CRITICAL POINTS OF QUADRATIC MAPS OF THE PLANE

J.L. GARRIDO, N. ROMERO, A. ROVELLA, AND F. VILAMAJÓ

Abstract. In this article we classify the set of geometrically stable quadratic mappings of the plane, characterizing its set of critical points and critical values. We also complete the proof given in [2] of the fact that generic quadratic maps of the plane without fixed points have a Lyapunov function and hence empty limit set.

1. Introduction

Consider the set Q of self mappings of the real plane such that each coordinate is a quadratic polynomial, and endow it with the topology of coefficients. In [2], a six parameter family was found such that every F in an open and dense subset of Q is conjugated to a mapping of the given family. This open and dense set will be denoted by Q_g.

This construction was used there to prove that for generic quadratic maps of the plane without fixed points the α-limit and ω-limit sets of any point are empty. This constitutes a version for noninvertible mappings of the well known theorem of Brouwer stating that an orientation preserving homeomorphism of the plane having no fixed points has empty limit sets. This theorem can also be stated in terms of Lyapunov functions; the existence of a Lyapunov function for a map f implies that f has empty limit sets. This will be treated in the next section, together with the following result, strongly based in the proof of the main theorem in [2].

Theorem 1. *If a nondegenerate mapping $F \in Q$ has no fixed points, then there exists a Lyapunov function for F.*

A mapping $F \in Q$ is called degenerate if it is conjugated to a delay quadratic map, that is a map of the form $(x, y) \to (y, f(x, y))$ with f a quadratic polynomial. It was shown in [2] that the generic set Q_g does not contain degenerate mappings. Therefore the hypothesis of theorem 1 holds for an open and dense subset of the set of maps in Q without fixed points.

The main purpose of this work is to topologically describe the set of critical points and critical values of quadratic mappings, and the regions on which the number of preimages is constant. Let $f : M \to N$; suppose that all the points in a connected component of the set of regular values of f have the same finite number of preimages. In this case, say that the type of f is $(a_1, \dots, a_n)$ if the numbers a_i are the different numbers of preimages a regular point can have. Thus, to each component of the set of regular values, one of the numbers a_i is associated. In section 3 it will be shown that for a map in Q_g the point at ∞ is a fixed point.

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This implies that the hypothesis above always holds, that is two points in the same component of the set of regular values have the same number of preimages.

The set of critical points of a C^∞ self mapping of the plane is generically a (possibly nonconnected) manifold. There are (also generically) only two kind of critical points: folds and cusps, both having simple local cannonical forms. Folds are locally conjugated to $(x, y) \to (x^2, y)$ and so the mapping is locally of type $(0, 2)$. Cusps are isolated and have a cannonical form (due to Whitney): $(x, y) \to (y, -x^3 + x^2y)$. It follows that locally at a cusp point the mapping is of type $(1, 3)$; see section 3.

It will be used the following concept of geometric equivalence and stability for mappings between manifolds, (see [7]): Two C^∞ mappings $f, g : M \to N$ are said *(geometrically) equivalent* if there exist C^∞ diffeomorphisms $\varphi : M \to M$ and $\psi : N \to N$ such that $f\varphi = \psi g$. The mapping f is said *(geometrically) stable* if there exists a strong C^∞ neighborhood of f such that every g in that neighborhood is equivalent to f.

The topology of the set of critical points, the number of cusp points, the type of the map and the absolute value of the degree of the map are invariants under geometric equivalence.

Theorem 2. *For the open and dense subset Q_g of Q the following properties hold:*

(1) *For every $G \in Q_g$, the point at ∞ is an attractor.*
(2) *Every map in Q_g is geometrically stable.*
(3) *There exist only two classes of geometric equivalence in Q_g, these are denoted by Q_h and Q_e.*
(4) *For every $f \in Q_h$, the set of critical points is a hyperbola containing exactly one cusp point. Mappings in Q_h are of type $(0, 2, 4)$ and have degree 0.*
(5) *For every $f \in Q_e$, the set of critical points is an ellipse that contains exactly three cusp points. Every map in Q_e is of type $(2, 4)$ and has degree ± 2.*

See the figures in section 3 where the typical sets of critical values are shown in each case. This result gives a geometric description of the mapping. However, only the first item says something about the dynamics of the map. On the other hand, as theorem 1 proves, nontrivial dynamics can only appear when the map has fixed points. If this is not the case, it will be of interest to determine if at least one of the fixed points belongs to the boundary of the immediate basin of ∞. This would start the machinery developed in [10] for the quadratic family to describe the boundary of the basin of ∞.

It is worth noting that the list of discrete dynamical systems given by quadratic maps is huge. This is particularly the case of models in engineering, economics and biology.

2. Lyapunov functions

In this section we analyze the existence of Lyapunov functions for mappings without fixed points.

Definition 1. *Let h be a continuous mapping in a topological space X. A continuous function $L : X \to \mathbb{R}$ is a Lyapunov function for h if $L(h(x)) - L(x) > 0$ for every $x \in X$. L is a strict Lyapunov function for h if there exists an $\alpha > 0$ such that $L(h(x)) - L(x) \geq \alpha$ for every $x \in X$.*

Lemma 1. *If there exists a Lyapunov function for h, then every α-limit and ω-limit set of h is empty.*

The proof of this lemma is very easy, see for example [2]. For an invertible mapping h, $\omega(x) = \emptyset$ for every x is equivalent to $\alpha(x) = \emptyset$ for every x; however, there are examples of noninvertible mappings on the plane having all its ω-limit sets empty, but nonempty α-limit sets.

Consider the problem of finding classes of mappings for which the absence of fixed points implies trivial dynamics (empty limit sets). To begin, say that a class of mappings A satisfies property P if the following statement is a theorem:

If $f \in A$ and f has no fixed points, then it has a Lyapunov function.

The first and obvious example of a class of mappings satisfying property P is $End^0(\mathbb{R}, \mathbb{R})$, the set of continuous functions on the real line. For an f in this class having no fixed points, one of the functions $L(x) = x$ or $L(x) = -x$ is a Lyapunov function.

The most remarkable example of a family satisfying property P is $Hom_+(R^2)$, the class of orientation preserving homeomorphisms of the plane. This example will be treated again below.

Also in higher dimensions there exist other classes satisfying property P. For example, say that a mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is a *delay endomorphism* if there exists a function $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that $f(x) = f(x_1, \ldots, x_n) = (x_2, \ldots, x_n, \varphi(x))$. Denote by $D^r(\mathbb{R}^n)$ the set of delay endomorphisms of class C^r, and by $CD^r(\mathbb{R}^n)$ the set of C^r delay endomorphisms for which the function φ is strictly convex. It was proved in [12] that $CD^0(\mathbb{R}^n)$ satisfies the property P. Another class of mappings, see [11], satisfying this property is $VD^r(\mathbb{R}^n)$ $(r \geq 2)$, the set of delay endomorphisms for which the function φ is vertical, that is $\partial_{11}\varphi \geq K|\partial_{ij}\varphi|$ for every $(i, j) \neq (1, 1)$ and some constant $K > 0$. In [4] and [5] other examples of families satisfying property P are given, these constitute subfamilies of the set of circular cellular automata of $\mathbb{R}^n$, a class of mappings characterized by the commutation with a cyclic orthogonal map of $\mathbb{R}^n$.

Clearly in this context theorem 1 says that the set of nondegenerate quadratic maps of the plane satisfies property P. Examples can be readily constructed showing that the whole set Q does not satisfy property P, for example within $(x, y) \to (y, ax^2 + by^2 + cxy + y + d)$ one can find mappings with two periodic points but without fixed points.

For the class of preserving orientation homeomorphisms of the plane, there is a lot of work done after the theorem of Brouwer; indeed, Brouwer itself proved the Plane Translations Theorem, stating that for a mapping in this class and having no fixed points, there exists a cover of the plane by invariant simply connected open sets on each of which the map is conjugated to a translation. Those interested in some problems arised can search into the references Guillou [8] and Beguin and Le Roux [1].

The following result implies that the class $Hom_+(\mathbb{R}^2)$ satisfies property P, it is just, in fact, another way to state the theorem of Brouwer:

Theorem 3. *Let h be an orientation preserving homeomorphism of the plane.*

(a) *h has no fixed points if and only if there exists a Lyapunov function for h.*
(b) *There exists a strict Lyapunov function for h if and only if h is conjugated to a translation.*

Proof. The first item is Franks's proof of Brouwer's theorem, see [6].

The proof of the non obvious direction of the second item was communicated to us by F. Le Roux, and goes as follows:
Define the set of singular pairs of h as the set of (x, y), x and y not in the same orbit, such that given neighborhoods U of x and V of y there exists an $n \in \mathbb{Z}$ such that $h^n(U) \cap V \neq \emptyset$. It was proved by Kérékjárto [9] that h has no singular points if and only if it is conjugated to a translation. Now we show that if h has a strict Lyapunov function, then it cannot have singular pairs: indeed, let x and y be points in different orbits, and assume that $L(y) - L(x) \geq 0$. Then there exist neighborhoods U of x and V of y and a positive integer m such that $m\alpha > L(y') - L(x') > -\alpha$ for every $y' \in V$ and $x' \in U$. This implies that the values of L at the points in $h^n(U)$ are different from those at V whenever $n > m$ or $n < 0$, so $h^n(U)$ does not intersect V for these values of n. Diminishing if necessary U and V one can assume that $h^j(U)$ does not intersect V for every $0 \leq j \leq m$, because x and y belong to different orbits. □

In almost all the cases the proof given in [2] that the fixed point free nondegenerate quadratic maps of the plane have trivial dynamics was supported on the construction of linear and strict Lyapunov functions. There is a case which does not admit linear Lyapunov function and in [2] the proof was done directly, without constructing Lyapunov functions. So to complete the assertion in theorem 1, it must be shown that a Lyapunov function exists also in this case. The Lyapunov function will not be strict.

Lemma 2. *Let $F(x, y) = (x^2 + x, f(x, y))$ be a generic quadratic map of the plane having no fixed points. Then there exists a Lyapunov function for F.*

Proof. Observe that F has fixed points if and only if $f(0, y) = y$ for some $y \in \mathbb{R}$. So we can assume that $f(0, y) > y$ for every y, the other case being analogous. This implies also that there exists a constant $\tau > 0$ such that $f(0, y) - y > \tau$ for every y. If $f(x, y) = ax^2 + by^2 + cxy + dx + ey + k$, the above inequality is equivalent to $by^2 + (e-1)y + k > \tau$ for every y. Assuming the generic condition $b \neq 0$, it follows that there exists an $\epsilon_0 > 0$ such that $by^2 + \rho(x)y + k > \frac{3}{4}\tau$ for every y if $\rho(x)$ is a function such that $|\rho(x) - (e-1)| < \epsilon_0$ for every x. Therefore,

$$f(x, y) - y = [(ax + d)x] + [by^2 + (cx + e - 1)y + k] \geq \tfrac{3}{4}\tau - |(a\epsilon + d)\epsilon| > \tfrac{1}{2}\tau$$

for every y if $|x| < \epsilon$, ϵ a suficiently small positive number.

We construct now the Lyapunov function. Let $\psi : (-\epsilon, \epsilon) \to \mathbb{R}$ be a deacresing homeomorphism (for example $-\tan\frac{\pi x}{2\epsilon}$), then define $L(x, y) = x$ for $|x| \geq \epsilon$ and $L(x, y) = -\psi^{-1}(y - \psi(x))$ if $|x| < \epsilon$. It is clear that L is continuous.

To prove that L is a Lyapunov function for F, observe that for $x \geq \epsilon$, the first coordinate of F is strictly increasing, $(x^2 + x) - x > \epsilon^2$. Let $(x_1, y_1) = F(x, y)$; if $x < -\epsilon$ then, either $x_1 \leq -\epsilon$, in which case $LF(x, y) = x_1 > x = L(x, y)$, or $x_1 \in (-\epsilon, 0)$ and $L(x_1, y_1) \in (-\epsilon, 0)$, and this is greater than x. For the last case remaining, take $|x| < \epsilon$. If $x_1 \geq \epsilon$ the conclusion is again trivial, and if $|x_1| < \epsilon$, then $LF - L$ is positive if and only if:

$$y_1 - \psi(x_1) - (y - \psi(x)) > 0,$$

because $-\psi^{-1}$ is increasing. But $y_1 - y > \frac{\tau}{2}$ as result of $|x| < \epsilon$, and $\psi(x) - \psi(x_1) > 0$ because ψ is decreasing and $x_1 < x$. See figures 1(a) where the level curves of L

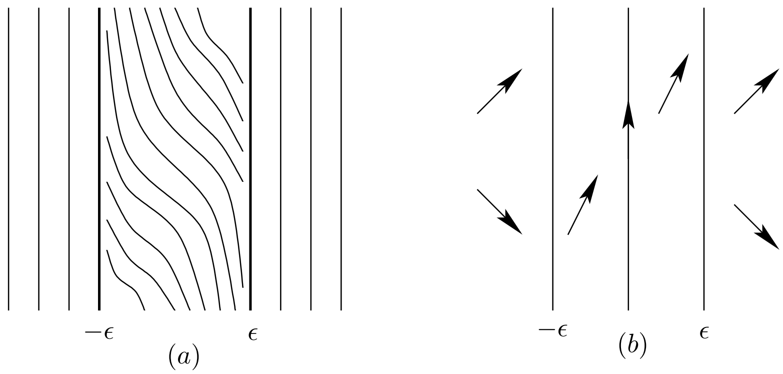

FIGURE 1. Level sets for the Lyapunov function L and locations of the vectors $F(x, y) - (x, y)$.

are shown and 1(b) where the vectors $F(x, y) - (x, y)$ are sketched. Finally, the nongeneric case $b = 0$ is treated in a similar way. □

This lemma and the results of [2] imply theorem 1.

3. Critical sets and critical values

It was proved (see corollary 5.1 in [2]) that the set Q_g of quadratic mappings conjugated to a map of the following family is open and dense in the set of quadratic maps of the plane:

$$G(x, y) = (pxy + L_1(x, y), rx^2 + sy^2 + txy + L_2(x, y)),$$

where $p \neq 0$ and $r^2 + s^2 \neq 0$ and L_1 and L_2 are affine functions. We consider the subset (also open and dense) where $rs \neq 0$.

In this section we analyze the possible geometric features of the set of critical points and critical values for the generic family of quadratic mappings of the plane, in fact we prove theorem 2. We need first recall some definitions and properties of the set of critical points of a smooth map. A complete reference for the nontrivial assertions is the book by Golubitsky and Guillemin [7].

(1) Let f be a mapping of a manifold M. A point x is a critical point of f if the differential of f at x is noninvertible. The set of critical points of f will be denoted by $\ell(f)$, in general, $\ell_k(f)$ will denote the set of points x for which the kernel of the differential of f at x has dimension k.
(2) Generically for C^2 mappings, the set $\ell_k(f)$ is a codimension k^2 submanifold of M. It follows that in dimension 2, $\ell = \ell_1$ generically.
(3) Within $\ell_1(f)$ there exist two kind of points: those for which the kernel of f is tangent to ℓ_1 (called cusp points) and those for which it is not (called fold points). It is easy to prove that under generic conditions (in C^3 topology) the set of points $\ell_{11}(f)$ for which the first case holds is either the empty set or a codimension one submanifold of $\ell_1(f)$. It follows that in dimension 2, $\ell_{11}(f)$ is generically a discrete set.
(4) There exist normal forms for both fold and cusp points in dimension 2. Indeed, it can be proved that if p is a fold point, then there exist changes of coordinates (one in a neigborhood of p and the other in a neighborhood

of $f(p)$) such that the map f, in these new coordinates, is $(x, y) \to (x^2, y)$. It follows that if p is a fold point of f, then there exist neighborhoods U of p and V of $f(p)$ such that the set of critical values separates V into two components, one of them having no preimages in U, and the other one having exactly two preimages, one at each side of ℓ_1.

(5) When p is a cusp point, there exist coordinates carrying p to $(0, 0)$ and $f(p)$ to $(0, 0)$ such that f is the mapping $(x, y) \to (x^3 - xy, y)$ in a neighborhood of the origin. It follows that there exist neighborhoods U of p and V of $f(p)$ such that the set of critical values separate V into two components; the points in one of them have one preimage in U and the points in the other one have three preimages in U. Moreover, $f^{-1}(f(\ell_1))$ separates U into four regions; restricted to each one of them the map f is injective.

Proof of Theorem 2. All the statements in theorem 2 are invariant under conjugation. So it suffices to prove the theorem for the family generic family:

$$G(x, y) = (pxy + ax + by + c, rx^2 + sy^2 + txy + dx + ey + k), \tag{1}$$

where $p > 0$ and $rs \neq 0$. Let $|(x, y)| = \max\{|x|, |y|\}$; we show that there exists $K_0 > 0$ depending only on G such that, for $K > K_0$ the condition $|(x, y)| > K$ implies $|G(x, y)| > 2K$. So it is clear that ∞ is an attracting fixed point for G. Indeed, if we assume that $|(x, y)| > K$ and $|x| \geq |y|$, then $|pxy + ax + by + c| < 2K$, then

$$\begin{aligned} |y| &< \frac{2K + |ax + c|}{|px + b|} \leq \frac{2K + |a||x| + |c|}{p|x| - |b|} \\ &\leq \frac{3K}{p|x|} + \frac{2|a|}{p} \leq \frac{3 + 2|a|}{p}. \end{aligned}$$

This inequality implies that:

$$|G(x, y)| \geq |r|x^2 - \left(\left|\frac{t(3 + 2|a|)}{p} + d\right|\right)|x| - \frac{s(3 + 2|a|)^2}{p^2} - \frac{|e|(3 + 2|a|)}{p} - k;$$

since $r \neq 0$ it follows that $|G(x, y)| \geq 2K$ if K is sufficiently large and $|x| > K$. The proof for the case $|y| \geq |x|$ is similar. This proves part (1) of theorem 2.

To prove the remaining parts still need some previous results, the first one is general for generic maps.

Lemma 3. *The restriction of a map $G \in Q_g$ to any component of the complement of $G^{-1}(G(\ell_1))$ is a covering map whose image is a component of the complement of $G(\ell_1)$.*

Proof. Denote by B^c the complement of a subset B of $\mathbb{R}^2$. Let A be a component of $(G^{-1}(G(\ell_1)))^c$. If $y \in G(A)$, then $G^{-1}(y)$ is finite: it cannot have accumulation points (the genericity of G implies that locally at every point x the numbers of preimages is finite), nor a divergent subsequence since ∞ is an attractor. As the points in A are noncritical, it follows that there exists a neighborhood of y whose preimage intersected with A is the union of a finite number of disjoint neighborhoods of the preimages of y on each of which G is an homeomorphism. It remains to show that the image of A is a component of $G(\ell_1)^c$. It is contained in such a component since it is connected and does not intersect $G(\ell_1)$. Moreover, if y belongs to the boundary of $G(A)$, then there exists a sequence $\{x_n\}$ in A such that $G(x_n) \to y$. But $\{x_n\}$ cannot diverge, and cannot converge to a point in A because this would imply that y is interior to $G(A)$. Therefore the sequence $\{x_n\}$ must converge to a point in the boundary of A, that is contained in $G^{-1}(G(\ell_1))$. It follows that y is a critical value, and then the image of A contains a whole component of the complement of the set of critical values. $\square$

The following two lemmas are needed to prove theorem 2, however, as it involve a lot of calculations, we put it in the last section.

Lemma 4. *Let G be as given by in equation (1); if $rs > 0$ then the set of critical points $\ell_1 = \ell_1(G)$ is a hyperbola containing exactly one cusp point; and if $rs < 0$, then ℓ_1 is an ellipse having exactly three cusp points.*

Denote by Q_h the set of maps having $rs > 0$ and by Q_e those having $rs < 0$.

Lemma 5. *If G is a map as in equation (1), then the restriction of G to the set of critical points is an injective map.*

Now we proceed to the proof of the remaining of theorem 2. First we consider the case $rs < 0$. By lemma 4, ℓ_1 is an ellipse having three cusp points c_1, c_2 and c_3. Let D be the bounded component of the complement of ℓ_1. It is clear by lemma 5 that $G(D)$ is the (unique) bounded component of the complement of $G(\ell_1)$. From lemma 3 it follows that the restriction of G to D is a homeomorphism onto $G(D)$, so $G(D)$ is a topological discs.

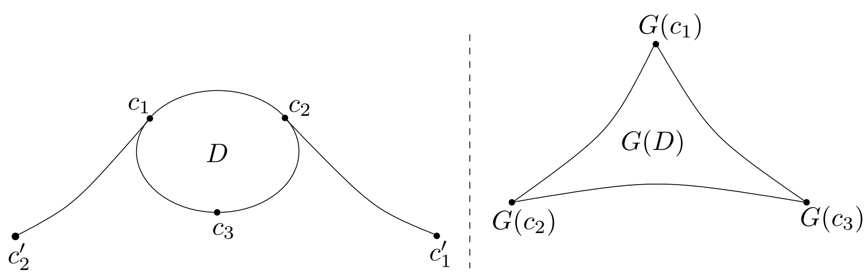

FIGURE 2. Critical set of G when $rs < 0$, its critical values and a piece of its preimages.

Claim: *Every point in the unbounded component of the complement of $G(\ell_1)$ has two preimages.*
Indeed, by lemma 3, it suffices to show that one point outside $G(D)$ has two preimages. This can be done using the equation of the map G. The first coordinate of G can be made $pxy + ay + b$ by a translation in the second variable. The preimage of a point (b, v) (with v large enough), satisfies $y = 0$ or $x = -a/p$. Substituting $y = 0$ in the second equation a degree two polynomial is obtained, and assuming v large, there exist solution if and only if $rv < 0$; on the other hand, substituting $x = -a/p$ in the second equation, and taking v large, it comes that a solution exists if and only if $sv < 0$. As exactly one of these inequalities hold, the claim follows.

A point z close to $G(c_1)$ but not in the closure of $G(D)$ has two preimages: a_1 and a_2; one of them, say a_1, is close to c_1, but the other is located far from D and converges to a point a' as z approaches $G(c_1)$. It follows that $G(c_1)$ has also two preimages: the cusp c_1 and a point c'_1 outside the closure of D. The same occurs for the other cusp points c_2 and c_3. It follows that the preimage of the segment s joining $G(c_1)$ and $G(c_2)$ in $G(\ell_1)$ is a curve given by three segments: from c'_1

to c_2 outside $G(D)$, from c_2 to c_1 in ℓ_1 and from c_1 to c_2'; see the figure 2. The same arguments on the other segments in $G(\ell_1)$ conducts to the picture of figure 3. It is clear now, invoking again lemma 3, that the restrictions of G to the regions A, B, C and D are homeomorphisms onto $G(D)$. Moreover, the restriction of G to the unbounded component E of the complement of $G^{-1}(G(\ell_1))$ is a two-to-one covering of the unbounded component of the complement of $G(\ell_1)$. It remains to calculate the degree of G. It suffices to calculate the degree of one regular value, so take as in the claim the point (b, v) with $v > 0$ large. If $r > 0$ then the preimages of (b, v) are $(x_\pm, 0)$. The determinant of DG at these points has the same sign of $-pv$, so the degree is $-2sign(p)$ if $r > 0$. If $r < 0$ then the preimages of (b, v) are $(-a/p, y_\pm)$ and at these points the determinant of DG has the same sign of pv; so in this case the degree of the map is $2sign(p)$. It follows that the degree of G is $-2sign(pr) = 2sign(ps)$. The assertion (5) of the theorem follows.

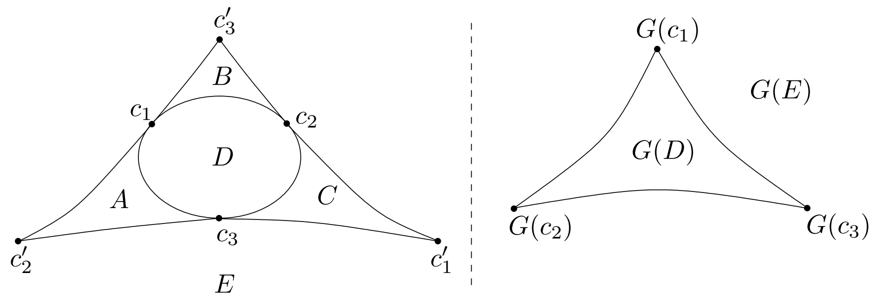

FIGURE 3. Regions A, B, and D are homeomorphic to $G(D)$ while region E cover two-to-one $G(E)$.

Now consider the case $rs > 0$. The critical set ℓ_1 of G is a hyperbola. One of the components, say ℓ, has a cusp point c_1 and the other one, ℓ', has only fold points. As $G(\ell)$ and $G(\ell')$ are disjoint and each separates the plane (see lemma 5) it follows that the set of regular values has three components: Y_0, Y_2 and Y_4. The same calculation of the claim above shows that if $rs > 0$, then there exists a region of points without preimages, we denote this region by Y_0; observe that the boundary of Y_0 is $G(\ell')$, this follows from the fact that in the neighborhood of the image of a cusp point, every point has at least one preimage. Let Y_2 be the region such that $\partial Y_2 = G(\ell) \cup G(\ell')$. Points in Y_2 have two preimages because passing through $G(\ell')$ from Y_0 to Y_2 means an increasing in two units of the number of preimages. Finally, Y_4 is the region whose boundary is $G(\ell)$; points in Y_4 must be have at least three preimages because it contains part of a neighborhood of the cusp point and locally at a cusp point the map is of the type $(1, 3)$. And again, crossing from Y_2 to Y_4 through a fold point means an increasing in two units the number of preimages.

As ℓ has a cusp point c_1, it follows that $G^{-1}(G(\ell))$ contains the union of ℓ and a curve ℓ'' tangent to ℓ at c_1. Moreover, it contains also a curve ℓ''' in the other component of the complement of ℓ'. We claim that ℓ'' is contained in the region bounded by ℓ and ℓ'. Take a curve $\gamma \subset Y_2$ joining $G(\ell)$ with $G(\ell')$. Consider γ without its extreme points: it is contained in Y_2, so its preimages cannot intersect ℓ'. However, one of the extreme points of $G^{-1}(\gamma)$ belongs to ℓ'', so the claim follows. Therefore, the complement of $G^{-1}(G(\ell_1))$ is the union of six regions: A, B, C, D, E

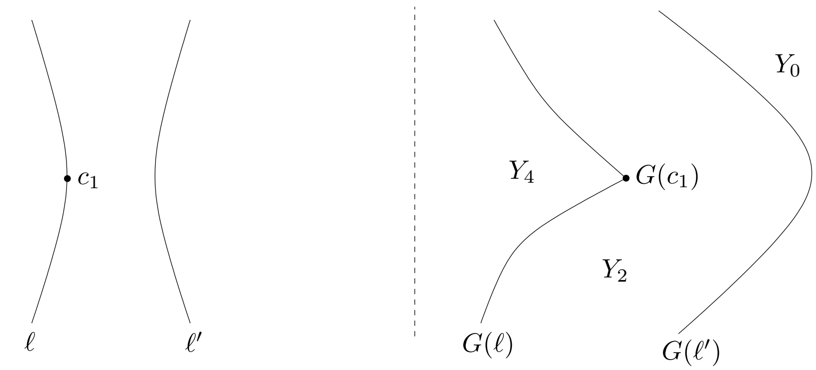

FIGURE 4. Regions Y_0, Y_2 and Y_4 bounded by the image of the critical set of G when $rs > 0$.

and F. The restriccions of G to A, B, C and F are homemorphisms onto the region I, and the restriccions of G to D and E are homeomorphisms onto the region H; see figure 5. Finally, it is obvious that the degree of G is 0. This proves assertion (4) of the theorem 2.

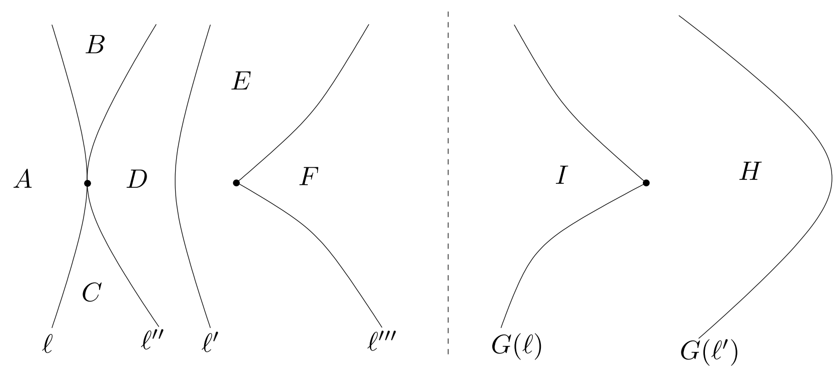

FIGURE 5. Regions A, B, C, D, E and F bounded by the preimage of the critical values of G when $rs > 0$.

To prove the remaining parts of the theorem, observe first that if $\widetilde{G}$ is a strong C^∞ perturbation (not necessarily quadratic) of a mapping $G \in Q_g$, then the point at ∞ is still an attractor for $\widetilde{G}$. It is clear that $\widetilde{G}$ has the same type of critical points, and that diffeomorphisms φ and ψ can be constructed in such a way that $\psi G = \widetilde{G}\varphi$; this construction can be given by first making φ carry critical points of G to critical points of $\widetilde{G}$, carrying cusps into cusps and then carrying diffeomorphically a region in $G^{-1}(G(\ell_1(G)))$ to a region in $\widetilde{G}^{-1}(\widetilde{G}(\ell_1(\widetilde{G})))$, taking care of making two points in $\widetilde{G}^{-1}(y)$ to be carried by φ into points having the same image under G, thus the equation $G\varphi\widetilde{G}^{-1}$ defines a map ψ that is clearly a diffeomorphism. This proves part (2) of the theorem 2, and part (3) follows in identical way. $\square$

4. Proofs of lemma 4 and lemma 5

Consider an arbitrary endomorphism in generic normal form:

$$F(x,y) = (pxy + ax + by + k_1, rx^2 + sy^2 + txy + cx + dy + k_2), \tag{2}$$

with $prs \neq 0$.

Proof of Lemma 4. The first step is to calculate $\ell_{11}(F)$, the set of cusp points of F. As the Jacobian of F at (x,y) is given by:

$$J(x,y) = ad - bc + (at - cp - 2br)x + (dp + 2as - bt)y - 2prx^2 + 2psy^2,$$

then $\ell_1 = \ell_1(F)$ can be written as:

$$\ell_1 = \{(x,y) : -r(x - v_1)^2 + s(y - v_2)^2 - \mu = 0\},$$

where

$$V = (v_1, v_2) = \left(\frac{-cp - 2br + at}{4pr}, \frac{-dp - 2as + bt}{4ps} \right),$$

$$W = (w_1, w_2) = \left(\frac{cp - 2br - at}{4pr}, \frac{dp - 2as - bt}{4ps} \right) \text{ and } \mu = -rw_1^2 + sw_2^2.$$

It follows immediately that ℓ_1 is an ellipse if $rs < 0$ and a hyperbola if $rs > 0$. From now on, we will assume the generic condition $\mu \neq 0$.

Notice that $V = (v_1, v_2)$ is the unique point at which the gradient of the Jacobian vanishes. This point does not belong to ℓ_1. The cusp points are those for which the vector tangent to ℓ_1 belongs to the kernel of DF. A vector tangent to ℓ_1 at a point (x,y) is orthogonal to the gradient of the Jacobian. If $\nabla J(x,y)$ denotes the gradient of the Jacobian at the point (x,y) and $\nabla^{\perp} J(x,y)$ is orthogonal to it, then it follows that the solutions of the equation

$$DF(x,y)(\nabla^{\perp} J(x,y)) = 0$$

are V and the cusp points of F. Simple evaluation shows that:

$$DF(x,y).\nabla^{\perp} J(x,y) = (h_1(x,y), h_2(x,y)) = (0,0),$$

where:

$$\begin{aligned} h_1(x,y) &= 4p^2rx^2 + 4p^2sy^2 + (cp^2 + 6bpr - apt)x \\ &\quad + (dp^2 + 6aps - bpt)y + bcp + adp + 2b^2r + 2a^2s - 2abt \end{aligned}$$

and

$$\begin{aligned} h_2(x,y) &= 4prtx^2 + 4psty^2 + 16prsxy + (6dpr + 4ars + cpt - at^2)x \\ &\quad + (6cps + 4brs + dpt - bt^2)y + 2cdp + 2bdr + 2acs - bct - adt. \end{aligned}$$

Change the second equation $h_2(x,y)$ by the linear combination

$$\begin{aligned} h_3(x,y) &= th_1(x,y) - ph_2(x,y) \\ &= -16p^2rsxy + (-6dp - 4as + 6bt)prx + (-6cp - 4br + 6at)psy \\ &\quad + 2bcpt + 2adpt + 2b^2rt + 2a^2st - 2abt^2 - 2cdp^2 - 2bdpr - 2acps, \end{aligned}$$

and solve the system $(h_1(x,y), h_3(x,y)) = (0,0)$. Put the origin at V with the change of variables $(x,y) = (X + v_1, Y + v_2)$. In these new variables one has

$H_1(X,Y) = h_1(x+v_1, y+v_2)$ and $H_3(X,Y) = h_3(x+v_1, y+v_2)$, where:

$$H_1(X,Y) = 4p^2\left(rX^2 - \frac{cp-2br-at}{4p}X + sY^2 - \frac{dp-2as-bt}{4p}Y\right),$$

$$H_3(X,Y) = -16p^2rs\left(XY + \frac{dp-2as-bt}{8ps}X + \frac{cp-2br+at}{8pr}Y\right).$$

Using the values of $W = (w_1, w_2)$ these functions can be expressed as:

$$H_1(X,Y) = 4p^2(rX^2 - rw_1X + sY^2 - sw_2Y) \text{ and}$$
$$H_3(X,Y) = -16p^2rs(XY + \frac{w_2}{2}X + \frac{w_1}{2}Y).$$

Assume now that $w_1w_2 \neq 0$, and consider $(H_1(X,Y), H_3(X,Y)) = (0,0)$, that is:

$$\left\{\begin{array}{l} rX^2 - rw_1X + sY^2 - sw_2Y = 0 \\ XY + \frac{w_2}{2}X + \frac{w_1}{2}Y = 0 \end{array}\right. . \tag{3}$$

From the second equation of this system, it comes that:

$$Y = -\frac{w_2X}{2X+w_1}. \tag{4}$$

Substituting this value into the first equation and simplifying, it follows that:

$$\frac{4rX}{(2X+w_1)^2}P_3(X) = 0,$$

where $P_3(X) = X^3 + \frac{3\mu}{4r}X + 2\frac{\mu w_1}{4r} = 0$. Observe that $P_3(-\frac{w_1}{2}) = -\frac{s}{8r}w_1w_2^2$ is always nonzero by hypothesis; thus if X is a nonzero root of P_3, then (X,Y) belongs to $\ell_{11}(F)$, where Y is given by equation (4).

Now we study the roots of the polynomial P_3 depending on the sign of $r\mu$.

Case 1: $r\mu > 0$.

In this case the derivative of P_3 is always positive and there exists only one nonzero solution of $P_3(X) = 0$ and $rs > 0$.

Case 2: $r\mu < 0$.

In this case $P_3'(X) = 0$ if and only if $X = \pm\sqrt{-\frac{\mu}{4r}}$, and the product of their images is:

$$P_3\left(\sqrt{-\frac{\mu}{4r}}\right)P_3\left(-\sqrt{-\frac{\mu}{4r}}\right) = \frac{s}{r}\left(\frac{\mu w_2}{4r}\right)^2.$$

Therefore if $rs > 0$, $P_3(X) = 0$ has one nonzero solution, and if $rs < 0$, it has three different nonzero solutions.

It remains to consider the case when one of the components of W is zero. Suppose $w_1 = 0$. Then the system (3) can be expressed as:

$$\left\{\begin{array}{l} rX^2 + sY^2 - sw_2Y = 0 \\ XY + \frac{w_2}{2}X = 0 \end{array}\right. .$$

The solutions of this system are:

$$(X,Y) = (0,0),\ (X,Y) = (0,w_2) \text{ and } (X,Y) = \frac{w_2}{2}\left(\pm\sqrt{\frac{-3s}{r}}, -1\right)$$

and the result follows. The case $w_2 = 0$ is similar, so the proof of lemma 4 is finished. □

Proof of Lemma 5. Consider an arbitrary endomorphism in generic normal form as in equation (2), that is:

$$F(x,y) = (pxy + ax + by + k_1, rx^2 + sy^2 + txy + cx + dy + k_2),$$

with $prs \neq 0$. The proof will be direct: first we give a parametric equation for ℓ_1 giving rise to a parametric expression for $F(\ell_1)$ and the we prove that the map is one-to-one over ℓ_1. We begin with the case $rs > 0$.

Making the following linear change of variables $(X,Y) = (\sqrt{rs}x, sy)$, the endomorphism can be expressed as:

$$F(X,Y) = (PXY + AX + BY + K_1, X^2 + Y^2 + TXY + CX + DY + K_2).$$

Consider V, W and μ as in the proof of lemma 4. Now putting the origin of coordinates at the point V, the endomorphism becomes:

$$F(x,y) = \left(pxy + ax + by + k_1, x^2 + y^2 + txy + \frac{at-2b}{p}x + \frac{bt-2a}{p}y + k_2\right).$$

We assume the case $|a| > |b|$, the proof when $|a| < |b|$ is similar. The case $|a| = |b|$ is nongeneric and is obtained when $\mu = 0$.

In the case $|a| > |b|$ we consider the values (a,b) expressed in hyperbolic coordinates, that is:

$$(a,b) = (\rho\cosh(\alpha), \rho\sinh(\alpha));$$

when $|a| < |b|$, $(a,b) = (\rho\sinh(\alpha), \rho\cosh(\alpha))$.

As the Jacobian is $-2px^2 + 2py^2 + 2\dfrac{b^2-a^2}{p}$, it follows that the hyperbola of critical points has two branches:

$$\ell_{1_+}(x,y) = \frac{\rho}{p}(\sinh(\omega), \cosh(\omega)) \text{ and } \ell_{1_-}(x,y) = -\ell_{1_+}(x,y),$$

where ω varies in $(-\infty, \infty)$.

Surprisingly, the image of this hyperbola has a simple expression:

$$F(\ell_{1_+}(x,y)) = (k_1,k_2) + \frac{\rho^2}{2p^2}(\varphi_+(\omega)(p,t) + \psi_+(\omega)(0,-2)) \text{ and}$$
$$F(\ell_{1_-}(x,y)) = (k_1,k_2) + \frac{\rho^2}{2p^2}(\varphi_-(\omega)(p,t) + \psi_-(\omega)(0,-2)),$$

where the functions $\varphi_\pm$ and $\psi_\pm$ are given by:

$$(\varphi_+(\omega), \psi_+(\omega)) = (\sinh(2\omega) + 2\sinh(\omega+\alpha), -\cosh(2\omega) + 2\cosh(\omega+\alpha)), \text{ and}$$
$$(\varphi_-(\omega), \psi_-(\omega)) = (\sinh(2\omega) - 2\sinh(\omega+\alpha), -\cosh(2\omega) - 2\cosh(\omega+\alpha)).$$

As the endomorphism is nondegenerate, (p,t) and $(0,-2)$ are linearly independent vectors, and the proof of the injectivity condition be can reduced to show that the curves $\omega \to (\varphi_+(\omega), \psi_+(\omega))$ and $\omega \to (\varphi_-(\omega), \psi_-(\omega))$ are injective and disjoints.

Take points ω and ν and define:

$$\begin{aligned}(\xi_+, \eta_+) &= (\varphi_+(\omega), \psi_+(\omega)) - (\varphi_+(\nu), \psi_+(\nu)) \\ &= (\sinh(2\omega) + 2\sinh(\omega+\alpha), -\cosh(2\omega) + 2\cosh(\omega+\alpha)) \\ &\quad -(\sinh(2\nu) + 2\sinh(\nu+\alpha), -\cosh(2\nu) + 2\cosh(\nu+\alpha)).\end{aligned}$$

Using the conversion formulas it comes that:

$$\xi_+ = 2\sinh(\omega-\nu)\cosh(\omega+\nu) + 4\sinh\left(\frac{\omega-\nu}{2}\right)\cosh\left(\frac{\omega+\nu+2\alpha}{2}\right)$$
$$\eta_+ = -2\sinh(\omega-\nu)\sinh(\omega+\nu) + 4\sinh\left(\frac{\omega-\nu}{2}\right)\sinh\left(\frac{\omega+\nu+2\alpha}{2}\right).$$

From these expressions it follows that:

$$\xi_+ = 4\sinh\left(\frac{\omega-\nu}{2}\right)\left(\cosh\left(\frac{\omega+\nu+2\alpha}{2}\right) + \cosh\left(\frac{\omega-\nu}{2}\right)\cosh(\omega+\nu)\right)$$
$$\eta_+ = 4\sinh\left(\frac{\omega-\nu}{2}\right)\left(\sinh\left(\frac{\omega+\nu+2\alpha}{2}\right) - \cosh\left(\frac{\omega-\nu}{2}\right)\sinh(\omega+\nu)\right).$$

Now, if $\omega \neq \nu$, then $(\xi_+, \eta_+) = (0,0)$ only if the following equations are verified:

$$\cosh\left(\frac{\omega+\nu+2\alpha}{2}\right) + \cosh\left(\frac{\omega-\nu}{2}\right)\cosh(\omega+\nu) = 0,$$
$$\sinh\left(\frac{\omega+\nu+2\alpha}{2}\right) - \cosh\left(\frac{\omega-\nu}{2}\right)\sinh(\omega+\nu) = 0.$$

It follows that:

$$\cosh\left(\frac{\omega-\nu}{2}\right) = -\frac{\cosh(\frac{\omega+\nu+2\alpha}{2})}{\cosh(\omega+\nu)} = \frac{\sinh(\frac{\omega+\nu+2\alpha}{2})}{\sinh(\omega+\nu)}, \tag{5}$$

Therefore:

$$\cosh\left(\frac{\omega+\nu+2\alpha}{2}\right)\sinh(\omega+\nu) + \sinh\left(\frac{\omega+\nu+2\alpha}{2}\right)\cosh(\omega+\nu) = 0;$$

observe that this equation can be simplified to:

$$\sinh\left(\frac{3\omega+3\nu+2\alpha}{2}\right) = 0.$$

Now it is clear that $(\xi_+, \eta_+) = (0,0)$ implies $\omega+\nu = -\frac{2}{3}\alpha$. But substituting these values in the equation (5) it comes that:

$$\cosh\left(\frac{\omega-\nu}{2}\right) = -\frac{\cosh(\frac{2\alpha}{3})}{\cosh(-\frac{2\alpha}{3})} = \frac{\sinh(\frac{2\alpha}{3})}{\sinh(-\frac{2\alpha}{3})} = -1,$$

which has no solution. Then $(\xi_+, \eta_+) = (0,0)$ implies that $\omega = \nu$. This proves that the curve $\omega \to (\varphi_+(\omega), \psi_+(\omega)$ is injective. The same proof can be made to prove that $\omega \to (\varphi_-(\omega), \psi_-(\omega)$ is also injective. It remains to prove that the image of one component of ℓ_1 does not intersect the image of the other one; the proof of this is similar. Consider the difference of two points, each one in the image of a different component of ℓ_1:

$$\begin{aligned}(\xi,\eta) &= (\varphi_+(\omega), \psi_+(\omega)) - (\varphi_-(\nu), \psi_-(\nu))\\ &= (\sinh(2\omega) + 2\sinh(\omega+\alpha), \cosh(2\omega) - 2\cosh(\omega+\alpha))\\ &\quad -(\sinh(2\nu) - 2\sinh(\nu+\alpha), -\cosh(2\nu) - 2\cosh(\nu+\alpha)).\end{aligned}$$

Using the conversion formulas one has:

$$\xi = 2\sinh(\omega-\nu)\cosh(\omega+\nu) + 4\sinh\left(\frac{\omega+\nu+2\alpha}{2}\right)\cosh\left(\frac{\omega-\nu}{2}\right)$$
$$\eta = -2\sinh(\omega+\nu)\sinh(\omega-\nu) + 4\cosh\left(\frac{\omega+\nu+2\alpha}{2}\right)\cosh\left(\frac{\omega-\nu}{2}\right).$$

Using again these formulas, in all cases there exists a common factor $4\cosh\left(\frac{\omega+\nu}{2}\right)$; it comes that:

$$\xi = 4\cosh\left(\frac{\omega-\nu}{2}\right)\left(\sinh\left(\frac{\omega+\nu+2\alpha}{2}\right) + \sinh\left(\frac{\omega-\nu}{2}\right)\cosh(\omega+\nu)\right)$$
$$\eta = 4\cosh\left(\frac{\omega-\nu}{2}\right)\left(\cosh\left(\frac{\omega+\nu+2\alpha}{2}\right) - \sinh\left(\frac{\omega-\nu}{2}\right)\sinh(\omega+\nu)\right).$$

However, as cosh is always nonzero, from $(\xi,\eta)=(0,0)$ it comes that:

$$\sinh\left(\frac{\omega-\nu}{2}\right) = -\frac{\sinh(\frac{\omega+\nu+2\alpha}{2})}{\cosh(\omega+\nu)} = \frac{\cosh(\frac{\omega+\nu+2\alpha}{2})}{\sinh(\omega+\nu)}.$$

Therefore:

$$\cosh\left(\frac{\omega+\nu+2\alpha}{2}\right)\cosh(\omega+\nu) + \sinh\left(\frac{\omega+\nu+2\alpha}{2}\right)\sinh(\omega+\nu) = 0.$$

Finally, simplifying these equations, it follows that:

$$\cosh\left(\omega+\nu+\frac{\omega+\nu+2\alpha}{2}\right) = \cosh\left(\frac{3\omega+3\nu+2\alpha}{2}\right) = 0.$$

But this equation never has solution. This finishes the proof of the lemma for the case $rs>0$.

We begin now with the proof of lemma 5 for the case $rs<0$. Consider an arbitrary endomorphism in generic normal form with $rs<0$:

$$F(x,y) = (pxy + ax + by + k_1, rx^2 + sy^2 + txy + cx + dy + k_2).$$

With the linear change of variables $(X,Y) = (\sqrt{-rs}x, -sy)$ the endomorphism F can be expressed as:

$$F(X,Y) = (PXY + AX + BY + K_1, X^2 - Y^2 + TXY + CX + DY + K_2);$$

putting the origin of coordinates at the point V, the endomorphism becomes:

$$F(x,y) = \left(pxy + ax + by + k_1, x^2 - y^2 + txy + \frac{at-2b}{p}x + \frac{bt+2a}{p}y + k_2\right).$$

In this case consider the values a and b expressed in trigonometric coordinates, that is $(a,b) = (\rho\cos(\alpha), \rho\sin(\alpha))$. Then parametrize the set of critical points

$$\ell_1 = \left\{(x,y) : x^2 + y^2 = \frac{\rho^2}{p^2}\right\}$$

as

$$\ell_1 = \left\{\frac{\rho}{p}(\sin(\omega), \cos(\omega)) : \omega \in [0, 2\pi)\right\}.$$

The image of this ellipse can be expressed as:

$$F(\ell_1) = (k_1, k_2) + \frac{\rho^2}{2p^2}(\varphi(\omega)(p,t) + \psi(\omega)(0,-2)),$$

where $\omega \in [0, 2\pi)$ and the functions φ and ψ are defined as

$$(\varphi(\omega), \psi(\omega)) = (\sin(2\omega) + 2\sin(\omega + \alpha), \cos(2\omega) - 2\cos(\omega + \alpha)).$$

As the endomorphism is nondegenerate, (p, t) and $(0, -2)$ are linearly independent vectors, and the injectivity condition can be reduced to prove that the map $\omega \to (\varphi(\omega), \psi(\omega))$ is injective.

Take numbers ω and ν and define

$$\begin{array}{rcl}(\xi, \eta) & = & (\varphi(\omega), \psi(\omega)) - (\varphi(\nu), \psi(\nu)) \\ & = & (\sin(2\omega) + 2\sin(\omega + \alpha), \cos(2\omega) - 2\cos(\omega + \alpha)) \\ & & -(\sin(2\nu) + 2\sin(\nu + \alpha), \cos(2\nu) - 2\cos(\nu + \alpha)).\end{array}$$

Using the conversion formulas it comes that:

$$\xi = 2\sin(\omega - \nu)\cos(\omega + \nu) + 4\sin\left(\frac{\omega - \nu}{2}\right)\cos\left(\frac{\omega + \nu + 2\alpha}{2}\right)$$
$$\eta = -2\sin(\omega - \nu)\sin(\omega + \nu) + 4\sin\left(\frac{\omega - \nu}{2}\right)\sin\left(\frac{\omega + \nu + 2\alpha}{2}\right).$$

As above it follows that:

$$\xi = 4\sin\left(\frac{\omega - \nu}{2}\right)\left(\cos\left(\frac{\omega - \nu}{2}\right)\cos(\omega + \nu) + \cos\left(\frac{\omega + \nu + 2\alpha}{2}\right)\right)$$
$$\eta = 4\sin\left(\frac{\omega - \nu}{2}\right)\left(-\cos\left(\frac{\omega - \nu}{2}\right)\sin(\omega + \nu) + \sin\left(\frac{\omega + \nu + 2\alpha}{2}\right)\right).$$

Now, if $\omega \neq \nu \in [0, 2\pi)$, the common factor is always different from zero, and $(\xi, \eta) = (0, 0)$ only if the following equations are verified:

$$\cos\left(\frac{\omega - \nu}{2}\right) = -\frac{\cos(\frac{\omega+\nu+2\alpha}{2})}{\cos(\omega + \nu)} = \frac{\sin(\frac{\omega+\nu+2\alpha}{2})}{\sin(\omega + \nu)}. \tag{6}$$

Therefore:

$$\cos\left(\frac{\omega + \nu + 2\alpha}{2}\right)\sin(\omega + \nu) + \sin\left(\frac{\omega + \nu + 2\alpha}{2}\right)\cos(\omega + \nu) = 0.$$

But observe that we can simplify this equation as:

$$\sin\left(\omega + \nu + \frac{\omega + \nu + 2\alpha}{2}\right) = \sin\left(\frac{3\omega + 3\nu + 2\alpha}{2}\right) = 0.$$

So it follows that for $(\xi, \eta) = (0, 0)$ it is necessary that $\omega + \nu = -\frac{2}{3}\alpha$. Now substituting these values into the equation (6), it comes that:

$$\cos\left(\frac{\omega - \nu}{2}\right) = -\frac{\cos(\frac{2\alpha}{3})}{\cos(-\frac{2\alpha}{3})} = \frac{\sin(\frac{2\alpha}{3})}{\sin(-\frac{2\alpha}{3})} = -1.$$

Observe that the unique solution is $\omega = \nu$, and the curve is injective. □

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