

# Robustly expansive homoclinic classes

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## Abstract

Let  $f : M \rightarrow M$  be a diffeomorphism defined in a 3-dimensional compact boundary-less manifold  $M$ . We prove that for an open dense set,  $C^1$ -robustly expansive homoclinic classes  $H(p)$  for  $f$  are hyperbolic. A diffeomorphism  $f$  is  $\alpha$ -expansive on a compact invariant set  $K$  if there is  $\alpha > 0$  such that for all  $x, y \in K$ , if  $\text{dist}(f^n(x), f^n(y)) \leq \alpha$  for all  $n \in \mathbb{Z}$  then  $x = y$ . By robustly we mean that there is  $\alpha > 0$  such that for all nearby diffeomorphism  $g$  the homoclinic class  $H(p_g)$  of the continuation of  $p$  is  $\alpha$ -expansive.

## 1 Introduction

Let  $M$  be a compact connected boundary-less Riemannian 3-dimensional manifold and  $f : M \rightarrow M$  a homeomorphism. Let  $K$  be a compact invariant subset of  $M$  and  $\text{dist} : M \times M \rightarrow \mathbb{R}$  a metric on  $M$  and  $\alpha > 0$ . We say that  $f$  restricted to  $K$  is  $\alpha$ -expansive when  $\text{dist}(f^n(x), f^n(y)) \leq \alpha$  for all  $x, y \in K$  and all  $n \in \mathbb{Z}$  implies  $x = y$ . The number  $\alpha > 0$  is called a constant of expansiveness for  $f$  and  $K$ , and sometimes we say that  $f$  is expansive in  $K$  if  $\alpha$  is fixed. Expansiveness is a property shared by a large class of dynamical systems exhibiting chaotic behavior. Roughly speaking, an expansive dynamical system is one in which two different trajectories can be distinguished by an observer with an instrument capable to distinguish points at a distance greater than a certain constant  $\alpha > 0$  (constant of expansiveness). Examples of expansive systems are for instance diffeomorphisms  $f : M \rightarrow M$  acting hyperbolically in an  $f$ -invariant compact subset  $\Lambda$  of a manifold  $M$ . This includes for instance Anosov systems or the non-wandering set of Axiom A diffeomorphisms. Other types of examples are the pseudo-Anosov homeomorphisms in surfaces of genus greater than 1 [HT]. Another examples are given by sub-shifts

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of finite type defined in  $N^{\mathbb{Z}}$  endowed with the Tychonov topology, where  $N$  is a finite set of symbols. In this paper we show that for an open dense subset,  $C^1$ -robustly expansive homoclinic classes are hyperbolic. By *homoclinic class* we mean the closure of the transversal homoclinic points associated to a hyperbolic periodic orbit  $p$ . By *robustly expansive* homoclinic classes we mean that for all nearby diffeomorphism  $g$ , the homoclinic class of the continuation of  $p$  is expansive with the same constant of expansiveness as the one for  $f$  restricted to  $H(p)$ . To state our results in a precise way, let us introduce some notations and definitions.

Throughout  $M$  is a compact boundary-less 3-dimensional manifold,  $\text{Diff}^r(M)$ ,  $r \geq 1$ , is the space of  $C^r$  diffeomorphisms on  $M$  with the  $C^r$  topology. If  $x \in M$  then  $\mathcal{O}_f(x)$  denotes the orbit of  $x$  by  $f$ ,  $W^s(x, f)$  and  $W^u(x, f)$  denote the stable and unstable manifolds of  $x$  respectively, and  $H(p, f)$  denotes the homoclinic class of a periodic point  $p$ . When no ambiguity is possible we denote these sets by  $\mathcal{O}(x)$ ,  $W^s(x)$ ,  $W^u(x)$ , and  $H(p)$  respectively. Similarly we denote by  $W_\epsilon^s(x)$  and  $W_\epsilon^u(x)$  to the  $\epsilon$ -local stable and unstable manifolds of points  $x \in M$  respectively. When  $\epsilon > 0$  is fixed we omit to mention it and just say local stable manifold, etc.

When these local manifolds are one-dimensional the concepts of diameter and length are usually enough for us to describe their properties. But if they are  $k$ -dimensional,  $k > 1$ , we need another concept.

Let us assume that  $W_\epsilon^\sigma(x)$ ,  $\sigma = s, u$  is an embedded topological  $k$ -disk,  $k \geq 1$ . Let  $\Gamma$  be the family of all parameterized continuous arcs  $\gamma : [0, 1] \rightarrow W_\epsilon^\sigma(x)$  joining  $x$  with the boundary of  $W_\epsilon^\sigma(x)$ .

**Definition 1.1.** We call the size of  $W_\epsilon^\sigma(x)$  to  $\inf\{\text{diam}(\gamma) / \gamma \in \Gamma\}$ .

In the case that  $H(p)$  has topological dimension zero (like occurs in Smale's horseshoe, see example 2.1) the fact that  $H(p)$  is expansive has very poor implications from the hyperbolic point of view. For instance we may have non-hyperbolic periodic points in  $H(p)$  (see example 2.2). So to obtain nice conclusions from expansiveness on  $H(p)$ , we need to assume that it holds not only for the homoclinic class of a single diffeomorphism  $f$ , but that it is shared by diffeomorphisms in a neighborhood of  $f$  (see examples 2.2 and 2.3). That is, we need the homoclinic class to be robustly expansive. Let us precise this notion.

**Definition 1.2.**  $H(p)$  is  $C^r$ -robustly expansive ( $r \geq 1$ ) iff there exist  $\alpha > 0$  and a  $C^r$ -neighborhood  $\mathcal{U}(f)$  of  $f$  such that for all  $g \in \mathcal{U}(f)$ , there exists a continuation  $p_g$  of  $p$  such that  $g$  is  $\alpha$ -expansive in  $H(p_g)$ .

To state in a precise way our results we need to introduce the following definitions.

**Definition 1.3.** We say that a compact  $f$ -invariant set  $\Lambda$  admits a dominated splitting if the tangent bundle  $T_\Lambda M$  has a continuous  $f$ -invariant splitting  $E \oplus F$  and there exist  $C > 0$ ,  $0 < \lambda < 1$  such that

$$\|Df^n|E(x)\| \cdot \|Df^{-n}|F(f^n(x))\| \leq C\lambda^n \forall x \in \Lambda, n \geq 0.$$

**Definition 1.4.** We say that a compact  $f$ -invariant set  $\Lambda$  is hyperbolic if the tangent bundle  $T_\Lambda M$  has a continuous  $f$ -invariant splitting  $E \oplus F$  and there exist  $C > 0$ ,  $0 < \lambda < 1$  such that

$$\|Df^n|E(x)\| \leq C\lambda^n \forall x \in \Lambda, n \geq 0,$$

and

$$\|Df^{-n}|F(x)\| \leq C\lambda^n \forall x \in \Lambda, n \geq 0.$$

Our main results are

**Theorem A.** Let  $f \in \text{Diff}^r(M)$ ,  $r \geq 1$ , with a hyperbolic periodic point  $p$  such that its homoclinic class  $H(p)$  is robustly expansive. Then for an open dense subset  $\mathcal{N}$  in a neighborhood of  $f$  in the  $C^1$ -topology,  $\mathcal{U}(f)$ , it holds that if  $g \in \mathcal{N}$  then  $H(p_g)$  has a co-dimension one dominated splitting  $E \oplus F$ .

**Theorem B.** Let  $f \in \text{Diff}^r(M)$ ,  $r \geq 1$ , with a hyperbolic periodic point  $p$  such that its homoclinic class  $H(p)$  is robustly expansive and has a co-dimension one dominated splitting  $E \oplus F$ . Then for an open dense subset  $\mathcal{N}$  in a neighborhood of  $f$  in the  $C^1$ -topology,  $\mathcal{U}(f)$ , it holds that if  $g \in \mathcal{N}$  then  $H(p_g)$  is hyperbolic.

Both theorems above give

**Theorem C.** Generically,  $C^1$ -robustly expansive homoclinic classes are hyperbolic.

## 1.1 Comments

Ricardo Mañé studied the case in which for  $f : M \rightarrow M$  there is a  $C^1$ -neighborhood  $\mathcal{U}$  of  $f$  such that for  $g \in \mathcal{U}$ ,  $g$  is expansive (i.e.:  $f$  is in the  $C^1$ -interior of the expansive diffeomorphisms, see [Ma1]). He proved that in that case  $f$  is quasi-Anosov, that is  $Df : TM \rightarrow TM$  is expansive (equivalently: if  $0 \neq v \in TM$  then  $\|Df^n(v)\| \rightarrow \infty$  either for  $n \rightarrow +\infty$  or for  $n \rightarrow -\infty$ ). It follows (see [Ma1]) that  $f$  is Axiom A and verifies the no-cycle condition. In particular all the homoclinic classes are hyperbolic.

In our case we do not assume that  $f$  is expansive in the whole manifold, but that there is a homoclinic class  $H(p)$  of an  $f$ -periodic point  $p$  such that  $f$  is expansive

in  $H(p)$  and that the same is valid for  $g$   $C^1$ -close to  $f$  restricted to  $H(p_g)$  the homoclinic class of  $p_g$  the continuation of  $p$ .

Without loss of generality we may assume that  $H(p)$  is the homoclinic class of an  $f$ -hyperbolic fixed point  $p$  of index 2, that is,  $\dim W^s(p) = 2$ . Observe that this hypothesis about the index of  $H(p)$  is not a restriction and that the requirement  $p$  hyperbolic seems to be relevant to have uniqueness of  $p_g$ , the continuation of  $p$ .

Moreover we will see in §2.3 that if  $p$  is not hyperbolic then the class  $H(p)$  cannot be  $C^1$ -robustly expansive. This result will follow applying Lemma 3.1. Although we sketch the proof only in an example it can be extended easily to a general case. We remark that Lemma 3.1 is false in the  $C^r$ -topology,  $r \geq 2$ , as has been proved in [PS2]. Thus it is not necessarily true that if  $H(p)$  is  $C^r$ -robustly expansive with  $r \geq 2$  then  $p$  (or another periodic point  $q \in H(p)$ ) has to be hyperbolic.

Once we assume that  $p$  is hyperbolic it seems natural to assume also that  $H(p) \neq \{p\}$ , otherwise the dynamics should be trivial. Thus we assume that  $p$  is not a hyperbolic repeller nor a hyperbolic attractor. On the other hand, in these cases  $H(p)$  is trivially robustly expansive.

If we consider diffeomorphisms on surfaces then Theorems A and B follow from [PS1, Theorem A, Theorem B], taking into account that we cannot have an invariant circle in  $H(p)$  if  $f|_{H(p)}$  is expansive.

Observe that it is supposed in [PS1] that  $f$  is  $C^2$  and that all the periodic points are hyperbolic of saddle type. We also have to assume in part of our arguments that  $f$  is  $C^2$ . But, as  $C^2$ -diffeomorphisms are dense in the  $C^1$  topology we finally obtain that such properties hold densely in the  $C^1$  topology and this is enough to achieve our result.

One of the problems we have to face is the following: Let  $q$  be a periodic point in  $H(p)$ . If it were true that *always* for  $q$  we have  $W^s(q) \cap W^u(p) \neq \emptyset$  and  $W^u(q) \cap W^s(p) \neq \emptyset$  then we can show using small  $C^1$ -perturbations that all periodic points have to be hyperbolic and all the intersections have to be transverse. But if we do not have this property, when perturbing  $f$  it may happen that the point  $q$  "escapes" from the homoclinic class. To avoid this problem we assume certain  $C^1$ -generic properties (see Theorems 3.6 and 3.8). On the other hand, it seems natural to try to remove these assumptions.

## 1.2 Sketch of the proof

We now give an idea of the proofs of Theorem A and Theorem B. We first construct a dominated splitting for  $H(p)$ . For this we start by proving that robustly expansive homoclinic classes do not have Lyapunov stable points and hence there are not sources or sinks on them. Moreover, we cannot have invariant arcs or closed curves in a compact invariant set  $\Lambda$  if  $f|_{\Lambda}$  is expansive. And if  $\Lambda = H(p)$  we cannot have an

arc contained in the intersection of the stable manifold of a periodic point  $q \in H(p)$  with the unstable manifold of a periodic point  $r \in H(p)$  with an heteroclinic connection. Otherwise the use of blenders as in [BD] will imply that this curve has to be contained in  $H(p)$ , what cannot occur by expansiveness (see Lemma 3.10). Thus periodic points homoclinically related have the same index. Moreover, robustly expansive homoclinic classes cannot have tangency between stable and unstable manifolds of their periodic points (see Lemma 3.9). This will imply that the angle between those manifolds is uniformly bounded away from zero (see §4.2). Moreover  $C^1$ robustness of expansiveness will imply that there exist  $1 > \delta > 0$  such that if  $q$  is a periodic point of period  $\tau$  then if  $\lambda$  is a contracting eigenvalue of  $Df_q^\tau$  then  $|\lambda| < (1 - \delta)^\tau$  and if  $\mu$  is an expanding eigenvalue of  $Df_q^\tau$  then  $|\mu| > (1 + \delta)^\tau$ . All these facts together implies the existence of a dominated splitting  $E \oplus F$  (with  $F$  being one-dimensional (see §4.3)). This will prove Theorem A.

The next step is to prove that this splitting is hyperbolic for a residual subset. First we note that robust expansiveness prevents  $H(p)$  from having a periodic point  $q$  such that  $Df_q^\tau$  has a poor rate of contraction in  $E_q$ . If we had that  $Df$  expands uniformly in  $F$  then a result of Mañé [Ma3, Theorem II.1, p. 173 ] would imply hyperbolicity of  $E \oplus F$ . Since we can not assume this we prove next that the center-unstable manifolds, which are tangent to  $F$ , have good dynamical properties as diameter bounded away from zero and that they are true unstable manifolds when restricted to periodic points (see §4.1). It is here that we need, by technical reasons, to assume that  $f$  is  $C^2$  in order to make use of [PS1, Proposition 3.1, p. 987] that enables us to control the  $\omega$ -limit of certain points which rest in a neighborhood of  $H(p)$  for all positive iterates. The properties of the local unstable manifolds replace that of uniform expansion in  $F$  there, and together with the fact that  $H(p)$  has no periodic points where  $Df$  contracts weakly in the period in  $E$ , are enough to obtain that  $E$  is a contracting bundle (see §5.3). Finally, using the result of Mañé mentioned above, we obtain that  $E \oplus F$  is hyperbolic. Once we have this, it follows that the same is true for a  $C^1$ -neighborhood of  $f$  and we can conclude Theorem B.

This paper is organized as follows. In Section 2 we give examples and counterexamples of robustly expansive homoclinic classes. In Section 3 we recall some results proved elsewhere and that will be used to prove our results and establish some basic properties of robustly expansive homoclinic classes. In Section 4 we construct a dominated splitting for robustly expansive homoclinic classes proving Theorem A, and in Section 5 we conclude that densely this dominated splitting is hyperbolic proving our Theorem B.

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## 2 Examples and counterexamples

### 2.1 The horseshoe: $C^r$ robustly expansive for all $r \geq 1$

The usual Smale's horseshoe is hyperbolic and therefore an example of a  $C^r$ -robustly expansive homoclinic class for all  $r \geq 1$ . It may be seen as the homoclinic class of any of its periodic points which are all hyperbolic. This example exhibits two fixed points,  $p$ ,  $q$ . See Figure 1. Multiplying this 2-dimensional horseshoe by a strong linear expansion in the  $Oz$  direction we obtain examples of robustly expansive homoclinic classes in any 3-dimensional manifold.

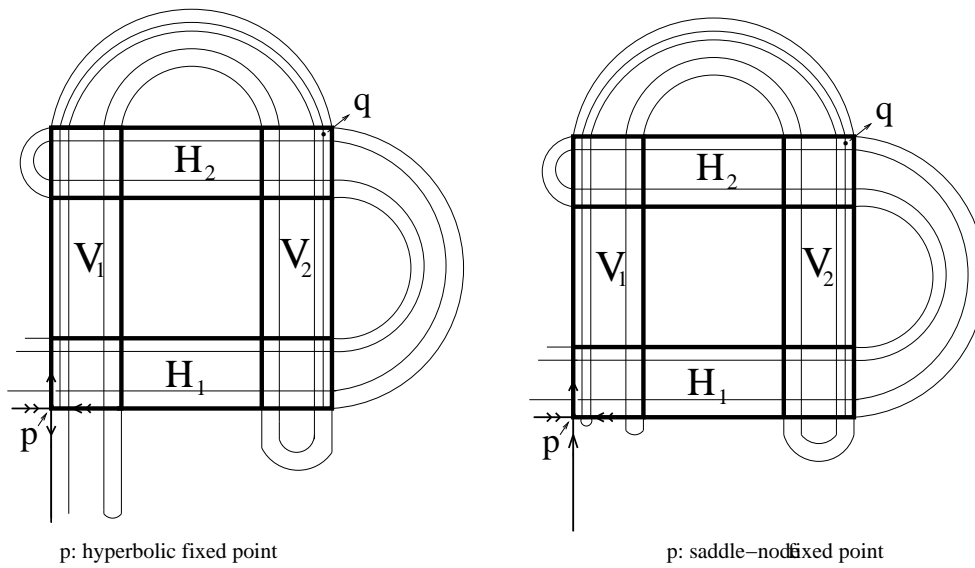


Figure 1: Horseshoes,  $p$  hyperbolic and saddle-node

### 2.2 Critical saddle-node horseshoe: neither $C^2$ nor $C^1$ robustly expansive

Observe that in Smale's example, the class itself does not require the hyperbolicity of the fixed point  $p$ . To see this consider a square  $C$  of sides  $b, t, l, r$  of length 1, with two horizontal sub-rectangles  $H_1, H_2$  and two vertical sub-rectangles  $V_1, V_2$  as in Figure 1. The length of the horizontal sides of  $H_1, H_2$  is 1 and the vertical sides of  $H_1$  and  $H_2$  have length  $1/10$ .  $H_1$  has its bottom coinciding with  $b$ , the bottom of  $C$ , and  $H_2$  has its top coinciding with  $t$ , the top of  $C$ .  $V_1, V_2$  have horizontal sides of length  $1/10$  and vertical sides of length 1.  $V_1$  has its left side coinciding with the

left side,  $l$ , of  $C$  and  $V_2$  has its right side coinciding with  $r$ , the right side of  $C$ . Let  $f : S^2 \rightarrow S^2$  be a diffeomorphism such that  $f(H_1) = V_1$ ,  $f(H_2) = V_2$ ,  $f$  contracts linearly horizontal fibers in  $H_1$ , but  $f$  has eigenvalue 1 at the fixed point  $p$  in the leftmost bottom corner intersection of  $H_1$  with  $V_1$ . Moreover, it is not difficult to prove that all the other periodic points of this horseshoe are hyperbolic, see [DRV, Theorem B]. Furthermore, it is possible to construct this example in such a way that any  $C^2$ -perturbations  $g$  of  $f/H(p)$ , preserving the fixed point  $p$ , is conjugate to Smale's horseshoe (2.1). Hence  $H(p_g)$  is expansive. As a matter-of-fact, the constant of expansiveness is equal to the minimum gap between horizontal and vertical rectangles.

But if we perturb in such a way that the fixed point  $p$  disappears then we may lose expansiveness (see Figures 1 and 2) due to the appearance of a tangency. Of course, in this case we cannot speak of the class of  $p$ . But the class persists: it may be seen, for instance, as the homoclinic class of  $q$ , the other fixed point in the example.

Thus a horseshoe with a *critical saddle-node*, that we call saddle-node horseshoe, is not  $C^2$  robustly expansive and therefore is not  $C^1$  expansive.

As in the example given above (2.1) we can transform the saddle-node horseshoe in a three dimensional one.

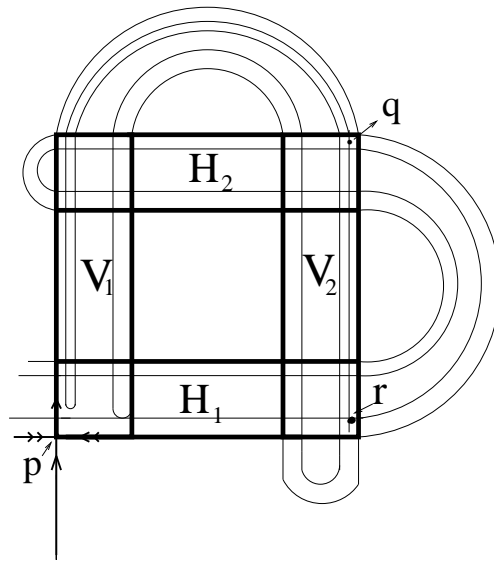


Figure 2: Perturbing the critical saddle-node

### 2.3 Non-critical saddle-node horseshoe: $C^2$ -robustly expansive but not $C^1$ -robustly expansive

Consider a non-critical saddle-node horseshoe  $H(p)$  like that of Figure 3, where below  $p$  we have another saddle  $r$ . It is possible to construct this example in such a way that for any  $C^2$  perturbation such that the fixed point  $p$  disappears, all the periodic points of the perturbed system are hyperbolic, and, if  $\mu$  is an expanding eigenvalue of a periodic point  $q$  of period  $\tau$  then  $|\mu| > (1 + \delta)^\tau$ , for a fixed  $\delta > 0$ . Indeed, to assure this it is enough to build the example with a Markov partition such that the horizontal sub-rectangles have vertical diameter sufficiently small, and the vertical sub-rectangles have horizontal diameter sufficiently small. Since all the periodic points are hyperbolic for the perturbed system and it has a dominated splitting, using [PS1, Theorem 3.1] we obtain that the perturbed system is hyperbolic. Note that by construction, the perturbed system is conjugated to a Smale's horseshoe (2.1). Hence, the non-critical saddle-node horseshoe gives an example of a  $C^2$  robustly expansive homoclinic class.

Let us see that this example is not  $C^1$  robustly expansive. Using transitions maps as in Proposition 4.4 (see [BDP]), we can find a hyperbolic periodic point  $\bar{q} \in H(p)$  with a normalized eigenvalue arbitrarily close to 1 accumulated by points in  $H(p)$  from both sides of  $W_\epsilon^u(\bar{q})$ . Thus we can unfold the periodic point  $\bar{q}$  into three periodic points  $q_1, \bar{q}, q_2$ , with  $q_1, q_2 \in H(p)$  and with the same period as  $\bar{q}$ , and such that  $\text{dist}(f^n(q_1), f^n(q_2)) \leq \epsilon$  for all  $n \in \mathbb{Z}$ , contradicting robust expansiveness.

### 2.4 Generalized pseudo-Anosov of $S^2$ : a non expansive homoclinic class

The following is an example in which the homoclinic class is maximal and almost all periodic points are hyperbolic, but it is not expansive for any  $\alpha > 0$ . In particular, it is not robustly expansive.

If we consider in  $S^2$  a generalized pseudo-Anosov map like that given in [BLJ, Chapter 9], then for all  $\epsilon > 0$  the  $\epsilon$ -local stable and unstable sets of infinitely many periodic points are not locally connected as has been pointed out in [Le1]. As a consequence of this lack of local connectedness the homoclinic class (which for all periodic point is all  $S^2$ ) is  $\epsilon$ -expansive for no  $\epsilon > 0$  (in fact this is consequence of the general classification of expansive homeomorphisms on surfaces see ([Le1, Hi]).

Let us show that the  $\epsilon$ -local stable and unstable sets are not locally connected. For this we proceed as follows.

Take in  $\mathbb{R}^2$  the action given by the matrix  $\begin{pmatrix} 13 & 8 \\ 8 & 5 \end{pmatrix}$  and pass it to the quotient  $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ . This gives us a very well known Anosov diffeomorphism  $a : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ .



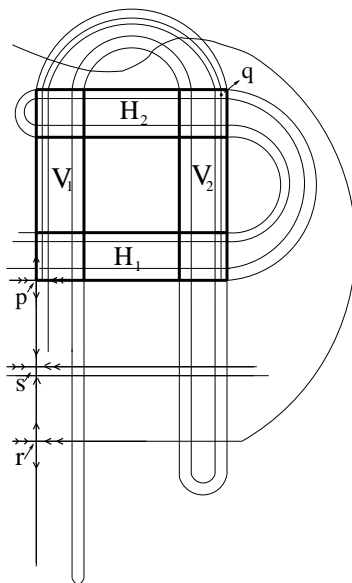


Figure 3: Non-critical saddle-node

In  $\mathbb{T}^2$  (which is an abelian group) we consider the relation  $X \sim -X$ . This is the same as to consider this relation in a fundamental domain in  $\mathbb{R}^2$ . The quotient  $\mathbb{T}^2 / \sim$  gives the sphere  $S^2$ . Moreover the relation  $X \sim -X$  is compatible with the Anosov map  $a$ , i. e.:  $a(X) \sim -a(X) = a(-X)$  by linearity, and therefore projects to  $S^2$  as a generalized pseudo-Anosov map  $g : S^2 \rightarrow S^2$ ,  $g(x) = \Pi(a(\Pi^{-1}(x)))$  that has 1-prongs (that is, points  $P \in S^2$  in which the local stable and unstable "manifolds" are arcs finishing at  $P$  (see Figure 4 where  $O$  is a point with 1-prongs). Observe that the projection  $\Pi : \mathbb{T}^2 \rightarrow S^2$  is a branched covering and that the definition of  $g$  does not depend on the pre-image of  $x$  by  $\Pi^{-1}$ .

One of the 1-prongs in  $S^2$  is the image  $O$  (via the projection map from  $\mathbb{T}^2$  to  $S^2$ ) of  $(0, 0) \in \mathbb{T}^2$  which is fixed by  $g$ . It is not difficult to see that the unstable manifold of  $(0, 0)$  projects to  $S^2$  as an arc ending at  $O$  (because  $x \sim -x$ ). The stable and unstable manifolds of the points in  $\mathbb{T}^2$  near  $(0, 0)$  projects to points in  $S^2$  near  $O$  like in Figure 4. The intersection of the stable and unstable manifolds of the points  $(0, x)$  and  $(0, -x)$  consists of four points identified by pairs by the relation  $X \sim -X$ . If  $X \in \mathbb{T}^2$  projects to  $x \in S^2$ , let us call  $s_x$  and  $u_x$  to the projections of the local stable and unstable manifolds respectively of the point  $X$ .

One may suppose at first sight that these projections are the local stable and unstable sets of the points in  $S^2$  under the action given by  $g$ , but this is not true. To see this, let us choose a 1-prong  $O$  and let  $p$  be a periodic point very close to

$O$  of, say, period  $k$ . More precisely, given  $\epsilon > 0$  choose  $p \neq O$  a periodic point satisfying  $\text{dist}(p, O) < \epsilon/4$ . Such a point exists since periodic points are dense for the Anosov diffeomorphism  $a$  defined on  $\mathbb{T}^2$  and projects on  $S^2$  as periodic points for  $g$ . Let  $\{p, p'\} = s_p \cap u_p$ . We take  $\epsilon$  sufficiently small in order to have  $p' \in W_{\epsilon/2}^u(p) \cap W_{\epsilon/2}^s(p)$ . Therefore there is a small arc  $\sigma \subset u_p, p'$  belonging to the interior of  $\sigma$ , such that  $\sigma \subset W_{3\epsilon/4}^u(p)$ . Then there is a sub-arc  $\gamma_1 \subset u_{g^k(p')}$ ,  $g^k(p') \in \gamma_1$  such that  $\gamma_1 \subset W_\epsilon^u(p, g^k)$ . Iterating by  $g^k$  we see that the same is true for  $u_{g^{kn}(p')}$  for all integer  $n > 0$ . That is, for all  $n$ , there is an arc  $\gamma_n \subset W_\epsilon^u(p, g^k)$ . Let us see that the union of these arcs is not locally connected. To see it observe that there is a residual set  $X$  of points in the torus  $\mathbb{T}^2$  whose orbits under  $a$  are dense and therefore these orbits project in  $S^2$  as orbits dense under the action of  $g$ . Thus given any point between two arcs  $\gamma_n$  and  $\gamma_{n+1}$  and close to  $p$ , there is a point  $x \in X$  between  $\gamma_n$  and  $\gamma_{n+1}$ . This point cannot be in  $W_\epsilon^u(p, g^k)$  because its orbit is dense. Hence the arc  $u_x \not\subset W_\epsilon^u(p, g^k)$  separates  $\gamma_n$  from  $\gamma_{n+1}$  and therefore  $W_\epsilon^u(p, g^k)$  is not locally connected.

Arguing similarly we see that the same is true for  $W_\epsilon^s(p)$ , i.e.: there are infinitely many disjoint arcs  $\beta_n \subset W_\epsilon^s(p, g^k)$ .

We conclude that the orbit of the points  $x_{n,m}, x'_{n,m}$  defined by the intersection of the arc  $\gamma_n \subset W_\epsilon^u(p, g^k)$  with the arc  $\beta_m \subset W_\epsilon^s(p, g^k)$  are  $\epsilon$ -close to the orbit of  $p$  for all the iterates by  $g$  (positives and negatives).

As  $\epsilon > 0$  is arbitrary, it follows that for no  $\epsilon > 0$  we have that  $g$  is  $\epsilon$ -expansive. Moreover,  $g$  is transitive and hyperbolic periodic points are dense on  $S^2 = \Omega(g)$ . In fact all periodic points are hyperbolic except points like those identified with  $O$ . But there are only four points like  $O$ . As all the stable and unstable manifolds of the Anosov  $a : T^2 \rightarrow T^2$  are dense in  $T^2$  it follows that in  $S^2$  homoclinic classes  $H(p)$  of all the periodic points  $p$  coincide and are equal to  $S^2$ . Thus the homoclinic class  $H(p)$  is maximal for all  $p$  periodic (hyperbolic or not).

There are real analytic models for  $g : S^2 \rightarrow S^2$  (see [Ge], and [LL]).

### 3 Preliminary results

We assume in the sequel that  $p$  is hyperbolic and that  $Df_p$  has one eigenvalue  $\mu$  of modulus greater than 1 and two  $\lambda_1$  and  $\lambda_2$ , with moduli less than 1.

We recall that in [Nh, Lemma 2.3], it is proved that there exists a residual subset in  $H(p)$  of points  $x \in H(p)$  such that its forward and backward orbits are dense in  $H(p)$ . In particular  $H(p)$  is always transitive. Moreover hyperbolic periodic points are dense in  $H(p)$  ([Sm]).

A fundamental tool to be used is the following result due to J. Franks.

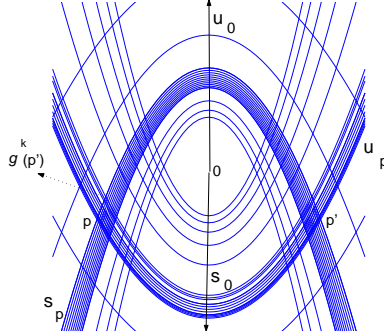


Figure 4: Generalized pseudo-Anosov

**Lemma 3.1.** *Let  $M$  be a closed  $n$ -manifold and  $f : M \rightarrow M$  be a  $C^1$  diffeomorphism. There exists  $\epsilon_0(f) > 0$  such that for all  $0 < \epsilon < \epsilon_0(f)$  there exists  $\delta = \delta(\epsilon, f)$  such that if  $S \subset M$  is a finite set,  $S = \{p_1, p_2, \dots, p_m\}$  and  $L_i, i = 1, \dots, m$  are linear maps  $L_i : TM_{f(p_i)} \rightarrow TM_{f(p_i)}$  satisfying  $\|L_i - Id\| \leq \delta, i = 1, \dots, m$  there exists  $g \in \text{Diff}^1(f)$  satisfying  $D_{p_i}g = L_i \circ D_{p_i}f$  and  $d_1(f, g) < \epsilon$ . Moreover, if  $U$  is a neighborhood of  $S$  then we may choose  $g$  so that  $g(x) = f(x)$  for all  $x \in \{p_1, p_2, \dots, p_m\} \cup (M \setminus U)$ .*

*Proof.* See [Fr, Lemma 1.1]. The statement given there is slightly different from that above, but the proof of our statement is contained in [Fr].  $\square$

**Lemma 3.2.** *Let  $p$  be a hyperbolic fixed point for  $f$  of index 2 and  $H(p)$ , its homoclinic class, robustly  $\alpha$ -expansive. If  $x \in W^s(p) \cap W^u(p)$  then  $W^s(p)$  intersects transversally  $W^u(p)$  at  $x$ .*

*Proof.* By our assumptions  $W^s(p)$  is an Euclidean plane and  $W^u(p)$  a line both immersed in  $M$ . If the intersection at  $x$  of  $W^s(p)$  and  $W^u(p)$  is not transversal we should have that  $T_x W^u(p) \subset T_x W^s(p)$ . Let  $D_x \subset W_\epsilon^s(x) \cap W^s(p)$  a small disk and  $N > 0$  be such that  $f^N(D_x) \subset W_\epsilon^s(p)$ , and let  $L_x$  be a small arc in  $W_\epsilon^u(x) \cap W^u(p)$  such that  $f^{-N}(L_x) \subset W_\epsilon^u(p)$ . A small  $C^1$  perturbation, like those in Lemma 3.1, will produce a diffeomorphism  $g \in \mathcal{U}(f)$  in which the corresponding stable and unstable manifolds of  $p_g$  will intersect each other in two points  $x', x''$  close enough to  $x$  and such that  $D_{x',g} \ni x'', L_{x',g} \ni x''$ . Choosing  $0 < \epsilon < \alpha/4$  and  $g$  sufficiently close to  $f$  in order to have that  $g^N(D_{x',g}) \subset W_\epsilon^s(p_g)$ , and  $g^{-N}(L_{x',g}) \subset W_\epsilon^u(p_g)$  and  $\text{dist}(g^j(x'), g^j(x'')) < \epsilon, \forall j : j = -N, \dots, N$ , this would imply that  $x', x'' \in H(p_g)$  and at the same time  $\text{dist}(g^n(x'), g^n(x'')) \leq \alpha$  for all  $n \in \mathbb{Z}$  contradicting robust expansiveness.  $\square$

A hyperbolic periodic point  $p$  is  $C^1$ -far from homoclinic tangency if every diffeomorphism  $g$   $C^1$ -close to  $f$  does not have homoclinic tangency associated to  $p_g$ , the continuation of  $p$ .

**Corollary 3.3.** *Under the hypothesis of Lemma 3.2 we have that  $p$  is  $C^1$ -far from homoclinic tangency.*

**Definition 3.1.** *We say that  $x$  is a Lyapunov stable point in  $H(p)$  if for all  $\epsilon > 0$  there is  $\delta > 0$  such that there is a neighborhood  $V(x)$  in  $H(p)$  of diameter less than  $\delta > 0$  such that for all  $n \geq 0$  we have  $\text{diam}(f^n(V(x))) < \epsilon$ .*

**Lemma 3.4.** *There are no Lyapunov stable points in  $H(p)$  if  $H(p)$  is  $\alpha$ -expansive.*

*Proof.* Observe that if  $x, y \in H(p)$  verify  $\text{dist}(f^n(x), f^n(y)) < \epsilon$  for all  $n \geq 0$  then  $\text{dist}(f^n(x), f^n(y)) \xrightarrow[n \rightarrow \infty]{} 0$ . For otherwise we may take subsequences  $\{f^{n_k}(x)\}$ ,  $\{f^{n_k}(y)\}$  converging to points  $z, w$  respectively such that  $\text{dist}(z, w) > 0$  and  $\text{dist}(f^n(z), f^n(w)) \leq \epsilon$  for all  $n \in \mathbb{Z}$ , thus contradicting expansiveness of  $f$  in  $H(p)$  if  $\epsilon < \alpha$ .

Now assume that  $x \in H(p)$  is Lyapunov stable. Choose  $0 < \epsilon < \alpha/4$  and the corresponding  $\delta > 0$  and  $V(x)$ . As  $x \in H(p)$  there exists  $y \in W^u(p) \cap W^s(p)$  in  $V(x)$ . Let  $N$  be such that  $\text{dist}(f^n(y), p) < \epsilon$  if  $n \geq N$ . It follows that  $\text{dist}(f^n(x), p) < 2\epsilon < \alpha$  and therefore by the observation at the beginning of the proof  $\text{dist}(f^n(x), p) \xrightarrow[n \rightarrow \infty]{} 0$ . Thus  $x \in W^s(p)$ . Hence there is  $N' > 0$  such that both  $f^{N'}(y)$  and  $f^{N'}(x)$  are in  $W_\epsilon^s(p)$ . As  $x, y \in H(p)$  they are accumulated by points in  $H(p)$  and therefore there is  $y' \in W_\epsilon^u(y) \cap W^s(p)$  different from  $y$  and  $x$ , as close to  $y$  as we wish,  $y' \in V(x)$ , and with  $\epsilon' > 0$  so small that  $f^{N'}(y') \in W_\epsilon^u(f^{N'}(y))$ . By expansiveness it follows that there is  $n > N'$  such that  $\text{dist}(f^n(y'), p) > \alpha$ . But then  $\text{dist}(f^n(x), f^n(y')) > \alpha - \epsilon > \epsilon$  contradicting that  $\text{diam}(f^n(V(x))) < \epsilon$  for all  $n \geq 0$ .  $\square$

We denote by sink any attracting periodic point and by source any repelling periodic point of  $f$ .

**Remark 3.5.** *It follows from Lemma 3.4 that there are no sinks in  $H(p)$ . Similarly, arguing with  $f^{-1}$ , we may prove that there are no sources in  $H(p)$ .*

The following result is in [CMP].

**Theorem 3.6.** *There exists a residual set  $\mathcal{T}$  in  $\mathcal{U}(f)$  such that for  $g \in \mathcal{U}(f)$ ,  $\text{Cl}(W^s(p_g, g)) \cap \text{Cl}(W^u(p_g, g)) = H(p_g)$  and all the periodic points in  $H(p_g)$  are hyperbolic, i. e. ,  $g$  is Kupka-Smale. Moreover, all homoclinic classes are maximal transitive in the sense that if  $H$  is a homoclinic class and  $\Lambda$  is a transitive set such that  $\Lambda \cap H \neq \emptyset$  then  $H \supset \Lambda$ .*

**Remark 3.7.** For  $g \in \mathcal{T}$ ,  $\text{Cl}(\text{Per}(g))|_{H(p_g)} = H(p_g)$  and the periodic points are hyperbolic.

Given  $f \in \text{Diff}^1(M)$  denote by  $\text{Per}_h(f)$  the set of hyperbolic points of  $f$ . The following theorem is contained in [Ar].

**Theorem 3.8.** *There exists a residual set in  $\text{Diff}^1(M)$  such that if  $q \in \text{Per}_h(f) \cap H(p)$  and  $\dim(W^s(p)) \geq \dim(W^s(q))$  then there exists a heteroclinic connection between  $p$  and  $q$ .*

*Proof.* See [Ar, Proposition 20]. □

From now on we assume that  $f$  is in the residual subset  $\mathcal{G}$  of  $\mathcal{U}(f)$  in which Theorems 3.6 and 3.8 hold.

**Lemma 3.9.** *Let  $p$  and  $H(p)$  be as in Lemma 3.2 and  $q$  a hyperbolic periodic point in  $H(p)$ . If  $2 = \text{index}(p) = \text{index}(q)$  and  $x \in W^s(p) \cap W^u(q)$  then  $W^s(p)$  intersects transversally  $W^u(q)$  at  $x$ . Analogously, if  $y \in W^u(p) \cap W^s(q)$  then  $W^s(q)$  intersects transversally  $W^u(p)$  at  $y$ .*

*Proof.* Arguing by contradiction suppose that  $W^u(p)$  is tangent to  $W^s(q)$  at a point  $y$ . Iterating by  $f$  we may assume that the tangency  $y \in W_\epsilon^s(q)$ . We also may assume that there is a transverse intersection close to  $y$ . By the  $\lambda$ -Lemma a disk in  $W^s(p)$  converges in the  $C^1$  topology to  $W^s(q)$ . With another small  $C^1$ -perturbation we may create a homoclinic tangency between  $W^s(p)$  and  $W^u(p)$  contradicting Lemma 3.2. The arguments are similar if we suppose that  $W^s(p)$  is tangent to  $W^u(q)$ . □

**Lemma 3.10.** *If  $f \in \mathcal{G}$  and  $H(p)$  is  $\alpha$ -expansive then  $\text{index}(q) = \text{index}(p)$  for all periodic  $q \in H(p)$ .*

*Proof.* Assume, by contradiction, that  $\text{index}(q) = 1$  for some  $q \in \text{Per}(f) \cap H(p)$ . By a small  $C^1$  perturbation we may create an intersection between  $W^s(p)$  and  $W_\epsilon^u(q)$  containing a  $C^1$ -circle or a compact arc  $\gamma$ . By Theorem 3.8 we have a heteroclinic connection and using blenders (see [BD]) we may assume that  $\gamma \subset H(p)$ . Let  $x \in \gamma$ ,  $x \neq p$  and  $x \neq q$ . Then there is  $N > 0$  such that  $f^N(x) \in W_\epsilon^s(p)$  and  $f^{-N}(x) \in W_\epsilon^u(\mathcal{O}_f(q))$ . By continuity there is  $\delta > 0$  such that if  $\text{dist}(x, y) < \delta$  then  $\text{dist}(f^j(x), f^j(y)) < \epsilon$  for  $|j| \leq N$ . Hence, taking  $y \in \gamma$  such that  $\text{dist}(x, y) < \delta$  we contradict  $\alpha$ -expansiveness of  $f$  if  $0 < \epsilon < \alpha/2$ . □

**Corollary 3.11.** *Let  $f \in \mathcal{G}$ . If  $q \in H(p)$  is a periodic point then there are no homoclinic tangency associated to  $q$ .*

*Proof.* Similar to that of Lemma 3.9. □

**Lemma 3.12.** *Let  $q$  be a hyperbolic periodic point in  $H(p)$ ,  $H(p)$   $\alpha$ -expansive, such that  $W^s(p)$  cuts  $W^u(q)$  and  $W^s(q)$  cuts  $W^u(p)$ . If the intersections are transversal then there exists  $\mathcal{U}(q, f) \subset \mathcal{U}(f)$  such that  $q_g \in H(p_g)$  for all  $g \in \mathcal{U}(q, f)$ .*

*Proof.* Follows from transversality. □

## 4 Proof of Theorem A

In this section we will see that generically  $C^1$ -robustly expansive homoclinic classes  $H(p)$  have a co-dimension one dominated splitting  $E \oplus F$ ,  $\dim(F) = 1$ . To prove this we will use that the stable and unstable subspaces of the periodic points in  $H(p)$  have constant dimension and that the angle they form is bounded away from zero. Moreover, we will see that the eigenvalues of these periodic points are far from the unit circle. These results will allow us to use results of [We] that give the desired dominated splitting.

By technical reasons we split the proofs of the results about the eigenvalues in two cases: one in which there exists a periodic point in  $H(p)$  with complex eigenvalues, and another in which for all  $q$  periodic in  $H(p)$  its eigenvalues are real. In the first case we will include as "complex" the case of real eigenvalues  $\lambda_1, \lambda_2$  of a periodic point  $q \in H(p)$  such that  $|\lambda_1 - \lambda_2| \approx 0$  and also the case of a single eigenvalue with geometric multiplicity 1 and algebraic multiplicity 2 ( $D_q f$  not diagonalizable). The reason for treating these eigenvalues as complex is that by a slight  $C^1$  perturbation of the diffeomorphism we will obtain a diffeomorphism  $g$  with  $q$  a periodic point with (true) complex eigenvalues.

First we reject the possibility of having a periodic point  $q \in H(p)$  such that  $Df_q$  has more than one contracting eigenvalue of modulus close to 1.

**Lemma 4.1.** *With  $p$  and  $H(p)$  as in Lemma 3.2 and  $f \in \mathcal{G}$ , for all periodic point  $q$  in  $H(p)$  of index 2 we have that at most one of its contracting eigenvalues,  $|\lambda_1| \leq |\lambda_2| < 1$ , is close to 1 in modulus. In particular there are not complex eigenvalues of modulus close to 1.*

*Proof.* Let  $q \in H(p)$  be a periodic point of period  $\tau$  and index 2 and assume first that  $Df_q$  has one eigenvalue of modulus greater than 1 and two complex eigenvalues of modulus close to 1. Since  $q \in H(p)$ , there is a sequence  $\{x_n\} \subset W^s(p) \cap W^u(p)$  with  $x_n \rightarrow q$  as  $n \rightarrow \infty$ . By assumption  $W_\epsilon^u(x_n) \cap W_\epsilon^s(q) = z_n$  and, as  $W_\epsilon^s(q)$  is accumulated by  $W^s(p)$ , by the  $\lambda$ -lemma it follows that  $z_n \in H(p)$ . By forward iteration by  $f^\tau$ ,  $\tau$  being the period of  $q$ , the points  $z_n$  accumulate in  $q$ .  $C^1$ -perturbing  $f$  we may assume that the eigenvalues of  $Dg_q^\tau$ , where  $g$  is the perturbation of  $f$ , are  $\rho e^{\pm i\theta\pi}$  with  $\theta$  irrational,  $\rho < 1$  but close to 1. As in Hopf's

bifurcation we let  $\rho \rightarrow 1$  obtaining an irrational rotation in a small circle  $\mathcal{C}$  around  $q$ , contained in  $W_\epsilon^s(q, f)$  such that the orbit of  $z_n$  by  $g$  accumulates in  $\mathcal{C}$  which therefore is contained in  $H(p_g)$ . This contradicts expansiveness of  $g|_{H(p_g)}$ . The same is valid for the case in which both eigenvalues are real, of modulus close to 1 and of the same sign; after an arbitrarily small  $C^1$ -perturbation we fall in the case in which we have complex eigenvalues close to 1. The case in which we have contracting eigenvalues one close to 1 and the other close to -1 can be treated analogously.  $\square$

Let  $q$  be a hyperbolic periodic point homoclinically related to  $p$  and of period  $k$ , such that  $Df_q^k/E^s(q)$  has two real eigenvalues  $\lambda_1, \lambda_2$ , with  $1 > |\lambda_1| > (1 - \frac{1}{n})^k$ ,  $|\lambda_2| < (1 - \delta)^k$ ,  $0 < \delta < 1$  fixed. There exists two invariant lines  $ss$  and  $ws$  in  $W_\epsilon^s(q)$  tangent to the directions given by the eigenvectors corresponding to  $\lambda_1$  and  $\lambda_2$  respectively.  $ss$  separates  $W_\epsilon^s(q)$  in two regions  $B_1$  and  $B_2$ . Let  $\{z_i\}, \{z'_i\}$  be sequences of points in  $W^u(p) \cap B_1$ , and in  $W^u(p) \cap B_2$  respectively.

**Lemma 4.2.** *There exists  $g$  near  $f$  with  $q$  a periodic point of  $g$  of period  $k$  and such that  $g$  has two more periodic points in  $ws$ ,  $q_1, q_2$ , arbitrarily close one to another, with  $q_1, q_2$  separated by  $q$  in the line  $ws$ , and with the same period of that of  $q$  or with period the double of the period of  $q$ . Moreover the perturbation can be done such that  $z_i \in W^u(q', g) \cap B_i$ ,  $i = 1, 2$ , and such that  $g^{kn}(z_1) \rightarrow q_1$  and  $g^{kn}(z_2) \rightarrow q_2$  when  $n \rightarrow +\infty$ .*

*Proof.* Let  $\nu > 0$  be such that  $|\lambda_1| = (1 - \nu)^k$ , and let  $v_i$  be the eigenvector associated to  $\lambda_i$ ,  $i = 1, 2$  and  $u$  the eigenvector associated to the eigenvalue  $\mu$  of modulus greater than 1. Let  $Q_q = \bigoplus_{i=0}^{k-1} TM_{f^i(q)}$  and let us consider a basis of  $TM_{f^i(q)}$  formed by  $\{Df_q^i(v_1), Df_q^i(v_2), Df_q^i(u)\}$ . The union of these basis form a basis of  $Q_q$ . Let  $s \in [1 - \nu, 1]$  and define  $\varphi_i(s) : TM_{f^i(q)} \rightarrow TM_{f^i(q)}$ ,  $i = 0, \dots, k-1$  by the matrix

$$\begin{pmatrix} s(1 - \nu)^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We obtain a linear map depending on  $s$   $\varphi(s) = (\varphi_0(s), \dots, \varphi_{k-1}(s))$  defined on  $Q_q$ . Exploiting that  $TM_{f^k(q)} = TM_q$  the composition  $F_i(s) = df_{f^i(q)} \circ \varphi_i(s)$  gives a linear map  $G(s) = (F_0(s), \dots, F_{k-1}(s))$  defined on  $Q_q$ .

It follows that  $G^k = G^k(1)$  has  $k$  eigenvalues of modulus 1 with eigenvectors  $v_1, \dots, Df_q^{k-1}(v_1) \in Q_q$ . Moreover, the eigenvalues of  $G^k$  are real and hence the eigenvalue associated to  $v_1$  is  $\pm 1$ . If it is 1 then  $G^k(v_1) = v_1$ , otherwise  $G^{2k}(v_1) = v_1$ . Clearly for all  $\epsilon_0 > 0$  there exists  $\nu > 0$  such that  $\|G(s) - Df_q\| < \epsilon_0/10$  if  $\nu > 0$  is sufficiently small. By Lemma 3.1, there exists  $g(s) : M \rightarrow M$  such that  $\|f - g(s)\| < \epsilon_0$  and  $Dg(s)/Q_q = G(s)/Q_q$ . In particular  $Dg(1)|_{E_q^s} = G|_{E_q^s}$ .

Let us recall how it is proved Lemma 3.1. First it is assumed that  $M$  is embedded in  $\mathbb{R}^N$  for certain  $N$  and that the Riemannian metric on  $M$  comes from that embedding and the usual metric of  $\mathbb{R}^N$ . This allows us to assume that  $\exp_x(v) \in M$  and  $v \in TM_x$  are both in  $\mathbb{R}^N$ . Then it is chosen  $\delta > 0$  to guarantee that the exponential map  $\exp : TM \rightarrow M$  when restricted to  $\hat{B}_i = \{v \in TM_{f^i(q)} / \|v\| \leq \delta\}$  is near the identity and it is defined

$\hat{f} : \hat{B}_i \rightarrow TM_{f^{i+1}(q)}$  by  $\hat{f}(v) = \exp^{-1}(f(\exp(v)))$ . After that it is chosen a  $C^\infty$  function  $\sigma : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that for all  $x \in \mathbb{R}^+$ :  $0 \leq \sigma(x) \leq 1$ ,  $\sigma(x) = 0$  if  $x \geq \delta$ ,  $\sigma(x) = 1$  if  $x \leq \delta/4$ , and  $0 \leq |\sigma'(x)| \leq 2/\delta$  for all  $x \in \mathbb{R}^+$ . Choosing  $\rho : TM \rightarrow \mathbb{R}^+$  by  $\rho(v) = \sigma(\|v\|)$  and defining the auxiliary function  $\hat{g}(v) = \rho(v)G(v) + (1 - \rho(v))\hat{f}(v)$  we finally obtain that putting  $g(x) = \exp(\hat{g}(\exp^{-1}(x)))$  if  $x \in B_i = \exp(\hat{B}_i)$  and  $g(x) = f(x)$  elsewhere,  $g(x)$  is  $C^1$ -near  $f$  and  $Dg/Q_q = G/Q_q$ .

Let  $\Delta = \max_{0 \leq i \leq 2k} \|G^i\|$  and  $I = \{tv_1/0 \leq |t| \leq \frac{\delta\Delta}{4}\}$ . It follows by construction of  $g$  that for all  $u \in L = \exp_q(I)$ ,  $u$  is a periodic point of  $g$  of period equal to  $k$ .

Observe that  $L$  is normally hyperbolic for  $g$ . The plane  $W^{cs}(q_g)$  defined locally by the strong stable foliation given by the direction defined by  $v_2$  and  $L$  is accumulated by the stable manifold of  $p_g$  which is transversal to  $W^u(q_g)$ . Moreover, the unstable manifold of  $q_g$  cuts the stable manifold of  $p_g$  and so, by the  $\lambda$ -lemma, the unstable manifold of  $q_g$  accumulates  $C^1$  in the unstable manifold of  $p_g$ . Hence there is a horseshoe  $Z$  associated to  $W^u(q_g)$  and  $W^{cs}(q_g)$ . If  $x \in Z$  then  $x \in H(p_g)$ . Forward iterations of the orbit of  $x$  will converge to a point  $q' \in L$ , an end-point of  $L$ . If we have points of  $W^u(p)$  in both,  $B_1$  and  $B_2$  we can obtain points converging to both end-points  $q_1, q_2$ . As  $W^s(p)$  contains a disk converging in the  $C^1$  topology to  $W^{cs}(q_g)$ , we have that  $q_1$  and  $q_2$  are in  $H(p_g)$  concluding the proof of the Lemma.  $\square$

#### 4.1 Estimates of the eigenvalues

Let  $A = Df_p^{\tau_p}$ ,  $B = Df_q^{\tau_q}$  where  $p$  and  $q$  are hyperbolic periodic points homoclinically related of periods  $\tau_p$  and  $\tau_q$  respectively. In [BDP] it is proved that there are matrices  $T_1 : T_p M \rightarrow T_q M$  and  $T_2 : T_q M \rightarrow T_p M$ , called *transition matrices*, that give sense to the composition of  $A$  and  $B$ . Moreover the corresponding contracting and expanding sub-spaces of  $T_p M$  and  $T_q M$  are preserved. Using these transition matrices we may compose a finite number of times

$$A^{m_1} \circ T_2 \circ B^{n_1} \circ T_1 \circ A^{m_2} \circ T_2 \circ B^{n_2} \dots$$

obtaining a matrix  $C$  which represents the differential map of a periodic point  $r \in H(p)$  that spends  $m_1$  iterates near the orbit of  $p$ , then  $n_1$  iterates near  $\mathcal{O}(q)$ , then  $m_2$  iterates near  $\mathcal{O}(p)$  and so on. As big is  $m_j$  as more and more close is an



iterate of  $r$  to  $p$  and the period  $l$  of  $r$  is about  $m_1 + n_1 + m_2 + n_2 + \dots$ . Furthermore, the eigenvalues of  $r$  are determined by those of  $p$  and  $q$ . More precisely, when  $m_j$  and  $n_j$  growth without bounds the influence of  $T_1$  and  $T_2$  become negligible and the differential map  $Df_r^l$  is given approximately by

$$A^{m_1} \circ B^{n_1} \circ A^{m_2} \circ B^{n_2} \circ \dots,$$

see [BDP].

We shall use these transition matrices in the proof of Propositions 4.3 and 4.4 below.

Let us assume  $f \in \mathcal{G}$  and that for all  $q$  hyperbolic periodic point in  $H(p)$  it holds that its eigenvalues are real. As usual we assume that  $\text{index}(p) = 2$ . Observe that we are saying that for a  $C^1$ -neighborhood  $\mathcal{N}$  of  $f$  it holds that all eigenvalues of all periodic point of  $g \in \mathcal{N}$  are real.

**Proposition 4.3.** *There exist  $0 < \delta < 1$  such that for all hyperbolic periodic point  $q$  in  $H(p)$  it holds that the contracting eigenvalues  $\lambda_1, \lambda_2$  of  $Df_q$ ,  $0 < |\lambda_1| < |\lambda_2|$  verify  $|\lambda_2| < (1 - \delta)^\tau$ , where  $\tau$  is the period of  $q$ .*

*Proof.* First observe that if we have a sequence  $q_m$  of hyperbolic periodic points such that both contracting eigenvalues  $\lambda_1, \lambda_2$  verify that  $(1 - 1/m)^{\tau_m} < |\lambda_j(q_m)| < 1$ ,  $j = 1, 2$  we should have that the period of  $q_m$  goes to infinity with  $m$  or we will obtain as a limit point a non-hyperbolic periodic point. In the case in which the period goes to infinity we will increase the compressing properties of one of the eigenvalues, say  $\lambda_1(q_m)$  perturbing along the orbit of  $q_m$  multiplying in the direction of  $Df_{f^j(q)}v_1$ , where  $v_1$  is the eigenvector corresponding to  $\lambda_1$ , by  $(1 - 1/m)$ . If  $1/m$  is small enough we can apply Lemma 3.1 obtaining a diffeomorphism  $g$  for which  $q_m$  has an eigenvalue  $\lambda_1(q_m, g)$  such that  $|\lambda_1(q_m, g)| < (1 - 1/m)^{\tau_m}$ . We then may apply the arguments below.

Taking into account the preceding paragraph we assume that there exists  $\{q_m\}$  a sequence of hyperbolic periodic points in  $H(p)$  such that

$$(1 - 1/m)^{\tau_m} < |\lambda_2(q_m)| < 1$$

where  $\tau_m$  is the period of  $q_m$ . Let  $q$  be a hyperbolic periodic point in  $H(p)$  such that  $1 > |\lambda_2(q)| = |\lambda_2| = (1 - \nu)^\tau > (1 - 1/n)^\tau$ .

Assume that  $q$  has positive eigenvalues and let  $B$  be the matrix of  $Df_q^\tau$  which, in a convenient basis, can be written as

$$B = \begin{pmatrix} (1 - \nu)^\tau & 0 & 0 \\ 0 & \lambda_1^\tau & 0 \\ 0 & 0 & \mu^\tau \end{pmatrix}.$$

Let  $v_1$  be the eigenvector corresponding to  $\lambda_1$ , the stronger contracting eigenvalue and let  $v_2$  be the eigenvector corresponding to  $1 - \nu$ . In  $W_\epsilon^s(q)$  it is defined a local foliation by arcs transverse to  $v_2$  and of the direction of  $v_1$ . At  $q$  there are defined lines  $ss$  and  $ws$  tangent to  $v_1$  and  $v_2$  respectively. The arc  $ss$  separates  $W_\epsilon^s(q)$  in two regions (half-disks)  $B_1$  and  $B_2$ .

If there are points of  $H(p)$  in both regions  $B_1$  and  $B_2$  different from  $q$  then we will have points  $z_1$  and  $z_2$  in different separatrices of the line that under forward iterates will converge to  $q$ . In this case by Lemma 4.2 there is  $g$  near  $f$  with  $q$  a periodic point of  $g$  and such that  $g$  has two more periodic points in  $ws$ ,  $q_1, q_2$ , arbitrarily close one to another, with  $q_1, q_2$  separated by  $q$ , and with the same period that of  $q$  or with period the double of the period of  $q$ , and such that  $g^{\tau n}(z_1) \rightarrow q_1$  and  $g^{\tau n}(z_2) \rightarrow q_2$  when  $n \rightarrow +\infty$ . Observe that by hypothesis  $W^s(q)$  intersects transversely  $W^u(p)$  and  $W^s(p)$  intersects transversely  $W^u(q)$ . By the  $\lambda$ -Lemma there exists a horseshoe  $C$  associated to  $q$  and hence we have that  $W^s(q) \cap W^u(q)$  is contained in  $H(p)$ , in fact  $H(q) = H(p)$ . There is a periodic point  $R$  associated to the horseshoe  $C$  close to a point  $S \in W_\epsilon^s(q) \cap W^u(p)$ . By forward iterations by  $f$  we obtain points  $f^{k\tau}(S)$  closer to  $q$  than  $S$ . Let  $R_1$  be a hyperbolic periodic point close to  $f^{k\tau}(S)$ . Generically, the point  $S \notin ss$  but  $S \in B_1 \cup B_2$ . If  $S \in ss$  we may perturb  $f$  to obtain  $S \notin ss$ . Hence the same is true for  $R$  and  $R_1$ , that is,  $R, R_1 \notin ss$ .

Using transition matrices we may compose a finite number of times  $Df_q$  with  $Df_R$  and  $Df_{R_1}$  in order to obtain a hyperbolic periodic point  $T \in H(p)$  of period  $\tau_T$  arbitrarily close to  $f^{k\tau}(S)$  and with matrix  $Df_T^{\tau_T}$  near

$$D = \begin{pmatrix} (1 - \nu)^{\tau_T} & 0 & 0 \\ 0 & \lambda_1^{\tau_T} & 0 \\ 0 & 0 & \mu^{\tau_T} \end{pmatrix}.$$

We analyze the relative positions between  $q$  and  $T$ . As the value of the normalized eigenvalues are about the same for  $q$  and  $T$ , the size of their local stable and unstable manifold is also the same. We can split (and do) the local stable manifold of  $T$  with the strong local stable manifold  $ss(T)$  in two regions  $B_1(T)$  and  $B_2(T)$  where there are segments of  $ws(T)$ .

We may assume that  $f^{k\tau}(S)$  is very close to  $q$  and therefore  $T$  and  $q$  are close, their stable directions are almost parallel and the same is true with respect to the unstable directions. Thus  $W_\epsilon^u(q)$  intersects  $W_\epsilon^s(T)$ , say at  $y$ , and  $W_\epsilon^u(T)$  intersects  $W_\epsilon^s(q)$  at  $y'$ . Also the unstable manifold  $W^u(p)$  accumulates in these points. Close to the point  $R \in W_\epsilon^s(q) \cap W^u(p)$  there is a point  $R' \in W_\epsilon^s(T) \cap W^u(p)$  located in  $B_1(T)$  if  $R$  is in  $B_1$  and in  $B_2(T)$  if  $R$  is in  $B_2$  (we assume that locally we can orient  $ws$  and  $ss(T)$  in the same way). Forward iterations of  $R'$  and  $y'$  go to the orbit of

$T$  and the same is true for  $y$  and  $R$  with respect to  $q$ . Thus  $T$  is close to  $q$  but the relative position of  $y$  and  $y'$  are different. That is, if  $y \in B_1$  then  $y' \in B_2(S)$ . It follows that either  $q$  or  $T$  are accumulated from both sides of  $W_\epsilon^s(\cdot) \setminus ss(\cdot)$ . Here  $\cdot$  means either  $q$  or  $T$ . By Lemma 4.2 we may perturb to obtain new periodic points  $q_1$  and  $q_2$  in  $ws(\cdot)$  as close one to another as we wish, both of the same period, contradicting expansiveness and we finish the proof of Proposition 4.3.  $\square$

Let us assume now that the hyperbolic fixed point  $p$  has either complex eigenvalues or two real eigenvalues  $\lambda_1, \lambda_2$  of modulus less than 1 and such that  $\lambda_1 - \lambda_2 \approx 0$  or that  $Df_p$  is not diagonalizable (recall that we are assuming that  $\text{index}(p) = 2$ ).

**Proposition 4.4.** *There exists  $0 < \delta < 1$  such that for all hyperbolic periodic point  $\hat{q}$  in  $H(p)$  it holds that if  $\lambda$  is a contracting eigenvalue of  $Df_{\hat{q}}$  then  $|\lambda| < (1 - \delta)^\tau$ , where  $\tau$  is the period of  $\hat{q}$ .*

*Proof.* First, if  $Df_p$  has real contracting eigenvalues  $\lambda_1, \lambda_2$ , such that  $\lambda_1 - \lambda_2 \approx 0$  then under a small  $C^1$ -perturbation we obtain true complex contracting eigenvalues. The same is true if  $Df_p$  is not diagonalizable. So, without loss of generality, we assume that  $Df_p$  has contracting eigenvalues  $\lambda_1, \lambda_2 \notin \mathbb{R}$ . Arguing by contradiction let us assume that there exists a sequence  $q_N$  of hyperbolic periodic points in  $H(p)$  such that they have contracting eigenvalues  $\lambda_N$  verifying  $(1 - \frac{1}{N})^{\tau_N} < |\lambda_N| < 1$ . Both contracting eigenvalues verify this condition and so if  $\{\tau_N\}$  has a bounded infinite subsequence, say by  $T$ , then we should have a periodic point of period less or equal than  $T$  with two eigenvalues of modulus 1 contradicting Lemma 4.1. Otherwise the sequence  $\tau_N$  goes to infinity and only one of the contracting eigenvalues is weak and therefore both are real. Assume that the eigenvalues at  $p$  are  $\rho e^{i\phi}$  and  $\rho e^{-i\phi}$  where  $\phi$  defines an irrational rotation (i. e.:  $\phi \neq \frac{m}{n}\pi$  with  $m, n \in \mathbb{Z}$ ) and  $0 < \rho < 1$ . Let  $A_s = Df_p|_{E_p^s}$  and  $B_s = Df_{\hat{q}}|_{E_{\hat{q}}^s}$ . We complete these matrices with the corresponding eigenvalues greater than 1 in modulus in the corresponding unstable subspaces in order to have them acting in the whole tangent space and call them  $A$  and  $B$  respectively. In convenient basis we have

$$A = \begin{pmatrix} \rho \cos \phi & \rho \sin \phi & 0 \\ -\rho \sin \phi & \rho \cos \phi & 0 \\ 0 & 0 & \mu_1 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 - \nu & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu_2 \end{pmatrix}$$

where, assuming that the eigenvalues are positive,  $0 < |\nu| < 1/n$ .

Now we use transitions matrices to obtain a periodic point  $q$  with differential map  $C = A^m \circ T_2 \circ B^n$ . Choose  $n = m^2$  and let  $m$  be sufficiently big to have that the influence of  $T_2$  is negligible. In this case, the matrix  $C$  is near  $A^m \circ B^n$ . Take  $m$  such that  $m\phi \approx 0 \pmod{2\pi}$ . Recall  $\phi$  is irrational with respect to  $\pi$ . Then  $A^m$  is close to the matrix given by

$$\begin{pmatrix} \rho^m & 0 & 0 \\ 0 & \rho^m & 0 \\ 0 & 0 & \mu_1^m \end{pmatrix}$$

and  $B^n$  is given by

$$B^n = \begin{pmatrix} (1-\nu)^n & 0 & 0 \\ 0 & \lambda^n & 0 \\ 0 & 0 & \mu_2^n \end{pmatrix}.$$

The composition of both matrices above yields  $C$  with contracting eigenvalues  $c_1 = \rho^m(1-\nu)^n$  and  $c_2 = \rho^m\lambda^n$ . The period of  $q$  is  $m+n$  and hence

$$c_1^{\frac{1}{m+n}} = \rho^{\frac{m}{m+n}}(1-\nu)^{\frac{n}{m+n}}.$$

As  $m \rightarrow \infty$  we obtain that  $c_1^{\frac{1}{m+n}} \rightarrow (1-\nu)$ , taking into account that  $n = m^2$ . Thus  $q$  behaves like  $\hat{q}$  and we may assume that the eigenvalues of  $Df_q$  are  $\lambda$ ,  $1-\nu$ , and  $\mu_1$ , with  $0 < \nu < 1/n$ . But  $q$  has the advantage that it has iterates as close as we wish to  $p$  (just take  $m$  big), and therefore we will have that  $W^u(p)$  intersects  $W^s(q)$  and  $W^s(p)$  intersects  $W^u(q)$ . Thus they are homoclinically related. By assumption  $Df_p$  has contracting complex eigenvalues, a homoclinic point in  $W_\epsilon^s(p)$  will describe an spiral in  $W_\epsilon^s(p)$  which projects via the corresponding segments of the unstable manifold  $W^u(p)$  (almost parallel to  $W_\epsilon^u(p)$  by the  $\lambda$ -Lemma) into  $W_\epsilon^s(q)$ . This implies that  $W^u(p) \cap W_\epsilon^s(q) \subset H(p)$  accumulates in both separatrices  $u^+$  and  $u^-$  of  $W_\epsilon^s(q)$  corresponding to the weak eigenvalue  $(1-\nu)$ . That is, there are points  $z_1 \in H(p) \cap u^+$  and  $z_2 \in H(p) \cap u^-$ ,  $z_1, z_2$  different from  $q$ .

Analogously to what we have done in Proposition 4.3, we may perturb  $f$  to obtain  $g$  with  $q$  a periodic point of  $g$  and such that  $g$  has two more periodic points in  $u$ ,  $q_1, q_2$ , arbitrarily near one to another, with  $q_1, q_2$  separated by  $q$ , and with the same period that  $q$  or with period the double of the period of  $q$ , and such that  $g^{\tau n}(z_1) \rightarrow q_1$  and  $g^{\tau n}(z_2) \rightarrow q_2$  when  $n \rightarrow +\infty$ , see Lemma 4.2.

The existence of  $q_1, q_2$  as above contradicts robust expansiveness concluding the proof of Proposition 4.4.  $\square$

**Remark 4.5.** *It follows from the proof of Proposition 4.4 that it doesn't matter if the point with complex eigenvalues is another point  $p'$  different from  $p$  as far as  $H(p) = H(p')$ .*

**Proposition 4.6.** *Let  $f \in \mathcal{G}$  and  $p$  and  $H(p)$  be as in Lemma 3.2. Let  $\mu$  be the (real) expanding eigenvalue of a periodic point  $q \in H(p)$  of period  $k$ . Then there exists  $\delta > 0$  such that  $|\mu|^k > (1 + \delta)^k$ .*

*Proof.* Similar to those of Propositions 4.3 and 4.4. □

**Remark 4.7.** *Propositions 4.3, 4.4, and 4.6 imply in particular that we cannot have periodic points in  $H(p)$  homoclinically related to  $p$  and which are not hyperbolic whenever  $H(p)$  is  $C^1$ -robustly expansive.*

## 4.2 Estimates of the angles

We define as in [We] wandering rates for linear contractions and expansions.

**Definition 4.1.** *Let  $V$  be a finite dimensional normed linear space and  $A : V \rightarrow V$  a linear contraction or expansion. Given  $v \neq 0$ , a ball  $B(v, r)$  is  $A$ -wandering if  $A^n(B(v, r)) \cap B(v, r) = \emptyset$  for all integer  $n \neq 0$ . Define the  $A$ -wandering rate of  $v$  with respect to  $A$  as*

$$\omega(v, A) = \sup\{\lambda > 0; B(v, \lambda\|v\|) \text{ is } A\text{-wandering}\}$$

and the wandering rate  $\omega(A)$  of  $A$  to be

$$\omega(A) = \inf\{\omega(v, A); v \neq 0\}.$$

The following lemma is in [We]. It doesn't require  $q$  to be hyperbolic.

**Lemma 4.8.** *Let  $f$  be a diffeomorphism and  $\epsilon > 0$  be a real number. For any periodic point  $q$  of  $f$  of period  $\tau$ , if there is a  $Df^\tau$ -contracting subspace  $V^s$  and a  $Df^\tau$ -expanding subspace  $V^u$  of  $T_qM$  such that*

$$\frac{\angle(V^s, V^u)}{\min\{\omega(Df^\tau|_{V^s}), \omega(Df^\tau|_{V^u})\}} < \epsilon$$

then there is  $g$ ,  $C^1$   $\epsilon$ -close to  $f$  that differs from  $f$  only in an arbitrarily small ball near  $q$  but disjoint from its orbit such that  $g$  exhibits homoclinic tangency associated to  $q_g = q$ .

*Proof.* See [We, Lemma 4.2]. □

**Proposition 4.9.** *For  $f \in \mathcal{G}$  there exist  $m_0$  and  $\gamma > 0$  such that for all  $q \in H(p)$  of period  $\tau$  greater than  $m_0$  it holds that  $\angle(E_q^s, E_q^u) > \gamma$ .*

*Proof.* It suffices to prove that for all 1-dimensional subspace  $L^s \subset E_q^s = E^s$ , if the period of  $q$  is greater than  $m_0$  then  $\angle(L^s, L^u) > \gamma$  for certain  $\gamma > 0$ , where  $E_q^u = L^u$ . We can assume that  $Df_q^k|_{E^s}$  has real eigenvalues. Otherwise, using Lemmas 3.1 and 3.12, we obtain a nearby diffeomorphism  $g$  such that  $q$  is a periodic point with the same period as for  $f$  and such that for certain iterate  $k$  multiple of  $\tau$ ,  $Df_q^k|_{E^s}$  has real eigenvalues. The proof is similar to that of [We, Lemma 5.1] or [PS1, Lemma 2.2.2].

Let  $\epsilon > 0$  be so small that any  $C^1$   $\epsilon$ -perturbation of  $f$  is in the neighborhood  $\mathcal{U}(f)$ , and  $C_0 = \sup\{\|Dg_x\|; g \in \mathcal{U}(f), x \in M\}$  and let  $\epsilon'_0 = \epsilon_0/C_0$  where  $\epsilon_0$  is given by Lemma 3.1. We take  $\gamma < \epsilon/3$  and  $m_0 > \frac{12}{\epsilon\epsilon'_0}$ . Assume without loss of generality that  $\omega(Df^\tau|_{L^u}) < \omega(Df^\tau|_{L^s})$ . Suppose that the proposition does not hold and  $\beta = \angle(L^s, L^u) < \gamma$ . Fix  $\delta = \beta\epsilon'_0/4$ . Let  $T_j : Df^j(T_q M) \rightarrow Df^j(T_q M)$  be a linear map such that  $T_j|_{L^s} = (1 - \delta)id$  and  $T_j|_{L^u} = (1 + \delta)id$ , where  $id$  stands for the identity map. In the rest of the space  $T_j$  acts as the identity. Hence  $T_j$  compresses  $L^s$  and stretches  $L^u$ . The norm of  $T_j - id$  is less than (see [Ma2])

$$\begin{aligned} \|T_j - id\| &\leq \frac{1 + \beta}{\beta} (\|(T_j - id)|_{L^s}\| + \|(T_j - id)|_{L^u}\|) = \\ &= \frac{1 + \beta}{\beta} 2\delta = \frac{1 + \beta}{\beta} 2\beta\epsilon'_0/4 \leq \frac{1 + \beta}{2} \epsilon'_0 < \epsilon'_0. \end{aligned}$$

Hence we may apply Lemma 3.1, and assume that  $T_j \circ Df_{f^j(q)} = Dg_{g^j(q)}$  (i. e.: the orbit of  $q$  remains unchanged). Observe that we have not changed the stable and unstable sub-spaces of  $q$  and so  $q \in H(p)$ . Moreover, if  $|\sigma| > 1$  is the eigenvalue of  $Df_q^\tau|_{L^u}$  then the eigenvalue of  $Dg_q^\tau|_{L^s}$  is  $\sigma_1 = (1 + \delta)^\tau \sigma$ .

Since  $L^u$  is 1-dimensional the wandering rate of  $Dg_q^\tau$  restricted to  $L^u$  is  $\omega_u = \frac{\sigma_1 - 1}{\sigma_1 + 1}$ . Moreover, we still have  $\omega(Dg^\tau|_{L^u}) < \omega(Dg^\tau|_{L^s})$  since we have compressed the stable sub-bundle by  $(1 - \delta)$  while stretched  $L^u$  by  $(1 + \delta)$ . If  $\lambda$  is the eigenvalue of  $Df_q^\tau$  restricted to  $L^s$  then the condition on the wandering rates is equivalent to  $|\lambda\sigma| < 1$  and this still holds if  $\lambda$  is multiplied by  $(1 - \delta)$  and  $\sigma$  is multiplied by  $(1 + \delta)$ . If  $\sigma_1 \geq 2$  then  $\frac{\sigma_1 - 1}{\sigma_1 + 1} \geq \frac{1}{3}$ . Hence, in this case,

$$\begin{aligned} \frac{\beta}{\min\{\omega(Dg^\tau|_{L^s}), \omega(Dg^\tau|_{L^u})\}} &= \frac{\beta}{\omega_u} = \\ &= \frac{\beta(\sigma_1 + 1)}{\sigma_1 - 1} < \frac{\gamma(\sigma_1 + 1)}{\sigma_1 - 1} < \epsilon. \end{aligned}$$

On the other hand, if  $\sigma_1 < 2$  then

$$\frac{\sigma_1 - 1}{\sigma_1 + 1} \epsilon > \frac{\sigma_1 - 1}{3} \epsilon =$$

$$= \frac{(1 + \delta)^\tau \sigma - 1}{3} \epsilon \geq m_0 \delta \sigma \frac{\epsilon}{3} \geq \gamma \geq \beta.$$

Hence, by Lemma 4.8 we can create a homoclinic tangency between  $W^s(q_g)$  and  $W^u(q_g)$ , contradicting expansiveness since  $q \in H(p_g)$ .  $\square$

**Corollary 4.10.** *Let  $f \in \mathcal{G}$ . There exists  $\gamma > 0$  such that for all periodic point  $q$  in  $H(p)$  it holds that  $\angle(E_q^s, E_q^u) > \gamma$ .*

*Proof.* By Proposition 4.9 this is true for  $q$  periodic of period greater than  $m_0$ . On the other hand, due to the expansive properties of  $H(p)$  there are only finitely many periodic points of period less or equal than  $m_0$  on  $H(p)$ . The result follows.  $\square$

### 4.3 Constructing a dominated splitting

**Lemma 4.11.** *Let  $V$  be an inner product space with  $\dim(V) \geq 2$ . For any  $\epsilon > 0$  there is  $N_0 \geq 2$  such that if a contracting linear map  $A : V \rightarrow V$  has norm  $\|A\| \geq N_0$  then there is a linear map  $T : V \rightarrow V$  with  $\|T - id\| < \epsilon$  such that  $T \circ A$  has its largest eigenvalue in absolute value greater or equal to 1. In fact it may be taken to be exactly of modulus 1.*

*Proof.* See [We, Lemma 3.3].  $\square$

**Lemma 4.12.** *Assume that  $f \in \mathcal{G}$ ,  $p$  and  $H(p)$  as in Lemma 3.2. Then there are a  $C^1$ -neighborhood  $\mathcal{V}(f) = \mathcal{V}$  of  $f$ , a number  $K \geq 2$  and a number  $0 < \lambda < 1$  such that for any hyperbolic periodic point  $q \in H(p_g)$  of any period  $\tau$  and any  $g \in \mathcal{V}$  it holds, for all  $n \geq 0$ ,*

$$\|Dg^{-\tau}|_{E^u(q,g)}\| \leq K \quad \text{and} \quad \|Dg^{-\tau n}|_{E^u(q,g)}\| \leq K\lambda^{\tau n}, \quad (1)$$

$$\|Dg^\tau|_{E^s(q,g)}\| \leq K \quad \text{and} \quad \|Dg^{\tau n}|_{E^s(q,g)}\| \leq K\lambda^{\tau n}. \quad (2)$$

*Proof.* Observe that  $\|Dg^{-\tau}|_{E^u(q,g)}\| \leq K$  is trivial since  $E^u(q, g)$  is 1-dimensional. Moreover, Proposition 4.6 proves

$$\|Dg^{-\tau}|_{E^u(q,g)}\| \leq K\lambda^\tau.$$

And so we have proved (1).

To prove (2) we proceed as follows. By Proposition 4.9 and Corollary 4.10 we have that  $\angle(E_q^u, E_q^s) > \gamma > 0$ . Let  $\epsilon > 0$  be so small that any  $C^1$   $\epsilon$ -perturbation of  $f$  is in the neighborhood  $\mathcal{U}(f)$ . Recall that all  $g \in \mathcal{U}(f)$  is  $\alpha$  expansive in  $H(p_g)$ .

Let  $C_0 = \sup\{\|Dg_x\|; g \in \mathcal{U}(f), x \in M\}$  and  $\epsilon'_0 = \epsilon_0/C_0$  where  $\epsilon_0$  is given by Lemma 3.1. This gives a neighborhood  $\mathcal{V}(f)$  of  $f$  which we may assume contained in

$\mathcal{U}(f)$ . Let  $N_0$  be that given in Lemma 4.11 to achieve that the perturbation  $T$  giving the eigenvalue equal to 1 in modulus verifies  $\|T - id\| < \epsilon'_0$ . Take  $K \geq \max\{2, N_0\}$  and let  $\delta' = \delta/C_0$  where  $\delta > 0$  is that given by Lemma 3.1, with respect to  $\mathcal{V}(f)$ . Let  $\delta_1 > 0$  be so small that

$$\frac{1 + \gamma}{\gamma}(2\delta_1) \leq \delta'.$$

Finally let  $\lambda = \frac{1}{1+\delta_1}$ . Assume that there is  $g$  and  $q \in H(p_g)$  hyperbolic periodic point for  $g$  of period  $\tau$  such that  $\|Dg^\tau|_{E^s(q,g)}\| > K$ . By Lemma 4.11 there exists  $T : E^s(q, g) \rightarrow E^s(q, g)$  such that  $\|T - id\| < \epsilon'_0$  and such that  $T \circ Df|_{E^s(q,g)}$  has its largest eigenvalue in absolute value equal to 1. We extend  $T$  to all of  $T_q M$  as the identity in  $E^u(q, g)$ . As the angle between  $E^s(q, g)$  and  $E^u(q, g)$  is bounded away from zero, we can assume  $\|T - id\| < \epsilon'_0$ . By Lemma 3.1 we can think that  $T \circ Dg_q^\tau$  is  $Dg_{1_q}^\tau$ . There is an isotopy  $G(t)$ ,  $t \in [0, 1]$ , close to the identity, connecting  $g = G(0)$  to  $g_1 = G(1)$ . For values close to  $t = 1$ ,  $G(t)$  has a contracting eigenvalue of modulus close to 1 associated to a hyperbolic periodic point  $q_t$  contradicting Propositions 4.3 and 4.4. Thus  $\|Dg^\tau|_{E^s(q,g)}\| \leq K$ .

Next we prove  $\|Dg^\tau|_{E^s(q,g)}\| \leq K\lambda^\tau$  and  $\|Dg^{-\tau}|_{E^u(q,g)}\| \leq K\lambda^\tau$ .

For this, let us prove first that

$$\|Dg^\tau|_{E^s(q,g)}\| \leq K\lambda^\tau.$$

Given  $q$  periodic in  $H(p)$  of period  $\tau$  define  $T_j : Dg^j(T_q M) \rightarrow Dg^j(T_q M)$  by  $T_j|_{Dg^j(E^s(q,g))} = (1 + \delta_1)id$ . Thus  $T_j$  stretches the  $E^s$  direction. As  $\angle(E^u, E^s) > \gamma$  for all periodic point  $q$ , we have that

$$\|T_j - id\| \leq \frac{1 + \gamma}{\gamma}(\delta_1) \leq \delta'.$$

Thus we may assume, using again Lemma 3.1, that  $T_j \circ Dg_{g^j(q)} = Dg_{g_1^j(q)}$ . We claim that  $E^s(q, g_1)$  is contracted by  $Dg_1^\tau$ . Otherwise we may choose  $t \in [0, 1]$  to get  $T_j(t)$  that expands  $E^s$  by a factor  $(1 + t\delta)$  obtaining for suitable  $t$  an eigenvalue of modulus arbitrarily close to 1 associated to an eigenvector in  $E^s$ , and this contradicts Propositions 4.4 or 4.3, proving the claim. Observe that since we do not alter the  $E^u$  direction and  $\angle(E^u, E^s) > \gamma$  we also get that  $E^u(q, g_1)$  is expanded by  $Dg_1^\tau$ . Hence, applying (1) to  $g_1$  we obtain

$$\|Dg_1^\tau|_{E^s(q,g_1)}\| = (1 + \delta_1)^\tau \|Dg^\tau|_{E^s(q,g)}\| \leq K.$$

From this inequality we obtain

$$\|Dg^\tau|_{E^s(q,g)}\| \leq \frac{1}{(1 + \delta_1)^\tau} K = \lambda^\tau K$$

concluding the proof of (2). The proof of Lemma 4.12 is complete.  $\square$



**Theorem 4.13.** *Let us assume that there are numbers  $\gamma > 0$ ,  $K \geq 2$  and  $0 < \lambda < 1$  such that  $\angle(E^s, E^u) \geq \gamma$  for all periodic point  $q \in H(p_g)$  of  $g \in \mathcal{V}(f)$ . Assume furthermore that*

$$\|Df_q^\tau|_{E^s}\| \leq K \quad \text{and} \quad \|Df_q^{-\tau}|_{E^s}\| \leq K$$

for all periodic point  $q$  for all period  $\tau$  and that either

$$\|Df_q^\tau|_{E^s}\| \leq K\lambda^\tau$$

or

$$\|Df_q^{-\tau}|_{E^u}\| \leq K\lambda^\tau.$$

Then there exists a dominated splitting  $E \oplus F$  in the homoclinic class  $H(p)$  with  $\dim(E) = 2$  and  $\dim(F) = 1$ . If  $x = q$  is a periodic point then  $E(q) = E^s(q)$  and  $F(q) = E^u(q)$ .

*Proof.* It follows from Lemma 4.12 as in [We, Proposition 3.7] or [PS1, Lemma 2.0.1].  $\square$

The proof of Theorem A is complete.

## 5 Proof of Theorem B

In this section we prove that generically  $H(p)$  is hyperbolic. We shall follow the steps of [Ma3]. In that article, Mañé used  $\gamma$ -strings (see Definition 5.1 below) to prove that, if in a dominated splitting  $E \oplus F$ ,  $F$  is a uniform expanding bundle then either  $E$  is a uniform contracting bundle or there is a periodic point  $q$  such that  $Df_q$  contracts in the period with a rate arbitrarily close to 1. In our case we cannot assume *a priori* that  $F$  is an expanding bundle. So, we replace the former property by the fact that points in a neighborhood of  $H(p)$  have local unstable manifolds with both separatrices of diameter bounded away from zero.

### 5.1 Hyperbolic times

**Definition 5.1.** *We say that a pair of points  $(x, f^n(x))$  contained in  $H(p)$ ,  $n > 0$ , is a  $\gamma$ -string,  $0 < \gamma < 1$ , if*

$$\prod_{j=1}^n \|Df/E(f^j(x))\| \leq \gamma^n.$$

*We say that  $(x, f^n(x))$  is a uniform  $\gamma$ -string if  $(x, f^k(x))$  is a  $\gamma$ -string for all  $0 \leq k \leq n$ .*

**Lemma 5.1.** (Pliss' Lemma) *Let  $0 < \gamma_1 < \gamma_2 < 1$  and  $(x, f^n(x))$  be a  $\gamma_1$ -string. There exist a positive integer  $N = N(\gamma_1, \gamma_2, f)$ ,  $c = c(\gamma_1, \gamma_2, f) > 0$  such that if  $n \geq N$  then there exist numbers*

$$0 \leq n_1 \leq n_2 \leq \dots \leq n_l \leq n$$

*such that  $(f^{n_r}(x), f^n(x))$  are uniform  $\gamma_2$ -strings for all  $r = 1, 2, \dots, l$ , with  $l \geq cn$ . That is,*

$$\prod_{i=n_r}^j \|Df/E(f^i(x))\| \leq \gamma_2^{j-n_r}; \quad r = 1, 2, \dots, l; \quad n_r \leq j \leq n.$$

*Proof.* The proof of this lemma can be found in [P11, P12]. See also [Ma3, Lemma II.3].  $\square$

**Remark 5.2.** *We call the sequence  $n_1, n_2, \dots$  hyperbolic times for  $x$ .*

**Corollary 5.3.** *Given  $0 < \gamma_1 < \gamma_2 < 1$  and  $x \in H(p)$  such that for some  $m$*

$$\prod_{i=0}^n \|Df/E(f^i(x))\| \leq \gamma_1^n \text{ for all } n \geq m$$

*there exists a sequence  $0 \leq n_1 < n_2 < \dots$  such that*

$$\prod_{i=n_r}^j \|Df/E(f^i(x))\| \leq \gamma_2^{j-n_r} \quad \forall j \geq n_r, \quad r = 1, 2, \dots$$

**Definition 5.2.** *We say that  $(x, f^n(x))$  is an  $(N, \gamma)$ -obstruction,  $0 < \gamma < 1$ ,  $0 < N \leq n$  if  $(f^m(x), f^n(x))$  is not a  $\gamma$ -string for all  $n - N \leq m \leq n$ .*

**Lemma 5.4.** *Let us take  $0 < \gamma_0 < \gamma_3 < 1$  and  $0 < \gamma_2 < \gamma_3$  and let  $(x, f^n(x))$  be a  $\gamma_0$ -string. Moreover let  $0 \leq n_1 < n_2 < \dots < n_k \leq n$  be the set of integers such that  $(f^{n_r}(x), f^n(x))$  is a uniform  $\gamma_3$ -string and let  $N = N(\gamma_2, \gamma_3, f)$  be the positive integer given by Lemma 5.1. Then, for all  $1 \leq i < k$  either  $n_{i+1} - n_i \leq N$  or  $(f^{n_i}(x), f^{n_{i+1}}(x))$  is a  $(N, \gamma_2)$ -obstruction. Moreover, either  $n_1 \leq N$  or  $(x, f^{n_1}(x))$  is a  $(N, \gamma_2)$ -obstruction.*

*Proof.* See [Ma3, Lemma II.4].  $\square$

## 5.2 Unstable manifolds have uniform size

Let  $f : M \rightarrow M$  be a  $C^2$ -diffeomorphism and assume that there is a dominated splitting  $E \oplus F$  in  $H(p)$ . Moreover assume that  $f$  is Kupka-Smale. This splitting can be uniformly extended to a neighborhood  $V(H(p))$  of  $H(p)$  (see [PS1, page 985]). We point out that this extension is not necessarily invariant. Nevertheless points whose  $f$ -orbits stay in  $V(H(p))$ , belonging or not to  $H(p)$ , have a dominated splitting.

For points  $x$  in  $V(H(p))$  we may define  $W_\epsilon^{cs}(x)$  and  $W_\epsilon^{cu}(x)$   $\epsilon$ -center-stable manifolds and  $\epsilon$ -center-unstable manifolds respectively such that for  $y \in W_\epsilon^{cs}(x)$   $T_y(W_\epsilon^{cs}(x)) = E(y)$  and for  $y \in W_\epsilon^{cu}(x)$   $T_y(W_\epsilon^{cu}(x)) = F(y)$ . The local center stable manifold is a disk and the local center unstable manifold is an arc. These manifolds vary continuously in  $H(p)$ .

In [PS1, Lemmas 3.3.1 and 3.3.2] it is proved the following result related to the center unstable manifolds.

**Lemma 5.5.** *For all  $\epsilon \leq \delta_0$  there exists  $\gamma = \gamma(\epsilon)$  such that for any  $x$  with orbit in a neighborhood of  $H(p)$  and for all  $n \geq 0$ :*

1.  $f^{-n}(W_\gamma^{cu}(x)) \subset W_\epsilon^{cu}(f^{-n}(x))$ ,
2. For every  $\gamma \leq \gamma(\delta_0)$ , either
  - (a)  $\ell(f^{-n}(W_\gamma^{cu}(x))) \rightarrow 0$  when  $n \rightarrow \infty$ ,
  - (b) or  $x \in W_\gamma^{cu}(p')$  for some  $p' \in \text{Per}(f|_{H(p)})$  such that  $p' \in W_\gamma^{cu}(x)$  and there exists another periodic points  $q \in \text{Cl}(W_\gamma^u(p'))$  which is a sink or a non-hyperbolic periodic point.

**Remark 5.6.** *As a matter-of-fact Lemma 5.5 was proved for diffeomorphisms on surfaces [PS1] but it can be extended to the co-dimension-one case in the same way.*

**Corollary 5.7.** *If  $x \in W^s(p) \cap W^u(p)$  then  $W_\epsilon^{cu}(x) = W_\epsilon^u(x)$ .*

*Proof.* By expansiveness if  $x \in W^s(p) \cap W^u(p)$  then  $W^s(p)$  intersects  $W^u(p)$  transversely and  $W_\epsilon^{cu}(x) \subset W^u(p)$ . Therefore  $W_\epsilon^{cu}(x)$  is a true unstable manifold and the result follows.  $\square$

**Lemma 5.8.** *There exist  $\eta > 0$  and a neighborhood  $V'(H(p)) \supset H(p)$ ,  $V'(H(p)) \subset V(H(p))$  such that  $\text{diam}(W_\epsilon^{u,\sigma}(p')) > \eta$  for all periodic point  $p'$  such that  $\mathcal{O}(p') \subset V(H(p))$ ,  $\sigma = +, -$ .*

*Proof.* Assume that the result is not true. Then there exists a sequence  $p_n$  accumulating in  $H(p)$  such that at least one of its separatrices (say  $W_\epsilon^{u,+}(p_n)$ ) has diameter going to zero with  $n \rightarrow \infty$  and hence Lemma 5.5 together with the fact that  $f$  is Kupka-Smale give that there is a sink  $q_n \in \text{Cl}(W_\epsilon^{u,+}(p_n))$ .

Using again that  $f$  is Kupka-Smale we obtain that the period of  $p_n$  goes to infinity with  $n$ .

Since  $\text{diam}(W_\epsilon^{u,+}(p_n)) \xrightarrow{n \rightarrow \infty} 0$ ,  $\{q_n\}$  also accumulates in  $H(p)$  and the same is valid to the orbits of both  $q_n$  and  $p_n$ .

Using [PS1, Lemma 3.2.2 and Corollary 3.3] we may assume that  $W_\epsilon^s(p_n)$  has size greater than some fixed  $\delta_2 > 0$ . Moreover, the same is true for all points in  $\text{Cl}(W_\epsilon^{u,+}(p_n))$ .

Let  $z$  be an accumulation point of  $\{q_n\}$ . The point  $z$  belongs to  $H(p)$ , and as  $q_n \in \text{Cl}(W_\epsilon^{u,+}(p_n))$  we conclude that the local stable manifold of  $z$ , tangent to  $E$ , has size bounded away from zero. Moreover, the separatrix  $W_\epsilon^{cu,+}(z)$  from the side in which  $z$  is accumulated by  $q_n$ , is in fact part of the local unstable manifold of  $z$ . Otherwise  $z$  would be in the center-unstable manifold of a periodic point  $p'$  with a source  $q' \in \text{Cl}(W_\epsilon^{u,+}(z))$ , see Lemma 5.5. By forward iteration points like  $q_n$  will fall in the basin of attraction of  $q'$  contradicting that  $q_n$  itself is periodic. Thus  $W_\epsilon^{cu,+}(z)$  is part of the local unstable manifold of  $z$ .

Let  $q_{n_1}$  be a sink close to  $z$ . Its strong local stable manifold,  $W_\epsilon^{ss}(q_{n_1})$ , (tangent to  $E$ ) cuts the local unstable manifold of  $z$ . And the local center-unstable manifold of  $q_{n_1}$  cuts the local stable manifold of  $z$ . As  $q_{n_1}$  is a sink, it should exist a saddle-type periodic point  $p'_{n_1}$  in the center unstable manifold of  $q_{n_1}$  such that  $p'_{n_1} \in H(p)$ . For,  $W_\epsilon^{cu}(q_{n_1}) \cap W_\epsilon^s(z) \neq \emptyset$  and  $W_\epsilon^{ss}(q_{n_1}) \cap W_\epsilon^u(z) \neq \emptyset$ . But  $z \in H(p)$  implies that it is accumulated by points  $x \in W^s(p) \cap W^u(p)$ . If  $x$  is such a point near to  $z$  then its local stable manifold  $W_\epsilon^s(x) \subset W^s(p)$  has size bounded away from zero. Therefore  $W_\epsilon^{cu}(q_{n_1}) \cap W^s(p) \neq \emptyset$  and  $W^u(p) \cap W_\epsilon^{ss}(q_{n_1}) \neq \emptyset$ . Backward iteration by multiples of the period of  $q_{n_1}$  gives that there is a saddle periodic point in its center-unstable manifold. Indeed, otherwise  $q_{n_1} \in H(p)$  which is an absurd.

The same argument shows that there is a saddle-type periodic point  $p'_{n_1} \in W_\epsilon^{cu}(q_{n_1}) \cap H(p)$ . As before the stable manifold of  $p'_{n_1}$  has size bounded away from zero.

Let us repeat similar arguments with a sink  $q_{n_2}$  between the local stable manifold of  $z$  and that of  $p'_{n_1}$ . It may be seen that one of the separatrices of the local center-unstable manifold of  $q_{n_2}$  cuts the stable manifold of  $z$  while the other one cuts the stable manifold of  $p'_{n_1}$ . Therefore we obtain two different periodic points  $p'_{n_2}, p''_{n_2} \in W_\epsilon^{cu}(q_{n_2}) \cap H(p)$ . It follows that for all  $k \in \mathbb{Z}$ ,  $\text{dist}(f^k(p'_{n_2}), f^k(p''_{n_2})) \leq 2\epsilon < \alpha$  contradicting the expansive properties of  $f$  in  $H(p)$ .  $\square$

**Corollary 5.9.** *There exists  $\eta > 0$  such that for every point  $x \in H(p)$  it holds that  $\text{diam}(W_\epsilon^{u,\sigma}(x)) \geq \eta$ ,  $\sigma = +, -$ .*

*Proof.* Hyperbolic periodic points are dense in  $H(p)$  and they are in any neighborhood of  $H(p)$ .  $\square$

### 5.3 Hyperbolic splitting

Our next target is to prove that generically  $E \oplus F$  is hyperbolic. For this, we shall prove first that  $E$  is contracting, and this is done following Mañé's proof of [Ma3, Theorem I.4].

**Lemma 5.10.** *Let  $f : M \rightarrow M$  be a  $C^1$  diffeomorphism such that there exist a horseshoe associated to some hyperbolic periodic point  $p$ . Then  $h$ , the topological entropy of  $f|_{H(p)}$ , is positive.*

*Proof.* As  $H(p)$  contains a hyperbolic periodic point with a transverse homoclinic intersection the result follows from [Sm].  $\square$

From Lemma 5.10 and the variational equation for the metric entropy there exists an ergodic measure  $\mu$  in  $H(p)$  such that  $h_\mu > 0$ . Using Ruelle's inequality

$$\sum_{\chi_j > 0} \chi_j \geq h_\mu(f)$$

where  $\chi_1, \dots, \chi_k$  are the Lyapunov exponents associated to  $(f|_{H(p)}, \mu)$  we conclude that there exists a positive Lyapunov exponent  $\mu$ -ae and arguing with  $f^{-1}$  instead of  $f$  a negative Lyapunov exponent  $\mu$ -ae. We have

**Lemma 5.11.** *There exist  $c > 0$  and a dense subset of  $H(p)$  such that for  $x$  in this subset we have*

$$\liminf_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=1}^n \log \|Df/E(f^{-j}(x))\| \leq -c.$$

Lemma 5.11 ensures that there exists a dense subset of points  $x$  in  $H(p)$  where  $\prod_{j=1}^n \|Df/E(f^{-j}(x))\|$  converges to 0 exponentially fast.

Let  $\gamma_0$  be such that  $0 < e^{-c} < \gamma_0 < 1$  where  $c > 0$  is given by Lemma 5.11. It follows that there exists  $D$  dense in  $H(p)$  such that if  $x \in D$  there are infinitely many values of  $n$  satisfying

$$\prod_{j=1}^n \|Df/E(f^j(x))\| < \gamma_0^n.$$

Take  $\gamma_1, \gamma_2, \gamma_3, \gamma_4$  such that

$$0 < \gamma_0 < \gamma_1 < \gamma_2 < \gamma_3 < \gamma_4 < 1.$$

Let  $N' = (N(\gamma_3, \gamma_4))$  given by Lemma 5.1.

**Lemma 5.12.** *If  $E|_{H(p)}$  is not a contracting bundle then for all  $\epsilon > 0$  there exist a compact invariant set  $\Lambda(\epsilon) \subset H(p)$  and  $N = N(\epsilon)$  such that every  $x \in \Lambda(\epsilon)$  has the following property: there exist  $x_0$  arbitrarily near  $x$ ,  $n_0 \geq 0$  and  $y \in \Lambda(\epsilon)$  such that  $\text{dist}(f^{n_0}(x_0), y) < \epsilon$ ,  $(y, f^n(y))$  is an  $(N', \gamma_2)$ -obstruction for all  $n \geq N = N(\epsilon)$  and if  $n_0 > 0$  then  $(x_0, f^{n_0}(x_0))$  is a uniform  $\gamma_4$ -string. Moreover,  $\Lambda(\epsilon)$  is the closure of its interior.*

*Proof.* See [Ma3, Lemmas II.6 and II.7]. □

Fix  $0 < \gamma < \gamma_0$  and let  $\gamma_1, \gamma_2, \gamma_3, \gamma_4$  be as in Lemma 5.12. Choose  $k_0 \in (0, 1)$  such that  $\gamma < k_0^2 \gamma_1$  and  $k_0^{-1} \gamma_4 < 1$ , i.e.,  $1 > k_0 > \max\{\gamma_4, \sqrt{\gamma/\gamma_1}\}$ .

**Proposition 5.13.** *If  $E$  is not contracting, for all  $\epsilon > 0$  there exist sequences  $\{x_i\} \subset \Lambda(\epsilon/4)$ ,  $(\Lambda(\epsilon))$  as in Lemma 5.12) and  $n_i > 0$  such that*

- (a)  $\text{dist}(f^{n_i}(x_i), x_{i+1}) < \epsilon$
- (b)  $(x_i, f^{n_i}(x_i))$  is a uniform  $\gamma_4$ -string but if  $i$  is even  $(x_i, f^{n_i}(x_i))$  is not a  $\gamma_1$ -string.
- (c) If  $K = \min\{\|Df/E(x)\|; x \in H(p)\}$  then  $\gamma_1^{n_i} K^{n_i-1} \geq (k_0 \gamma_1)^{n_i+n_{i+1}}$  for all even  $i$ .
- (d) There exist odd numbers  $k, l$ ;  $k > l$  such that  $\text{dist}(x_l, x_k) < \epsilon/2$ .

*Proof.* See [Ma3, page 179] for the proof of (a), (b), (c). Item (d) follows from compactness. □

**Corollary 5.14.** *For all  $\epsilon > 0$  there exist a sequence  $x_1, x_2, \dots, x_k$  and uniform  $\gamma_4$ -strings  $(x_i, f^{n_i}(x_i))$  which are not  $\gamma_1$ -strings, such that  $\text{dist}(x_i, x_{i+1}) < \epsilon$  and  $\text{dist}(f^{n_k}(x_k), x_1) < \epsilon$  for all  $i = 1, 2, \dots, k-1$ .*

**Lemma 5.15.** *Let us assume that  $(x, f^n(x))$  is a uniform  $\gamma$  string,  $0 < \gamma < 1$ ,  $n > 0$ . Then there exist positive  $\epsilon$  and  $\eta$  such that if  $\text{diam}(f^n(W_c^{cu, \sigma}(x))) \geq 2\eta$  for  $n > n_0$  and  $\text{diam}(f^j(W_c^{cu, \sigma}(x))) < 2\eta$ ;  $j = 0, \dots, n-1$  then  $\text{diam}(f^n(W_c^{cu, \sigma}(y))) \geq \eta$ , for all  $y \in W_c^{cs}(x)$ ,  $\sigma = +, -$ .*

*Proof.* Let  $1 > \gamma_1 > \gamma$ . There exists  $\epsilon > 0$  such that if  $(z, f^n(z))$  is a uniform  $\gamma_1$  string then  $\text{dist}(f^j(z), f^j(y)) \leq \gamma_2^j \epsilon$  for some  $1 > \gamma_2 > \gamma_1$  and for all  $y \in W_\epsilon^{cs}(z)$ . Let  $\epsilon, \eta$  be such that if  $\text{dist}(z, w) \leq \epsilon + 2\eta$  then

$$1 - c < \frac{\|Df|_{E(w)}\|}{\|Df|_{E(z)}\|} < 1 + c \quad (3)$$

and

$$1 - c < \frac{\|Df|_{F(w)}\|}{\|Df|_{F(z)}\|} < 1 + c. \quad (4)$$

We choose  $c > 0$  below and further assume that  $\epsilon < \eta/2$ . Arguing by contradiction assume that for some  $y \in W_\epsilon^{cs}(x)$  and for all  $j = 0, 1, \dots, n$  we have that  $\text{diam}(f^j(W_c^{cu,\sigma}(y))) < 2\eta$  and  $\text{diam}(f^n(W_c^{cu,\sigma}(y))) < \eta$  then  $\text{dist}(f^j(x), f^j(y')) < 2\eta + \epsilon$  for all  $y' \in W_c^{cu,\sigma}(y)$ . Thus there exists  $c > 0$  such that if (3) and (4) hold then  $(y', f^n(y'))$  is a uniform  $\gamma_1$  string and therefore  $\text{dist}(f^j(z'), f^j(y')) \leq \gamma_2^j \epsilon$  for all  $z' \in W_\epsilon^{cs}(y')$ . Moreover by the domination property there is  $\delta > 0$  such that for all  $x' \in W_c^{cu,\sigma}(x)$  there is  $y' \in W_c^{cu,\sigma}(y)$  such that  $x' \in W_\epsilon^{cs}(y')$  if  $\text{dist}(x, y) < \delta$ . Consequently

$$\begin{aligned} \text{dist}(f^n(x), f^n(x')) &\leq \text{dist}(f^n(x), f^n(y)) + \text{dist}(f^n(y), f^n(y')) + \text{dist}(f^n(y'), f^n(x')) < \\ &< \epsilon \gamma_2^n + \eta + \epsilon \gamma_2^n < 2\eta \end{aligned}$$

contradicting the hypothesis.

On the other hand, if there is  $0 < k < n$  such that  $\text{dist}(f^k(y), f^k(y')) > 2\eta$  and  $\text{dist}(f^n(y), f^n(y')) < \eta$ ,  $n \geq n_0$ , then let us assume that  $k$  is the maximum  $j \in [0, n]$  with that property and let us find  $y''$  in the center-unstable arc  $[y, y']$  such that  $\text{dist}(f^k(y), f^k(y'')) = 2\eta$ . Then by the arguments above, interchanging the roles of  $y$  and  $x$ , we conclude that  $\text{dist}(f^k(x), f^k(x')) \geq \eta$ . Let  $\hat{y} \in W_\eta^{cu,\sigma}(f^k(y)) \cap W_\epsilon^{cs}(f^k(x'))$ . Then  $\hat{y} \in [f^k(y), f^k(y'')]$  and we may argue with  $\hat{y}$  as we have done with  $y'$  in the first part of the proof. Thus  $\text{diam}(f^n(W_c^{cu,\sigma}(y))) \geq \eta$  finishing the proof.  $\square$

The following proposition is similar to a lemma proved in [Li] (see also [Ma3]). The proof there uses that  $Df|_F$  is uniformly expanding. We, instead, use that the diameter of the unstable manifolds are uniformly bounded away from zero (Lemma 5.8).

**Proposition 5.16.** *Let  $f$  be a  $C^2$  diffeomorphism. Given  $\delta > 0$ ,  $0 < \gamma < 1$  and  $(x_i, f^{n_i}(x_i))$  a sequence of uniform  $\gamma$ -strings in  $H(p)$ ,  $i = 1, \dots, k$ , then there exist  $\mu = \mu(\gamma, \delta) > 0$  and  $N_0 > 0$  such that if  $\text{dist}(x_i, x_{i+1}) < \mu$  and  $\text{dist}(f^{n_k}(x_k), x_1) < \mu$  for all  $i = 1, 2, \dots, k-1$  and  $n_1 + n_2 + \dots + n_k \geq N_0$  then there exists a periodic*

point  $q$  of  $f$  with period  $N = n_1 + n_2 + \dots + n_k$  such that  $\text{dist}(f^n(q), f^n(x_1)) < \delta$  for all  $0 \leq n \leq n_1$  and setting  $N_i = n_1 + \dots + n_i$   $\text{dist}(f^{N_i+n}(q), f^n(x_{i+1})) < \delta$  for all  $0 \leq n \leq n_{i+1}$ ,  $1 \leq i \leq k-1$ .

*Proof.* Let us first assume  $k = 1$ . Let  $x \in H(p)$  and  $W_\epsilon^{cs}(x)$  be the local invariant manifold given by the dominated splitting which is an embedded disk transverse to  $F$ . It is proved in [PS2] that for any  $\epsilon > 0$  there is  $r = r(\epsilon) > 0$  such that  $\text{size}(W_\epsilon^{cs}(x)) \geq r$ . Moreover, if  $x = x_1$  then the fact that  $Df|_E$  contracts  $n_1$  iterates implies that  $W_\epsilon^{cs}(x_1)$  behaves as a stable manifold for that iterates. That is, there is  $\gamma_1, \gamma < \gamma_1 < 1$  such that for  $j = 0, 1, \dots, n_1$ , if  $y \in W_\epsilon^{cs}(x_1)$  then  $\text{dist}(f^j(x_1), f^j(y)) < \gamma_1^j \epsilon$ . Here  $\alpha/2 > \epsilon > 0$ , where  $\alpha > 0$  is a constant of expansiveness. On the other hand, by Corollary 5.9, it holds that there is  $\eta > 0$  such that  $\text{diam}(W_\epsilon^{u,\sigma}(x_1)) \geq 2\eta$  where  $\sigma = +, -$  indicates anyone of the separatrices of  $W_\epsilon^u(x_1)$ . Let us consider local center-unstable manifolds of the points  $x \in D_1 \subset W_\epsilon^{cs}(x_1)$  where  $D_1$  is a 2-disk centered in  $x_1$ . These center-unstable manifolds are coherent with the local unstable manifolds of points of  $H(p) \cap W_\epsilon^{cs}(x_1)$ . By expansiveness of  $f$  in  $H(p)$  for all  $c_0 > 0$  there exists  $N_0 > 0$  such that  $\text{diam}(f^n(W_c^u(y))) > 2\eta$  for all  $y \in H(p)$ ,  $c \geq c_0$  and  $n \geq N_0$ . Hence, by Lemma 5.15 we have that  $\text{diam}(f^{n_1}(W_c^{cu,\sigma}(x))) \geq \eta$  because  $\text{diam}(f^{n_1}(W_c^{u,\sigma}(x_1))) \geq 2\eta$ .

Let  $\mathcal{B}_1$  be a cylinder centered in  $x_1$  given by  $\mathcal{B}_1 = \cup_{x \in D_1} U_x$  where  $U_x \subset W_\epsilon^{cu}(x)$  has diameter  $\eta$  and is centered in  $x$ . Let  $\mathcal{C}_1 \subset \mathcal{B}_1$  be defined by

$$\mathcal{C}_1 = \cup_{x \in D_1} (W_c^{cu}(x)).$$

Take  $\mu > 0$  small enough such that when  $\text{dist}(f^{n_1}(x_1), x_1) < \mu$  then  $f^{n_1}(W_\epsilon^{cs}(x_1))$  is contained in the interior of  $\mathcal{B}_1$  and moreover  $W_\epsilon^{cu}(x)$  cuts  $W_\epsilon^{cs}(y)$  whenever  $\text{dist}(x, y) < \mu$  with  $x, y \in V(H(p))$ . Hence  $f^{n_1}(\mathcal{C}_1)$  intersects  $\mathcal{C}_1$  and any point in the boundary of  $\mathcal{C}_1$  is not fixed by  $f^{n_1}$ . Therefore, by a standard argument of index theory, see [Do], there exist a fixed point  $q$  of  $f^{n_1}$  in  $\mathcal{C}_1 \cap f^{n_1}(\mathcal{C}_1)$ . That is,  $q$  is a periodic point of  $f$ . Observe that by Lemma 5.8 there is  $z$  in  $W_\epsilon^u(q) \cap f^{n_1}(D_1)$ . By backward iterations the distance between  $f^{-j}(z)$  and  $f^{-j}(q)$  is bounded by  $\epsilon$  while  $\text{dist}(f^{-j}(z), f^{n_1-j}(x_1)) < \text{diam}(\gamma_1^{n_1-j} r)$  where  $\gamma_1^{n_1-j} r = \text{diam}(f^{n_1-j}(D_1))$ . Therefore  $\text{dist}(f^j(q), f^j(x_1)) \leq \epsilon + r$  for all  $j = -1, -2, \dots, -n_1$ . Choosing  $r < \epsilon < \delta/2$  we conclude that the orbit of  $q$   $\delta$ -shadows the orbit of  $x_1$  for all  $j = 1, 2, \dots, n_1$ , proving Proposition 5.16 when  $k = 1$ .

For  $k = 2$  we proceed as follows. Take a small disk  $D_2 \subset W_\epsilon^{cs}(x_2)$  and  $\mathcal{B}_1$  as in the previous case and set  $\mathcal{B}_2 = \cup_{x \in D_2} (W^{cu}(x))$  and  $\mathcal{C}_2 = \cup_{x \in D_2} W_\beta^{cu}(x)$ , where  $\beta$  is such that  $\text{diam}(f^{n_2}(W_\beta^{cu}(x))) > \eta$ . Find  $\mu > 0$  such that  $\text{dist}(f^{n_1}(x_1), x_2) < \mu$  and  $\text{dist}(f^{n_2}(x_2), x_1) < \mu$  imply  $f^{n_1}(D_1) \subset \text{int}(\mathcal{B}_2)$  and  $f^{n_2}(D_2) \subset \text{int}(\mathcal{B}_1)$ , where  $\text{int}(A)$  stands for the interior of  $A$ . Then  $f^{n_2}(f^{n_1}(\mathcal{C}_1) \cap \mathcal{C}_2)$  is a small cylinder that cuts  $\mathcal{C}_1$  and no point of the boundary of  $\mathcal{C}_1 \cap f^{-n_1}(\mathcal{C}_2)$  is fixed by  $f^{n_1+n_2}|_{(\mathcal{C}_1 \cap f^{-n_1}(\mathcal{C}_2))}$ .



Hence, arguing as before, there is a fixed point  $q$  of  $f^{n_1+n_2}$  and reasoning as in the case  $k = 1$  we conclude the proof for  $k = 2$ .

The general case follows by induction.  $\square$

**Lemma 5.17.** *Let  $\gamma_0 < \gamma_1 < \gamma_2 < \gamma_3 < \gamma_4 < 1$  be as in Lemma 5.12. Moreover let  $0 < \gamma < \gamma_0$  and take  $0 < k_0 < 1$  such that  $\gamma < k_0^2 \gamma_1$  and  $k_0^{-1} \gamma_4 < 1$ . Then the periodic point  $q$  given by Lemma 5.16 satisfies*

$$\gamma^N < \prod_{n=1}^N \|Df/E(f^n(q))\| < (k_0^{-1} \gamma_4)^N.$$

*Proof.* The proof is given in detail in [Ma3, Section II], pages 179-181 (the reader should be aware that in Mañé's article our  $\gamma_3$  is denoted  $\bar{\gamma}_2$  and  $\gamma_4$  is denoted by  $\gamma_3$ ).  $\square$

**Remark 5.18.** *The number  $0 < \gamma < 1$  is arbitrary since the choice of  $\gamma_0 \in (e^{-c}, 1)$  is arbitrary. On the other hand, in the construction of [Ma3, section II], it is not pointed out that there exists  $0 < \gamma_5 = k_0^{-1} \gamma_4 < 1$  such that*

$$\prod_{n=1}^N \|Df/E(f^n(q))\| < \gamma_5^N.$$

*But the proof of this fact is given there. To put it in evidence we introduce the rather artificial number  $k_0$  above as it appears in the proof given by Mañé.*

**Proposition 5.19.** *We may construct the periodic point  $q$  of Proposition 5.16 such that it belongs to  $H(p)$ .*

*Proof.* By Proposition 5.17 if  $\delta > 0$  is small enough we ensure that  $Df_q/E(q)$  contracts in a rate similar to that of  $Df_{x_j}/E(x_j)$ ,  $j = 1, 2, \dots, k$  if  $\text{dist}(f^j(q), f^j(x_j)) < \delta$  for all  $j = 0, \dots, N$ , in the local center-stable manifold of  $q$ . More precisely we have

$$\gamma^N < \prod_{n=1}^N \|Df/E(f^n(q))\|.$$

And for all  $j \in [1, N]$ :

$$\prod_{n=1}^j \|Df/E(f^n(q))\| < (k_0^{-1} \gamma_4)^j.$$

As  $q$  is periodic of period  $N$  this implies that its center-stable manifold is a stable manifold, of size about the same of that of the center-stable manifold of  $x_j$ . Hence

the local unstable manifold of a periodic point  $q' \in H(p)$  close to  $x_1$  intersects the local stable manifold of  $q$ . Therefore  $W^u(p)$  accumulates in  $W^u(q)$ . If we have that for some  $j = 1, \dots, k$ ,  $i = 0, \dots, n_j$ ,  $W_\epsilon^{cs}(f^i(x_j))$  is a true stable manifold then we are done. For in that case  $W^s(p)$  and  $W^u(p)$  would accumulate in  $W^s(q)$  and in  $W^u(q)$  respectively and we obtain that  $q \in H(p)$ , concluding the proof of Proposition 5.19.

As we cannot assume that, we have to proceed in a different way. Choose  $V_l(H(p))$ ,  $l \geq 1$ , a sequence of admissible neighborhoods of  $H(p)$  such that

$$\text{Cl}(V_{l+1}(H(p))) \subset V_l(H(p)) \text{ and } \bigcap_{l \geq 1} V_l(H(p)) = H(p).$$

For any  $V_l(H(p))$  we may find  $q_l$  such that  $q_l$  shadows the pseudo-orbit given by a sequence  $(x_j, f^{n_j}(x_j))$ ,  $j = 1, \dots, k_l$  of uniform  $\gamma_4$ -strings that are not  $\gamma$ -strings as in Proposition 5.13. Take an accumulation point  $x$  of  $\{q_l\}$ . If there is a subsequence  $\{q_{l_h}\}$  of  $\{q_l\}$  such that the periods of  $q_{l_h}$  are uniformly bounded then  $x \in H(p) \cap \text{Per}(f)$  and  $x$  is the desired periodic point. Otherwise the periods of  $q_l$  are unbounded and  $x \in H(p)$  is a uniform  $\gamma_4$ -string for all  $n \geq 0$ . Therefore by arguments similar to those used in [PS1, Corollary 3.3] we obtain a stable manifold for  $x$  that the unstable manifold of  $q_l$  will cut for  $l \geq l_0$ . Thus  $q_l$  is homoclinically related with  $p$  completing the proof.  $\square$

#### *Proof of Theorem B*

By Propositions 5.16, 5.19, and Lemma 5.17, if  $E$  is not a contracting sub-bundle there exists  $q \in H(p) \cap \text{Per}(f)$  such that  $DF_q$  contracts in  $E$  rather weakly in the period contradicting Lemma 4.12. Therefore  $E$  is a uniform contracting bundle. This in turn implies, using the arguments of [Ma3, section II], that  $F$  is a uniform expanding bundle. Thus  $E \oplus F$  is a hyperbolic splitting. This implies that the same is true for  $g$   $C^1$ -close to  $f$ . As the set of  $g \in \text{Diff}^2(M)$  are dense in the  $C^1$  neighborhood in which the homoclinic class is robustly expansive and taking into account that Kupka-Smale diffeomorphisms are residual in  $C^r$ -topology for all  $r \geq 1$ , the result follows.  $\square$

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