate $\hat{\phi} = \sqrt{\frac{l\pi}{4(m-1)}}$. It continues (after a breaking point) almost horizontally until the boundary line AB (the last part corresponds to points that have a bounce on a_4a_1).

The distribution of V_{-m} in C_n is very irregular. The "upper horizontal" part of C_n (the region "over" the line $\phi = -\pi/2 + r$) is crossed by arcs of V_{-m} with m between $k_1 = 1 + \left[\sqrt{\frac{\pi n}{4l}}\right]$ and n. Between the lines $\phi = \frac{-\pi}{2} + r$ and $\mathcal{C}$, the arcs of V_{-m} that intersect C_n have m between n and $k_2 = 2 + \left[\frac{4l}{\pi}(n+1)^2\right]$. The "lower horizontal" part of C_n (between $\mathcal{C}$ and ∂M ($\phi = \frac{-\pi}{2} + \frac{r}{2}$)) is crossed by at most $\frac{8l}{\pi}n$+const. smooth arcs V_{-m}.

If x is in a n-ccell, $\cos\phi(x) \approx$ const.n^{-1}, and const.$n^{-1/2} \ge \cos\phi(Fx)$, $\cos\phi(F^{-1}x) \ge$ const.n^{-1}. Then, in the stadium, the μ-measure of a n-ccell is $\approx$const.n^{-4}.

If there exist a trajectory leaving one circle and arriving tangent to the other one, ccells exist, but they are not close to the boundaries $\phi = \pm\pi/2$. Then, the μ-measure of these n-ccells is $\approx$const.n^{-3}. These remark is very important to understand the results in Theorems 4 and 5.

We need subtle estimates of unstable derivatives on points of C_n. For any point $x \in M$, $v \in C^u(x)$,

$$\frac{|DT^m(v)|_p}{|v|_p} = \frac{|DT(v)|_p}{|v|_p} \cdot \frac{|DT^{m-1}(Tv)|_p}{|DT(v)|_p}.$$

The following computation is done for $x_0 \in M_1$ and the next m bounces on the other semicircle. From (4) it follows that if there is one intermediate bounce (x_1) on the straight lines, $d\phi_2/ds_2 = K(s_2) + \cos\phi_2(t(x_0) + t(x_1) + \cos\phi_0(d\phi/ds + K(x_0))^{-1})^{-1}$. This means that for the computation of invariant directions, we can omit the bounce on the straight lines. Let $y_k = (s_k, \phi_k) = T^k(s_0, \phi_0), 1 \le k < m, \phi_k \approx -\pi/2 + \pi/2m, t(y_k) = 2\cos\phi_k$, represents successive bounces on the same semicircle. The estimates of the derivatives for large values of l are quite simple. So we will especially discuss them for small l.

From (4), (9) and the estimates of $\frac{d\phi_n}{dr_n}$ in (10), we have, for $l < \sqrt{3}$,

$$\frac{d\phi_1}{ds_1} \le -1 + \frac{3}{3 + l^2} \ \text{ and, } \quad \frac{d\phi_k}{ds_k} \le \frac{-1}{2k + 1 + 3/l^2}\ , \quad \text{for} \quad k \ge 2. \quad \text{Hence, from (7),}$$

$$\frac{|DT(v)|_p}{|v|_p} \ge \max\left\{1 + \frac{2}{2n + 1 + 3/l^2}, \frac{2ln}{\pi} - 1\right\} = D_{n,1} \quad \text{if} \quad n \ge 2, \text{ and} \tag{11}$$

$$\ge 1 + \frac{2l^2}{3 + l^2} = D_{1,1} \qquad \text{if} \quad n = 1. \tag{12}$$

We also have, on the other semicircle ($m \ge 2$ consecutive bounces),

$$\frac{|DT^{m-1}(Tv)|_p}{|DT(v)|_p} \ge D_1 \prod_{i=2}^{m-1}\left[-1 + \frac{t(x_i)}{\cos\phi_i}\left(1 + \frac{1}{2k + 1 + 3/l^2}\right)\right] = D_{1,1}\ \frac{2m + 1 + 3/l^2}{5 + 3/l^2} = D_m.$$

If $m = 1$, for $l < \sqrt{3}$, it results $n \leq 2$. Hence, for points $x \in C_n$, we have

$$\frac{|DT^m(v)|_p}{|v|_p} \geq D_{n,1}D_m = D_{n,m} \quad \text{if } n \geq 2,\ m \geq 2; \tag{13}$$
$$\geq D_{1,1}\, D_m = D_{1,m} \quad \text{if } n = 1,\ m \geq 2; \tag{14}$$
$$\geq D_{2,1} \quad \text{if } n = 2,\ m = 1; \quad \text{and} \tag{15}$$
$$\geq D_{1,1} \quad \text{if } n = m = 1. \tag{16}$$

What happens with unstable derivatives when the trajectory leaves the circle after having m bounces on it? We have that $\cos\phi_m \approx$ const. m^{-1}, $d\phi_m/dr_m \approx$ -const. $/m$, and then $|D_xTu|_p/|u|_p \geq$ const. m. These remarks show that the unstable p-derivative of T^{m+1} for points in $T^{-m}C_m$ is of order m^2.

3.2. Curves of type b) accumulate on D (or B). They are of two classes. Curves of order n of each class, cross each other on the symmetric line DC' to DC with respect to $\phi = 0$. The first class is formed by decreasing curves that go from DC to DC' and are the direct image of the vertices on the other half semicircle after n bounces on the parallel lines. The prolongation of these curves goes from $(0, \arctan(\frac{l}{2(n+1)}))$ to $(2\arctan\frac{l}{2n}, -\arctan\frac{l}{2n})$. The second class is formed by decreasing curves that go from $r = 0$ to DC' and are trajectories that have bounced n times on the parallel lines and once on the semicircle. These strips P_n between the lines of order n and $n+1$ are the sets of points that have n consecutive bounces on the parallel lines. We call them n-**pcells** and denote by P_n^1 (P_n^2) the n-pcell of the first (second) class. We must distinguish trajectories that have one bounce on a parallel line but do not cross the major axis of the Stadium. We will include this (sliding) trajectories in P_0.

For large n, the μ-measure of a n-pcell is $\approx$ const.n^{-3}. In the Stadium with half semicircles, these n-pcells are crossed by lines of the set R_{-1} of order from $n/3+$ const. to $3n+$ const. Then, any LUM in a n-pcell intersects at most $\frac{8}{3}n+$ const. singularity curves V_{-m} (const. is 2 if n is big enough). The width between the lines of these curves also is $\approx$ const.n^{-2} Then, LUM's contained in a n-pcell will get divided, before leaving M_0, by these discontinuity lines. If γ_2 is a LUM in a n-pcell that is not divided by V_{-m}, then $p(\gamma_2) \leq$const.n^{-2}.

The coefficients of expansion on LUM's under the action of T^m in a n-pcell are different depending on which class of pcell is involved. If the trajectory leaves a n-pcell of the first class and arrives to a m-pcell of the first (second) class the p-derivative is of order $4m$ ($12m$). If it leaves a n-pcell of the second class the values are respectively $\frac{8}{3}m$ and $8m$. These estimates may seem not important from a "theoretical" point of view but are essential –with some modifications– to obtain the bounds in formula (26). They follow from computations based on case (a) in Subsection 2 and the fact that if T^ix has m iterations ($i = 1, \cdots, m$) on neutral components, then the product of derivative matrices (2) gives a new matrix that results from substituting t_0 by $\Sigma_{i=0}^{n-1} t(T^ix)$ and K_1 by zero in (2).

A point of a pcell of the second class has had a previous bounce on the same circle (the line joining these last bouncing points is very close to either the lines a_1a_2 or a_3a_4). On this last part of its trajectory the p-derivative increases in a factor very close to three.

4. Accumulation of singularity lines. In this Subsection we include two important remarks related to the pattern of accumulation of singularity lines and, so, with the shape of the billiards.

First. There might be multiple intersections of the curves in $\tilde{V}_{-m} = \cup_{i=1}^m V_{-i}$. Denote by K_m the maximal number of smooth curves in $\tilde{V}_{-m}$ that intersect or terminate at any point $x \in M$. We will assume that $K_m \leq Am$ for some fixed constant $A \geq 3$, only depending on Q. The shapes of our billiard tables are very simple, so this assumption is quite natural. The same bound was assumed in [BSC91] (Theorem 1.1. and Subsection 2.1., Condition B). Let be $\tilde{D} = \min\{D_{n,m}$ in (13) - (16)$\}$. We fix an $m_1 = m_1(l) \geq 1$ such that $3(Ak+1) < \tilde{D}^k$ for any $k > m_1$.

Second. If the billiard is not the stadium, we will assume that the sliding trajectories are the only "disturbing factors" we have referred at the end of Section 2. In Subsection 3.1 we have carefully studied the way the curves V_n, V_{-m} accumulate in the stadium. In this case there are trajectories that after sliding on one semicircle, go directly to slide on the other one: images of n-ccells of one arc intersect m-ccells of the other arc. In some sense this is a good but rare situation because in a "generic" billiard of our type both types of cells do not intersect for $m, n \geq N_0$, for some $N_0 > 0$, but some unstable manifolds may be cut by an infinite number of singularity lines that are the boundaries of $T^{-m}C_m$ for $m \geq N_0$. This is the case if there is a trajectory leaving one circle and entering tangent to the other one. We will use the observations at the end of Subsection 3.1 to handle this problem. In fact it is possible to do almost all the computations without using these observations, but we will need to define a different pseudometric to resolve other computational difficulties. On points of $T^{-m}C_m$, the new pseudometric will obtained multiplying the p-metric by $m/2$. A different method can be used to handle this problem: to change the phase space (and the induced map) in order to consider *entering* trajectories to the circles.

4 Exponential decay of the induced map F

We will define an induced map F on M, and prove its exponential decay of correlations if the assumptions in Subsection 4 of the previous Section are satisfied. We are going to apply Theorem 2.1 in [Ch99]. In this Theorem all the estimates are expressed with respect to the Riemannian metric on local unstable manifolds (LUM); and s-manifolds that are used in Section 4 of [Ch99] have their tangent vectors contained in stable cones. We will use the pseudometric defined in Section 2. This pseudometric is a metric restricted to stable and unstable vectors. Then the whole method and estimates of Chernov's paper are valid in our situation. Although some of the estimates needed to apply Chernov's results are not satisfied if we use directly the pseudometric p. For this reason we must do some changes that look quite artificial.

We define a new psueudometric q, on unstable manifolds of points of M. It is the pseudometric p in every point with the following exceptions (the factors 2 and 1/2 can be suitably changed):
on points of $T^{-m-1}P_m^2 \cap P_n^1$, $|\cdot|_q = 1/2 \, |\cdot|_p$;
on points of $T^{-m}P_m^1 \cap P_n^2$, $|\cdot|_q = 2 \, |\cdot|_p$; and
on points of $T^{-m}C_m \cap C_n$, $|\cdot|_q = m/2 \, |\cdot|_p$.
The new metric continues being increasing on unstable cones as a consequence of the observations at the end of Subsections 3.1 and 3.2 of Section 3.

1. We remark that if $x \in P_n \cap T^{-m}P_m, n, m > 0$, then $\frac{|DT^m(v)|_q}{|v|_q} \geq \frac{8}{3}m$, and that for l close to zero, the lower bounds in (13) - (16) may be only slightly over one. So, for points in P_0 we must take care simultaneously of the small values of the derivative and the accumulation of singularity lines: for each l there exists $m_0 = m_0(l) \geq m_1$ such that for $x \in P_0$, $v \in C^u(x)$, $\frac{|D_xT^m(v)|_q}{|v|_q} \geq 3(Am+1) = \Lambda(m)$ for $m \geq m_0$.

Then we define the induced map $F : M \to M$ by $F(x) = T^{B(x)}$, where

$$B(y) = \inf\{i \geq m_0 : T^i y \in M\}, \text{ if } y \in P_0; \text{ and } B(y) = \inf\{i \geq 1 : T^i y \in M\}, \text{ otherwise.}$$

This induced map preserves the measure μ. For points in P_0 we will also need the following observation: if $y \in P_0 \cup T^{-m}C_m$ for $m \geq m_0$, then both definitions of $B(y)$ coincide. For $v \in C^u(x)$ the derivative satisfies

$$\begin{aligned} \frac{|D_xF(v)|_q}{|v|_q} &\geq E_{i,j}\, m \text{ for any } x \in P_n^j \cap F^{-1}P_m^i, \; n, m > 0, \; i, j = 1, 2, && (17) \\ &\geq \Lambda(B(x)) \text{ for any } x \in P_0 \cap C_n \cap F^{-1}C_k, \; k < m_0, \quad \text{and} && (18) \\ &\geq \frac{k}{2} D_{n,k} \text{ for } x \in C_n \cap F^{-1}C_k, \; k \geq m_0. && (19) \end{aligned}$$

where $E_{1,1} = 4, \; E_{1,2} = 6, \; E_{2,1} = \frac{16}{3}, \; E_{2,2} = 8.$

Properties of this induced map F (for $m_0 = 1$) were carefully studied in [BSC90] and, specially, in [BSC91], pp 58 - 60, 97 - 99. We have obtained other necessary properties in the previous Section.

Let $R_{-1} = F^{-1}V_0$ and $R_1 = FV_0$. They are the discontinuity curves for F and F^{-1}, respectively. R_1 (R_{-1}) is the union of C^2-curves contained in V_n (V_{-n}). Each of these curves is called a regular component of R_1 (R_{-1}); it is of order n if it is contained in V_n (V_{-n}). Let $R_0 = V_0|_M$, $R^{(n)} = R_0 \cup F^{-1}R_0 \cup \cdots \cup F^{-n+1}R_0$.

2. The LUMs γ contained in P_n, $n > 0$ that do not cut lines of the set R_{-1} satisfy the following condition: γ has had n successive bounces on the parallel lines and $F(\gamma)$ is not fractioned by discontinuity curves. If γ is contained in $C_n \cap P_0$ and is not cut by boundary lines of the set $T^{-m-1}C_m$ it may be cut into no more than $A\,B(x)+1$ pieces.

From (8) and (19), we deduce that the two families of cones $C^u(x), C^s(x)$ (defined in Section 2) satisfy $DF(C^u(x)) \subset C^u(Fx)$, $DF(C^s(x)) \supset C^s(Fx)$ (strictly) whenever DF exists, and that there exists $\Lambda_q > 1$ such that at any point $x \in M$

$$|DF(v)|_q \geq \Lambda_q |v|_q \quad \forall v \in C^u(x), \qquad |DF^{-1}(v)|_q \geq \Lambda_q |v|_q \quad \forall v \in C^s(x). \tag{20}$$

Hence, the map $F : M \to M$ is uniformly hyperbolic in the p-metric. In particular, the strict invariance of the field of cones implies that there is a total measure set of points in M with nonzero Lyapunov exponents for F.

Alignment also holds for F and infinitely many components of R_{-1} can only accumulate near ∂M (this property is usually referred as **continuation**).

3. The proof by Chernov requires that the sectional curvature of any LUM is uniformly bounded. In our billiard tables this is a standard result that follows from the bounds of the continuous fractions (5), (6) studied in Section 3 of [Bu79].

Then, almost all the hypothesis required in Theorem 2.1 of [Ch99] have been shown to hold, with the exception of the following two: distortion estimates for the unstable q-derivative and growth of unstable manifolds with respect to the q-metric.

4. Distortion bounds. If $x \in W$, denote by J^i_x the q-derivative on unstable direction on $T^i(x)$. We know –formula (7)– that

$$J^i_x = |1 + t^i_x B^i_x| \quad \text{where} \quad t^i_x = t(T^i x), \quad B^i_x = k^u(T^i x).$$

If x, y are in a connected component of $W \setminus R^n$, then the following distortion estimate holds

$$\left| \prod_{i=0}^{n-1} \frac{J^i_x}{J^i_y} - 1 \right| \leq \text{const.}\ [q(T^n x, T^n y)]^c \tag{21}$$

for some positive numbers const. and c.

This result is a consequence of important results in [BSC91]. We recall Lemma 3.7 and Proposition A1.1 of this paper.

Lemma 1. *Let W be a LUM such that $T^n W$ is not broken up by discontinuity lines, then for any $a, b \in W$, and any $k \geq 1$,*

$$\left| \prod_{i=-k}^{-1} \frac{J^i_a}{J^i_b} - 1 \right| \leq E\beta^n \ , \qquad \left| \frac{k^s(a)}{k^s(b)} - 1 \right| \leq E\beta^n \quad \text{and} \quad \left| \frac{k^u(a)}{k^u(b)} - 1 \right| \leq E\beta^n \tag{22}$$

for some positive number E and $0 < \beta < 1$.

However, a careful analysis of the arguments in A.1.5 and A.1.1. of [BSC91] reveals that in fact more has been proved there. If $a = T^{n-1}x, b = T^{n-1}y, k = n$, inequality (21) was proved to hold for $c = 1/3$. The arguments are lengthy and we will not repeat the details.

5. Growth of unstable manifolds. W is a δ_0-LUM if its q-length is $\leq \delta_0$. For an open subset $V \subset W$ and $x \in V$ denote the connected component of V containing the point x by $V(x)$. Let $n \geq 0$. We will call an open subset $V \subset W$ a (δ_0, n)-subset if $V \cap (\cup_{i=0}^{-n+1} F^i R_0) = \emptyset$ (the map F^n is defined on V) and $q(F^n V(x)) \leq \delta_0$ for every $x \in V$. Note that $F^n V$ is the union of δ_0-LUM's. Define the function $r_{V,n}(x)$ as $r_{V,n}(x) = q(F^n x, \partial F^n V(x))$. Hence, $r_{V,n}(x)$ is the radius of the largest open ball in $F^n V(x)$ centered at $F^n x$. In particular, $r_{W,0}(x) = q(x, \partial W)$, and if TV is not cut by a singularity line, then $\{x : r_{V,1}(x) < \epsilon\}$ is the set of points whose images Fx are ϵ-close to ∂FV.

Following the main ideas in [Ch99] we will prove the following estimates on the growth of unstable manifolds. There exists constants $\alpha_0 \in (0,1)$ and $\beta_0, D_0, \sigma, \xi > 0$ with the following property. For any sufficiently small $\delta_0, \delta > 0$ and any δ_0-LUM W, there is an open $(\delta_0, 0)$-subset $V_\delta^0 \subset W \cap \mathcal{U}_\delta$, and an open $(\delta_0, 1)$-subset $V_\delta^1 \subset W \setminus \mathcal{U}_\delta$ (one of these may be empty; $\mathcal{U}_\delta$ is the δ-neighbourhood of $R_{-1} \cup \partial M$ in the q-distance) such that $q(W \setminus (V_\delta^0 \cup V_\delta^1)) = 0$ and for every $\epsilon > 0$

$$q\{r_{V_\delta^1,1} < \epsilon\} \leq \alpha_0 \Lambda_q \cdot q\{r_{W,0} < \epsilon/\Lambda_q\} + \epsilon \beta_0 \delta_0^{-1} q(W), \tag{23}$$

$$q\{r_{V_\delta^0,0} < \epsilon\} \leq D_0 \delta^{-1} q\{r_{W,0} < \epsilon\} \tag{24}$$

$$q(V_\delta^0) \leq D_0 q\{r_{W,0} < \xi \delta^\sigma\} \tag{25}$$

These estimates ensure that FV_δ^1 is large enough, V_δ^0 is small enough, and the boundaries ∂V_δ^1 and ∂V_δ^0 are regular enough. Note that the above assumptions only involve one iterate of F.

Due to Transversality, the angles between W and the curves of R_{-1} that cross W are uniformly bounded away from zero. A LUM can be cut into many pieces by curves in R_{-1}. In Subsection 3.3. we have studied the accumulation of such curves close to the vertices of M_1 and the estimates of unstable derivatives on each cell.

For each connected component $\Delta \subset W \setminus R_{-1}$, put $\Delta^0 = \Delta \cap \mathcal{U}_\delta$ and $\Delta^1 = \text{int}(\Delta \setminus \mathcal{U}_\delta)$, The set Δ^0 consists of two subintervals adjacent to the endpoints of Δ (they may overlap and cover Δ). The set Δ^1 is either empty or a subinterval of Δ. We put $W^1 = \cup_{\Delta \subset W \setminus R_{-1}} \Delta^1$. For each Δ^1, the set $F(\Delta^1 \cap \{r_{W^1,1} < \epsilon\})$ is the union of two subintervals of $F\Delta^1$ of length ϵ adjacent to the endpoints of $F\Delta^1$.

If $W \subset C_n \cap T^{-m} C_m, m \geq m_0$ it will be cut by components of R_{-1} of order from k_1 to k_2+ const.n (definitions of k_1, k_2, depending on n are in 3.1. of Section 3). We fix n_0 such that $k_1(n_0) \geq m_0$; then,

$$q\{r_{W^1,1} < \epsilon\} < 2\epsilon \sum_{k=k_1}^{k_2+\text{const}.n} \frac{2}{k} D_{n,k}^{-1} \leq 2\epsilon\alpha_1, \quad \alpha_1 = D_{n,1}^{-1}\left(\frac{3}{2}\log n + \text{const.}\right).$$

If $n \geq N_0 \geq n_0$, it results $\alpha_1 < 1$. *We have maintained the computation as if the factor $2/k$ does not exist, to emphasize that this is possible in the case of the stadium, but the convergence of the sume and the bound for α_1 are valid in all the cases we are considering.*

If $W \subset P_q^i$ (recall that there are two classes of k-pcells) we have

$$q\{r_{W^1,1} < \epsilon\} < 2\epsilon A^i \sum_{m=k/3+\text{const.}}^{3k+\text{const.}} \frac{1}{m} < 2\epsilon \, \frac{5}{12} \log 9 = 2\epsilon\alpha_2 \text{ with } \alpha_2 < 1, \text{ if } q \geq N_1. \tag{26}$$

where $A^1 = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$, $A^2 = \frac{3}{16} + \frac{1}{8} = \frac{5}{16}$.

In $C_n \cup P_p$, $1 \le n < N_0$, $0 \le p < N_1$, for δ_0 small enough (these values depend only on l; hence, they are constant for a given stadium) W may be divided by at most $Am + 2$ ($m \ge m_0$) components of R_{-1}, one of them perhaps being $T^{-1}a_4$ (this is the curve $\mathcal{C}$) or $T^{-1}a_3$. Then, the worst possibility is to have $Am + 3$ subsegments and

$$q\{r_{W^1,1} < \epsilon\} < 2\epsilon \left(\frac{1}{3} + \frac{Am+1}{\Lambda(m)} + \frac{2}{(m+1)D_{n,m+1}}\right) < 2\epsilon\alpha_3, \quad \text{with } \alpha_3 < 1.$$

Let be $1 > \alpha_0 > \max\{\alpha_1, \alpha_2, \alpha_3, \Lambda_q^{-1}\}$. We get $q(r_{W^1,1} < \epsilon) \le \min\{q(W), 2\alpha_0\epsilon\}$. The first term on the right hand side of (23) is equal to $\alpha_0\Lambda_q \min\{q(W), 2\epsilon\Lambda_q^{-1}\} = \min\{\alpha_0\Lambda_q q(W), 2\alpha_0\epsilon\}$. Since $\alpha_0\Lambda_q > 1$, we get

$$q\{r_{W^1,1} < \epsilon\} < \alpha_0\Lambda_q \cdot q\{r_{W,0} < \epsilon\Lambda_q^{-1}\}.$$

Next, to obtain an open $(\delta_0, 1)$-subset V_δ^1 fo W^1, one needs to further subdivide intervals $\Delta^1 \subset W$ such that $q(F\Delta^1) > \delta_0$, into s_Δ equal subintervals of length $\le \delta_0$, with $s_\Delta \le q(F\Delta^1)/\delta_0$. If $q(F\Delta^1) \le \delta_0$, then we set $s_\Delta = 0$ and leave Δ^1 unchanged. The union of the preimages under F of the above intervals will make V_δ^1. Now we must estimate the measure of the ϵ -neighbourhoods of the additional endpoints of the subintervals of $F\Delta^1$. This gives

$$q\{r_{V_\delta^1,1} < \epsilon\} - q\{r_{W^1,1} < \epsilon\} \le \sum_{\Delta \subset W \setminus R_{-1}} 2s_\Delta \epsilon E_1 q(\Delta^1)/q(F\Delta^1)$$

$$\le \sum_{\Delta \subset W \setminus R_{-1}} 2\epsilon E_1 q(\Delta^1)/\delta_0 \le 2E_1\epsilon\delta_0^{-1} q(W).$$

Here $E_1 = \exp\{\text{const.}(q(W)_{\max})^c\}$ is an upper bound on distortions on LUM's, see (22). This inequality completes the proof of (23) with $\beta_0 = 2E_1$.

We now prove (24). It is enough to consider $\epsilon < q(W)/2$, so that the right hand side of (24) equals $2D_0\delta^{-1}\epsilon$. We can put $V_\delta^0 = W \setminus \overline{V_\delta^1}$. Then, the left hand side of (24) does not exceed $2J_\delta\epsilon$ where J_δ is the number of nonempty connected components of the set $\overline{V_\delta^0}$, which at most equals the number of connected components of $W \setminus R_{-1}$ of length $> 2\delta$. Hence, clearly $J_\delta \le q(W)/\delta \le \delta_0/\delta$. This proves (24).

Last, we prove (25). Let $\Delta, \Delta^0, \Delta^1$ be defined as above. Since the angles between any LUM Z and the vertical lines are bounded away from zero,
$q(Z) \le \text{const.}\cos\phi \cdot \{\text{Euclidean length of } Z\}$, where ϕ is the angle coordinate of any point of Z. The right hand side of (23) equals $D_0 \min\{q(W), 2\xi\delta^\sigma\}$. So, it is enough to show that $q(V_\delta^0) \le \text{const.}\delta^\sigma$ for some $\sigma > 0$.

If $W \subset C_n$, using the computations in 3.1. of Section 3, we obtain

$$q(V_\delta^0) \le 2\delta\left((n - k_1) + \text{const.}n\right) + \frac{1}{n^3} \le q(W) \le \frac{\text{const.}}{n^2},$$

the last term in the sum corresponding to the elements between $\phi = \frac{\pi}{2}$ and $\mathcal{C}$. This relation implies $\delta \le \text{const.}n^{-3}$. Hence, if $n \le \text{const.}\delta^{-1/3}$,

$$q(V_\delta^0) \le 2\delta\left((n - k_1) + (k_2 - n) + \text{const.}n\right) \le 2\delta(k_2 + \text{const.}n) \le \text{const.}\,\delta^{1/3}$$

If $W \subset P_n$, $q(V_\delta^0) \le 2\delta(\frac{8}{3}n + \text{const.}) \le q(W) \le \text{const.}n^{-1}$. This relation implies $\delta \le \text{const.}n^{-2}$. Hence, if $n \le \text{const.}\delta^{-1/2}$, $q(V_\delta^0) \le 2\delta\,\text{const.}\,\delta^{-1/2} \le \text{const.}\,\delta^{1/2}$. We have proved (25) with $\sigma = 1/3$.

6. Define

$$M^+ = \{x \in M : F^n x \notin R_0, n \ge 0\}, \quad M^- = \bigcap_{n>0} F^n(M \setminus R^{(n)}), \quad M^0 = M^+ \cap M^-.$$

The sets M^+ and M^- consist, respectively, of points whose future and past iterations by F are defined, and M^0 is the set of points where all the iterations by F are defined.

For any $k = 1, 2, \cdots,\ h > 0$ let be

$$M^{\pm}_{h,k} = \{x \in M^{\pm} : q(F^{\pm n}x, R_{\mp} \cup \partial M) > k^{-1}e^{-nh} \quad \forall n \geq 0\}, \quad M^0_{h,k} = M^+_{h,k} \cap M^-_{h,k}$$

The following result is a standard consequence of the theory of maps with nonzero Lyapunov exponents and transversality of invariant manifolds. See, for example [Pe92]. There exist $h,\ \epsilon = \epsilon(k)$ such that for every $x \in M^-_{h,k}$ there exists a LUM $W^u(x)$ such that $q(x,\ \partial W^u(x)) \geq \epsilon$. Similarly, LSM $W^s(x),\ q(W^s(x)) \geq 2\epsilon$ is defined for every $x \in M^+_{h,k}$. For the fixed h we will only consider k's such that $\mu(M^0_{h,k}) > 0$.

7. Now we define **rectangles** as in [Ch99]. A subset $R \subset M^0$ is called a rectangle if there exists $h > 0$ such that for any $x, y \in R$ there is a LSM $W^s(x)$ and a LUM $W^u(y)$ both of diameter $\leq h$, that meet in exactly one point $z = [x, y]$, which also belongs in R. A subrectangle $R' \subset R$ is called a **s-subrectangle** if $W^s(x) \cap R = W^s(x) \cap R'$ for every $x \in R'$. Similarly, **u-subrectangles** are defined. We say that a rectangle R' **u-crosses** another rectangle R if $R \cap R'$ is a u-subrectangle in R and a s-subrectangle in R'. The unstable **disk** of radius ϵ, through x, $W^u_\varepsilon(x)$, is the LUM which is a ε-ball centered at x in the q-metric. We assume that δ_0, δ_1 are small enough, so that $A_{\delta_1} = \{x \in M :$ the unstable disks $W^u_{\delta_1}(x)$ exists$\} \neq \emptyset$. If $W,\ W'$ are two δ_0-LUM's, we say that W' **overshadows** W if, roughly speaking, the stable cone constructed at any point of W has a common point with W'. In this case we can define the s-distance from W to W': $q^s(W, W') = \sup_{x \in W} q^s(x, W')$.

Given a rectangle R, fix $\tilde{z} \in R$; since the rectangle has the structure of a direct product, it can be represented in this form: $R = [W^u_R(\tilde{z}), W^s_R(\tilde{z})]$, where $W^{u,s}_R(\tilde{z}) = W^{u,s}(\tilde{z}) \cap R$. $W^u(\tilde{z}), W^s(\tilde{z})$ are called **coordinate axes** in R. Let Γ^u_R, Γ^s_R be the canonical projections of R onto the axes $W^u(\tilde{z}), W^s(\tilde{z})$, respectively. Then, by analogy, $R = [\Gamma^u_R, \Gamma^s_R]$. For almost every point $z \in R$, the Jacobian $J^u(z)\ (J^s(z))$ of the canonical isomorphism of the set $W^u_R(z)\ (W^s_R(z))$ onto its projection $\Gamma^u_R\ (\Gamma^s_R)$ is defined. Using the differential equations satisfied by LUM's and LSM's (see results that are in between formulas (5) and (6)) it is possible to prove that

$$\mu(R) = c_\mu \int_R (k^u(z) + k^s(z)) d\mu_R(z) = c_\mu \int_{\Gamma^u_R} dq(x) \int_{\Gamma^s_R} dq(y) (k^u(z) + k^s(z)) J^u(z) J^s(z), \qquad (27)$$

where $d\mu_R(z)$ denotes the measure on R which is the direct product of the measures $q(.)$ on the unstable and stable manifolds $W^u_R(z), W^s_R(z)$, c_μ is a normalizing factor, and $x \in \Gamma^u_R, y \in \Gamma^s_R$ are points such that $z = [x, y]$ (see [BSC91], pp. 61-62).

8. $\delta - \Lambda -$ **Filtration.** Let $\delta_0, \delta > 0$, and W be a δ_0-LUM. Two sequences of open subsets $W = W^1_0 \supset W^1_1 \supset W^1_2 \ldots$ and $W^0_n \subset W^1_n \setminus W^1_{n+1}, n \geq 0$ are said to make a $\delta - \Lambda -$ **filtration of W**, denoted by $\{W^1_n, W^0_n\}$, if $\forall n \geq 0$, F^n is defined on W^1_n and W^0_n, each **connected** component of $F^n W^i_n$ has q-length $\leq \delta_0, i = 0, 1$ and they are constructed inductively in such a way that the segments W^0_n are taken out from W^1_n if its n-iterate comes too close ($\delta\Lambda^{-n}$ close) to singularity lines at time n, not earlier. So, points in W whose images come too q-close to the singularity lines will be set apart and no longer iterated under F. This will create countably many gaps in W in which stable manifolds fail to be long enough. Let be $W^1_\infty = \cap_{n \geq 0} W^1_n$.

9. At the same time we can vary all the small parameters $\delta_i, i \geq 1$ that appear in the sequel preserving the specified relations between them. For any $z \in A_{\delta_1}$ let be $W(z) = W^u_{\delta_1/3}(z)$ and define a **canonical rectangle** $R(z)$ as follows: $y \in R(z)$ iff $y = W^s_{\delta_2}(x) \cap W$ for some $x \in W^1_\infty(z)$ and for some LUM W that overshadows $W(z)$, and such that $q^s(W(z), W) \leq \delta_3$. In Section 4 of [Ch99], it was observed that if $\delta_3/\delta_2 < c'$, where $c' > 0$ is determined by the minimum angle between the stable and unstable cone families, then every W that overshadows $W(z)$ and is δ_3-close to it in the above sense will meet all stable disks $W^s_{\delta_2}(x)$, for $x \in W^1_\infty(z)$. In that case $R(z)$ will be a rectangle, indeed.

For any connected subdomain $V \subset W(z)$, the set $R_V(z) = \{y \in R(z) : W^s(y) \cap V \neq \emptyset\}$ is an s-subrectangle in $R(z)$ "based on V". For $n \geq 1$, the partition of $W^1_n(z)$ into connected components $\{V\}$, induces a partition of $R(z)$ into s-subrectangles $\{R_V(z)\}$ that are based on those components. If $R_V(z)$ is one of those s-subrectangles, then $F^n R_V(z)$ is a rectangle.

If δ_0 is small enough, then there exists $z_1 \in A_{\delta_1}$ such that $\mu(R(z_1)) > 0$. We fix such a δ_0 and one such z_1. We then denote, for brevity, $R = R(z_1), W = W(z_1), W^1_\infty = W^1_\infty(z_1)$. Let $\mathcal{Z} = \{z_1, z_2, \cdots, z_p\}$ be a finite δ_4-dense subset of A_{δ_1} containing the above point z_1. We call $\mathcal{R} = \cup_i R(z_i)$ the **rectangular structure**. It is a finite union of rectangles that are likely to overlap and are far away from S_0. This means, in particular, that on its points the density $\cos\phi$ is bounded away from zero.

10. We will partition the set W^1_∞ into a countable collection of subsets $W^1_{\infty,k}, k \geq 0$. For every $k \geq 1$ there is a r_k such that for the s-subrectangle $R_k = \{x \in R = R(z_1) : W^s(x) \cap W^1_\infty \subset W^1_{\infty,k}\}$ based on $W^1_{\infty,k}$, the set $F^{r_k}(R_k)$ will be a u-subrectangle of some $R(z_i)$. We say that R_k **has returned into** $\mathcal{R}$ under r_k iterations of F. The **return time function** $r(x)$ is defined on W^1_∞ by $r(x) = r_k$ for $x \in W^1_{\infty,k}$, $k \geq 1$ and $r(x) = \infty$ if $x \in W^1_{\infty,0}$ (**leftover set**). In fact, the return time function is defined on small segments of W that "return" over the elements of $\mathcal{R}$ at the same time (see Section 5 in [Ch99]).

11. We say that (F, ν) has **exponential decay of correlations for Hölder continuous functions** if $\forall \eta > 0$, there exists $\beta(\eta) \in (0,1)$ such that for every η-Hölder functions f, g there exists $C_1 = C_1(f,g) > 0$ such that

$$\left| \int_M (f \circ F^n) g d\nu - \int_M f d\nu \int_M g d\nu \right| \leq C_1 \beta^{|n|} \quad \forall n \in \mathbb{Z}.$$

We say that (F, ν) satisfies the **Central Limit Theorem for Hölder continuous functions** if $\forall \eta > 0$, and for every η-Hölder function f, there exists $\sigma = \sigma_f \geq 0$ such that for every interval $A \subset \mathbb{R}$

$$\nu\left(\left\{x \in M : \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} f(F^i(x)) - \int f d\nu \in A\right\}\right) \to \frac{1}{\sqrt{2\pi}\sigma} \int_A e^{\frac{-t^2}{2\sigma^2}} dt \text{ as} \to \infty$$

(Convergence in distribution to $\mathcal{N}(0, \sigma^2)$). Furthermore, $\sigma_f \neq 0$ iff $f \neq g \circ F - g$ for any $g : M \to \mathbb{R}$.

The proofs in [Ch99] can be followed step by step in order to study the measure $p\{r(x) > n\}$. Its exponential decay is enough to deduce the exponential decay of correlations and the central limit theorem, using [LSY98].

Theorem 2. *There exist $C > 0$ and $0 < \Theta < 1$ such that $q\{r(x) > n\} \leq C\Theta^n$. (F, μ) has exponential decay of correlations and satisfies the central limit theorem for Hölder continuous functions on M*

5 Polynomial decay of T.

We will verify the hypothesis of Theorems 3 and 4 of [LSY99]: a distortion estimate related with the return times and bounds for the p-measure of points that have large return times ("tail" bound of $q\{S > N\}$).

1. We will deduce the following distortion estimate (see Subsection 4.2.): if x, y are in a regular component of W with the same return time S, then

$$\left| \prod_{i=0}^{S-1} \frac{J^i_x}{J^i_y} - 1 \right| \leq \text{const. } \beta^s \tag{28}$$

for some positive number const. and $0 < \beta < 1$; $s = s(T^S x, T^S y)$ is the smallest n such that $(T^S)^n x$, $(T^S)^n y$ lie in components with different return times $(T^S x = T^{S(x)} x)$.

This result is also a consequence of the first inequality in (22). Take $a = T^{S(x)}x, b = T^{S(y)}$ and $k = S(x) = S(y)$. Our result on distortion estimates follows from the fact that $s(T^{S(x)}x, T^{S(y)}y)$ is the maximum n that satisfies the hypothesis of Lemma 1.

2. We want to study $q\{S > N\}$ where $S(x)$ is the return time by T of points $x \in W$, defined by means of $r(x)$ (see Subsection **4.11**) in the following way: $S(x) = r(x) + K_1 + K_2 \cdots + K_r,\ r = r(x)$ where $K_i = K_i(x) \geq m_0$, is the number of iterations by T that are included in $F^i(x)$ As a consequence of the remarks in Subsection **4.2**, every $x \in W^1_\infty$ belongs to a regular component of W all whose points have the same F-return time.

Let A be the set of points $x \in W^1_\infty$ such that $S(x) > N$ and $r(x) > n \geq \alpha \log N$, for some positive α that will be fixed later. Then, as a consequence of Theorem 1, we have

$$q(A) \leq \text{const.}\Theta^n \leq \text{const.}\, e^{-a\alpha \log N} \leq \text{const.}\, N^{-a\alpha}.$$

3. Let now be $S(x) > N, r(x) < \alpha \log N$. We define the stopping times B_i in the following way:

$$B_1(x) = B(x), \quad B_i(x) = B_{i-1}(x) + B(T^{B_{i-1}}x).$$

The subindices i in $B_i(x)$ satisfy: if $B_i(x) \leq j < B_{i+1}(x)$, then $\frac{|DT^j(v)|_q}{|v|_q} \geq \Lambda_q^i$ for $v \in E^u(x)$.

We recall that for large m $(k_1, k_2$ defined in 3.1. of Section 3),
if $y \in P_n \cap F^{-1}P_m$, then $B(y) = m$, $\frac{1}{3}n + \text{const.} \leq m \leq 3n + \text{const.}$, and
if $y \in C_n \cap F^{-1}C_m$, then $B(y) = m$, $k_1 \leq m \leq k_2 + \text{const.}\, n$.

Since the iteration by p converges weakly to the invariant measure and all the images of a given part of a LUM may come into one cell, we have the following estimates
$q\{x : B_j(x) - B_{j-1}(x) > e\} \leq \text{const.}\, \nu(P_{[e]})$ if $T^{B_j(x)}x \in P_{[e]}$, and
$q\{x : B_j(x) - B_{j-1}(x) > e\} \leq \text{const.}\, \nu(C_{[e]})$ if $T^{B_j(x)}x \in C_{[e]}$, for $e \geq n_0$.

The last estimate is very important, because in those ergodic billiards whose boundaries are arcs of circles in the relative position of the arcs of the stadium and non parallel lines (without trajectories with infinitely many successive bounces on the straight lines –this condition will be referred as (*) at the end of this Subsection), the decay of correlations will be smaller.

Then

$$q\{S > N\} = \sum_{i \leq [\alpha \log N]} q\{S > N; B_{i-1} \leq N < B_i\} + q\{S > N; B_{[\alpha \log N]} \leq N\}.$$

The second term is smaller than $q\{S > B_{[\alpha \log N]}\} \leq \Lambda_q^{-[\alpha \log N]} < N^{-c_1 \alpha}$. Now we estimate the first term. For each fixed i, we write

$$q\{S > N; B_{i-1} \leq N < B_i\} \leq q\{S > B_{i-1}; N < B_i\} \leq \sum_{j=1}^{i} q\{S > B_{i-1}; B_j - B_{j-1} > N/i\}.$$

Each of the terms in this sum is $\leq \text{const.}\nu(\cup_{K \geq N} P_{[K/i]})$.

Finally, we take α such that $c_1\alpha,\ a\alpha > 4$. Hence, we have proved the following tail bound for the billiard in the Stadium:

$$q\{S > N\} < \text{const.}\, N^{-4} + \sum_{i=1}^{[\alpha \log N]} \text{const.}\, \frac{i^2}{N^2} \leq \text{const.}\, \frac{\log^3 N}{N^2}.$$

If there are not parallel lines and condition (*) is satisfied, then $q\{S > N\} < \text{const.}\, (\log N)^4 N^{-3}$. *In this case, we point out that for large* N, $N/(\alpha \log N) > n_0$.

We also remark that if there are trajectories that leave one of the circles with angle different from $\pm\pi/2$ and arrive tangent to the other one, then $q\{S > N\} <$ const. $(\log N)^3 N^{-2}$

4. Now we briefly describe the "tower" abstract method [LSY99], adapted to our system.

Put $\Delta = \{(y, n) \in R(z_1) \times \{0, 1, \cdots\} : y \in W^s_{\delta_2}(z), z \in W^1_\infty, n < S(z)\}$ where $S : W^1_\infty \to \mathbb{Z}^+$ is the return function. This definition makes sense because the return times S are the same for all elements in a stable fiber of $R(z_1)$., cf. Subsections 4.10 and 4.11. Return times are constant (S_i) on each s-subrectangle R_i based on $W^1_{\infty,i} = W_i$. If $\Delta_l = \{(y, l) : y \in R_i,\ l < S_i\}$ is the l-th level of the tower, and $\Delta_{l,i} = \Delta_l \cap \{y \in R_i\}$ then $\Delta_{S_i-1,i}$ is the top level of the tower exactly above of R_i. $G : \Delta \to \Delta$ is defined by $G(z, l) = (z, l + 1)$ if $l + 1 < S(z)$ and $G(\Delta_{S_i-1,i}) = (F^{S_i}R_i, 0)$ bijectively. $F^{S_i}R_i$ is a u-subrectangle in some $R(z_j)$. We identify Δ_0 with each one of these u-subrectangles and define $G^R : \Delta_0 \to \Delta_0$ by $G^R z = G^{R(z)} z$. Then $G(z, R(z) - 1) = (G^R z, 0)$. Let be $\pi : \Delta \to M$, the projection defined by $\pi(z, l) = T^l z$. It satisfies $T \circ \pi = \pi \circ G$.

We assume that all the sets mentioned above are $\mathcal{A}$-measurable for some σ-algebra $\mathcal{A}$ of subsets of Δ, and that there is a reference measure $\hat{m}, \hat{m}(\Delta_0) < \infty$. In fact $\hat{m}$ is first defined on Δ_0 and then carried by G from $\Delta_{l,i}$ to $\Delta_{l+1,i}$ for $l + 1 < S_i$; its Jacobian on the top levels satisfy the distortion estimates studied in Subsection 1. In our case $\hat{m}|_{\Delta_0} = \nu|_{\Delta_0}$. As shown in [LSY98], Section 2, a finite G-invariant Borel measure is naturally defined as an extension of $\hat{m}$. Then $\nu = [\tilde{\nu}(\Delta)]^{-1}\pi_*\tilde{\nu}$ is the usual (SRB) billiard measure on M. If $g : M \to \mathbb{R}$, let be $\tilde{g} = g \circ \pi$ the lift of g to $\Delta :\ \tilde{g}(z, l) = g(T^l z)$. Then we have $C_n(f, g, T, \nu) = C_n(\tilde{f}, \tilde{g}, G, \tilde{\nu})$.

The polynomial decay of the map T is deduced from the study of the return times $\hat{S}$ of the map G: $\hat{S}(w) = \min\{n \geq 0 :\ G^n w \in \Delta_0\}$. Then $\hat{m}\{\hat{S} > N\} \approx \delta_2$ const. $\sum_{S_i>N}(S_i - N)\, q(W_i)$; see (27). Theorem 3 in [LSY99] and results in Section 4 of [LSY98] establish that, if

$$\hat{m}\{\hat{S} > N\} \leq CN^{-\alpha} \log^\omega n, \quad \text{then} \quad C_N(\tilde{f}, \tilde{g}, G, \tilde{\mu}) \leq C_1 N^{-\alpha} \log^\omega N.$$

Let K be such that $S_{K-1} \leq N < S_K$ and $a_i = q(W_i)$; then

$$q\{S > N\} = \sum_{S_i>N} a_i < CN^{-\zeta} \log^\omega N, \quad \text{and} \quad \hat{m}\{\hat{S} > N\} \leq$$

$$\leq \sum_{S_i>N} S_i a_i = \sum_{i\geq K} S_i a_i = S_K \sum_{i\geq K} a_i + \sum_{j\geq K}(S_{j+1} - S_j) \sum_{i\geq j+1} a_i \leq C\frac{2\zeta - 1}{\zeta - 1} N^{-\zeta+1} \log^\omega N.$$

So $\hat{m}\{\hat{S} > N\} \leq CN^{-\alpha} \log^\omega N$ with $\alpha = \zeta - 1$.

5. In Theorem 3 of [LSY99], it was proved that if $\hat{m}\{\hat{S} > N\} \leq CN^{-\alpha}$ then $C_n(f, g, T, \nu) \leq C_1|N|^{-\alpha}$. All the estimates in Subsection 4.1 (pp. 171 -173) of that paper can be repeated with our tail bound. The main "new" point is the following observation that is used at least twice in the proof of the polynomial decay: if

$$\hat{m}\{\hat{S} > N\} \leq C\, N^{-\alpha}\, \log^\omega N \ \text{ and } \ i \leq \frac{1}{2}\left[\frac{n}{n_0}\right], \ \text{ then } \ \hat{m}\{\hat{S} > \frac{N}{i} - n_0\} \leq \text{const.} \left(\frac{i}{N}\right)^\alpha \log^\omega N.$$

We have proved (the hypothesis of all the theorems include the assumptions on accumulation of singularity lines made en Subsection 4 of Section 3)

Theorem 3. *If (M', T, ν) is the stadium dynamical system, then (T, ν) has polynomial decay of correlations with $\gamma = 1$*

We have also proved

Theorem 4. *If the boundary of the billiard table contains two arcs not larger than semicircles -in a "generic" relative position (see Subsection 3.4)- and straight lines, and there are not trajectories with infinitely many bounces on the straight lines, then (T, ν) has polynomial decay of correlations with $\gamma = 1$*

Moreover, in a very special case, when both arcs are not in the "generic" position, we have a "better" result:

Theorem 5. *If the boundary is as in Theorem 4, both arcs are semicircles but the billiard is not the stadium and the tangents at the end of both semicircles coincide, then the system has polynomial decay of correlations with $\gamma = 2$, and it satisfies the Central Limit Theorem.*

The conclusion on the Central Limit Theorem follows from the following remark: if the polynomial decay of correlations is satisfied with $\gamma > 1$, then, as a consequence of Theorem 4 in [LSY99], the Central Limit Theorem holds for Hölder continuous functions (see also the last part of Section 6 in [LSY99]).

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